Chapter 1
Introduction

Intelligent Transportation Systems (ITS) is considered possessing the potential to solve the traffic congestion problem. By providing road users with traffic information and then helping them to make better travel decisions, ITS promises to enhance the utilization of existing transportation facilities and to mitigate traffic congestion. A considerable amount of resources have been concentrated on the academic research of fundamental issues as well as the empirical deployments of ITS infrastructures in Taiwan and worldwide.

For examples, the national master plan and system architecture had been accomplished by the Institute of Transportation, Ministry of Transportation and Communication. The follow-up project of ITS regional architecture has been kept on going by the same authority. There are several traffic control systems and integrated traffic information system deployed on the national freeway system by the National Freeway Bureau and the Institute of Transportation, Ministry of Transportation and Communications. There are also many ITS-related long-term projects funded by the National Science Council and implemented by the Ministry of Transportation and Communications. Some ITS initiatives of advanced information system for bus transit operations had been carried out in metropolitan areas of Taiwan.

One of the key features to functionalize the effectiveness and the efficiency of ITS operations is the interaction between travel information provision and the corresponding response of road users. Therefore, the ability of predicting how the travel information predicted and provided by ITS influences the time trajectory of network flows is an essential issue in the viewpoints of both theoretical analysis and empirical improvements.

This dissertation is developed to concentrate on the theoretical analysis of network states interacted among users, system performances, and travel information. The methodology proposed in this thesis is expected to form the preliminary foundation of the further elaborations for the operational analysis and planning applications of transportation system under the scenario of ITS services

1.1 Problem Statement

Several desirable properties of ITS deployment are outlined as the following. ITS should be based on projected future demand to evaluate traffic control strategy, anticipate traffic variations, reduce overall delays, control and eliminate overreaction, improve travel information reliability, and maintain credibility to achieve traveler compliance. In order to reach these benchmark, encapsulating the inter-dependence of network flows and travel information provision into the ITS functionality is a fundamental and critical issue for traffic researchers.

As pointed out by historical studies (Friesz, 1985, Friesz *et al*., 1996), the above statement can be realized implicitly if the problem is reduced to the so-called dynamic network design problem, a formulation showing that how to determine a temporal management (or control) plan, which recognizes that flow perturbations generated from the new treatments bring about disequilibria that might adjust toward equilibrium. It is commonly observed that there exists a subset (or a sub level programming) to formulate the network flow pattern as the network design problem is solved in both static and dynamic scenario. The mentioned flow patterns construct the basic feasible solution set for the corresponding problem but each of them meets a specific criterion that depends on the methodology or behavioral assumption it assumes.

Therefore, the problem concerned in the dissertation is focused on modeling the effects of travel information on the network flow evolution under the maneuvers of ITS. And the results of this research can be applied to the dynamic network design problem and to address the network flow evolution involving the implementation of traffic improvement alternatives.

In addition, the scenario of ITS operations mentioned above assumes that ITS solutions must be able to record the traffic volumes and to predict and distribute travel information defined in this research. No specific technologies are preferred to deliver these functions and services of ITS. The users discussed in this study are limited to the home-to-work commuters with private mode.

1.2 Motivations

The motivations of this research are distributed into two parts. The first one is the research problem mentioned above is valuable. A network flow evolution model is a core device for the traffic analysis of operational and planning applications. It determines the quality of predicted travel information and the effectiveness of traffic management alternatives. The ITS infrastructure won't work itself better without a well-predicted flow pattern.

The other motivation is that the research problem mentioned above is a fundamental and challenging work. The excellently established Wardrop's principle (1952) dominates the static theory of traffic assignment. However, one aspect that is unanimous among traffic researchers is that the general dynamic traffic assignment (DTA) problem is inherently specified by ill-behaved system characteristics that are imposed by the need to represent traffic realism and user manners adequately. And that's the reason why researchers have become increasingly aware that the theory of DTA is still relatively undeveloped, which necessitates new approaches that account for challenges from the application domains as well as for the fundamental questions related to tractability and realism.

Another common feature of the existing studies is that they depart from the standard static assignment assumptions to deal with time-varying flows and none of them presently provides a universal solution for general networks.

1.3 Objectives

The purpose of this research is to develop an analytical approach capable of modeling the interacted relationships between network flows and travel information and characterizing the theoretical issues of existence, uniqueness, and stability for the proposed model with a thoroughly mathematical foundation. It is expected that the proposed theory will build an analytical linkage between the flow evolution and the empirical adaptability of travel preference under the operations of Intelligent Transportation Systems.

1.4 Methodology

Based on the behavioral assumptions of minimal travel time seeking and daily learning and adaptive process of travel decision, a theory of day-to-day network dynamics under ITS operations is developed. The proposed theory of day-to-day network dynamics is composed of path flow dynamics, predicted minimal travel time dynamics, and their interactions.

Then, a mathematical formulation for the theory is presented by using continuous-time dynamical system approach. The existence and uniqueness of solution is characterized by the fundamental theorem of ordinary differential equations. The property of equilibrium solution for the proposed model is analyzed and proved to be asymptotic stable by the so-called second method of Lyapunov.

1.5 Contributions

This research primarily concerns the interacted network dynamics under ITS operations with the behavioral assumptions of minimal travel time seeking and daily learning and adaptive process of travel decision. Several significant contributions are noted in the following:

(1) The author develops a new theory of day-to-day network dynamics under ITS operations to guide the behavior of flow evolution.

(2) Two mathematical formulations of the proposed theory for both uniform and inhomogeneous user classes are successfully accomplished.

(3) A Lipschitz Lemma is generated and proved to claim the existence and uniqueness of the presented dynamical systems by means of the fundamental theorem of ordinary differential equations.

(4) A strict Liapunov function is built to claim the asymptotic stability of the equilibrium solution for the proposed theory and models.

1.6 Dissertation Outline

The dissertation is composed of eight chapters. They are briefly narrated as follows.

The first chapter gives an introduction of the problem statement, motivations, objectives, methodology, and contributions of this research.

A literature review is presented in chapter 2. Historical studies are grouped into two categories, disequilibrium approaches and equilibrium-like methods.

In chapter 3, the theory of day-to-day network dynamics with the behavioral assumption of minimal travel time seeking and daily learning and adaptive process of travel decision is developed. The conceptual framework and general model of day-to-day network dynamics are derived to form the basis for the further invention of modeling and theoretic analysis.

Path flow dynamics, predicted minimal travel time dynamics, and their inter-dependence are formulated in chapter 4. Both uniform and inhomogeneous user classes are considered in the formulations.

Issues of existence and uniqueness are provided in chapter 5. A Lipschitz Lemma is generated and proved to claim the existence and uniqueness of the proposed dynamical systems by means of the fundamental theorem of ordinary differential equations.

Chapter 6 gives the analysis of steady state and its stability by the so-called second method of Lyapunov. After introducing the theorem of Lyapunov stability, a strict Lyapunov function is created to specify the asymptotic stability of the equilibrium solution.

Two examples of the proposed models, homogeneous-user model and multiple-user-class model, solved by the high-order Runge-Kutta method are illustrated to show the network dynamics numerically with a simple network in chapter 7.

Finally, conclusions and prospects are summarized in chapter 8.

Chapter 2

Literature Review

This chapter provides literature reviews of DTA related studies by classifying the various theories into two broad methodological groups: disequilibrium methods and equilibrium-like approaches. The disequilibrium methods simulate the time passage of network flows in a day-to-day time scale. No equilibrium-like behavioral assumption is enforced on the theory and modeling process of these kind approaches. A further sub label can be imposed on these approaches to make the distinction between analytical methods and simulation-based methods. For the equilibrium-like approaches, a vast body of literature had been developed over the past two decades. Brief reviews on mathematical programming, optimal control, and variational inequality are also provided in this chapter.

2.1 Disequilibrium Methods

The question of whether equilibrium actually takes place or is a mathematical construct is a very old issue and probably precedes even the definition of traffic equilibrium itself. The context of disequilibrium network flows provides opportunities to relax the restrictive steady-state equilibrium assumptions and to model the phenomena of evolving disequilibria. This type of evolution models is able to capture the transition states of system and consequently might reach an equilibrium as the simulation duration is long enough. However, if the duration of the disruption and the time it takes for the system to reach equilibrium are such that the system stays longer in a non-equilibrium state rather than in equilibrium, it is important to catch the inter-transition process of traffic diversion. These are the most important and common features of disequilibrium methods. The following three sub sections introduce these approaches with a daily temporal scaling.

2.1.1 Simulation-based Method - DYNASMART

Probably the only experimental evidence of user decision-making behavior is a real test that involved 100 travelers over a 24-day period performed by Chang and Mahmassani's (1988), also see Mahmassani, Chang, and Herman (1986), Mahmassani (1990). The results of these serial researches revealed that route choice behaviors of commuters had indicated that the learning and adaptive process for this choice may take weeks, partly because of the dynamic feedbacks from the traffic system, and can indeed be lengthened by complex switching that resulted from the provision of better information.

Mahmassani (2001) gave an overview of the DYNASMART simulation-assignment logic, the principle formulations that it is intended to support in the context of ITS network applications, and the specific DTA procedures developed for these formulations. A DTA system for advanced traffic network management was built around the traffic simulation-assignment modeling framework, which describes the evolution of traffic patterns in the network for given traffic loading under particular control measures and route guidance information supply strategies to individual motorists. The simulator was also embedded in an interactive search algorithm to determine optimal route guidance instructions to motorists. The related historical references can also be found in Mahmassani and Peeta (1993), Jayakrishnan et al. (1994), Hu and Mahmassani (1995, 1997).

2.1.2 Deterministic Adjustment Process

Smith (1979) gave a new interpretation of Wardrop's first principle to formulate a dynamical system conceptually with its equilibria being exactly the same as Wardrop's equilibria. His principle states that a single driver may use the same route tomorrow. However if he does change a route then he must change to a route, which today was cheaper than the one he actually used today. This point in Smith's paper should be highlighted that cost comparisons, which are supposed to influence tomorrow's behavior, are based on today's actual route-costs, the latest information.

Friesz et al. (1994) addressed the experiment of Chang and Mahmassani (1988) theoretically by introducing a tatonnement process in micro economic theory for modeling the transition of disequilibria from one state to another. They presented both a qualitative analysis of stability and numerical studies, which show that such an approach provides a reliable means for determining static user equilibria. They also described circumstances for which these models depict day-to-day adjustments from one realizable disequilibrium state to another and how these adjustment processes differ depending on the quality of the information being provided by traveler information system.

In particular, Friesz et al. facilitated certain elementary properties of variational inequality problems and fixed point problems to prove the steady state of the adjustment process satisfying static Wardropian user equilibria on congested network with fully general demand and cost structure. Under appropriate regularity conditions, they gave the proof of asymptotic stability for equilibrium solution in the sense of Lyapunov. However, the presented flow adjustment process was impassive to congestion level. And the non-uniqueness of solutions raises the methodological and practical issues in flow prediction.

2.1.3 Stochastic Adjustment Process

In general, stochastic adjustment process utilizes a stochastic mechanism to determine the route choice probability of traffic users. A simplified formula is applied to estimate the predicted travel time by giving each historical travel time a weight.

Horowitz (1984) developed an investigation of the stability of stochastic equilibrium in a two-link network involving a route choice decision-making over time. It had been shown that even when equilibrium is unique, link volumes may converge to their equilibrium values, oscillate about equilibrium perpetually, or converge to values that may be considerably different from the equilibrium ones, depending on the details of the route choice decision-making process.

The serial studies of Cascetta and Cantarella (1991), Cantarella and Cascetta (1995), and V. Astarita et al. (1999) provided a doubly dynamic assignment model for a general network. They assumed that users' choices are based on information about travel times and generalized transportation costs occurred in a finite number of previous days. The information are supplied and managed by an information system. This model follows a non-equilibrium approach in which both within-day and day-to-day flow fluctuations were formulated as a stochastic process. They also presented a dynamic network loading method for computing within-day variable arc flows from path flows. The model deals explicitly with queuing at oversaturated intersections and can be denoted as a fixed point problem. The authors proved that the process admits a unique stationary probability distribution with an equilibrium probability vector if some sufficient conditions hold. Numerical results of an application of the proposed procedure to a realistic network were also described in Cascetta and Cantarella (1991) and V. Astarita et al. (1999).

Watling (1999) considered the stochastic approach to design a day-to-day traffic assignment process on a simple two-link network. The model consists of a multinomial split of the day-specific demand among those drivers that follow information, those that follow habitual choices, and those that choose a route based on previously experienced costs. The optimal split in the information are computed according to three criteria:

(1) user optimal routing, in which the user equilibrium proportions are used;

(2) stochastic user optimal routing, in which the stochastic user equilibrium proportions for a logit-based model with dispersion parameter are used; and

(3) system optimal routing, in which the proportions that minimize total travel cost are used.

Based on the numerical tests of this study, the total travel time is a largely decreasing function of the probability of drivers following information. In order to overcome the correlations among the overlapping route segments, Watling suggested that it is more usually assumed that link cost perceptual errors follow independent Normal distributions, which implies a Multivariate Normal distribution of cost perceptual errors and leads to a probit choice model.

2.2 Equilibrium-like Approaches

Some other researchers are concerned about the formulations and solution methods of dynamic traffic assignment problem to compute the flow pattern satisfying system optimum (SO), dynamic user equilibrium (DUE), or dynamic user optimum (DUO). These studies do not simulate the evolutions of network flows but only a unique flow solution with some optimal criteria.

2.2.1 Mathematical Programming Formulations

Merchant and Nemhauser (1978) represent the first attempt to formulate the DTA problem as a mathematical program. The model was limited to the single-destination, single-commodity, and SO case. A link exit function to propagate traffic and a static link performance function to represent the travel cost as a function of link volume were provided. The model was shown to give a proper generalization of the conventional static SO assignment problem, and the global solution was obtained by solving a piecewise linear version of the model.

Janson (1991) represents one of the earliest attempts to model DUE assignment problem as a mathematical programming. This approach seeks an equilibrium solution with experienced path travel times instead of the instantaneous travel time. Non-linear mixed integer constraints are proposed in the formulation to ensure temporal continuity of origin-destination (O-D) flows, though they may be violated in the solution procedure specified which is a straightforward extension of the well-known incremental assignment heuristic for static formulations.

Ziliaskopoulos (2000) introduced a linear programming formulation for the single destination SO assignment problem based on the cell transmission model for traffic propagation. H.K. Lo and W.Y. Szeto (2002) presented a cell-based dynamic traffic assignment model that follows the ideal dynamic user optimal principle. These studies circumvents the need for link performance functions as the flow propagation according to the cell transmission model (Daganzo, 1994), thereby being more sensitive to traffic realities.

2.2.2 Optimal Control Models

Friesz et al. (1989) discussed link-based optimal control formulations for both SO and DUE objectives on a single destination case. The models assume that adjustment s from one system state to another may occur concurrently as the network conditions change; that is, the routing decisions are made based on current network conditions, but can be continuously modified as conditions change. The SO model is a temporal ectension of the static SO model, and proves that at the optimal solution the instaneous flow marginal costs on the used paths for an O-D pair are identical and less than or equal the ones on the unused paths.

In the study of Ran et al. (1993), several link-based SO models and instantaneous DUE optimal control models were proposed for an urban transportation network with multiple origins and destinations. They use linear exit functions and quadratic link performance functions so as to reduce the computational burden for a time-space decomposition solution procedure that can only handle very small network problems.

2.2.3 Variational Inequality Methods

Friesz et al. (1993) formulated a continuous time path-based VI model to solve for the departure time/route choice by equilibrating the experienced travel times. Wei et al. (1995) introduced a discretized VI formulation for the simultaneous route-departure equilibrium problem to enable computational tractability, and proposed a heuristic algorithm to approximately solve it. They show solution existence under certain regularity conditions.

Ran and Boyce (1996) proposed a link-based discretized VI formulation with fixed departure times to conquer the problems with path-based VI models. Chen and Hsueh (1998) presented a link-based VI formulation for the DUE assignment problem. They show that without loss of generality, travel time on a link can be represented as a function of link in flow only (instead of function of link inflow, exit flow, and number of vehicles on the link). A solution algorithm based on the nested diagonalization procedure was also designed.

2.3 Summary and Discussions

It is obvious that the equilibrium-like approaches are unable to simulate the traffic patterns when perturbations of the traffic system create disequilibria. And they are invalid to depict the interactions between flow dynamics and information provision.

The simulation-based method of disequilibrium approaches is the most flexible approach to characterize the complicated system behaviors and their inter-dependence. However, this type of method can’t provide a satisfied theoretical analysis to sustain the issues of existence, uniqueness, and stability. Stability of the DTA solution is an important operational issue for the control of dynamic traffic network. This is because inappropriate assignment proportions may lead to increased unpredictability, or ill-behaved consequences for the system. Conceptually, the notion of stability implies that all solutions are bounded and converge to the time-dependent desirable states. The practical implication is that a stable solution minimizes or limits the deterioration of system performance (Peeta and Ziliaskopoulos, 2001).

However, even the analytical approaches of disequilibrium methods, the theoretical issues are not easily derived (Friesz et al., 1994). The stochastic adjustment models proposed by Cascetta and Cantarella (1991, 1995) failed due to the over-simplified updating process of anticipated cost. Under the operations of ITS, the predicted travel information should be specified in a more systematic viewpoints rather than a plain weighted mechanism.

Chapter 3
Theory of Day-to-day Network Dynamics

In this chapter, some essential assumptions and specific notation are employed to reflect the defined network dynamics and evolution focus. After the structure of day-to-day network dynamics theory is identified, the consequent conceptual framework and general model are also presented to form the preliminary foundation for the further modeling of network dynamics.

3.1 Assumptions and Notation

The main assumptions in this research are the minimal travel time seeking and the daily learning and adaptive process for both road users and operator of ITS. These assumptions are commonly accepted and applied in the existing disequilibrium approaches. The information provided by ITS include the actual path travel times and the predicted minimal travel time of previous day to help users making travel decision. Operator of ITS predicts minimal travel time based on the traffic volume detected by ITS and the predicted (or given) demand of previous day in a system-wide standpoint.

In addition, each link cost function is assumed to be a smooth and strict monotone function of link flow and is used to estimate the actual travel time for all paths. Travel demand is presumably fixed in this study without the loss of generality under the supposition of no structural changes from competing transportation facilities over the whole period of interest. The main concern is that the variations of path flow and path travel time are much more sensitive than that of the O-D demand if the travel information provided by ITS is the only perturbation of transportation system. It is also assumed that travel demands of all O-D pairs do not jointly violate any capacity constraints of all the links. En-route information provision and the corresponding user response are not considered in this research, neither.

Some notation based on the typical equilibrium models of commuter route choice are employed and augmented to meet the concerns of following context. In particular, all vectors are assumed to be column vectors. Vectors and matrices are expressed in boldface.

Table 3.1 Notation

Notation	Descriptions	units
t	time index	day
W	the full set of O-D pairs with $\overline{W}$ O-D pairs	
P	the full set of paths with $\overline{P}$ paths	
A	the full set of links with $\overline{A}$ links	
P_w	the set of paths connecting O-D pair *w* with $\overline{P}_w$ paths	
h_p^t	the non-negative peak-flow of path *p* at day *t*	vehicle/hour
h_w^t	the sum of h_p^t $\forall p \in P_w$	vehicle/hour
$\mathbf{h}^t$	the full vector of path flows at day *t*	
f_a^t	the non-negative peak volume of link *a* at day *t*, $f_a^t = \sum_p \delta_{ap} h_p^t$, where $\delta_{ap} = 1$ if link *a* belongs to path *p*, otherwise $\delta_{ap} = 0$	vehicle/hour
k_a	the capacity of link *a*;	vehicle/hour
$c_a(f_a^t)$	the unit average travel time on link *a* at day *t*, a smooth and strict monotone function of f_a^t	hour
c_p^t	the unit average travel time on path *p* at day *t*, without considering node travel time, $c_p^t = \sum_a \delta_{ap} c_a(f_a^t)$	hour
c_w^t	the unit minimal travel time on O-D pair *w* at day *t* predicted and provided by ITS	hour

$\boldsymbol{c}^t$	the full vector of c_w^t at day t	
D_w	the travel demand of O-D pair w over whole time period	vehicle/hour
α_p	the positive and path-specific parameter to denote the propensity of path flow dynamics	vehicle/hour^2
β_w	the positive and O-D-specific parameter to denote the sensitivity of predicted travel time dynamics	hour^2/vehicle
$\dot{x}$	the ordinary derivative of x with respect to t	
$\boldsymbol{x}'$	the transpose of vector (or matrix) $\boldsymbol{x}$	

3.2 Basic Structure of Network Dynamics

Conceptually, a dynamical system is to describe the future states unambiguously as the evolution rules and initial situations are specified. In order to meet this end, a way to formulate the time passage of all the points for a given space E mathematically is the first and most important thing before advancing further analysis. Consider a first-order dynamical system

$$\begin{aligned} \dot{x}_1 &= G_1(x_1, \cdots, x_n) \\ &\vdots \\ \dot{x}_n &= G_n(x_1, \cdots, x_n), \end{aligned} \tag{1}$$

this n-dimensional nonlinear system of first-order ordinary differential equations can be expressed in vector form as

$$\dot{\boldsymbol{v}} = \boldsymbol{G}(\boldsymbol{v}), \tag{2}$$

where $\boldsymbol{v} = (x_1, \cdots, x_n)'$ is a vector of system variables. There are some useful fundamental results for equation (2) to articulate the issues of existence and uniqueness analytically if a well-behaved slop field $\boldsymbol{G}$ were provided. As the system dynamics of interest could be successfully translated into the same format as equation (2) behaves, the analysis of asymptotic behavior will be very helpful to evaluate the prediction of system evolution.

The goal of this study mentioned in chapter one is to provide a theory capturing the effects of travel information on network dynamics, especially day-to-day interactions over system variables, network performances, and travel information. Let us start from the learning and adjustable behavior process assumed to guide daily travel decisions.

This process is specified as that travel information, the actual path travel times and the predicted minimal travel time of previous day, are provided to all path users of each O-D pair. Then travelers with actual travel time less than predicted minimal travel time should have perceived a (pseudo) travel time saving on the traveling of previous day. On the other hand, users with actual travel time that is more than the predicted minimal travel time should feel like a (pseudo) travel time loss on the traveling of previous day. These deviations result in the path flows adjustments of the next day to coincide with the behavioral assumption of minimal travel time seeking.

On the side of ITS operator, by recording actual path traffic volumes, ITS operator evaluates the difference between the fixed travel demand and the sum of the path flows for each O-D pair. In order to reflect the relative scarcity (or surplus) of transportation facilities, it is assumed that the predicted minimal travel time of an O-D pair adjusts accordingly as the difference between the fixed travel demand and the sum of the corresponding path flows is prompted.

Now we are ready to construct our dynamical system in the sense of (1). Taking time derivatives of the predicted minimal travel time of an O-D pair *w* and the corresponding path flows as the left hand side (LHS) of (1), we have the general form as

$$\begin{aligned} \dot{h}_p^t &= G_p((\boldsymbol{h}^t)', c_w^t) \\ \dot{c}_w^t &= G_w((\boldsymbol{h}^t)', (D_w)'), \end{aligned} \tag{3}$$

$\forall\, t \in T,\ p \in P_w,\ w \in W$. As mentioned in previous paragraph, the functions $G_{,p}$,$\forall\ p \in P$, in (3) are dominated by actual path travel time, which is a function of time-varying path flow pattern, and the predicted minimal travel time of O-D pair *w* at day *t*. On the other hand, the functions G_w ,$\forall\ w \in W$, in (3) are determined by the real happened path flows and the corresponding predicted demand of O-D pair *w*. It is obvious now that the network dynamics identified here includes path flow dynamics, predicted minimal travel time dynamics, and their interactions. This conceptual framework of network dynamics is illustrated as Fig. 3-1.

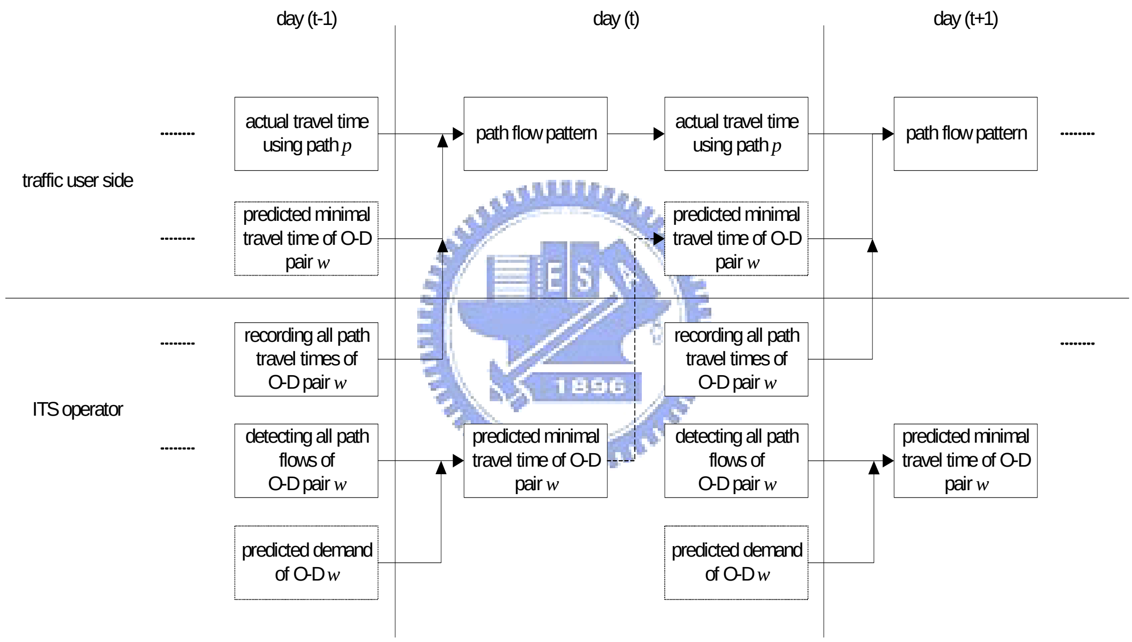

Fig. 3-1 Conceptual framework of network dynamics

Chapter 4
Modeling Network Dynamics

The contents of this chapter are distributed into two parts. The first is path flow dynamics that interact with ITS information, which are formulated in section 4.1. The second is predicted minimal travel time evolutions of O-D pairs that is predicted by ITS, which are analyzed in section 4.2. The whole network dynamics is a combination of these two components.

4.1 Path Flow Dynamics

The basic behavioral assumption of least travel time seeking has been mentioned in chapter 3. Before giving a mathematical relationship between path flow dynamics and this supposition, the author defines the value of total perceived travel time loss (or saving) for path $p \in P_w$ at day t as

$$PTTL_{p,w}^{t} \equiv h_p^t (c_p^t - c_w^t) . \quad (4)$$

The perceived travel time loss (or saving) is measured by multiplying path flow with the difference between users' average experienced travel time for a path estimated by link cost functions and the predicted travel time provided by ITS for the corresponding O-D pair. This quantity can be an estimation of the total travel time loss (or saving) perceived by travelers driving path p at day t. The meaning of positive (negative) $PTTL_{p,w}^{t}$ is that the average path travel time the users actually underwent is greater (less) than the travel time predicted by ITS. Travelers might be motivated to change their route due to this difference. Alternatively, $PTTL_{p,w}^{t}$ can be viewed as a measure of path performance for the previous time point and as a key factor to formulate the consequent shift of path flow at the next time point. In addition, the congestion effect is embedded in $PTTL^t_{p,w}$ implicitly by including the current state of path flow. It means that the values of $PTTL^t_{p,w}$ will be different at various congestion levels even under the condition that the average path travel time experienced by the users is equal to the travel time predicted by ITS. The predicted travel time of O-D pair w at day t, c^t_w, is updated by another dynamic process that will be presented in section 4.2. To continue the development it is useful to postulate that future path flow is established through the tuning of the present state at a rate proportional to the value of $PTTL^t_{p,w}$. That is

$$h_p^{t+\Delta t} \equiv h_p^t - \alpha_p (PTTL^t_{p,w})\Delta t\,, \tag{5}$$

with constraints

$$0 < \alpha_p < \inf_{c^t_p - c^t_w > 0}\left(\frac{1}{c^t_p - c^t_w}\right), \tag{6a}$$

and

$$\sum_{\substack{\forall p\in P \\ c^t_p - c^t_w < 0}} -\delta_{ap}\alpha_p PTTL^t_{p,w} \le k_a - \sum_{\forall p\in P}\delta_{ap}h^t_p\,, \tag{6b}$$

for all OD pair $w\in W$, $p\in P_w$, $a\in A$ at day t. Inequalities (6a) and (6b) ensure to avoid infeasibilities of non-negative flow and path capacity constraints. It is obvious that inequalities (6a) and (6b) is naturally satisfied if α_p is carefully calibrated from the empirical data in which meets the assumptions and conditions defined previously. By taking the limit of (5) as Δt approaches to zero and substituting (4) into (5), our path flow dynamics follow immediately as

$$\dot{h}_p^t = \frac{dh_p^t}{dt} = -\alpha_p h_p^t (c_p^t - c_w^t) \tag{7}$$

for all OD pair $w \in W$, $p \in P_w$ at day t. The physical meaning of (7) is that the time change rate of the flow for path p at day t is equal to the negative product of the propensity of path flow shift and the value of $PTTL_{p,w}^t$.

4.2 Predicted Minimal Travel Time Dynamics

To forward the network traffic to a steady status is the major intent of ITS. This goal is realized by successfully capturing predicted minimal travel time dynamics and delivering this information to users. In this section, we will focus on the formulation of predicted minimal travel time dynamics. To start the task, we first define the excess travel demand of an O-D pair w at day t as

$$ETD_w^t \equiv D_w - h_w^t, \tag{8}$$

where D_w denotes the time-invariant travel demand of an O-D pair w over whole time period of interest. ETD_w^t is considered to be the difference between the travel demand and the sum of the corresponding path flows for an O-D pair conceptually similar to Carey (1980) and Friesz et al. (1994). Positive (negative) excess travel demand means that the rate at which users desire to depart from an origin to a destination is greater than (less than or equal to) that where such movements are actually occurring. Predicted minimal travel time is then adjusted due to this deviation. Consequently, the predicted minimal travel time dynamics for all O-D pairs $w \in W$ at day t can be written as

$$c_w^{t+\Delta t} \equiv c_w^t + \beta_w (D_w - h_w^t)\Delta t, \tag{9}$$

where

$$0 < \beta_w < \min\left\{ \inf_{\forall\left(D_w - h_w^t < 0\right)} \left(\frac{\breve{c}_w - c_w^t}{D_w - h_w^t} \right), \inf_{\forall\left(D_w - h_w^t > 0\right)} \left(\frac{\widehat{c}_w - c_w^t}{D_w - h_w^t} \right) \right\}, \tag{10}$$

if we further suppose that the minimal travel time prediction of the next time point for an O-D pair is transformed from the current status at a rate scaled to ETD_w^t. Inequality (10) guarantees that all predicted travel times stay in the feasible region without greater than $\widehat{c}_w$, the travel time at maximal flow, or less than $\breve{c}_w$, the travel time at free flow, for OD pair w. And we define

$$\breve{c}_w \equiv \min_{j \in P_w} \left(\sum_a \delta_{aj} c_a (f_a = 0) \right) \tag{10a}$$

and

$$\widehat{c}_w \equiv \max_{j \in P_w} \left(\sum_a \delta_{aj} c_a (f_a = k_a) \right). \tag{10b}$$

After taking the limit of (9) the predicted minimal travel time dynamics can be written in differential form as

$$\dot{c}_w^t = \frac{dc_w^t}{dt} = \beta_w (D_w - h_w^t) \tag{11}$$

for all O-D pairs $w \in W$ at day t. Equation (11) reveals that the time change rate of the predicted minimal travel time for O-D pair w at day t is equal to the product of the sensitivity of predicted minimal travel time dynamics and ETD_w^t. The whole version of network dynamics under operations of ITS is accomplished as

$$
\begin{cases}
\dot{h}_p^t = -\alpha_p h_p^t (c_p^t - c_w^t) \\
\dot{c}_w^t = \beta_w (D_w - h_w^t) \\
0 < \alpha_p < \inf\limits_{c_p^t - c_w^t > 0} \left(\dfrac{1}{c_p^t - c_w^t} \right), \text{ and } \sum\limits_{w \in W} \sum\limits_{\substack{p \in P_w \\ c_p^t - c_w^t < 0}} \left(-\delta_{ap} \alpha_p PTTL_{p,w}^t \right) \le k_a - \sum\limits_{p \in P} \delta_{ap} h_p^t \\
0 < \beta_w < \min \left\{ \inf\limits_{\forall (D_w - h_w^t < 0)} \left(\dfrac{\breve{c}_w - c_w^t}{D_w - h_w^t} \right), \inf\limits_{\forall (D_w - h_w^t > 0)} \left(\dfrac{\hat{c}_w - c_w^t}{D_w - h_w^t} \right) \right\}
\end{cases} \quad (12)
$$

for all O-D pairs $w \in W$, $p \in P_w$ at day t.

4.3 Network Dynamics Involving Heterogeneous User Classes

In this section road users are classified into n sub-groups to consider the effects of heterogeneous adjustments due to the various sensitivities of multi-class users on path flow dynamics. That is to say the parameter α_p in (7) is not only path specific but also user specific. Consequently there are n path flow dynamics for path $p \in P_w$ at day t and (7) is augmented as

$$
\dot{h}_{ip}^t = \frac{dh_{ip}^t}{dt} = -\alpha_{ip} h_{ip}^t (c_p^t - c_w^t) \quad (13)
$$

for all O-D pair $w \in W$, $p \in P_w$, i=1,2,…,n, at day t. The physical meaning of (13) is that the time change rate of the flow for user class i using path p at day t is equal to the negative product of the propensity of path flow shift for user class i and the value of perceived travel time loss (saving) for user class i, $PTTL_{ip,w}^t$. If we further suppose that there are two options for ITS to provide predicted O-D travel time, i.e. a homogeneous forecast for all users or individual predictions according to multi-class users, the predicted minimal travel time dynamics should be reformulated as

$$\dot{c}_w^t = \frac{dc_w^t}{dt} = \beta_w (D_w - h_w^t), \qquad (14)$$

or

$$\dot{c}_{iw}^t = \frac{dc_{iw}^t}{dt} = \beta_{iw} (D_{iw} - h_{iw}^t) \qquad (15)$$

for all O-D pairs $w \in W$, i=1,2,…,n, at day t respectively. In (14) the definition of h_w^t is the same as mentioned before but calculated in a different way and expressed as

$$h_w^t = \sum_{p \in P_w} \sum_i h_{ip}^t \ . \qquad (16)$$

All components in (15) are re-indexed from (11) by adding a subscript i for the corresponding user class in particular with

$$h_{iw}^t = \sum_{p \in P_w} h_{ip}^t \ . \qquad (17)$$

Equation (15) tells us that the time change rate of the predicted minimal travel time for user class i traveling on O-D pair w at day t is equal to the product of the sensitivity of predicted minimal travel time dynamics of user class i and the value of excess travel demand of user class i, ETD_{iw}^t. Now the multi-class users dynamical systems for all O-D pairs $w \in W$, $p \in P_w$, i=1,2,…,n, at day t are proposed as

$$
\begin{cases}
\dot{h}_{ip}^{t} = -\alpha_{ip} h_{ip}^{t} (c_{p}^{t} - c_{w}^{t}) \\
\dot{c}_{w}^{t} = \beta_{w} (D_{w} - h_{w}^{t}) \\
0 < \alpha_{ip} < \inf\limits_{c_p^t - c_w^t > 0} \left(\dfrac{1}{c_{p}^{t} - c_{w}^{t}} \right), \text{ and } \sum\limits_{w \in W} \sum\limits_{i \in N} \sum\limits_{\substack{p \in P_w \\ c_p^t - c_w^t < 0}} \left(-\delta_{ap} \alpha_{ip} PTTL_{ip,w}^{t} \right) \le k_a - \sum\limits_{\substack{p \in P \\ i \in N}} \delta_{ap} h_{ip}^{t} \\
0 < \beta_{w} < \min \left\{ \inf\limits_{\forall (D_w - h_w^t < 0)} \left(\dfrac{\breve{c}_{w} - c_{w}^{t}}{D_{w} - h_{w}^{t}} \right), \inf\limits_{\forall (D_w - h_w^t > 0)} \left(\dfrac{\widehat{c}_{w} - c_{w}^{t}}{D_{w} - h_{w}^{t}} \right) \right\}
\end{cases}
\quad (18)
$$

for the case of providing a uniform prediction of O-D travel time by ITS and as

$$
\begin{cases}
\dot{h}_{ip}^{t} = -\alpha_{ip} h_{ip}^{t} (c_{p}^{t} - c_{iw}^{t}) \\
\dot{c}_{iw}^{t} = \beta_{iw} (D_{iw} - h_{iw}^{t}) \\
0 < \alpha_{ip} < \inf\limits_{c_p^t - c_w^t > 0} \left(\dfrac{1}{c_{p}^{t} - c_{iw}^{t}} \right), \text{ and } \sum\limits_{w \in W} \sum\limits_{i \in N} \sum\limits_{\substack{p \in P_w \\ c_p^t - c_w^t < 0}} \left(-\delta_{ap} \alpha_{ip} PTTL_{ip,w}^{t} \right) \le k_a - \sum\limits_{\substack{p \in P \\ i \in N}} \delta_{ap} h_{ip}^{t} \\
0 < \beta_{iw} < \min \left\{ \inf\limits_{\forall (D_{iw} - h_{iw}^t < 0)} \left(\dfrac{\breve{c}_{w} - c_{iw}^{t}}{D_{iw} - h_{iw}^{t}} \right), \inf\limits_{\forall (D_{iw} - h_{iw}^t > 0)} \left(\dfrac{\widehat{c}_{w} - c_{iw}^{t}}{D_{iw} - h_{iw}^{t}} \right) \right\}
\end{cases}
\quad (19)
$$

for the case of individually supplying user-definite O-D travel time respectively.

Empirically, it is easy to be accomplished for such a hybrid system by providing a multi-access information inquiry system. However, we are concerned about the asymptotic behavior of the steady state and whether it is in complete accord with the well-known Wardrop's user equilibrium. These topics are the major components in chapter 6.

Chapter 5

Existence and Uniqueness

A major property of a dynamical system is to specify the future states unambiguously when the evolution rules and initial situations are specified. In this section, we will briefly illustrate the issue of existence and uniqueness of a uniform-user model, i.e. (12), by the fundamental theorem of ordinary differential equations in section 5.1. For the part of multi-class models, brief statements of refinement from the results of the identical-user model are provided in section 5.2. Time indices, day *t*, and inequalities (6) and (10) are omitted for conciseness in the subsequent chapters, then (12) is rewritten as

$$\begin{cases} \dot{h}_p = -\alpha_p h_p (c_p - c_w) \\ \dot{c}_w = \beta_w (D_w - h_w) \end{cases} \tag{20}$$

5.1 Homogeneous User Model

A dynamical system is a way of describing the time passage of all the points for a given space *E*. Mathematically, the space *E* might be an Euclidean space, *R*, or a subset of *R*. For the network dynamics mentioned in chapter 4, the set of possible non-negative path flows and predicted minimal travel time is clearly a convex subset of $R_+^{\bar{P}+\bar{W}}$, denoted as *S* even the constraints of path capacity are added. From the following theorem (Perko, 1996),

Theorem 1. Suppose that $G \in C^1(E)$ and that $G(\boldsymbol{x})$ satisfies the global Lipschitz condition

$$|G(\boldsymbol{x}) - G(\boldsymbol{y})| \le L|\boldsymbol{x} - \boldsymbol{y}| \tag{21}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in E$. Then for $\boldsymbol{x_0} \in R^n$, the initial value problem

$$\dot{\boldsymbol{x}} = G(\boldsymbol{x}) \text{ with } \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{22}$$

has a unique solution $\boldsymbol{x}(t)$ defined for all $t \in R$.

The existence and uniqueness of (20) can be claimed if (20) $\in C^1(R_+^{\overline{P}+\overline{W}})$ and (20) satisfies the global Lipschitz condition. The assumptions of the smooth and strict monotone function of link travel time ensure that (20) is $C^1(R_+^{\overline{P}+\overline{W}})$. Then, the proof of the global Lipschitz of $G(\boldsymbol{x})$ is given by following lemma. For convenience, $G: S \to R_+^{\overline{P}+\overline{W}}$ in (20) is re-indexed as

$$G(\boldsymbol{h}', \boldsymbol{c}') \equiv \begin{cases} g_j(\boldsymbol{h}', \boldsymbol{c}') = -\alpha_j h_j (c_j - c_w),\ \forall\, w \in W,\ j \in \{1, 2, ..., \overline{P}\} \text{ if path } j \in P_w \\ g_j(\boldsymbol{h}', \boldsymbol{c}') = \beta_w [D_w - h_w],\ \forall\, w \in W,\ j \in \{1 + \overline{P}, 2 + \overline{P}, ..., w + \overline{P}\} \end{cases} \tag{23}$$

and the norm $|\boldsymbol{y}| = \sum_{l=1}^{\overline{P}+\overline{W}} |y_i|$ is used $\forall\, \boldsymbol{y} \in R_+^{\overline{P}+\overline{W}}$. Now, we give the lemma of the Lipschitz condition for (23) and prove it.

Lemma. $G(\boldsymbol{h}', \boldsymbol{c}')$ defined in (23) satisfies the global Lipschitz condition with a Lipschitz constant $L = \max\{\max_{\forall w}\{\beta_w\}, L_p\}$, where

$$L_p = \max_{1 \le j \le \overline{P}} \left\{ \sup_{j \in P_w} \left(\alpha_j \left((\hat{c}_w - \breve{c}_w) + h_j \left. \frac{\partial c_j}{\partial h_j} \right|_{h_j = \hat{h}_j} \right) \right), \sup_{j, l \ne j} \left(\alpha_j \hat{h}_j \left. \frac{\partial c_j}{\partial h_l} \right|_{h_l = \hat{h}_l} \right), \sup_j \left(\alpha_j \hat{h}_j \right) \right\}. \tag{24}$$

Proof. From the assumptions and definitions of the link cost function mentioned in previous sections, G in (23) is obviously continuously differentiable on S. Let $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{y}$ and $\boldsymbol{y}, \boldsymbol{x}$ are two given points in S, i.e. $\boldsymbol{y}' = (\boldsymbol{h}'_y, \boldsymbol{c}'_y)$ and $\boldsymbol{x}' = (\boldsymbol{h}'_x, \boldsymbol{c}'_x)$. Since convexity of S, the points

$\boldsymbol{v} = \boldsymbol{y} + b\boldsymbol{u}$ are also in S for all $0 \le b \le 1$. Defining the function $H:[0,1] \to R_+^{\overline{P}+\overline{W}}$ by

$$H(b) = G(\boldsymbol{v}), \tag{25}$$

and by the chain rule, we have

$$\frac{dH(b)}{db} = \sum_{j=1}^{\overline{P}+\overline{W}} \sum_{l=1}^{\overline{P}+\overline{W}} \frac{\partial g_j(\boldsymbol{v})}{\partial v_l} \frac{dv_l}{db} = \sum_{j=1}^{\overline{P}+\overline{W}} \sum_{l=1}^{\overline{P}+\overline{W}} \frac{\partial g_j(\boldsymbol{v})}{\partial v_l} (x_l - y_l)$$

and hence

$$\left|\frac{dH(b)}{db}\right| \le \sum_{j=1}^{\overline{P}+\overline{W}} \sum_{l=1}^{\overline{P}+\overline{W}} \left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| \left|\frac{dv_l}{db}\right| = \sum_{j=1}^{\overline{P}+\overline{W}} \sum_{l=1}^{\overline{P}+\overline{W}} \left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| \left|x_l - y_l\right|. \tag{26}$$

There are three conditions in (26) that keep $\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right|$ from vanishing for the path flow dynamics, i.e. $\forall\ g_j$, $1 \le j \le \overline{P}$. If we let $\boldsymbol{v}' = (\boldsymbol{h}'_v, \boldsymbol{c}'_v)$, they are

$$\begin{cases} \left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_j \left((c_j - c_w) + h_j \frac{\partial c_j}{\partial h_l}\right)\right| \text{, if } l = j \text{ and } 1 \le l \le \overline{P} \\ \left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_j h_j \frac{\partial c_j}{\partial h_l}\right| \text{, if } l \ne j \text{, path } l \text{ overlaps partly with path } j \text{ and } 1 \le l \le \overline{P} \\ \left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_j h_j\right| \text{, } \forall\ 1 + \overline{P} \le l \le \overline{P} + \overline{W}, l = \overline{P} + w, j \in P_w \end{cases} . \tag{27}$$

The upper bound of $\left|\frac{\partial g_j}{\partial v_l}\right|$ in condition (27) can be decided by

$$\max_{1 \le j \le \overline{P}} \left\{ \sup_{j \in P_w} \left(\alpha_j \left((c_j - c_w) + h_j \frac{\partial c_j}{\partial h_j} \right) \right), \sup_{j, j \ne l} \left(\alpha_j h_j \frac{\partial c_j}{\partial h_l} \right), \sup_j \left(\alpha_j h_j \right) \right\}. \tag{28}$$

We know that h_p and c_w are bounded globally by constraints (6a), (6b), (10), (10a) and (10b). Hence, (27) is also bounded as the boundary conditions of (6a), (6b), (10), (10a) and (10b) take place. That is to say the upper bound of $\left|\frac{\partial g_j}{\partial v_l}\right|$ in condition (27) is shown as

$$\begin{aligned} L_p &= \max_{1\le j\le \bar{P}}\left|\frac{\partial g_j}{\partial v_l}\right| \\ &= \max_{1\le j\le \bar{P}}\left\{\sup_{j\in P_w}\left(\alpha_j\left((\hat{c}_w-\breve{c}_w)+h_j\left.\frac{\partial c_j}{\partial h_j}\right|_{h_j=\hat{h}_j}\right)\right), \sup_{j,l\ne j}\left(\alpha_j\hat{h}_j\left.\frac{\partial c_j}{\partial h_l}\right|_{h_l=\hat{h}_l}\right), \sup_j\left(\alpha_j\hat{h}_j\right)\right\}, \end{aligned} \tag{29}$$

where

$$\hat{h}_p \equiv \min_{\forall a\in p}\{k_a\}. \tag{30}$$

By similar treatments, there is only one condition that $\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right|$, $\forall\ g_j$ and $\bar{P}+1\le j\le \bar{P}+\bar{W}$, in (26) is not equal to zero, it is

$$\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \beta_w \ ,\ \forall\ 1\le l\le \bar{P}\ \text{ if path } l\in P_w\ . \tag{31}$$

The results of (29) and (31) lead us to set

$$L = \max\{\max_{\forall w}\{\beta_w\}, L_p\} \tag{32}$$

and hence

$$\left|\frac{dH(b)}{db}\right| \le \sum_{j=1}^{\bar{P}+\bar{W}}\sum_{l=1}^{\bar{P}+\bar{W}}\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right|\left|\frac{dv_l}{db}\right| = \sum_{j=1}^{\bar{P}+\bar{W}}\sum_{l=1}^{\bar{P}+\bar{W}}\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right|\left|x_l-y_l\right| \le L\sum_{l=1}^{\bar{P}+\bar{W}}\left|x_l-y_l\right| = L\left|\boldsymbol{x}-\boldsymbol{y}\right|. \tag{33}$$

Now from the relation $G(\boldsymbol{x}) - G(\boldsymbol{y}) = H(1) - H(0) = \int_0^1 H'(z)dz$, we find that $\left|G(\boldsymbol{x}) - G(\boldsymbol{y})\right| \le \int_0^1 \left|H'(z)\right| dz \le L\left|\boldsymbol{x} - \boldsymbol{y}\right|$. Thus $G(\boldsymbol{h}', \boldsymbol{c}')$ in (23) satisfies the global Lipschitz condition and the corresponding Lipschitz constant L can be determined by (33). After proving the above lemma, the global existence and uniqueness of (20) is standard and we refer readers to Perko (1996) for a proof of the fundamental global theorem.

5.2 Multiple User Classes Model

For the cases of multi-class users, (23) is amplified as

$$G(\boldsymbol{h}', \boldsymbol{c}') \equiv \begin{cases} g_j(\boldsymbol{h}', \boldsymbol{c}') = -\alpha_{ip} h_{ip} (c_p - c_w) \, , \forall \, w, p \in P_w \, , \; j = \overline{P}(i-1) + p \\ g_j(\boldsymbol{h}', \boldsymbol{c}') = \beta_w [D_w - h_w] \; , \forall \, w \in W \, , \, j = n\overline{P} + w \end{cases} \tag{34}$$

and

$$G(\boldsymbol{h}', \boldsymbol{c}') \equiv \begin{cases} g_j(\boldsymbol{h}', \boldsymbol{c}') = -\alpha_{ip} h_{ip} (c_p - c_{iw}) \, , \forall \, w, p \in P_w \, , \, j = \overline{P}(i-1) + p \\ g_j(\boldsymbol{h}', \boldsymbol{c}') = \beta_{iw} [D_{iw} - h_{iw}] \; , \forall \, w \in W , \; j = n\overline{P} + \overline{W}(i-1) + w \end{cases} , \tag{35}$$

where i = 1,2,…,n to fit (18) and (19) respectively. The dimension of $G(\boldsymbol{h}', \boldsymbol{c}')$ and $(\boldsymbol{h}', \boldsymbol{c}')$ in (34) are the same and denoted as $n\overline{P} + \overline{W}$ and in (35) as $n(\overline{P} + \overline{W})$. By the similar treatments, condition (27) for path flow dynamics with $1 \le j \le n\overline{P}$ is modified as

$$\begin{cases} \left|\dfrac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_{ip}\left((c_p - c_w) + h_{ip}\dfrac{\partial c_p}{\partial h_{ip}}\right)\right| \; , \forall \, l \in [1, n\overline{P}] \, , \text{if } \, l = \overline{P}(i-1) + p \\ \left|\dfrac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_{ip} h_{ip} \dfrac{\partial c_p}{\partial h_{ez}}\right| \; , \forall \, l \in [1, n\overline{P}] \, , \text{if } \, l \ne \overline{P}(i-1) + p \text{ and } l = \overline{P}(e-1) + z \\ \left|\dfrac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_{ip} h_{ip}\right| \; , \forall \, l \in [1 + n\overline{P}, n\overline{P} + \overline{W}] \, , p \in P_w \, , \text{if } \, l = nP + w \end{cases} . \tag{36}$$

and

$$\begin{cases} \left|\dfrac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_{ip}\left((c_p - c_{iw}) + h_{ip}\dfrac{\partial c_p}{\partial h_{ip}}\right)\right| \ , \forall\ l \in [1, n\overline{P}]\ , \text{if}\ l = \overline{P}(i-1)+p \\ \left|\dfrac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_{ip} h_{ip}\dfrac{\partial c_p}{\partial h_{ez}}\right| \ , \forall\ l \in [1, n\overline{P}]\ , \text{if}\ l \neq \overline{P}(i-1)+p \text{ and } l = \overline{P}(e-1)+z \\ \left|\dfrac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \left|\alpha_{ip} h_{ip}\right| \ , \forall\ l \in [1+n\overline{P}, n\overline{P}+n\overline{W}]\ , \ p \in P_w\ , \text{if}\ l = n\overline{P} + \overline{W}(i-1)+w \end{cases} \quad .(37)$$

for the two cases of multi-class users model respectively. And the corresponding non-zero parts of partial derivative of predicted minimal travel time dynamics can be similarly derived as

$$\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \beta_w \ , \ \forall\ 1 \le l \le n\overline{P} \ \text{ if } l = \overline{P}(i-1)+p \text{ and } p \in P_w \qquad (38)$$

$\forall\ g_j\ ,\ j \in [\,1+n\overline{P}, n\overline{P}+\overline{W}\,]$ in (34), and

$$\left|\frac{\partial g_j(\boldsymbol{v})}{\partial v_l}\right| = \beta_{iw} \ , \ \forall\ 1 \le l \le n\overline{P} \ \text{ if } l = \overline{P}(i-1)+p \text{ and } p \in P_w \qquad (39)$$

$\forall\ g_j\ ,\ j \in [\,1+n\overline{P}, n\overline{P}+n\overline{W}\,]$ in (35). Eventually the Lipschitz constant can be expressed as $L = \max\{\max_{\forall w}\{\beta_w\}, L_{Ip}\}$, where

$$L_{Ip} = \max_{\substack{1\le i,e\le n \\ 1\le p,z\le \overline{P}}}\left\{ \sup_{i,p\in P_w}\left(\alpha_{ip}\left((\hat{c}_w - \breve{c}_w) + h_{ip}\left.\frac{\partial c_p}{\partial h_{ip}}\right|_{h_{ip}=\hat{h}_{ip}}\right)\right), \sup_{\substack{i,p,e,z \\ l\neq \overline{P}(i-1)+p \\ l=\overline{P}(e-1)+z}}\left(\alpha_{ip}\hat{h}_{ip}\left.\frac{\partial c_p}{\partial h_{ez}}\right|_{h_{ez}=\hat{h}_{ez}}\right), \sup_{i,p}\left(\alpha_{ip}\hat{h}_{ip}\right)\right\} \quad (40)$$

for (34) and as $L = \max\{\max_{\forall\, i,w}\{\beta_{iw}\}, L_{Ip}\}$, where

$$L_{Ip} = \max_{\substack{1\le i,e\le n \\ 1\le p,z\le \bar{P}}} \left\{ \sup_{i,p\in P_w} \left(\alpha_{ip} \left((\hat{c}_{iw} - \breve{c}_{iw}) + h_{ip} \left. \frac{\partial c_p}{\partial h_{ip}} \right|_{h_{ip}=\hat{h}_{ip}} \right) \right), \sup_{\substack{i,p,e,z \\ l\neq \bar{P}(i-1)+p \\ l=\bar{P}(e-1)+z}} \left(\alpha_{ip} \hat{h}_{ip} \left. \frac{\partial c_p}{\partial h_{ez}} \right|_{h_{ez}=\hat{h}_{ez}} \right), \sup_{i,p} \left(\alpha_{ip} \hat{h}_{ip} \right) \right\} \quad (41)$$

for (35).

Chapter 6

Analysis of Steady State and Its Stability

The steady state of (12) is analyzed to be in agreement with Wardrop's user equilibrium in section 6.1 followed by the proof of the associated stability theorems using Lyapunov's direct method in section 6.2.

6.1 Analysis of Steady State

It is useful to recall the definition of Wardrop's static user equilibrium in our terms before elaborating on the steady state of the proposed network dynamics. If the symbol, "$\bar{\ }$", is used to denote steady-state or equilibrium point, the Wardrop's user equilibrium can be described as

$$\begin{cases} \bar{h}_p > 0 \rightarrow c_p = \bar{c}_w \\ c_p > \bar{c}_w \rightarrow \bar{h}_p = 0 \\ \bar{h}_w = D_w \end{cases}, \quad \forall\ w \in W \text{ and } p \in P_w. \tag{42}$$

Condition (42) states a condition that is stable only when no traveler can improve his travel time by unilaterally changing paths. All path travel times of the same O-D pair are equal and minimal at this status.

The steady state of (20) implies $\dot{h}_p = 0$ and $\dot{c}_w = 0$ for all O-D pair $w \in W$, $p \in P_w$. After some algebraic reasoning, we get the conditions below jointly equivalent to the steady state of path flow dynamics in (20)

$$\begin{cases} h_p > 0 \rightarrow c_p = c_w \quad \text{or} \quad h_p < 0 \rightarrow c_p = c_w \\ c_p > c_w \rightarrow h_p = 0 \quad \text{or} \quad c_p < c_w \rightarrow h_p = 0 \\ c_w > 0 \rightarrow D_w = h_w \end{cases} \tag{43}$$

for all O-D pair $w \in W$, $p \in P_w$. The second sub-case of the first part in condition (43) never happens because of the violating non-negative flow constraint. The second sub-case of the second relationship in condition (43) will never happen if initial conditions with positive path flows are provided. For the positive nature of the predicted minimal travel time even at zero path flow level, the last equilibrium state in condition (43) is evidently held on. Finally, we abstract the critical components in condition (43) as

$$\begin{cases} h_p > 0 \rightarrow c_p = c_w \\ c_p > c_w \rightarrow h_p = 0 \\ D_w = h_w \end{cases} \tag{44}$$

for all O-D pair $w \in W$, $p \in P_w$. It is easy to infer that the actual average travel time is equal to the predicted minimal travel time by ITS and is minimal among all paths of an O-D pair simultaneously in condition (44). And the travel demand of an O-D pair is equal to the sum of corresponding path flows. Based on these results, we can claim that the steady state of (20) is identical to Wardrop's user equilibrium.

Similarly, the equilibrium state of multi-class-user models are derived as

$$\begin{cases} h_{ip} > 0 \rightarrow c_p = c_w \\ c_p > c_w \rightarrow h_{ip} = 0 \\ D_w = h_w \end{cases} \tag{45}$$

in (18) and as

$$\begin{cases} h_{ip} > 0 \to c_p = c_{iw} \\ c_p > c_{iw} \to h_{ip} = 0 \\ D_{iw} = h_{iw} \end{cases} \tag{46}$$

in (19) for all O-D pair $w \in W$, $p \in P_w$, i = 1,2,…,n respectively. In condition (45) h_w is defined in (16). And because c_p is not user-specific we have that path travel times with positive path flow are equal to the predicted minimal travel time of O-D pair w simultaneously and this path travel time is the minimal for all paths connecting O-D pair w. Travel demand of an O-D pair is equal to the sum of flows distributed over all user classes and corresponding paths. In condition (46), similarly the predicted minimal travel times of O-D pair w for all user classes are the same and equal to the path travel times simultaneously. The demand of an O-D pair is divided into n parts for n user classes and each part is equal to the sum of corresponding path flows.

6.2 Analysis of Stability

In this section, our interest is in showing that model (12) is asymptotically stable. The definitions and theorem of stability in the sense of Lyapunov are employed as the following statements (Alligood et al., 1997).

Definition 1. Let $\bar{\boldsymbol{v}}$ be a steady state of a dynamical system, $G \in C^1(E)$. A function $L: E \to R$ is called a strict Lyapunov function for $\bar{\boldsymbol{v}}$ if the following conditions are satisfied:

(1) $L(\bar{\boldsymbol{v}}) = 0$, and $L(\boldsymbol{v}) > 0 \ \ \forall \ \boldsymbol{v} \neq \bar{\boldsymbol{v}}, \boldsymbol{v} \in E$;

(2) $\dot{L}(\boldsymbol{v}) < 0 \ \ \forall \ \boldsymbol{v} \neq \bar{\boldsymbol{v}}, \boldsymbol{v} \in E$.

Theorem 2. Let $\bar{\boldsymbol{v}}$ be a staedy state of $\dot{\boldsymbol{v}} = G(\boldsymbol{v})$. If there exists a strict Lyapunov function

$\forall\ \boldsymbol{v} \neq \bar{\boldsymbol{v}}, \boldsymbol{v} \in E$, then $\bar{\boldsymbol{v}}$ is asymptotically stable.

Accordingly theorem 2, the stability of proposed dynamical system is illustrated on theorem 3.

Theorem 3. Let $\begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix}' = (\bar{h}_1, \bar{h}_2, ..., \bar{h}_{\bar{P}}, \bar{c}_1, \bar{c}_2, ..., \bar{c}_{\bar{W}})$ be a steady state of (12) and $\begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix}$ is asymptotically stable.

For the sake of conciseness, the following definitions are introduced to rewrite (20) in vector and matrix form. Let $\hat{\boldsymbol{h}} \equiv \begin{pmatrix} h_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_{\bar{P}} \end{pmatrix}$ and rearrange it into several diagonal matrices as

$$\hat{\boldsymbol{h}}_r \equiv \begin{pmatrix} \boldsymbol{h}_+ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{h}_- \end{pmatrix}, \tag{47}$$

where the components of $\boldsymbol{h}_+$ and $\boldsymbol{h}_-$ are path flows with $(h_p - \bar{h}_p)(c_p - c_w) \geq 0$ and $(h_p - \bar{h}_p)(c_p - c_w) < 0$ respectively. Moreover, let

$$\bar{\boldsymbol{h}}_\varepsilon \equiv \begin{pmatrix} \bar{\boldsymbol{h}}_{\varepsilon+} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{h}}_{\varepsilon-} \end{pmatrix}, \tag{48}$$

where $\bar{\boldsymbol{h}}_{\varepsilon+}$ and $\bar{\boldsymbol{h}}_{\varepsilon-}$ are two diagonal matrices and there exists $\varepsilon_p^+ \in R^+$ and $\varepsilon_p^- \in R^+$ such that the elements of $\bar{\boldsymbol{h}}_{\varepsilon+}$ and $\bar{\boldsymbol{h}}_{\varepsilon-}$ are $\bar{h}_p^*\ \varepsilon_p^+$ and $\bar{h}_p^*\ \varepsilon_p^-$ respectively with

$$\frac{h_p}{\bar{h}_p^* \ \varepsilon_p^+} > 1, \forall h_p \in \boldsymbol{h}_+ \tag{49}$$

and

$$\frac{h_p}{\bar{h}_p^* \ \varepsilon_p^-} < 1, \forall h_p \in \boldsymbol{h}_-, \tag{50}$$

where $\bar{h}_p^* = \sup_{\forall h_p \in \boldsymbol{h}} (\bar{h}_p)$ and $\bar{h}_p$ is the steady state of h_p. $\mathbf{0}$ denotes the zero matrix with suitable dimension. Furthermore, if we let $\boldsymbol{s} \equiv \begin{pmatrix} \Delta \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix}$, $\boldsymbol{M}(\boldsymbol{s}) \equiv \begin{pmatrix} \boldsymbol{\Delta}' \boldsymbol{c}_a(\boldsymbol{\Delta h}) \\ -\boldsymbol{O} \end{pmatrix}$, $\boldsymbol{\Omega} \equiv \begin{pmatrix} \mathbf{0} & -\boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}' & \mathbf{0} \end{pmatrix}$, and $\varphi \equiv \begin{pmatrix} \boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta} \end{pmatrix}$, where $\boldsymbol{I}$, $\boldsymbol{c}_a(\boldsymbol{\Delta h})$, $\boldsymbol{\Delta}$, $\boldsymbol{O}$, and $\boldsymbol{\Gamma}$ denote identity matrix, full link-cost vector, link-path incident matrix, full OD pair demand vector, and path-OD pair incident matrix respectively. α and β are diagonal matrices with all α_p and β_w as diagonal elements respectively. And all the elements of the mentioned vectors and matrices are ordered in accordance with $\hat{\boldsymbol{h}}_r$. Then we rewrite (20) as

$$\begin{pmatrix} \dot{\boldsymbol{h}} \\ \dot{\boldsymbol{c}} \end{pmatrix} = -\begin{pmatrix} \hat{\boldsymbol{h}}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix} \varphi \begin{pmatrix} \boldsymbol{\Delta}' \boldsymbol{c}_a(\boldsymbol{\Delta h}) - \boldsymbol{\Gamma c} \\ \boldsymbol{\Gamma}' \boldsymbol{h} - \boldsymbol{O} \end{pmatrix} = -\begin{pmatrix} \hat{\boldsymbol{h}}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix} \varphi \left(\boldsymbol{M}(\boldsymbol{s}) + \boldsymbol{\Omega} \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \right). \tag{51}$$

Now we are ready to prove theorem 3.

Proof. Let $L: E \to R$ be a C^1 map and

$$L\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \equiv \frac{1}{2} \left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \bar{\boldsymbol{h}}_\varepsilon & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}^{-1} \varphi^{-1} \left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right), \tag{52}$$

where $\begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix}$ is an equilibrium point of the proposed model. It is obvious that $L\begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} = 0$ and

$L\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} > 0 \;\; if \;\; \forall \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \neq \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \in E$.

$$\dot{L}\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} = \left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \bar{\boldsymbol{h}}_\varepsilon & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}^{-1} \varphi^{-1} \begin{pmatrix} \dot{\boldsymbol{h}} \\ \dot{\boldsymbol{c}} \end{pmatrix}$$

$$= -\left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \hat{\boldsymbol{h}}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{h}}_\varepsilon & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}^{-1} \varphi^{-1} \varphi \left(\boldsymbol{M}(\boldsymbol{s}) + \boldsymbol{\Omega} \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \right)$$

$$= -\left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \hat{\boldsymbol{h}}_r (\bar{\boldsymbol{h}}_\varepsilon)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \left(\boldsymbol{M}(\boldsymbol{s}) + \boldsymbol{\Omega} \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \right)$$

$$= -\left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \boldsymbol{h}_+ (\bar{\boldsymbol{h}}_{\varepsilon+})^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{h}_- (\bar{\boldsymbol{h}}_{\varepsilon-})^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \left(\boldsymbol{M}(\boldsymbol{s}) + \boldsymbol{\Omega} \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} \right)$$

Because $\boldsymbol{c}_a(\boldsymbol{\Delta h})$ is a strict monotone function, we have

$$(\boldsymbol{s} - \bar{\boldsymbol{s}})' \left(\begin{pmatrix} \boldsymbol{c}_a(\boldsymbol{\Delta h}) \\ -\boldsymbol{O} \end{pmatrix} - \begin{pmatrix} \boldsymbol{c}_a(\boldsymbol{\Delta} \bar{\boldsymbol{h}}) \\ -\boldsymbol{O} \end{pmatrix} \right) > 0, \; \forall \boldsymbol{s} \neq \bar{\boldsymbol{s}} = \begin{pmatrix} \boldsymbol{\Delta} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \text{ and } \boldsymbol{s} \in E,$$

i.e. $(\boldsymbol{s} - \bar{\boldsymbol{s}})' \begin{pmatrix} \boldsymbol{c}_a(\boldsymbol{\Delta h}) \\ -\boldsymbol{O} \end{pmatrix} > (\boldsymbol{s} - \bar{\boldsymbol{s}})' \begin{pmatrix} \boldsymbol{c}_a(\boldsymbol{\Delta} \bar{\boldsymbol{h}}) \\ -\boldsymbol{O} \end{pmatrix}$, and this implies

$$\left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \boldsymbol{\Delta}' \boldsymbol{c}_a(\boldsymbol{\Delta h}) \\ -\boldsymbol{O} \end{pmatrix} > \left(\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{c} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{c}} \end{pmatrix} \right)' \begin{pmatrix} \boldsymbol{\Delta}' \boldsymbol{c}_a(\boldsymbol{\Delta} \bar{\boldsymbol{h}}) \\ -\boldsymbol{O} \end{pmatrix}.$$

So we have

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\boldsymbol{M}(s)>\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\boldsymbol{M}(\bar{s})$$

From the analysis of the steady state, we also have

$$\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}'\left(\boldsymbol{M}(\bar{s})+\boldsymbol{\Omega}\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)=0 \quad \text{and} \quad \begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}'\left(\boldsymbol{M}(\bar{s})+\boldsymbol{\Omega}\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)\geq 0,$$

this implies

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\boldsymbol{M}(\bar{s})\geq\left(\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}-\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)'\boldsymbol{\Omega}\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}.$$

Hence we have

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\boldsymbol{M}(s)>\left(\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}-\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)'\boldsymbol{\Omega}\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}. \tag{53}$$

By adding $\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}$ to both sides of inequality (53) gives

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\left(\boldsymbol{M}(s)+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)>\left(\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}-\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)'\left(\boldsymbol{\Omega}\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}-\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right),$$

i.e.

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\left(\boldsymbol{M}(s)+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)>\left(\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}-\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)'\boldsymbol{\Omega}\left(\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}-\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right).$$

For further calculating, we have

$$\begin{pmatrix}\bar{\boldsymbol{h}}-\boldsymbol{h}\\ \bar{\boldsymbol{c}}-\boldsymbol{c}\end{pmatrix}'\begin{pmatrix}\mathbf{0} & -\boldsymbol{\Gamma}\\ \boldsymbol{\Gamma}' & \mathbf{0}\end{pmatrix}\begin{pmatrix}\bar{\boldsymbol{h}}-\boldsymbol{h}\\ \bar{\boldsymbol{c}}-\boldsymbol{c}\end{pmatrix}=\begin{pmatrix}(\bar{\boldsymbol{c}}-\boldsymbol{c})'\boldsymbol{\Gamma}'\\ (\bar{\boldsymbol{h}}-\boldsymbol{h})'(-\boldsymbol{\Gamma})\end{pmatrix}'\begin{pmatrix}(\bar{\boldsymbol{h}}-\boldsymbol{h})\\ (\bar{\boldsymbol{c}}-\boldsymbol{c})\end{pmatrix},$$

this implies

$$(\bar{\boldsymbol{c}}-\boldsymbol{c})'\boldsymbol{\Gamma}'(\bar{\boldsymbol{h}}-\boldsymbol{h})-(\bar{\boldsymbol{h}}-\boldsymbol{h})'\boldsymbol{\Gamma}(\bar{\boldsymbol{c}}-\boldsymbol{c})=0,$$

i.e.

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\left(\boldsymbol{M}(\boldsymbol{s})+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)>0$$

and equivalently

$$-\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\left(\boldsymbol{M}(\boldsymbol{s})+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)<0.$$

Now recall

$$\dot{L}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}=-\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\begin{pmatrix}\boldsymbol{h}_+(\bar{\boldsymbol{h}}_{\varepsilon+})^{-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{h}_-(\bar{\boldsymbol{h}}_{\varepsilon-})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{I}\end{pmatrix}\left(\boldsymbol{M}(\boldsymbol{s})+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right),$$

by the definitions of $\boldsymbol{h}_+$, $\boldsymbol{h}_-$, $\bar{\boldsymbol{h}}_{\varepsilon+}$, and $\bar{\boldsymbol{h}}_{\varepsilon-}$, in (47)-(50) we have

$$\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\begin{pmatrix}\boldsymbol{h}_+(\bar{\boldsymbol{h}}_{\varepsilon+})^{-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{h}_-(\bar{\boldsymbol{h}}_{\varepsilon-})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{I}\end{pmatrix}\left(\boldsymbol{M}(\boldsymbol{s})+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right)$$

$$>\left(\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}-\begin{pmatrix}\bar{\boldsymbol{h}}\\ \bar{\boldsymbol{c}}\end{pmatrix}\right)'\left(\boldsymbol{M}(\boldsymbol{s})+\boldsymbol{\Omega}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix}\right).$$

Accordingly, we have $\dot{L}\begin{pmatrix}\boldsymbol{h}\\ \boldsymbol{c}\end{pmatrix} < 0$.

Hence, (52) is affirmed as a strict Lyapunov function of dynamical system (12). Asymptotic stability is then immediate from theorem 2.

For the multi-class users model shown in (19), we replace (47) and (48) with

$$\hat{\boldsymbol{h}}_{ir} \equiv \begin{pmatrix}\boldsymbol{h}_{i+} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{h}_{i-}\end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{h}}_{Nr} \equiv \begin{pmatrix}\hat{\boldsymbol{h}}_{1r} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \hat{\boldsymbol{h}}_{nr}\end{pmatrix} \tag{54}$$

and

$$\bar{\boldsymbol{h}}_{N\varepsilon} \equiv \begin{pmatrix}\bar{\boldsymbol{h}}_{1\varepsilon} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{h}}_{n\varepsilon}\end{pmatrix},$$

with

$$\bar{\boldsymbol{h}}_{i\varepsilon} \equiv \begin{pmatrix}\bar{\boldsymbol{h}}_{i\varepsilon+} & \mathbf{0}\\ \mathbf{0} & \bar{\boldsymbol{h}}_{i\varepsilon-}\end{pmatrix}, \tag{55}$$

where $\hat{\boldsymbol{h}}_{ir}$, $\bar{\boldsymbol{h}}_{i\varepsilon+}$, and $\bar{\boldsymbol{h}}_{i\varepsilon-}$ denote diagonal matrices of user class *i* defined by the same rule as $\hat{\boldsymbol{h}}_{r}$, $\bar{\boldsymbol{h}}_{\varepsilon+}$, and $\bar{\boldsymbol{h}}_{\varepsilon-}$ in (47) and (48) respectively. Now the whole system dynamics can be rewritten as

$$\dot{\boldsymbol{s}}_N = -\begin{pmatrix} \hat{\boldsymbol{h}}_{Nr} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_N \end{pmatrix} \varphi_N \begin{pmatrix} \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) - \boldsymbol{\Gamma c}_1 \\ \vdots \\ \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) - \boldsymbol{\Gamma c}_n \\ \boldsymbol{\Gamma}'\boldsymbol{h}_1 - \boldsymbol{O}_1 \\ \vdots \\ \boldsymbol{\Gamma}'\boldsymbol{h}_n - \boldsymbol{O}_n \end{pmatrix}$$

(56)

$$= -\begin{pmatrix} \hat{\boldsymbol{h}}_{Nr} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_N \end{pmatrix} \varphi_N \left(\boldsymbol{M}_N(\boldsymbol{s}_N) + \boldsymbol{\Omega}_N \boldsymbol{s}_N\right)$$

where

$$\boldsymbol{s}_N \equiv \begin{pmatrix} \boldsymbol{h}_1 \\ \vdots \\ \boldsymbol{h}_n \\ \boldsymbol{c}_1 \\ \vdots \\ \boldsymbol{c}_n \end{pmatrix}, \quad \boldsymbol{M}_N(\boldsymbol{s}_N) \equiv \begin{pmatrix} \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) \\ \vdots \\ \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) \\ -\boldsymbol{O}_1 \\ \\ -\boldsymbol{O}_n \end{pmatrix},$$

$$\boldsymbol{\Omega}_N \equiv \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & -\boldsymbol{\Gamma}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \boldsymbol{0} & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\Gamma}_n \\ \boldsymbol{\Gamma}'_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \boldsymbol{0} & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Gamma}'_n & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix}$$

with $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}$, and

$$\varphi_N \equiv \begin{pmatrix} \boldsymbol{\alpha}_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & \ddots & & \iddots & \\ & \ddots & \boldsymbol{\alpha}_n & 0 & & \vdots \\ \vdots & & 0 & \boldsymbol{\beta}_1 & \ddots & \\ & \iddots & & \ddots & \ddots & 0 \\ 0 & & \cdots & & 0 & \boldsymbol{\beta}_n \end{pmatrix},$$

where $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ are diagonal matrices with their elements to be α_{ip} and β_{iw} respectively, $\boldsymbol{h}_i$, $\boldsymbol{c}_i$, $\boldsymbol{O}_i$ denote path flows, predicted OD travel times and OD demands respectively of user class i. Then a strict Lyapunov function of the multi-class user dynamical system described as (56) can be proposed as

$$L_N(\boldsymbol{s}_N) \equiv \frac{1}{2}(\boldsymbol{s}_N - \bar{\boldsymbol{s}}_N)' \begin{pmatrix} \bar{\boldsymbol{h}}_{N\varepsilon} & 0 \\ 0 & \boldsymbol{I}_N \end{pmatrix}^{-1} \varphi_N^{-1}(\boldsymbol{s}_N - \bar{\boldsymbol{s}}_N). \tag{57}$$

It is easy to check (57) by satisfying the two conditions in definition 1 in a similar way used in the proof of theorem 3. Finally the multi-class user dynamical system described as (18) can be reformulated as

$$\begin{aligned} \dot{\boldsymbol{s}}_N &= -\begin{pmatrix} \hat{\boldsymbol{h}}_{Nr} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_N \end{pmatrix} \varphi_N \begin{pmatrix} \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) - \boldsymbol{\Gamma c} \\ \vdots \\ \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) - \boldsymbol{\Gamma c} \\ \left(\sum_{i=1}^{n} \boldsymbol{\Gamma}'\boldsymbol{h}_i\right) - \boldsymbol{O} \end{pmatrix} \\ &= -\begin{pmatrix} \hat{\boldsymbol{h}}_{Nr} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_N \end{pmatrix} \varphi_N\left(\boldsymbol{M}_N(\boldsymbol{s}_N) + \boldsymbol{\Omega}_N \boldsymbol{s}_N\right) \end{aligned} \tag{58}$$

with the same function form of (57) to be a strict Lyapunov function but newly defined as

$$\boldsymbol{s}_N \equiv \begin{pmatrix} \boldsymbol{h}_1 \\ \vdots \\ \boldsymbol{h}_n \\ \boldsymbol{c} \end{pmatrix}, \quad \boldsymbol{M}_N\left(\boldsymbol{s}_N\right) \equiv \begin{pmatrix} \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) \\ \vdots \\ \boldsymbol{\Delta}'\boldsymbol{c}_a\left(\sum_{i=1}^{n} \boldsymbol{\Delta h}_i\right) \\ -\boldsymbol{O} \end{pmatrix},$$

$$\boldsymbol{\Omega}_N \equiv \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & -\boldsymbol{\Gamma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & -\boldsymbol{\Gamma}_n \\ \boldsymbol{\Gamma}'_1 & \cdots & \boldsymbol{\Gamma}'_n & \boldsymbol{0} \end{pmatrix}, \text{ and } \quad \varphi_N \equiv \begin{pmatrix} \boldsymbol{\alpha}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \boldsymbol{\alpha}_n & 0 \\ 0 & \cdots & 0 & \boldsymbol{\beta} \end{pmatrix}$$

with $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}$.

Chapter 7

Numerical Illustration of a Simple Network

A simple network with four nodes and five links illustrated as Fig. 7-1 is used to show the numerical results of the proposed models. There is only one OD pair w = {node 1, node 4} which is connected by three paths denoted as path 1 = {link 1, link 4}, path 2 = {link 2, link 5}, and path 3 = {link 1, link 3, link 5} respectively. The parameters of the link cost functions are set in Table 7.1 with the same function form as $c_a\left(f_a^t\right) = A_a + B_a\left(\frac{f_a^t}{k_a}\right)^4$.

Table 7.1 Parameters of link cost function

Links	A_a	B_a	k_a
1	40	20	80
2	60	30	80
3	20	10	120
4	50	25	80
5	30	15	80

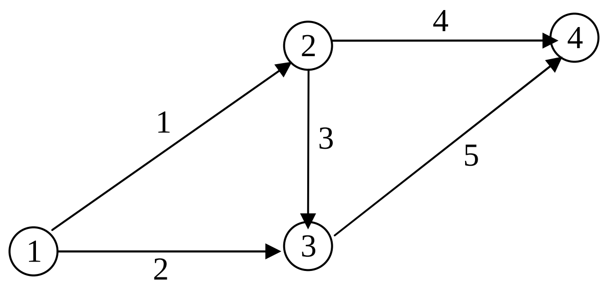

Fig. 7-1 Graph of numerical example

The following three examples (M1, M2, and M3) were solved using the high-order Runge-Kutta numerical method with the O-D demand fixed as 120. The initial conditions, assumed identical parameters of the propensity of path flow dynamics, and parameters to denote the sensitivity of predicted travel time dynamics for the homogeneous user model (M1), i.e. (11), are given as $(h_1^0, h_2^0, h_3^0, c_1^0, \alpha, \beta) =$ (40,50,30,125,-0.0006,0.1). The second example is set for the multiple user classes model with identical predicted minimal O-D travel time by ITS (M2), i.e. (17) as n=2. Parameters of the propensity of path flow dynamics for the two classes are assumed to be different but the same within a user class. They are –0.0006 and –0.003 for user class 1 and user class 2 respectively. The other inputs of the second example are the same as the previous case. Finally, the third case is prepared for the two-user classes model with the user-specific predicted minimal O-D travel time by ITS (M3), i.e. (18) as n=2. All inputs are the same as the second example but each user class is provided with a dedicated minimal O-D travel time predicted by ITS and an equal O-D demand, i.e. $D_{11} = D_{21} = 60$.

7.1 Numerical Results of Homogeneous User Model

Table 7.2 shows the dynamics of flows, travel times, and predicted minimal O-D travel times by ITS at three different states for the first example. It is clear that the steady state satisfies the Wardrop's user equilibrium and the predicted minimal O-D travel time is equal to the path travel times of which path flows are positive simultaneously. Numerical results by the evolutions of network dynamics illustrated from Fig. 7-2 to Fig. 7-5 also show that the path flow increases (decreases) as path travel time is less (more) than the predicted minimal O-D travel time and predicted minimal O-D travel time increases (decreases) as O-D demand is more (less) than the sum of path flows.

Table 7.2 Results of numerical example for homogeneous user model (M1)

	Flow			Travel time		
	Initial	State at t=200	Steady state	Initial	State at t=200	Steady state
Path 1	40	51.06	56.16	103.29	103.84	103.79
Path 2	50	53.13	56.95	109.58	104.05	103.79
Path 3	30	15.69	6.89	116.76	107.91	103.80
Link 1	70	66.75	63.05	51.72	49.69	47.72
Link 2	50	53.13	56.95	64.58	65.84	67.70
Link 3	30	15.69	6.89	20.04	20.01	20.00
Link 4	40	51.06	56.16	51.56	55.15	56.07
Link 5	80	68.82	63.84	45.00	38.22	36.08
Predicted minimal O-D travel time by ITS				125.00	104.25	103.79

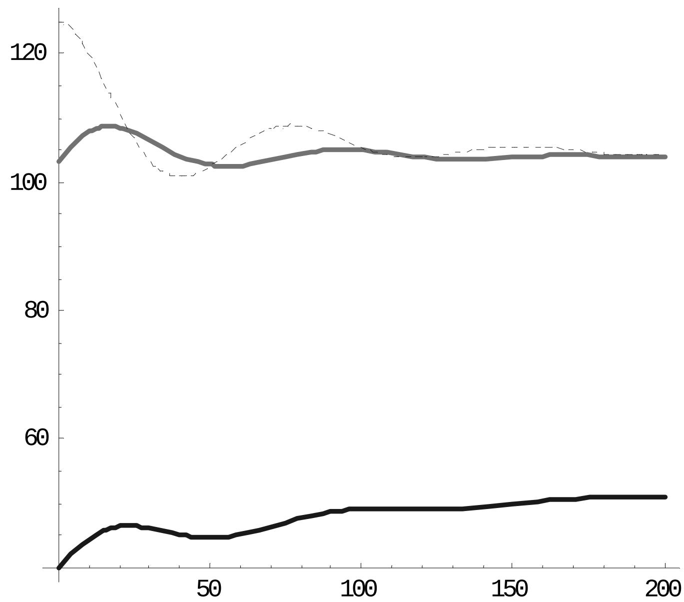

Fig. 7-2 Evolutions of path 1 flow dynamics and unit $PTTL^{t}_{path1,w}$ (M1)

(dashed line: predicted minimal O-D travel time by ITS; gray line: travel time of path 1; black line: flow of path 1)

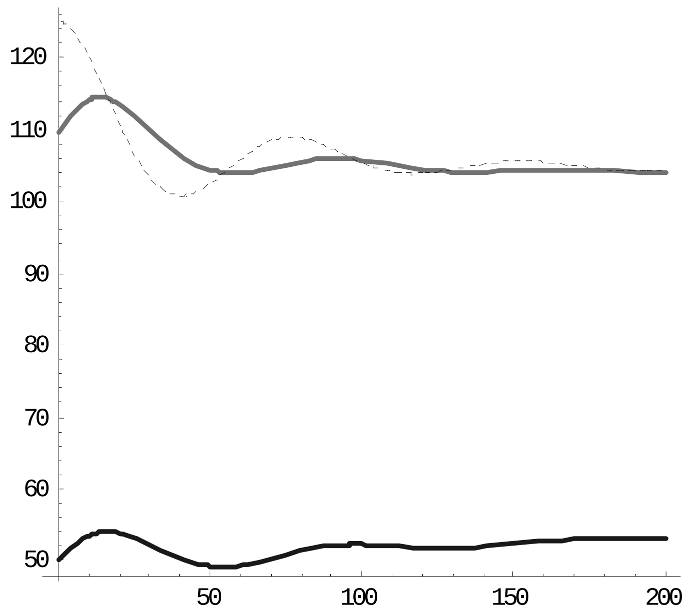

Fig. 7-3 Evolutions of path 2 flow dynamics and unit $PTTL^{t}_{path\,2,w}$ (M1)

(dashed line: predicted minimal O-D travel time by ITS; gray line: travel time of path 2; black line: flow of path 2)

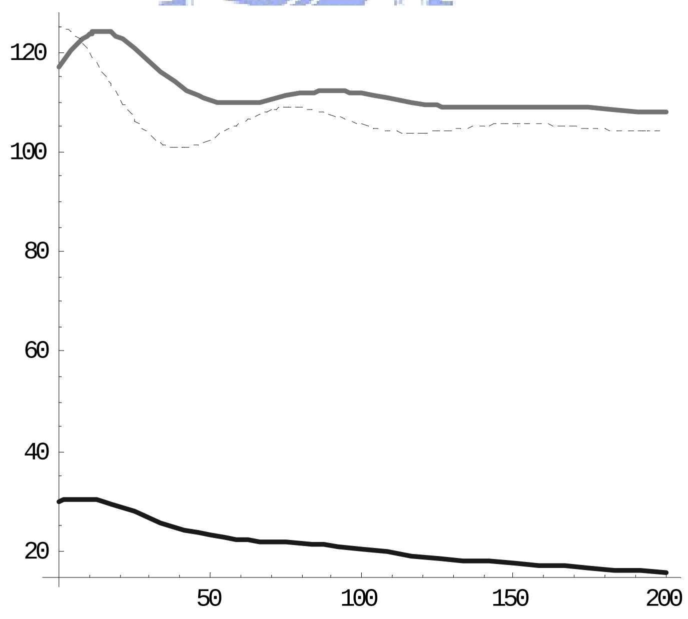

Fig. 7-4 Evolutions of path 3 flow dynamics and unit $PTTL^{t}_{path\,3,w}$ (M1)

(dashed line: predicted minimal O-D travel time by ITS; gray line: travel time of path 3; black line: flow of path 3)

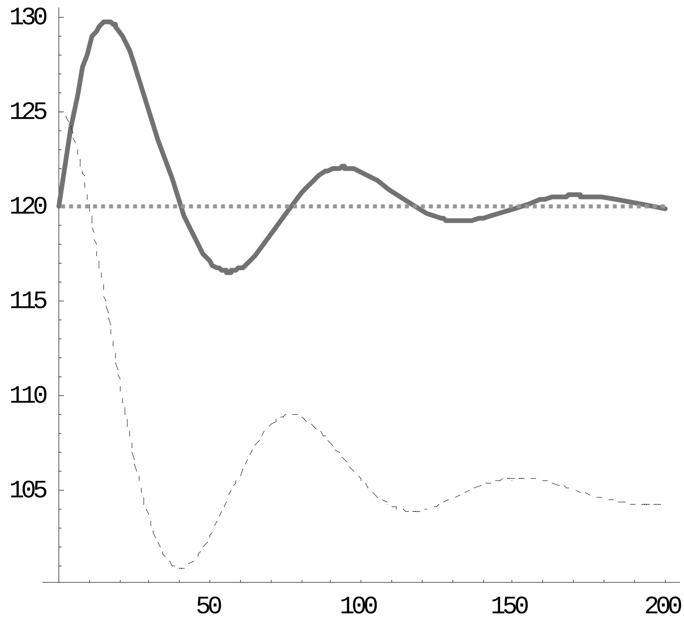

Fig. 7-5 Evolutions of predicted minimal O-D travel time by ITS and ETD_w^t (M1)

(dashed line: predicted minimal O-D travel time by ITS; dotted line: O-D demand; gray line: sum of path flows)

7.2 Numerical Results of Multiple User Classes Model

The resultss reveal that the steady state is qualified as a Wardrop's user equilibrium. The embedded mechanisms of adjustments are also sustained by the results in Table 7.3 and from Fig. 7-6 to Fig. 7-9. However, two additional facts are found that sensible users occupy better routes sooner than less sensible users in the evolution process until equilibrium is reached, and the ratio of 1st class users to 2nd class users for three paths at a steady state is different from that at the initial status. The latter one can be interpreted as the result of the former and evidently due to various parameters of the propensity of path flow dynamics. Moreover, we remind readers that the shares of O-D demand for two user classes might be changed in the processes.

Unsurprisingly the outcome, shown in Table 7.4 and from Fig. 7-10 to Fig. 7-15, follows the asymptotic behaviors claimed in the previous section. It is also observed that the adjustment speeds in M3 are less than that in M2 from the initial conditions to the 200th time state. It is in that the path flow dynamics are limited by a fixed and smaller O-D demand for user class two in M3. However, it seems to be reasonable that the O-D demand ratio of user class one to user class two should not be overly distorted in the evolution processes even though the total O-D demand is kept unchanged. This point should be a necessary check when M2 is implemented into the empirical study.

Table 7.3 Results of numerical example for M2*

		Flow			Travel time		
		Initial	State at t=200	Steady state	Initial	State at t=200	Steady state
Path 1	C1	20	21.95	22.23	103.29	103.79	103.79
	C2	20	31.84	33.94			
Path 2	C1	25	25.81	26.08	109.58	103.80	103.79
	C2	25	29.30	30.88			
Path 3	C1	15	9.55	6.62	116.76	105.72	103.79
	C2	15	1.57	0.25			
Link 1	C1	35	31.50	28.85	51.72	48.67	47.71
	C2	35	33.41	34.19			
Link 2	C1	25	25.81	26.08	64.58	66.76	67.71
	C2	25	29.30	30.88			
Link 3	C1	15	9.55	6.62	20.04	20.00	20.00
	C2	15	1.57	0.25			
Link 4	C1	20	21.95	22.23	51.56	55.11	56.08
	C2	20	31.84	33.94			
Link 5	C1	40	35.36	32.70	45.00	37.05	36.08
	C2	40	30.87	31.13			
Predicted minimal O-D travel time by ITS					125.00	103.88	103.79

* Two-class users model with identical predicted minimal O-D travel time for two user classes, i.e. (18) with n=2; C1 denotes user class one and C2 user class two.

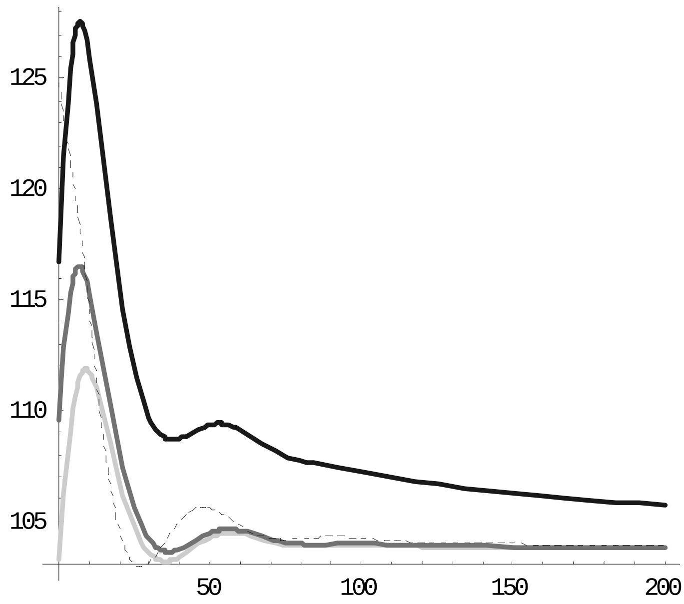

Fig. 7-6 Travel time evolutions of M2

(dashed line: predicted minimal O-D travel time; light gray line: path 1; medium gray line: path 2; black line: path 3)

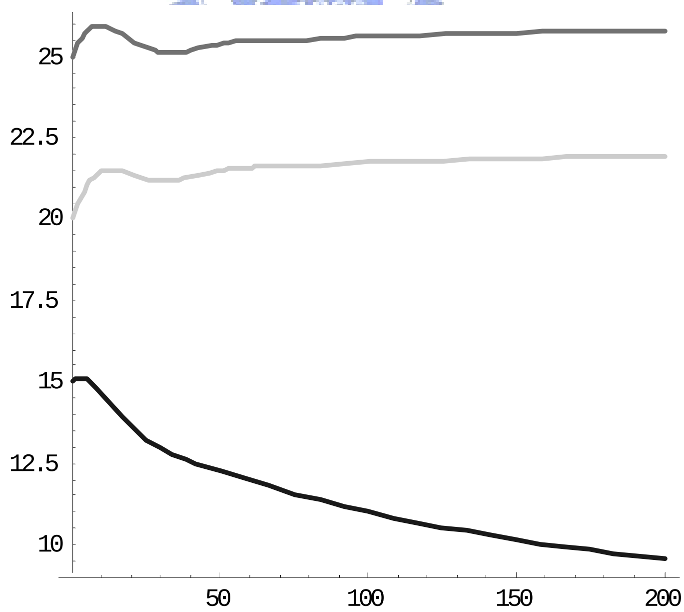

Fig. 7-7 Path flow evolutions of user class one in M2

(light gray line: path 1; medium gray line: path 2; black line: path 3)

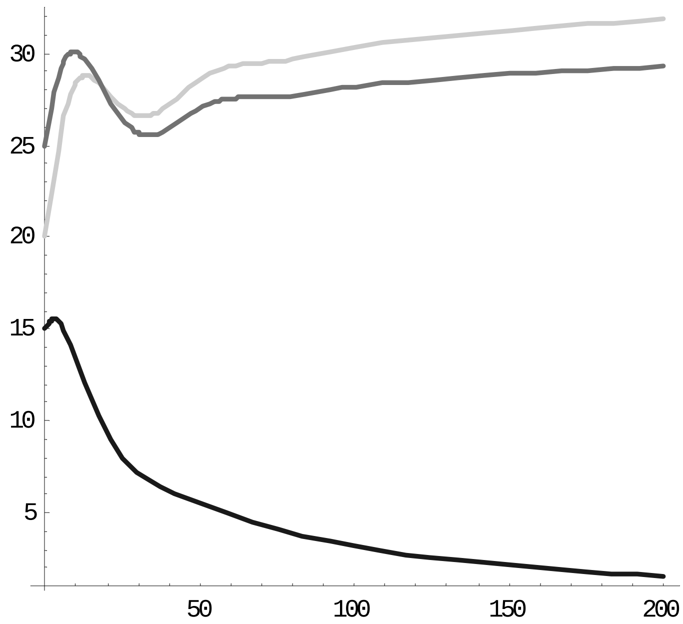

Fig. 7-8 Path flow evolutions of user class two in M2

(dashed line: predicted minimal O-D travel time; light gray line: path 1; medium gray line: path 2; black line: path 3)

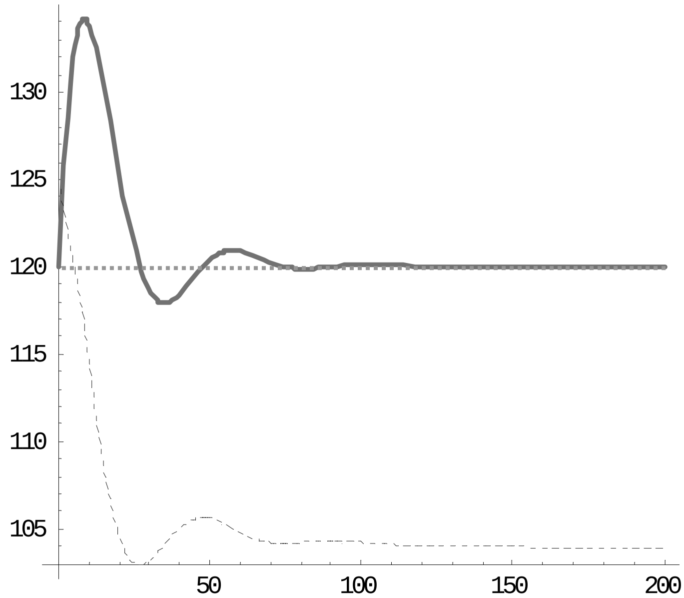

Fig. 7-9 Evolutions of predicted O-D travel time by ITS and ETD_w^t (M2)

(dashed line: predicted minimal O-D travel time by ITS; dotted line: O-D demand; gray line: sum of path flows)

Table 7.4 Results of numerical example for M3#

		Flow			Travel time		
		Initial	State at t=200	Steady state	Initial	State at t=200	Steady state
Path 1	C1	20	22.62	24.56	103.29	103.04	103.78
	C2	20	30.40	31.60			
Path 2	C1	25	26.56	28.71	109.58	103.09	103.78
	C2	25	27.80	28.24			
Path 3	C1	15	9.66	6.72	116.76	104.91	103.78
	C2	15	1.36	0.15			
Link 1	C1	35	32.28	31.28	51.72	48.21	47.71
	C2	35	31.76	31.75			
Link 2	C1	25	26.56	28.71	64.58	66.40	67.70
	C2	25	27.80	28.24			
Link 3	C1	15	9.66	6.72	20.04	20.00	20.00
	C2	15	1.36	0.15			
Link 4	C1	20	22.62	24.56	51.56	54.82	56.07
	C2	20	30.40	31.60			
Link 5	C1	40	36.22	35.43	45.00	36.69	36.08
	C2	40	29.16	28.39			
Predicted minimal O-D travel time				C1	125.00	106.02	103.79
				C2	125.00	102.99	103.79

Two-class users model with user-specific predicted minimal O-D travel time for two user classes, i.e. (19) with n=2; C1 denotes user class one and C2 user class two.

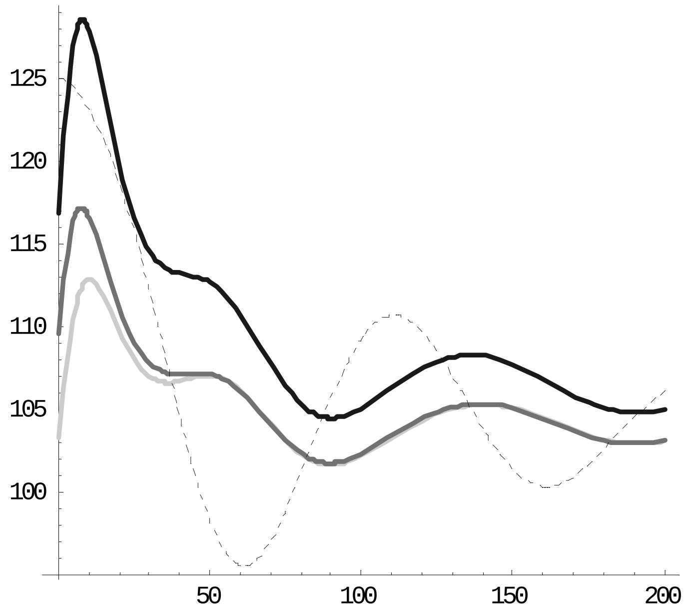

Fig. 7-10 Evolutions of path travel times and C1 predicted minimal O-D travel times for M3

(dashed line: C1 predicted minimal O-D travel time; light gray line: path 1; medium gray line: path 2; black line: path 3)

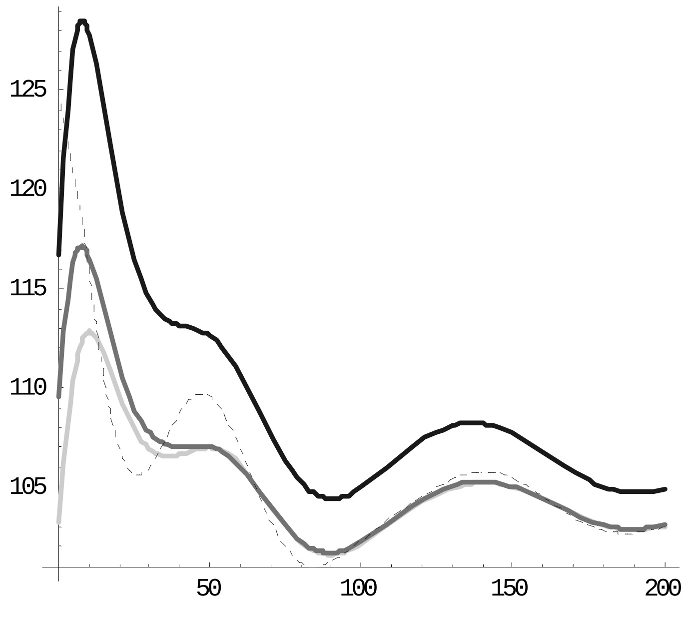

Fig. 7-11 Evolutions of path travel times and C2 predicted minimal O-D travel times for M3

(dashed line: C2 predicted minimal O-D travel time; light gray line: path 1; medium gray line: path 2; black line: path 3)

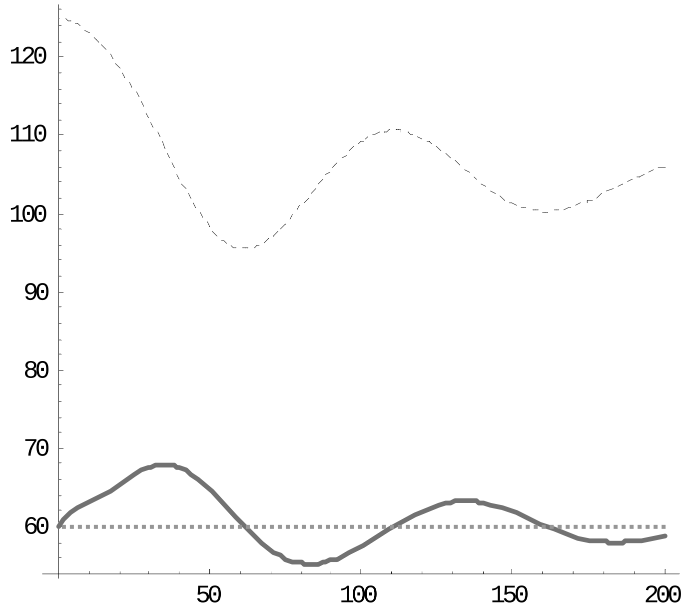

Fig. 7-12 Evolutions of C1 predicted minimal O-D travel time by ITS and ETD_{1w}^{t} (M3)

(dashed line: C1 predicted minimal O-D travel time by ITS; dotted line: O-D demand; gray line: sum of path flows for C1)

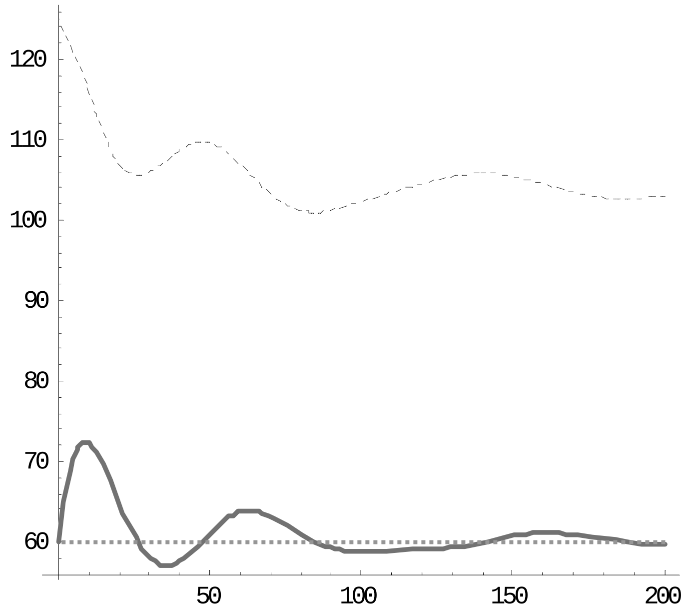

Fig. 7-13 Evolutions of C2 predicted minimal O-D travel time by ITS and ETD_{2w}^{t} (M3)

(dashed line: C2 predicted minimal O-D travel time by ITS; dotted line: O-D demand; gray line: sum of path flows for C2)

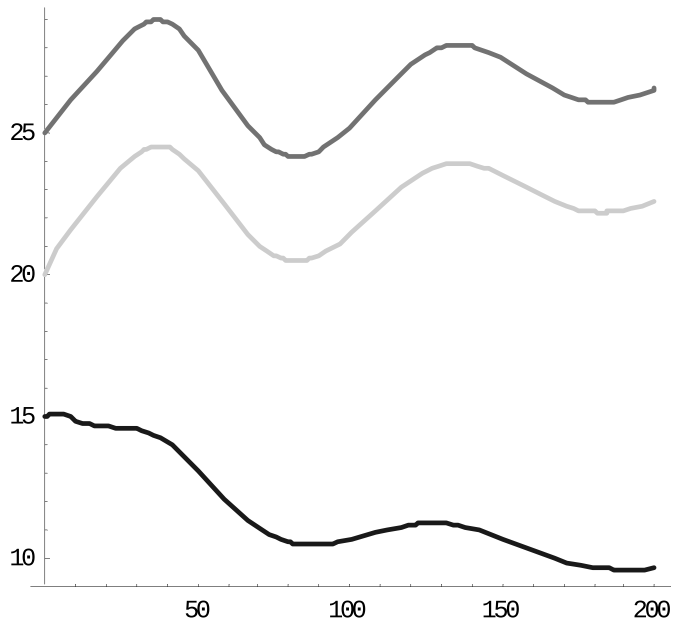

Fig. 7-14 Path flow evolutions of C1 in M3

(light gray line: path 1; medium gray line: path 2; black line: path 3)

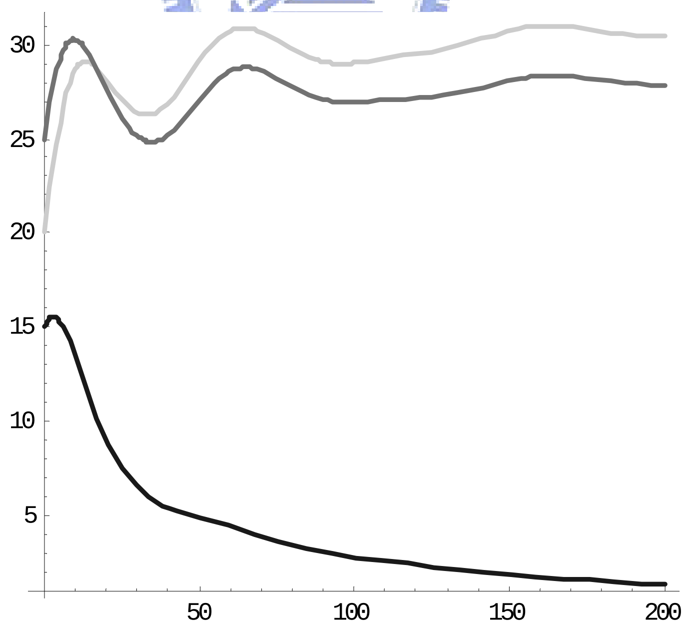

Fig. 7-15 Path flow evolutions of C2 in M3

(light gray line: path 1; medium gray line: path 2; black line: path 3)

Chapter 8

Conclusions and Prospects

8.1 Conclusions

In this research, the author deals with vehicular network dynamics in a day-to-day time scale by using a nonlinear dynamical system approach. Based on the learning and adaptive behavioral assumption and ITS operations, a new theory of network dynamics is developed. The structure of day-to-day network dynamics is identified as the path flow dynamics, predicted travel time dynamics, and their interactions. Two inhomogeneous user classes dynamical systems are formulated in the sense of different sensitivity of path flow dynamics due to the total perceived travel time loss (or saving) and information supply strategies. Incorporating the total perceived travel time loss (or saving), path flow dynamics are generated on a flow-weighted base to prevent them from being insensible of flow level.

The equilibrium solutions of presented models are analyzed to satisfied the Wardropian equilibria under come conditions. And the stability of the equilibrium solution and are proved to be asymptotically stable in the standpoints of Lyapunov. The Lipschitz condition for the proposed dynamical system is a key lemma in the proof of existence and uniqueness by way of the fundamental theorem of ordinary differential equations. Based on these results, the proposed models build an analytical linkage among day-to-day network dynamics, the Wardrop's user equilibrium, and the empirical adaptability of route preference under the operations of Intelligent Transportation Systems.

8.2 Prospects

The developed methodology built a new paradigm of dynamic traffic assignment problem. However, there are still some promising issues that should be further improved to enrich this research field. They are outlined in the following.

(1) Encapsulating within-day network dynamics will make the proposed framework more complete. For example, traffic dynamics could be replaced with a traffic flow model; en-route travel information could also be considered in future analysis. However, it can be inferred that the effects within-day network dynamics will cause the link cost operator much more complicated. And a more challenging task to derive the existence, uniqueness, and stability is very possibly unavoidable.

(2) The parameter calibration for the sensitivity of network dynamics will make the proposed models more powerful in practical applications. It is also a valuable issue in the research filed of transportation demand analysis under the operations of ITS.

(3) To formulate some problems of operational and planning applications as a dynamic network design problem in this style, will help traffic operator to understand the perturbations caused by a new management alternative. That provides a different viewpoint from conventional equilibrium-based approaches.

(4) With the further treatments mentioned above, a well potential could be expected to deploy a traffic simulator based on the proposed methodology.

References

Astarita, V., V. Adamo, G.E. Cantarella and E. Cascetta, "A doubly dynamic traffic assignment model for planning applications," *Proceedings of 14th International Symposium on Transportation and Traffic Theory*, (Edited by A. Ceder), pp. 373-386, Pergamon Press, London, (1999).

Cantarella, G.E., and E. Cascetta, "Dynamic processes and equilibrium in transportation networks: towards a unifying theory," *Transportation Science* 29, 305-329, (1995).

Carey, M., "Stability of competitive regional trade with monotone demand/supply functions," *Journal of Regional Science* 20, 489-501, (1980).

Cascetta, E., and G.E. Cantarella, "A day-to-day and within day dynamic stochastic assignment model," *Transportation Research* 25A, 277-291, (1991).

Chang, G.L., and H.S. Mahmassani, "Travel time prediction and departure time adjustment behavior dynamics in a congested traffic system," *Transportation Research* 22B, 217-232, (1988).

Chen, H.K., and C.F. Hsueh, "A model and an algorithm for the dynamic user-optimal route choice problem," *Transportation Research* 32B, 219-234, (1998).

Daganzo, C.F., "The cell-transmission model: A simple dynamic representation of highway traffic," *Transportation Research* 28B, 269-287, (1994).

Friesz, T.L., D. Bernstein, N.J. Mehta, R.L. Tobin and S. Ganjalizadeh, "Day-to-day dynamic network disequilibria and idealized traveler information systems," *Operations Research* 42, 1120-1136, (1994).

Friesz, T.L., D. Bernstein, N.J. Mahta, R.L. Tobin and S. Ganjalizadeh, "Dynamic network traffic assignment considered as a continuous time optimal control problem," *Operations Research* 37, 893-901, (1989).

Friesz, T.L., D. Bernstein, T.E. Smith, R.L. Tobin and B.W. Wie, "A variational inequality formulation of the dynamic network user equilibrium problem," *Operations Research* 41, 179-191, (1993).

Friesz, T.L., "Transportation network equilibrium, design and aggregation," *Transportation Research* 19A, 413-427, (1985).

Friesz, T.L., D. Bernstein and R. Stough, "Dynamic systems, variational inequalities and control theoretic models for predicting time-varying urban network flows," *Transportation Science* 30, 14-31, (1996).

Horowitz, J.L., "The stability of stochastic equilibrium in a two link transportation network," *Transportation Research* 18B, 13-28, (1984).

Hu, T.Y., and H.S. Mahmassani, "Evolution of network flows under real-time information: a day-to-day dynamic simulation-assignment framework," *Transportation Research Record* 1493, 46-56, (1995).

Hu, T.Y., and H.S. Mahmassani, "Evolution of network flows under real-time information and responsive signal control systems," *Transportation Research* 5C, 51-69, (1997).

Janson, B.N., "Dynamic traffic assignment for urban road networks," *Transportation Research* 25B, .143-161, (1991).

Jayakrishnan, R., H.S. Mahmassani, and T.Y. Hu, "An evaluation tool for advanced traffic information and management systems in urban networks," *Transportation Research* 2C, 129-147, (1994).

Lo, H.K., and W.Y. Szeto, "A cell-based dynamic traffic assignment model: Formulation and properties," *Mathematical and Computer Modelling* 35, 849-865, (2002).

Mahmassani, H.S., "Dynamic network traffic assignment and simulation methodology for advanced system management applications," *Networks and Spatial Economics* 1(3/4), 267-292, 2001

Mahmassani, H.S., G.L. Chang, and R. Herman, "Individual decisions and collective effects in simulated traffic system," *Transportation Science* 20, 258-271, (1986).

Mahmassani, H.S., "Dynamic models of commuter behavior: experimental investigation and application to the analysis of planned traffic disruptions," *Transportation Research* 24A, 465-484, (1990).

Mahmassani, H.S., and S. Peeta, "Network performance under system optimal and user equilibrium dynamic assignment: implications for ATIS," *Transportation Research Record* 1408, 83-93, (1993).

Merchant, D. K., and G.L. Nemhauser, "A model and an algorithm for the dynamic traffic assignment problems," *Transportation Science* 12, 183-199, (1978).

Peeta, S., and A.K. Ziliaskopou;os, "Foundations of dynamic traffic assignment: the past, the present and the future," *Networks and Spatial Economics* 1(3/4), pp.233-266, 2001.

Ran, B., D. Boyce and L. LeBlanc, "Toward a new class of instantaneous dynamic user-optimal traffic assignment models," *Operations Research* 41, 192-202, (1993).

Ran, B., R.W. Hall, and D. Boyce, "A link-based variational inequality model for dynamic departure time/route choice," *Transportation Research* 30B, 31-46, (1996).

Smith, M.J., "The existence,uniqueness, and stability of traffic equilibria," *Transportation Research* 13B, 295-304, (1979).

Wardrop, J.G., "Some theoretical aspects of road traffic research," *Proceedings of the Institute of Civil Engineers* Part II, pp. 325-378, (1952).

Watling, D., "A stochastic process model of day-to-day traffic assignment and information," *Behavioural and Network Impacts of Driver Information Systems*, (Edited by R. Emmerink and P. Nijkamp), pp. 115-139, Ashgate Publishing Ltd., (1999).

Wei, B.W., R.L. Tobin, T.L. Friesz, and D. Berstein, "A dicrete time, nested cost operator approach to the dynamic network user equilibrium problem," *Transportation Science* 29(1), pp.79-92, (1995).

Ziliaskopoulos, A.K., "A linear programming model for the single destination system optimum dynamic traffic assignment problem," *Transportation Science* 34, 37-49, (2000).

Vita

Ming-Chorng Hwang was born in Taiwan, R.O.C., on June 19, 1966. He received Bachelor degree in Transportation Engineering and Management from the National Chiao Tung University (NCTU) in June, 1989. And he received Master degree in Traffic and Transportation Engineering from NCTU in June 1991. After his two-year military service in the 46th Transportation Command Post of Army General Headquarter, he was an traffic engineer in the Taiwan Area National Expressway Engineering Bureau from December 1993 to December 2000. Then he transferred to be a certified traffic researcher in the Institute of Transportation, Ministry of Transportation and Communication, Taiwan from December 2000 to April 2002. In September 1997, he joined the Ph. D. program in Transportation Technology and Management at NCTU.

During the period of Ph. D. study, he participated the project of Intelligent Transportation Systems – Human Centered Technology, Program for Promoting University Academic Excellence sponsored by the Ministry of Education and several ITS-related fundamental research projects funded by the National Science Council that jointly resulted in the following publications:

A. Refereed Journal Papers.

1. Cho, Hsun-Jung and Ming-Chorng Hwang, "Day-to-day vehicular flow dynamics in intelligent transportation network", *Mathematical and Computer Modelling* (Accepted), 2004. (SCI) (NSC 91-2211-E-009-050)
2. Cho, Hsun-Jung and Ming-Chorng Hwang, "A stimulus-response model of day-to-day network dynamics", *IEEE Transactions on Intelligent Transportation Systems* (Accepted), 2004. (SCI, EI) (NSC 91-2622-E-009-014)

B. Refereed Conference Papers.

1. Cho, Hsun-Jung and Ming-Chorng Hwang, "Quasi User Equilibrium and Its Stability",

IEEE International Conference on Systems, Man & Cybernetics, October 10–13, 2004, Hague, Netherlands. (EI) (NSC-91-2415-H-009-002)

2. Jou, Yow-Jen Ming-Chorng Hwang and Jia-Ming Yang, "A Traffic Simulation Interacted Approach for the Estimation of Dynamic Origin-Destination Matrix" *IEEE International Conference on Networking, Sensing & Controls*, March 21-24, 2004, Taipei, Taiwan. (EI) (NSC-92-2622-E-009-011-CC3)

3. Cho, Hsun-Jung, Ming-Chorng Hwang and Du-Hwan Lin, "A Traffic Simulation Optimization Mehtod for the Layout Design of Working Zone", *IEEE International Conference on Networking, Sensing & Controls*, March 21-24, 2004, Taipei, Taiwan. (EI)

4. Cho, Hsun-Jung, Ming-Chorng Hwang, and Hsien-Hung Shih, "A Day-to-day and Within-day Network Dynamics Encapsulating Hyperbolic Traffic Behavior", *IEEE International Conference on Networking, Sensing & Controls*, March 21-24, 2004, Taipei, Taiwan. (EI) (NSC 92-2415-H-009-004)

5. Cho, Hsun-Jung and Ming-Chorng Hwang, "A Stimulus-Response Model of Day-to-day Network Dynamics", *IEEE International Conference on Intelligent Transportation Systems*, Oct. 12-15 2003, Shanghai, China. (EI) (NSC 91-2622-E-009-014).

6. Cho, Hsun-Jung, Hsien-Hung Shih and Ming-Chorng Hwang, "Numerical Simulation of a Multilane Drift-Diffusion Traffic Model", *IEEE International Conference on Systems, Man & Cybernetics*, October 5–8, 2003, Washington, D.C., USA. (EI)

7. Jou, Yow-Jen Ming-Chorng Hwang, Yu-H. Wang and Chih-H Chang, "Estimation of Dynamic Origin-Destination by Gaussian State Space Model with Unknown Transition Matrix", *IEEE International Conference on Systems, Man & Cybernetics*, October 5–8, 2003, Washington, D.C., USA. (EI) (NSC-91-2415-H-009-002)

8. Cho, Hsun-Jung, Ming-Chorng Hwang, and T. J., Jou, "A Finite Difference Method with Adaptive Time Mesh for Hyperbolic Traffic Flow," *The IEEE 5th International Conference on Intelligent Transportation Systems*, 3-6 Sep. 2002, Singapore.(EI)