國立交通大學

統計研究所

碩 士 論 文

被測物理量的測量不確定度

Measurement Uncertainty of Measurand

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摘要

被測量物的不確定分析在量測科學上是一個重要的課題。然而，被測量物的意義常被誤解，導致所求得的不確定性區間的意義造成混淆。被測量物的真值應屬於參數，而測量結果應屬於隨機變數。通常，在測量科學上估計參數會比預測變數的未來值有意義的多。古典不確定性分析的理論都是將被測量物定義為隨機變數而發展的，但這會喪失不確定性區間對於真值的準確性。我們在這篇文章中會討論被測量物的統計分析。

Measurement Uncertainty of Measurand

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Abstract

Uncertainty analysis of measurement of measurand is an important topic in metrology. However, vague statistical concept of measurand results in inefficient inference uncertainty for the true measurand. Measurand and the variable representing its measurement are completely different in probability concept; one is an unknown distributional parameter and the other is a random variable. Generally, a parameter may be estimated more efficiently than the prediction of the future observation of a random variable. The classical uncertainty analysis in literature is developed based on the structure that a measurand is a random variable. This misspecification of statistical model costs serious price of sacrificing efficiency in constructing uncertainty interval for gaining the knowledge of the true measurand. We formally formulate a statistical analysis for measurement of measurand.

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Contents

1. Introduction

An experiment for measuring the measurand, the quantity to be measured, is a method through a process that tries to gain or discover knowledge of the measurand. Measurements always have errors and therefore uncertainties. The practice of measurement science has made us realize that the comparisons of measured values require, in addition to the proper value, a statement of the reliability and quality of that value. General rules for evaluating and reporting uncertainty in measurement has been published by the most important and internationally widespread metrological publication-ISO (the International Standards Organization) *Guide to the Expression of Uncertainty in Measurement* (GUM, 1993). According to the GUM, the measurement result should be reported with a specified confidence as an uncertainty interval defining the range of values that could reasonably be attributed to the measurand. Unfortunately, uncertainty analysis of measurement is, in our opinion, totally inappropriate for missing the aim of gaining knowledge of real measurand due to the conceptual understanding of measurand.

Those conflicting views on statistical concepts of interpreting a measurand result in the inappropriateness. That is, a parameter and a random variable are misleadingly interchangeable used to represent the measurand. The efficiencies of predicting an unknown parameter and a random variable are remarkably different. In fact, it is more capable, from point of probability, in prediction of an unknown parameter than it of a future random variable. In GUM B.2.9, the measurand is defined as a particular quantity subject to measurement. On the other hand, GUM 3.3.1 also admits that measurand could have a true value. Baratto (2008) proposed a new but precise and comprehensive definition of measurand guiding that it is a specific quantity that one intends to measure. The terms particularly and specially are to specify that a quantity is existed with restrictive conditions or assumptions. Hence, it is generally accepted that a measurand is an unknown constant to be predicted.

The uncertainty analysis becomes confusing for the fact that although a measurand is known as an unknown parameter, it is measurable. However, in classical statistical models, a parameter in the model is not measurable and is involved in the distribution of a measurable random variable. Hence, it is also generally accepted that the variable in measuring the unknown measurand be termed a measurand. Hence measurand or measurement quantity is used simultaneously to represent a parameter to be predicted and the variable for that its observations are used to predict the unknown measurand.

With the confusion, classically, the study of the uncertainty interval for measurement is based on a statistical model of random measurand. Let Y denotes the random measurand and predicted value $\hat{y}$ computed from observations of its input quantities. ISO GUM proposes pooling estimated variance components for all sources of error with its square root, saying $u_c(y)$ termed the combined uncertainty. It then reports uncertainty interval in the form of expanded uncertainty, U, as

$$Y = \hat{y} \pm U \tag{1.1}$$

where U is termed as

$$U = k_p u_c(y) \tag{1.2}$$

and where k_p is a coverage factor so that this uncertainty interval may cover the distribution of the random measurand Y with a fixed confidence, saying 0.95. Is this type of uncertainty interval appropriate in gaining proper knowledge of uncertainty in prediction of the constant measurand? Or more specific, does an interval covering the random measurand with probability 0.95 also cover the unknown constant measurand with the same probability?

For the common constant measurand, metrologists use different methods of measurement and analysis to define different random measurands. There are reported values and uncertainty intervals constructed from these different random measurands being communicated to other places and other times. These communications are not via the shared understanding or knowledge investigation of the common constant measurand, since each reported value and uncertainty interval is predictions of its corresponding random measurand determined by one method of measurement. The constant measurand should be the truth assumed to be unperturbed by variations in methods and instruments. Hence, in a course of discussion the perception of the method and analysis could differ from metrologist to metrologist, but they must talk about the same thing, the prediction of the measurand. Much information and knowledge must be lost if we use one reported value and uncertainty interval to explain the unknown constant measurand. Why shouldn't the metrologists develop uncertainty interval to interpret the uncertainty of predicted value of the unknown measurand?

Here, in this paper, we want to express the other treatment of the measurand. Since the true value of it is a constant, we are supposed to look on it as a parameter. Then the uncertainty should be analyzed in terms of parameter. Of course, the outcomes are shorter and more meaningful than that of the random variable.

In section 2, we define the statistical model for random and constant measurand, and bring up an example to explain the difference of the uncertainty intervals between these two measurands. In section 3 and 4, we take input qualities into consideration. We define the statistical models for random and constant functional measurand, respectively. Analyzing the uncertainty in different operations, such as addition, subtraction, multiplication and division, we find the variances of the random measurand and MSE's of the constant measurand. In section 5, we carry out four examples. We rewrite the measurement function with constant measurand and find the uncertainty intervals. Besides, we also compare them with the intervals of random measurand method.

2. Uncertainty intervals for Random Measurand and Constant Measurand

For the measurand, a particular quantity to be measured, its true value and the measurement of this true value are conceptually different in statistics. One is an unknown parameter and one is a random variable with a probability distribution and then their statistical inferences are with remarkably different efficiencies. We call the true value as the constant measurand and the measurement as the random measurand. It is supposed that we want to measure the amount of gas in a container. There is an unknown and fixed amount of gas contained in this specific container and it is the constant measurand. When we have made measurements several times with different results, the variable representing the measurement is the random measurand that also represents the amount of gas in this container. However, it is not a fixed number. We define statistical models for these two types of measurand.

In our opinion, the uncertainty analysis of the measurement of the constant measurand is more important for metrologists to analyze, not that of the random measurand. We study the uncertainty intervals for these two targets separately. The simplest experiment of measurement is that we have a random measurand Y and we want to predict it with a random sample $Y_1,...,Y_n$.

Definition 2.1 The statistical model for random measurand includes:

(a) Random measurand: Y with distribution F_y,

(b) Probability model: $Y_1,...,Y_n$ are random sample drawn from distribution F_y.

For a random measurand Y with probability density function $f_y(y)$, the aim in developing uncertainty interval is to search an interval (u,v), nonrandom or random, that satisfies

$$\int_u^v f_y(y)dy = 0.95. \tag{2.1}$$

Unfortunately, the pdf f_y is generally not (completely) known so that a nonrandom uncertainty interval is not available. With the statistical model from random measurand, the observations $y_1,...,y_n$ are used for computing uncertainty interval for random measurand Y. This idea works for computing a random type or even approximate random type uncertainty interval.

The random measurand represents the measurement variable to measure the constant measurand. Next, we consider a model that deals with the true measurand value that has a sample of random measurand for prediction.

Definition 2.2. The statistical model for constant measurand includes:

(a) Constant measurand model: θ_y is an unknown parameter that is measurable,

(b) Probability model: Random variable Y measuring θ_y has distribution F_y,

(c) Sampling model: There are random sample $Y_1,...,Y_n$ drawn from distribution F_y.

The interest of uncertainty interval for constant measurand is to develop random interval $(T_1, T_2) = (t_1(Y_1,...,Y_n), t_2(Y_1,...,Y_n))$ such that

$$0.95 = P_{\theta_y}\{T_1 \le \theta_y \le T_2\}. \tag{2.2}$$

This may be done by the classical statistical inferences of confidence interval.

Example 1. Suppose that there is a pencil on a table and we would like to measure its length. This pencil is the quantity to be measured. Along our definitions, the measured length is a random variable called the random measurand and the true length of this pencil is the constant measurand. It is supposed that we have random variables $Y_1,...,Y_n$ representing n measurements of the constant measurand (true length of the pencil). We also assume that the instrument for measurement reveals that these random variables are independent and identically distributed with normal distribution $N(\mu_y, \sigma^2)$. The best estimate of the random measurand Y is $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, and a 95% confidence interval for Y is $(\bar{Y} - t_{0.025}(n-1)\cdot s, \bar{Y} + t_{0.025}(n-1)\cdot s)$ where S is the sample standard deviation with $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ and $t_{0.025}(n-1)$ is the 0.975 quantile of the t-distribution $T(n-1)$ where $n-1$ is degree of freedom. Suppose that we have a sequence of 5 measurements (mm) as

41.12, 41.08, 41.10, 41.14, 41.06.

These observations are the sample realization of the random measurand. The average of these measurements is $\bar{y}$ = 41.10mm and sample standard deviation is s = 0.032mm. The uncertainty interval for the random measurand is

$$41.10mm \pm 2.776 \times 0.032mm = 41.10mm \pm 0.089mm .$$

This uncertainty interval indicates that the "next" realization of random measurand Y will between 41.011 mm and 41.189 mm with probability 0.95. This is not a direct connection to the true length of the pencil. Isn't it weird?

The constant measurand θ_y represents the true length of the pencil. With normal assumption, $100(1-\alpha)\%$ confidence interval for θ_y is $\bar{y} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$. Hence, a 95% uncertainty interval for the constant measurand θ_y is $\hat{\theta}_y \pm t_{0.025}\frac{s}{\sqrt{n}}$. In this case, it is

$$41.10mm \pm 2.776 \times \frac{0.032}{\sqrt{5}} mm = 41.10mm \pm 0.040mm .$$

The uncertainty intervals of random measurand and constant measurand are with the same center point $\hat{y} = \hat{\theta}_y = 41.10mm$. However, the expanded uncertainty for the constant measurand is 0.040 mm, which is significant smaller than 0.089 mm, the expanded uncertainty for the random measurand. This uncertainty interval indicates that we have 95% confidence with true length of the pencil to be between 41.06 mm and 41. 14 mm.

In this example, it is obviously that the primary interest is the true length of the pencil on desk, not the next measurement of the length. If we study the random measurand, we would stray from the main purpose. Therefore, we should be clear about what we are concerned.

3. Statistical Methods for Random Functional Measurand

The GUM was also developed under the assumption that the random measurand Y can not be measured directly, but is determined from several input (influence) qualities (also random variables) $X_1,...,X_k$ through a known functional relation as

$$Y = h(X_1,...,X_k) \tag{3.1}$$

where variables X_j's are measurements of some other qualities. Any measurement for quantity X_j is subject to errors such as offset of a measuring instrument, drift in its characteristics, and personal bias in reading. This random effect shows the variation in repeated measurements. Hence, this measurement function represents a relationship for measturement variable not only a physical law but also a measurement process.

It is assume that there are results $X_{ji}, i = 1,...,n_j$, a random sample drawn from the distribution of variable X_j, that may be observed during the jth experiment.

What have been done in literature in dealing with random measurand?

Definition 3.1. The statistical model for random functional measurand includes:

(a) Measurement variables for measurand: $Y = h(X_1,...,X_k)$,

(b) Probability model: $X_1,...,X_k$ are input quantities (variables) with joint distribution function $F_{1,...,k}(x_1,...,x_k)$,

(c) Sampling model: For each $j, j = 1,...,k, X_{j1},...,X_{jn_j}$ is a random sample corresponding with random variable X_j.

The classical uncertainty analysis is developed based on this model. Let $\hat{x}_j$ be the prediction estimate, classically it is the sample mean $\bar{x}_j$, of variable X_j from the observations $x_{ji}, i = 1,\ldots,n_j$. The prediction of future random measurand is

$$\hat{y} = h(\hat{x}_1,\ldots,\hat{x}_k). \tag{3.2}$$

This provides a predictor of future observation of the random measurand Y, not an estimate of the unknown true value of the constant measurand. It is not complete to provide a predictor of Y without an indication of precision. This classical way in developing the uncertainty interval of predictor $\hat{y}$ is stated below. Let's denote $Var(X_j) = \sigma_j^2, j = 1,\ldots,k$. In the construction of uncertainty interval for the random measurand Y, it is generally assumed that $\hat{x}_j$ is the expected value of the distribution of input variable X_j, so that $\sigma_j^2 = E[(X_j - \hat{x}_j)^2]$ and this hold for all *j*'s. With predicted value $\hat{y} = h(\hat{x}_1,\ldots,\hat{x}_k)$, the first-order Taylor series approximation to the measurement variable Y about the estimates $(\hat{x}_1,\ldots,\hat{x}_k)$ gives

$$Y \approx \hat{y} + \sum_{j=1}^{k} b_j (X_j - \hat{x}_j) \tag{3.3}$$

where $b_j = \frac{\partial h(x_1,\ldots,x_k)}{\partial X_j}\Big|_{x_h=\hat{x}_h, h=1,\ldots,k}$, called the uncertainty coefficient with respect to influence quantity X_j. The combined standard uncertainty of the random measurand is defined as the square root of its variance, which is approximated as

$$\sigma_y^2 \approx \sum_{j=1}^{k} b_j^2 \sigma_j^2 + \sum_{j \neq l} b_j b_l \sigma_{jl} \tag{3.4}$$

where $\sigma_{jl} = Cov(X_j, X_l)$.

The uncertainty interval is defined as

$$Y = \hat{y} \pm k_p u_c(y)$$

with $u_c(y) = \sigma_y$ and k_p is the coverage factor so that this uncertainty interval may cover the possible values of random measurand Y with a fixed probability, saying $1-\alpha$. Interpreted by Willink (2006), in a potential series of equally reliable independently-determined intervals, this uncertainty interval encloses the value of the random measurand Y, on an average, in $100(1-\alpha)$ out of every 100 measurements.

Uncertainty in Sum and Differences

Suppose that we have the measurement variable as

$$Y = X_1 + \ldots + X_k .$$

The uncertainty of the random measurand is the square root of its variance.

The variance is

$$\sigma_y^2 = Var(X_1 + \ldots + X_k) = \sum_{j=1}^{k} Var(X_j) + \sum_{j \neq l} Cov(X_j, X_l)$$

where $Cov(X_j, X_l) = E[(X_j - \hat{x}_j)(X_l - \hat{x}_l)], j \neq l, j,l = 1,\ldots,k$, representing the covariance of variables.

Now, suppose that we have the measurement variable as

$$Y = X_1 + \ldots + X_k - (Z_1 + \ldots + Z_m).$$

The variance of the random measurand is

$$\begin{aligned} \sigma_y^2 &= \sum_{j=1}^{k} Var(X_j) + \sum_{j=1}^{m} Var(Z_j) + \sum_{j \neq l,\, j,l=1,\ldots,k} Cov(X_j, X_l) \\ &+ \sum_{j \neq l,\, j,l=1,\ldots,m} Cov(Z_j, Z_l) - \sum_{j \neq l,\, j=1,\ldots,k,\, l=1,\ldots,m} Cov(X_j, Z_l) \end{aligned} .$$

Uncertainty in Multiplication and Division

Suppose that we have the measurement variable as

$$Y = X_1 \ldots X_k.$$

The variance of the measurement variable is

$$\sigma_y^2 = \sum_{j=1}^{k} \left(\frac{Y}{X_j} \right)^2 Var(X_j) + \sum_{j \neq l} \left(\frac{Y}{X_j X_l} \right) Cov(X_j, X_l).$$

Suppose that we have the measurement variable as

$$Y = \frac{1}{X_1 \ldots X_k}.$$

Then, we have variance as

$$\sigma_y^2 = \sum_{j=1}^{k} \left(\frac{-Y}{X_j} \right)^2 Var(X_j) + \sum_{j \neq l} \left(\frac{Y}{X_j X_l} \right) Cov(X_j X_l).$$

Suppose that we have the true measurand as

$$Y = \frac{X_1 \ldots X_k}{X_{k+1} \ldots X_w}.$$

Then, we have variance as

$$\begin{aligned}\sigma_y^2 &= \sum_{j=1}^{w} \left(\frac{Y}{X_j} \right)^2 Var(X_j) + \left[\sum_{j \neq l, j, l = 1, \ldots, k} + \sum_{j \neq l, j, l = k+1, \ldots, w} \right] \left(\frac{Y}{X_j X_l} \right) Cov(X_j, X_l) \\ &+ \sum_{j \neq l, j = 1, \ldots, k, l = k+1, \ldots, w} \left(\frac{-Y}{X_j X_l} \right) Cov(X_j, X_l)\end{aligned}.$$

4. Statistical Inferences for Statistical Model for Constant Measurand

For measurement of the unknown measurand, there are input variables $X_1, \ldots, X_k$ having joint distribution with sample space Θ_j for every variable X_j. There exist unknown parameters $\theta_1,...,\theta_k$ in their corresponding parameter spaces Θ_j's, such that the unknown measurand θ_y may be formulated as

$$\theta_y = h(\theta_1,...,\theta_k), \theta_j \in \Theta_j, j = 1,...,k \tag{4.1}$$

where h is a known specified function. Unknown parameter θ_y is the true value of the measurand, which is the target to be estimated. Input variables are measurements for Θ_j's. They are random variables since the measurements are subject to measurement errors. This formulation serves direct way in studying estimation and uncertainty analysis for estimation of the constant measurand.

Definition 4.1 The statistical model for constant functional measurand includes:

(a) Constant measurand model: $\theta_y = h(\theta_1,...,\theta_k), \theta_j \in \Theta_j$, where $\theta_1,...,\theta_k$ are measurable,

(b) Probability model: $X_1,...,X_k$ are input quantities representing the measurement variables, respectively, for parameters $\theta_1,...,\theta_k$ with joint distribution $F_{1,...,k}(x_1,...,x_k)$,

(c) Sampling model: For each j, $j = 1,...,k$, $X_{j1},...,X_{jn_j}$ is a random sample drawn from distribution $F_{1,...,k}(x_1,...,x_k)$.

This statistical model needs to be further clarified as follows:

(1) In the measurand model, there exist true parameter values $\theta_{10},...,\theta_{k0}$ such that the true measurand θ_y is $\theta_{y0} = h(\theta_{10},...,\theta_{k0})$. So, in the statistical inferences, we aim to predict true $\theta_1,...,\theta_k$. If there are efficient inferences for these unknown parameters, then the resulted inferences for measurand θ_y should also be efficient.

(2) There is distinction between parameters in measurand model and the parameter in

classical statistical model. In the classical statistical models, unknown parameter is not measurable; but in this statistical model for the measurand, the parameters $\theta_1,...,\theta_k$ and even θ_y are measurable.

(3) The relationship between $\theta_1,...,\theta_k$ and joint distribution function $F_{1,...,k}(x_1,...,x_k)$ has to be practically investigated and specified.

The statistical model specifies the information for establishing theory of statistical inference procedures for measurand θ_y. Practically there are sample realizations $\{x_{j1},...,x_{jn_j}\}, j=1,...,k$ for drawing inference conclusions. What has been done in literature to deal with θ_y? And what has been done to deal with measurand θ_y by the use of these samples $X_{ji}, i=1,...,n_j; j=1,...,k$?

The inference results depend crucially on the correct relation between parameters $\theta_1,...,\theta_k$ and joint distribution function $F_{1,...,k}(x_1,...,x_k)$. Hence, without involving the relation in the construction of inference techniques, any technique in influences of measurand θ_y, such as those proposed in literature, based on realization of random samples may provides very biased conclusions.

Let $\hat{\theta}_j = \hat{\theta}_j\left(x_{j1},...,x_{jn_j}\right)$ be appropriate estimate of θ_j based on observations $\{x_{ji}, i=1,...,n_j\}$, for $j=1,...,k$. The point estimate of the unknown measurand θ_y may then be constructed by estimate of θ_j's.

Definition 4.2 By letting $\hat{\theta}_j = \hat{\theta}_j\left(X_{j1},...,X_{jn_j}\right), j=1,...,k$, the point estimator of θ_y is defined as

$$\hat{\theta}_y = h(\hat{\theta}_1, \ldots, \hat{\theta}_k). \tag{4.1}$$

If we replace $\hat{\theta}_j = \hat{\theta}_j(X_{j1}, \ldots, X_{jn_j})$ by its sample realization $\hat{\theta}_j = \hat{\theta}_j(x_{j1}, \ldots, x_{jn_j})$, then this $\hat{\theta}_y$ is called the point estimate of measurand θ_y.

Two comments are needed to clarify the measurand estimator:

(a) The estimator $\hat{\theta}_y$ is for estimation of parameter θ_y. On the other hand, the classical estimator $h(\hat{X}_1, \ldots, \hat{X}_k)$ represents a prediction of future variable Y. Suppose that we let $\hat{X}_j = \hat{\theta}_j$ for all j's. Then estimate $\hat{\theta}_y$ and predictor $\hat{y}$ are identical. However, their roles are different, one is predicting a future observation and one is estimated value of a parameter and then their uncertainty intervals are also different.

(b) In the attempt of gaining knowledge of the true measurand θ_y, an estimate $\hat{\theta}_y$ is incomplete without an indicator of its precision since there is no confidence that we can say for $\hat{\theta}_y$ to be equal to θ_y. Uncertainty interval for estimate of θ_y is an appropriate choice to explain the confidence of possible values of θ_y.

The mean square error (MSE) of estimator $\hat{\theta}_y$ is

$$MSE_{\theta_y} = E\left[\left(\hat{\theta}_y - \theta_y\right)^2\right]. \tag{4.2}$$

In this constant measurand statistical model, the uncertainty in the result of constant measurand estimator $\hat{\theta}_y$ is an estimate of the MSE of $\hat{\theta}_y$. Whenever $\hat{\theta}_y$ is unbiased for true value θ_y of the measurand, the MSE of $\hat{\theta}_y$ is equal to the variance of $\hat{\theta}_y$ as $MSE_{\theta_y} = \sigma^2_{\theta_y}$. Under statistical model for measurand, the first order Taylor's expansion for measurand function h on estimators $(\hat{\theta}_1, \ldots, \hat{\theta}_k)$ yields

$$\hat{\theta}_y = \theta_y + \sum_{j=1}^{k} c_j \left(\hat{\theta}_j - \theta_j\right) + R_{\hat{\theta}_y} \tag{4.3}$$

where we let $c_j = \frac{\partial h\left(\hat{\theta}_1,\ldots,\hat{\theta}_k\right)}{\partial \hat{\theta}_j}\Big|_{\hat{\theta}_i=\theta_i, i=1,\ldots,k}$, the sensitivity coefficient with respect to influence parameter θ_j, and $R_{\hat{\theta}_y}$ is the remainder expressed by

$$\begin{aligned} R_{\hat{\theta}_y} = \frac{1}{2}\Bigg[& \sum_{j=1}^{k} \frac{\partial^2 h\left(\hat{\theta}_1,\ldots,\hat{\theta}_k\right)}{\partial \hat{\theta}_j^2}\Big|_{\hat{\theta}_i=\theta_i+\delta(\hat{\theta}_i-\theta_i)} \left(\hat{\theta}_j - \theta_j\right)^2 \\ & + \sum_{j\neq l} \frac{\partial^2 h\left(\hat{\theta}_1,\ldots,\hat{\theta}_k\right)}{\partial \hat{\theta}_j \partial \hat{\theta}_l}\Big|_{\hat{\theta}_i=\theta_i+\delta(\hat{\theta}_i-\theta_i)} \left(\hat{\theta}_j - \theta_j\right)\left(\hat{\theta}_l - \theta_l\right)\Bigg] \end{aligned}$$

with $0 < \delta < 1$. When $\hat{\theta}_i \to \theta_i$, for all i, the remainder term approaches zero more quickly than the first terms in (4.3) and all the higher terms are generally neglected, provided that the uncertainties in $\hat{\theta}_i$'s are small and $\hat{\theta}_i$'s are, respectively, close to θ_i's.

The MSE can be substituted into equation (4.3) to yield

$$\begin{aligned} MSE_{\theta_y} &\approx E\left\{\left[\sum_{j=1}^{k} c_j\left(\hat{\theta}_j - \theta_j\right)\right]^2\right\} \\ &= E\left\{\sum_{j=1}^{k} c_j^2\left(\hat{\theta}_j - \theta_j\right)^2 + \sum_{j\neq l} c_j c_l \left(\hat{\theta}_j - \theta_j\right)\left(\hat{\theta}_l - \theta_l\right)\right\} \\ &= \sum_{j=1}^{k} c_j^2 MSE_{\theta_j} + \sum_{j\neq l} c_j c_l MSE_{\theta_{jl}} \end{aligned} \tag{4.4}$$

where $MSE_{\theta_j} = E\left[\left(\hat{\theta}_j - \theta_j\right)^2\right]$ and, $MSE_{\theta_{jl}} = E[(\hat{\theta}_j - \theta_j)(\hat{\theta}_l - \theta_l)]$ called the co-mean square error between estimates θ_j and θ_l which is not necessary to be nonnegative. This formulation introduces the uncertainty of estimate $\hat{\theta}_y$, as a linear combination of

MSE's and co-MSE's associated with parameter estimates $\hat{\theta}_j$'s. The expected values are considered to be the best estimate for parameters.

There are several comments on this decomposition of MSE of estimator of constant measurand:

(a) MSE_{θ_j} is the MSE of the constant measurand estimate contributed by the estimate of parameter θ_j.

(b) $MSE_{\theta_{jl}}$ is the co-mean square error associated with estimates of parameters θ_j and θ_l, and that contributes the MSE of the constant measurand estimate.

(c) From (4.4), the standard uncertainty of the estimate, $\hat{\theta}_y$, of the measurand that is attributed to the input quantity parameter estimates is a function of the estimated MSE_{θ_j}'s and their estimates of $MSE_{\theta_{jl}}$'s.

Uncertainty in Sum and Differences

Suppose that we have the true meaurand as

$$\theta_y = \theta_1 + ... + \theta_k .$$

We also have their estimator as $\hat{\theta}_j, j = 1,...,k$. We then have

$$MSE_{\theta_y} = \sum_{j=1}^{k} MSE_{\theta_j} + \sum_{j \neq l} B_{\theta_j} B_{\theta_l}$$

where $B_{\theta_j} = E[\hat{\theta}_j - \theta_j], j = 1,...,k$, representing the bias of estimators $\hat{\theta}_j$'s.

Now, if we have true measurand as

$$\theta_y = \theta_1 + ... + \theta_k - (\delta_1 + ... + \delta_m)$$

the MSE of estimator of true measurand is

$$MSE_{\theta_y} = \sum_{j=1}^{k} MSE_{\theta_j} + \sum_{j=1}^{m} MSE_{\delta_j} + \sum_{j \neq l} B_{\theta_j} B_{\theta_l} + \sum_{j \neq l} B_{\delta_j} B_{\delta_l} - \sum_{j \neq l} B_{\theta_j} B_{\delta_l}.$$

Uncertainty in Multiplication and Division

Suppose that we have the true measurand as

$$\theta_y = \theta_1 \ldots \theta_k .$$

Then, we have

$$MSE_{\theta_y} = \sum_{j=1}^{k} \left(\frac{\theta_y}{\theta_j} \right)^2 MSE_{\theta_j} + \sum_{j \neq l} \frac{\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l} .$$

Suppose that we have the true measurand as

$$\theta_y = \frac{1}{\theta_1 \ldots \theta_k} .$$

Then, we have

$$MSE_{\theta_y} = \sum_{j=1}^{k} \left(\frac{\theta_y}{\theta_j} \right)^2 MSE_{\theta_j} + \sum_{j \neq l} \frac{\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l} .$$

Suppose that we have the true measurand as

$$\theta_y = \frac{\theta_1 \ldots \theta_k}{\theta_{k+1} \ldots \theta_w} .$$

Then, we have

$$MSE_{\theta_y} = \sum_{j=1}^{w} \left(\frac{\theta_y}{\theta_j} \right)^2 MSE_{\theta_j} + \left[\sum_{j \neq l,\, j,l=1,\ldots,k} + \sum_{j \neq l,\, j,l=k+1,\ldots,w} \right] \frac{\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l} + \sum_{j \neq l,\, j=1,\ldots,k,\, l=k+1,\ldots,w} \frac{-\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l}.$$

Here we list a table to compare the constant and random functional measurands.

		MSE of the constant measurand	Variance of the random variable
Uncertainty in Sum and Differences	$\theta_y = \theta_1 + \ldots + \theta_k$	$MSE_{\theta_y} = \sum_{j=1}^{k} MSE_{\theta_j} + \sum_{j \neq l} B_{\theta_j} B_{\theta_l}$	$\sigma_y^2 = \sum_{j=1}^{k} Var(X_j) + \sum_{j \neq l} Cov(X_j, X_l)$
	$\theta_y = \theta_1 + \ldots + \theta_k - (\delta_1 + \ldots + \delta_m)$	$MSE_{\theta_y} = \sum_{j=1}^{k} MSE_{\theta_j} + \sum_{j=1}^{m} MSE_{\delta_j} + \sum_{j \neq l} B_{\theta_j} B_{\theta_l} + \sum_{j \neq l} B_{\delta_j} B_{\delta_l} - \sum_{j \neq l} B_{\theta_j} B_{\delta_l}$	$\sigma_y^2 = \sum_{j=1}^{k} Var(X_j) + \sum_{j=1}^{m} Var(Z_j) + \sum_{j \neq l, j,l=1,\ldots,k} Cov(X_j, X_l) + \sum_{j \neq l, j,l=1,\ldots,m} Cov(Z_j, Z_l) - \sum_{j \neq l, j=1,\ldots,k, l=1,\ldots,m} Cov(X_j, Z_l)$
Uncertainty in Multiplication and Division	$\theta_y = \theta_1 \ldots \theta_k$	$MSE_{\theta_y} = \sum_{j=1}^{k} \left(\frac{\theta_y}{\theta_j} \right)^2 MSE_{\theta_j} + \sum_{j \neq l} \frac{\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l}$	$\sigma_y^2 = \sum_{j=1}^{k} \left(\frac{Y}{X_j} \right)^2 Var(X_j) + \sum_{j \neq l} \left(\frac{Y}{X_j X_l} \right) Cov(X_j, X_l)$
	$\theta_y = \frac{1}{\theta_1 \ldots \theta_k}$	$MSE_{\theta_y} = \sum_{j=1}^{k} \left(\frac{\theta_y}{\theta_j} \right)^2 MSE_{\theta_j} + \sum_{j \neq l} \frac{\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l}$	$\sigma_y^2 = \sum_{j=1}^{k} \left(\frac{-Y}{X_j} \right)^2 Var(X_j) + \sum_{j \neq l} \left(\frac{Y}{X_j X_l} \right) Cov(X_j X_l)$
	$\theta_y = \frac{\theta_1 \ldots \theta_k}{\theta_{k+1} \ldots \theta_w}$	$MSE_{\theta_y} = \sum_{j=1}^{w} \left(\frac{\theta_y}{\theta_j} \right)^2 MSE_{\theta_j} + \left[\sum_{j \neq l, j,l=1,\ldots,k} + \sum_{j \neq l, j,l=k+1,\ldots,w} \right] \frac{\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l} + \sum_{j \neq l, j=1,\ldots,k, l=k+1,\ldots,w} \frac{-\theta_y}{\theta_j \theta_l} B_{\theta_j} B_{\theta_l}$	$\sigma_y^2 = \sum_{j=1}^{w} \left(\frac{Y}{X_j} \right)^2 Var(X_j) + \left[\sum_{j \neq l, j,l=1,\ldots,k} + \sum_{j \neq l, j,l=k+1,\ldots,w} \right] \left(\frac{Y}{X_j X_l} \right) Cov(X_j, X_l) + \sum_{j \neq l, j=1,\ldots,k, l=k+1,\ldots,w} \left(\frac{-Y}{X_j X_l} \right) Cov(X_j, X_l)$

5. Example

The measurement of Y is generally assumed to follow normal or t distribution in GUM. However, Sim and Lim (2008) claim that Y actually follows asymmetric distribution, not just normal or t distribution. The random measurand can be stated as $Y \approx \hat{y} + \sum_{j=1}^{k} b_j \left(X_j - \hat{x}_j \right) = \hat{y} + \sum_{j=1}^{k} b_j \left(X_j - \overline{x_j} \right)$, and $\overline{x_j}$ is always regarded as the true value of X_j. The distribution of $Y-\hat{y}=\sum_{j=1}^{k} b_j\left(X_j-\overline{x_j}\right)$ is not always normal, if the distribution of X_j may not be normal. Hence, Sim and Lim consider other distributions and the coverage factors corresponding to different distribution. The uncertainty interval is $Y=y\pm k_p u_c(Y)$. The coverage factor, k_p, can be obtained by the coefficient of skewness and kurtosis in these distributions. Meanwhile, the central moments of random variable are involved to identify those two coefficients. So, Sim and Lim focus on coverage factors, corresponding to four asymmetric distributions: the Pearson family of distribution, the Tukey's lambda- distribution, the Tukey's gh-distribution, and the GS-distribution.

Here, we focus on the difference between parameter and variable, which lead to different standard uncertainties and uncertainty intervals. In the following examples, we'll express the measurands as the constant ones, if they are actually constants, and express the constant measurand function by first-order Taylor expansion. We assume that the estimates are unbiased for true value of measurand. Therefore, with MSE of the estimate of the measurands, we can obtain a 95% uncertainty interval for the constant measurand.

Example 1.

The first example is drawn from Sim and Lim (2008), and Willink (2006). The measurand Y is the velocity of a type of wave in some medium and measurement function is

$$Y=\frac{X_1}{X_2}$$

where X_1 is the distance from a transmitter to a receiver and X_2 is the time of flight. However, this measurand, velocity of a wave, is a particular quantity to be measured. The distance between a transmitter and a receiver should be fixed, and it is a constant to be measured. Besides, the time of the flight of the wave is a constant as well. Therefore, both can be expressed as parameter. We use θ_1 and θ_2 to replace X_1 and X_2 respectively, and θ_y replace Y. So, the measurement function is

$$\theta_y = \frac{\theta_1}{\theta_2} .$$

The first-order Taylor expansion of measurand function is

$$\hat{\theta}_y = \frac{\hat{\theta}_1}{\hat{\theta}_2} = \frac{\theta_1}{\theta_2} + \frac{1}{\theta_2}\left(\hat{\theta}_1 - \theta_1\right) - \frac{\theta_1}{\theta_2^{\,2}}\left(\hat{\theta}_2 - \theta_2\right).$$

It is supposed that $\hat{\theta}_1 - \theta_1$ is distributed as $-2\times 10^{-4}\theta_1 V_1$, where V_1 follows a chi-square distribution; $\hat{\theta}_2 - \theta_2$ is distributed as $U_2 V_2$, where U_2 follows a uniform distribution $U[4.5\times 10^{-6}, 5.5\times 10^{-6}]$ and V_2 follows an exponential distribution with mean as 1. θ_1 is taken as 4.931 mm, and θ_2 as 10.9×10^{-3} s.

The expected value is

$$E(\hat{\theta}_y) = \frac{\theta_1}{\theta_2} + \frac{1}{\theta_2} E\left(\hat{\theta}_1 - \theta_1\right) - \frac{\theta_1}{\theta_2^{\,2}} E\left(\hat{\theta}_2 - \theta_2\right)$$

while

$$E\left(\hat{\theta}_1 - \theta_1\right) = -2\times 10^{-4}\theta_1 \cdot E(V_1) = -2\times 10^{-4}\theta_1 \cdot 1$$

$$E\left(\hat{\theta}_2 - \theta_2\right) = E(U_2)\cdot E(V_2) = \left(\frac{4.5\times 10^{-6} + 5.5\times 10^{-6}}{2}\right)\cdot 1 .$$

Hence, $E(\hat{\theta}_y) = 452.087$ which is close to $\frac{\theta_1}{\theta_2} = 452.385$.

The MSE of the estimate of measurand is

$$MSE_{\theta_y} = \frac{1}{\theta_2^{\,2}} MSE_{\theta_1} + \left(\frac{\theta_1}{\theta_2^{\,2}} \right)^2 MSE_{\theta_2} .$$

The MSE's are:

$$MSE_{\theta_1} = \frac{\sigma^2_{\hat{\theta}_1}}{n} = \frac{2}{n} \cdot (-2 \times 10^{-4} \theta_1)^2 .$$

$$MSE_{\theta_2} = \frac{\sigma^2_{\hat{\theta}_2}}{n} = \frac{1}{n} \left(\frac{5\alpha^2 + 5\beta^2 + 2\alpha\beta}{12} \right), \text{ where } \alpha = 4.5 \times 10^{-6}, \beta = 5.5 \times 10^{-6}.$$

Hence, a 95% uncertainty interval for the constant measurand θ_y is

$$\left(\frac{\hat{\theta}_1}{\hat{\theta}_2} - z_{0.025} \sqrt{MSE_{\theta_y}}, \frac{\hat{\theta}_1}{\hat{\theta}_2} + z_{0.025} \sqrt{MSE_{\theta_y}} \right).$$

Since sample size is needed in the uncertainty interval, we assume that n = 1, 5, 10, 30 and 50. The uncertainty interval of the method for constant measurand is named as U_M; that of the method for random measurand is named as U_{SL} . In the table below is the comparison of U_{SL} from Sim and Lim (2008) and U_M corresponding to different sample size.

Methods for uncertainty interval for random measurand	U_{SL}, 95% uncertainty interval (n is not involved)	Methods for uncertainty interval for constant measurand	U_M, 95% uncertainty interval (n is involved)
GUF (GUM uncertainty framework)	(451.91, 452.86)	n=1	(451.9063, 452.8643)
Pearson family of distribution	(451.96, 452.93)	n=5	(452.1711, 452.5995)
Tukey's lambda-distribution	(451.95, 452.93)	n=10	(452.2339, 452.5368)

Tukey's gh-distribution	(451.96, 452.93)	n=30	(452.2979, 452.4728)
GS-distribution (Generalized S-distribution)	(451.95, 452.93)	n=50	(452. 3176, 452.4531)
MCM (Monte Carlo method)	(452.06, 453.09)		

In left columns shows few different methods of uncertainty intervals for random measurand. Besides GUM framework, Sim and Lim have tried five other methods. There are only slight differences between uncertainty intervals from five other methods. In the right columns shows the uncertainty interval for constant measurand. We have some finding as below. Firstly, when rounding to hundredth digit, U_M, with n as 1, is the same as U_{SL} of GUF. This clearly indicates that, the sample size is not taken into consideration in Sim and Lim's method. Secondly, in our method, the 95% uncertainty interval for constant measurand θ_y is shorter than the results of Sim and Lim.

Example 2.

Another example is from Sim and Lim (2008) and Willink (2006). The measurand Y is an intensity measured by a circular sensor from a simple measurement function

$$Y = \frac{\exp(X_1)}{X_2^{\ 2}}$$

where X_1 is the ideal reading of the sensor which follows a normal N (1.044, 0.0225) distribution and X_2 is the diameter of the sensor that follows a uniform U (0.57, 0.61) distribution. However, the "ideal" reading of the sensor should be the most accurate one, and it should be a constant. The diameter of the sensor is fixed, and it's a constant as well. Since the reading of the sensor and the diameter of the sensor are particular quantities that can be measured. So, we use θ_1 and θ_2 to replace X_1 and X_2 respectively, and θ_y to replace Y. So, the measurement function is

$$\theta_y = \frac{\exp(\theta_1)}{\theta_2^{\ 2}}.$$

The first-order Taylor expansion of measurand function is

$$\frac{e^{\hat{\theta}_1}}{\hat{\theta}_2^{\ 2}} = \frac{e^{\theta_1}}{\theta_2^{\ 2}} + \frac{e^{\theta_1}}{\theta_2^{\ 2}}\left(\hat{\theta}_1 - \theta_1\right) - 2\frac{e^{\theta_1}}{\theta_2^{\ 3}}\left(\hat{\theta}_2 - \theta_2\right).$$

The MSE of the estimate of measurand is

$$MSE_{\theta_y} = \left(\frac{e^{\theta_1}}{\theta_2^{\ 2}}\right)^2 MSE_{\theta_1} + \left(2\frac{e^{\theta_1}}{\theta_2^{\ 3}}\right)^2 MSE_{\theta_2}$$

where $MSE_{\theta_1} = \dfrac{\sigma_{\theta_1}^2}{n}$ and $MSE_{\theta_2} = \dfrac{\sigma_{\theta_2}^2}{n}$. We can obtain $\sigma_{\theta_1}^2$ =0.0225, and $\sigma_{\theta_2}^2 = \dfrac{(0.61-0.57)^2}{12}$ from the given distribution. The question doesn't indicate the value for θ_1 and θ_2. So, we assume they are unbiased, and estimate them with their expected value. We take $\hat{\theta}_1$ as $E(\theta_1) = 1.044$ and $\hat{\theta}_2$ as $E(\theta_2) = \dfrac{1.044 + 0.0225}{2} = 0.59$.

Hence, a 95% uncertainty interval for the constant measurand θ_y is

$$\left(\frac{e^{\hat{\theta}_1}}{\hat{\theta}_2^{\ 2}} - z_{0.025}\sqrt{MSE_{\theta_y}}, \frac{e^{\hat{\theta}_1}}{\hat{\theta}_2^{\ 2}} + z_{0.025}\sqrt{MSE_{\theta_y}}\right).$$

The 95% uncertainty interval for the measurand, [6.001, 11.025], is obtained from Wiilink(2006). In the table below is the comparison of U_{SL} from Sim and Lim (2008) and U_M corresponding to different sample size n = 1, 5, 10, 30 and 50

Methods for uncertainty interval for random measurand	U_{SL}, 95% uncertainty interval (n is not involved)	Methods for uncertainty interval for constant measurand	U_M, 95% uncertainty interval (n is involved)
GUF (GUM uncertainty framework)	(5.7143, 10.7437)	n=1	(5.680745, 10.63960)
Pearson family of distribution	(6.0020, 11.0242)	n=5	(7.05134, 9.269008)
Tukey's lambda-distribution	(5.9948, 11.0111)	n=10	(7.37611, 8.944238)
Tukey's gh-distribution	(6.0016, 11.0164)	n=30	(7.707494, 8.612854)
GS-distribution (Generalized S-distribution)	(5.9835, 11.0143)	n=50	(7.80953, 8.510818)
MCM (Monte Carlo method)	(5.9980, 11.0210)		

In the left columns are uncertainty intervals of random variables. Among them, U_{SL} of Tukey's lambda-distribution and U_{SL} of Pearson family are shorter than others. In the right columns is the uncertainty interval of constant measurand method. Obviously, the 95% uncertainty interval for constant measurand θ_y is shorter than the results from Sim and Lim (2008) and Willink (2006). Since we can obtain shorter and meaningful uncertainty interval for measurand, why not try this method?

Example 3.

An example comes from Semyon Rabinovich (1995). The current I, generated by γ rays from standards of unit radium mass, is defined by the expression $I = \frac{CU}{\tau}$. C is the capacitance of the capacitor which helps the ionization current compensate, U is the initial voltage on the capacitor, and τ is the compensation time (seconds). The measurement is performed by making 27 repeated observations in Semyon Rabinovich (1995), with certain fixed conditions: C =4006.3 pF and U=7V is established, and compensation time is measured. Under same circumstances, the compensation time should be fixed, and it should be a constant. The current, the measurand, generated by γ rays should be constant as well. So, we replace τ with θ_τ, and I with θ_I. The formula of the current is expressed as $\theta_I = \frac{CU}{\theta_\tau}$.

The first-order Taylor expansion of measurand function is

$$\hat{\theta}_I = \frac{CU}{\hat{\theta}_\tau} = \frac{CU}{\theta_\tau} + \left(-\frac{CU}{\theta_\tau{}^2}\right)\left(\hat{\theta}_\tau - \theta_\tau\right).$$

The MSE of the estimated measurand is

$$MSE_{\theta_I} = \left(-\frac{CU}{\theta_\tau{}^2}\right)^2 MSE_{\theta_\tau}$$

where $MSE_{\theta_\tau} = \frac{\sigma^2_{\theta_\tau}}{n} = 0.0003904189 s^2$, $\sigma^2_{\theta_\tau}$ =0.01054131 s^2, and $\overline{\tau} = 74.41481$ s as best estimate for θ_τ. So, $MSE_{\theta_I} = 0.01001328 \times 10^{-24} A^2$.

Therefore, we can obtain a 95% uncertainty interval for the constant measurand θ_I:

$$\left(\frac{CU}{\hat{\theta}_\tau} - z_{0.025}\sqrt{MSE_{\theta_I}}, \frac{CU}{\hat{\theta}_\tau} + z_{0.025}\sqrt{MSE_{\theta_I}}\right) = \left(376.6657\times10^{-12}, 377.0579\times10^{-12}\right)A$$

$$= \left(3.766657\times10^{-10}, 3.770579\times10^{-10}\right)A.$$

The 95% uncertainty interval from Semyon Rabinovich (1995) is

$$I = (3.761 \pm 0.009)\times10^{-10} = \left(3.752\times10^{-10}, 3.77\times10^{-10}\right)A.$$

Example 4.

An example from Semyon Rabinovich (1995). The density of a solid body is measured by $\rho = \frac{m}{V}$, where m is the mass of the body and V is the volume of the body. Under the same circumstances, the mass and the volume of the solid body wouldn't be changed no matter how many times we measured. They are both particular quantities to be measured, so they are constant measurand. We should replace m with θ_m, v with θ_v and ρ with θ_ρ, in order to show them in terms of parameter. Then we express the formula as $\theta_\rho = \frac{\theta_m}{\theta_v}$. Please see the 11 repeated measurements in Semyon Rabinovich (1995). We have $\overline{\theta_m} = 252.9120\times10^{-3}kg$, and $\overline{\theta_v} = 195.3798\times10^{-6}m^3$.

The first-order Taylor expansion of the measurand function is

$$\hat{\theta}_\rho = \frac{\hat{\theta}_m}{\hat{\theta}_v} = \frac{\theta_m}{\theta_v} + \left(\frac{1}{\theta_v}\right)\left(\hat{\theta}_m - \theta_m\right) + \left(\frac{-\theta_m}{\theta_v^{\,2}}\right)\left(\hat{\theta}_v - \theta_v\right).$$

The MSE of the estimate of the measurand θ_ρ is

$$MSE_{\theta_\rho} = \left(\frac{\partial\theta_\rho}{\partial\theta_m}\right)^2 MSE_{\theta_m} + \left(\frac{\partial\theta_\rho}{\partial\theta_v}\right)^2 MSE_{\theta_v}$$

$$= \left(\frac{1}{\theta_v}\right)^2 MSE_{\theta_m} + \left(-\frac{\theta_m}{\theta_v^{\,2}}\right)^2 MSE_{\theta_v}$$

where $\overline{\theta_m}$ and $\overline{\theta_v}$ are regarded as the best estimate of θ_m and θ_v respectively.

$MSE_{\theta_m} = \dfrac{\sigma^2_{\theta_m}}{n}$, $\sigma^2_{\theta_m} = 2.130545 \times 10^{-12} kg^2$ and $MSE_{\theta_v} = \dfrac{\sigma^2_{\theta_v}}{n}$,

$\sigma^2_{\theta_v} = 1.802727 \times 10^{-18} m^6$, and $\hat{\theta}_\rho = \dfrac{\hat{\theta}_m}{\hat{\theta}_v} = 1.294463 \times 10^3 \, {kg}/{m^3}$.

Therefore, a 95% uncertainty interval for the constant measurand θ_ρ is

$$\left(\frac{\hat{\theta}_m}{\hat{\theta}_v} - z_{0.025}\sqrt{MSE_{\theta_\rho}}, \frac{\hat{\theta}_m}{\hat{\theta}_v} + z_{0.025}\sqrt{MSE_{\theta_\rho}}\right) = (1.294456 \times 10^3, 1.294470 \times 10^3) \, \mathrm{kg}/\mathrm{m}^3 .$$

In our method, expanded uncertainty is $z_{0.025}\sqrt{MSE_{\theta_\rho}} = 0.007 = 7 \times 10^{-3} \, {kg}/{m^3}$.

Now, let's compare the result with those in Semyon Rabinovich (1995). In this case, Semyon Rabinovich tried two different ways to find the expanded uncertainty. One is under student t distribution with degree of freedom as 10: the relative error is $6.0 \times 10^{-4}\%$, which means the expanded uncertainty is

$$6.0 \times 10^{-4}\% \times 1.294463 \times 10^3 = 7.76678 \times 10^{-3} \, {kg}/{m^3} .$$

The other is under student t distribution with the effective estimate of the degree of freedom, $v_{eff} = 19$: the relative error is $5.7 \times 10^{-4}\%$, which means the expanded uncertainty is

$$5.7 \times 10^{-4}\% \times 1.294463 \times 10^3 = 7.37844 \times 10^{-3} \, {kg}/{m^3} .$$

With the method of constant measurand, the expanded uncertainty 7×10^{-3} is smaller than the other two. So, the uncertainty interval of the measurand is shorter. With this method, we can provide shorter and more meaningful interval. Why don't we use it?

6. Conclusion

When the measurand is an unknown, but measurable constant, it should be regarded as a parameter. We consider the true value of the measurand. With first-Taylor expansion and Central limit theorem, we have MSE's of the input quantities, and obtain an uncertainty interval for the constant measurand, which covers the true value of the measurand.

In this paper, we clarify the misinterpretation of the constant and variable measurand. Before analyzing uncertainty, we should know what we are concerned? The prediction of the variable or the estimation of the parameter? If we are dealing with an unknown parameter, then we have a more efficient way to analyze the true value. In our method, the uncertainty interval of the true value is shorter than that of the random measurand method.

However, criticizing others is not what we want to do; we just have different idea about the treatment of the measurand. We just provide a meaningful and thoughtful concept about the measurand. Hope this will be helpful to get more precise knowledge about the quantity to be measured.

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