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Centers of Chordal Graphs*

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Abstract. In a graph $G = (V, E)$, the *eccentricity* $e(S)$ of a subset $S \subseteq V$ is $max_{x \in V} min_{y \in S} d(x, y)$; and $e(x)$ stands for $e(\{x\})$. The *diameter* of G is $max_{x \in V} e(x)$, the *radius* $r(G)$ of G is $min_{x \in V} e(x)$ and the *clique radius* $cr(G)$ is $min\, e(K)$ where K runs over all cliques. The *center* of G is the subgraph induced by $C(G)$, the set of all vertices x with $e(x) = r(G)$. A *clique center* is a clique K with $e(K) = cr(G)$. In this paper, we study the problem of determining the centers of chordal graphs. It is shown that the center of a connected chordal graph is distance invariant, biconnected and of diameter no more than 5. We also prove that $2cr(G) \leq d(G) \leq 2cr(G) + 1$ for any connected chordal graph G. This result implies a characterization of a biconnected chordal graph of diameter 2 and radius 1 to be the center of some chordal graph.

1. Introduction

In a graph $G = (V, E)$, the *distance* $d(x, y)$ from vertex x to vertex y is the minimum number of edges in a path from x to y. The *eccentricity* $e(x)$ of a vertex x is the maximum distance from x to any vertex in G. The *diameter* $d(G)$ of G is the maximum eccentricity of a vertex in G and *radius* $r(G)$ the minimum eccentricity. Denote by $C(G)$ the set of all vertices whose eccentricities are equal to $r(G)$. The *center* of G is the subgraph $\langle C(G) \rangle$ induced by $C(G)$.

It was shown in [7] that the center of a graph lies within a single block (biconnected component), but need not be a block. As described in [1], Hedetniemi proved that any graph H is isomorphic to the center of some graph G which is of diameter 4 and radius 2. In fact, G can be obtained from H by adding four new vertices u, v, w, x such that v and w are adjacent to all vertices of H, u is adjacent only to v and x only to w. However, the centers of some special graphs are restricted. The oldest result is Jordan's well-known theorem for trees [8]: the center of a tree is either K_1 or K_2. As an easy generalization we can say that the center of a connected block graph, i.e. a graph whose blocks are complete graphs, is either a cut-vertex or a block. Proskurowski [10] proved that the center of a maximal

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outplanar graph is one of seven special graphs. As a generalization, in [11] he found all possible centers of 2-trees, and showed that the center of a 2-tree is biconnected.

A graph is *chordal* (*triangulated* or *rigid circuit*) if every cycle of length greater than three possesses a chord, i.e. an edge joining two nonconsecutive vertices of the cycle. Chordal graphs were first introduced by Hajnal and Surányi [6] and then studied extensively by many people, see [5] for general results. The class of chordal graphs contains trees, block graphs, maximal outerplanar graphs and 2-trees. It was shown in [9] that the center of a chordal graph is connected. The main purpose of this paper is to study the centers of chordal graphs and to answer a part of the question given by Duchet [4]: determine the centers of chordal graphs.

Section 2 introduces the idea of *clique radius* $cr(G)$ of a graph G, and proves a main theorem: $2cr(G) \leq d(G) \leq 2cr(G) + 1$ for any connected chordal graph G. This result is used in Section 3 as the key for a characterization of centers of some chordal graphs.

Section 3 studies necessary and sufficient conditions for the centers of chordal graphs. In particular, we prove that the center of a chordal graph is distance invariant, biconnected and of diameter no more than 5. Finally, by using the main theorem in Section 2, we give a necessary and sufficient condition for a biconnected chordal graph of diameter 2 and radius 1 to be the center of some chordal graph.

2. Clique Centers of Chordal Graphs

A *clique* of a graph is a set of pairwise adjacent vertices. In a graph $G = (V, E)$, the distance $d(x, S)$ from a vertex x to a set $S \subseteq V$ is $min_{y \in S}\, d(x, y)$. The *eccentricity* $e(S)$ of a set S of vertices is the maximum distance from any vertex to S. A *clique center* of G is a clique with minimum eccentricity which is called the *clique radius* of G and is denoted by $cr(G)$. This idea is similar to *bi-center* which is an edge with minimum eccentricity; see Theorem 4.2 in [3].

The main result of this section is the relation between clique radius and diameter of a connected chordal graph. It is the keystone for determining the necessary and sufficient conditions of a chordal graph of diameter 2 and radius 1 to be the center of some chordal graph.

Theorem 2.1. $2cr(G) \leq d(G) \leq 2cr(G) + 1$ *for any connected chordal graph* $G = (V, E)$.

Before proving Theorem 2.1, we first list some definitions and results from [2] which are needed in this paper. Lemma 2.4 is from [9].

If $d(x, y) = k$ is finite and $0 \leq m \leq k$, then $Bet(x, m, y)$ denotes the set of all vertices z *between* x and y such that $d(x, z) = m$ and $d(z, y) = k - m$. An *n-sun* is a chordal graph of $2n$ vertices with a Hamiltonian cycle $(y_1, z_1, y_2, z_2, \ldots, y_n, z_n, y_1)$ and each y_i is of degree two. Equivalently, an n-sun is a chordal graph $G = (V, E)$ whose vertex set V can be partitioned into $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ such that the following three conditions hold.

(S1) Y is a stable set in G.

(S2) $(z_1, \ldots, z_n, z_1)$ is a cycle in G.

(S3) $(y_i, z_j) \in E$ if and only if $i = j$ or $i = j + 1 \pmod{n}$.
In the above definition, if Z is a clique, then we call the n-sun a *complete* n-sun.

Lemma 2.2. *If C is a cycle in a chordal graph, then for every edge (u, v) of C there is a vertex w of C which is adjacent to both u and v.*

Lemma 2.3. [2] *If G is chordal and $d(x, y) = k$, then $Bet(x, m, y)$ is a clique for any $0 \leq m \leq k$.*

Lemma 2.4. [9] *In a chordal graph G, if K is a clique and x a vertex such that $d(x, y) = k$ is a constant for all $y \in K$, then $Bet(z, 1, x)$ and $Bet(w, 1, x)$ are comparable for any $z, w \in K$ (i.e. one is a subset of the other); consequently, $\bigcap_{y \in K} Bet(y, 1, x)$ is not empty.*

Theorem 2.5. [2] *$2r(G) - 2 \leq d(G) \leq 2r(G)$ for any connected chordal graph G. Moreover, if $2r(G) - 2 = d(G)$, then G has a 3-sun as an induced subgraph.*

We can also prove two slightly more general results as follows.

Theorem 2.6. *If X and Y are two cliques in a chordal graph G such that $d(x, y) = k$ is a constant for all $x \in X$ and $y \in Y$, then $Bet(X, m, Y) \equiv \bigcup \{Bet(x, m, y)\colon x \in X$ and $y \in Y\}$ is a clique for any $0 \leq m \leq k$.*

Proof. Consider the graph G^* obtained from G by adding two new vertices u and v which are adjacent to all vertices in X and Y respectively. Then G^* is a chordal graph and $d(u, v) = k + 2$. The theorem follows from Lemma 2.3 and the fact that $Bet(X, m, Y) = Bet(u, m + 1, v)$. □

Theorem 2.7. *In a chordal graph G, if K is a clique and x is a vertex such that $d(x, y) = k$ is a constant for all $y \in K$, then $Bet(z, m, x)$ and $Bet(w, m, x)$ are comparable for any $1 \leq m \leq k$ and $z, w \in K$; consequently, $\bigcap_{y \in K} Bet(y, m, x)$ is not empty.*

Proof. The theorem is true for $m = 1$ by Lemma 2.4. Suppose it is true for $m - 1$. Let $Z = Bet(z, m - 1, x)$ and $W = Bet(w, m - 1, x)$. $Z \subseteq W$ or $W \subseteq Z$ by the induction hypothesis. Theorem 2.6 implies that $Z \cup W$ is a clique. By Lemma 2.4, $Bet(y, 1, x)$'s are comparable for all $y \in Z \cup W$. $Bet(z, m, x) = \bigcup_{y \in Z} Bet(y, 1, x)$ and $Bet(w, m, x) = \bigcup_{y \in W} Bet(y, 1, x)$ then imply that $Bet(z, m, x)$ and $Bet(w, m, x)$ are comparable. □

The graph in Fig. 1 shows that $Bet(x_1, 1, y_1)$ and $Bet(x_2, 1, y_2)$ are not comparable, so we cannot get a generalization of Theorem 2.7 or Lemma 2.4 by replacing vertex x by a cliue.

Now we are ready to prove the main theorem of this section.

Proof of Theorem 2.1. Choose a clique center K such that $S(K) = \{x \in V\colon d(x, K) = cr(G)\}$ has smallest number of vertices. Suppose $x, y \in V$ are such that $d(x, y) = d(G)$. Choose $x^*, y^* \in K$ with $d(x, x^*) = d(x, K)$ and $d(y, y^*) = d(y, K)$. Then

$$d(G) = d(x, y) \leq d(x, x^*) + d(x^*, y^*) + d(y^*, y) \leq 2cr(G) + 1.$$

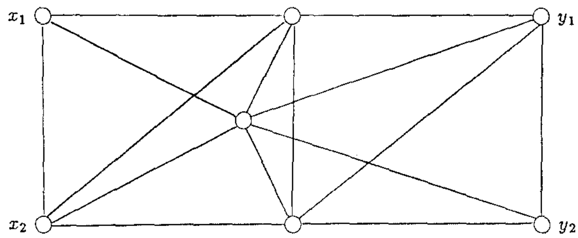


Fig. 1. $Bet(x_1, 1, y_1)$ and $Bet(x_2, 1, y_2)$ are not comparable

Suppose that $d(G) \le 2cr(G) - 1$. Choose a fixed vertex $w \in S(K)$. Let $K^* = \{w^* \in K\colon d(w, w^*) = cr(G)\}$. Suppose $K = K^*$, i.e. $d(w, w^*) = cr(G)$ for all $w^* \in K$. By Lemma 2.4, there is a vertex $x \in \bigcap_{w^* \in K} Bet(w^*, 1, w)$. Then $K \cup x$ is a clique center with $S(K \cup x) \subsetneq S(K) - w$, a contradiction to the minimality of $|S(K)|$. So K^* is a proper subset of K.

Next, we consider the set $T = \{x \in V\colon cr(G) - 1 \le d(x, K) < d(x, K^*)\}$. If $T = \varnothing$, then K^* is a clique center with $S(K^*) = S(K)$. The same arguments in the second paragraph lead to a contradiction. So $T \ne \varnothing$. For any $x \in T$ and $w^* \in K^*$, choose $x^* \in K - K^*$ with $d(x, x^*) = d(x, K)$. By the definitions of K, K^* and T, we have

$$cr(G) - 1 \le d(x, x^*) = d(x, w^*) - 1 \tag{2.1}$$

and

$$cr(G) = d(w, w^*) = d(w, x^*) - 1. \tag{2.2}$$

Choose shortest paths $P(x, x^*)$, $P(w, w^*)$, $P(x, w)$ from x to x^*, w to w^*, and x to w respectively as in Fig. 2; where y (resp. z) is the vertex in $P(x, x^*) \cap P(x, w)$ (resp. $P(w, w^*) \cap P(x, w)$) with largest distance from x (resp. w).

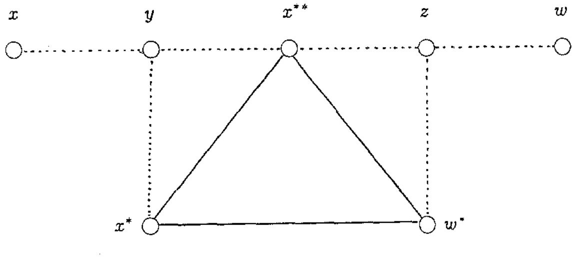


Fig. 2.

Suppose $y = x^*$, then $d(G) \ge d(x, w) = d(x, x^*) + d(x^*, w) \ge 2cr(G)$ by (2.1) and (2.2), a contradiction. Hence $y \ne x^*$. Similarly, $z \ne w^*$. In the cycle $(y, \ldots, x^*, w^*, \ldots, z, \ldots, y)$, by Lemma 2.2, there is a vertex x^{**} adjacent to both x^* and w^*. Note that x^{**} is not between y and x^* (similarly, not between z and w^*) otherwise $d(x, w^*) \le d(x, x^*)$, which contradicts (2.1). (2.1) also implies

$$d(x, x^{**}) \ge d(x, w^*) - 1 \ge cr(G) - 1 \tag{2.3}$$

and (2.2) implies

$$d(w, x^{**}) \geq d(w, x^*) - 1 \geq cr(G). \tag{2.4}$$

(2.3) and (2.4) together with the assumption $2cr(G) - 1 \geq d(G) \geq d(x, w)$ imply that all inequalities in (2.3) and (2.4) are in fact equalities; and so $x^{**} \in Bet(x^*, 1, w)$.

Now consider $K^{**} = K^* \cup \{x^{**}: x \in T\}$. It is easy to see that $e(K^{**}) \leq e(K)$. Since $K^{**} \subseteq Bet(K - K^*, 1, w)$, by Theorem 2.6, K^{**} is a clique. So K^{**} is a clique center; in fact, is one with $S(K^{**}) \subseteq S(K)$ and $d(w, w^{**}) = cr(G)$ for all $w^{**} \in K^{**}$. The same arguments as in the second paragraph lead to a contradiction. This shows that $d(G) \geq 2cr(G)$ and completes the proof of the theorem. □

As a consequence of Theorems 2.1 and 2.5, and the trivial inequalities $cr(G) \leq r(G) \leq cr(G) + 1$, for any connected chordal graph G exactly one of the following holds: $2r(G) = d(G) = 2cr(G)$, $2r(G) - 1 = d(G) = 2cr(G) + 1$ and $2r(G) - 2 = d(G) = 2cr(G)$. For the case of block graphs, the last case is impossible since a block graph contains no 3-sun. For a block graph G with $d(G) = 2cr(G)$ (resp. $d(G) = 2cr(G) + 1$), $C(G)$ is a cut-vertex (resp. block); in any case $C(G)$ is always a clique center.

3. Graphs Which Are Centers of Chordal Graphs

A *chord* of a path is an edge joining two nonconsecutive vertices of the path.

Lemma 3.1. *In a chordal graph G, all vertices of a chordless path joining two vertices of $C(G)$ entirely belongs to $C(G)$.*

Proof. Suppose $x, y \in C(G)$ and $P(x, y) = (x = v_0, v_1, \ldots, v_n = y)$ is a chordless x-y path. Let v_i be the first vertex of $P(x, y)$ which is not in $C(G)$. Choose a vertex z such that $d(v_i, z) > r(G)$ and $j > i$ as small as possible with $d(v_j, z) \leq r(G)$. Then $d(v_k, z) > r(G)$ for $i \leq k < j$ and $d(v_{i-1}, z) = d(v_j, z) = r(G)$. Let w be the last common vertex on shortest paths $P(z, v_{i-1})$ and $P(z, v_j)$. Then $C = P(w, v_{i-1}) \cup P(v_{i-1}, v_j) \cup P(v_j, w)$ is a cycle. By Lemma 2.2, C has a vertex u adjacent to both v_{i-1} and v_i. Since $P(x, y)$ is chordless, $u \in P(w, v_{i-1})$ or $u \in P(v_j, w)$. In the former case, $d(z, v_i) \leq d(z, u) + 1 = r(G)$ which is impossible. In the latter case, $r(G) < d(z, v_i) \leq d(z, u) + 1 \leq d(z, v_j) - 1 + 1 = r(G)$, a contradiction. So the lemma holds. □

Theorem 3.2. *The center of a connected chordal graph G is a distance invariance induced subgraph of G.*

Proof. Since any shortest x-y path is chordless, the theorem follows from Lemma 3.1. □

Theorem 3.3. *The center of a connected chordal graph G is biconnected.*

Proof. Suppose z is a cut vertex of the center $\langle C(G) \rangle$. Let x and y be two vertices in different components of $\langle C(G) \rangle - z$. There are two disjoint x-y paths in G since $C(G)$ lies in a biconnected component of G as shown in [7]. Take chords, if there is any, to shorten these two paths until two chordless x-y paths are found. By Lemma 3.1, all vertices of these two paths are in $C(G)$. These two paths then both contain the vertex z, a contradiction. So the center is biconnected. □

Note that Theorem 3.3 was proved for 2-trees in [11]. By Theorem 2.5, for any connected chordal graph G, there are three cases: $d(G) = 2r(G)$, $d(G) = 2r(G) - 1$, $d(G) = 2r(G) - 2$. We shall derive some restrictions on diameters and radii of centers of chordal graphs according to these cases.

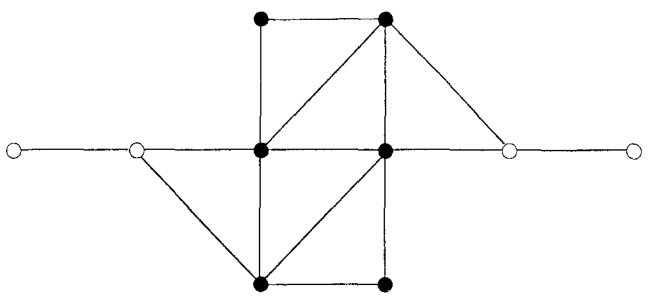
Theorem 3.4. *$C(G)$ is a clique for any connected chordal graph G with $d(G) = 2r(G)$.*

Proof. Choose two vertices x and y such that $d(x, y) = d(G)$. For any $z \in C(G)$, we have

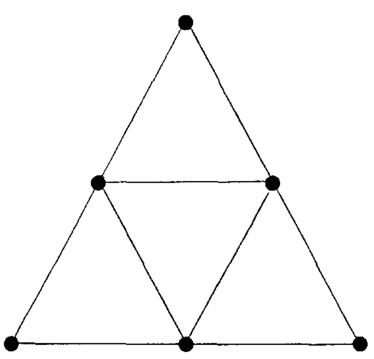
$$2r(G) = d(G) = d(x, y) \leq d(x, z) + d(z, y) \leq r(G) + r(G),$$

which imply that $d(x, z) = d(z, y) = r(G)$ and so $z \in Bet(x, r(G), y)$. Thus $C(G) \subseteq Bet(x, r(G), y)$. The theorem then follows from Lemma 2.3. □

In general, $C(G)$ is not necessarily a clique for the case of $d(G) = 2r(G) - 1$ or $2r(G) - 2$, see Fig. 3.



(a) $d(G) = 2r(G) - 1 = 5$ and $d(C(G)) = 3$.



(b) $d(G) = 2r(G) - 2 = 2$ and $d(C(G)) = 2$.

Fig. 3. Black vertices form $C(G)$

Theorem 3.5. *$d(C(G)) \leq 3$ for any connected chordal graph G with $d(G) = 2r(G) - 1$.*

Proof. Choose x and y such that $d(x, y) = d(G)$. For any $z \in C(G)$, we have

$$2r(G) - 1 = d(G) = d(x, y) \leq d(x, z) + d(z, y) \leq 2r(G).$$

Hence $d(x, z)$ and $d(z, y)$ are either $r(G) - 1$ or $r(G)$ but not both $r(G) - 1$. We shall prove that either $z \in Bet(x, r(G), y)$ or is adjacent to some vertex in $Bet(x, r(G), y)$. Since $Bet(x, r(G), y)$ is a clique by Lemma 2.3, every two vertices of $C(G)$ are of distance at most three in G and hence in $C(G)$.

For the case of $d(x, z) = r(G)$ and $d(z, y) = r(G) - 1$, $z \in Bet(x, r(G), y)$. For the case of $d(x, z) = r(G) - 1$ and $d(z, y) = r(G)$, $z \in Bet(x, r(G) - 1, y)$ and so is adjacent to some vertex in $Bet(x, r(G), y)$.

Suppose $d(x, z) = d(z, y) = r(G)$. Choose shortest paths $P(x, y)$, $P(y, z)$, $P(z, x)$ which pairwise meet at x^*, y^*, z^* as in Fig. 4. Since $r(G) \geq d(x, t)$, $r(G) \geq d(z, y)$ and $d(x, z^*) + d(z^*, y) \geq d(x, y) \geq 2r(G) - 1$, we must have $z = z^*$. By Lemma 2.2, there is a vertex w in the cycle $C = (x^*, \ldots, u, z^*, v, \ldots, y^*, \ldots, x^*)$ which is adjacent to both z^* and v. If $w = u$, then $d(x, u) = d(v, y) = r(G) - 1$ and $(u, v) \in E$ imply $v \in Bet(x, r(G), y)$; and so z is adjacent to vertex v in $Bet(x, r(G), y)$. If $w \neq u$, then w is between x^* and y^* as in Fig. 4. Note that $r(G) = d(x, z^*) \leq d(x, w) + 1$, i.e. $d(x, w) \geq r(G) - 1$. Similarly $d(w, y) \geq r(G) - 1$. So $d(x, w) = r(G)$ or $r(G) - 1$. In the former case, z is adjacent to $w \in Bet(x, r(G), y)$. In the latter case, z is adjacent to $v \in Bet(x, r(G), y)$. This completes the proof of the theorem. □

Similar arguments as in the proof of Theorem 3.5 lead to the following result.

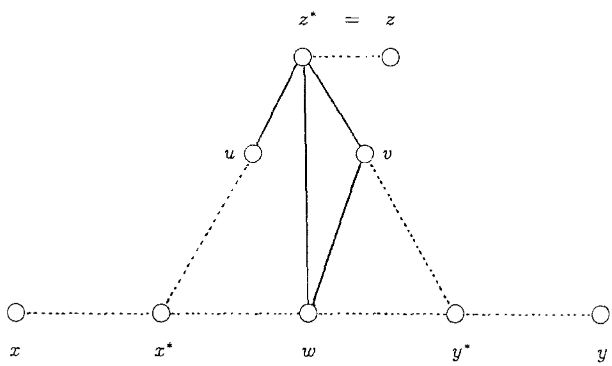


Fig. 4.

Theorem 3.6. *$d(C(G)) \leq 5$ for any connected chordal graph G with $d(G) = 2r(G) - 2$.*

By observing many examples, we have the following conjecture.

Conjecture: $d(C(G)) \leq 2$ for any connected chordal graph G with $d(G) = 2r(G) - 2$.

Next we study sufficient conditions for a biconnected chordal graph H to be the center of some chordal graph G. If $d(H) = 1$ or $d(H) = r(H) = 2$, then H is the center of itself. For the case of $d(H) = 2$ and $r(H) = 1$, we have the following result by using the main theorem of Section 2.

Theorem 3.7. *Suppose $H = (U, F)$ is a biconnected chordal graph with $d(H) = 2$, $r(H) = 1$ and $x \in C(H)$. H is the center of some chordal graph $G = (V, E)$ if and only if $d(H - x) \leq 3$.*

Proof. ($\Rightarrow$) Suppose H is the center of G, i.e. $U = C(G)$. Choose $w \in V$ such that $d(x, w) = r(G)$. For any $z \in U - x$, z is adjacent to x and $d(z, w) \leq r(G)$. Hence either $d(z, w) = r(G) - 1$ and so $z \in Bet(x, 1, w)$, or else $d(z, w) = r(G)$. For the latter case, $d(z, w) = d(x, w) = r(G)$ imply, by Lemma 2.4, that $Bet(z, 1, w) \cap Bet(x, 1, w) \neq \emptyset$ and so z is adjacent to some vertex in $Bet(x, 1, w)$. This is true for all $z \in H$. Since $Bet(x, 1, w)$ is a clique, $d(H - x) \leq 3$.

($\Leftarrow$) Suppose $d(H - x) \leq 3$. By Theorem 2.1, $cr(H - x) = 1$. Let K be a clique center of $H - x$. Then every vertex in $U - x$ is adjacent to x; and every vertex in $U - K$ is adjacent to some vertex in K. Consider the graph G obtained from H by adding two new vertices u and v such that u is adjacent to x and v is adjacent to all vertices of K. It is straightforward to check that G is a chordal graph and H is the center of G. □

We close this paper by the following summary of results: a graph H is the center of some chordal graph if and only if

(1) H is chordal and biconnected and

(2) $d(H) = 1$, or
$d(H) = r(H) = 2$, or
$d(H) = 2$, $r(H) = 1$ and $d(H - x) \leq 3$ for any $x \in C(H)$, or
$d(H) = 3$, $r(H) = 2$ and "some conditions we still do not know", or
$d(H) = 4$ or 5 (we conjecture that this case is impossible).

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