國立交通大學

物理研究所

碩 士 論 文

旋轉玻色－愛因思坦凝結中的巨形漩渦的

基態與激發能階研究

Ground State and Excitation Spectrum of Giant Vortex

in Rotating Bose－Einstein Condensates

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摘 要

近年，旋轉束縛在簡諧位能加四次方位能中的玻色－愛因思坦凝結是一個很熱門的題目，並且目前的研究指出：在這樣的系統中，當旋轉速度夠快時，會一種新型態的漩渦產生，就是「巨形漩渦」。在這篇論文中，我們研究這種巨形漩渦的一些基本性質，包括它的基態能量和激發能階。最後，我們發現這種巨形漩渦並不穩定。它可能只是某種不穩定的平衡態，而不是真正的基態。

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ABSTRACT

Rapidly rotating BEC in harmonic plus quartic potential is a popular subject recently, and giant vortex is a new anticipated phenomenon in this system when rotation frequency is high enough. In this thesis, we study some basic properties of the giant vortex, including ground state energy and excitation spectrum. Finally, we find that the giant vortex is not stable. It may be just a metastable state rather than a real ground state.

誌 謝

首先我要謝謝我的指導老師江進福老師和程思誠老師，除了感謝兩位老師在研究及論文的指導外，更重要的是我從他們身上看到做研究的態度及熱情，我後來之所以決定繼續往研究之路邁進，一部份是受到兩位老師的影響。接著我要感謝我的研究室伙伴，徐煥鈞、張研俞、黃盈靜、溫智媛、盧佳均、謝宏慶，(依筆畫排列 XD)，和分散在實驗室的麻吉同學，張展源、翁靖勛、羅中廷，(依筆畫排列 XD)，還有其他的班上同學，想起和大家在一起討論問題，無論是課業、研究、或生活上，出遊，聊八卦，在教室偷看電影……等等的一切快樂生活點滴，總是會不自覺地揚起嘴角，說真的，能夠認識大家真好。最後要感謝我的爸爸、媽媽和姊姊在經濟上的支援和精神上的支持，謝謝。

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Chapter 1：Introduction

After the realization of BEC on dilute gaseous atoms in experiment in 1995, the study on BEC becomes energetic. (In the following, we will use the word “BEC” to stand for rotating dilute, atomic, gaseous BEC for simplicity.) And both the theoretical research and experimental technique promote rapidly.

Recently, rotating BEC becomes one of the most popular subjects in the field of ultracold atoms. Because rotating BEC has a special property analogous to the superfluid under rotation. That is the appearance of vortices. The mechanism for their similar phenomena is that they both condensate to the BEC state. The only different is that superfluid has much stronger interactions between atoms than the gases atomic BEC. The stronger interactions cause the theoretical study on superfluid more difficult. However, the dilute atomic gases are relatively easy to deal with theoretically and even experimentally. Thus, physicists expect to use dilute atomic BEC to simulate the more difficult superfluid.

On the other hand, the Hamiltonian of the rotating BEC has the form analogous to the electron moving in the magnetic field known as the "quantum hall effect". Rotation here plays a role as the magnetic field in Quantum Hall effect. So, people anticipate that there shall be some phenomena occurring in rotating BEC analogous to the Quantum Hall effect, and they are devoted to find it.

Quantum Hall effect is a purely two dimensional effect. In order to reach this regime, we need to increase the rotating speed. Particles under rotation will experience a centrifugal force, and will be pushed out. As a result, the distribution of the condensate will spread out, becomes a shape like pancake, and finally achieves a quasi-two dimensional regime which also calls the "Quantum Hall regime".

Experimentally, we confine the particles with a harmonic potential. But, the particles will become unbounded and fly out when the rotation frequency exceed the frequency of the harmonic potential which is caused by the centrifugal force. So, we need an additional potential. The potential of the quartic form is one of the choices, and has been produced experimentally. Thus, harmonic plus quartic trap become the basic trap to explore the rapidly rotating BEC. Since the confined potential changes, from original harmonic trap to harmonic plus quartic trap, one can anticipate some new phenomena occur in this kind of potential. And, giant vortex is the new phenomena existing in this new trap.

In this thesis, we study the ground state and the excitations of the giant vortex, and this thesis is organized as follows. In part two, Gross–Pitaevskii equation will be introduced simply, because GP equation is the main tool to study the BEC. In part three, there is a simple argument to see why vortices occur. In part four, I will show the phenomena occur in the harmonic plus quartic trap by other group's simulation result. In part five, we calculate the ground state of the giant vortex. In part six, we calculate the excitations of the giant vortex. Finally, make some discussion about the result of calculations.

Chapter 2：Gross – Pitaevskii Equation

$\hat{\psi}(\vec{r})$ is the field operator describing a many-particles system. Of course, we discuss bosons here. $\hat{\psi}(\vec{r})$ can be expanded in terms of a complete set of single particle wavefunctions.

$$\hat{\psi}(\vec{r}) = \hat{a}_0\phi_0 + \sum_{i\neq 0}\hat{a}_i\phi_i \tag{1}$$

$\hat{a}_i$ and $\hat{a}_i^{\dagger}$ are annihilation and creation operators of a particle in the state ϕ_i, and they obey the commutation relations

$$\left[\hat{a}_i, \hat{a}_j^{\dagger}\right] = \delta_{ij} \qquad \left[\hat{a}_i, \hat{a}_j\right] = 0 \qquad \left[\hat{a}_i^{\dagger}, \hat{a}_j^{\dagger}\right] = 0 \tag{2}$$

If there are a large number of particles in the ground state, we can replace $\hat{a}_0$, $\hat{a}_0^{\dagger}$ with the c-number $\sqrt{N_0}$. This is the Bogoliubov approximation which is equivalent to ignore the noncommutativity of the operators. Since

$$N_0 = \left\langle \hat{a}_0^{\dagger}\hat{a}_0 \right\rangle \gg 1 \tag{3}$$

So,

$$\left\langle\left[\hat{a}_0, \hat{a}_0^{\dagger}\right]\right\rangle = \left\langle \hat{a}_0\hat{a}_0^{\dagger}\right\rangle - \left\langle \hat{a}_0^{\dagger}\hat{a}_0\right\rangle = 1 \approx 0 \tag{4}$$

Bogoliubov approximation is a good approximation for describing the BEC. Under this approximation, the field operator is replaced by the classical field, and eqn (1) can be rewritten as

$$\hat{\psi}(\vec{r}) = \psi_0(\vec{r}) + \delta\hat{\phi}(\vec{r}) \tag{5}$$

$$\psi_0(\vec{r}) = \sqrt{N_0}\,\phi_0(\vec{r}) \tag{6}$$

$\psi_0(\vec{r})$ is called the condensate wavefunction.

Next, we investigate the matrix element of $\hat{\psi}(\vec{r},t)$ between N particles state and N-1 particles state

$$\left\langle N-1\left|\hat{\psi}(\vec{r},t)\right|N\right\rangle = \left\langle N-1\left|e^{\frac{iE(N-1)t}{\hbar}}\hat{\psi}(\vec{r})e^{\frac{-iE(N)t}{\hbar}}\right|N\right\rangle = \psi_0(\vec{r})e^{\frac{-i\mu t}{\hbar}} \tag{7}$$

$\mu \approx E(N)-E(N-1)$ is the chemical potential.

We find that condensate wavefunction evolve in time with $e^{\frac{-i\mu t}{\hbar}}$.

Now, we find the equation which governing the field operator for a dilute, trapped BEC system. The Hamiltonian operator of the system can be written as

$$\hat{H} = \int\left(\frac{\hbar^2}{2M}\nabla\hat{\psi}^{\dagger}\nabla\hat{\psi}\right)dr + \int\hat{\psi}^{\dagger}V_{ext}\hat{\psi}dr + \frac{1}{2}\int\hat{\psi}^{\dagger}\hat{\psi}^{\dagger}{}'V(r-r')\hat{\psi}\hat{\psi}'dr'dr \tag{8}$$

The first term of the Hamiltonian operator is the kinetic energy term, second term is the external trapped potential, and the last term is the interaction between two particles. From the Heisenberg's equation of motion

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(\vec{r},t) = \left[\hat{\psi}(\vec{r},t),\hat{H}\right] \tag{9}$$

We get the time-dependent Schrodinger equation for the field operator,

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(\vec{r},t) = \left[\frac{-\hbar\nabla^2}{2M} + V_{ext} + \int\hat{\psi}^{\dagger}(r',t)V(r-r')\hat{\psi}(r',t)dr'\right]\hat{\psi}(\vec{r},t) \tag{10}$$

At very low temperature, a large number of particles condense to ground state, as discuss above, we can replace the field operator $\hat{\psi}(\vec{r},t)$ with the classical field $\psi_0(\vec{r},t)$. Also, at very low temperature, only s-wave scattering between two particles is important, so the interaction between two particles can be specified by the s-wave scattering length a_S . Under the diluteness condition, $na_S \ll 1$, the average distance between particles is larger than the scattering length a_S . Thus the actual form of the potential becomes not important. Potentials that give the same value of the s-wave scattering length a_S have the same physical property macroscopically. Then we can change the potential to a smooth one, so that $\psi_0(\vec{r},t)$ varies slowly on distances of order of the range of the interatomic force. So, Eqn (10) simplify to the form

$$i\hbar\frac{\partial}{\partial t}\psi_0(\vec{r},t)=\left[\frac{-\hbar\nabla^2}{2M}+V_{ext}+g\left|\psi_0(\vec{r},t)\right|^2\right]\psi_0(\vec{r},t) \tag{11}$$

$$g=\int V_{eff}(r)dr \tag{12}$$

According to scattering theory,

$$g=\frac{4\pi\hbar a}{M} \tag{13}$$

and

$$\psi_0(\vec{r},t)=\psi_0(\vec{r})e^{\frac{-i\mu t}{\hbar}} \tag{14}$$

Finally, we obtain the equation of motion describing the dilute, trapped BEC.

$$\left[\frac{-\hbar\nabla^2}{2M}+V_{ext}+g\left|\psi_0(\vec{r})\right|^2\right]\psi_0(\vec{r})=\mu\psi_0(\vec{r}) \tag{15}$$

This is the famous GP equation.

Chapter 3：Circulation Quantized and vortex

In this section, we introduce the interesting phenomenon of the rotating superfluid and the rotating BEC due to their coherent property of condensate wavefunction.

Assume the condensate wavefunction has the form,

$$\psi_0 = \sqrt{n}e^{i\theta} \qquad (16)$$

Insert to the current density

$$j = \frac{1}{2i}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) \qquad (17)$$

We get

$$j = n\nabla\theta \qquad (18)$$

And $j = nV$ by definition

Obviously,

$$V = \nabla\theta \qquad (19)$$

Notice that the velocity field is the gradient of the phase of the condensate wavefunction, so the velocity field of the condensate is also called "potential flow". This velocity field has two properties. First, it is irrotational.

$$\nabla \times V = \nabla \times \nabla\theta = 0 \qquad (20)$$

Second, the circulation of the velocity around a singularity, a vortex for example, is quantized.

$$\oint_c ds \cdot V = \frac{\hbar}{M}\oint_c ds \cdot \nabla\theta = \Delta\theta = \frac{h}{M}m \qquad (21)$$

M is the mass of the individual particle, m is integer.

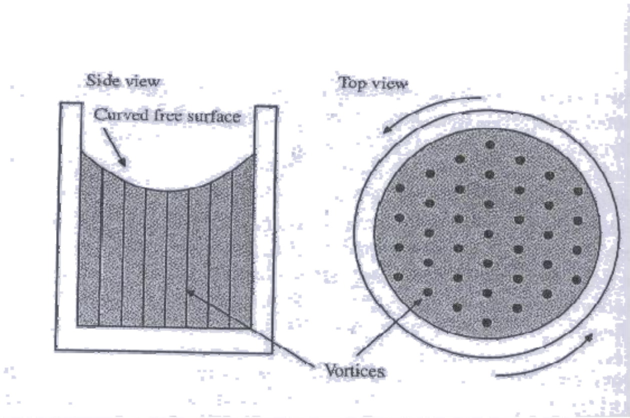


Figure 3-1 Rotate a bucket of superfluid

If we rotate a bucket of superfluid, the velocity of the fluid should be zero due to the irrotational property of the condensate velocity field.

$$\oint_c ds \cdot V = \int_s (\nabla \times V) \cdot d\sigma = 0 \quad \Rightarrow \quad V=0 \qquad (22)$$

This means that if we rotate a bucket of superfluid, it is impossible to have a macroscopic flow without some singularity points such as "vortices". In another word, if we can let the superfluid have a definitely circular flow, the vortices will be produced.

Chapter 4：Rotating BEC in a harmonic plus quartic trap

The harmonic plus quartic trap has the form

$$V_{ext} = \frac{M}{2}\omega_0^2 r^2 + \frac{M}{2}\omega_0^2 \lambda \frac{r^4}{d^2} \qquad (23)$$

The first term is harmonic term, and the last term is quartic term. d is the harmonic length $\sqrt{\hbar/M\omega_0}$. λ is the parameter characterized the quartic potential strength. Let the angular velocity direct along the z axis, and the equation of motion of the system can be expressed as

$$i\hbar\frac{\partial}{\partial t}\psi_0(\vec{r},t) = \left[\frac{-\hbar\nabla^2}{2M} + V_{ext} + g\left|\psi_0(\vec{r},t)\right|^2 - \Omega L_z\right]\psi_0(\vec{r},t) \qquad (24)$$

The numerical works done by Fetter's group are shown as follow

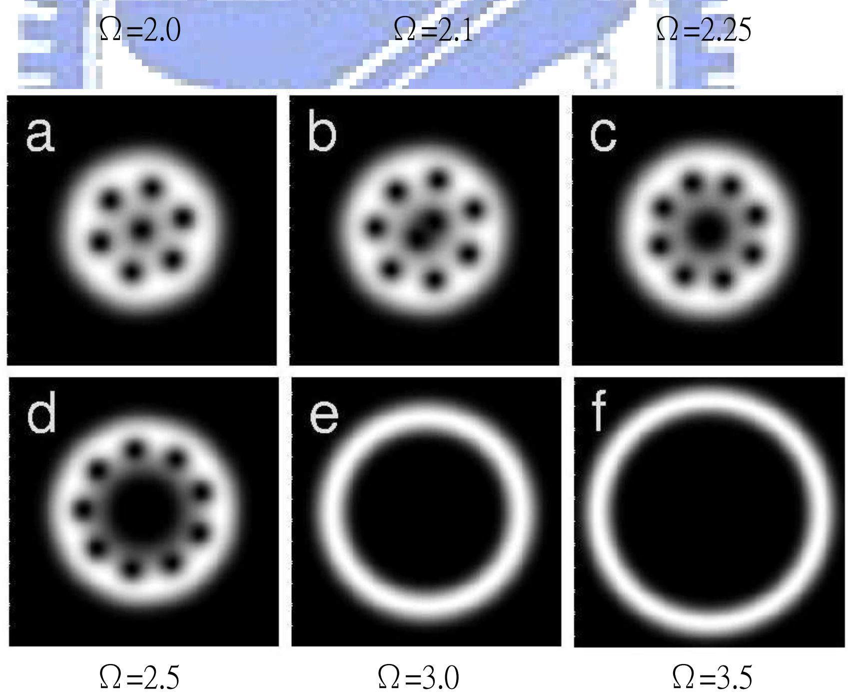


Figure 4-1 Density distribution of rotating BEC confined

in a harmonic plus quartic trap for g=80

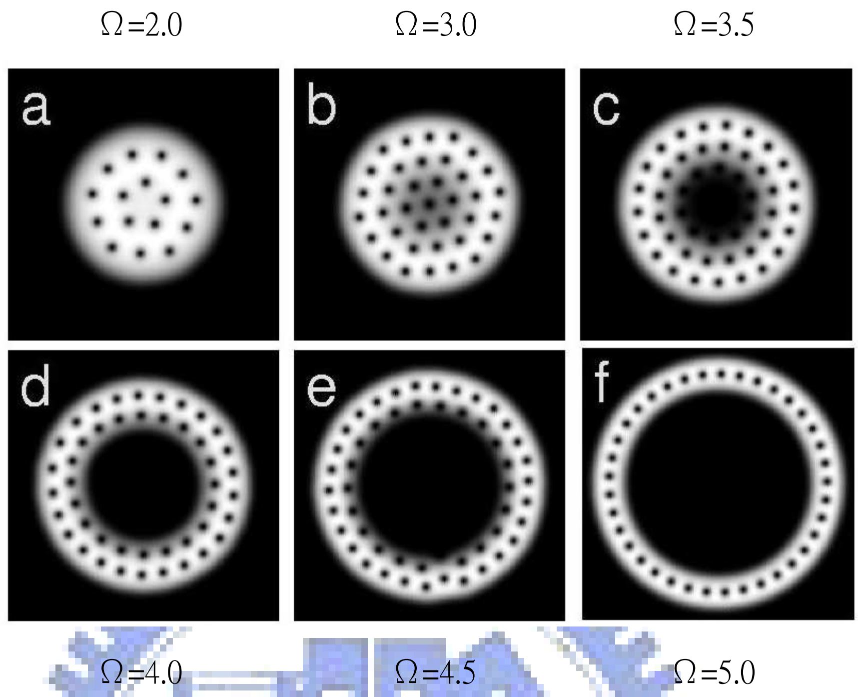


Figure 4-2 Density distribution of rotating BEC confined

in a harmonic plus quartic trap for g=1000

For g=80, when Ω=2, (our unit is $\hbar = M = d = \omega_0 = 1$), there is a singly quantized vortex locating at the center of the condensate. Other six singly quantized vortices are surrounding the center ones. The vortices form a vortex lattice. When Ω=2.1 there is an additional singly quantized vortex appearing. When Ω=2.5, the center vortex becomes doubly quantized. We call this state "the vortex lattice with a hold". Keeping increasing the rotation speed, all vortices merge to form a multiple quantized vortex at the center, and other vortices disappear. We call this kind of vortex "giant vortex". For g=1000, the condensate has the same tendency, transition from vortex lattice to vortex lattice with a hold. However, in the g=1000 case, they can not find the giant vortex even when Ω =7. Exceeding Ω=7, the numerical work becomes difficult to go on. It is believe that vortex lattice with a hold should be transition to a giant vortex at some greater rotation speed.

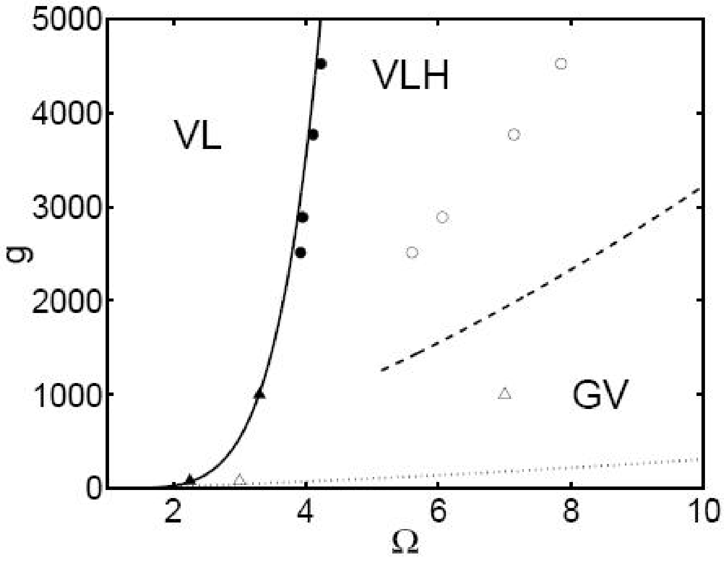


Figure 4-3 phase diagram of rotating BEC with angular velocity versus interaction strength g, **with** λ =0.5.

By collecting the data, they get a g-Ω phase diagram which separate the three kinds of state. The triangles are the numerical result of Fetter's group. The circles are analytic result done by Baym's group, while lines are Fetter's analytic result. VL means vortex lattice. VLH means vortex lattice with a hold. GV means giant vortex. Here, we employ the numerical result done by Fetter's group to define the region where the giant vortex exists. And, we study the giant vortex in this region.

Chapter 5：Ground state of Giant Vortex

5.1 Giant Vortex Wavefunction

Start from the circulation quantized condition

$$\oint_c \vec{V}\cdot d\vec{r} = m\frac{h}{M} \tag{25}$$

$$\vec{V} = \frac{\hbar}{M}\nabla S \tag{26}$$

For the giant vortex, it is natural to assume the velocity field is symmetry about the rotating axis. Then the velocity field is independent of variable θ. Eqn (25) becomes

$$\oint_c \vec{V}\cdot d\vec{r} = \oint_c V\cdot r d\theta = 2\pi r V = m\frac{h}{M} \tag{27}$$

So the velocity field has the form

$$V = \frac{m\hbar}{Mr} \tag{28}$$

Insert the velocity field back to eqn (26) and integrate it, we get the phase of the condensate wavefunction S = mθ. So the condensate wavefunction for the giant vortex can be taken the form

$$\phi_g(r) = \sqrt{n}e^{im\theta} \tag{29}$$

5.2 Thomas -Fermi density profile

Hamiltonian of the rotating, trapped bosons system is

$$H = \frac{1}{2M}P^2 + V_{ext} + \frac{1}{2}g_{2D}\left|\phi(r)\right|^2 - \Omega L_z \tag{30}$$

$$V_{ext} = \frac{1}{2}M\omega_0^{\,2}r^2 + \frac{1}{2}\lambda M\omega_0^{\,2}\frac{r^4}{d^2} \tag{31}$$

Write down the free energy

$$G = -\mu N + E \tag{32}$$

$$E=\int d^2r\phi_g^* H\phi_g \tag{33}$$

$$=\int d^2r\left[\frac{\hbar^2}{2M}\left|\nabla\phi_g\right|^2+\frac{M\omega_0^2}{2}(r^2+\frac{\lambda r^4}{d^2})\left|\phi_g\right|^2+\frac{1}{2}g\left|\phi_g\right|^4-\Omega\phi_g^* L_z\phi_g\right] \tag{34}$$

$$N=\int d^2r\left|\phi_g\right|^2 \tag{35}$$

Insert the giant vortex wavefunction

$$G=-u\int n(r)dr+\int n(r)\left(\left|\nabla\sqrt{n(r)}\right|^2+\frac{m^2\hbar^2}{2Mr^2}+\frac{M}{2}\omega_0^2r^2+\frac{M}{2}\lambda\omega_0^2\frac{r^4}{d^2}-\Omega\hbar m\right)dr$$

$$+\frac{g_{2D}}{2}\int n^2(r)dr\Big] \tag{36}$$

If the system contains sufficiently large number of particles and the interactions between the particles are large, the density of the condensate will varies slowly over the whole condensate, excluding the boundaries. The condition can be wrote mathematically as

$$\frac{Na}{d}\gg 1 \tag{37}$$

As a result, the quantum pressure term $\left|\nabla\sqrt{n(r)}\right|^2$ in the free energy functional can be neglected. This is the so called "Thomas–Fermi approximation". So the free energy becomes

$$G=-u\int n(r)dr+\int n(r)\left(\frac{m^2\hbar^2}{2Mr^2}+\frac{M}{2}\omega_0^2r^2+\frac{M}{2}\lambda\omega_0^2\frac{r^4}{d^2}-\Omega\hbar m\right)dr$$

$$+\frac{g_{2D}}{2}\int n^2(r)dr \tag{38}$$

Variation of free energy with respect to n(r), we get Thomas–Fermi density profile

$$\frac{\delta G}{\delta n}=0$$

$$\Rightarrow\ n(r)=\frac{1}{g_{2D}}\left[\mu'-\frac{M}{2}\omega_0^2r^2-\frac{M}{2}\lambda\omega_0^2\frac{r^4}{d^2}-\frac{m^2\hbar^2}{2Mr^2}\right] \tag{39}$$

$$=\frac{\hbar\omega_0\lambda}{2g_{2D}}\left[\frac{2\mu'}{\lambda\hbar\omega_0}-\frac{\rho^2}{\lambda}-\rho^4-\frac{m^2}{\lambda\rho^2}\right] \tag{40}$$

$$\mu' = \mu + m\hbar\Omega \qquad \rho = \frac{r}{d} \tag{41}$$

5.3 calculation of the ground state energy of the giant vortex

First, find the boundary of the condensate. Let n(r) = 0 when r = R.

$$n(R) = 0 \tag{42}$$

$$\Rightarrow \rho^6 + \frac{\rho^4}{\lambda} - \frac{2\mu'}{\lambda\hbar\omega_0}\rho^2 + \frac{m^2}{\lambda} = 0 \tag{43}$$

$$\Rightarrow x^3 + \frac{x^2}{\lambda} - \frac{2\mu'}{\lambda\hbar\omega_0}x + \frac{m^2}{\lambda} = 0 \qquad x = \rho^2 \tag{44}$$

$$\Rightarrow x^3 + a_1 x^2 + a_2 x + a_3 = 0 \tag{45}$$

$$a_1 = \frac{1}{\lambda} \qquad a_2 = -\frac{2\mu'}{\lambda\hbar\omega_0} \qquad a_3 = \frac{m^2}{\lambda} \tag{46}$$

Use Carden's formula to find the three root of eqn (44).

$$q = \frac{3a_2 - a_1^2}{9} = -\left(\frac{2\mu'}{3\lambda\hbar\omega_0} + \frac{1}{9\lambda^2}\right) \tag{47}$$

$$r = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54} = -\frac{1}{6}\left(\frac{2\mu'}{\lambda^2\hbar\omega_0} + \frac{3m^2}{\lambda}\right) - \frac{1}{27\lambda^3} \tag{48}$$

$Q = q^3 + r^2 < 0$ condition for existence of two real solutions (49)

$$Z_1 = \left[r + \sqrt{Q}\right]^{\frac{1}{3}} \tag{50}$$

$$Z_2 = \left[r - \sqrt{Q}\right]^{\frac{1}{3}} \tag{51}$$

$$x_1 = (Z_1 + Z_2) - \frac{1}{3\lambda} \tag{52}$$

$$x_2 = -\frac{1}{2}(Z_1 + Z_2) + \frac{i\sqrt{3}}{2}(Z_1 - Z_2) - \frac{1}{3\lambda} \tag{53}$$

$$x_3 = -\frac{1}{2}(Z_1 + Z_2) - \frac{i\sqrt{3}}{2}(Z_1 - Z_2) - \frac{1}{3\lambda} \tag{54}$$

After some testing, we find x_3 and x_1 are the two real, positive roots we want. And $x_1 > x_3$, so $\sqrt{x_1}$ is the outer radius of the annulus while $\sqrt{x_3}$ is the inner radius of the annulus.

$$\text{Let} \quad R_+ = \sqrt{x_1} \qquad R_- = \sqrt{x_3} \tag{55}$$

The ground state energy of the giant vortex is

$$E = \int_{R_-}^{R_+} d^2 r \phi_g^* H \phi_g \tag{56}$$

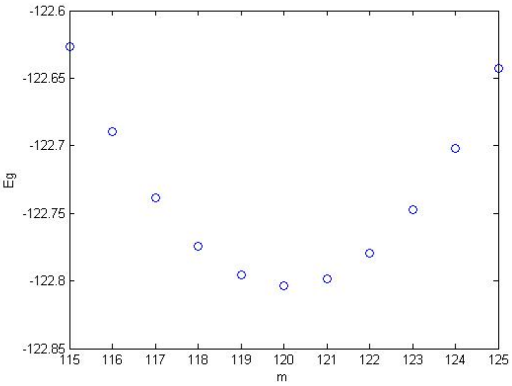
$$= \int_{R_-}^{R_+} d^2 r \cdot n(r) \left(\frac{m^2 \hbar^2}{2Mr^2} + \frac{M}{2} \omega_0^2 r^2 + \frac{M}{2} \lambda \omega_0^2 \frac{r^4}{d^2} - \Omega \hbar m \right) + \frac{g_{2D}}{2} n^2(r) \tag{57}$$

Insert eqn (39) to above equation, we can express E as

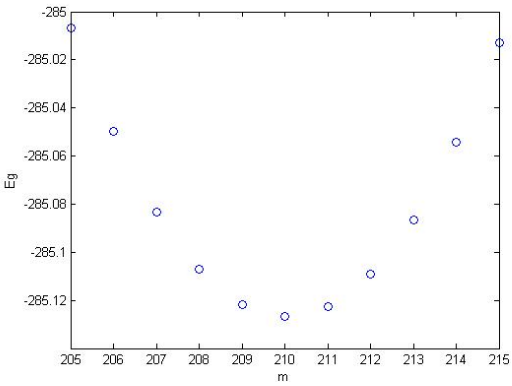
$$\Rightarrow \frac{E}{\hbar \omega_0 N} = \frac{\mu}{\hbar \omega_0} - \frac{\lambda^2}{32\sigma} \int_{R_-^2}^{R_+^2} \left(\frac{2\mu'}{\lambda \hbar \omega_0} - \frac{x}{\lambda} - x^2 - \frac{m^2}{\lambda x} \right)^2 dx \tag{58}$$

5.4 result

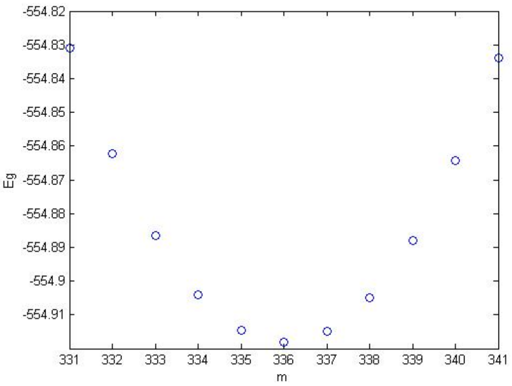
For a given angular frequency, the ground state energy varies with m, the angular momentum or circulation quantum number of the giant vortex. We find the lowest energy for some angular frequencies for g=1000, λ=0.5. The results are listed below, we also show the plot of m corresponding to the lowest energy versus Ω, and a table listed some characteristic value of the giant. Here, we take $d = \sqrt{\hbar / M\omega_0}$, the harmonic oscillator length, as the length scale, and take $N\hbar\omega_0$ as the unit of energy.



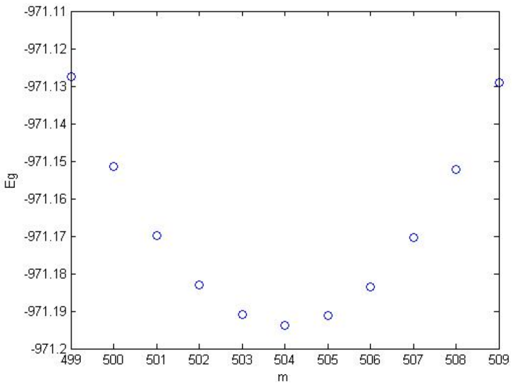
Ω=5



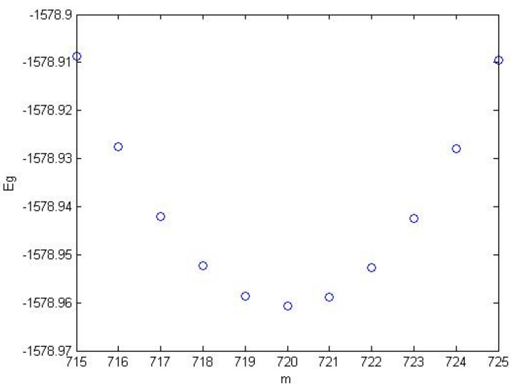
Ω=6



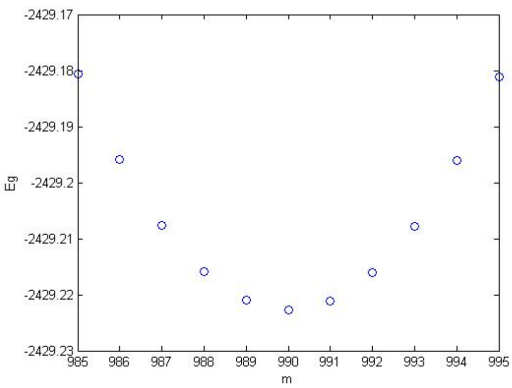
Ω=7



Ω=8



Ω=9



Ω=10

Figure 5-1 diagrams of Eg-Ω, show that we actually find the lowest energy of giant vortex.

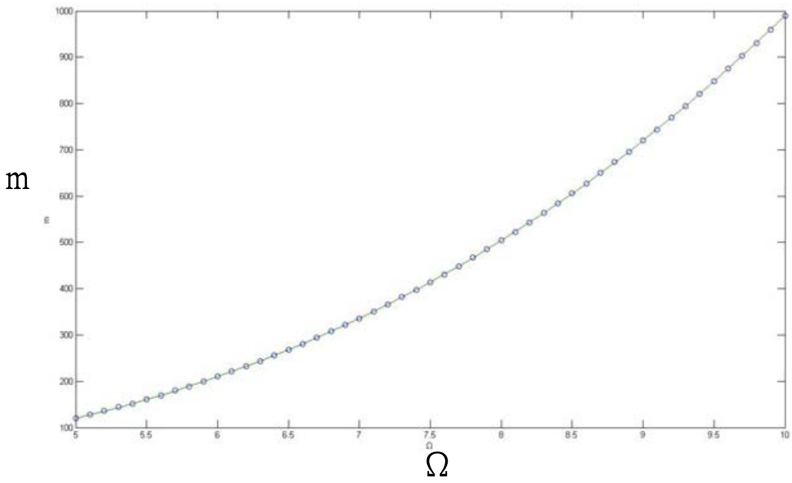


Figure 5-2 diagram of m-Ω

Ω (ω_0)	m ($\hbar$)	μ' ($N\hbar\omega_0$)	E ($N\hbar\omega_0$)	r (d)	R (d)
5	120	-108.6569	-122.8039	1.3718	4.9162
6	210	-271.0368	-285.1269	1.1432	5.9257
7	336	-540.8592	-554.9181	0.9793	6.9342
8	504	-957.1537	-971.1937	0.8564	7.9412
9	720	-1564.9	-1579.0	0.7608	8.9470
10	990	-2145.2	-2429.2	0.6844	9.9519

Table5-1 Numerical value of the calculation result. Ω is the rotation frequency, m is the angular momentum of the system in the rotation direction. μ' is the effective chemical potential, E is the energy of the condensate, r is the difference between the outer and inner radius of the condensate, R is the mean radius of the outer and inner radius of the condensate .

Finally, compare the energy with the numerical result done by Fetter's group.

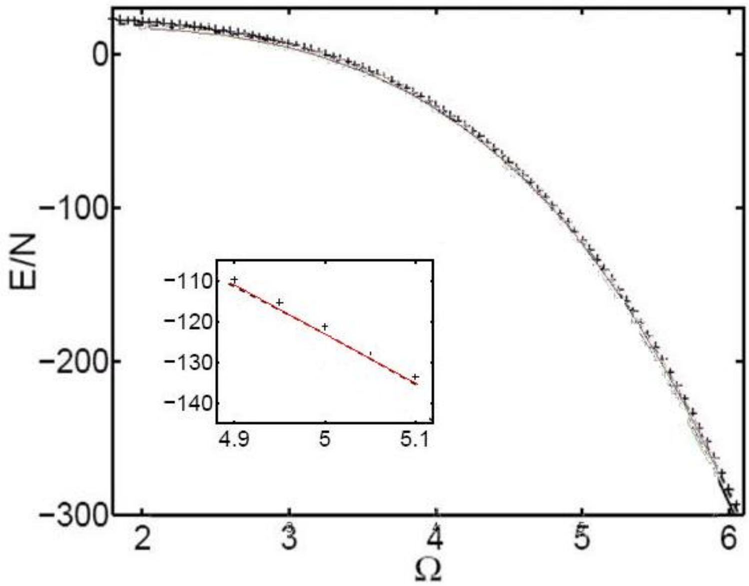


Figure 5-3 diagram of Eg-Ω. **Cross signs are the numerical result for the giant vortex, the red line is our analytical result.**

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Chapter 6：Excitations of Giant Vortex

To obtain the excitations of the giant vortex, we allow the wavefunction of the giant vortex to oscillate about its equilibrium value. And, take the perturbed wavefunction as the form

$$\psi\left(\vec{r},t\right)=\psi_0(\vec{r})e^{\frac{-i\mu t}{\hbar}}+\left[u\left(\vec{r}\right)e^{-i\omega t}-v^*\left(\vec{r}\right)e^{i\omega t}\right]e^{\frac{-i\mu t}{\hbar}} \qquad (59)$$

Insert eqn（59）into the time-dependent Schrodinger's equation,

$$i\hbar\frac{\partial\psi\left(\vec{r},t\right)}{\partial t}=H_0\psi\left(\vec{r},t\right)+g_{2D}\left|\psi\right|^2\psi\left(\vec{r},t\right) \qquad (60)$$

$$H_0=\frac{P^2}{2M}+\frac{1}{2}M\omega_0^2r^2+\frac{M\omega_0^2}{2d}\lambda r^4-\Omega L_z \qquad (61)$$

For unperturbed state,

$$\left(H_0+g_{2D}\left|\psi_0\right|^2\right)\psi_0=\mu\psi_0 \qquad (62)$$

Colleting terms, we get the following two equations,

$$\left(H_0-2g_{2D}\left|\psi_0\right|^2-\mu\right)u\left(\vec{r}\right)+g_{2D}\psi_0^2v\left(\vec{r}\right)=\hbar\omega u\left(\vec{r}\right) \qquad (63)$$

$$\left(H_0^+-2g_{2D}\left|\psi_0\right|^2-\mu\right)v\left(\vec{r}\right)+g_{2D}\psi_0^{*2}u\left(\vec{r}\right)=-\hbar\omega v\left(\vec{r}\right) \qquad (64)$$

In order to solve the two coupling equations, we try

$$u\left(\vec{r}\right)=ue^{ik\theta}\psi_0 \qquad (65)$$

$$v\left(\vec{r}\right)=ve^{ik\theta}\psi_0^* \qquad (66)$$

Insert into eqn(63) and eqn(64), we get

$$\left[\frac{-ik\hbar^2}{Mr^2}\frac{\partial\psi_0}{\partial\theta}+\left(\frac{\hbar^2k^2}{2Mr^2}-\hbar\Omega k+g_{2D}\left|\psi_0\right|^2\right)\psi_0\right]u-g_{2D}\left|\psi_0\right|^2\psi_0v=\hbar\omega\psi_0u \qquad (67)$$

$$\left[\frac{-ik\hbar^2}{Mr^2}\frac{\partial\psi_0^*}{\partial\theta}+\left(\frac{\hbar^2k^2}{2Mr^2}+\hbar\Omega k+g_{2D}\left|\psi_0\right|^2\right)\psi_0^*\right]v-g_{2D}\left|\psi_0\right|^2\psi_0^*u=-\hbar\omega\psi_0^*v$$

(68)

Let $\psi_0 = \psi_G = \sqrt{n(r)}e^{im\theta}$ (69)

$$n(r) = \frac{1}{g_{2D}}\left[\mu' - \frac{1}{2}M\omega_0^2 r^2 - \frac{M^2\omega_0^3}{2h}\lambda r^4 - \frac{m^2\hbar^2}{2Mr^2}\right] \tag{70}$$

$$\mu' = \mu + m\hbar\Omega \tag{71}$$

Multiply eqn (67) and eqn (68) by ψ_G^* and ψ_G respectively and integration. We get

$$\left[\frac{-ik\hbar^2}{M}(im)\int_{R-}^{R+}\frac{1}{r^2}n(r)2\pi rdr + \frac{\hbar^2k^2}{2M}\int_{R-}^{R+}\frac{1}{r^2}n(r)2\pi rdr - \hbar\Omega kN + g_{2D}\int_{R-}^{R+}n^2(r)2\pi rdr\right]u$$
$$-g_{2D}\int_{R-}^{R+}n^2(r)2\pi rdrv = \hbar\omega Nu$$

(72)

$$\left[\frac{-ik\hbar^2}{M}(-im)\int_{R-}^{R+}\frac{1}{r^2}n(r)2\pi rdr + \frac{\hbar^2k^2}{2M}\int_{R-}^{R+}\frac{1}{r^2}n(r)2\pi rdr + \hbar\Omega kN + g_{2D}\int_{R-}^{R+}n^2(r)2\pi rdr\right]v$$
$$-g_{2D}\int_{R-}^{R+}n^2(r)2\pi rdru = -\hbar\omega Nv$$

(73)

Let

$$\frac{1}{N}\int_{R-}^{R+}\frac{1}{r^2}n(r)2\pi rdr = A \tag{74}$$

$$\frac{g_{2D}}{N}\int_{R-}^{R+}n^2(r)2\pi rdr = B \tag{75}$$

eqn (72) and eqn (73) can be expressed as

$$\left[\frac{k\hbar^2m}{M}A + \frac{\hbar^2k^2}{2M}A - \hbar\Omega k + B\right]u - Bv = \hbar\omega u \tag{76}$$

$$\left[\frac{-k\hbar^2m}{M}A + \frac{\hbar^2k^2}{2M}A + \hbar\Omega k + B\right]v - Bu = -\hbar\omega v \tag{77}$$

For the nontrivial solutions of u and v, the determinant of the coefficients of eqn (76) and eqn (77) must vanish, so

$$\begin{pmatrix} \left(\dfrac{\hbar^2 k^2}{2M}A+B\right)+\left(\dfrac{k\hbar^2 m}{M}A-\hbar\Omega k-\hbar\omega\right) & -B \\ -B & \left(\dfrac{\hbar^2 k^2}{2M}A+B\right)-\left(\dfrac{k\hbar^2 m}{M}A-\hbar\Omega k-\hbar\omega\right) \end{pmatrix}=0 \tag{78}$$

$$\Rightarrow \left(\frac{\hbar^2 k^2}{2M}A+B\right)^2-\left(\frac{k\hbar^2 m}{M}A-\hbar\Omega k-\hbar\omega\right)^2-B^2=0 \tag{79}$$

Finally, we get the dispersion relation

$$\omega_{\pm}=\frac{k\hbar m}{M}A-\Omega k\pm\frac{1}{\hbar}\sqrt{\left(\frac{\hbar^2 k^2}{2M}A+B\right)^2-B^2} \tag{80}$$

Because the small perturbations violate the rotational symmetry, the two solution should both take into account.

Here is the numerical results for A and B versus Ω for g=1000, λ=0.5,

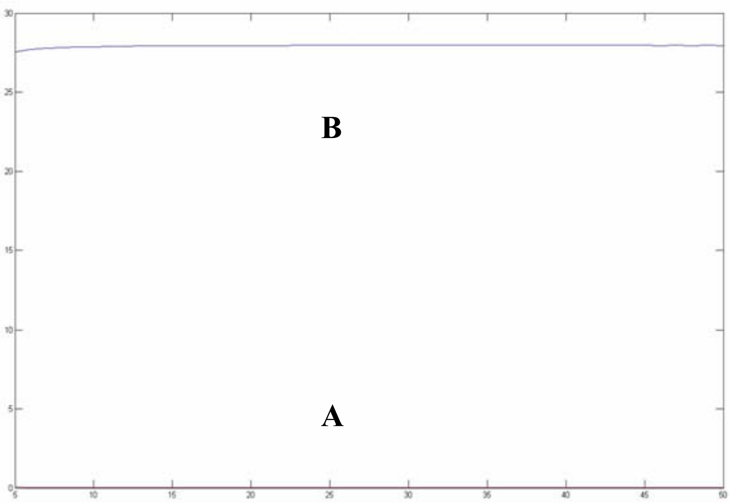

Figure 6-1 Numerical value of A and B versus Ω for g=1000, λ=0.5.

And, plot the spectrum of the excitations for k= ±1, ±2, ±3.

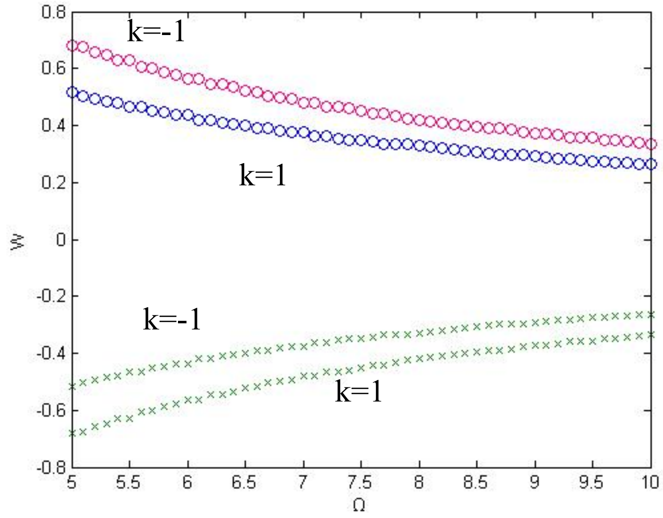

Figure 6-2 w-Ω diagram for k=±1, g=1000, λ=0.5.

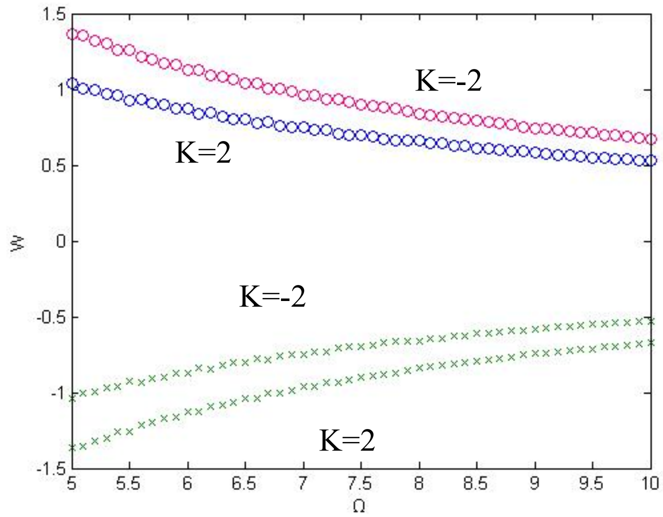

Figure 6-3 w-Ω diagram for k=±2, g=1000, λ=0.5.

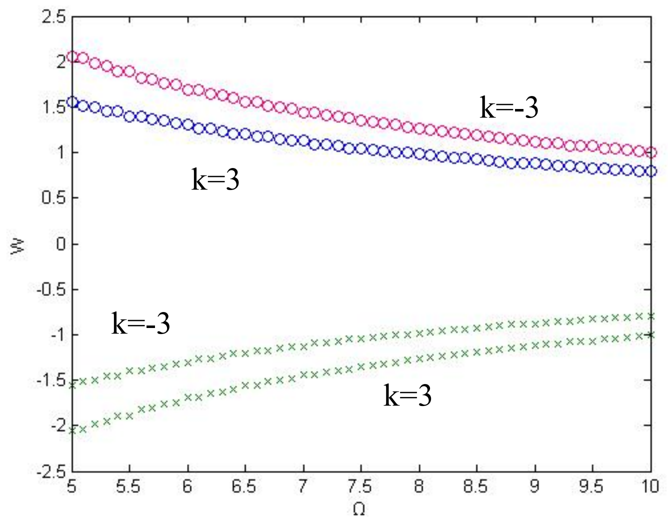

Figure 6-4 w-Ω diagram for k=±3, g=1000, λ=0.5.

We find that, for each k, there is a positive energy spectrum companied with a negative energy spectrum. The excitations with negative energy will make the system unstable. Back to the results simulated by Fetter's group, for g=1000, λ=0.5, they can not find the giant vortex even the angular frequency is increasing to 7, while the giant vortex has appeared before this frequency predicted by other analytic result. This means that the giant vortex may be a metastable state rather than a real ground state. Although, in the g=80, λ=0.5 case, they find the giant vortex, this may be just a local minimum rather than a real ground state. Giving some perturbation, the giant vortex may transition to an actual ground state which is likely to be a hole with a ring of singly quantized vortex surrounding it. This still needs to study further.

Under other conditions, g=1000, λ=0.005, we find the energy spectrum both become positive for k=-1,-2,-3.

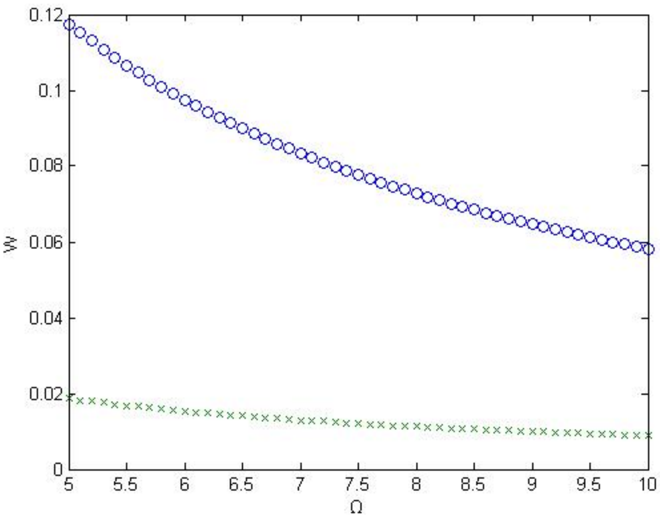

Figure 6-5 w-Ω diagram for k=-1, g=1000, λ=0.005.

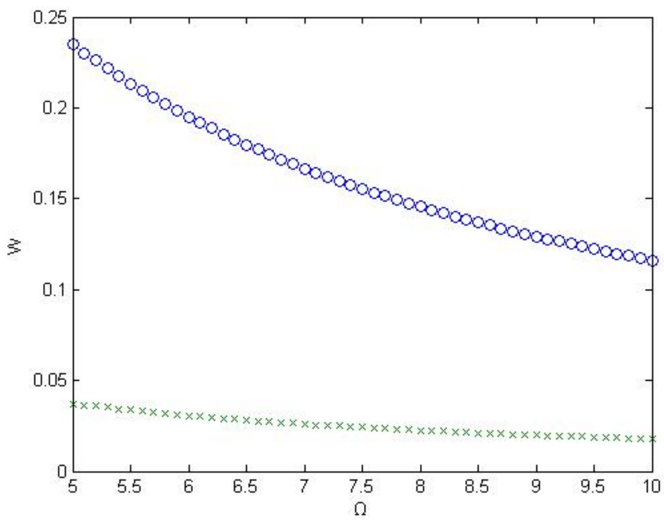

Figure 6-6 w-Ω diagram for k=-2, g=1000, λ=0.005.

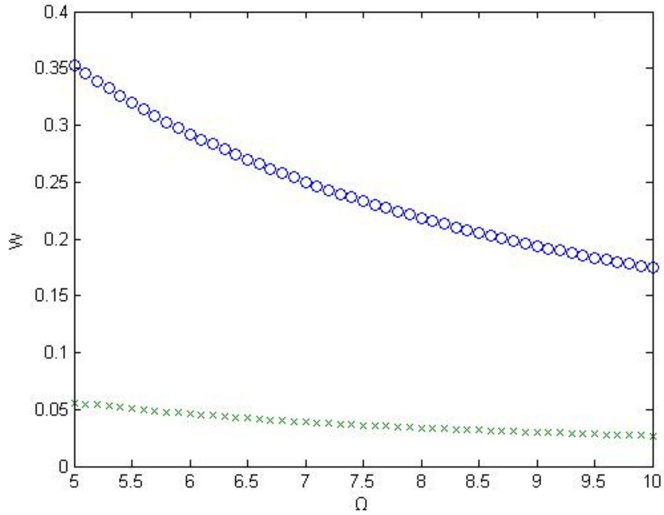

Figure 6-7 w-Ω diagram for k=-3, g=1000, λ=0.005.

But, the energy spectrum both become negative for k=1, 2, 3.

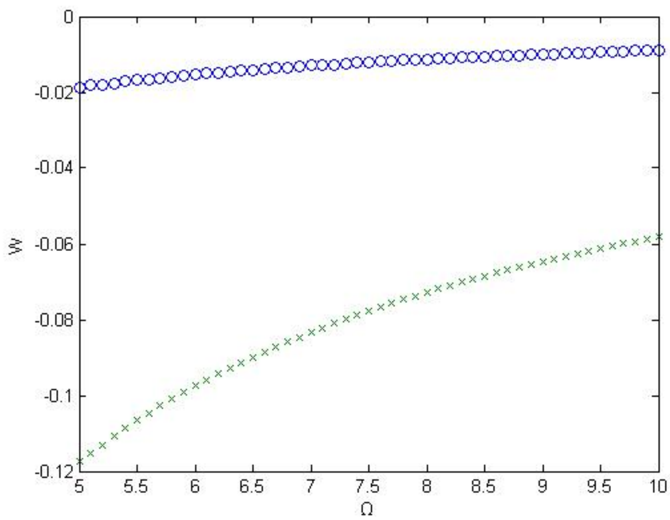

Figure 6-8 w-Ω diagram for k=1, g=1000, λ=0.005.

As a result, the giant vortex is still unstable.

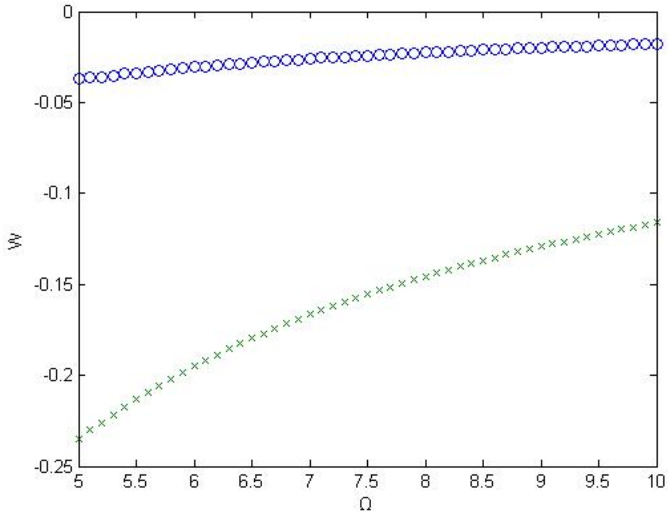

Figure 6-9 w-Ω diagram for k=2, g=1000, λ=0.005.

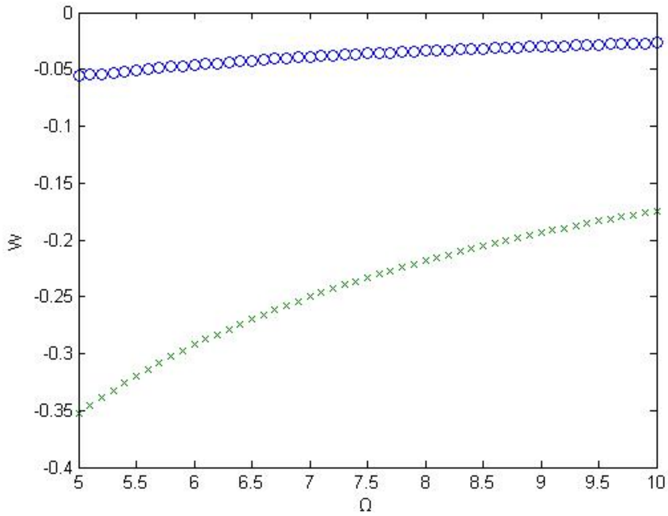

Figure 6-10 w-Ω diagram for k=3, g=1000, λ=0.005.

As a result, the giant vortex is still unstable.

Chapter 7：Conclusion

According to our analysis, we find that giant vortex may not be the ground state of the rapidly rotating BEC in a harmonic plus quartic trap.

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