

Fig. 2. Ratio of $E(v)$ and σ .

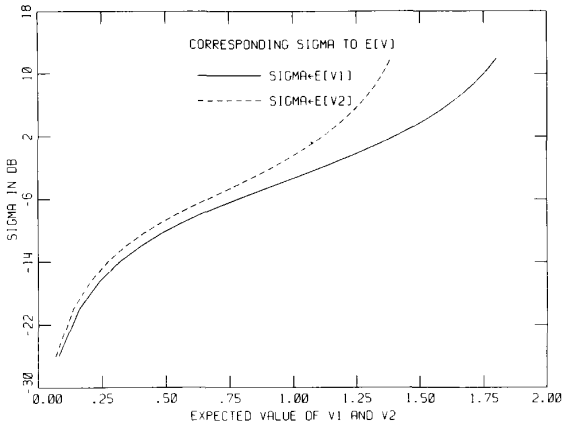


Fig. 3. Expected values of v_1 and v_2 .

A/D output from both channels. Assuming a Gaussian input, we derived the relationship between σ of the input of the A/D converter and $E(|x| + |y|)$ and $E(\max(|x|, |y|) + \frac{1}{2} \min(|x|, |y|))$ for the quantized inphase and quadrature output x and y . Numerical results obtained using the derived expression and the statistical data obtained through simulation show an excellent agreement. Because of its simplicity, the cubic equation obtained by fitting the numerical results should be useful in practical purposes.

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On The Rotation Vector Differential Equation

A direct and simple derivation of the differential equation of the rotation vector is provided. The property that the time derivative of the rotation vector and the angular velocity have equal component along the direction of the rotation vector is also derived.

I. INTRODUCTION

The computation of vehicle attitude is a vital problem in strapdown inertial navigation systems. The rotation/orientation vector concept is one way for improving the accuracy of attitude determination. Bortz [1] has shown that the time derivative of the rotation vector includes two components: the angular velocity vector and the noncommutativity rate vector. In a strapdown inertial system, gyroscopes are used to measure the angular velocity of the body frame with respect to inertial frame. The gyro outputs are fed into the navigation computer to calculate the attitude parameters which can be ideally determined from a simple rotation vector information. Therefore the main difficulty in describing rigid body motions lies in the noncommutativity of finite rotations. Hence, one way to design an efficient attitude updating algorithm is based on the kinematics of the rotation vector. This motivates us to study the properties of the rotation vector.

Bortz's derivation of the rotation vector differential equation is straightforward but rather lengthy. Nazarov [2] presented the result with a much simpler approach via the Euler-Rodrigues parameters which are directly related to the rotation vector. Based upon the quaternion updating equation and transferring from discrete to continuous form, Savage [3] offered another derivation.

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We first show that both the time rate of change of the rotation vector and the angular velocity have equal component along the direction of the rotation vector. This enables us to provide an alternative approach for obtaining the differential equation of the rotation vector readily. No more parameter is needed.

II. PRELIMINARIES

As a consequence of the Euler theorem [4], the rotation vector is the eigenvector of the direction cosine transformation matrix associated with eigenvalue +1, i.e.,

$$(C - I)\phi = 0 \quad (1)$$

where $\phi = [\phi_x, \phi_y, \phi_z]^T$ denotes the rotation vector, C represents the direction cosine matrix, and I is the identity matrix.

The rotation vector is defined by Bortz [1] as a vector whose orientation and magnitude corresponds, respectively, to the axis and angle of the rotation. In terms of the rotation vector, the direction cosine matrix which transforms body frame vectors into the reference frame can be written as [1, 4]

$$C = I + \frac{\sin \phi}{\phi} [\phi \times] + \frac{(1 - \cos \phi)}{\phi^2} [\phi \times]^2 \quad (2)$$

where

$$[\phi \times] = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix} \quad (3)$$

represents the skew symmetric matrix equivalent to the vector cross-product operation and

$$\phi = (\phi^T \phi)^{1/2} \quad (4)$$

denotes the magnitude (angle) of the rotation. It is noticed, from (2), that the matrix $(C - I)$ is singular.

Let $\omega = [\omega_x, \omega_y, \omega_z]^T$ denote the angular velocity of the body frame with respect to the reference frame. Then the time rate of change of the direction cosine matrix is related to the angular velocity via the well-known equation

$$\dot{C} = C[\omega \times] \quad (5)$$

where $[\omega \times]$ is defined in similar manner to (3). From (2) it is seen that the trace of the matrix is

$$\text{tr}(C) = 1 + 2 \cos \phi. \quad (6)$$

Differentiating (6) gives

$$\text{tr}(\dot{C}) = -2 \sin \phi \dot{\phi}. \quad (7)$$

Substitution of (2) into (5), it can be found that

$$\text{tr}(\dot{C}) = -\frac{2 \sin \phi}{\phi} \phi^T \omega. \quad (8)$$

Combine (7) and (8) to yield

$$\dot{\phi} = \frac{\phi^T \omega}{\phi}. \quad (9)$$

Next, differentiating (4) gives

$$\dot{\phi} = \frac{\phi^T \dot{\phi}}{\phi}. \quad (10)$$

Obviously, from (9) and (10), we get the important relationship

$$\phi^T \dot{\phi} = \phi^T \omega \quad (11)$$

which shows that both $\dot{\phi}$ and ω have equal magnitude along the direction of ϕ .

Three basic properties of the matrix $[\phi \times]$ which should be used for the derivation are given below. Using the matrix identity

$$[\phi \times]^2 = \phi \phi^T - \phi^2 I \quad (12)$$

and making use of the fact $[\phi \times] \phi = 0$, we have

$$[\phi \times]^3 = -\phi^2 [\phi \times] \quad (13)$$

and

$$[\phi \times]^4 = -\phi^2 [\phi \times]^2. \quad (14)$$

III. ROTATION VECTOR DIFFERENTIAL EQUATION

Differentiate (1) with respect to time and use (5) to yield

$$(C - I)\dot{\phi} = -C[\omega \times] \phi = C[\phi \times] \omega \quad (15)$$

where the last equality follows from the vector product $\phi \times \omega = -\omega \times \phi$. It is evident that one cannot obtain $\dot{\phi}$ by taking matrix inversion in (15) since $(C - I)$ is singular. However, from (2) and (12), $(C - I)$ can be written as

$$C - I = -(1 - \cos \phi)I + \frac{\sin \phi}{\phi} [\phi \times] + \frac{(1 - \cos \phi)}{\phi^2} \phi \phi^T. \quad (16)$$

Substitution of (16) into (15) gives

$$\left\{ -(1 - \cos \phi)I + \frac{\sin \phi}{\phi} [\phi \times] \right\} \dot{\phi} + \frac{(1 - \cos \phi)}{\phi^2} \phi \phi^T \dot{\phi} = C[\phi \times] \omega. \quad (17)$$

Define

$$D = I - \frac{\sin \phi}{\phi(1 - \cos \phi)} [\phi \times]. \quad (18)$$

Now, introducing (11) and (18) into (17) and dividing both sides by $-(1 - \cos \phi)$, we have

$$D \dot{\phi} = \left\{ \frac{1}{\phi^2} \phi \phi^T - \frac{1}{1 - \cos \phi} C[\phi \times] \right\} \omega. \quad (19)$$

Using (2), (12), and (13) in (19) gives

$$D\dot{\phi} = \left\{ I - \frac{\cos\phi}{1-\cos\phi} [\phi \times] + \frac{1}{\phi^2} \left(1 - \frac{\phi \sin\phi}{1-\cos\phi} \right) [\phi \times]^2 \right\} \omega. \quad (20)$$

The matrix D is equal to the identity matrix minus a skew symmetric matrix. Hence it is always invertible.

$$D^{-1} = I + \frac{\sin\phi}{2\phi} [\phi \times] + \frac{\sin^2\phi}{2\phi^2(1-\cos\phi)} [\phi \times]^2. \quad (21)$$

Finally, premultiplying (20) by (21) and using (13) and (14) yield the desired result:

$$\dot{\phi} = \left\{ I + \frac{1}{2} [\phi \times] + \frac{1}{\phi^2} \left(1 - \frac{\phi \sin\phi}{2(1-\cos\phi)} \right) [\phi \times]^2 \right\} \omega. \quad (22)$$

IV. AN EXAMPLE

The classical coning motion is chosen as example of the application of the rotation vector differential equation. Let the coning rotation vector be

$$\phi = \begin{bmatrix} \theta \sin \Omega t \\ \theta \cos \Omega t \\ 0 \end{bmatrix} \quad (23)$$

where θ denotes the cone half-apex angle and Ω denotes the coning angular frequency. Then (4) shows that

$$\dot{\phi} = \theta \Omega. \quad (24)$$

Substituting (23) and (24) into (22) yields

$$\begin{bmatrix} \theta \Omega \cos \Omega t \\ -\theta \Omega \sin \Omega t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \left(1 - \frac{\theta \sin \theta}{2(1-\cos\theta)} \right) \cos^2 \Omega t & \left(1 - \frac{\theta \sin \theta}{2(1-\cos\theta)} \right) \sin \Omega t \cos \Omega t & \frac{\theta}{2} \cos \Omega t \\ \left(1 - \frac{\theta \sin \theta}{2(1-\cos\theta)} \right) \sin \Omega t \cos \Omega t & 1 - \left(1 - \frac{\theta \sin \theta}{2(1-\cos\theta)} \right) \sin^2 \Omega t & -\frac{\theta}{2} \sin \Omega t \\ -\frac{\theta}{2} \cos \Omega t & \frac{\theta}{2} \sin \Omega t & \frac{\theta \sin \theta}{2(1-\cos\theta)} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}. \quad (25)$$

The angular velocity ω which describes the classical coning motion can be obtained by solving the above equation:

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \Omega \sin \theta \cos \Omega t \\ -\Omega \sin \theta \sin \Omega t \\ \Omega(1-\cos\theta) \end{bmatrix}. \quad (26)$$

In fact, it is easier to obtain ω from (22) directly as

$$\omega = \left\{ I - \frac{1-\cos\phi}{\phi^2} [\phi \times] + \frac{1}{\phi^2} \left(1 - \frac{\sin\phi}{\phi} \right) [\phi \times]^2 \right\} \dot{\phi} \quad (27)$$

$$= \begin{bmatrix} \Omega \sin \theta \cos \Omega t \\ -\Omega \sin \theta \sin \Omega t \\ \Omega(1-\cos\theta) \end{bmatrix}. \quad (28)$$

V. CONCLUSION

The kinematics of the rotation vector is useful in establishing vehicle attitude computational algorithms. We also show that the time derivative of the rotation vector and the angular velocity have equal magnitude along the direction of the rotation vector. The derivation of the rotation vector differential equation is simple and direct.

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