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序列複合選擇權之
評價、分析、計算與應用

The Sequential Compound Options:
Valuation, Analysis, Computation and Applications

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摘要

在專案鑑價方法的需求之下，本文提出序列複合選擇權(Sequential Compound options，SCOs)、它們的一般化評價公式以及敏感度分析。傳統專案鑑價的評價方法忽略了複雜專案的內在本質，例如內部高度交互作用或是多層堆疊，使得這些方法不適用，進而誤導策略制定。基於專案的特質，本研究提出序列複合選擇權，以提升專案鑑價的效能。

文獻中大部分的複合選擇權，大多是參數固定的簡單兩層選擇權。在多層複合選擇權的現有研究，也只侷限在序列複合買權(Sequential Compound CALL options，SCCs)。本研究提出多層的序列複合選擇權(SCOs)，定義為以(複合)選擇權為標的的選擇權，而它們每一層的買權(call)或賣權(put)性質是可以任意指定。此外，隨機利率與隨時間改變之資產價格波動度讓模型更加彈性。評價公式是由 risk-neutral 方法與 change of numéraire 方法分別推導而得到。一個多維度常態積分的偏微分關係，可以被視為萊布尼茲法則(Leibnitz's Rule)的推廣，也在本研究裡推導而得，並且被用來推導序列複合選擇權(SCOs)的敏感度分析。

序列複合選擇權(SCOs)的計算，比起其他傳統的選擇權還要複雜許多。傳統歐式選擇權與(兩層或更多層)複合選擇權在演算上的差異，在於約當資產價格(Equivalent Asset Prices，EAPs)的槽套迴圈計算以及常態積分的維度。本研究克服這些困難，提出序列複合選擇權(SCOs)的演算法與三層複合選擇權的數值例。

序列複合選擇權(SCOs)可以強化並增廣複合選擇權理論在專案鑑價、風險管理與財務衍生性商品定價領域的應用。對於里程碑專案(例如新藥開發)而言，里程碑專案的達成代表擁有選擇進入下一個階段與否的權利，因此這類專案可以用序列複合選擇權(SCOs)來評價。擁有擴張、縮小規模、中止、放棄、轉換或成長選擇權在裡面交互作用的複雜專案，也可以運用序列複合選擇權(SCOs)來評價。序列複合選擇權(SCOs)的優點，包括較便宜的權利金、允許決策後延、費用分期支付、較高的彈性，可以提高風險控管的效果。一些金融機構所關心的最重要議題，例如波動度風險、抵押貸款提前還款風險與天氣風險，也可以透過序列複合選擇權(SCOs)而得到良好的控管。此外，序列複合選擇權(SCOs)也可以被運用於財務衍生性商品的定價，例如新奇美式選擇權。

本文提出序列複合選擇權(SCOs)的數值範例，包括政府營收保證評估與外匯避險運用。另外，以序列複合選擇權(SCOs)為核心的資訊系統也被提出，以作為專案與衍生性商品的評價。

關鍵字： 複合選擇權，專案鑑價，實質選擇權，萊布尼茲法則，選擇權定價，風險管理

The Sequential Compound Options:
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Abstract

This paper proposes the sequential compound options (SCOs), their generalized pricing formula and sensitivity analysis under the necessity from project valuation. Traditional methods for project valuation ignoring complicated projects' intrinsic properties, such as highly internal interacting or multiple-fold stacks, are far beyond the adequacy and will cause misleading for strategy-making. Based on project's characteristics, this study propose SCOs in order to have better effectiveness for project valuation.

Most compound options described in literatures are simple 2-fold options whose parameters are constant over time. Existing research on multi-fold compound options has been limited to sequential compound CALL options (SCCs). The multi-fold sequential compound options (SCOs) proposed in this study are defined as compound options on (compound) options where the call/put property of each fold can be arbitrarily assigned. Besides, the random interest rate and time-dependent variance of asset price make the model more flexible. The pricing formula is derived by risk-neutral method and change of numéraire method. The partial derivative of a multivariate normal integration, a extension case of Leibnitz's Rule, is derived in this study and used to derive the SCOs sensitivities.

Evaluations of SCOs are more complicated than those of conventional options. The computation differences between European options and compound options (2-fold or more) lie in the equivalent asset prices (EAPs) evaluation with nested loops and the dimension of normal integrals. This study overcomes these difficulties and proposes the computing algorithm for SCOs and the numerical illustration of 3-fold SCOs.

SCOs can enhance and broaden the use of compound option theory in the study of project valuation, risk management and financial derivatives valuation. For milestone projects (e.g., the new drug development), the milestone completion has the choice to enter the next stage or not, and hence the projects can be pricing by SCOs. Complex projects, within which expansion, contraction, shutting down, abandon, switch and or growth option interacting, can also be evaluated by the SCOs. Several most important issues, such as volatility risk, prepayment risk of mortgage and weather risk, concerned by the finance institutions can be well controlled through SCOs. The advantages of SCOs, including the cheaper premium, permission of decision postponement, split-fee and better flexibility, can enhance the risk management effectiveness. In addition, the SCOs can also be applied for the pricing of financial derivatives, e.g. exotic American options.

The numerical examples of SCOs are proposed, including evaluation of government revenue guarantee and currency hedging. In addition, the information management system with SCOs as its core module is also proposed in order to evaluating projects and financial derivatives.

Keywords: compound option; project valuation; real option; Leibnitz's Rule; option pricing; risk management

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To the Almighty

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李 孟 育 敬筆

Chapter 1 Introduction

§1.1 Motivation

This paper proposes and analyzes the sequential compound options (SCOs) from the demand for project valuation (or appraisal).

There are seven different methods for project valuation (Razgaitis, 1999), including discounted cash flow (DCF), industry standards, rule of thumb, rating and ranking, Monte Carlo simulation, auctions and real options. Different methods should be applied to different types of projects according to their distinctions. For example, the DCF method is suitable for projects with certain and predictable incomes, while the real option approach is proper for those with high uncertainty. The DCF methods for projects with high risk and growth potential will to tend to reject the investment decision and consequently lose many opportunities. The popular real option approach is more suitable for valuations of projects with high potential and risk, such as new drug developments (NDAs), oil exploration, etc.. Judy Lewent, the Chief of Financial Officer of Merck, even claims that "*all kinds of business decisions are options*" and can be dealt with by the real option approach (Nichols, 1994).

However, the conventional real option approach for valuations of projects with evolutionary sophisticated structure is not enough. The sophisticated structure of derivative pricing and its wide deployment in the real options field have revealed the limitations of the current methodology. 2-fold compound options cannot be used as further building blocks to model other financial innovations, but results concerning multi-fold compound options so far have focused only on sequential compound calls. Although Remer et al. (2001, p.97) mention that "*… in practice, different project phases often have different risks that warrant different discount rates,"* the important feature of time-dependent (or fold-dependent) parameters is rarely taken into account by current methodologies. In order to enhance flexibility, many projects are embedded with different types of options, such as growth, switch, abandon, shutting down, contraction or expansion (Trigeorgis 1993, 1996). Nevertheless, the flexibility is accompanied by the difficulty of valuation and hence results in misleading decisions by existing methods.

The SCOs, defined as (compound) options on options, are proposed to valuate complex projects according to their intrinsic structures. For example, a project is usually valuated as a European call option, thus its expansion and abandon options can be regarded as a call on call and a put on call respectively. Therefore the abandon option on the expansion option is a typical 3-fold put on call on call option. In contrast with the traditional real option approach which considers the project as one option, the complex project can be deconstructed as distinct essential options and their interactions, and all of them can be evaluated by different SCOs. This kind of method decomposes the project according to its special structure and provides decision makers with better understanding of it. The decomposition method by SCOs offers a more logic and rational way for complex project valuation. Besides, projects with different milestones can be regarded as special cases of the complex projects. Thus the SCOs can be applied for valuation.

In addition to project valuation, SCOs also can offer several advantages for financial derivative applications. The SCO buyers pay a few premiums at the initial time and own the privilege to pay again when they exercise the right to gain the next fold SCOs. The SCOs will be discarded when they are not worth holding in sacrificing previous payment. This split-fee property lets the SCO owners pay proportionally according to available information at that time, instead of sinking option premium at the beginning. Thus the decision-making can be postponed under indefinite environments and more flexibility is offered to SCO holders. The feature with high potential under constrained cost can provide greater leverage and yield enhancement for SCO owners. SCOs can be tailored for financial institutions as risk management, such as hedging or mortgage pipeline risk (Bhattacharya, 2005).

§1.2 Result Sketch

This paper, using vanilla European options as building blocks, extends the compound option theory to multi-fold sequential compound options (SCOs) with random or fold-wise parameters as well as alternating puts and calls arbitrarily (see Table 1). An SCO is defined as a (compound) option written on another compound option, where the call/put feature of each fold can be assigned arbitrarily. The SCOs presented in this study also allow parameters (such as volatility, interest rate) to vary over time or fold. This study derives an explicit valuation formula for SCOs by the risk-neutral method and change of numéraire method (Geman et al., 1995; Shreve, 2004) respectively, and performs the sensitivity analysis on the result. The option price is measured in units of a numéraire asset to make the derivation simple. Compared with the P.D.E. method, more financial intuition is gained by the change of numéraire derivation. Nonetheless, the partial derivative of a multivariate normal integration (a special case of Leibnitz's Rule), is also derived here for the sensitivity analysis.

Multi-fold SCOs with alternating puts and calls and random parameters can greatly enhance the number of practical applications for compound options, especially in the real option field. Real world cases can often be expressed in terms of multiple interacting options (Trigeorgis, 1993, 1996) of different types, such as expansion, contraction, shutting down, abandon, switch, and/or growth. The interaction between different types of options could be evaluated by the SCOs. For example, a highway or utility construction build-operate-transfer (BOT) project could be regarded as a vanilla call option. The simple expansion or extension privilege, which allows only once at a certain date, hence can be evaluated using the 2-fold compound option: call on call; the abandon for the main construction project could be appraised by the put on put. Similarly, the abandon option on the expansion or extension right could be viewed as a 3-fold compound option: the put on call on call. While the expansion or extension is flexible, such as been allowed to launch within a time period or perform for two or more times, the compound option evaluated for the privilege is exotic (Agliardi, 2006). Consequently, the valuation formula of the abandon on the expansion is also a exotic SCO.

The SCOs discussed in this study make the evaluation of exotic multiple interacting options possible. The SCOs can also be applied to the existing real option applications, such as the competing technology adoption (Kauffman and Li, 2005), joint ventures behavior analysis (Kogut, 1991) and strategic project examination (Bowman and Moskowitz, 2001). Furthermore, the pricing of exotic financial derivatives, such as exotic chooser options and capletions, can also be accomplished using SCOs.

The numerical examples of SCOs are proposed, including evaluation of government revenue guarantee and currency hedging. In addition, the information management system with SCOs as its core module is also proposed in order to evaluating projects and financial derivatives.

§1.3 Contribution

The contribution of this study are listed as follows.

- Enable the realistic and flexible valuation for complex projects, such as the BOT, new drug applications (NDAs).
- Define and analysis of SCOs.
- Derive of the partial derivative of the multivariate normal integral, which can be applied widely for the sensitivity of financial derivatives.
- Broaden the financial derivative pricing.
- Enhance Risk Management.

§1.4 Dissertation Structure

This dissertation is arranged as the follows. Chapter 2 presents the knowledge roadmap of related literatures. Chapter 3 derives the pricing formula of SCOs. Chapter 4 presents the partial derivative of multivariate normal integrand. Chapter 5 derives some comparative statistics of SCOs. The recursive computing algorithm and numerical examples are presented in Chapter 6. Chapter 7 presents further SCOs applications, including milestone project valuation, complex project valuation, derivatives pricing, hedging of volatility risk, mortgage pipeline risk and weather risk. Chapter 8 illustrates two numerical SCOs examples. Chpater 9 exhibits the framework of information management system with SCOs evaluation as its core module. The paper ends with the conclusion.

Chapter 2 The Knowledge Roadmap

This chapter describes the knowledge roadmap (Figure 2.1) of related literatures. The project valuation methods mentioned in Chapter 1 are concluded by Razgaitis (1999). The following paragraphs focus on compound option and real option methodology.

Compound options, initiating by Geske (1977; 1979), are options with other options as underlying assets. The *fold number* of a compound option counts the number of option layers tacked directly onto underlying options. The original closed form of compound option is proposed by Geske (1977; 1979) and constitutes as a precedent with respect to later works. Specific multi-fold compound option pricing formulas are proposed by Geske and Johnson (1984a) and Carr (1988) while the pricing formula sequential compound call (SCC) is proved by Thomassen & Van Wouwe (2001) and Chen (2003). Chen (2002) and Lajeri-Chaherli (2002) simultaneously derive the price formula for 2-fold compound options through the risk-neutral method. Agliardi & Agliardi (2003) generalize that results to 2 fold compound calls with time-dependent parameters, while Thomassen & Van Wouwe (2003) and Agliardi & Agliardi (2005) extend the multi-fold compound call options to parameters varying with time. The evolution of compound methodology is listed in Table 2.1.

Financial applications based on compound option theory are widely employed. Geske and Johnson (1984a) derived an analytic multi-fold exotic compound option formula for the American put option, while Carr (1988) presented the pricing formula for sequential exchange options. Corporate debt (Chen, 2003; Geske & Johnson, 1984b) and chooser options (Rubinstein, 1992), as well as *capletions* and *floortions* (options on interest rate options) (Musiela & Rutkowski, 1998) are also priced by compound options.

In addition to the pricing of financial derivatives, compound option theory is widely used in the study. This approach originate from Myers (1977) and follow by Brennan and Schwartz (1985), Pindyck (1988), Trigeorgis (1993, 1996) and so forth. Examples include project valuation of new drugs (Cassimon et al., 2004), production and inventory (Cortazar & Schwartz, 1993) and capital budget decision (Duan et al., 2003). Compound options turn out to be very common, and the theory is versatile enough to treat many real-world cases (Copeland and Antikarov, 2003). Some interesting topics of compound options are left for readers, such as stochastic volatility (Fouque and Han, 2005), stochastic interest rates (Thomassen and Van Wouwe, 2003; Lee et al., 2007), options with extendible maturities (exotic compound options, Longstaff, 1990), modular derivation (Zhu, 2000).

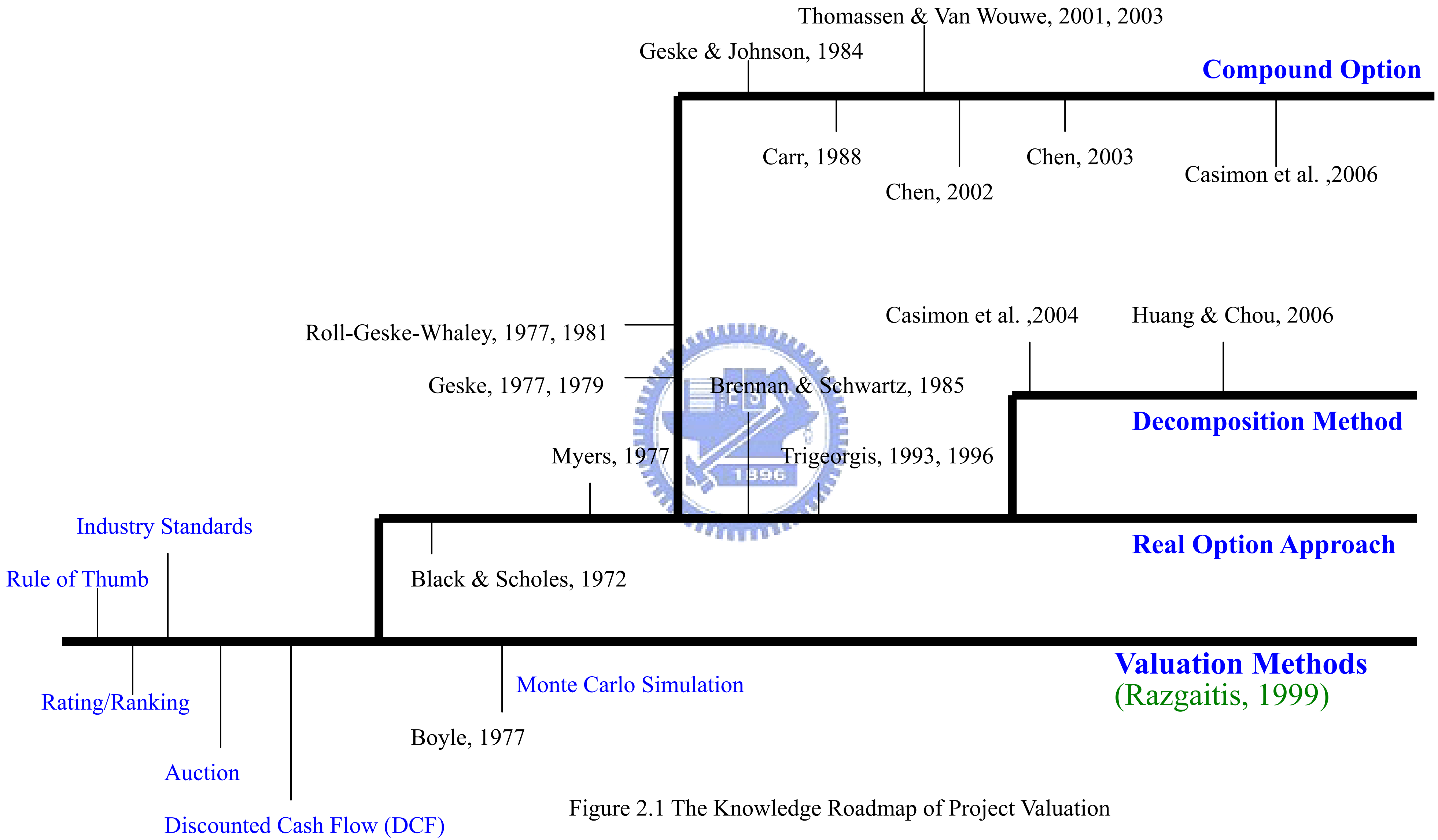


Figure 2.1 The Knowledge Roadmap of Project Valuation

Table 2.1 Evolutions of Compound Option Theory

Reference	Fold Number	Approach	Generalization	
			Put-Call alternating	time-dependent parameters
Geske (1977; 1979)*	2	PDE	Put/Call	No
Agliardi & Agliardi (2003)	2	PDE	Call	Yes
Chen (2002);Lajeri-Chaherli (2002)	2	Risk-neutral	Put/Call	No
Carr (1988), Chen (2003)	Multiple	Risk-neutral	Call	No
Thomassen & Van Wouwe (2001)	Multiple	PDE	Call	No
Thomassen & Van Wouwe (2003)	Multiple	PDE	Call	Yes
Agliardi & Agliardi (2005)	Multiple	Risk-neutral	Call	Yes
This Study	Multiple	Risk-neutral & Change of numéraire	Put/Call	Yes

*: The seminal compound option paper series.

Chapter 3 Valuation of the Sequential Compound Options

This chapter derives the analytic formula for the SCOs in both cases of fold-wise parameters and random parameters. Section 3-1 describes the notation and fundamental theorems used for derivations. Section 3-2 derives the closed-form price of SCOs with fold-wise parameters by the risk-neutral method. The analytic pricing formula of the generalzed SCOs, in which the interest rate and variance of asset price are random, are derived by the change of numéraire method in Section 3-3. Section 3-4 explains the implication of these pricing formulas. The main results of this chapter are available in Lee et al. (2007).

§3-1 Notation and Foundations

This section describes notation and foundation theorems used for SCOs valuation. Denote the correlation matrix $\mathbf{Q}_k := [Q_{\{k\},g,h}]_{k\times k}$, where $Q_{\{k\},g,h}$ is the symmetric (g, h) entry of the matrix $\mathbf{Q}_k$, $\forall 1\le g\le h\le k$. Similarly, $d_{\{k\},g}$ is the g-th entry of the vector $\left[d_{\{k\},g}\right]_{k\times1}$. $\left([Q_{\{k\},g,h}]_{k\times k}\right)^{(-i,-j)}$ is the (k–1) by (k–1) matrix which excludes the i-th row and the j-th column of $[Q_{\{k\},g,h}]_{k\times k}$. Define the function $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$. The k-variate normal integral with upper bound limit vector $\left[d_{\{k\},g}\right]_{k\times1}$ and correlation matrix $\mathbf{Q}_k$ is characterized as

$$\mathbf{N}_k\left\{\left[d_{\{k\},g}\right]_{k\times1};\mathbf{Q}_k\right\}=\int_{-\infty}^{d_{\{k\},1}}\int_{-\infty}^{d_{\{k\},2}}\cdots\int_{-\infty}^{d_{\{k\},k}}\frac{1}{(2\pi)^{\frac{k}{2}}\sqrt{|\mathbf{Q}_k|}}e^{-\frac{1}{2}\mathbf{Z}'\mathbf{Q}_k^{-1}\mathbf{Z}}\mathrm{d}z_k\mathrm{d}z_{k-1}\cdots\mathrm{d}z_1,$$

where $\mathbf{Z}'=[z_1,z_2,\cdots,z_k]$, and $\mathbf{N}_0\equiv1$. The following theorem is the statement about the construction of multivariate normal integrals.

Theorem 3.1

(a) The relationship between the (k-1) and k–variate normal integrals (Curnow & Dunnett, 1962)

$$\forall 1\le v\le k,\ \mathbf{N}_k\left\{\left[d_{\{k\},g}\right]_{k\times1};\mathbf{Q}_k\right\}=$$

$$\int_{-\infty}^{d_{\{k\},v}} f(z_v)\mathbf{N}_{k-1}\left\{\left(\left[\frac{d_{\{k\},g}-Q_{\{k\},v,g}z_v}{\sqrt{1-\left(Q_{\{k\},v,g}\right)^2}}\right]_{k\times1}\right)^{(-v,)};\left(\left[\frac{Q_{\{k\},g,h}-Q_{\{k\},v,g}Q_{\{k\},v,h}}{\sqrt{1-\left(Q_{\{k\},v,g}\right)^2}\sqrt{1-\left(Q_{\{k\},v,h}\right)^2}}\right]_{k\times k}\right)^{(-v,-v)}\right\}\mathrm{d}z_v$$

(b) The decomposition of a multivariate normal integral (Schroder, 1989)

$$\mathbf{N}_k\left\{\left[d_{\{k\},g}\right]_{k\times1};\mathbf{Q}_k\right\}=\int_{-\infty}^{d_{\{k\},v}}\mathbf{N}_{v-1}\left\{\left[\frac{d_{\{k\},g}-Q_{\{k\},g,v}z_v}{\sqrt{1-Q^2_{\{k\},g,v}}}\right]_{(v-1)\times1};\left[\frac{Q_{\{k\},g,h}-Q_{\{k\},g,v}Q_{\{k\},h,v}}{\sqrt{1-Q^2_{\{k\},g,v}}\sqrt{1-Q^2_{\{k\},h,v}}}\right]_{(v-1)\times(v-1)}\right\}$$

$$\times\mathbf{N}_{k-v}\left\{\left[\frac{d_{\{k\},v+g}-Q_{\{k\},v,v+g}z_v}{\sqrt{1-Q^2_{\{k\},v,v+g}}}\right]_{(k-v)\times1};\left[\frac{Q_{\{k\},v+g,v+h}-Q_{\{k\},v,v+g}Q_{\{k\},v,v+h}}{\sqrt{1-Q^2_{\{k\},v,v+g}}\sqrt{1-Q^2_{\{k\},v,v+h}}}\right]_{(k-v)\times(k-v)}\right\}f(z_v)\mathrm{d}z_v$$

where $\mathbf{Q}_k$ is the correlation matrix , $\forall 1\le v\le k$.

In Theorem 3.1, (a) reveals that the k-variate normal integral can be constructed from the (k–1)-variate by adding another dimension to the upper limit vector and correlation matrix. (b) states that the specific multivariate normal integral can be partitioned into two integrals of lesser variates. This result can extend the current compound option methodology from 2-fold to multi-fold by induction, while Chen (2003) just "observe a pattern" to generalize the SCC. Before applying this theorem to sequential compound option pricing, more pieces of notation are introduced in next section.

§3-2 Sequential Compound Options

This section derives the closed-form formula of sequential compound option prices with fold-wise parameters by the risk-neutral method, in which the asset price is assumed follows the geometric Brownian Motions process.

Let $T_{u\text{-}1} < T_u$, $\forall u \ge 1$. The asset price at time T_u is donated as S_u. The instantaneous volatility of the asset price is given as $\sigma(u)$. The instantaneous interest rate and dividend rate are denoted as $r(u)$ and $q(u)$, respectively. The dividend rate q_u can also be considered as the depreciation rate (Remer et al., 2001).

The fold numbers in this study come in reverse order. Denote $\Psi_i(T_0)$ as the i-fold SCO with strike price K_1 and it starts at time T_0 and expires at time T_1. The (i–1)-fold SCO $\Psi_{i-1}(T_1)$, active from T_1 to T_2, is the underlying asset of $\Psi_i(T_0)$. Provided that the last fold SCO starts from T_0, the underlying SCO $\Psi_{i-u+1}(T_{u-1})$ is valid from $T_{u\text{-}1}$ to T_u with fold number (i–u+1) and strike price K_u and has parameters σ_u^2, r_u, and q_u. $\Psi_1(T_{i-1})$ is the first fold option and a vanilla option with the asset as its underlying asset.

The notation for an arbitrary i-fold SCO starting from T_0 is exhibited in Figure 3.1. For any $u \ge 1$, the *option feature* $\Lambda_{u,u}$ characterizes the put or call attribute of the (underlying) SCO with fold number (i–u+1) ranging from $T_{u\text{-}1}$ to T_u. If the SCO of this fold is a put, $\Lambda_{u,u}$=–1, otherwise the feature $\Lambda_{u,u}$= 1 is for a call. For example, a put on a call (a 2-fold compound option) starting at T_0 has the option features $\Lambda_{1,1}$=−1 and $\Lambda_{2,2}$=1. Denote $\Lambda_{1,0} \equiv 1$ and $\Lambda_{h,g} = \prod_{u=g}^{h} \Lambda_{u,u}, \forall 1 \le g \le h$.

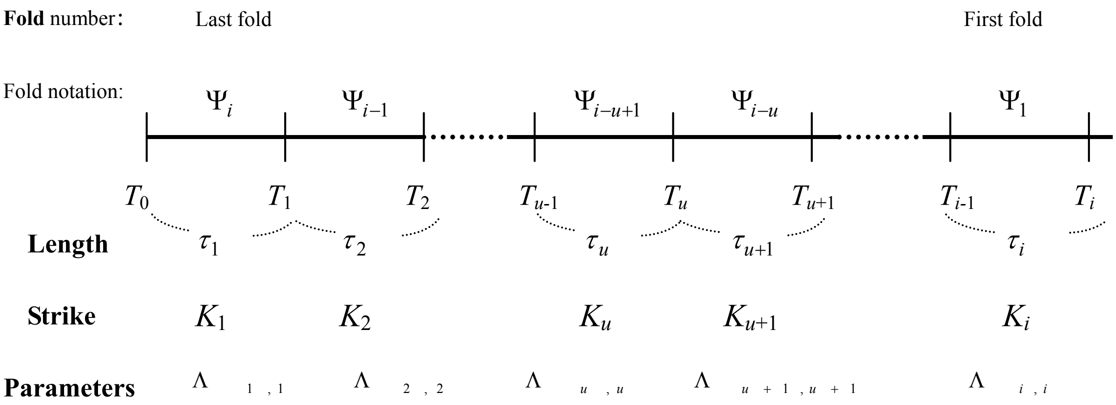


Figure 3.1: The Notation of the *i*-fold Sequential Compound Option

With the same assumptions as Thomassen and Van Wouwe (2001) except for "parameters constant in each fold" here, the following Theorem 3.1 derives the pricing formula of an i-fold SCO at time T_0, $\Psi_i(T_0)$, with arbitrary calls and puts by the risk-neutral method under a perfect market. Without loss of generality, the SCO in this theorem is assumed to start from T_0.

The assumptions of the SCOs are listed as follows.

1. No arbitrage.
2. The asset is tradable with any quantity.
3. Perfect market.
4. Perfect hedge.
5. No liquidity concerns.
6. No credit risk.
7. The asset price follows a geometric Brownian Motion process.
8. Assume there is no drift for the forward price under the risk neutral measure.
9. The volatility of the forward price is constant.
10. Any zero coupon prices are available.

Based on these assumptions, the SCO price is derived as a analytic form.

Theorem 3.2: Pricing Formula of Sequential compound option with Fold-wise Parameters

Denote Denote

$$(a)\, a_{i,g,*v} \equiv a_{i,g,*v}(S_v) = \frac{\ln\left(\frac{S_v}{S^{®}_{\#v+g,v+i}}\right) + \int_{T_v}^{T_{v+g}} \left[r(u) - q(u) + \frac{1}{2}\sigma^2(u) \right] du}{\sqrt{\int_{T_v}^{T_{v+g}} \sigma^2(u)du}}, \quad \forall g \geq 1$$

$$(b)\, b_{i,g,*v} \equiv b_{i,g,*v}(S_v) = a_{i,g,*v}(S_v) - \sqrt{\int_{T_v}^{T_{v+g}} \sigma^2(u)du}, \quad \forall g \geq 1$$

$$(c)\, \tilde{\rho}_{g,h,*v} = \Lambda_{v+h-1,v+g}\rho_{g,h,*v}, \forall h > g \geq 1;\ \rho_{g,g,*v} = 1, \forall g;\ \rho_{g,h,*v} = \rho_{h,g,*v}, \forall h, g;$$

$$\rho_{g,h,*v} = \sqrt{\frac{\int_{T_v}^{T_{v+g}} \sigma^2(u)du}{\int_{T_v}^{T_{v+h}} \sigma^2(u)du}}, \forall 1 \leq g < h.$$

$$(d)\, a_{i,g,\#v} \equiv a_{i,g,*v}(S_{\#v,i});\ b_{i,g,\#v} \equiv b_{i,g,*v}(S_{\#v,i})$$

$$(e)\, a_{i,g} \equiv a_{i,g,*0};\ b_{i,g} \equiv b_{i,g,*0};\ \rho_{i,g} \equiv \rho_{i,g,*0};\ \tilde{\rho}_{g,h} \equiv \tilde{\rho}_{g,h,*0}$$

(f) Equivalent asset price of the underlying (EAP)

$$S_{\#g,i} = \begin{cases} K_i, \text{ for } g = i \\ \text{The asset price which makes } \Psi_{i-g}(T_g) = K_g, \ \forall 1 \leq g < i \end{cases}$$

then

$$\Psi_i\,(T_0) = \Lambda_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} S_0 \mathbf{N}_i \left\{ \left[\Lambda_{i,g} a_{i,g}\right]_{i\times 1}; \left[\tilde{\rho}_{g,h}\right]_{i\times i} \right\} - \sum_{j=1}^{i} \Lambda_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \mathbf{N}_j \left\{ \left[\Lambda_{i,g} b_{i,g}\right]_{j\times 1}; \left[\tilde{\rho}_{g,h}\right]_{j\times j} \right\}$$

......(3.2.1)

under the assumption that the EAP ($S_{\#g,i}$) exists, $\forall 1 \leq g \leq i$.

Proof: see Appendix A.

§3-3 The Existence of Equivalent Asset Price

This section proves the existence conditions of the equivalent asset price. Denote $\tilde{\Psi}_{i,2\&3}(T_0) = \sum_{j=1}^{i} \Lambda_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \mathbf{N}_j \left\{ \left[\Lambda_{i,g} b_{i,g}\right]_{j\times 1}; \left[\tilde{\rho}_{g,h}\right]_{j\times j} \right\}$, which is the second component of the SCO pricing formula in Equation (3.2.1). Note that $\tilde{\Psi}_{i,2\&3}(T_0)$ may be negative or positive but all SCO prices $\Psi_i(T_0)$ are always nonnegative.

Lemma 3.1: The Sufficient Condition for the Existence of Existence of Equivalent Price (EAP)

Given g ($1 \le g \le i-1$), the $S_{\#g,i}$ exists if

(a) $S_{\#\ell,i}$ exists for all $g-1 \le \ell \le i-1$,

and either the following condition stand.

(b) $\Lambda_{i-g,1} = +1$;

(c) $\Lambda_{i-g,1} = -1$ and $K_g \le -\widetilde{\Psi}_{i-g,2\&3}(T_g)$.

Proof: see Appendix B.

The condition (a) of Lemma 3.1 reveals that the existence conditions is also derived based essentially on the induction, by which the multi-fold SCO price is available in Theorem 3.2. If the EAPs of previous folds exist, the EAP existence of the current fold is discussed according to the different sign of the cumulative option future $\Lambda_{i-g,1}$. The condition (c) states that the strike price of the current fold K_g is limited by a maximum because the asset price has opposite direction against the current fold SCO price. The opposite direction is represented by the negative cumulative option feature. For the case of positive cumulative option feature (condition (b)), there is no restriction for the strike price. The non-existing EAP will incur the zero SCO price.

The SCO price (Ψ_i) is monotone with respect to the asset price and hence the equivalent asset price (EAP, $S_{\#g,i}$) is unique if it exists. According to the sensitivity analysis in Theorem 5.1, the Delta ($\frac{\partial \Psi_i(T_0)}{\partial S(T_0)}$) is a strictly monotone function. Its increasing or decreasing nature depends on the cumulative option feature ($\Lambda_{i,1}$). Therefore the EAP, defined as the asset price making the SCOs price equal to a specific strike price, is unique if it exists. The EAP may not exist due to the range limitation of a decreasing SCO price.

§3-4 The Generalized Sequential Compound Options

This section derives the closed-form formula of sequential compound option prices with random parameters, including the interest rate, the variance of asset price. Comparing with the cases of fold-wise parameters, it is named as the generalized SCOs due to of random parameters. It is assumed that there is no depression rate here.

Assume the asset price and instantaneous variance of asset price at time t are given as $S(t)$ and $\sigma^2(t)$, respectively. Denote the interest rate process $r(t)$, $T_0 \le t \le T_i$

and the discount process $D(t) = e^{-\int_{T_0}^{t} r(u)du}$. Let $B(t, T)$ be the zero coupon bond price at time t that matures at time T. The bond price $B(T_0, T_j)$ can be represented by the stochastic interest rate $r(t)$, $B(T_0, T_j) = e^{-\int_{T_0}^{T_j} r(u)du}$. In other words, the bond price is determined by the interest rate $r(t)$. Note the bond prices act as the representation of stochastic interest rate and hence there is no need to specify the interest rate dynamics in this study. Denote the τ-forward price of T-maturing zero coupon price at time t as $F(t, \tau, T)$, $\forall t \le \tau \le T$.

Denote as $\Psi_i^{®}(T_0)$ the i-fold generalized SCO price starting at time T_0 and expiring at time T_1, with strike K_1. The notation ® stands for "stochastic interest rate and random variance of asset price". Its underlying asset is the (i–1)-fold SCO $\Psi_{i-1}^{®}(T_1)$, which is active from T_1 to T_2. Under the assumption that the last fold SCO starts from T_0, the underlying SCO with fold number (i–u+1), $\Psi_{i-u+1}^{®}(T_{u-1})$, is valid from T_{u-1} to T_u with strike price K_u. The first fold option, $\Psi_1^{®}(T_{i-1})$, is a vanilla option with the asset as its underlying asset. It should be noted that fold numbers come in the reverse order. The notation for an arbitrary i-fold generalized SCO starting from T_0 is exhibited in Figure 3.2.

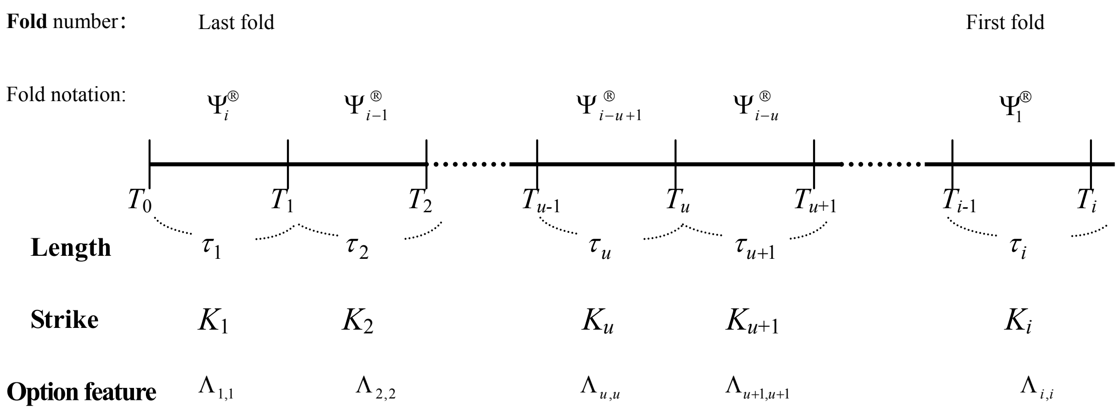


Figure 3.2: The Notation of the *i*-fold Generalized Sequential Compound Option

Under all the assumptions of Thomassen and Van Wouwe (2001), except for random parameters of interest rate and variance of asset price, the following theorem derives the pricing formula of an i-fold SCO with alternating arbitrarily calls and puts by the change of numéraire method. Although the SCOs presented in later sections can start at any time T_u, the SCO in this theorem is starting from T_0 without loss of generality. The symbol " $*v$ ", meaning "start from time T_v", is used to indicate time shift in the sensitivity derivation. Following the above notation, $\Psi_i^{\circledR}(T_0)$ is denoted as the SCO price at the time T_0.

Theorem 3.3: The generalized sequential compound option pricing with random parameters

Denote

$$\text{(a) } a_{i,g,*v}^{\circledR}(S(T_v)) = \frac{\ln\left(\dfrac{S(T_v)}{B(T_v,T_g)S_{\#g,i}^{\circledR}}\right) + \dfrac{1}{2}\displaystyle\int_{T_v}^{T_{v+g}} \sigma^2(t)dt}{\sqrt{\displaystyle\int_{T_v}^{T_{v+g}} \sigma^2(t)dt}}, \quad \forall g \ge 1$$

$$\text{(b) } b_{i,g,*v}^{\circledR}(S(T_v)) = a_{i,g,*v}^{\circledR}(S(T_v)) - \sqrt{\int_{T_v}^{T_{v+g}} \sigma^2(t)dt}, \quad \forall g \ge 1$$

$$\text{(c) } \tilde{\rho}_{g,h,*v}^{\circledR} = \Lambda_{v+h-1,v+g}\,\rho_{g,h,*v}^{\circledR}, \forall h > g \ge 1;\ \rho_{g,g,*v}^{\circledR} = 1, \forall g;\ \rho_{g,h,*v}^{\circledR} = \rho_{h,g,*v}^{\circledR}, \forall h, g;$$

$$\rho_{g,h,*v}^{\circledR} = \sqrt{\frac{\displaystyle\int_{T_v}^{T_{v+g}} \sigma^2(t)dt}{\displaystyle\int_{T_v}^{T_{v+h}} \sigma^2(t)dt}}, \forall 1 \le g < h.$$

$$\text{(d) } a_{i,g,\#v}^{\circledR} \equiv a_{i,g,*v}^{\circledR}(S_{\#v,i}^{\circledR});\ b_{i,g,\#v}^{\circledR} \equiv b_{i,g,*v}^{\circledR}(S_{\#v,i}^{\circledR})$$

$$\text{(e) } a_{i,g}^{\circledR} \equiv a_{i,g,*0}^{\circledR};\ b_{i,g}^{\circledR} \equiv b_{i,g,*0}^{\circledR};\ \rho_{i,g}^{\circledR} \equiv \rho_{i,g,*0}^{\circledR};\ \tilde{\rho}_{g,h}^{\circledR} \equiv \tilde{\rho}_{g,h,*0}^{\circledR}$$

(f) Equivalent asset price

$$\text{(EAP) } S_{\#g,i}^{\circledR} = \begin{cases} K_i, \text{ for } g = i \\ \text{The asset price which makes } \Psi_{i-g}^{\circledR}(T_g) = K_g, \ \forall 1 \le g < i \end{cases}$$

then

$$\Psi_i^{\circledR}(T_0) = \Lambda_{i,1}S(T_0)\mathbf{N}_i\left\{\left[\Lambda_{i,g}a_{i,g}^{\circledR}\right]_{i\times 1};\left[\tilde{\rho}_{g,h}^{\circledR}\right]_{i\times i}\right\} - \sum_{j=1}^{i}\Lambda_{j,1}B(T_0,T_j)K_j\mathbf{N}_j\left\{\left[\Lambda_{i,g}b_{i,g}^{\circledR}\right]_{j\times 1};\left[\tilde{\rho}_{g,h}^{\circledR}\right]_{j\times j}\right\} \quad \ldots\ldots(3.4.1)$$

under the assumption that the EAP ($S^{®}_{\#g,i}$) exists, $\forall 1 \le g \le i$.

Proof: see Appendix C.

The SCO price in Equation (3.4.1) is derived by the change of numéraire method, which is also known as the forward measure approach because it makes a second change of measure from the risk-neutral measure to a forward measure. Equation (C.5) is the key of derivation. It means that the current asset price is the expectation of the future price with interest rate discount. Both of $D(T_1)$ and $\max[\Lambda_{1,1}\Psi_i^{®}(T_1)-\Lambda_{1,1}K_1]$ within the expectation contain the interest rate $r(t)$, which is a adopted stochastic process, and they can not be dealt separately. Thus the asset price and bond price are regarded as numéraires to overcome the difficulty.

§3-5 Interpretation of the Formula

This section interprets the implication of pricing formula of SCOs (Equation (3.2.1) and (3.4.1)).

According to Equation (3.2.1) & (3.4.1), the price of an i–fold SCO can be expressed as the weighted asset price minus the weighted sum of the strike prices of the i folds with different underlying assets. The weights consist of three factors: the cumulative option features, the bond prices, and the in-the-money probabilities. The cumulative option feature is obtained by synthesizing the option features from the current fold to the last fold. The bond price is a deduction made due to interest rate compounding. The in-the money probabilities are assessed under different probability measures by multivariate normal integrations. The factors $a_{i,g} / a^{®}_{i,g}$ and $b_{i,g} / b^{®}_{i,g}$ in the integration are similar to the "d_1" and "d_2" appearing in conventional pricing formulas for vanilla options. The correlation matrices of SCOs are similar to those of the sequential compound calls, except for a sign change due to the cumulative option features. Within these 3 weighting factors, the parameters of the last fold have the widely impact on the pricing formula.

These formulas of SCOs are more general than those derived for vanilla options, 2–fold compound options, and sequential compound calls, all of which can be regarded as special cases of SCOs. The main difference between SCOs and sequential compound calls lies in the freedom to alternate calls and puts, which is represented by a sign changes in the cumulative option features $\Lambda_{h,g}$, $\forall 1 \le g \le h$. In other words, the option prices will depend on the fold features $\Lambda_{h,g}$. Moreover, allowing the parameters to vary over time makes the integrated variance and discounting bond price of an SCO quite different from that derived by Thomassen & Van Wouwe

(2001). Setting all $\Lambda_{h,g}$ to +1 in an SCO results in a SCC.

Chapter 4 Partial Derivative of the Multivariate Normal Integral

This chapter derives the partial derivative of the multivariate normal integral by induction. The result (Theorem 4.2) can be regard as the special case of Leibnitz's rule and will be applied during the derivation of SCOs' sensitivity analysis (see Chapter 5).

First of all, the Leibnitz's Rule is listed.

Theorem 4.1 (Leibnitz's Rule) (Casella & Berger 2002, p.69)

If $f(x,\theta)$, $a(\theta)$, $b(\theta)$ are differentiable with respect to θ, then

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)\mathrm{d}x = f(b(\theta),\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}b(\theta) - f(a(\theta),\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)}\frac{\partial}{\partial\theta}f(x,\theta)\mathrm{d}x .$$

It is noted that if $a(\theta)$, $b(\theta)$ are constant, we have a special case of Leibnitz's Rule:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{a}^{b} f(x,\theta)\mathrm{d}x = \int_{a}^{b}\frac{\partial}{\partial\theta}f(x,\theta)\mathrm{d}x .$$

According to the Leibnitz's Rule, the partial derivative of the multivariate normal integral is derived.

Theorem 4.2: Partial Derivative of the Multivariate Normal Integral

Let $d_{\{k\},g}(G_1, G_2,\ldots, G_p) \equiv d_{\{k\},g}$, representing a function of $G_1, G_2,\ldots, G_p$.

$$\forall 1 \le k, 1 \le \ell \le p,\ \frac{\partial \mathbf{N}_k([d_{\{k\},g}]_{k\times 1};[Q_{\{k\},g,h}]_{k\times k})}{\partial G_\ell}$$

$$= \sum_{j=1}^{k} f\left(d_{\{k\},j}\right)\frac{\partial d_{\{k\},j}}{\partial G_\ell}\mathbf{N}_{k-1}\left\{\left(\left[\frac{d_{\{k\},g} - d_{\{k\},j}Q_{\{k\},j,g}}{\sqrt{1-Q^2_{\{k\},j,g}}}\right]_{k\times 1}\right)^{(-j,)};\left(\left[\frac{Q_{\{k\},g,h} - Q_{\{k\},j,g}Q_{\{k\},j,h}}{\sqrt{\left(1-Q^2_{\{k\},j,g}\right)\left(1-Q^2_{\{k\},j,h}\right)}}\right]_{k\times k}\right)^{(-j,-j)}\right\}$$

......(4.1.1)

where $[Q_{\{k\},g,h}]_{k\times k}$ is a correlation matrix that is not a function of G_ℓ.

Proof: see Appendix D.

Theorem 4.2 shows that the partial derivatives of a (k+1)–variate normal integration can be represented as the k+1 weighted sum of k–variate normal integrations. As Equation (4.1.2) shows, the Leibnitz's rule can be used to decompose the partial derivative into two parts. The first term is a k–variate normal integration with a weighting factor. The second part is an integration of a partial derivative of the (k–1)-variate normal. Theorem 4.2 proves, however, that this second part can be represented in the same form as the first term. This means that Theorem 4.2 extends the Leibnitz's rule to multivariate normal cases and it can be regarded as the special case of Leibnitz's rule.

The specific partial derivatives presented in Thomassen & Van Wouwe (2002) can be viewed as a special case of Theorem 4.2. If the elements of the correlation matrix in Equation (4.1.1) as specified as $Q_{\{k\},g,h} = Q_{\{k\},h,g} = \sqrt{\frac{\tau_g}{\tau_h}}$, for $1 \le g \le h$, then

$\frac{Q_{\{k\},g,h} - Q_{\{k\},j,g} Q_{\{k\},j,h}}{\sqrt{(1 - Q^2_{\{k\},j,g})(1 - Q^2_{\{k\},j,h})}}$ =0, for $g<j<h$ or $h<j<g$.

Chapter 5 Greeks: the Sensitivity Analysis

This chapter probes the sensitivity analysis of SCOs. A short Lemma is discussed before the analysis.

A feature of multivariate normal integrations will be presented after the following notation has been defined. Let

$$\aleph_{v-1,b} \equiv \mathbf{N}_{v-1}\left\{ \left[\frac{\Lambda_{i,g} b^{\circledR}_{i,g} - \Lambda_{i,v} b^{\circledR}_{i,v}\tilde{\rho}^{\circledR}_{g,v}}{\sqrt{1-(\tilde{\rho}^{\circledR}_{g,v})^2}} \right]_{(v-1)\times 1} ; \left[\frac{\tilde{\rho}^{\circledR}_{g,h} - \tilde{\rho}^{\circledR}_{v,g}\tilde{\rho}^{\circledR}_{v,h}}{\sqrt{[1-(\tilde{\rho}^{\circledR}_{v,g})^2][1-(\tilde{\rho}^{\circledR}_{v,h})^2]}} \right]_{(v-1)\times(v-1)} \right\},$$

$$\aleph_{j-1,b,-v} \equiv \aleph_{j-1,b,-v}(b^{\circledR}_{i,g}) = \mathbf{N}_{i-1}\left\{ \left(\left[\frac{\Lambda_{i,g} b^{\circledR}_{i,g} - \Lambda_{i,v} b^{\circledR}_{i,v}\tilde{\rho}^{\circledR}_{v,g}}{\sqrt{1-(\tilde{\rho}^{\circledR}_{v,g})^2}} \right]_{i\times 1} \right)^{(-v,)} ; \left(\left[\frac{\tilde{\rho}^{\circledR}_{g,h} - \tilde{\rho}^{\circledR}_{v,g}\tilde{\rho}^{\circledR}_{v,h}}{\sqrt{[1-(\tilde{\rho}^{\circledR}_{v,g})^2][1-(\tilde{\rho}^{\circledR}_{v,h})^2]}} \right]_{i\times 1} \right)^{(-v,-v)} \right\},$$

$$\aleph_{i-1,a,-v} \equiv \aleph_{i-1,a,-v}(a^{\circledR}_{i,g})$$

Lemma 5.1 shows that the multivariate integrals for SCO sensitivities can be factored into two separated normal integrals.

Lemma 5.1

(a) $\aleph_{i-1,a,-v} = \aleph_{v-1,b} \times \mathbf{N}_{i-v}\left\{ \left[\Lambda_{i,v+g} a_{i,g,\#v}\right]_{(i-v)\times 1} ; \left[\tilde{\rho}_{g,h,*v}\right]_{(i-v)\times(i-v)} \right\}$

(b) $\aleph_{j-1,b,-v} = \aleph_{v-1,b} \times \mathbf{N}_{j-v}\left\{ \left[\Lambda_{i,v+g} b_{i,g,\#v}\right]_{(j-v)\times 1} ; \left[\tilde{\rho}_{g,h,*v}\right]_{(j-v)\times(j-v)} \right\}$

(c) $\aleph^{\circledR}_{i-1,a,-v} = \aleph^{\circledR}_{v-1,b} \times \mathbf{N}_{i-v}\left\{ \left[\Lambda_{i,v+g} a^{\circledR}_{i,g,\#v}\right]_{(i-v)\times 1} ; \left[\tilde{\rho}^{\circledR}_{g,h,*v}\right]_{(i-v)\times(i-v)} \right\}$

(d) $\aleph^{\circledR}_{j-1,b,-v} = \aleph^{\circledR}_{v-1,b} \times \mathbf{N}_{j-v}\left\{ \left[\Lambda_{i,v+g} b^{\circledR}_{i,g,\#v}\right]_{(j-v)\times 1} ; \left[\tilde{\rho}^{\circledR}_{g,h,*v}\right]_{(j-v)\times(j-v)} \right\}$

Sketch Proof:

The left-hand sides of Lemma 3.1 (a) and (b) are identical, hence the integrand of the left hand sides. Lemma 5.1 can be proved according to the above result. Lemma 5.1 can also be proved directly through a multivariate normal integration whose correlation matrix can be partitioned into "four quadrants". The top-right and the bottom-left quadrants are zero matrices, so the integrals can be represented as the product of two uncorrelated normal integrals (Bickel and Doksum, 2001, Theorem B.6.4). **Q.E.D.**

Note that the same factor $\aleph_{v-1,b}$ appears on the right-hand side of Lemma 5.1 (a) & (b), and the same factor $\aleph^{\circledR}_{v-1,b}$ appears on the right-hand side of Lemma 5.1 (c) & (d).

The sensitivity analysis of SCOs is now possible thanks to the two results

(Theorem 4.2 and Lemma 5.1) demonstrated in the preceding section and chapter. Thomassen and Van Wouwe (2002) derived the sensitivities of SCCs. Theorem 5.1 extends their analysis to SCOs with the possibility of alternating calls and puts arbitrarily based on Theorem 4.2. Theorem 5.1 also shows the interest rate sensitivity under the special case of interest rate fold-wise.

Theorem 5.1: Sensitivities of SCOs

(a) Delta: $\dfrac{\partial \Psi_i(T_0)}{\partial S_0} = \Lambda_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} \mathbf{N}_i \left\{ \left[\Lambda_{i,g} a_{i,g}\right]_{i\times 1}; \left[\widetilde{\rho}_{g,h}\right]_{i\times i} \right\}$

(b) Gamma: $\dfrac{\partial^2 \Psi_i(T_0)}{\partial S_0^2} = \sum_{v=1}^{i} \dfrac{\Lambda_{v-1,1} e^{-\frac{1}{2}a_{i,v}^2 - \int_{T_0}^{T_i} q(u)du}}{S_0 \sqrt{2\pi \int_{T_0}^{T_v} \sigma^2(u)du}} \aleph_{i-1,a,-v}$

(c) Let the interest rate and the variance of asset price be fold-wise constant. In other words, $r(t) = r_u, \sigma(t) = \sigma_u, \forall T_{u-1} \le t < T_u, 1 \le u \le i$. Under this simplification, the "underscore" labels are added to the corresponding pieces of notation. The SCO price, the correlation matrix and the two upper limit vectors are denoted as $\underline{\Psi_i}$, $\underline{\widetilde{\rho}_{g,h}}$, $\underline{a_{i,g}}$ and $\underline{b_{i,g}}$, respectively. Thus, the interest rate sensitivity Rho is:

$$\forall 1 \le \ell \le i, \frac{\partial \underline{\Psi_i}(T_0)}{\partial r_\ell} = r_\ell \sum_{j=\ell}^{i} \Lambda_{j,1} K_j e^{-\sum_{u=1}^{j} r_u \tau_u} \mathbf{N}_j \left\{ \left[\Lambda_{i,g} \underline{b_{i,g}}\right]_{j\times 1}; \left[\underline{\widetilde{\rho}_{g,h}}\right]_{j\times j} \right\}.$$

(d) $\dfrac{\partial \Psi_i^{®}(T_0)}{\partial S_0} = \Lambda_{i,1} \mathbf{N}_i \left\{ \left[\Lambda_{i,g} a_{i,g}^{®}\right]_{i\times 1}; \left[\widetilde{\rho}_{g,h}^{®}\right]_{i\times i} \right\}$

(e) $\dfrac{\partial^2 \Psi_i^{®}(T_0)}{\partial S_0^2} = \sum_{v=1}^{i} \dfrac{\Lambda_{v-1,1} e^{-\frac{1}{2}(a_{i,v}^{®})^2}}{S_0 \sqrt{2\pi \int_{T_0}^{T_v} \sigma^2(t)dt}} \aleph_{i-1,a,-v}$

Proof: see Appendix

As SCOs pricing formulas (Theorem 3.2 and Theorem 3.3) generalize previous results for vanilla options and SCCs, the SCOs sensitivities given in Theorem 5.1 are also extension of these previous works intuitively. Again, the sequence of option features will affect the signs of the sensitivities. According to Theorem 5.1 (a) and (d), the value of a SCO is monotonic with respect to the current asset price $S(T_0)$, hence the EAP is unique if it exists.

Chapter 6 Computation Algorithm

This chapter explains the recursive computation algorithm of SCOs and illustrates the computing results of 3-fold SCOs.

§6.1 Computation Algorithm

The computation differences between European options and compound options (2-fold or more) lie in the EAP and the dimension of normal integrals. By definition, the EAP is the asset price which makes the (compound) option price equivalent to a specific strike price. Similar to the concept of implied volatility, the EAP can be regarded as the "implied asset price", solving by the known (compound) option price (given as the strike price) and other conventional option parameters except the asset price itself. Thus there is no EAP concern in the 1-fold option computation and it is calculated only for the 2 or more fold compound options. It seems that *i*-1 EAPs ($S_{\#g,i}, \forall 1 \le g < i$) are calculated during the *i*-fold SCO price computation. However, more EAPs are calculated because they are solving by the bisection method in this study and the higher-fold EAPs are obtained based on the lower-fold EAPs. Many EAPs are figured just for another and are not used straight for the SCOs price calculation. Hence the nested algorithms, using the lower-fold SCO pricing formula for EAPs while seeking for the higher-fold one, are time-consuming.

The other computing dissimilarity between European and compound options is the normal integrals. The highest dimension of normal integrals of the SCO equals its fold number. Precise computation of the multivariate normal integration needs more work than that of a univariate case. Besides, the precise approximation of multivariate normal integrals with arbitrary dimension and integration range is neither easy nor convenient, although the univariate, bivariate and trivariate cases are disclosed explicitly (Denz, 2004). Lin (2004) compares 3 computing methods for the multivariate normal integral, including the improved Gauss quadrature method, Monte Carlo method and Lattice method, to evaluating the 4-fold SCCs.The Monte Carlo integration is applied here for normal integral computation in case the higher fold SCOs are adopted. Casimon et al. (2004) even use the SCCs up to 6 fold!

The recursive computing algorithm of the SCO price, calculating from the first fold to the last fold, include 5 looped steps and are exhibited in Figure 6.1. The computing algorithm do not encompass any estimation or calibration of parameters, which should be ready when the algorithm begin. In the flow chart, the rhombuses represent decision symbols where a decision must be made, while the rectangles symbolize the actions. The details of the chart are explained as follows.

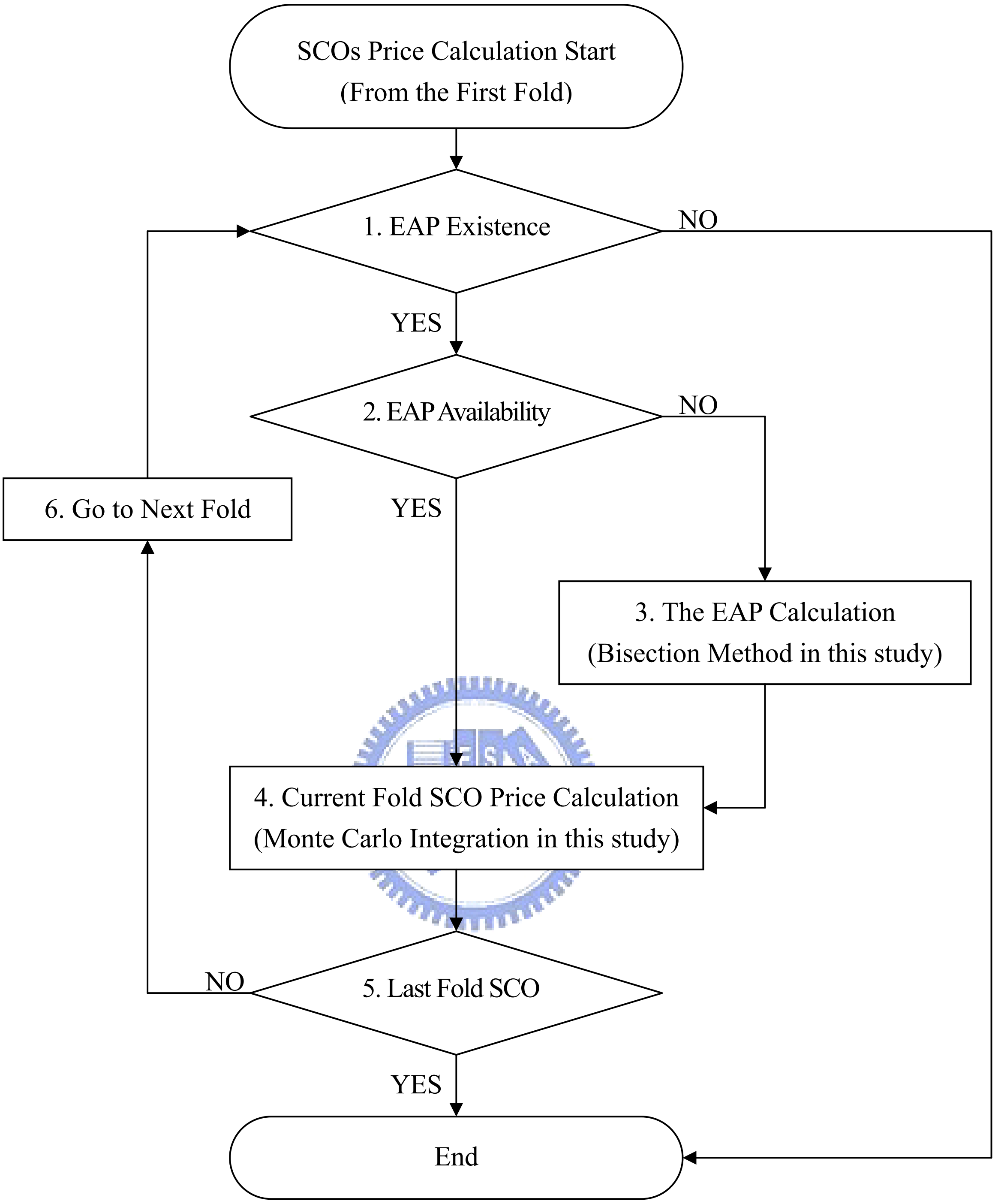


Figure 6.1 The Nested Computing Algorithm of the SCOs

Step 1: Check the EAP existence of the current fold. If EAP exists, go to Step 2, otherwise terminate the algorithm. The EAP may not exist because of the non-negative range limitation of the decreasing SCO price. There is no need to calculate EAP for the 1-fold option because it is for compound options only.

Step 2: Check the EAP availability. If the desired EAP is available, skip to Step 4, otherwise go to Step 3. The EAP calculation is time-consuming, thus it can be used repeatedly to save time if the same one was solved before.

Step 3: The EAP Calculation. Since the EAP is like "implied asset price", it is solved according to Theorem 3.2 (d) or Theorem 3.3 (f) to by the bisection method in this study. Within this step, it is necessary to calculate the lower-fold SCO prices, which is the main target of the computation algorithm. Hence it causes the processes to be nested and sophisticated.

Step 4: The Current Fold SCO Price Calculation. The SCO price is computed according to Equation (3.2.1) or (3.3.1) if all the EAPs are available. The cumulative probabilities of multivariate normal density are acquired by Monte Carlo integration in this study.

Step 5: Check whether the current fold is last fold. If yes, the last SCO price is the final result, otherwise go to Step 6.

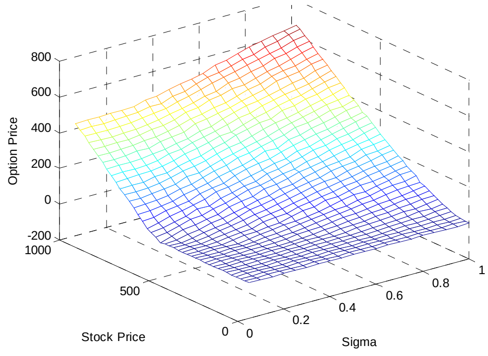
Step 6: Go to the next fold. If the current fold is not the last one, go to the next fold and results so far are bases to calculate the next fold SCO price. Compared with the current fold case, the dimension and fold number are increased by one to enter the next loop.

In the SCOs evaluation algorithm, there are one recursive loop and three decision nodes. The recursive loop occurs in the EAP calculation (Step 3), while the decision nodes take place in determining whether the current fold is last (Step 5), EAP existence (Step 1) and availability (Step 2), respectively. The recursive loop involved in the bisection method together with decision nodes makes the computation sophisticated.

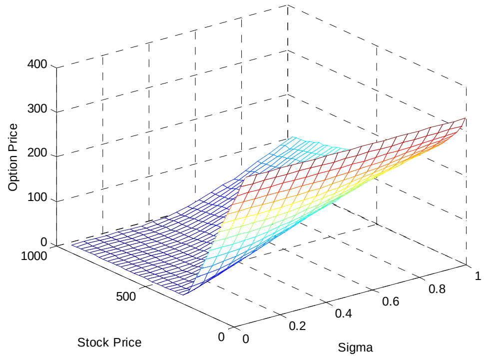
The numerical methods mentioned above, such as the bisection method for EAPs in Step 3 and Monte Carlo integration for multivariate normal integrals in Step 4, can be substituted by other suitable methods. The conventional options just need the "Step 4" to calculate the option price straightforwardly. By contrast, the looped and nested computation algorithm of SCO prices, involving some numerical techniques, are more complicated.

§6.2 Three-Fold SCOs Illustration

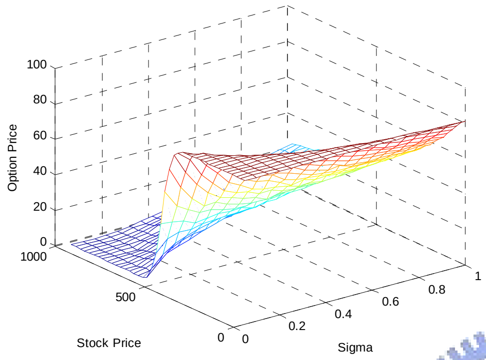
This subsection illustrates the 8 cases of 3-fold sequential compound options, including the call on call on call (CCC), call on call on put (CCP), call on put on call (CPC), call on put on put (CPP), put on call on call (PCC), put on call on put (PCP), put on put on call (PPC), put on put on put (PPP). The parameters of these SCOs are identical for comparison in the numerical examples. The time to maturity of 3 folds are all equal to one, and the strikes K_1, K_2, K_3 are 10, 100, 500 respectively. Assume the volatility and dividend rate keeps constant in these three folds. Figure 6.2 exhibits the SCOs price along the volatility and asset price.



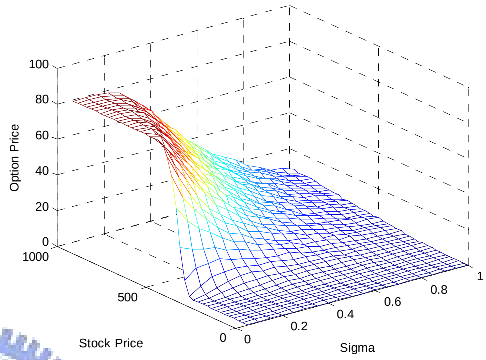
(a) Call on call on call



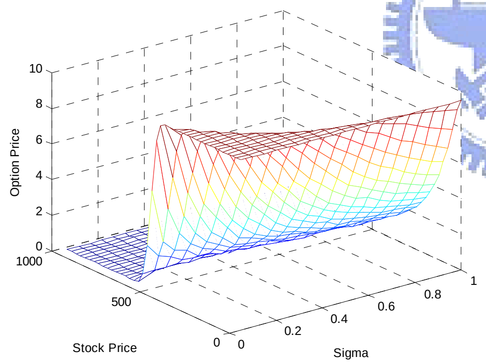
(b) Call on call on put



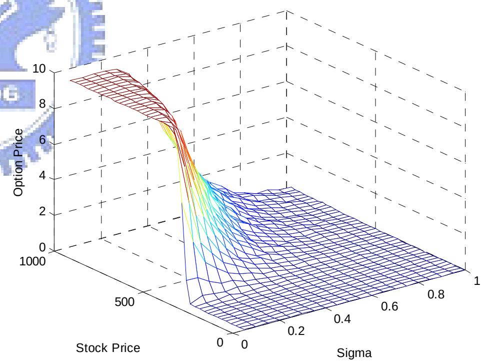
(c) Call on put on call



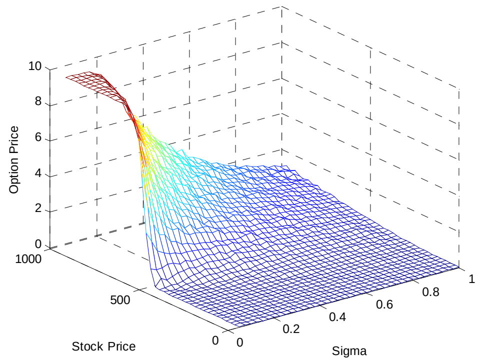
(d) Call on put on put



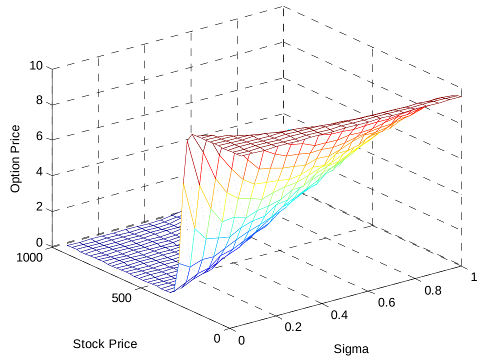
(e) Put on call on call



(f) Put on call on put



(g) Put on put on call



(h) Put on put on put

Figure 6.2: The Price Surface of 3-fold SCOs

The price surface of PCC is presented in Table 6.1 and Figure 6.2 (e). The follows explain the PCC to understand the feature of SCOs. The max price of PCC is about 9.51 because the last fold put option strikes with 10. The PCC price drops as the stock price hikes under the same volatility due to the put feature of the underlying asset. Although with different underlying assets, the PPP also has a similar phenomenon due to the same reason. This reason also supports the fact that the PCC price descends with the volatility (sigma) increasing under the same stock price. Theoretically, the SCOs are monotone with respect to the asset prices (Thomassen and Van Wouwe, 2002; Lee et al., 2007). However, the integrals evaluated by Monte Carlo simulation result in subtle non-monotonicity.

Table 6.1: Prices of the 3-fold SCO (Put on Call on Put)

SCOs Price		Volatility of Asset Price																			
		0.05	0.10	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
Asset Price	1	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51
	26	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.50	9.49	9.46	9.41	9.34	9.24	9.13	9.00	8.86
	51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.49	9.45	9.36	9.23	9.05	8.85	8.62	8.38	8.12	7.88	7.65
	76	9.51	9.51	9.51	9.51	9.51	9.51	9.51	9.50	9.46	9.35	9.18	8.93	8.62	8.29	7.98	7.65	7.33	7.02	6.76	6.51
	101	9.51	9.51	9.51	9.51	9.51	9.51	9.49	9.43	9.27	9.01	8.66	8.23	7.79	7.37	7.02	6.65	6.32	6.02	5.78	5.55
	126	9.51	9.51	9.51	9.51	9.51	9.50	9.42	9.23	8.90	8.44	7.94	7.38	6.87	6.43	6.09	5.74	5.43	5.16	4.95	4.75
	151	9.51	9.51	9.51	9.51	9.51	9.44	9.23	8.85	8.32	7.71	7.10	6.48	5.97	5.55	5.25	4.94	4.67	4.43	4.26	4.10
	176	9.51	9.51	9.51	9.51	9.48	9.30	8.85	8.28	7.59	6.88	6.24	5.62	5.14	4.76	4.51	4.25	4.02	3.83	3.69	3.56
	201	9.51	9.51	9.51	9.50	9.38	9.00	8.28	7.56	6.78	6.03	5.41	4.82	4.40	4.08	3.88	3.66	3.48	3.32	3.21	3.11
	226	9.51	9.51	9.51	9.46	9.17	8.52	7.55	6.74	5.94	5.21	4.65	4.11	3.76	3.49	3.34	3.17	3.02	2.89	2.81	2.74
	251	9.51	9.51	9.51	9.35	8.78	7.86	6.70	5.89	5.13	4.45	3.96	3.48	3.20	2.99	2.88	2.75	2.63	2.53	2.47	2.42
	276	9.51	9.51	9.48	9.09	8.19	7.05	5.81	5.06	4.38	3.78	3.36	2.95	2.73	2.56	2.49	2.39	2.30	2.22	2.19	2.15
	301	9.51	9.51	9.39	8.61	7.40	6.16	4.93	4.29	3.70	3.18	2.84	2.49	2.33	2.20	2.16	2.08	2.02	1.96	1.94	1.92
	326	9.51	9.50	9.14	7.88	6.48	5.25	4.10	3.59	3.10	2.67	2.40	2.11	1.99	1.90	1.88	1.83	1.78	1.74	1.73	1.72
	351	9.51	9.46	8.64	6.93	5.49	4.37	3.37	2.97	2.59	2.24	2.03	1.78	1.70	1.64	1.64	1.60	1.57	1.55	1.55	1.54
	376	9.51	9.28	7.82	5.82	4.52	3.57	2.72	2.44	2.15	1.87	1.71	1.51	1.46	1.42	1.44	1.41	1.39	1.38	1.39	1.39
	401	9.51	8.74	6.68	4.69	3.62	2.87	2.18	2.00	1.78	1.56	1.44	1.28	1.26	1.23	1.26	1.25	1.24	1.23	1.25	1.26
	426	9.49	7.61	5.36	3.62	2.82	2.27	1.73	1.62	1.47	1.30	1.22	1.09	1.08	1.07	1.11	1.11	1.11	1.11	1.13	1.15
	451	9.20	5.91	4.03	2.68	2.15	1.77	1.36	1.32	1.21	1.08	1.03	0.92	0.93	0.93	0.98	0.98	0.99	1.00	1.03	1.05
	476	7.67	3.99	2.83	1.92	1.61	1.37	1.07	1.06	1.00	0.90	0.87	0.79	0.81	0.82	0.86	0.88	0.89	0.90	0.93	0.96
	501	4.37	2.32	1.87	1.33	1.18	1.05	0.83	0.86	0.82	0.75	0.74	0.67	0.70	0.71	0.77	0.78	0.80	0.82	0.85	0.87
	526	1.41	1.16	1.17	0.89	0.86	0.79	0.65	0.69	0.68	0.63	0.63	0.57	0.61	0.63	0.68	0.70	0.72	0.74	0.78	0.80
	551	0.24	0.50	0.69	0.59	0.61	0.60	0.50	0.55	0.56	0.53	0.53	0.49	0.53	0.55	0.61	0.63	0.65	0.67	0.71	0.74
	576	0.02	0.19	0.39	0.38	0.43	0.45	0.39	0.44	0.46	0.44	0.45	0.42	0.46	0.49	0.54	0.57	0.59	0.62	0.65	0.68
	601	0.00	0.07	0.21	0.24	0.30	0.33	0.30	0.36	0.38	0.37	0.39	0.36	0.40	0.43	0.48	0.51	0.54	0.56	0.60	0.63
	626	0.00	0.02	0.11	0.15	0.21	0.25	0.23	0.29	0.31	0.31	0.33	0.31	0.35	0.38	0.43	0.46	0.49	0.51	0.55	0.58
	651	0.00	0.01	0.05	0.09	0.14	0.18	0.18	0.23	0.26	0.26	0.28	0.27	0.31	0.34	0.39	0.42	0.45	0.47	0.51	0.54
	676	0.00	0.00	0.03	0.05	0.10	0.14	0.14	0.19	0.21	0.22	0.24	0.23	0.27	0.30	0.35	0.38	0.41	0.43	0.47	0.50
	701	0.00	0.00	0.01	0.03	0.07	0.10	0.10	0.15	0.18	0.18	0.21	0.20	0.24	0.27	0.32	0.35	0.37	0.40	0.44	0.47
	726	0.00	0.00	0.01	0.02	0.05	0.07	0.08	0.12	0.15	0.16	0.18	0.17	0.21	0.24	0.29	0.31	0.34	0.37	0.40	0.43
	751	0.00	0.00	0.00	0.01	0.03	0.05	0.06	0.10	0.12	0.13	0.15	0.15	0.19	0.22	0.26	0.29	0.31	0.34	0.37	0.40
	776	0.00	0.00	0.00	0.01	0.02	0.04	0.05	0.08	0.10	0.11	0.13	0.13	0.17	0.19	0.23	0.26	0.29	0.31	0.35	0.38
	801	0.00	0.00	0.00	0.00	0.01	0.03	0.04	0.06	0.08	0.09	0.11	0.11	0.15	0.17	0.21	0.24	0.27	0.29	0.32	0.35
	826	0.00	0.00	0.00	0.00	0.01	0.02	0.03	0.05	0.07	0.08	0.10	0.10	0.13	0.16	0.19	0.22	0.25	0.27	0.30	0.33
	851	0.00	0.00	0.00	0.00	0.01	0.02	0.02	0.04	0.06	0.07	0.09	0.09	0.12	0.14	0.18	0.20	0.23	0.25	0.28	0.31
	876	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.03	0.05	0.06	0.07	0.08	0.11	0.13	0.16	0.19	0.21	0.23	0.26	0.29
	901	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.03	0.04	0.05	0.07	0.07	0.09	0.11	0.15	0.17	0.19	0.22	0.25	0.27
	926	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.03	0.04	0.06	0.06	0.08	0.10	0.13	0.16	0.18	0.20	0.23	0.26
	951	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.03	0.04	0.05	0.05	0.08	0.09	0.12	0.14	0.17	0.19	0.22	0.24
	976	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.03	0.04	0.05	0.07	0.09	0.11	0.13	0.15	0.18	0.20	0.23

Chapter 7 Further Applications

This chapter demonstrates the applications of SCOs in project valuation, risk management and financial derivatives pricing.

§7.1 Milestone Project Valuation

This section proposes the Milestone Projection Valuation (MPV) method for multi-stage projects. The projects setting some critical milestones which should be achieved sequentially are called milestone projects (see Figure 7.1 for example). The milestone projects fail if any one of the serial milestones is not completed. The milestone projects are very common in real situations, including R&D management, manufactures, technology development, etc.. Originally, the milestone projects are valuated by methods including the net present values (NPV) and decision trees. The NPV method valuates a project under a rigorous assumption that all future cash flows are certain. Obviously, the uncertainty is ignored in the NPV method and results in symmetric underestimates. Recently, the popular real option approach is applied for flexible consideration and reasonable explanation. Under the framework of financial option theory, the real option approach decomposes the project valuation as several parameters, including the present value, costs, time to maturity, value uncertainty (volatility) and interest rate. Most of the existing real option studies for multi-stage milestone project valuations use one-fold options, while others apply multi-fold options under the assumption of constant parameters through the whole process (Casimon et al., 2004). However, the parameters often change due to the milestone completion and the project values will be misestimated if parameters are assumed constant through all the time. The one-fold real option approach is even inadequate for a multi-stage project.

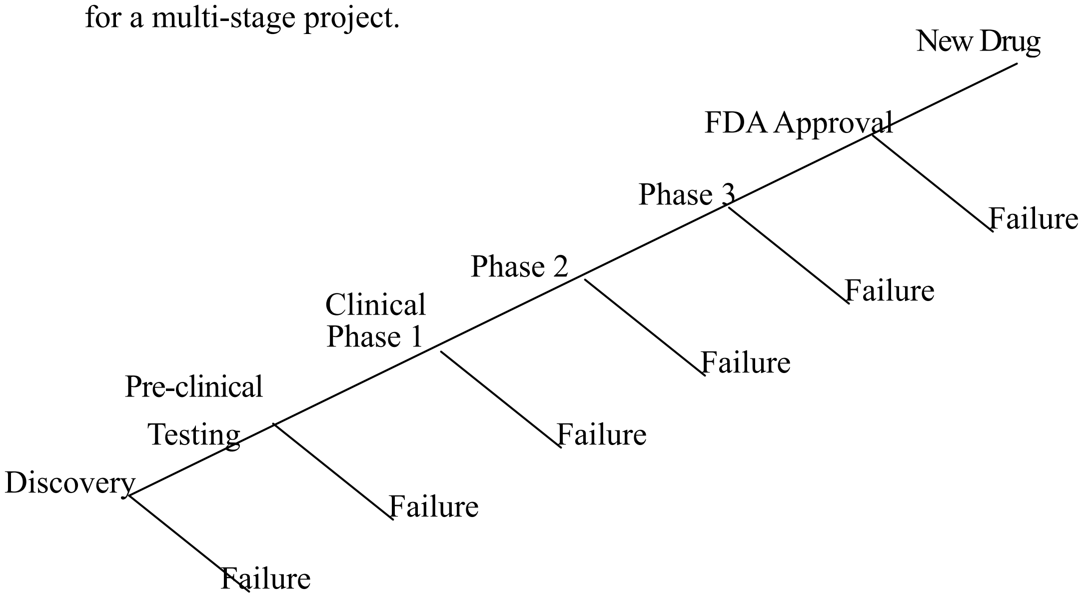


Figure 7-1 : A Milestone Project Example: the New Drug Development (NDA)

Based on Theorem 3.2 & 3.3, this paper proposes a method called Milestone Projection Valuation (MPV) for multi-stage project valuation. Each milestone completion has the choice to enter the next stage or not, and the sequential project milestone can be viewed by the sequential compound CALL options. The MPV method adopts the results of SCOs and the project is regarded as the corresponding asset in SCOs. Under the same denotations as Theorem 3.2, the MPV valuation formula is expressed as

$$MPV_i(T_0) = e^{-\sum_1^i q_u \tau_u} S_0 \mathbf{N}_i \left\{ \left[a_{i,g}\right]_{i\times 1}; \left[\tilde{\rho}_{g,h}\right]_{i\times i} \right\} - \sum_{j=1}^{i} e^{-\sum_1^j r_u \tau_u} K_j \mathbf{N}_j \left\{ \left[b_{i,g}\right]_{j\times 1}; \left[\tilde{\rho}_{g,h}\right]_{j\times j} \right\},$$

......(7.1.1)

where the strikes represent the cost at different stages; the volatilities come from the project value fluctuation and the dividend rates are replaced by the depression rates. The option features ($\Lambda_{i,g}$) equal one (for any *i*, *g*) due to the underlying compound calls, hence disappear in the MPV pricing formula.

Compared with the literature, the MPV not only applies multi-fold compound option theory, but also allows the piece-constant parameters to vary with the distinct stages. The different parameters of different stages can adapt to the change of project nature after the milestone completion. More phenomena can be discovered from the parameter comparisons. Under the MPV model, the implicit "project valuation experience" is decomposed as the parameter estimation.

The new drug applications (NDAs) may be the most famous and significant milestone projects. Under the consideration of human health, the NDAs are well-regulated including the stages of pre-clinical trial, phase 1, phase 2, phase 3 and approval phase. Each stage has a definitive milestone. The time- and cost-consuming NDAs are the cores of the pharmaceutical companies because the R&D results from NDAs dominate their future! The MPV model can enhance the NDA valuation under a more reasonable framework and improve the R&D management of these companies.

§7.2 Complex Project Valuation

Projects with tremendous amounts value often have great contribution and impact to the society and catch a lot of public attention. In order to make sure of being executed smoothly, these projects tend to enhance project flexibility by insetting many options, such as growth, switch, abandon, shutting down, contraction or expansion (Trigeorgis 1993, 1996). However, these embedded options will also make the project structure complicated. For this kind of projects, the valuation by real option approach regarding the project still as only a one-fold option is not reasonable. Realistic and rational valuation should take the project structure into consideration.

The different options and their interactions can be evaluated separately by different SCOs. The MPV case discussed in the previous subsection is just a special case of complex project valuation using SCOs. The effect of revenue guarantee, for example, in a build-operate-transfer (BOT) project of utility construction can be evaluated by SCOs. A company signs the BOT contract with the government to build and operate the construction while related revenue belongs to the company during operating period. The guarantee promised by government ensures the company's minimum revenue. If the actual revenue is less than the minimum, the deficit is subsidized by the government. The company hence owns the operating revenue and the put option written by the government. The put option, with the guarantee amount as its strike price, can enhance the incentives for the BOT project. At the preparation period time prior to construction, the put option can be considered as a 2-fold compound option, call on put. The add-in call option, with the construction cost as its strike price, represents the right to participate in the construction and share the potential revenue.

Similarly, the revenue guarantee of the expansion can be regarded as a 3-fold SCO, call on call on put, at the preparation period. Assume the government will offer corresponding revenue guarantee for the expansion if there is an expansion right embedded in the BOT project. The revenue guarantee of the expansion can be viewed as another put option with its own guarantee amount as the strike price. At the main construction time, the put option can be considered as a 2-fold compound option, call on put. This add-in call option, with the expansion cost as it strike price, stands for the expansion right. At the preparation time, the right can be evaluated as a 3-fold SCO: call on call on put. The last add-in call option, with the proportional main construction cost as it strike price, represents the right to participate in the main construction. Note that the main construction cost is divided proportionally as the strike prices of both call options for the guarantee of main and expansion construction. The call on call, stacked on the put option, represents the sequential feature that the expansion right exists only when the main construction is executed. The SCOs discussed in this study make the evaluation of complex options possible.

The project valuation considering the intrinsic structure is more logical and acceptable, and is applied gradually (eg: Huang and Chou, 2006). The adoption of SCOs for project valuation can broaden and expand the real option application. Besides, the sensitivity analysis is more visible under this situation. The change of risk source (such as the asset price, its variance, interest rate) will have different impacts on different parts of the project. The impacts can be quantified by the Greeks of SCO's (Thomassen and Van Wouwe, 2002; Lee et al., 2007) and can be applied for risk management.

§7.3 American Options

The American options and their exotic styles can be also valuated analytically with SCOs. As is widely known, the optimal time to exercise the American call is only at the time immediately before ex-dividend of the underlying stock. Hence the valuations of American calls in the absence dividends are similar to that of European ones. The Roll-Geske-Whaley model (Roll, 1977; Geske, 1979b, 1981; Whaley, 1981) gives the explicit form for American calls with single dividend by replication of European calls and a 2-fold conventional compound option, while Cassimon et al. (2007) extend their results to the cases of multiple dividends. Geske and Johnson (1984) propose the American put's closed form formula, which is actually an exotic SCO form.

The SCOs can be adopted for the derivation of pricing formulas for American puts on stocks paying multiple dividends. Besides, the closed form of exotic 2-fold compound options, such as European option on American options, can be proposed explicitly with SCOs.

§7.4 Risk Management

SCOs applied as the instruments for risk control are discussed in this subsection, such as for volatility risk, mortgage pipeline risk and weather risk.

(1) Volatility Risk

Volatility risk, the unobservable but crucial variable, determines the option premiums and the order of the financial system. The notorious Long Term Capital Management (LTCM) crash is just one of the evidences (Lowenstein, 2000). Originally, the volatility index is designed for hedging volatility fluctuation, e.g. the CBOE Volatility index (VIX). Brenner et al. (2006) and Zsembery (2004) propose exotic 2-fold compound options, the option on a forward-start straddle, in order to improve the efficiency and tradability of volatility hedging.

Under this framework, SCOs can enhance the effectiveness of volatility hedging. The plain straddle could be replaced with exotic straddles (different strikes and maturities) or complex chooser option (Rubinstein, 1992). Under identical conditions, the complex chooser option is cheaper than the straddle and thus more attractive. The compound option written on these exotic straddles or chooser options can be valuated precisely through SCOs.

(2) Mortgage Pipeline Risk

Mortgage pipeline risk, the unexpected irregular payment caused mainly by interest rate fluctuation, has been widely hedged by compound option (Bhattacharya, 2005). If the interest rates rise, the mortgage loans fall out of the pipeline and resulting in the lenders' loss. The pipeline risk will be amplified when the loans have been sold. The 2-fold calls on put options provide mortgage corporate the rights to buy put options with cheaper cost. The put options allow the lender to sell the mortgage with higher strike prices to cover loss. There is no need to exercise the put options while the interest rate decreases and this save the cost accordingly.

This kind of pipeline risk hedge can be enhanced through the long position of a pool of SCOs. The central banks (e.g., the FED in the U.S.) often take sequential actions of interest rate hiking to overcome inflation, so the pools of SCOs (combination with 2-fold, 3-fold, etc.) let the lenders make decisions depending on the up-to-date situations with cheaper expenses. The SCOs can also be used for valuating the mortgage with prepayment under the option-adjusted-spread (OAS) framework.

(3) Weather Risk

Insurers and reinsurers pay more attention to the ecosystem evolution than single accidents. They also are more willing to provide long-term management of weather risk than most trading houses. Thus compound options are introduced to give reinsurers (e.g., Swiss Re) the rights to buy an option on the weather risk at a later time (Gakos, 1999). The split-fee feature of compound options can reduce the cost the long-maturity and high-amount hedge of weather risk.

SCOs can offer better hedge effects than 2-fold compound options. Through the combination of different fold/ different maturities SCOs, insurers with a slight up-front premium can lock in coverage at different future exercise dates with additional lower premiums.

Chapter 8 Numerical Examples

This chapter illustrates 2 examples of SCO applications, including the flexibility evaluation of revenue guarantee and currency hedging by the pool of SCOs.

§8.1 Revenue Guarantee

This section illustrates a SCO application of BOT (Build-Operate-Transfer) project valuation. In this example, a 2-fold option (call on put) and a 3-fold SCO (call on call on put) are used to evaluate the promise value of revenue guarantee.

§8.1.1 Description for the Revenue Guarantee

Assume the government issues a BOT project of electric power plants in order to increase power supply. Assume the government and the company sign the contract at starting time T_0. The project starts from 1.5 years of preparation period, which follows with 4-year construction. After the construction, there will be operation period of 30 years. If the project is constructed and operated well, there can be an expansion at the 24-year. The expansion, with scale size α_{a1} proportional to the original one, takes 2 years and won't extend the operation time of the project. The Figure 8.1 illustrates the schedule of the BOT project.

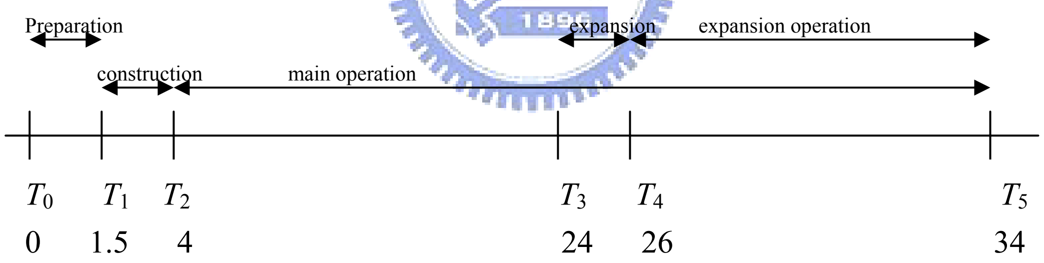


Figure 8-1: The time intervals of the example of revenue guarantee

For the government, the BOT project can increase the power supply without huge construction cost at one time. Hence the government will try its best to increase the project's investment incentives, such as the annual revenue guarantee. Revenue guarantee is the minimum revenue promised by the government. If the annual operation revenue is less than the guarantee amount, then the deficit is subsided by the government. Compared with the once huge construction cost, the payment of the revenue guarantee from the government is less and distributed over many years, and causes less burdens to the government. In addition, the guarantee can strengthens companies' incentives toward the BOT project, thus increase the plausibility of project execution.

The revenue guarantee can be regarded as the put option written by the government and owned by the company. Their Payoffs are exhibited in Figure 8.2 (a) and (b) respectively.

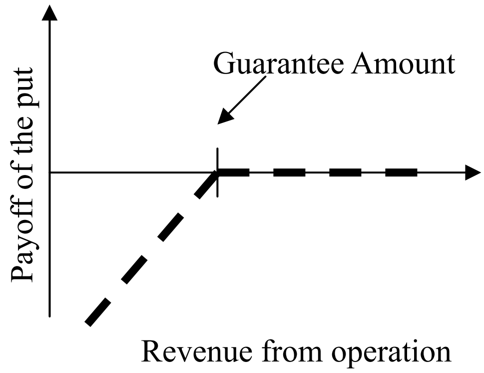


Figure 8.2 (a) The payoff of the put option written by the government

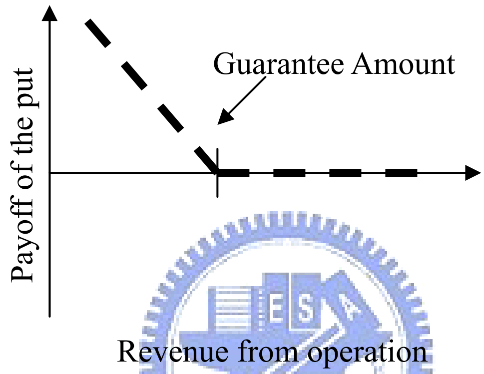


Figure 8.2 (b) The payoff of the put option owned by the company

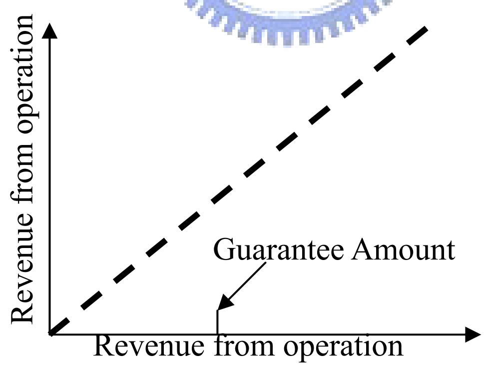


Figure 8.2 (c) The operation payoff of the company

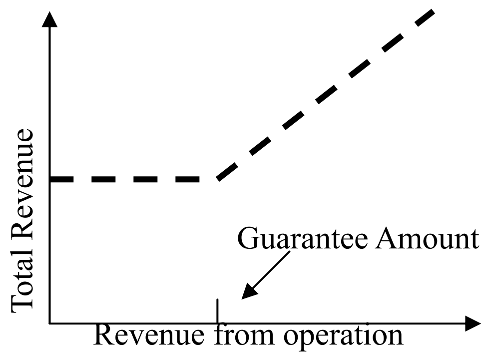


Figure 8.2 (d) The total payoff of by the company

The payoff from the operation is shown in Figure 8.2 (c), which means that the company hold simultaneously downside risk and upside potential. Downside risk can be eliminated by holding the revenue guarantee and thus the company can retain the upside potential (Figure 8.2(d)).

The value of the revenue guarantee, which essentially is the value of the put option, can be regarded as the expected value of the company's extra gain or the expected value of the government's payment. However, the revenue guarantee should be considered as a multi-fold SCO to coincide with preceding sequential decisions. The following section describes details about the revenue guarantee with numerical examples.

§8.1.2 Guarantee Evaluation

Assume the government promises every year's revenue guarantee (K_2^{a1}) to the company, which lasts for 30 years. The expansion part also has every year's revenue guarantee (K_3^{a2}) for 8 years. All the construction costs are paid by the company and the revenue are belong to the company. The preparation period is 1.5 years before the 2.5-year's construction period. Assume there is no inflation. In other words, the inflation is accounting as parts of the risk-free rate. No depression rate (q=0) in this example. The construction payment occurs at its end time. The parameter setting is listed as follows.

The main construction cost (at Time T_2): K_1^{a1}=50,000,000,000.

The expansion construction cost (at Time T_3): $K_2^{a2} = \alpha_{a1} K_1^{a1} * 1.05$.

The each year's guarantee revenue of original construction (at time T_1): K_2^{a1}=100,000,000.

The each year's guarantee revenue of expansion construction (at time T_3): $K_3^{a2} = \alpha_{a1} K_1^{a1} * 1.05$

The expansion scale coefficient (comparing to the original scale): α_{a1}=0.3.

The initial time: T_0=0

The start time of main construction: T_1=1.5

The end time of main construction and the start time of operation: T_2=4

The start time of expansion construction: T_3=24.

The end time of expansion construction and the start time of the expansion operation: T_4=26.

The end time of all operation and transfer the plant to the government: T_5=34.

The risk-free rate (through time): r=3.5%.

The annual volatility of the underlying revenues (through time): σ=0.5.

The estimated average revenue of each year (at time T_0): S_0=2,000,000,000.

For the revenue guarantee of the main construction at time T_1, it can be regarded as the guarantee revenue plus 30 1-fold forward-start put options ($\sum_{u=1}^{30} \Psi_{1,(u)}^{a1}$) written by the government to the company. These forward-start options can be regarded as European options because their dividend rates are zero. The company should pay the construction cost (K_1^{a1}) as the "option premium". It should be noted that the main construction cost K_1^{a1} is shared as the strike price of both the main guarantee and the expansion one. The strike price of the option ($\Psi_{1,(u)}^{a1}$) is the revenue guarantee (K_2^{a1}).

At time T_0, the revenue guarantee of the main construction can be considered as 2-fold compound options (call on put) $\sum_{u=1}^{30} \Psi_{2,(u)}^{a1}$, whose strike prices are the proportional construction cost $\frac{K_2^{a1}}{\tau_{a1}} \times \frac{\tau_{a1}}{\tau_{a1} + \tau_{a2}\alpha_{a1}}$. Thus the payoff of a individual option at time T_1 is $\Psi_{2,(u)}^{a1}(T_1) = \max\left[0, \Psi_{1,(u)}^{a1} - \frac{K_2^{a1}}{\tau_{a1}} \times \frac{\tau_{a1}}{\tau_{a1} + \tau_{a2}\alpha_{a1}}\right]$. At time T_0, the payoff is $\Psi_{2,(u)}^{a1}(T_0) = \tilde{E}(\max\left[0, \Psi_{1,(u)}^{a1} - \frac{K_2^{a1}}{\tau_{a1}} \times \frac{\tau_{a1}}{\tau_{a1} + \tau_{a2}\alpha_{a1}}\right])$.

Similarly, the revenue guarantee of the expansion construction can be regarded as 3-fold SCOs (call on call on put). At time T_3, the SCOs can be regarded as 8 1-fold forward-start put options ($\sum_{u=1}^{8} \Psi_{1,(u)}^{a2}$) written by the government to the company. The company pays the main construction cost and the expansion cost as the "option premium". The strike price of the option ($\Psi_{1,(u)}^{a2}$) is the revenue guarantee amount of the expansion (K_3^{a2}). At time T_1, the revenue guarantee of expansion can be considered as 2-fold SCOs (call on put), whose strike price is the proportional cost of expansion construction K_2^{a2}. It means that the company should pay the expansion cost in order to gain the revenue guarantee. Thus the payoff of the individual option is $\Psi_{2,(u)}^{a2}(T_1) = \tilde{E}(\max\left[0, \Psi_{1,(u)}^{a2} - K_3^{a2}\right])$. At time T_0, the payoff can be regarded as 3-fold SCOs (call on call on put), whose strike price is proportional cost of main construction $\frac{K_2^{a1}}{\tau_{a2}} \times \frac{\tau_{a2}\alpha_{a1}}{\tau_{a1} + \tau_{a2}\alpha_{a1}}$. It means that the company should pay the cost of main construction in order to gain the expansion right. The payoff of the individual option is $\Psi^{a2}_{3,(u)}(T_0) = \widetilde{E}(\max\left[0, \Psi^{a2}_{2,(u)} - \frac{K_2^{a1}}{\tau_{a2}} \times \frac{\tau_{a2}\alpha_{a1}}{\tau_{a1} + \tau_{a2}\alpha_{a1}}\right])$.

At time T_0, the main revenue guarantee is 4.437 billion worth, which is evaluated by 30 2-fold options (call on put). Thus the company is expected to gain 4.437 billion from the government to eliminate the downside risk of main operation. The company can get at least 6 billion in 30 years. Similarly, the revenue guarantee of expansion is 0.287 billion worth, which is evaluated by 8 3-fold SCOs (call on call on put). In other words, the company is expected to gain 0.287 billion from the government according to the revenue guarantee of expansion construction. There the company can get at least 1.6 billion in last 8 years due to expansion.

The sensitivity analysis is listed as follows. Table 8.1 represents the guarantee amount sensitivity. The annual guarantee revenue of the expansion construction is associated with that of main construction. The 30-year guarantee revenue is the summation of 30 years' annual guarantee. Guarantee worth (30-year) is the value of the revenue guarantee, which is evaluated by 2-fold compound options. It is found that guarantee worth decreases while the guarantee revenue increase. It means that the raise of guarantee amount can increase the guaranteed revenue hugely and thus results in the subtle reduce of the guarantee worth. In other words, the increase of the certainty part (guarantee amount) will diminish the uncertainty part (guarantee worth). Similarly, the opposite direction of the guarantee amount and guarantee worth also appears in the expansion construction.

Table 8.1: The Guarantee Sensitivity of the Guarantee Example (Unit: 10^9 NT)

Main Construction	Annual Guarantee Revenue	0.100	0.200	0.300	0.500
	30-year Guarantee Revenue	3.000	6.000	9.000	15.000
	Guarantee Worth(30yr)	6.096	4.436	3.162	1.451
Expansion Construction	Annual Guarantee Revenue	0.0315	0.063	0.094 5	0.158
	8-year Guarantee Revenue	0.252	0.504	0.756	1.260
	Guarantee Worth(8yr)	0.884	0.794	0.708	0.544

Table 8.2 represents the sensitivity analysis of volatility. The volatility of the annual revenue is assumed constant through time. The table shows that the guarantee worth, which is evaluated as option summation, increases with the volatility. The results correspond with general intuition.

Table 8.2: The Volatility Sensitivity of the Guarantee Example (Unit: $10^{\wedge 9}$ NT)

Volatility	0.300	0.400	0.500	0.600
Guarantee Worth (30yr) of Main Construction	0.932	2.467	4.436	6.631
Guarantee Worth (8yr) of Expansion Construction	0.001	0.137	0.794	1.517

The sensitivity of estimated annual revenue is tabulated in Table 8.3. The guarantee worth decreases while the estimated annual revenue increases, which is consistent with put's behavior. In other words, the increase of the estimated annual revenue will also increase the certainty of high revenue, thus causes reduce of the uncertainty (guarantee worth).

Table 8.3: The Sensitivity of Estimated Annual Revenue of the Guarantee Example (Unit: $10^{\wedge 9}$ NT)

Estimated Annual Revenue S_0	0.500	1.000	2.000	3.000
Guarantee Worth (30yr) of Main Construction	22.581	12.972	4.436	1.691
Guarantee Worth (8yr) of Expansion Construction	1.772	1.321	0.794	0.496

Table 8.4 exhibits the interest rate sensitivity. The hike of interest rate results in the guarantee worth decrease because the discounting of high interest rate will diminish the guarantee's value.

Table 8.4: The Interest Rate Sensitivity of the Guarantee Example

(Unit: $10^{\wedge 9}$ NT)

Interest rate r	2.5%	3.0%	3.5%	4.0%
Guarantee Worth (30yr) of Main Construction	4.815	4.623	4.436	4.256
Guarantee Worth (8yr) of Expansion Construction	2.207	1.440	0.794	0.309

§8.2 Currency Hedging

Assume an American company participates in a project auction and may have to buy Japanese products sequentially in the future. The company wants to hedge the appreciation risk of Japanese Yen. It can take a pool of SCOs, instead of a strip of futures or a stack of futures. The pool including a 1-fold European put, a 2-fold compound option (call on put) and a 3-fold SCO (call on call on put). Compared with the strip/stack of futures, the SCO pool is a better risk management instrument because the downside risk is well protected.

Options in the pool are with the final strike price 110. The 2-fold and 3-fold option should pay the fold payment (5 Yen) when enter the next fold.

The parameters of this example are set as follows.

The current exchange rate= 123.8 (Yen/USD).

The final strike price =110.

Payment for each fold =5.

The domestic (US) risk-free interest rate: r=5%

The foreign (Japanese) risk-free interest rate: $q=r_f$=1%

The annual volatility of the exchange rate: σ=0.4.

The time interval for each fold: 0.5 yr.

The 1-fold put option is priced as 6.51 (Yen), while the 2-fold (call on put) and the 3-fold (call on call on put) are valued as 6.69 and 5.86, respectively. The following tables show the sensitivity analysis of this example.

Table 8.5 represents the exchange rate sensitivity. It is found that the value of the SCO pool decrease while the current exchange rate rises. The result corresponds with the behavior of put option. The volatility sensitivity is tabulated in Table 8.6. The table reflects the intuition that higher volatility causes higher option prices.

Table 8.5 The Exchange Rate Sensitivity of the Currency Example

(Unit:Yen)

S	115.00	123.80	130.00
1-fold	9.22	6.51	5.05
2-fold	9.04	6.69	5.38
3-fold	7.85	5.86	4.73

Table 8.7 and 8.8 exhibit the sensitivity of domestic and foreign interest rate, respectively. When the domestic (US) interest rate hikes, the US dollar becomes more strengthen and results in the exchange rate decrease. Nevertheless, the foreign (Japanese) interest rate raising will cause the exchange rate increase. The result can be

explained according to Interest Rate Parity (IRP).

Table 8.6 The Volatility Sensitivity of the Currency Example

(Unit:Yen)

Volatility	0.3	0.4	0.5
1-fold	3.77	6.51	9.42
2-fold	3.32	6.69	10.47
3-fold	2.38	5.86	10.03

Table 8.7 The Domestic (US) Interest Rate Sensitivity of the Currency Example (Unit:Yen)

r	4%	5%	6%
1-fold	6.71	6.51	6.32
2-fold	7.08	6.69	6.64
3-fold	6.37	5.86	5.37

Table 8.8 The Foreign (Japanese) Interest Rate Sensitivity of the Currency Example (Unit:Yen)

$r_f(q)$	0.5%	1%	2%
1-fold	6.43	6.51	6.68
2-fold	6.55	6.69	6.99
3-fold	5.67	5.86	6.23

The sensitivity of final strike and fold payment are shown in Table 8.9 and Table 8.10, respectively. The increase of final strike will result the value of the put-style SCO pool. The fold payment can be regarded as another premium payment. Table 8.10 prevails the fact that SCOs can support decision postponement, which is one of SCOs' advantages. The higher SCO premium payment at current time can enjoy less fold payment in the future.

Table 8.9 The Strike Sensitivity of the Currency Example

(Unit:Yen)

final Strike	100	110	130
1-fold	3.54	6.51	13.16
2-fold	3.72	6.69	13.05
3-fold	3.10	5.86	11.79

Table 8.10 The Fold Payment Sensitivity of the Currency Example

(Unit:Yen)

payment	1	5	10
2-fold	9.51	6.69	4.30
3-fold	11.32	5.86	2.12

Chapter 9 The Information Management System of Projects and Financial Derivatives Evaluation

This chapter illustrates the information management system of projects and financial derivatives evaluation. The system is designed based on the main idea that all projects and derivatives can be decomposed as and evaluated individually by other simple components. The system can be applied as knowledge management instruments for the price discovery.

The system includes six main steps, which are stated as follows.

Step 1: Project/Derivatives Decomposition. The target (such as projects or financial derivatives) is decomposed as different excluding parts in order to simplify the evaluation.

Step 2: Parameter Setting. The parameters of decomposed components are set in this step.

Step 3: Calculation of Individual Components. According to the parameters set in the previous step, individual components are evaluated separately here.

Step 4: Aggregation. All the evaluation results are aggregated in step 4. The interactions of disjoint parts are taken into consideration in this step. The values of projects or financial derivatives are available here.

Step 5: Sensitivity Analysis. The scenarios of the projects and derivatives are presented in order to enhance the risk management.

Step 6: Visualization. The results of previous steps are visualized to facilitate usage. The user interface is illustrated as Figure 9.1, while the six-step framework of the system is exhibited in Figure 9.2.

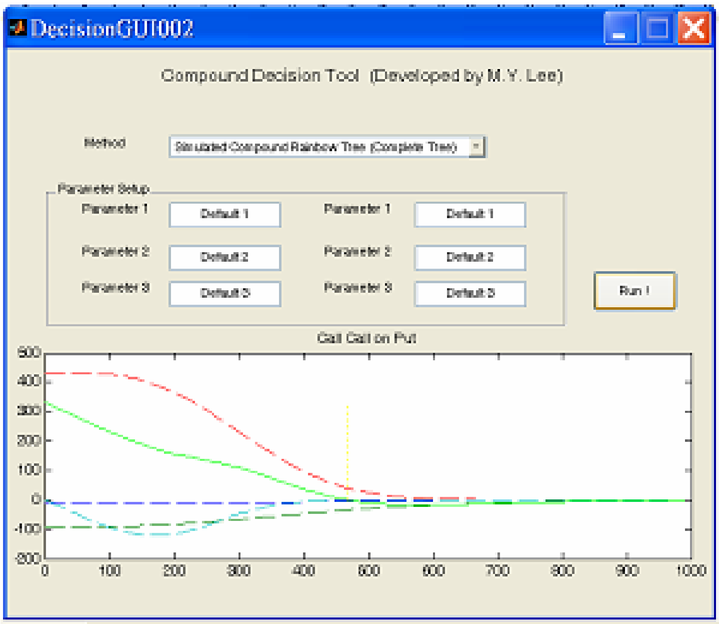


Figure 9.1 The illustrated User Interface of the System

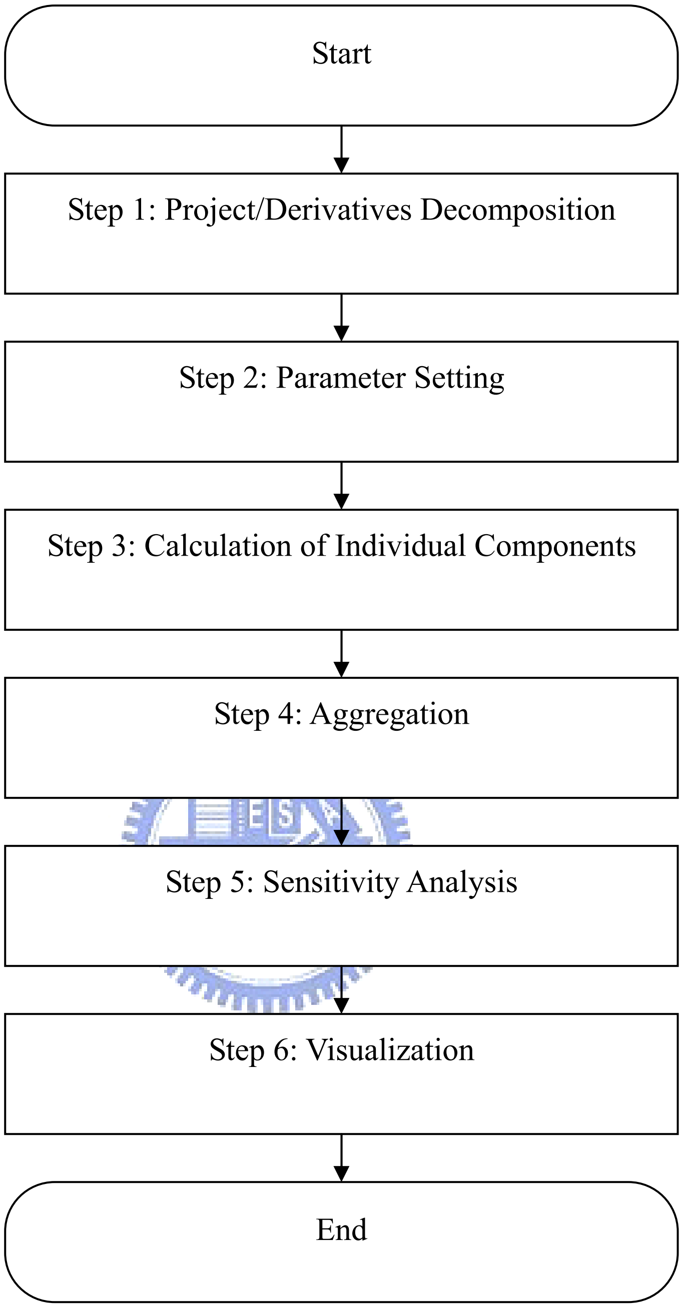


Figure 9.2: The Procedure of the Information Management System of Projects & Financial Derivatives Evaluation

Chapter 10 Conclusions

Motivated by the current inadequate methods for complex project valuation, the present study defines and derives the pricing formula of sequential compound options (SCOs), where the parameters vary over time/fold and each fold option may have different put/call attribute. The partial derivative of a multivariate normal integration is derived in this paper as a special case of Leibnitz's Rule, and is used to derive the sensitivities of SCOs. Besides, SCOs allow puts and calls alternating arbitrarily and are therefore suitable for project valuation with sophisticated structures such as internal options interaction and fold stack-up.

Previous results have analyzed 2-fold puts/calls-alternating compound options or multi-fold "sequential compound calls" where all options are of call-type. Fold-wise differences are rarely taken into consideration. The SCOs presented in this paper have the following qualities. First of all, multi-fold SCOs enable arbitrary option feature (call/put) assignments, greatly enhancing the range of practical applications that can be treated by compound option theory. Second, in real-world problems option parameters often vary over time; SCOs enabling random parameters can capture the "sequential" features. Third, SCOs can accommodate an arbitrary number of folds.

Furthermore, SCOs can be used to demonstrate some features of cumulative multivariate normal distributions, including a special form of Leibnitz's rule. The sensitivities of SCOs to asset price (and its change) and interest rate (under the case of interest rate fold-wise) are also derived.

SCOs not only generalize the methodology of European Options (Black-Scholes, 1973), 2-fold compound options (Geske, 1977; 1979) and sequential compound calls (Thomassen & Van Wouwe, 2001, 2003; Agliardi & Agliardi, 2005), but can be evaluated by a linear combination of the asset and strike prices weighted by different variate normal integrations. Corresponding to intuitions, an SCO can be seen as a multi-dimensional options extending from the work of Black-Scholes (1973) and Geske (1977; 1979). The changing numéraire method enriches the SCOs pricing formula derivation with more financial implications than P.D.E. method. The Leibnitz's rule can be used to decompose the partial differential of (k+1)-variate integration into two parts: a k-variate normal integration and an integration with the integrand of a partial derivative. This paper proves that, under the multivariate normal cases, these two parts can be presented in a unified form. Based on the above results, the sensitivities of SCOs can be expressed explicitly as a generalized version of those found by Black-Scholes (1973), Geske (1977; 1979) and Thomassen & Van Wouwe (2002).

The six-step recursive algorithm for SCOs evaluation is proposed in order to clarify the computing details. The evaluation of SCOs is not easy due to the nested loops, EAP calculation and the computation of multivariate normal integration. These difficulties all result from the multiple fold features of SCOs. Numerical examples of 3-fold SCOs are illustrated. The SCOs proposed in this study can extend the compound option methodology and broaden the popular real option applications.

SCOs can enhance and broaden the use of compound option theory in real option and financial derivative fields. Real options often incorporate multiple options of different types with sophisticated interactions, but such situations can be evaluated by aggregating various SCOs. Even milestone projects, which must decide whether or not a project has terminated according to the milestone achievement, can be evaluated by the use of fold-wise SCOs. Compared with the constant volatility assumed in in Casimon et al. (2004), allowing the volatilities and interest rates to vary for different periods makes this method of project valuation more precise and flexible.

Risk management is another SCO application. Volatility risk, prepayment risk of mortgage and weather risk are some the most important issues of concerned to finance institutions. The advantages of SCOs, including the cheaper premium, decision postponement, split-fee and flexibility, can enhance risk management effectiveness through SCO adoption.

Numerical examples of SCOs proposed in this study, including evaluation of government revenue guarantee and currency hedging, shows that SCOs can be applied widely in both real option and financial derivative field. Besides, the information management system with SCOs as its core module proposed here can support the evaluation of projects and financial derivatives.

Finally, a number of complex financial derivatives can be developed or evaluated using SCOs in the same way that chooser options and capletions can be priced by 2-fold compound options. These applications of SCOs with real-world cases will be the subject of probable future researches. Some topics for future study are listed below.

1. The uncertainty of the fold time intervals is an important issue for real option application. This problem can be solved by simulation.

2. The piecewise method can facilitate the computation procedures.

3. Discovery the forward rate by the piecewise method and optimization methodology. .

4. Examine the accuracy and consistency of SCOs routines.

5. Study the option with both European and American type in different time interval. This option is also common in real world application. According to Geske and Johnson (1984), the result seems like a exotic SCO.

6. Discovery the relation between simultaneous compound option and SCOs.

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Appendix A: Proof of Theorem 3.1

This theorem is proved by induction. When i=1, $\Psi_1(T_0)$ with $\Lambda_{1,1}$=1 and $\Lambda_{1,1}$=-1 are the vanilla call and put formulas respectively. When i=2, $\Psi_2(T_0)$ is the 2-fold compound option, such as call on call ($\Lambda_{1,1}$=1, $\Lambda_{2,2}$=1), put on call ($\Lambda_{1,1}$= -1, $\Lambda_{2,2}$=1), call on put($\Lambda_{1,1}$=1, $\Lambda_{2,2}$= -1), and put on put($\Lambda_{1,1}$=-1, $\Lambda_{2,2}$=-1). These generalized 2-fold cases can be extended easily from Chen (2002) and Lajeri-Chaherli (2002).

Assuming that Equation (3.2.1) is true for the i-fold compound option $\Psi_i(T_0)$, it will be shown that Equation (3.2.1) is also true for the (i+1)-fold compound option, for any $\Lambda_{g,g}$, $1 \le g \le i+1$.

Because the underlying asset of $\Psi_{i+1}(T_0)$ is $\Psi_i(T_1)$, instead of $\Psi_i(T_0)$, the start time of the i-fold compound option is shifted from T_0 to T_1. All pieces of notation for the i-fold compound option are changed simultaneously according to this time shift. (In other words, v=1).

Hence
$$\Psi_i(T_1) = \Lambda_{i+1,2} e^{-\int_{T_1}^{T_{i+1}} q(u)du} S_1 \mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1} a_{i,g,*1}\right]_{i\times 1};\left[\tilde{\rho}_{g,h,*1}\right]_{i\times i}\right\}$$
$$-\sum_{j=1}^{i} \Lambda_{j+1,2} e^{-\int_{T_1}^{T_{j+1}} r(u)du} K_{j+1} \mathbf{N}_j\left\{\left[\Lambda_{i+1,g+1} b_{i,g,*1}\right]_{j\times 1};\left[\tilde{\rho}_{g,h,*1}\right]_{j\times j}\right\} \qquad \text{......(A.1)}$$

At T_1, the maturity time of the i+1-fold compound option, the option price can be expressed as $\Psi_{i+1}(T_1) = \max[\Lambda_{1,1}\Psi_i(T_1) - \Lambda_{1,1}K_1]$. At its starting time T_0, the option price is given by

$$\Psi_{i+1}(T_0) = \tilde{E}\{e^{-\int_{T_0}^{T_1} r(u)du} \max[\Lambda_{1,1}\Psi_i(T_1) - \Lambda_{1,1}K_1] \| \boldsymbol{F}_0\}, \qquad \text{......(A.2)}$$

according to the fundamental theory of asset pricing (Baxter and Runie, 1996). $\tilde{E}$ is the expectation operator under the risk-neutral measure, and $\boldsymbol{F}_0$ denotes the information available at time T_0 from the asset price.

Under the assumption that the asset price follows a geometric Brownian motion, it can be expressed as

$$S_1 = S_0 e^{\int_{T_0}^{T_1}\left[r(u)-q(u)-\frac{1}{2}\sigma^2(u)\right]du + z\sqrt{\int_{T_0}^{T_1}\sigma^2(u)du}} \qquad \text{......(A.3)}$$

where z is a standard normal random number $z \sim N(0,1)$, with density function f.

$\Psi_{i+1}(T_0)$ is a function of S_0 and hence a function of z. Thus the SCO price can be represented as

$$\Psi_{i+1}(T_0)=e^{-\int_{T_0}^{T_1} r(u)du}\int_{-\infty}^{\infty}\max\left[\Lambda_{1,1}\Psi_i(T_1)-\Lambda_{1,1}K_1\right]f(z)dz .$$

Assume that $S_{\#1,i+1}$ is the equivalent asset price which makes $\Psi_i(T_1)-K_1=0$. The condition "$S_1=S_{\#1,i+1}$" is then equivalent to "$z=-b_{i+1,1}$", where

$$b_{i+1,1}=\frac{\ln(\frac{S_0}{S_{\#1,i+1}})+\int_{T_0}^{T_1}\left(r(u)-q(u)-\frac{1}{2}\sigma^2(u)\right)du}{\sqrt{\int_{T_0}^{T_1}\sigma^2(u)du}}$$

Because the integration range is either [-∞, $-b_{i+1,1}$] or [$-b_{i+1,1}$,∞], depending on $\Lambda_{i+1,1}$ (the sign of S_1), the compound option can be expressed in the unified form

$$\Psi_{i+1}(T_0)=e^{-\int_{T_0}^{T_1} r(u)du}\Lambda_{i+1,1}\int_{-b_{i+1,1}}^{\Lambda_{i+1,1}\infty}\{\Lambda_{1,1}\Psi_i(T_1)-\Lambda_{1,1}K_1\}f(z)dz .$$

Substituting Equation (A.1) into the previous equation, it can be obtained that

$$\begin{aligned}\Psi_{i+1}(T_0)&=e^{-\int_{T_0}^{T_1} r(u)du}\Lambda_{i+1,1}\Lambda_{i+1,1}\int_{-b_{i+1,1}}^{\Lambda_{i+1,1}\infty}e^{-\int_{T_1}^{T_{i+1}} q(u)du}S_1\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}a_{i,g,*1}\right]_{i\times1};\left[\tilde{\rho}_{g,h,*1}\right]_{i\times i}\right\}f(z)dz\\&-e^{-\int_{T_0}^{T_1} r(u)du}\Lambda_{i+1,1}\sum_{j=1}^{i}\Lambda_{j+1,1}\int_{-b_{i+1,1}}^{\Lambda_{i+1,1}\infty}e^{-\int_{T_1}^{T_{j+1}} r(u)du}K_{j+1}\mathbf{N}_j\left\{\left[\Lambda_{i+1,g+1}b_{i,g,*1}\right]_{j\times1};\left[\tilde{\rho}_{g,h,*1}\right]_{j\times j}\right\}f(z)dz\\&-e^{-\int_{T_0}^{T_1} r(u)du}\Lambda_{i+1,1}\Lambda_{1,1}\int_{-b_{i+1,1}}^{\Lambda_{i+1,1}\infty}K_1f(z)dz\\&\equiv\tilde{\Psi}_{i+1,1}-\tilde{\Psi}_{i+1,2}-\tilde{\Psi}_{i+1,3} .\end{aligned}$$

The following paragraphs derivates $\tilde{\Psi}_{i+1,1}$, $\tilde{\Psi}_{i+1,2}$ and $\tilde{\Psi}_{i+1,3}$ explicitly. By Equation (A.3), S_1 can be substituted by the representation of S_0 and thus

$$\tilde{\Psi}_{i+1,1}=\Lambda_{i+1,1}e^{-\int_{T_0}^{T_{i+1}} q(u)du}S_0\Lambda_{i+1,1}\int_{-b_{i+1,1}}^{\Lambda_{i+1,1}\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z-\sqrt{\int_{T_0}^{T_1}\sigma^2(u)du}\right)^2}\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}\tilde{a}_{i,g,*1}\right]_{i\times1};\left[\tilde{\rho}_{g,h,*1}\right]_{i\times i}\right\}dz,$$

where

$$\tilde{a}_{i,g,*1}=\frac{\ln(\frac{S_0}{S_{\#g+1,i+1}})+\int_{T_0}^{T_{g+1}}\left[r(u)-q(u)+\frac{1}{2}\sigma^2(u)\right]du+z\sqrt{\int_{T_0}^{T_1}\sigma^2(u)du}-\int_{T_0}^{T_1}\sigma^2(u)du}{\sqrt{\int_{T_1}^{T_{g+1}}\sigma^2(u)du}},\ \forall 1\le g\le i.$$

Let $z_2=z-\sqrt{\int_{T_0}^{T_1}\sigma^2(u)du}$, so that the above equation can be written as

$$\widetilde{\Psi}_{i+1,1}=\Lambda_{i+1,1}e^{-\int_{T_0}^{T_{i+1}}q(u)du}S_0\Lambda_{i+1,1}\int_{-a_{i+1,1}}^{\Lambda_{i+1,1}\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_2^2}\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}\bar{a}_{i,g,*1}\right]_{i\times 1};\left[\widetilde{\rho}_{g,h,*1}\right]_{i\times i}\right\}dz_2$$

where $\bar{a}_{i,g,*1}=\dfrac{a_{i+1,g+1}+z_2\rho_{1,g+1}}{\sqrt{1-\rho_{1,g+1}^2}}$, $\forall 1\le g\le i$.

Then denote $z_3=-\Lambda_{i+1,1}z_2$, hence

$$\widetilde{\Psi}_{i+1,1}=\Lambda_{i+1,1}e^{-\int_{T_0}^{T_{i+1}}q(u)du}S_0\int_{\infty}^{-\Lambda_{i+1,1}a_{i+1,1}}\frac{e^{-\frac{1}{2}z_3^2}}{\sqrt{2\pi}}\mathbf{N}_i\left\{\left[\frac{\Lambda_{i+1,g+1}a_{i+1,g+1}-\Lambda_{g,1}\widetilde{\rho}_{1,g+1}z_3}{\sqrt{1-\left(\Lambda_{g,1}\widetilde{\rho}_{1,g+1}\right)^2}}\right]_{i\times 1};\left[\widetilde{\rho}_{g,h,*1}\right]_{i\times i}\right\}dz_3$$

$$=\Lambda_{i+1,1}e^{-\int_{T_0}^{T_{i+1}}q(u)du}S_0\mathbf{N}_{i+1}\left\{\left[\Lambda_{i+1,g}a_{i+1,g}\right]_{(i+1)\times 1};\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}\right\}$$

The last equation is obtained by Theorem 1 (a). The following derivation will demonstrate that $\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}=\left[\hat{\rho}_{g,h}\right]_{(i+1)\times(i+1)}$.

According to Theorem 1 (a), $H_{0,1,1}=1$; $H_{0,1,g}=\Lambda_{h-1,1}\rho_{1,h}$, $\forall 2\le g\le i+1$; $H_{0,g,h}=H_{0,h,g}$; and $H_{0,g,g}=1$, $\forall 2\le g\le i+1$. $\forall 2\le g<h\le i+1$,

$$H_{0,g,h}=\Lambda_{g-1,1}\rho_{1,g}\Lambda_{h-1,1}\rho_{1,h}+\sqrt{1-(\Lambda_{g-1,1}\rho_{1,g})^2}\sqrt{1-(\Lambda_{h-1,1}\rho_{1,h})^2}\rho_{g-1,h-1,*1}$$

$$=\Lambda_{h-1,g}\frac{\sqrt{\int_{T_0}^{T_g}\sigma^2(u)du}}{\sqrt{\int_{T_0}^{T_h}\sigma^2(u)du}}=\Lambda_{h-1,g}\rho_{g,h}=\widetilde{\rho}_{g,h}.$$

According to the above statements, $\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}=\left[\widetilde{\rho}_{g,h}\right]_{(i+1)\times(i+1)}$ and hence

$$\widetilde{\Psi}_{i+1,1}=\Lambda_{i+1,1}e^{-\int_{T_0}^{T_{i+1}}r(u)du}S_0\mathbf{N}_{i+1}\left\{\left[\Lambda_{i+1,g}a_{i+1,g}\right]_{(i+1)\times 1};\left[\widetilde{\rho}_{g,h}\right]_{(i+1)\times(i+1)}\right\}.$$

By a similar method, $\widetilde{\Psi}_{i+1,2}$ and $\widetilde{\Psi}_{i+1,3}$ can be derived:

$$\widetilde{\Psi}_{i+1,2}=\sum_{j=2}^{i+1}\Lambda_{j,1}e^{-\int_{T_0}^{T_j}r(u)du}K_j\mathbf{N}_j\left\{\left[\Lambda_{i+1,g}b_{i+1,g}\right]_{j\times 1};\left[\widetilde{\rho}_{g,h}\right]_{j\times j}\right\}.$$

$$\widetilde{\Psi}_{i+1,3}=\Lambda_{1,1}e^{-\int_{T_0}^{T_1}r(u)du}K_1\mathbf{N}_1\left\{\Lambda_{i+1,1}b_{i+1,1}\right\}.$$

Equation (3.2.1) is true for any i+1-fold compound option, provided it is true for the i-fold compound option. Consequently, Theorem 3.2 is proved. **Q.E.D.**

Appendix B: Proof of Lemma 3.1

According to Theorem 3.2 (f), the $S_{\#g,i}$ will exist only when the EAPs of the previous folds ($S_{\#\ell,i}, g-1 \le \ell \le i-1$) exist. Thus the condition (a) holds. According to Theorem 5.1 (a), the option price $\tilde{\Psi}_{i-g}(T_g)$ is strict monotone and its sign is decided by $\Lambda_{i-g,1}$. Hence it is discussed as the cases of $\Lambda_{i-g,1} = +1$ (condition (b)) and $\Lambda_{i-g,1} = -1$ (condition (c)), respectively. For condition (b), $\Psi_{i-g}(T_g)$ has the same sign with the asset price and thus can ranges from zero to infinity to fit any nonnegative K_g. For condition (c), $\Psi_{i-g}(T_g)$ has the opposite sign with the asset price, then $\Psi_{i-g}(T_g)$ will reach the maximum $-\tilde{\Psi}_{i-g,2\&3}(T_g)$ while the asset price is zero. Therefore the strike price K_g can NOT exceed the maximum in order to keep $S_{\#g,i}$ exist. **Q.E.D.**

Appendix C: Proof of Theorem 3.3

This theorem is proved by induction. The dynamics of related securities are exhibited before the induction procedures. Let T be a fixed maturity date and $\widetilde{P}$ be the risk-neutral measure. Let $\widetilde{E}[\cdot]$ be the expectation operator under $\widetilde{P}$. Given T satisfying $T_0 \le T$, let $\widetilde{P}^T$ be the T-forward measure, that is,

$$d\widetilde{P}^T = \frac{D(T)}{B(T_0,T)} d\widetilde{P}, \text{ where } B(T_0,T) = \widetilde{E}[D(T)].$$

Under $\widetilde{P}^T$, it is known that the forward price, $F_S(t,T)$, is a martingale, for $T_0 \le t \le T$. Let $\widetilde{W}_t$ be the multiple dimensional Brownian Motion under $\widetilde{P}$. Assume that the dynamics of the bond price and the underlying asset are $dB(t,T) = B(t,T)\left[r(t)dt + \overline{\sigma}_B(t,T)d\widetilde{W}_t\right]$ and $dS(t) = S(t)\left[r(t)dt + \sigma_S(t)d\widetilde{W}_t\right]$, respectively, where $\overline{\sigma}_B(t,T) = -\int_t^T \sigma_B(t,u)du$. The forward price of the underlying asset, $F_S(t,T)$, equals $F_S(t,T) = \frac{S(t)}{B(t,T)}$, $\forall t \in [0,T]$. According to Itô Lemma,

$$dF_S(t,T) = F_S(t,T)\left[\overline{\sigma}_B(t,T)\sigma(t,T)dt + \widetilde{\sigma}(t,T)d\widetilde{W}_t\right], \qquad \text{......(C.1)}$$

where $\widetilde{\sigma}(t,T) = \left|\sigma_S(t) - \overline{\sigma}_B(t,T)\right|$. Because $F_S(t,T)$ is a martingale under $\widetilde{P}^T$, it can be set that $d\widetilde{W}_t^T = d\widetilde{W}_t - \overline{\sigma}_B(t,T)dt$ to cancel the drift term of Equation (C.1). Thus $dF_S(t,T) = F_S(t,T)\left[\widetilde{\sigma}(t,T)d\widetilde{W}_t^T\right]$. The stochastic differential equation of the logarithm forward price can be derived by Itô Lemma: $d\ln F_S(t,T) = \widetilde{\sigma}(t,T)d\widetilde{W}_t^T - \frac{1}{2}\widetilde{\sigma}^2(t,T)dt$. Therefore, the dynamics of the forward price is obtained,

$$F_S(t,T) = F_S(T_0,T)e^{\left\{-\frac{1}{2}\int_{T_0}^t \widetilde{\sigma}^2(u,T)du + \int_{T_0}^t \widetilde{\sigma}(u,T)d\widetilde{W}_u^T\right\}} \qquad \text{......(C.2)}$$

Take the asset price $S(t)$ as the numéraire, and the bond price is

$\dfrac{B(t,T)}{S(t)} = \dfrac{1}{F(t,T)}$ under $\widetilde{P}^T$. By Itô Lemma,

$d\left[\dfrac{1}{F(t,T)}\right] = \dfrac{-\widetilde{\sigma}(t,T)\left[d\widetilde{W}_t^T - \widetilde{\sigma}(t,T)dt\right]}{F(t,T)}$, for $T_0 \le t \le T$. Let

$$d\widetilde{W}_t^S = d\widetilde{W}_t^T - \widetilde{\sigma}(t,T)dt. \qquad \text{......(C.3)}$$

By Girsanov's Theorem, as $d\widetilde{P}^S$ satisfies $d\widetilde{P}^S = \dfrac{D(T)S(T)}{S(T_0)} d\widetilde{P}^T$, $\widetilde{W}_t^S$ is the Brownian Motion under $\widetilde{P}^S$. Hence, $d\left[\dfrac{1}{F(t,T)}\right] = \dfrac{-\widetilde{\sigma}(t,T)d\widetilde{W}_t^S}{F(t,T)}$ is a martingale under $\widetilde{P}^S$. By Itô Lemma again, $d\ln\left[\dfrac{1}{F(t,T)}\right] = -\dfrac{1}{2}\widetilde{\sigma}^2(t,T)dt - \widetilde{\sigma}(t,T)d\widetilde{W}_t^S$. Thus,

$$\frac{1}{F(t,T)} = \frac{1}{F(T_0,T)} e^{\left\{-\frac{1}{2}\int_{T_0}^{t}\widetilde{\sigma}^2(u,T)du + \int_{T_0}^{t}\widetilde{\sigma}(u,T)d\widetilde{W}_u^S\right\}}.$$

The exploited dynamics are used for the induction. The Equation (1) is true for i=1. For the case $\Lambda_{1,1}$=+1 is exhibited in Musiela and Rutkowski (1998, section 15.1.2) and Frey and Sommer (1998). The other case, $\Lambda_{1,1}$=−1, can be proved by the similar way.

Assume the Equation (1) is true for the i-fold compound option $\Psi_i^{®}(T_0)$, it is showed that the Equation (1) is also true for the i+1-fold compound option, for any $\Lambda_{g,g}$, $1 \le g \le i+1$.

Because the underlying asset of $\Psi_{i+1}^{®}(T_0)$ is $\Psi_i^{®}(T_1)$, instead of $\Psi_i^{®}(T_0)$, the start time of the i-fold compound option is shift from T_0 to T_1. All pieces of notation of the i-fold compound option are changed simultaneously according to the time shift. Hence

$$\Psi_i^{®}(T_1) = \Lambda_{i+1,2} S(T_1) \mathbf{N}_i \left\{ \left[\Lambda_{i+1,g+1} a_{i,g,*1}^{®}\right]_{i\times 1} ; \left[\widetilde{\rho}_{g,h,*1}^{®}\right]_{i\times i} \right\}$$
$$- \sum_{j=1}^{i} \Lambda_{j+1,2} B(T_1, T_{j+1}) K_{j+1} \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b_{i,g,*1}^{®}\right]_{j\times 1} ; \left[\widetilde{\rho}_{g,h,*1}^{®}\right]_{j\times j} \right\} \qquad \text{......(C.4)}$$

At the maturity time T_1 of the i+1-fold compound option, the SCO price is $\Psi_{i+1}^{®}(T_1) = \max[\Lambda_{1,1}\Psi_i^{®}(T_1) - \Lambda_{1,1}K_1]$; thus at the start time T_0, the option price is

$$\Psi_{i+1}^{®}(T_0) = \widetilde{E}\{D(T_1)\max[\Lambda_{1,1}\Psi_i^{®}(T_1) - \Lambda_{1,1}K_1] \| \boldsymbol{F}_0\}, \qquad \text{......(C.5)}$$

according to the fundamental theory of asset pricing (Baxter and Runie, 1996), where

$\boldsymbol{F}_0$ denotes the information available at time T_0 from the asset price.

$$\tilde{\Psi}^{\circledR}_{i+1}(T_0) = \tilde{E}\{D(T_1)\Lambda_{1,1}\Psi^{\circledR}_i(T_1)\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\} - \tilde{E}\{D(T_1)\Lambda_{1,1}K_1\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\}$$

$$= \Lambda_{1,1}\tilde{E}\left\{\Lambda_{i+1,2}D(T_1)S(T_1)\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}a^{\circledR}_{i,g,*1}\right]_{i\times 1};\left[\tilde{\rho}^{\circledR}_{g,h,*1}\right]_{i\times i}\right\}\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\right\}$$

$$-\sum_{j=1}^{i}\Lambda_{1,1}\tilde{E}\left\{D(T_1)\Lambda_{j+1,2}B(T_1,T_{j+1})K_{j+1}\mathbf{N}_j\left\{\left[\Lambda_{i+1,g+1}b^{\circledR}_{i,g,*1}\right]_{j\times 1};\left[\tilde{\rho}^{\circledR}_{g,h,*1}\right]_{j\times j}\right\}\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\right\}$$

$$-\Lambda_{1,1}B(T_0,T_1)K_1\tilde{E}\left\{\frac{D(T_1)}{B(T_0,T_1)}\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\right\}$$

$$\equiv \tilde{\Psi}^{\circledR}_{i+1,1} - \tilde{\Psi}^{\circledR}_{i+1,2} - \tilde{\Psi}^{\circledR}_{i+1,3}$$

$$\tilde{\Psi}^{\circledR}_{i+1,1}(T_0) = \Lambda_{i+1,1}S(T_0)\tilde{E}\left\{\frac{D(T_1)S(T_1)}{S(T_0)}\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}a^{\circledR}_{i,g,*1}\right]_{i\times 1};\left[\tilde{\rho}^{\circledR}_{g,h,*1}\right]_{i\times i}\right\}\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\right\}$$

$$= \Lambda_{i+1,1}S(T_0)\tilde{E}^S\left\{\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}a^{\circledR}_{i,g,*1}\right]_{i\times 1};\left[\tilde{\rho}^{\circledR}_{g,h,*1}\right]_{i\times i}\right\}\mathbf{1}_{\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1)>\Lambda_{1,1}K_1\}}\right\},$$

where $\tilde{E}^S[\cdot]$ is the expectation operator under $\tilde{P}^S$.

The condition $\{\Lambda_{1,1}\Psi^{\circledR}_i(T_1) > \Lambda_{1,1}K_1\}$, deciding whether the current fold SCO is worth exercising or not, is equivalent to

$$\{F_S(t,T) > S^{\circledR}_{1,i+1}\} \qquad \text{...... (C.6)}$$

and $\{z_{11} > -b^{\circledR}_{i+1,1}\}$, where z_{11} is the standard normal random variable. Frey and Sommer (1998) mention, under stochastic interest rate, that the volatility of asset price and bond price must be perfectly correlated to make sure the existence of compound options' equivalent price (EAP) (Equation C.6). The condition of volatility is generalized for SCOs as

$$\zeta_g\sigma_S(t) = \bar{\sigma}_B(t,T_{g+1}) - (1-\zeta_g)\bar{\sigma}_B(t,T_g),\ \forall t\in\left[T_{g-1},T_g\right],\ 1\le g\le i, \qquad \text{......(C.7)}$$

where ζ_g is constant. Because the integration range is either $\left[-\infty,-b^{\circledR}_{i+1,1}\right]$ or $\left[-b^{\circledR}_{i+1,1},\infty\right]$, depending on the sign of forward price ($\Lambda_{i+1,1}$), the $\tilde{\Psi}^{\circledR}_{i+1,1}(T_0)$ can be expressed as a unified form

$$\widetilde{\Psi}^{®}_{i+1,1}(T_0)=\Lambda_{i+1,1}S(T_0)\Lambda_{i+1,1}\int_{-b^{®}_{i+1,1}}^{\Lambda_{i+1,1}\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_{11}^2}\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}a^{®}_{i,g,*1}\right]_{i\times 1};\left[\widetilde{\rho}^{®}_{g,h,*1}\right]_{i\times i}\right\}dz_{11}\cdot$$

According to Equation (C.2) and (C.3), $F_S(T_1,T_{g+1})=F_S(T_0,T_{g+1})e^{\left\{-\frac{1}{2}\int_{T_0}^{T_1}\widetilde{\sigma}^2(u,T_{g+1})du+\int_{T_0}^{T_1}\widetilde{\sigma}(u,T_{g+1})d\widetilde{W}_u^{T_{g+1}}\right\}}$. Applying the last equation, it could be showed that

$$\widetilde{a}^{®}_{i,g,*1}=\frac{\ln\left[\frac{F(T_0,T_{g+1})}{S^{®}_{\#g+1,i+1}}\right]+\frac{1}{2}\int_{T_0}^{T_{g+1}}\widetilde{\sigma}^2(u,T_{g+1})du+\int_{T_0}^{T_1}\widetilde{\sigma}(u,T_{g+1})d\widetilde{W}_u^{T_{g+1}}}{\sqrt{\int_{T_1}^{T_{g+1}}\widetilde{\sigma}^2(u,T_{g+1})du}}\overset{d}{=}\frac{a^{®}_{i+1,g+1}+z_{12}\rho^{®}_{1,g+1}}{\sqrt{1-(\rho^{®}_{1,g+1})^2}}\equiv\overline{a}^{®}_{i,g,*1},\ \forall 1\le g\le i.$$

z_{12} is the standard normal random variable. Then,

$$\widetilde{\Psi}^{®}_{i+1,1}(T_0)=\Lambda_{i+1,1}S(T_0)\Lambda_{i+1,1}\int_{-a^{®}_{i+1,1}}^{\Lambda_{i+1,1}\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_{12}^2}\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}\overline{a}^{®}_{i,g,*1}\right]_{i\times 1};\left[\widetilde{\rho}^{®}_{g,h,*1}\right]_{i\times i}\right\}dz_{12}\cdot$$

Note that the lower limit of the integration is also changed by Equation (C.3).

Denote $z_{13}=-\Lambda_{i+1,1}z_{12}$, hence

$$\widetilde{\Psi}^{®}_{i+1,1}=\Lambda_{i+1,1}S(T_0)\int_{-\infty}^{\Lambda_{i+1,1}a^{®}_{i+1,1}}\frac{e^{-\frac{1}{2}z_{13}^2}}{\sqrt{2\pi}}\mathbf{N}_i\left\{\left[\frac{\Lambda_{i+1,g+1}a^{®}_{i+1,g+1}-\Lambda_{g,1}\rho^{®}_{1,g+1}z_{13}}{\sqrt{1-(\Lambda_{g,1}\rho^{®}_{1,g+1})^2}}\right]_{i\times 1};\left[\widetilde{\rho}^{®}_{g,h,*1}\right]_{i\times i}\right\}dz_{13}$$

$$=\Lambda_{i+1,1}S(T_0)\mathbf{N}_{i+1}\left\{\left[\Lambda_{i+1,g}a^{®}_{i+1,g}\right]_{(i+1)\times 1};\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}\right\}$$

The last equation is derived by Theorem 1 (a) and the following is to exhibit that $\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}=\left[\widetilde{\rho}^{®}_{g,h}\right]_{(i+1)\times(i+1)}$.

According to Theorem 1, $H_{0,1,1}=1$; $H_{0,1,g}=\Lambda_{h-1,1}\rho^{®}_{1,h},\ \forall 2\le g\le i+1$; $H_{0,g,h}=H_{0,h,g}$; $H_{0,g,g}=1,\ \forall 2\le g\le i+1$. $\forall 2\le g<h\le i+1$,

$$H_{0,g,h}=\Lambda_{g-1,1}\rho^{®}_{1,g}\Lambda_{h-1,1}\rho^{®}_{1,h}+\sqrt{1-(\Lambda_{g-1,1}\rho^{®}_{1,g})^2}\sqrt{1-(\Lambda_{h-1,1}\rho^{®}_{1,h})^2}\rho^{®}_{g-1,h-1,*1}$$

$$=\Lambda_{h-1,g}\frac{\sqrt{\int_{T_0}^{T_g}\widetilde{\sigma}^2(t,T_g)dt}}{\sqrt{\int_{T_0}^{T_h}\sigma^2(t,T_g)dt}}=\Lambda_{h-1,g}\rho^{®}_{g,h}=\widetilde{\rho}^{®}_{g,h}.$$

According to the above statements, $\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}=\left[\widetilde{\rho}^{®}_{g,h}\right]_{(i+1)\times(i+1)}$ and $\widetilde{\Psi}^{®}_{i+1,1}=\Lambda_{i+1,1}S(T_0)\mathbf{N}_{i+1}\left\{\left[\Lambda_{i+1,g}a^{®}_{i+1,g}\right]_{(i+1)\times 1};\left[\widetilde{\rho}^{®}_{g,h}\right]_{(i+1)\times(i+1)}\right\}$.

By the similar method, the $\widetilde{\Psi}^{®}_{i+1,2}$ and $\widetilde{\Psi}^{®}_{i+1,3}$ can be derived under the T-forward measure $\widetilde{P}^T$ with the corresponding expectation operator $\widetilde{E}^T[\cdot]$.

$$
\begin{aligned}
\widetilde{\Psi}^{\circledR}_{i+1,2} &= \sum_{j=1}^{i} \Lambda_{j+1,1} K_{j+1} B(T_0,T_{j+1}) \widetilde{E}\left\{ \frac{D(T_1)}{B(T_0,T_1)} \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b^{\circledR}_{i,g,*1}\right]_{j\times 1}; \left[\widetilde{\rho}^{\circledR}_{g,h,*1}\right]_{j\times j} \right\} \mathbf{1}_{\left\{\Lambda_{1,1}\Psi^{\circledR}_{i}(T_1) > \Lambda_{1,1}K_1\right\}} \right\} \\
&= \sum_{j=1}^{i} \Lambda_{j+1,1} K_{j+1} B(T_0,T_{j+1}) \widetilde{E}^{T} \left\{ \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b^{\circledR}_{i,g,*1}\right]_{j\times 1}; \left[\widetilde{\rho}^{\circledR}_{g,h,*1}\right]_{j\times j} \right\} \mathbf{1}_{\left\{\Lambda_{1,1}\Psi^{\circledR}_{i}(T_1) > \Lambda_{1,1}K_1\right\}} \right\}
\end{aligned}
$$

Substitute the (C.2) into the above equation and apply the similar way of $\widetilde{\Psi}^{\circledR}_{i+1,1}$, it can be derived that $\widetilde{\Psi}^{\circledR}_{i+1,2} = \sum_{j=2}^{i+1} \Lambda_{j,1} B(T_0,T_j) K_j \mathbf{N}_j \left\{ \left[\Lambda_{j,g} b^{\circledR}_{j,g}\right]_{j\times 1}; \left[\widetilde{\rho}^{\circledR}_{g,h}\right]_{j\times j} \right\}$.

$\widetilde{\Psi}^{\circledR}_{i+1,3}$ can be derived by the same method. $\widetilde{\Psi}^{\circledR}_{i+1,3} = \Lambda_{1,1} B(T_0,T_1) K_1 \mathbf{N}_1 \left\{ \Lambda_{i+1,1} b^{\circledR}_{i+1,1} \right\}$. Consequently, $\Psi^{\circledR}_{i+1}(T_0)$ is proved. **Q.E.D.**

Appendix D: Proof of Theorem 4.2

The theorem is proved by induction. For k=1, $\frac{\partial \mathbf{N}_1\{d_{\{1\},1}\}}{\partial G_\ell} = f\left(d_{\{1\},1}\right)\mathbf{N}_0$. The theorem thus stands for k=1.

By the result of k=1 and Leibnitz's rule, it is obtained that

$$\frac{\partial \mathbf{N}_2\left\{d_{\{2\},1}, d_{\{2\},2}; \left[Q_{\{2\},g,h}\right]_{2\times 2}\right\}}{\partial G_\ell} = -f\left(d_{\{2\},1}\right)\frac{\partial(-d_{\{2\},1})}{\partial G_\ell}\mathbf{N}_1\left\{\frac{d_{\{2\},2}-Q_{\{2\},1,2}d_{\{2\},1}}{\sqrt{1-Q^2_{\{2\},1,2}}}\right\} + \tilde{\mathbf{N}}_{2,1},$$

where $\tilde{\mathbf{N}}_{2,1} \equiv \int_{-d_{\{2\},1}}^{\infty} f(z)\frac{1}{\sqrt{2\pi}}\left\{\exp\left[-\frac{1}{2}\left(\frac{d_{\{2\},2}+Q_{\{2\},1,2}z}{\sqrt{1-\left(Q_{\{2\},1,2}\right)^2}}\right)^2\right]\right\}\frac{\partial\left(\frac{d_{\{2\},2}+Q_{\{2\},1,2}z}{\sqrt{1-Q^2_{\{2\},1,2}}}\right)}{\partial G}\mathrm{d}z$.

Denote $z_4 := \frac{z+d_{\{2\},2}Q_{\{2\},1,2}}{\sqrt{1-Q^2_{\{2\},1,2}}}$. Thus $\tilde{\mathbf{N}}_{2,1}$ can be rewritten as

$$\tilde{\mathbf{N}}_{2,1} = f\left(d_{\{2\},2}\right)\frac{\partial d_{\{2\},2}}{\partial G_\ell}\int_{\frac{-d_{\{2\},1}+Q_{\{2\},1,2}d_{\{2\},2}}{\sqrt{1-Q^2_{\{2\},1,2}}}}^{\infty} f(z_4)\mathrm{d}z_4 = f\left(d_{\{2\},2}\right)\frac{\partial d_{\{2\},2}}{\partial G_\ell}\mathbf{N}_1\left\{\frac{d_{\{2\},1}-Q_{\{2\},1,2}d_{\{2\},2}}{\sqrt{1-Q^2_{\{2\},1,2}}}\right\}$$

$$\frac{\partial \mathbf{N}_2(d_{\{2\},1}, d_{\{2\},2}; \left[Q_{\{2\},g,h}\right]_{2\times 2})}{\partial G_\ell} = \sum_{j=1}^{2} f\left(d_{\{2\},j}\right)\frac{\partial d_{\{2\},j}}{\partial G_\ell}\mathbf{N}_1\left\{\left(\left[\frac{d_{\{2\},g}-d_{\{2\},j}Q_{\{2\},j,g}}{\sqrt{1-Q^2_{\{2\},j,g}}}\right]_{2\times 1}\right)^{(-j,-j)}\right\}$$

Hence Equation (4.1.1) stands for k=2.

Assuming that Equation (4.1.1) is true for k, the following proves that it is also true for k+1. By Leibnitz's rule, $\frac{\partial \mathbf{N}_{k+1}\{[d_{\{k+1\},g}]_{k+1\times 1}; [Q_{\{k+1\},g,h}]_{(k+1)\times(k+1)}\}}{\partial G_\ell} =$

$$f(d_{\{k+1\},1})\frac{\partial d_{\{k+1\},1}}{\partial G_\ell}\mathbf{N}_k\left\{\left[\frac{d_{\{k+1\},g+1}-Q_{\{k+1\},1,g+1}d_{\{k+1\},1}}{\sqrt{1-Q^2_{\{k+1\},1,g+1}}}\right]_{k\times 1}; \left(\left[\frac{Q_{\{k+1\},g,h}-Q_{\{k+1\},1,g}Q_{\{k+1\},1,h}}{\sqrt{(1-Q^2_{\{k+1\},1,g})(1-Q^2_{\{k+1\},1,h})}}\right]_{(k+1)\times(k+1)}\right)^{(-1,-1)}\right\}$$

$$+\tilde{\mathbf{N}}_{k+1,1} \qquad \text{........ (D.1)}$$

$$\widetilde{\mathbf{N}}_{k+1,1} \equiv \int_{-d_{\{k+1\},1}}^{\infty} f(z)\frac{\partial}{\partial G_\ell}\left(\mathbf{N}_k\left\{\left[\frac{d_{\{k+1\},g+1}+Q_{\{k+1\},1,g+1}z}{\sqrt{1-Q^2_{\{k+1\},1,g+1}}}\right]_{k\times 1};\left[\frac{Q_{\{k+1\},g+1,h+1}-Q_{\{k+1\},1,g+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1-Q^2_{\{k+1\},1,g+1})(1-Q^2_{\{k+1\},1,h+1})}}\right]_{k\times k}\right\}\right)dz$$

Using the corresponding result for $\frac{\partial \mathbf{N}_k([d_{\{k\},g}]_{k\times 1};[Q_{\{k\},g,h}]_{k\times k})}{\partial G_\ell}$, by substituting

$\frac{d_{\{k+1\},g+1}+Q_{\{k+1\},1,g+1}z}{\sqrt{1-Q^2_{\{k+1\},1,g+1}}}$ and $\frac{Q_{\{k+1\},g+1,h+1}-Q_{\{k+1\},1,g+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1-Q^2_{\{k+1\},1,g+1})(1-Q^2_{\{k+1\},1,h+1})}}$ as $d_{k,g}$, $Q_{k,g,h}$ in

Equation (4.1.1) respectively and setting $Z_{k+1,j+1}=\frac{z+d_{\{k+1\},j+1}Q_{\{k+1\},1,j+1}}{\sqrt{1-Q^2_{\{k+1\},1,j+1}}}$, $\widetilde{\mathbf{N}}_{k+1,1}$ can

derived as

$$\widetilde{\mathbf{N}}_{k+1,1}=\sum_{j=1}^{k} f\left(d_{\{k+1\},j+1}\right)\frac{\partial d_{\{k+1\},j+1}}{\partial G_\ell}\int_{\frac{-d_{\{k+1\},1}+d_{\{k+1\},j+1}Q_{\{k+1\},1,j+1}}{\sqrt{1-Q^2_{\{k+1\},1,j+1}}}}^{\infty} f\left(Z_{k+1,j+1}\right)\mathbf{N}_{k-1}\left\{\widetilde{\mathbf{H}}_1;\widetilde{\mathbf{H}}_2\right\}dZ_{k+1,j+1} \quad \text{..... (D.2)}$$

The numerator and the denominator of $\widetilde{\mathbf{H}}_1$ are multiplied by $\frac{\sqrt{1-Q^2_{\{k+1\},1,g+1}}}{\sqrt{1-Q^2_{\{k+1\},j+1,g+1}}}$ in

order to match the format of Theorem 3.1. Therefore

$$\widetilde{\mathbf{H}}_1=\left(\left[\frac{\frac{d_{\{k+1\},g+1}-d_{\{k+1\},j+1}Q_{\{k+1\},j+1,g+1}}{\sqrt{1-Q^2_{\{k+1\},j+1,g+1}}}+Z_{k+1,j+1}H_{1,g}}{\sqrt{1-H^2_{1,g}}}\right]_{k\times 1}\right)^{(-j,)} \quad \text{and} \quad \widetilde{\mathbf{H}}_2=\left(\left[H_{2,g,h}\right]_{k\times k}\right)^{(-j,-j)},$$

where $H_{1,g}\equiv\frac{Q_{\{k+1\},1,g+1}-Q_{\{k+1\},1,j+1}Q_{\{k+1\},j+1,g+1}}{\sqrt{(1-Q^2_{\{k+1\},1,j+1})(1-Q^2_{\{k+1\},j+1,g+1})}}, \forall 1\le g\le k$;

$$H_{2,g,h}=\frac{\frac{Q_{\{k+1\},g+1,h+1}-Q_{\{k+1\},1,g+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1-Q^2_{\{k+1\},1,g+1})(1-Q^2_{\{k+1\},1,h+1})}}-\frac{Q_{\{k+1\},j+1,g+1}-Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,g+1}}{\sqrt{(1-Q^2_{\{k+1\},1,j+1})(1-Q^2_{\{k+1\},1,g+1})}}\frac{Q_{\{k+1\},j+1,h+1}-Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1-Q^2_{\{k+1\},1,j+1})(1-Q^2_{\{k+1\},1,h+1})}}}{\sqrt{\left[1-\left(\frac{Q_{\{k+1\},j+1,g+1}-Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,g+1}}{\sqrt{(1-Q^2_{\{k+1\},1,j+1})(1-Q^2_{\{k+1\},1,g+1})}}\right)^2\right]\left[1-\left(\frac{Q_{\{k+1\},j+1,h+1}-Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1-Q^2_{\{k+1\},1,j+1})(1-Q^2_{\{k+1\},1,h+1})}}\right)^2\right]}},$$

$\forall 1\le g,h\le k$.

The integration of $\widetilde{\mathbf{N}}_{k+1,1}$ can be performed by applying Theorem 3.1. Hence,

$$\widetilde{\mathbf{N}}_{k+1,1}=\sum_{j=1}^{k} f\left(d_{\{k+1\},j+1}\right)\frac{\partial d_{\{k+1\},j+1}}{\partial G_\ell}\mathbf{N}_k\left\{\left(\left[\frac{d_{\{k+1\},g}-d_{\{k+1\},j+1}Q_{\{k+1\},j+1,g}}{\sqrt{1-Q^2_{\{k+1\},j+1,g}}}\right]_{(k+1)\times 1}\right)^{(-j-1,1)};\widetilde{\mathbf{H}}_3\right\} \quad \text{......(D.3)}$$

where $\widetilde{\mathbf{H}}_3=\begin{bmatrix} 1 & \widetilde{\mathbf{H}}_4^{\mathrm{T}} \\ \widetilde{\mathbf{H}}_4 & \widetilde{\mathbf{H}}_5 \end{bmatrix}$, $\widetilde{\mathbf{H}}_4=\left(\left[\dfrac{Q_{\{k+1\},1,g+1}-Q_{\{k+1\},1,j+1}Q_{\{k+1\},j+1,g+1}}{\sqrt{(1-Q^2_{\{k+1\},1,j+1})(1-Q^2_{\{k+1\},j+1,g+1})}}\right]_{k\times 1}\right)^{(-j,)}$,

By Theorem 3.1, $\widetilde{\mathbf{H}}_3$ and $\widetilde{\mathbf{H}}_5$ are symmetric with diagonal elements equal to 1. For $1\le g<h$, $H_{5,g,h}=H_{1,g}H_{1,h}+\sqrt{(1-H_{1,g}^2)(1-H_{1,h}^2)}H_{2,g,h}$. Thus

$$\widetilde{\mathbf{H}}_5=\left(\left[\frac{Q_{\{k+1\},g+1,h+1}-Q_{\{k+1\},j+1,g+1}Q_{\{k+1\},j+1,h+1}}{\sqrt{(1-Q^2_{\{k+1\},j+1,g+1})(1-Q^2_{\{k+1\},j+1,h+1})}}\right]_{k\times k}\right)^{(-j,-j)} \quad \text{and}$$

$$\widetilde{\mathbf{H}}_3=\left(\left[\frac{Q_{\{k+1\},g,h}-Q_{\{k+1\},j+1,g}Q_{\{k+1\},j+1,h}}{\sqrt{(1-Q^2_{\{k+1\},j+1,g})(1-Q^2_{\{k+1\},j+1,h})}}\right]_{(k+1)\times(k+1)}\right)^{(-j-1,-j-1)}.$$

Substitute $\widetilde{\mathbf{H}}_3$ into Equation (D.3) and change the index j to obtain

$$\widetilde{\mathbf{N}}_{k+1,1}=\sum_{j=2}^{k+1}\frac{1}{\sqrt{2\pi}}f\left(d_{\{k+1\},j}\right)\frac{\partial d_{\{k+1\},j}}{\partial G_\ell}\times$$

$$\mathbf{N}_k\left\{\left(\left[\frac{d_{\{k+1\},g}-d_{\{k+1\},j}Q_{\{k+1\},j,g}}{\sqrt{1-Q^2_{\{k+1\},j,g}}}\right]_{(k+1)\times 1}\right)^{(-j,)};\left(\left[\frac{Q_{\{k+1\},g,h}-Q_{\{k+1\},j,g}Q_{\{k+1\},j,h}}{\sqrt{(1-Q^2_{\{k+1\},j,g})(1-Q^2_{\{k+1\},j,h})}}\right]_{(k+1)\times(k+1)}\right)^{(-j,-j)}\right\}$$

……(D.4)

Substituting the above result into Equation (D.1), the consequence is obtained: $\dfrac{\partial \mathbf{N}_{k+1}\{[d_{\{k+1\},g}]_{(k+1)\times 1};[Q_{\{k+1\},g,h}]_{(k+1)\times(k+1)}\}}{\partial G_\ell}$. **Q.E.D.**

Appendix E: Proof of Theorem 5.1

Proof of (a):

For part (a), $\frac{\partial \Psi_i(T_0)}{\partial S_0} = \Lambda_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} \mathbf{N}_i \left\{ \left[\Lambda_{i,g} a_{i,g}\right]_{i\times i}; \left[\tilde{\rho}_{g,h}\right]_{i\times i} \right\} + \tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2}$,

where $\tilde{\Psi}_{\partial S,1} \equiv \Lambda_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} S_0 \sum_{v=1}^{i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_{i,v}^2} (\Lambda_{i,v} \frac{\partial a_{i,v}}{\partial S_0}) \aleph_{i-1,a,-v}$,

$$\tilde{\Psi}_{\partial S,2} \equiv \sum_{j=1}^{i} \Lambda_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \sum_{v=1}^{j} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b_{i,v}^2} (\Lambda_{i,v} \frac{\partial b_{i,v}}{\partial S_0}) \aleph_{j-1,b,-v}$$

The sequential paragraphs demonstrate $\tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2} = 0$.

By definition, $e^{-\frac{1}{2}a_{i,v}^2} = \frac{S_{\#v,i}}{S_0} e^{-\frac{1}{2}b_{i,v}^2 - \int_{T_0}^{T_v} r(u)du}, \forall 1 \le v \le i$. ......(E.1)

The $\tilde{\Psi}_{\partial S,3}$ is denoted as $\tilde{\Psi}_{\partial S,3} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b_{i,v}^2} (\Lambda_{i,v} \frac{\partial b_{i,v}}{\partial S_0}) \aleph_{v-1,b} e^{-\int_{T_0}^{T_v} r(u)du}$ for convenience.

According to Lemma 5.1, Equation (E.1) and the fact that $\frac{\partial a_{i,v}}{\partial S_0} = \frac{\partial b_{i,v}}{\partial S_0}$, $\tilde{\Psi}_{\partial S,1}$ can be reformulated as

$$\tilde{\Psi}_{\partial S,1} = \sum_{v=1}^{i} \tilde{\Psi}_{\partial S,3} \left(\left[\mathbf{1}_{\{v<i\}} e^{-\int_{T_v}^{T_i} q(u)du} + \mathbf{1}_{\{v=i\}} \right] \Lambda_{i,1} S_{\#v,i} \mathbf{N}_{i-v} \left\{ \left[\Lambda_{i,v+g} a_{i,g,\#v}\right]_{(i-v)\times 1}; \left[\tilde{\rho}_{g,h,*v}\right]_{(i-v)\times(i-v)} \right\} \right).$$

$$\begin{aligned}
\tilde{\Psi}_{\partial S,2} &= \sum_{j=1}^{i} \Lambda_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \sum_{v=1}^{j} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b_{i,v}^2} (\Lambda_{i,v} \frac{\partial b_{i,v}}{\partial S_0}) \aleph_{v-1,b} \times \mathbf{N}_{j-v} \left\{ \left[\Lambda_{i,v+g} b_{i,g,\#v}\right]_{(j-v)\times 1}; \left[\tilde{\rho}_{g,h,*v}\right]_{(j-v)\times(j-v)} \right\} \\
&= \sum_{v=1}^{i} \tilde{\Psi}_{\partial S,3} \left(\Lambda_{v,1} K_v + \mathbf{1}_{\{v<i\}} \sum_{j=1}^{i-v} e^{-\int_{T_v}^{T_{v+j}} r(u)du} \Lambda_{v+j,1} K_{v+j} \mathbf{N}_j \left\{ \left[\Lambda_{i,v+g} b_{i,g,\#v}\right]_{(j-v)\times 1}; \left[\tilde{\rho}_{g,h,*v}\right]_{(j-v)\times(j-v)} \right\} \right)
\end{aligned}$$

The last equality is obtained by interchange of the two summations.

$$\tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2} = \tilde{\Psi}_{\partial S,3} \left(\Lambda_{i,1} S_{\#i,i} \mathbf{N}_0 - \Lambda_{i,1} K_i \right) + \sum_{v=1}^{i-1} \tilde{\Psi}_{\partial S,3} \left(\Lambda_{v,1} \tilde{\Psi}_{\partial S,4} - \Lambda_{v,1} K_v \right),$$

where $\widetilde{\Psi}_{\partial S,4} = e^{-\int_{T_v}^{T_i} q(u)du} \Lambda_{i,v+1} S_{\#v,i} \mathbf{N}_{i-v} \left\{ \left[\Lambda_{i,v+g} a_{i,g,\#v} \right]_{(i-v)\times 1} ; \left[\widetilde{\rho}_{g,h,*v} \right]_{(i-v)\times(i-v)} \right\}$

$$- \sum_{j=1}^{i-v} e^{-\int_{T_v}^{T_{v+j}} r(u)du} \Lambda_{v+j,v+1} K_{v+j} \mathbf{N}_j \left\{ \left[\Lambda_{i,v+g} b_{i,g,\#v} \right]_{(j-v)\times 1} ; \left[\widetilde{\rho}_{g,h,*v} \right]_{(j-v)\times(j-v)} \right\}.$$

By definitions, $S_{\#i,i}$=K$_i$, hence $\Lambda_{i,1} S_{\#i,i} \mathbf{N}_0 - \Lambda_{i,1} K_i = 0$. $\widetilde{\Psi}_{\partial s,4}$ is the (*i*-*v*)-fold compound option price with start time T_v (instead of T_0). In other words, $\widetilde{\Psi}_{\partial s,4} = \Psi_{i-v}(T_v)$ with initial asset price $S_{\#v,1}$. Thus, by definitions, $\widetilde{\Psi}_{\partial s,4} = K_v$, and $\widetilde{\Psi}_{\partial s,1} - \widetilde{\Psi}_{\partial s,2} = 0$. Part (b), (c), (d) and (e) can be proved by similar method to part (a). **Q.E.D.**