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博士論文

評選多項計畫的組合之高效能方法

An Efficient M[illegible]ting the Portfolios

of a [illegible] Projects

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摘　要

已知多個計畫的期望績效以多指標用來評量，從這些計畫之中選取若干項計畫所成的每個子集均被視為一個計畫組合。若對於包含 24 個計畫評選的問題，利用傳統的資料包絡分析法(DEA)對全部可能的計畫組合評量其相對效率，需要超過一天來求得績效高的計畫組合；本研究發[illegible]了計算時間，僅需要 37 秒。對於包含 37 個計畫的 2^{37} 個計畫組[illegible]直接使用傳統的 DEA，目前任何的數學規劃軟體均不能處理；本[illegible]能在一天內求解。本研究的第二個目的是評量每一個績效高的計[illegible]定度是指使該計畫組合仍維持高績效時，其中各項計畫在各指標[illegible]

關鍵詞：資料包絡法、多目標決策、計畫組合、績效評估、穩健性。

An Efficient Method for Selecting the Portfolios of a Large Number of Projects

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Abstract

We are selecting several projects out of a set of projects. Every subset of these projects is treated as a portfolio. Multiple indices are used to measure the expected performance of those projects. We employ Data Envelopment Analysis (DEA) to measure the relative efficiency of each port[illegible]ible portfolios. Our research has two major objectives. The first obje[illegible]orithm to reduce problem complexity and the required computation t[illegible]EA needs to generate all the possible portfolios first and then measu[illegible]ncy against all the portfolios. For the problem with 24 projects, it ne[illegible] obtain the efficient portfolios while our procedure needs only 37 seconds only. For our algorithm, a selection problem with 37 projects could be solved within one day in a personal computer. It is impossible to solve the problem with more than 2^{37} decision variables by any existing mathematical programming software if conventional DEA program is used. The second objective is to measure the stability of each identified efficient portfolio. The tolerance of its each individual index becomes worse could be measured for keeping its efficiency.

Keywords: data envelopment analysis (DEA), multiple criteria decision-making (MCDM), portfolio, performance evaluation, stability.

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Notations

MCDM

J : the number of objectives.

$f_j(\boldsymbol{x})$: the j^{th} objective of MCDM, $j = 1, \dots, J$.

$\boldsymbol{x}$: the feasible point of MCDM.

$\boldsymbol{S}$: the feasible set of MCDM.

MOBILP

K : the total number of projects.

k : the index of projects, $k = 1, \dots, K$.

s : the total number of output products.

r : the index of outputs, $r = 1, \dots, s$.

m : the total number of input resources.

i : the index of inputs, i …

c_{rk} : the amounts of produ… k, $r = 1, \dots, s$, and $k = 1, \dots, K$.

a_{ik} : the amounts of resou… k, $i = 1, \dots, m$, and $k = 1, \dots, K$.

Ω : the set of all feasible …

n : the total number of p…

P : the index of portfolio…

w_k : binary variables, w_k=1 if project k is selected and w_k=0 otherwise, k=1, … , K.

DEA

Ω_D : the set of all DMUs corresponding to portfolios in Ω.

DMU_P : the DMUs in Ω_D, $P \in \Omega$.

y_{rP} : the values of output r for portfolio P (DMU_P), $r = 1, \dots, s$, and $P \in \Omega$.

x_{iP} : the values of input i for portfolio P (DMU_P), $i = 1, \dots, m$, and $P \in \Omega$.

T : the portfolio currently under evaluation.

DMU_T : the DMU currently under evaluation.

y_{rT} : the values of output r for the evaluated portfolio T (DMU_T), $r = 1, \dots, s$.

x_{iT} : the values of input i for the evaluated portfolio T (DMU_T), $i = 1, \dots, m$.

DEA models

z_T : the objective of additive model when evaluating DMU_T.

η_T : the objective of BCC model when evaluating DMU_T.

ω_T : the objective of BCC dual model when evaluating DMU_T.

ξ_T : the objective of BCC ratio model when evaluating DMU_T.

ε : the infinitesimal constant.

θ_T : the proportional reduction applied to all inputs of DMU_T to improve efficiency.

λ_P : the variable for projecting DMU_P, $P \in \Omega$.

s_r^+ : the surplus in the amounts of output r, $r = 1, \dots, s$.

s_i^- : the slack in the amounts of input i, $i = 1, \dots, m$.

μ_r : the dual variable associated with the r^{th} output constraint, $r = 1, \dots, s$.

v_i : the dual variable associated with the i^{th} input constraint, $i = 1, \dots, m$.

u_0 : the intercept variable that reflect the impact of scale size of a DMU.

$\tilde{\mu}_r$: the weight assigned to output r, $r = 1, \dots, s$.

$\tilde{v}_i$: the weight assigned to input i, $i = 1, \dots, m$.

$\tilde{u}_0$: the constant assigned to maximize the output-input ratio of DMU_T.

Stability analysis of DEA

Δ_T : the radius of stability [illegible] for DMU_T.

Δ_r^O : the radius of stability [illegible]

Δ_i^I : the radius of stability [illegible]

Π : the reference set to e[illegible]f DMU_T.

λ_P : the variable for proje[illegible]

$\hat{c}_{rk}$: the r^{th} output coeffici[illegible]ge, $r = 1, \dots, s$, and $k = 1, \dots, K$.

$\hat{a}_{ik}$: the i^{th} input coefficient of project k after change, $i = 1, \dots, m$, and $k = 1, \dots, K$.

π : the increment in input of project k and portfolio T.

δ : the decrement in output of project k and portfolio T.

Γ : the value of increment in input and decrement in output simultaneously.

Ψ_1 : the set of changed portfolios when perturbed project k.

Ψ_0 : the set of unchanged portfolios when perturbed project k.

$\hat{y}_{rP}$: the r^{th} output of portfolio P after change, $r = 1, \dots, s$, and $P \in \Psi_1$.

$\hat{x}_{iP}$: the i^{th} input of portfolio P after change, $i = 1, \dots, m$, and $P \in \Psi_1$.

Identification of efficient portfolios

c_k : the value of single output of project k, $k = 1, \dots, K$.

a_k : the value of single input of project k, $k = 1, \dots, K$.

R_k : the ratio of single output to single input, $R_k = c_k / a_k$, for project k, $k = 1, \dots, K$.

R_k^{ri} : the ratio of the r^{th} output to i^{th} input, $R_k^{ri} = c_{rk} / a_{ik}$, for project k, $k = 1, \dots, K$.

h, p	: the indices of projects currently used.
e_k	: the unit row vector with 1 at the k^{th} component and 0 elsewhere.
H, G	: the indices of portfolios currently used.
$\boldsymbol{I}_P$	: the set consists of index of project k with $c_k > 0$ and $a_k > 0$.
$\boldsymbol{I}_N$	: the set consists of index of project k with $c_k < 0$ and $a_k < 0$.
$\boldsymbol{I}_0$	: the set consists of index of project k with $c_k > 0$ and $a_k \le 0$.
$\boldsymbol{I}_1$	: the set consists of index of project k with $c_k \le 0$ and $a_k > 0$.
$\boldsymbol{I}_C$	: the set consists of index of project k with $c_k < 0$ and $a_k = 0$.
$\boldsymbol{I}_A$	: the set consists of index of project k with $c_k = 0$ and $a_k < 0$.
Ω_0	: the subset of portfolios consists of P=(w_1, … , w_K) with w_k=0 for $k\in\boldsymbol{I}_0$.
Ω_1	: the subset of portfolios consists of P=(w_1, … , w_K) with w_k=1 for $k\in\boldsymbol{I}_1$.
Ω_A	: the subset of portfolios consists of P=(w_1, … , w_K) with w_k=0 for $k\in\boldsymbol{I}_A$.
Ω_C	: the subset of portfolios consists of P=(w_1, … , w_K) with w_k=1 for $k\in\boldsymbol{I}_C$.
Ω_{hp}	: the subset of portfolios consists of P=(w_1, … , w_K) with w_h=0 and w_p=1.
$\boldsymbol{E}$	: the set consists of efficient portfolios.
$\boldsymbol{N}$	: the set consists of ine[illegible]
$\lvert\cdot\rvert$	: the number of elemen[illegible]
Θ	: the union of index se[illegible]
$\overline{w}_k$	: the binary variable tr[illegible] $k\in\Theta$.
$\overline{c}_k$	: the coefficient transf[illegible]
$\overline{a}_k$	: the coefficient transf[illegible]
$\overline{\boldsymbol{I}}_P$	: the set consists of index of project k with $\overline{c}_k > 0$ and $\overline{a}_k > 0$.
$\overline{\boldsymbol{I}}_0$	: the set consists of index of project k with $\overline{c}_k > 0$ and $\overline{a}_k \le 0$.
$\overline{\boldsymbol{I}}_1$	: the set consists of index of project k with $\overline{c}_k \le 0$ and $\overline{a}_k > 0$.
N_P	: the number of elements in $\overline{\boldsymbol{I}}_P$.
N_0	: the number of elements in $\overline{\boldsymbol{I}}_0$.
N_1	: the number of elements in $\overline{\boldsymbol{I}}_1$.
$\Lambda(k)$	: the reference set of project k by using additive model.
n_j	: the number of portfolio flows in Phase II of filtering algorithm, j = 1, … , 10.
T_{current}	: the portfolio currently under evaluation in filtering algorithm.
T_{next}	: the next portfolio will being evaluated in filtering algorithm.

1. Introduction

1.1 Motivation and background

Decision-making problems involve both quantitative and non-quantitative factors. The non-quantitative factors are not usually well defined or are subjectively determined by the decision-maker. Such factors cannot be included in the mathematical models while the quantitative factors are modeled as multiple objective linear programming (MOLP). The coefficients in MOLP may obtainable, well defined, or not sensitive to the final solution. An example of MOLP may be projects of government investment, in which the minimization objective functions (inputs) may be manpower, machines, construction costs, operation costs, other controllable costs and uncontrollable costs while the maximization objective functions (outputs) may be revenues, rate of population growth, growth of economic improvement.

Project selection problems have received substantial attention in recent decades (Martino, 1995). This research [illegible] selection and evaluation of collective projects from a feasible set [illegible]any difficulties associated with the evaluation problems of collec[illegible]multiple conflicting objectives, non-quantitative objects, and the en[illegible]ble combinations. In this paper, each subset of the projects is trea[illegible]o, and evaluated against a relative production technology. Many researchers have proposed the evaluation and selection of projects in a portfolio (Oral et al., 1991; Cook & Green, 2000; Linton et al., 2002). It is desired to establish the portfolios of projects that can be justified as making the best use of available resources. It involves the evaluation, from a larger set of projects, of each portfolio to be undertaken. The problem discussed here falls firmly into the multiple criteria decision-making (MCDM) arena.

In MCDM, there are a number of alternatives among which a decision-maker must decide. Each alternative is described by its performance according to certain criteria, attributes, or objectives. Stewart (1996) defines a criterion as being a particular point of view according to which alternatives may be assessed and rank-ordered. An attribute is a particular feature of the alternative with which a numerical measure can be associated. An objective is a specific direction of preference defined in terms of an attribute. The aim of MCDM is to provide support to a decision-maker in making the best choice among alternatives, and to propose the 'optimal' solution under some form or preference ranking.

Data envelopment analysis (DEA) is a robust and valuable methodology for frontier estimation (Charnes et al., 1978). Based on mathematical programming techniques, it is particularly suited to estimating multiple input and output production correspondence. In the last two decades, DEA has become a popular method for analyzing the efficiency of various organization units (Norman & Stoker, 1991) which differ both in the quantities of inputs they consume and in the outputs they produce, and does not require any subjective or economic parameters (weights, prices, etc.). Many studies have been concerned with the efficiency of production. It is clear that DEA is now playing a wider role in management science. In particular, DEA approaches have assumed important status within the toolkits of investigators concerned with MCDM (Joro et al., 1998).

It is worthwhile to identify the role of our problem in the related academic studies. DEA and MCDM are two related techniques that have received considerable attention in the OR/MS literature. Many papers have proposed to analyze the links between DEA and MCDM (Belton & Vickers, 1993; Stewart, 1996; Joro et al., 1998; Sarkis, 2000). The success of DEA in the area of performance eval[illegible]gies between DEA and MCDM, has led some authors to propose DE[illegible] Doyle & Green, 1993; Stewart, 1994; Bouyssou, 1999; Liu et al., 200[illegible]eless, to be little interaction between these two sub-fields, despite th[illegible]ther similar problems. In general, the aim of DEA is not to select on[illegible]g unit (DMU), but rather to separate efficient DMUs from inefficient ones and to indicate the 'efficient peers' for each inefficient DMU. The MCDM and DEA formulations coincide (although their ultimate aims may still differ) if we view inputs and outputs as criteria or attributes for evaluating DEA, with minimized inputs and maximized outputs as associated objectives (Belton & Vickers, 1993).

Many researchers have discussed the project selection problems in various forms. Bunch et al. (1989) apply DEA additive model to solve the problems, Oral et al. (1991) depart from the DEA CCR model and propose a rather complex multi-stage collective evaluation and selection model, which is called the OKL point. Cook & Green (2000) follow the OKL point to solve the resource-constrained project selection problem by using mixed-integer programming.

1.2 Problem definition

Suppose a set of K candidate project proposals numbered $k = 1, \dots , K$ is somehow to be evaluated and selected. Project k consumes amounts of a_{ik}, $i = 1, \dots , m$ resources to produce c_{rk}, $r = 1, \dots , s$ products. A portfolio comprises a subset of the K feasible projects is denoted by $P = (w_1, \dots , w_K)$, where $w_k = 1$ if the k^{th} project belongs to portfolio P and $w_k = 0$ otherwise. Let Ω denote the set of all feasible portfolios where:

$$\Omega = \{P = (w_1, \dots , w_K) \mid w_k = 0 \text{ or } 1, k = 1, \dots , K.\}. \tag{1.1}$$

Let n be the number of total possible portfolios in set Ω under evaluation, $n = \|\Omega\| = 2^K$. It is assumed that the projects are neither synergistic nor interfering, and all portfolios are supportable since resource constraints are absent for a decision maker. If both projects were selected, the outputs produced would be the sum of their respective outputs, and so as the input resources used. The correspondence set of DMUs is:

$$\Omega_D = \{DMU_P = (y_{1P}, \dots , \tag{1.2}$$

where $y_{rP} = c_{r1}w_1 + \dots + c_{rK}w_K$, r $_1 + \dots + a_{iK}w_K$, $i = 1, \dots , m$. Then, the collective evaluation problem i ultiple objective binary integer linear programming (MOBILP):

Maximize $y_{rP} = c_{r1}w_1 +$. (M1)

Minimize $x_{iP} = a_{i1}w_1 + \dots + a_{iK}w_K$, $i = 1, \dots , m$.

Subject to $P \in \Omega$.

For solving model (M1), some different methods are proposed in Keeney & Raiffa (1976) and Steuer (1986). Difficulties arise due to disagreement between various interested parties concerning its form and detail. Instead of considering optimization of the criteria, a DEA-based approach circumvents these difficulties by allowing each portfolio to evaluate itself relative to all portfolios under consideration. DEA is intended to identify efficient portfolios, to characterize inefficient portfolios, and to assess from where inefficiencies arise.

However, DEA methodology is computationally intensive, requiring the solution of n mathematical programs when analyzing a data set that comprises n DMUs. As discussed in Ali (1990; 1992; 1994), identification of efficient and inefficient DMUs without solving a DEA program is very useful in streamlining the solution of DEA computations. In this study, we present mathematical properties to characterize the inherent relationships between efficiency of portfolios and data of projects. By using the output-input ratio of individual project, efficient and inefficient portfolios are identified prior to the DEA program. The frontier of the pre-identified efficient portfolios is developed as a filter and is used to characterize inefficient portfolios from the class of candidate efficiencies. Inefficiency of portfolios is identified with portfolios that lie within the DEA frontier. The case-based computer systems use linear programming (LP) with a small problem size to rapidly identify a large number of inefficient portfolios. Then, the remaining portfolios are evaluated by using DEA programs to identify efficient units and measure the stability of each efficient unit to rank all efficient units for the decision aim.

A large number of alternatives would be ruled out from final decision. There are many ways to use the solution of our method to obtain the final decision under the consideration of non-quantitative factors, such as follows: (i) Compare the super-efficiencies of all the efficient portfolios, (ii) Sensitivity analysis on the coefficients so that a specific extremely efficient portfolio becomes inefficient, and (iii) Sensitivity analysis on the coefficients so that a particular inefficient portfoli[illegible]efore, the effort for making the final decision is significant reduced.

The literatures of sensit[illegible]eal with only change values of input and/or output of one particula[illegible] the other DMUs are hold fixed, or change data of all efficient DM[illegible]ding some given rules. To investigate the stability of each efficient portfolio with respect to the coefficients of a specific project, the super-efficiency measure could not satisfy our requirements, since the portfolio consists of some projects. For a specific efficient portfolio, we are considering the stability of the portfolio while we are changing data of some portfolios through changing the coefficients in the inputs and outputs of a particular project. When all the stability measures are obtained, they are helpful to the final decision maker to possess the fine comparison of efficient portfolios.

1.3 Objectives of the research

Without predetermined the weights of the objectives, we use DEA to measure the efficiency and stability of each portfolio. The objective is to select and rank portfolios that are efficient in terms of the characteristic of DEA. The difficulty of the DEA analysis may spend more effort on computations while the number of portfolios (DMUs) tends to be large. In our problem, the total number of alternatives is 2^K, and it could be doubled when we added one more project to the MOBILP (M1). If use the conventional DEA model to assess each portfolio against the 2^K portfolios, one needs to solve a linear programming models with 2^K variables and (m+s) constraints. For instance, if K equals to 30, one needs a linear programming software package with the capacity to accommodate the 2^{30} variables. It may reach the capacity of existing software and the personal computers. The problem with K value beyond 30 would not be solved. The computation time is the other issue has to be conquered. In our experiment, for the case K=24, we spent more than one day to have final solution.

We develop an efficient method to identify the efficient portfolios for MOBILP with single minimization (input) and minimization (output) problem. One does not need to employ linear programming to obtain the solution. For the MOBILP with multiple minimization and maximization objective functions, an efficient and effective process for identifying inefficient portfolios is proposed to reduce the computation prior to the DEA programs, and identifying some efficient portfolios wh[illegible] implement the filtering algorithm. Therefore, all of the efficient [illegible] espondence efficiency measures are obtained by using the proposed [illegible]

The inputs and outputs [illegible] ectively obtained from the sum of the input and output of the selecte[illegible] anges of any one coefficient in (M1) would change a half number [illegible] at contain the changed project. The efficiency measures of those portfolios may be changed while the coefficient is perturbed. For instance, if the coefficient, say a_{ik}, is changed, all the portfolios with w_k=1 are changed respectively while the other half portfolios are remain unchanged. Our purpose concerns the perturbation of coefficients, a_{ik} and c_{rk}, of project k in an interested efficient portfolio to preserve its efficiency. We are considering the stability of an extremely efficient portfolio while we are changing the inputs and outputs of some portfolios through changing the coefficients of objective functions of a particular project (binary decision variable). The sensitivity analysis for the coefficients is modeled as a non-linear programming whose optimal values yield a stability region of an extremely efficient portfolio. Sufficient and necessary conditions are provided for upward variations of a_{ik} and downward variations of c_{rk} for a specific project such that an extremely efficient portfolio remains efficiency. A technique using linear programming to approximate the optimal solution to the non-linear programming also proposed.

1.4 Organization of the dissertation

The second chapter reviews the related literature in MCDM, DEA and its sensitivity analysis. Chapter three introduces an efficient process for constructing efficient frontier. The output-input ratio analysis for quickly identify dominated portfolios are proposed. Then, a filtering algorithm is used to solve the MOBILP (M1). Chapter four proposes the sensitivity analysis for DEA models. Non-linear models are proposed for finding the stability regions of efficient portfolios with respect to the data changed in project. The method that uses linear programming model to approximate non-linear programming stability model is also provided. Conclusion and discussion are presented in chapter five. The structure of this study is illustrated in Figure 1.

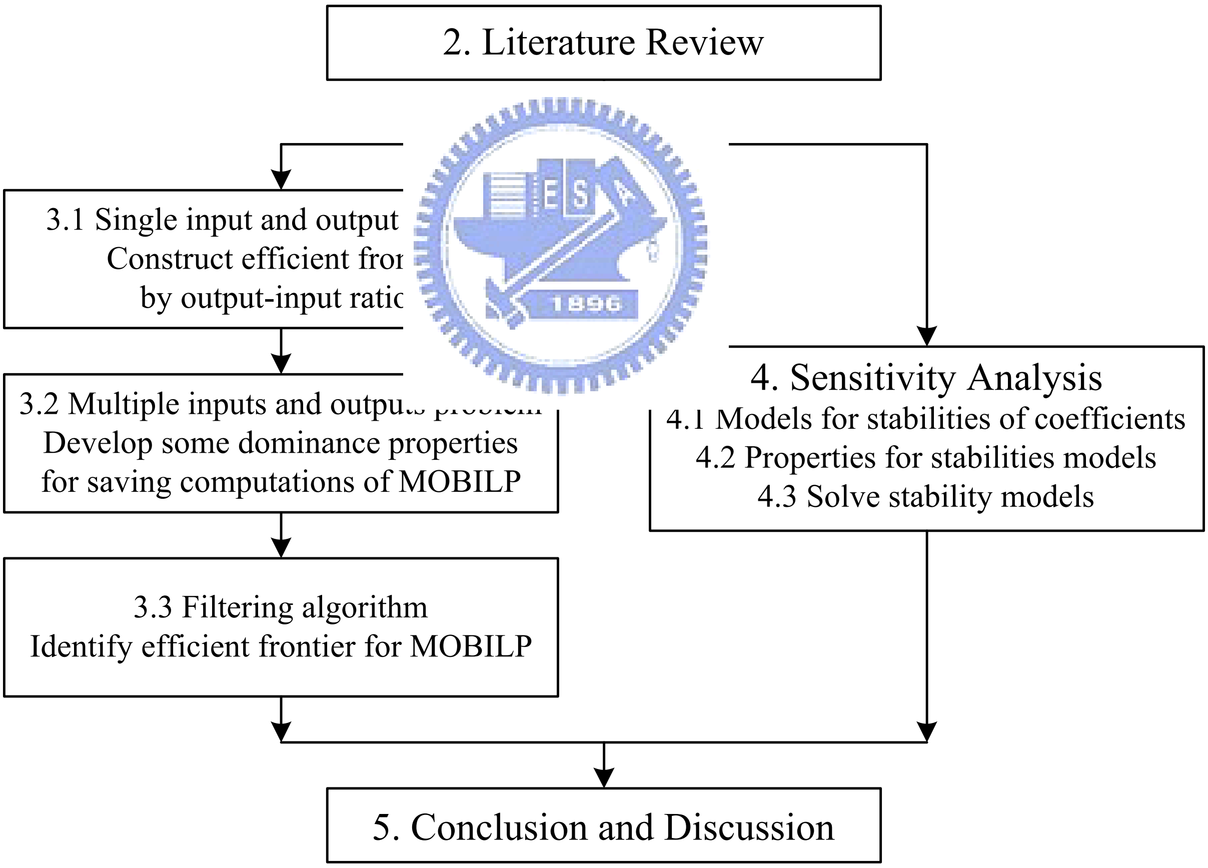

Figure 1. Organization of dissertation.

2. Literature Review

2.1 Multiple criteria decision making

The single objective mathematical programming problems are studied extensively in the past 40 years. However, single objective decision making methods reflected an earlier and simpler era. The world become more complex as we enter the information age. We find that almost every important real-world problem involves more than one objective, and decision makers find it imperative to evaluate solution alternatives according to multiple criteria. We now need to extend the single criterion problems to the multiple criteria problems. A MCDM mathematical programming is expressed as the following:

$$\text{Maximize} \quad \{f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_J(\boldsymbol{x})\} \qquad \text{(M2)}$$
$$\text{Subject to} \quad \boldsymbol{x} \in \boldsymbol{S}.$$

Where $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x})$, … ves whether it be linear, integer, or nonlinear, and $\boldsymbol{S}$ is the set of f objectives are all linear, it is called MOLP problem. In single obje must settle on a single objective such as minimizing cost or maximi the real-world applications, we will almost certainly be able to ider criteria. For example, the investment planning problems use the f sure as criteria: maximize {return, dividends} and minimize {risk, derivations from diversification goals}. A point in $\boldsymbol{S}$ is optimal if it maximizes the decision-maker's objectives. A point in $\boldsymbol{S}$ is 'efficient' if and only if its criterion vector is non-dominated. To be optimal, a point must be efficient. 'Inefficient' solutions are not candidates for optimality.

The success of DEA in the area of performance evaluation together with the formal analogue existing between DEA and MCDM have led some authors to propose to use DEA as a tool for MCDM. The DEA methodology is briefly reviewed in the following.

2.2 DEA models

As first developed by Charnes et al. (1978), DEA is a methodology used for assessing the relative efficiency of DMUs. DEA is a set of methods and models based on mathematical programming and used for characterizing efficiencies and inefficiencies of DMUs with the same multiple inputs and outputs. In this research, additive model and BCC model are used to identify the efficient portfolios. These models are briefly reviewed.

2.2.1 Additive model

The additive model, presents in Charnes et al. (1985a), is used to introduce the concepts of DEA. When portfolio $T \in \Omega$ is under evaluation, the model is set to evaluate its corresponding DMU, $DMU_T = (y_{1T}, \ldots, y_{sT}, x_{1T}, \ldots, x_{mT})$, as the following:

$$
\begin{aligned}
\text{Min } & z_T = -\sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_i^- \\
\text{s.t. } & \sum_{P\in\Omega} \lambda_P y_{rP} - s_r^+ = y_{rT}, \quad r = 1, \ldots, s, \\
& -\sum_{P\in\Omega} \lambda_P x_{iP} - s_i^- = -x_{iT}, \quad i = 1, \ldots, m, \\
& \sum_{P\in\Omega} \lambda_P = 1, \\
& \lambda_P \geq 0, P \in \Omega;\ s_r^+ \geq 0,\ r = 1, \ldots, s;\ s_i^- \geq 0,\ i = 1, \ldots, m.
\end{aligned}
\qquad \text{(M3)}
$$

The additive model rel[illegible]s to the economic concept of Pareto optimality. The optimal value, [illegible]rating that measures the distance that the particular DMU being rate[illegible]f the optimal value to model (M3) is equal to zero, then DMU_T is [illegible] The thinking seems to be that the observed portfolio T is in a se[illegible]er population of potential portfolios, whose input-output combinations are assumed to belong to a convex production possibility set (Charnes et al., 1985a). If the optimal value to model (M3) is non-zero, then DMU_T is not optimal for any linear aggregation of inputs and outputs, and is either dominated, or dominated by a convex combination of the inputs and outputs of two or more DMUs (i.e., convex-dominated). Thus, DMU_T is efficient if and only if z_T^*=0. The DMU_T is inefficient if it does not lie on the frontier. For example, if any component of the slack variables, s_i^{+*} or s_r^{-*} is not zero, the value of the nonzero component will identify the sources and amounts of inefficiency in the corresponding outputs and inputs.

The property of translation invariance for additive model is presented in Ali & Seiford (1990). They indicate that the efficient DMUs are preserved efficiency by varying input and/or output in the same value to all DMUs.

2.2.2 BCC model

The BCC model (Banker et al., 1984) separates the inefficiency into technical efficient and scale inefficiency. A new separate variable, u_0, is introduced which makes it possible to determine whether operations are conducted in regions of increasing, constant, decreasing return to scale in multiple input and output situations. The particular point of selected projection is dependent on the employed DEA model and the orientation. For instance, in an input orientation BCC model, one focuses on maximal movement toward the frontier through proportional reduction of inputs, whereas in an output orientation, one focuses on maximal movement via proportional augmentation of outputs. When portfolio $T \in \Omega$ is under evaluation, the BCC models with an input orientation are presented as the followings:

$$
\begin{aligned}
\text{Min } & \eta_T = \theta_T - \varepsilon \sum_{r=1}^{s} s_r^+ - \varepsilon \sum_{i=1}^{m} s_i^- \\
\text{s.t. } & \sum_{P \in \Omega} \lambda_P y_{rP} - s_r^+ = y_{rT}, \quad r = 1, \ldots, s, \\
& \theta_T \, x_{iT} - \sum_{P \in \Omega} \lambda_P x_{iP} \\
& \sum_{P \in \Omega} \lambda_P = 1, \\
& \lambda_P \ge 0, P \in \Omega; \; s_r^+ \quad i = 1, \ldots, m.
\end{aligned} \qquad \text{(M4)}
$$

Its dual form is as the fo

$$
\begin{aligned}
\text{Max } & \omega_T = \sum\nolimits_{r=1}^{s} \mu_r y_{rT} + u_0 \\
\text{s.t. } & \sum\nolimits_{i=1}^{m} v_i x_{iT} = 1, \\
& \sum\nolimits_{r=1}^{s} \mu_r y_{rP} - \sum\nolimits_{i=1}^{m} v_i x_{iP} + u_0 \le 0, \quad P \in \Omega, \\
& -v_i \le -\varepsilon, \quad i = 1, \ldots, m, \\
& -\mu_r \le -\varepsilon, \quad r = 1, \ldots, s, \\
& u_0 : \text{free in sign.}
\end{aligned} \qquad \text{(M5)}
$$

Several new constructions appear in this BCC model formulation. The variable θ_T appears in the primal problem and an infinitesimal constant, ε, appears both in the primal objective function and as a lower bound for the multipliers in the dual problem. The scalar variable θ_T is the proportional reduction applied to all inputs of DMU_T to improve efficiency. This reduction is applied simultaneously to all inputs and results in a radial movement toward the envelopment surface. The infinitesimal constant, ε, in the primal objective function effectively allows the minimization over θ_T to preempt the optimization involving the slacks. Evidently, the following two statements are equivalent:

1. A DMU is efficient if and only if the following two conditions are satisfied:

(a) the optimal θ_T^* =1, and

(b) all slacks and surpluses are zero.

2. A DMU is efficient if and only if $\omega_T^* = \eta_T^* = 1$.

Both Additive and BCC models are of the variables return to scale (VRS) DEA models (Charnes et al., 1994). Based on the DEA perspective, efficiency should be measured by the distance from the efficient frontier, as hinted by model (M3)–(M5). But, the usual DEA definition is based on the following BCC ratio form. When portfolio $T \in \Omega$ is under evaluation, the model is expressed as the following:

$$\text{Max } \xi_T = \frac{\sum_{r=1}^{s} \mu_r y_{rT} + u_0}{\sum_{i=1}^{m} v_i x_{iT}}$$

$$\text{s.t.} \quad \frac{\sum_{r=1}^{s} \mu_r y_{rP} + \ldots}{\sum_{i=1}^{m} v_i x_{iP}} \qquad \text{(M6)}$$

$$v_i \geq 0, \ i = 1 \ldots s; \ \text{and } u_0 \text{ free.}$$

If the optimal value to r… …e, then DMU_T is located on the VRS frontier. The ratio ξ_T is given by… …ve weights μ_r and v_i to multiply to its outputs and inputs, respectively. Essentially, each DMU_T is allowed to rate itself as highly as possible via ratio ξ_T and restrict no DMU to reach a rating greater than one under the given weights.

2.2.3 Output-input ratio and frontier

Chen & Ali (2002) use the output-input ratio to identify DEA frontier DMUs prior to the DEA calculation. They conclude that the output-input ratio with top-ranked performance is a DEA frontier DMU.

Theorem 2.1 If there exist weight combinations of $\tilde{v}_i \geq 0$, $i = 1, \ldots, m$, $\tilde{\mu}_r \geq 0$, $r = 1, \ldots, s$, and $\tilde{u}_0$, such that

$$\text{(i)} \quad \frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rT} + \tilde{u}_0}{\sum_{i=1}^{m} \tilde{v}_i x_{iT}} = \max_{P \in \Omega} \left\{ \frac{\sum_{r=1}^{s} \tilde{\mu}_r y_{rP} + \tilde{u}_0}{\sum_{i=1}^{m} \tilde{v}_i x_{iP}} \right\} \qquad (2.1)$$

or

$$\text{(ii)} \quad \frac{\sum_{i=1}^{m} \tilde{v}_i x_{iT} - \tilde{u}_0}{\sum_{r=1}^{s} \tilde{\mu}_r y_{rT}} = \max_{P \in \Omega} \left\{ \frac{\sum_{i=1}^{m} \tilde{v}_i x_{iP} - \tilde{u}_0}{\sum_{r=1}^{s} \tilde{\mu}_r y_{rP}} \right\}. \tag{2.2}$$

Then, DMU_T is located on the VRS frontier (Chen & Ali, 2002). ■

The properties allow using output-input ratio to identify the efficient DMUs without solving DEA mathematical programming problems. To illustrate the property, we consider the data set consists of 6 DMUs, D_1–D_6, each consuming one input, x_1, to produce two outputs, y_1 and y_2, as listed in Table 1. Columns 5-7 present the output-input ratios of y_1/x_1, y_2/x_1, and $(y_1+y_2)/x_1$, respectively. The ratios are calculated along with Theorem 2.1 by setting $\tilde{v}_1$=1, $\tilde{\mu}_1$=1, and $\tilde{\mu}_2$=1 to part (i).

Table 1. Data set with 6 DMUs.

	Outputs		Input				Efficient
DMU	y_1	y_2	[illegible]	[illegible]	y_2/x_1	$(y_1+y_2)/x_1$	classification
D_1	1	4	[illegible]	[illegible]	4^a	5	***F***
D_2	2	4	[illegible]	[illegible]	4^a	6	***E***
D_3	3	3.5	[illegible]	[illegible]	3.5	6.5	***E***
D_4	4	3	[illegible]	[illegible]	3	7^b	***E***
D_5	4	2	[illegible]	[illegible]	2	6	***F***
D_6	3	3	[illegible]	[illegible]	3	6	***N***

* ***E*** means efficient, ***F*** means inefficient on frontier, and ***N*** means inefficient inner frontier.

[a] The maximum ratio indicates the DMU is located on the frontier.

[b] The unique maximum ratio indicates the DMU is extremely efficient.

The ratio of y_1/x_1 indicates that D_4 and D_5 are located on the frontier, ratio of y_2/x_1 indicates that D_1 and D_2 are located on the frontier, and ratio of $(y_1+y_2)/x_1$ indicates that D_4 is located on the frontier. Hence, there are four DMUs, D_1, D_2, D_4, and D_5, locate on the efficient frontier. Unfortunately, the inefficient DMUs, D_1 and D_5, are also indicated. To avoid the misidentification of inefficient DMUs, Lai & Liu (2006) extend the property that allows using output-input ratio to identify the 'extremely' efficient DMUs without solving DEA programs. This following Corollary will indicate that the unique maximum value of ratio $(y_1+y_2)/x_1$ allows us to identify D_4 is VRS extremely efficient.

Corollary 2.1 If there exist weight combinations of $\tilde{v}_i \geq 0$, i=1, … , m, $\tilde{\mu}_r \geq 0$, r=1, … , s, and $\tilde{u}_0$ such that

(i) $\frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rT}+\tilde{u}_0}{\sum_{i=1}^{m}\tilde{v}_i x_{iT}} > \frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rP}+\tilde{u}_0}{\sum_{i=1}^{m}\tilde{v}_i x_{iP}}$ for all $P\in\Omega$ and $P\neq T$; (2.3)

or

(ii) $\frac{\sum_{i=1}^{m}\tilde{v}_i x_{iT}-\tilde{u}_o}{\sum_{r=1}^{s}\tilde{\mu}_r y_{rT}} < \frac{\sum_{i=1}^{m}\tilde{v}_i x_{iP}-\tilde{u}_o}{\sum_{r=1}^{s}\tilde{\mu}_r y_{rP}}$ for all $P\in\Omega$ and $P\neq T$. (2.4)

Then, DMU_T is VRS extremely efficient.

Proof: We first prove the part (i). For the weights of $\tilde{v}_i \geq 0$, i=1, … , m, $\tilde{\mu}_r \geq 0$, r=1, … , s, and $\tilde{u}_0$ we denote

$$t = \frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rT}+\tilde{u}_0}{\sum_{i=1}^{m}\tilde{v}_i x_{iT}}$$

Let $v_i = t\,\tilde{v}_i$, i=1, … , m, $\mu_r = \tilde{\mu}_r$, r=1, … , s, and $u_0 = \tilde{u}_0$ Then, we have

$$\xi_T = \frac{\sum_{r=1}^{s}\mu_r y_{rT}+u_0}{\sum_{i=1}^{m}v_i x_{iT}} = \text{[illegible]}\ {}_T+\tilde{u}_0 \ \text{[illegible]}\ {}_i x_{iT} = 1$$

and

$$\xi_P = \frac{\sum_{r=1}^{s}\mu_r y_{rP}+u_0}{\sum_{i=1}^{m}v_i x_{iP}} = \text{[illegible]}\ {}_P+\tilde{u}_0 \ \text{[illegible]}\ {}_i x_{iP} < 1, \text{ for all } P\in\Omega \text{ and } P\neq T.$$

It shows that weight combinations of v_i and μ_r takes the values to all constrains less than one, except the T^{th} constrains, and it has optimal value to one. Therefore, following the results of Charnes et al. (1991), DMU_T is VRS extremely efficient. The proof of part (ii) is analogous to part (i) and is omitted. ■

We observe that: there are m*s possible pairs of input i and output r, $i\in\{1, \ldots, m\}$ and $r\in\{1, \ldots, s\}$. If any one of the pairs satisfies the following Corollary, DMU_T is VRS extremely efficient (Lai & Liu, 2006).

Corollary 2.2 For any given pair of i' and r', $i'\in\{1, \ldots, m\}$ and $r'\in\{1, \ldots, s\}$. If there exists a weight combinations of $\tilde{v}_{i'} \geq 0$, $\tilde{\mu}_{r'} \geq 0$, and $\tilde{u}_0$, such that

$\frac{\tilde{\mu}_{r'} y_{r'T}+\tilde{u}_0}{\tilde{v}_{i'} x_{i'T}} > \frac{\tilde{\mu}_{r'} y_{r'P}+\tilde{u}_0}{\tilde{v}_{i'} x_{i'P}}$, for all $P\in\Omega$ and $P\neq T$. (2.5)

Then, DMU_T is VRS extremely efficient.

Proof: By taking $\tilde{v}_i = 0$, i=1, … , m, and i≠i', $\tilde{\mu}_r = 0$, r=1, … , s, and r≠r', we have:

$$\frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rT} + \tilde{u}_0}{\sum_{i=1}^{m}\tilde{v}_i x_{iT}} > \frac{\sum_{r=1}^{s}\tilde{\mu}_r y_{rP} + \tilde{u}_0}{\sum_{i=1}^{m}\tilde{v}_i x_{iP}} \quad \text{for all } P\in\Omega \text{ and } P\neq T.$$

Following the results of Corollary 2.1, DMU_T is VRS extremely efficient. ■

2.3 Sensitivity and stability analysis

DEA is non-parametric because it requires no assumption on the weights of the production function. Sensitivity and stability of DMUs is an important issue in DEA. Charnes et al. (1985b) first investigate the sensitivity of single output variation on the CCR model by updating the inverse of the optimal basis matrix. Charnes & Neralic (1990) use the same technique to explore the sensitivity of the additive model for a simultaneous change in all inputs and/or all outputs of an efficient DMU. Andersen & Petersen (1993) propose the 'extended DEA measure' (EDM) model for ranking the efficient units. The EDM model (is also called super-efficiency m[illegible]n the DEA sensitivity analysis. It is based on modifying DEA mode[illegible] is excluded from the reference set.

For DMU_T is under va[illegible] (1992; 1996) provide the following formulation to compute stabil[illegible]y classifications under the additive model:

$$\begin{aligned}
\Delta_T^* = \ & \text{Min } \ \Delta_T \\
s.t. \quad & \sum_{P\in\Pi, P\neq T}\lambda_P y_{rP} + \Delta_T \geq y_{rT}, \quad r = 1, 2, \ldots, s, \\
& \sum_{P\in\Pi, P\neq T}\lambda_P x_{iP} - \Delta_T \leq x_{iT}, \quad i = 1, 2, \ldots, m, \\
& \sum_{P\in\Pi, P\neq T}\lambda_P \qquad\qquad = 1, \\
& \Delta_T : \text{free}; \quad \lambda_P \geq 0, \quad P\in\Pi, \ P\neq T.
\end{aligned} \qquad \text{(M7)}$$

The optimal value Δ_T^* is the radius of stability under the ∞-norm. The absolute increase of inputs and absolute decrease of outputs are considered only for DMU_T. If we use different Δ_i^I and Δ_r^O and minimize $\sum_{i=1}^{m}\Delta_i^I + \sum_{r=1}^{s}\Delta_r^O$, then the optimal solution provides the radius of stability under the 1-norm. The sign of the optimal value indicates the classification of the test DMU_T (Charnes et al., 1992). In the event of set Π comprising the whole DMU being evaluated, negative identifies inefficient units while positive identifies efficient units.

In the event of set Π is a subset of Ω and DMU_T excludes in Π is under evaluation, negative also identifies inefficient; however, positive indicates that DMU_T is located above the frontier of Π, it means that DMU_T has the possibility to perform better than Π, and it is classified as an efficient candidate. Based on the results, our study suitably selects a class of portfolios with higher performance relative to the others, which is called an 'efficient candidate group' (ECG) within our proposed algorithm which is called the 'filtering algorithm' in this paper. The main frame of our filtering algorithm is:

(i) Using model (M7) to evaluate DMU_T, where Π is substituted by set ECG.

(ii) If $\Delta_T^* < 0$, DMU_T is identified as inefficient.

Otherwise, DMU_T joins to ECG as a new membership.

Zhu (1996) uses the super-efficiency model to determine necessary and sufficient conditions for preserving efficiency of the efficient DMUs under the CCR model when data of the test efficient DMU was changed, and Seiford & Zhu (1998a) generalize the method to yield the entire stability region [illegible] literatures of sensitivity and stability analysis deal with the situation [illegible] ns are only applied to the test DMU.

However, possible data [illegible] DMU simultaneously or individually. Thompson et al. (1994) utiliz[illegible] entary Slackness Condition (SCSC) multipliers to analyze the stabi[illegible] when the data for all efficient DMUs were worsened and data for all [illegible] improved simultaneously. Seiford & Zhu (1998b) discuss the stability of efficient DMU based on a worst-case scenario in which the efficiency of the test DMU was deteriorating while the efficiency of all other DMUs were improving. They use super-efficiency models to find a range of stability for each efficient DMU to preserve efficiency when data variations occurred in all DMUs simultaneously. In the real-world problems, uncertain conditions could occur not only in single DMU or in all of DMUs but also in a particular local or regional subset of DMUs. It means that the possible data errors may occur in a subset due to the situations of local uncertainty.

In this research, we are interested in the stability of a specific efficient DMU_T while the data of a particular subset of DMUs, including DMU_T, is deteriorated simultaneously in the same value. Since either an increase of any output or a decrease of any input cannot worsen an efficient DMU, we consider the data was changed by giving upward variations in inputs or giving downward variations in outputs in a subset of DMUs.

3. Identification of Efficient Portfolios

The difficulty for using DEA to assess and select portfolios of collective projects is that there are 2^K portfolios need to be evaluated. We must spend more effort on intensive DEA calculation. The papers Ali (1990; 1992; 1994) present some properties to allow identification of efficient and inefficient DMUs without solving a mathematical programming. To circumvent the time-consuming DEA computations, we also derive some properties to identify efficient and inefficient classes prior to the DEA calculation for streamlining the solution of DEA programs.

3.1 Single input and output problems

Now, let us first consider the special case that the projects have only one input and output. The two objectives BILP model is expressed as follows:

Maximize $y = c_1w_1 + c_2w$ … (M8)

Minimize $x = a_1w_1 + a_2w$ …

Subject to $w_k \in \{0, 1\}$, …

3.1.1 relationship between rati… ency

Let R_k denote the ratio … for project k. That is, $R_k = c_k / a_k$. The relationship of dominance between two projects by the output-input ratios is defined as follows:

Definition 3.1 Project h dominates project p, if $R_h > R_p$. ■

We shall show that if project p is dominated by project h, and a portfolio includes the dominated project p but excludes project h, then the portfolio is inefficient.

Lemma 3.1 If $\frac{c_1}{a_1} \geq \frac{c_2}{a_2}$, where a_1, a_2, c_1, and c_2 are all positive. Then, $\frac{c_1}{a_1} \geq \frac{c_1 + c_2}{a_1 + a_2} \geq \frac{c_2}{a_2}$. ■

This property shows that $c_1 / a_1 \geq (c_1w_1 + c_2w_2 + \dots + c_Kw_K)/(a_1w_1 + a_2w_2 + \dots + a_Kw_K)$, for all portfolio $P = (w_1, w_2, \dots, w_K)$ in Ω and $P \neq (0, 0, \dots, 0)$. That is, $P = (1, 0, \dots, 0)$ possesses the maximum output-input ratio among the 2^K possible portfolios. Note that $P = (0, 0, \dots, 0)$ and $P = (1, 0, \dots, 0)$ are evidenced as CCR efficiency (Ali, 1994). The following Theorem will be used to characterize inefficient portfolios. Let e_k denote the unit row vector with 1 at the k^{th} component and 0 elsewhere.

Theorem 3.1 $T=(w_1, w_2, \ldots, w_K)$ with $w_h=0$ and $w_p=1$, is inefficient if project h dominates project p.

Proof: Let portfolios $H=T-e_p$ and $G=T+e_h$. The DMUs corresponding to portfolios H, T, and G are expressed respectively as the followings:

$$DMU_H=(y_H, x_H),$$
$$DMU_T=(y_T, x_T)=(y_H+c_p, x_H+a_p),$$

and

$$DMU_G=(y_G, x_G)=(y_H+c_h+c_p, x_H+a_h+a_p).$$

Let us take constant $t=a_p/(a_h+a_p)$. It thus follows:

$$\begin{aligned}(1-t)\,x_H + t\,x_G &= (1-t)\,x_H + t\,(x_H+a_h+a_p) \qquad (3.1)\\ &= x_H + t\,(a \\ &= x_H + a_p \\ &= x_T\,,\end{aligned}$$

and

$$\begin{aligned}(1-t)\,y_H + t\,y_G &= (1-t)\,y_H \qquad (3.2)\\ &= y_H + t\,(c \\ &= y_H + a_p\,(c_h+c_p)/(a_h+a_p) \\ &> y_H + a_p\,(c_p/a_p) \qquad \text{(By Lemma 3.1)} \\ &= y_T\,.\end{aligned}$$

It shows that DMU_T is convex-dominated by DMU_H and DMU_G. Therefore, DMU_T is DEA inefficient and so does portfolio T. ■

This Theorem enables us to identify efficient and inefficient portfolios prior to the DEA calculation by comparing the output-input ratios of pair of projects.

3.1.2 Efficient portfolios

Without loss of generality, it is assumed that the indices of projects are arranged according to the descendant order of their output-input ratios, i.e., $R_1>R_2>\cdots>R_K$, and the strict inequality holds here. The following Corollary uses ratio analysis to characterize the dominated portfolios, and like their correspondent DMUs, they are inefficient.

Corollary 3.1 Portfolio T=(w_1, w_2, … , w_K) is inefficient if w_k=0 and w_{k+1}=1 for some k.

Proof: Since $R_k > R_{k+1}$ implies that project k dominates project (k+1). Then, the result follows from Theorem 3.1. ■

Corollary 3.1 indicates that a project with larger output-input ratio must be selected prior to the others. Based on the result, only the remaining (K+1) portfolios that have the possibility of VRS efficiency. They are listed in the followings:

Table 2. The portfolio lists of candidate efficiency.

Portfolio	w_1	w_2	$\cdots$	w_{K-1}	w_K
0	0	0	$\cdots$	0	0
1	1	0	$\cdots$	0	0
2	1	1	$\cdots$	0	0
$\vdots$	[illegible]	[illegible]	[illegible]	$\vdots$	$\vdots$
$K-1$	[illegible]	[illegible]	[illegible]	1	0
K	[illegible]	[illegible]	[illegible]	1	1

The null portfolio (0, 0,…, 0) v[illegible] is clearly VRS efficient (Ali, 1994). The other K portfolios will be [illegible]y employing model (M6) to evaluate their corresponding DMUs. The [illegible]as the followings:

$$DMU_T=(x_T, y_T)=(a_1+a_2+\ldots+a_T,\ c_1+c_2+\ldots+c_T),\ T=1, 2, \ldots, K. \tag{3.3}$$

Theorem 3.2 DMU_T, T=1, … , K, are all VRS extremely efficient.

Proof: For each T, we have:

$$c_T a_k > a_T c_k \text{ if } k<T \text{ and } c_T a_k < a_T c_k \text{ if } k>T. \tag{3.4}$$

Let model (M6) be set to evaluate DMU_T by taking μ=a_T, v=c_T, and u_0=$c_T x_T - a_T y_T$. It is shown that μ, v, and u_0 is feasible to model (M6) and attach the objective θ_T=1. For all $k<T$, we have:

$$\begin{aligned}\frac{\mu y_k+u_0}{vx_k} &= \frac{a_T y_k+(c_T x_T-a_T y_T)}{c_T x_k}\\ &= \frac{a_T(c_1+\ldots+c_k)+c_T(a_1+\ldots+a_T)-a_T(c_1+\ldots+c_T)}{c_T(a_1+\ldots+a_k)}\\ &= 1+\frac{c_T(a_{k+1}+\ldots+a_T)-a_T(c_{k+1}+\ldots+c_T)}{c_T(a_1+\ldots+a_k)}\\ &<1.\end{aligned} \tag{3.5}$$

For all $k > T$, we have:

$$
\begin{aligned}
\frac{\mu y_k + u_0}{v x_k} &= \frac{a_T y_k + (c_T x_T - a_T y_T)}{c_k x_k} \\
&= \frac{a_T(c_1 + \ldots + c_k) + c_T(a_1 + \ldots + a_T) - a_T(c_1 + \ldots + c_T)}{c_T(a_1 + \ldots + a_k)} \\
&= 1 + \frac{a_T(c_{T+1} + \ldots + c_k) - c_T(a_{T+1} + \ldots + a_k)}{c_T(a_1 + \ldots + a_k)} \\
&< 1.
\end{aligned}
\tag{3.6}
$$

The equality holds only for k=T. This indicates that the optimal value to (M6) is equal to one. Therefore, DMU_T is VRS extremely efficient for T=1, … , K. ■

Hence, there are (K+1) VRS efficient portfolios obtained by using ratio techniques. Ratio analysis is shown to be an effective method to identify the entire set of efficient portfolios for the single input and output problems. To illustrate this, let us consider the following example.

3.1.3 Example 1: single input

Suppose there are five p , 5, in a decision set. Their input and output are given in Table 3. W cts have been arranged in descendent order of output-input ratios. A nprise a subset of the 5 projects are evaluated by the following unc

Maximize $y = 6\,w_1 + 4.0\,w_2 + 7.2\,w_3 + 8\,w_4 + 1\,w_5$ (M9)

Minimize $x = 4\,w_1 + 2.8\,w_2 + 5.6\,w_3 + 9\,w_4 + 2\,w_5$

Subject to $w_k \in \{0, 1\}$, k=1, 2,…, 5.

According to the results of Theorem 3.2, six portfolios, (0,0,0,0,0), (1,0,0,0,0), (1,1,0,0,0), (1,1,1,0,0), (1,1,1,1,0), and (1,1,1,1,1) are identified as VRS efficient.

Table 3. The data of 5 projects for Example 1.

Project	Output (c_k)	Input (a_k)	Ratio (R_k)
1	6.0	4.0	1.500
2	4.0	2.8	1.429
3	7.2	5.6	1.286
4	8.0	9.0	0.889
5	1.0	2.0	0.500

3.1.4 Problems with non-positive coefficients

The assumption that the positive coefficients $a_k > 0$ and $c_k > 0$ for all $k=1, \dots, K$, could be violated. Now, let us consider that the projects be partitioned based on the following six sets of indices:

$$\boldsymbol{I}_P=\{ k \mid 1\le k\le K,\ c_k > 0 \text{ and } a_k > 0\}, \tag{3.7}$$

$$\boldsymbol{I}_N=\{ k \mid 1\le k\le K,\ c_k < 0 \text{ and } a_k < 0\}, \tag{3.8}$$

$$\boldsymbol{I}_0=\{ k \mid 1\le k\le K,\ c_k > 0 \text{ and } a_k \le 0\}, \tag{3.9}$$

$$\boldsymbol{I}_1=\{ k \mid 1\le k\le K,\ c_k \le 0 \text{ and } a_k > 0\}, \tag{3.10}$$

$$\boldsymbol{I}_C=\{ k \mid 1\le k\le K,\ c_k < 0 \text{ and } a_k = 0\}, \tag{3.11}$$

and

$$\boldsymbol{I}_A=\{ k \mid 1\le k\le K,\ c_k = 0 \text{ and } a_k < 0\}. \tag{3.12}$$

The problem can be handled according to the following theorems.

Theorem 3.3 Portfolio $H=(w_1,$ [illegible] s DEA inefficient if $k\in\boldsymbol{I}_0$.

Proof: Let $T=(w_1, \dots, w_{k-1}, 1, w$ [illegible]

$$(-x_T, y_T) = (-x_H - a_k, y_H + \text{[illegible]} \tag{3.13}$$

This implies that portfolio H is [illegible]

Theorem 3.4 Portfolio $H=(w_1, \dots, w_{k-1}, 1, w_{k+1}, \dots, w_K)$ is DEA inefficient if $k\in\boldsymbol{I}_1$.

Proof: Let $T=(w_1, \dots, w_{k-1}, 0, w_{k+1}, \dots, w_K)$. It follows

$$(-x_T, y_T) = (-x_H + a_k, y_H - c_k) > (-x_H, y_H). \tag{3.14}$$

This implies that portfolio H is DEA inefficient. ■

Theorem 3.3 and 3.4 indicate that a portfolio is inefficient if it excludes a project consuming non-positive input to produce positive output, or it includes a project consuming positive input to produce non-positive output. Therefore, we have the following subsets of portfolios are inefficient:

$$\Omega_0=\{P=(w_1, \dots, w_K) \mid w_k=0, \text{ for any } k\in\boldsymbol{I}_0\} \tag{3.15}$$

and

$$\Omega_1=\{P=(w_1, \dots, w_K) \mid w_k=1, \text{ for any } k\in\boldsymbol{I}_1\}. \tag{3.16}$$

For the case that both a_j and c_j are non-positive occurs in model (M8). We redefine all binary variables and coefficients of objectives as the followings:

$$\overline{w}_k = \begin{cases} 1 - w_k, & \text{if } k \in \Theta = \boldsymbol{I}_N \cup \boldsymbol{I}_C \cup \boldsymbol{I}_A \\ w_k, & \text{Otherwise} \end{cases} \tag{3.17}$$

$$\overline{c}_k = \begin{cases} -c_k, & \text{if } k \in \Theta \\ c_k, & \text{Otherwise} \end{cases} \tag{3.18}$$

$$\overline{a}_k = \begin{cases} -a_k, & \text{if } k \in \Theta \\ a_k, & \text{Otherwise} \end{cases} \tag{3.19}$$

Then, model (M8) can be rewritten as follows:

$$\begin{aligned} &\text{Max } \; y = \overline{c}_1 \overline{w}_1 + \ldots + \overline{c}_K \overline{w}_K + \sum_{k \in \Theta} c_k \\ &\text{Min } \; x = \overline{a}_1 \overline{w}_1 + \ldots + \overline{a}_K \overline{w}_K + \sum_{k \in \Theta} a_k \\ &\text{s.t. } \; \overline{w}_k \in \{0, 1\}, \; k = 1, 2, \ldots, K \end{aligned} \tag{M10}$$

The new MOBILP mod [illegible] with non-negative coefficients, either $\overline{c}_k \geq 0$ or $\overline{a}_k \geq 0$, correspondin [illegible]. We can construct the new sets of indices $\overline{\boldsymbol{I}}_0$, $\overline{\boldsymbol{I}}_1$, and $\overline{\boldsymbol{I}}_P$ correspo [illegible] nd it follows that $\boldsymbol{I}_C \subseteq \overline{\boldsymbol{I}}_0$, $\boldsymbol{I}_A \subseteq \overline{\boldsymbol{I}}_1$, and $\boldsymbol{I}_N \subseteq \overline{\boldsymbol{I}}_P$. Then, the following se [illegible] are characterized by using Theorem 3.3 and 3.4.

$$\Omega_A = \{P = (w_1, \ldots, w_K) \mid w_k = 0 \text{ if } k \in \boldsymbol{I}_A\} \subseteq \{P = (\overline{w}_1, \ldots, \overline{w}_K) \mid \overline{w}_k = 1 \text{ if } k \in \overline{\boldsymbol{I}}_1\}. \tag{3.20}$$

and

$$\Omega_C = \{P = (w_1, \ldots, w_K) \mid w_k = 1 \text{ if } k \in \boldsymbol{I}_C\} \subseteq \{P = (\overline{w}_1, \ldots, \overline{w}_K) \mid \overline{w}_k = 0 \text{ if } k \in \overline{\boldsymbol{I}}_0\} \tag{3.21}$$

However, the new model (M10) transforms the objectives to non-negative coefficients and all efficient portfolios can be determined by using Theorem 3.2–3.4.

3.1.5 Algorithm for identification of efficient classification

A complete algorithm for developing all efficient portfolios is presented as follows:

Step 1. Identify sets of indices $\boldsymbol{I}_P$, $\boldsymbol{I}_N$, $\boldsymbol{I}_0$, $\boldsymbol{I}_1$, $\boldsymbol{I}_C$, and $\boldsymbol{I}_A$ according to (3.7)–(3.12).

Step 2. Reset original indices of projects in $\boldsymbol{I}_N$, $\boldsymbol{I}_C$, and $\boldsymbol{I}_A$ according to equations (3.17)–(3.19).

Step 3. Identify sets of indices $\bar{\boldsymbol{I}}_P$, $\bar{\boldsymbol{I}}_0$, and $\bar{\boldsymbol{I}}_1$, and let N_P, N_0, and N_1 denote the number of elements in set $\bar{\boldsymbol{I}}_P$, $\bar{\boldsymbol{I}}_0$, and $\bar{\boldsymbol{I}}_1$, respectively.

Step 4. Re-index all projects and rewrite model:

Step 4.1 Re-indexed project, $\overline{w}_k$, from 1 to N_p for $k \in \bar{\boldsymbol{I}}_P$, from (N_P+1) to (N_P+N_0) for $k \in \bar{\boldsymbol{I}}_0$, and from (N_P+N_0+1) to $(N_P+N_0+N_1)$ for $k \in \bar{\boldsymbol{I}}_1$.

Step 4.2 Rearrange $\overline{w}_k$ according to $\overline{R}_1 > \overline{R}_2 > \cdots > \overline{R}_{N_p}$ for $k \in \bar{\boldsymbol{I}}_P$, where $\overline{R}_k = \overline{c}_k / \overline{a}_k$.

Step 4.3 Original problem (M8) is rewritten as (M10).

Step 5. Identify the set consists of N_P+1 efficient portfolios as follows:

$$\Omega_E=\{P=(\overline{w}_1,\ldots,\overline{w}_K) \mid \overline{w}_k \geq \overline{w}_{k+1} \text{ if } k<N_P,\ \overline{w}_k=1 \text{ if } k\in \bar{\boldsymbol{I}}_0, \text{ and } \overline{w}_k=0 \text{ if } k\in \bar{\boldsymbol{I}}_1\}. \qquad (3.22)$$

3.1.6 Example 2: general two objectives BILP

Suppose there are 10 pr[...], 10, in a decision set. The values of input and output are given in [...] of portfolio evaluation in the set is modeled as (M8). The efficient [...]d by according the following steps:

Step1. Sets of indices based on [...]d as follows:

$\boldsymbol{I}_P=\{1, 4, 6\}$, $\boldsymbol{I}_N=\{5, 9\}$, [...]}, and $\boldsymbol{I}_A=\{7\}$.

Step 2. Reset original data of projects 5, 9, 3, and 7 according to (3.17)–(3.19).

Step 3. Identify sets of indices $\bar{\boldsymbol{I}}_P$, $\bar{\boldsymbol{I}}_0$, and $\bar{\boldsymbol{I}}_1$, and number of elements in these sets are N_P=5, N_0=2, and N_1=3, respectively.

$\bar{\boldsymbol{I}}_P=\boldsymbol{I}_P \cup \boldsymbol{I}_N=\{1, 4, 6, 5, 9\}$, $\bar{\boldsymbol{I}}_0=\boldsymbol{I}_0 \cup \boldsymbol{I}_C=\{8, 3\}$, $\bar{\boldsymbol{I}}_1=\boldsymbol{I}_1 \cup \boldsymbol{I}_A=\{2, 10, 7\}$.

Step 4. Use Step 4.1 and 4.2 to re-index all projects as the followings:

$\bar{\boldsymbol{I}}_P=\{1, 2, 3, 4, 5\}$, $\bar{\boldsymbol{I}}_0=\{6, 7\}$, $\bar{\boldsymbol{I}}_1=\{8, 9, 10\}$.

The relationship between origin and transformed index is listed in Table 4. Then, use Step 4.3 to rewrite the original problem as the followings:

$$\begin{aligned}
&\text{Max } y_P = 6\overline{w}_1 + 4.0\overline{w}_2 + 7.2\overline{w}_3 + 8\overline{w}_4 + \overline{w}_5 + \overline{w}_6 + 1.6\overline{w}_7 - 3.2\overline{w}_8 - 3\overline{w}_9 - 9.8 \\
&\text{Min } x_P = 4\overline{w}_1 + 2.8\overline{w}_2 + 5.6\overline{w}_3 + 9\overline{w}_4 + 2\overline{w}_5 - 2.4\overline{w}_6 + 1.5\overline{w}_8 + 2\overline{w}_9 + 2.5\overline{w}_{10} - 10.1 \quad \text{(M11)} \\
&\text{s.t.} \quad P \in \Omega.
\end{aligned}$$

Table 4. The original and transformed data of 10 projects.

Original data of projects			Transformed data of projects			
Index (k of w_k)	Output (c_k)	Input (a_k)	Index (k of $\overline{w}_k$)	Output ($\overline{c}_k$)	Input ($\overline{a}_k$)	Ratio ($\overline{c}_k/\overline{a}_k$)
6	6.0	4.0	1	6.0	4.0	1.50
4	4.0	2.8	2	4.0	2.8	1.43
9	–7.2	–5.6	3	7.2	5.6	1.30
1	8.0	9.0	4	8.0	9.0	0.89
5	–1.0	–2.0	5	1.0	2.0	0.50
8	1.0	–2.4	6	1.0	–2.4	—
3	–1.6	0	7	1.6	0	—
2	–3.2	1.5	8	–3.2	1.5	—
10	–3.0	2.0	9	–3.0	2.0	—
7	0	–2.5	10	0	2.5	—

Step 5. Using Theorem 3.2–3.4, we have 6 efficient portfolios which is listed as follows:

$(\overline{w}_1,\ldots,\overline{w}_{10}) = (0,0,0,0,0$ [illegible] $= (0,0,1,0,0,0,0,1,0,0)$,

$(\overline{w}_1,\ldots,\overline{w}_{10}) = (1,0,0,0,0$ [illegible] $= (0,0,1,0,0,1,0,1,0,0)$,

$(\overline{w}_1,\ldots,\overline{w}_{10}) = (1,1,0,0,0$ [illegible] $= (0,0,1,1,0,1,0,1,0,0)$,

$(\overline{w}_1,\ldots,\overline{w}_{10}) = (1,1,1,0,0$ [illegible] $= (0,0,1,1,0,1,0,1,1,0)$,

$(\overline{w}_1,\ldots,\overline{w}_{10}) = (1,1,1,1,0,$ [illegible] $= (1,0,1,1,0,1,0,1,1,0)$,

$(\overline{w}_1,\ldots,\overline{w}_{10}) = (1,1,1,1,1,1,1,0,0,0) = (w_1, \ldots, w_{10}) = (1,0,1,1,1,1,0,1,1,0)$.

3.2 Multiple inputs and outputs problems

When there are m inputs and s outputs to MOBILP (M1). Since, ratio analysis is shown to be an efficient method to identify the entire set of efficient portfolios for the case of single input and output. Based on the results of Theorem 3.2 and Corollary 2.2, the ratio analysis is capable of identifying a subset of efficient portfolios for the cases of multiple inputs and outputs. The MOBILP can be decomposed to ($s\times m$) sub-problems by the pairs of one output and one input. There are (K+1) efficient portfolios identified by each sub-problem. Corollary 2.2 also indicates that those efficient portfolios are also efficient for the original model.

By removing the duplications, the efficient portfolios identified by employing the ($s\times m$) sub-problems are aggregated as a subset. The subset is called the ‘seed efficient class’

(SEC). In our filtering algorithm, the frontier of ECG is the filter for the algorithm and ECG consists of those elements in SEC initially.

3.2.1 Inefficiency with project dominance relationship (PDR)

Let R_k^{ri} denote the ratio of the r^{th} output value to i^{th} input value of project k, where $R_k^{ri} = c_{rk} / a_{ik}$. The dominance relationship between two projects by the output-input ratios is defined as follows:

Definition 3.2 Project h dominates project p, if $R_h^{ri} \geq R_p^{ri}$ for all pairs of r and i, $i = 1, \ldots, m$, and $r = 1, \ldots, s$, and strict inequality holds for at least one pair of indices. ■

The relationship between output-input ratios of projects and the efficiency of portfolio to the multiple inputs and outputs problems is shown in Liu & Lai (2005a).

Theorem 3.5 Portfolio T=($w_1, \ldots, w_K$) is inefficient if project h dominates project p and w_h =0 and w_p =1.

Proof: Let $H = T - e_p$ and $G = T$ … onding to portfolios H, T, and G are expressed as follows:

DMU_H=($x_{1H}, \ldots, x_{mH}, y_{1H}$ …

DMU_T=(x_{1H}+$a_{1p}, \ldots, x_{mH}$ … ,

and

DMU_G=(x_{1H}+a_{1h}+$a_{1p}, \ldots, x_{mH}$+a_{mh}+a_{mp}, y_{1H}+c_{1h}+$c_{1p}, \ldots, y_{sH}$+c_{sh}+c_{sp}).

Let us take constants β_1 and β_2 as follows:

β_1= max{$c_{rp}/(c_{rh} + c_{rp})$| r=1, … , s.}

and

β_2= min{$a_{ip}/(a_{ih} + a_{ip})$| i=1, … , m.},

where $\beta_1, \beta_2 \in (0,1)$. Then, there exist specific indices i and r such that

$$\beta_1 / \beta_2 = (c_{rp} / (c_{rh} + c_{rp})) / (a_{ip} / (a_{ih} + a_{ip}))$$

$$= (c_{rp} / a_{ip}) / ((c_{rh} + c_{rp}) / (a_{ih} + a_{ip}))$$

$$< 1. \qquad \text{(by Lemma 3.1)}$$

It indicates that $\beta_1 < \beta_2$. Let β be a constant between β_1 and β_2. We shall show that DMU_T is convex-dominated by DMU_H and DMU_G. Since,

$$\begin{aligned}(1-\beta)\, x_{iH} + \beta\, x_{iG} &= x_{iH} + \beta\, (a_{ih} + a_{ip}) \\ &\le x_{iH} + \beta_1\, (a_{ih} + a_{ip}) \\ &\le x_{iH} + a_{ip} \\ &= x_{rT}, \qquad \text{for all } i = 1, 2, \ldots, m,\end{aligned}$$

and

$$\begin{aligned}(1-\beta)\, y_{rH} + \beta\, y_{rG} &= y_{rH} + \beta\, (c_{rh} + c_{rp}) \\ &\ge y_{rH} + \beta_2\, (c_{rh} + c_{rp}) \\ &\ge y_{rH} + c_{rp} \\ &= y_{rT}, \qquad \text{for all } r = 1, 2, \ldots, s,\end{aligned}$$

and at least one inequality holds. It shows that DMU_T is dominated by $(1-\beta)DMU_H+\beta DMU_G$. Therefore, DMU_T is inefficient and so does portfolio T. ■

It has shown that if project p is dominated by project h and a portfolio includes the dominated project p but excludes project h, then the portfolio must be inefficient. This enables us to identify efficient a prior to the DEA calculation by using the output-input ratio of an indi

3.2.2 Example 3: use ratio ana

A simulated data set c rojects in a high tech corporation is listed in Table 5. These projects are proposed to promote the product quality for the company. Each project consumes two inputs to produce two outputs. The outputs are percentages of technical contributions to the products and direct economic contributions in product sales, while the inputs are percentages of manpower usage and finance usage with respect to the company. Suppose that the projects are neither synergistic nor interfering and the resources are fully supported. The decision-maker wants to select a class of portfolios, from all of the 128 ($=2^7$) feasible portfolios, play the best practice with respect to the others.

By comparing the output-input ratios of projects, we have project 7 being dominated by project 6. Following the results of Theorem 3.5, we conclude that a portfolio is identified as inefficient if it contains project 7 but excludes project 6. That is, a portfolio is inefficient if it is expressed as the following form.

Table 5. Data set of 7 R&D projects for Example 3.

R&D project (k)	Technical contribution (c_{1k})	Product sales (c_{2k})	Manpower usage (a_{1k})	Resource usage (a_{2k})	R_k^{11}	R_k^{21}	R_k^{12}	R_k^{22}
1	1.8	7.0	3.0	6.0	0.600	2.333	0.300	1.167
2	1.6	10.0	4.0	5.5	0.400	2.500	0.291	1.818
3	1.4	8.2	3.6	4.5	0.389	2.278	0.311	1.822
4	1.9	13.0	5.0	7.0	0.380	2.600	0.271	1.857
5	1.4	5.0	6.0	4.0	0.233	0.833	0.350	1.250
6	1.8	12.0	8.0	3.0	0.225	1.500	0.600	4.000
7	1.7	6.0	9.3	4.0	0.183	0.645	0.425	1.500

$$(w_1, w_2, w_3, w_4, w_5, 0, 1) \text{ for } w_k=0 \text{ or } 1, k=1, 2, 3, 4, 5. \quad (3.23)$$

Hence, 32 portfolios are characterized as inefficient by using ratio analysis. Now, we turn to identify efficient portfolios by using Theorem 3.2. The ratios of output 2 to input 1, say R_j^{21}, of projects are ranked as follow [illegible]

$$R_4^{21} > R_2^{21} > R_1^{21} > R_3^{21} > \text{[illegible]} \quad (3.24)$$

It indicates that the [illegible] 0,0), (0,0,0,1,0,0,0), (0,1,0,1,0,0,0), (1,1,0,1,0,0,0), (1,1,1,1,0,0,0), ([illegible] 1,0), and (1,1,1,1,1,1,1), are efficient. Similarly, we can rank the r[illegible] to identify efficient portfolios. By removing the duplications, our ratio analysis identifies 23 efficient portfolios. These techniques identify 23 efficient and 32 inefficient portfolios prior to the DEA programs. In total, we save 55 computations for solving linear program effectively and efficiently.

3.2.3 Inefficiency with inferior project combination (IPC)

Apply additive model (M3) or (M7) to evaluate a particular project h with respect to the original K projects. The reference set is defined as $\Lambda(h)=\{ k \mid \lambda_k^* > 0, k = 1, 2, \dots, K\}$. Then, a portfolio is identified to be inefficient if it composes project h and without any element in set $\Lambda(h)$. That is, the portfolio comprises only inferior projects. This portfolio is called as an inferior project combination (IPC).

Theorem 3.6. Portfolio $T = (w_1, w_2, \dots, w_K)$ is inefficient if $w_h = 1$ and $\sum_{k\in\Lambda(h),k\neq h} w_k = 0$.

Proof: The result is trivial and is omitted. ■

One can use this Theorem to pre-identify some inefficient portfolios: just use model (M3) or (M7) to evaluate the K projects. It is clear that the situation occurs only if project k is inefficient with respect to the original K projects.

3.2.4 Inefficiency with total dominated relationship (TDR)

Ali (1994) defined a total dominated relationship (TDR) between DMUs. A portfolio is totally dominated if its corresponding DMU is dominated by any other DMU in Ω_D.

Definition 3.3 Portfolio T is totally dominated by portfolio H if DMU_T is dominated by DMU_H, that is, $x_{iT} \geq x_{iH}$, for all $i = 1, \ldots, m$, $y_{rT} \leq y_{rH}$, for all $r = 1, \ldots, s$, and strict inequality holds for at least one index. ■

Theorem 3.7. If portfolio T is totally dominated by portfolio H for some H then portfolio T is inefficient.

Proof: The proof is omitted. ■

3.3 Filtering algorithm

We propose a forward [illegible] lgorithm to solve the unconstrained MOBILP (M1). To reduce th[illegible] lel (M7) and to identify inefficient portfolios effectively, we subst[illegible] by a group of portfolios ECG with higher performance throughout the algorithm. ECG is updated dynamically by using forward and backward filtering algorithms. An algorithm comprising three phases is presented below.

3.3.1 Phase I: initialization

Phase I contains three parts. First, we re-index these K projects according to their stability measures obtained by model (M7). Next, we build some sub-filters to identify inefficient portfolios based on Theorem 3.5 and 3.6. Third, ECG is initialized according to Theorem 3.2.

Step 1.0. Read data of the K projects: c_{rk}, $r = 1, \ldots, s$, and a_{ik}, $i = 1, \ldots, m$; $k = 1, \ldots, K$.

Step 1.1. Use model (M7) to evaluate the K projects. Reassign indices of projects according to their stability measures, such that $\Delta_1^* \leq \Delta_2^* \leq \cdots \leq \Delta_K^*$.

Step 1.2. Use model (M7) to obtain $\Lambda(k)$ for each project k with $\Delta_k^* < 0$, and generate IPC filter based on the relationship between project k and $\Lambda(k)$ (Theorem 3.6).

Step 1.3. According to Theorem 3.5, generate the PDR filter for any pair of projects h and p, and identify whether the dominance relationship between h and p exists, $h, p=1, \dots, K$.

Step 1.4. According to Theorem 3.2, identify efficient portfolios based on ratio analysis. For a pair of specific indices r and i, output-input ratios are arranged in descending order $R_{(1)}^{ri} > R_{(2)}^{ri} > \cdots > R_{(K)}^{ri}$, where (k) is the index of project with k^{th} largest ratio. Repeat the process $m \times s$ times to collect all the efficient portfolios in set SEC, for $i=1, 2, \dots, m$ and $r=1, 2, \dots, s$. Then, the initialized ECG is equal to SEC.

3.3.2 Phase II: forward filtering

Phase II is a forward filtering algorithm, assessing possible portfolios one after the other. When a current portfolio T is under evaluation, the rules of identification are: (i) skipped if the portfolio is alrea e sub-filters, PDR, IPC, and TDR, to identify inefficiency, or (iii) u te DMU_T by setting the reference Π equal to ECG. However, if the aluated to be efficient with respect to ECG, it indicates that the portf being VRS efficient, and is added to the ECG. Figure 1 depicts the e notations $n_1, \dots, n_{10}$ are the number of portfolios that flow through the arcs, respectively.

Step 2.0. Start classification with the portfolio T that comprises all projects, $T = (1, 1, \dots, 1)$.

Step 2.1. Use PDR to identify whether T is inefficient. If it is, then go to Step 2.5.

Step 2.2. Use IPC to identify whether T is inefficient. If it is, then go to Step 2.5.

Step 2.3. Use TDR to identify whether T is inefficient with respect to ECG. If it is, then go to Step 2.5.

Step 2.4. Use model (M7) to identify whether T is inefficient with respect to ECG. If it is, then go to Step 2.5. Otherwise, ECG is augmented by portfolio T.

Step 2.5. Generate the next portfolio, T_{next}, from Ω by perform binary subtraction to current portfolio, T_{current}, That is, $T_{\text{next}} = T_{\text{current}} - 1$.

Step 2.6. As all 2^K portfolios are all evaluated, then go to Phase III. Otherwise, go to Step 2.1.

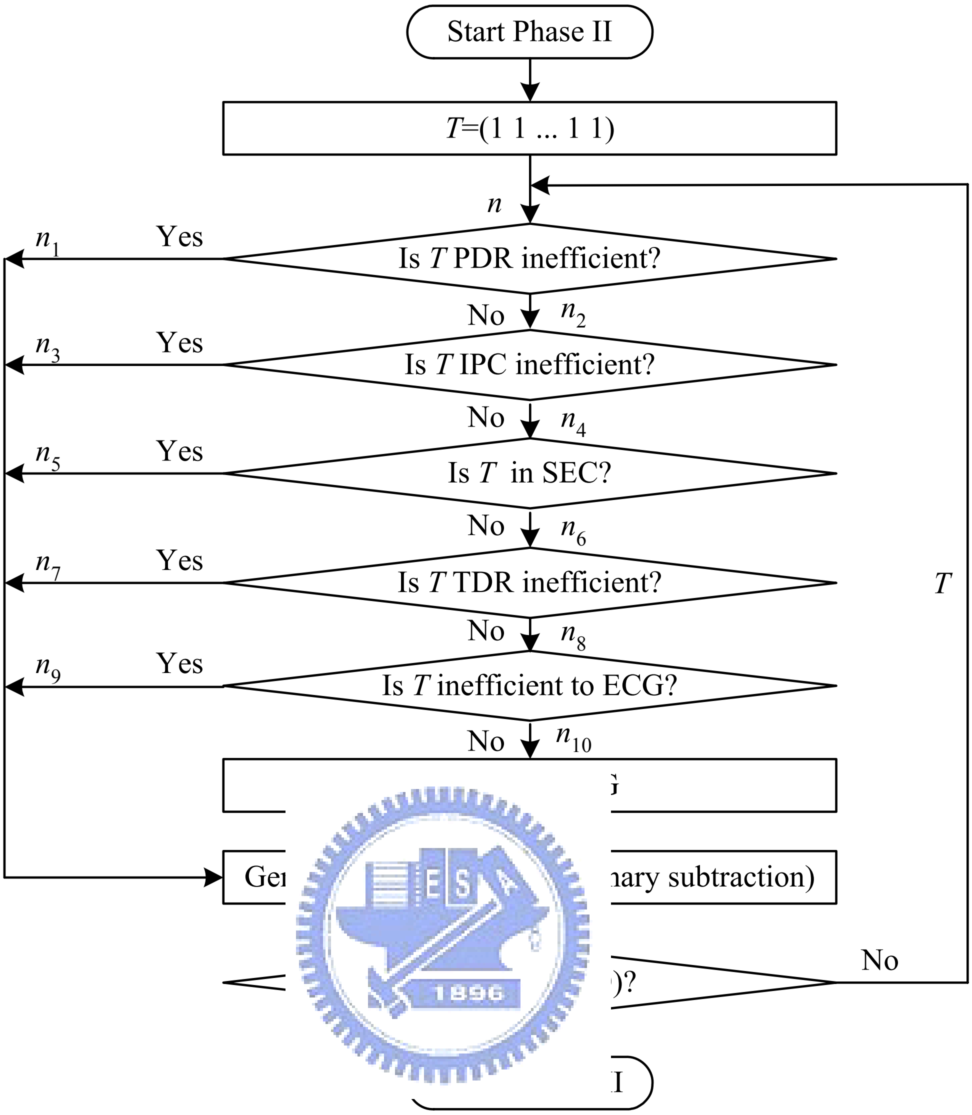

Figure 2. Flowchart of Phase II.

3.3.3 Phase III: reverse filtering

In this phase, we employ model (M7) to identify the efficiency of each portfolio, T, in ECG. A negative stability ($\Delta_T^* < 0$) indicates inefficient and T is erased from ECG, while a positive stability ($\Delta_T^* > 0$) indicates efficient and T remains in ECG. In case of $\Delta_T^* = 0$, we should perform the standard additive model (M3) to identify T. Finally, rank remaining portfolios in ECG according to their stability measures.

3.3.4 Design and computational issues

Phase I needs a little computation effort only. There are $K(K-1)/2$ pairs of projects to be checked to generate the PDR filter in Step 1.3. Each pair of projects h and p, needs $m \times s$

comparisons of output-input ratios. If project h dominates project p, then one quarter of the 2^K portfolios with w_h=0 and w_p=1 should be inefficient. The sub-filter PDR eliminates a large number of inefficient portfolios. Thus, a PDR filter is primarily used to reduce computation time and is therefore performed prior to the IPC and TDR filters.

To illustrate the fact, we consider the giving pair of projects, say h and p, and the subset of portfolios Ω:

$$\Omega_{hp} = \{P=(w_1, \ldots , w_K) \mid w_h = 0 \text{ and } w_p = 1.\} \tag{3.25}$$

The number of elements in Ω_{hp} is a quarter of total element in Ω. According to results of Theorem 3.5, all portfolios in class Ω_{hp} are identified as inefficient if project h dominates project p. Consequently, a quarter of the total portfolios could be saved from the computation of DEA evaluations. There are so many outcomes of the dominance relationship between projects. It is not worthwhile to list all of their savings in DEA computations.

Prior to the DEA calculation, Steps 2.1, 2.2 and 2.3 identified a large number of inefficient DMUs and a consid [illegible] nt DMUs. The DEA computations in Step 2.4 are effective and effici [illegible] decision variables is increased by one as an efficient DMU is identifi [illegible] GC and computational effort for the next evaluation to be increased [illegible]

The other major comput [illegible] backward filtering algorithm Phase III. In case of an inefficient portfolio is identified by the Step 3.2, it is deleted from ECG, the number of decision variables of model (M7) for the next evaluation is decreased by 1, and so the computational effort is reduced. The overall computational effort for the problem depends upon the problem size in terms of the values of K, s and m.

3.3.5 Performance of program MOBILP+

Table 6 depicts the performance of the algorithm. The first simulated data set D10 is the case of selecting the portfolios of projects K=10 with inputs, m=3, and outputs, s=2. The data c_{rk} and a_{ik} were randomly generated within the interval [10, 100]. For the cases consist of 15, 20, … , and 38 projects, the correspondence data sets are called D15, D20,…, and D38, respectively. We constructed a computer program *MOBILP+* coded in programming language *C++* to implement the algorithm and use the package, *CPLEX* (Ilog Inc., 2000), as linear programming solver.

Table 6. Number of portfolio flows and computing time of sample data sets.

Data set	K	n	PDR n_1	n_2	IPC n_3	n_4	SEC n_5	n_6
D10	10	2^{10}	769	255	7	248	47	201
D15	15	2^{15}	30329	2439	101	2338	80	2258
D20	20	2^{20}	994808	53768	2254	51514	108	51406
D22	22	2^{22}	4135987	58317	6360	51957	116	51841
D24	24	2^{24}	16705374	71842	14	71828	134	71694
D26	26	2^{26}	66995233	113631	172	113459	147	113312
D28	28	2^{28}	268018994	416462	25932	390530	160	390370
D30	30	2^{30}	1073456938	284886	41	284845	173	284672
D31	31	2^{31}	2146766030	717618	11887	705731	179	705552
D32	32	2^{32}	4294197549	769747	4049	765698	176	765522
D33	33	2^{33}	8588462091	1472501	35952	1436549	194	1436355
D34	34	2^{34}	17174422826	5446358	156788	5289570	195	5289375
D35	35	2^{35}	34357075944	2662424	6291	2656133	203	2655930
D36	36	2^{36}	68712503399	6973337	206324	6767013	203	6766810
D37	37	2^{37}	1374389276[illegible]	[illegible]	[illegible]	25837	206	25631
D38	38	2^{38}	2748736475[illegible]	[illegible]	[illegible]	4255382	221	4255161

Table 6. (*continued*)

Data Set	TDR n_7	n_8	(M[illegible] n[illegible]	[illegible]	[illegible]	Phase III $\|N\|$	$\|E\|$	Computing time* t_1	t_2
D10	17	184	11[illegible]	[illegible]	[illegible]	5	116	<1	<1
D15	681	1577	1252	325	405	28	377	1	702
D20	31051	20355	18680	1675	1783	424	1359	20	17216
D22	26247	25594	23494	2100	2216	663	1553	30	62150
D24	43821	27873	25886	1987	2121	594	1527	37	>24 hr
D26	47598	65714	61529	4185	4332	1785	2547	149	—
D28	227928	162442	156757	5685	5845	2503	3342	585	—
D30	184821	99851	95401	4450	4623	1690	2933	476	—
D31	503065	202487	194503	7984	8163	4180	3983	1487	—
D32	443881	321641	313035	8606	8782	3345	5437	2768	—
D33	972775	463580	451367	12213	12407	7310	5097	5140	—
D34	4752541	536834	521302	15532	15727	8664	7063	12267	—
D35	2037278	618652	606731	11921	12124	5776	6348	14801	—
D36	6005819	760991	742315	18676	18879	10563	8316	32833	—
D37	8314	17317	15761	1556	1762	717	1045	35788	—
D38	3332959	922202	904564	17638	17859	10603	7256	82891	—

* t_1 and t_2 are the computing time by using *MOBILP+* and using model (M7), respectively.

The program was executed on a Pentium IV-3.0 GHz desktop computer. The number of testing portfolios and computation time for each step are shown in Table 6. Column 3 shows that the number of feasible portfolios exponentially increased as the number of projects increased. The essential contribution of the four precedent filters could be observed from the number n_1, n_3, and n_7 of inefficient portfolios identified. The n_9 showed the numbers of inefficient portfolios identified by model (M7) in Step 2.4, and |***N***| and |***E***| are the numbers of inefficient and efficient portfolios identified by Step 3.2, respectively.

Step 2.4 is replicated n_6 times, that is, the number of optimization of model (M7) would be reduced to n_6 times, where (n–n_6) indicates the savings of computation from the three filters. More than 90% of portfolios are identified to be inefficient by the three filters prior to solving the DEA program. The benefit of using *MOBILP+* to streamline the computation of MOBILP (M1) can be easily seen from the sample data. The problem size in step 2.4 is also reduced significantly. The largest size of model (M7) in Step 2.4 is about n_9, which is less than 10% of por[illegible]s to optimize model (M7) is (n_6+n_9), which is also less than 10[illegible] procedure significantly reduces the computation time, especially f[illegible] and even less than 0.1% of time is needed.

The last two columns [illegible]omputing time required to execute *MOBILP+* and to solve mode[illegible] (|***E***|/n) indicates that the number of efficient portfolios to the total portfolios is very low. This allows the collective selection of projects to be handled effectively. Unfortunately, the results of D28 and D30 reveal that the computing time is data dependent, and D33, D34, D35 and D36 also indicate similar results. We found that the computing time is more dependent upon the number of efficient portfolios, |***E***|, but less dependent upon the number of projects, *K*.

The average and standard deviation of times to solve the 10 randomly generated data sets, each set comprised 20 simulated projects are listed in the first row of Table 7. The other 16 random samples, each sample also consists of 10 data sets, each data set comprised 21, 22,…, 36 simulated project was also solved. We discontinued the testing when the average time spend exceeds 24 hours. It seems that the expected computing times increase exponentially as the number of project *K* is increased. The algorithm would provide the solutions for selecting portfolios comprise 35 projects within one day. In our experiments, we observed that the standard deviations are highly relative to the mean, almost equal to the average for most of the cases. It indicates that the randomly generated data c_{rk} and a_{ik} have strongly affected the computation time.

Table 7. Average computational time of 10 random samples.

No. of project	Mean	SD	No. of project	Mean	SD
20	11	17	29	4939	9072
21	26	22	30	6068	7566
22	23	18	31	6251	5611
23	54	51	32	11191	9583
24	183	201	33	18198	18026
25	203	276	34	42648	62567
26	460	627	35	65311	61001
27	966	1211	36	>24 hr	>24 hr
28	2235	2737			

* Time unit: seconds.

4. Stability Analysis

As shown in model (M1), the input and output values of a portfolio are determined by summing the inputs and outputs of its performed projects, respectively. In this study, we focus on the perturbation of a particular coefficient a_{ik} (or c_{rk}) associated with a specific efficient portfolio with project k is performed (i.e. w_k=1). This research is focused on the stability of an efficient portfolio (DMU) by giving increase in an input, a_{ik}, or giving decrease in an output, c_{rk}, of a particular project k, if the portfolio remains efficient after the perturbation.

Let $\boldsymbol{I}$ and $\boldsymbol{O}$ denote the sets of indices of changed inputs and changed outputs, respectively. We consider the stability measures of coefficients a_{ik}'s and c_{rk}'s to preserve the efficiency of an efficient portfolio T, where project k is included in portfolio T. The data of project k is varied according to the following expressions:

$$\begin{cases} \hat{a}_{ik} = a_{ik} + \pi, & \pi \ge 0, \quad i \in \boldsymbol{I} \\ \hat{a}_{ik} = a_{ik}, & \end{cases} \tag{4.1}$$

and

$$\begin{cases} \hat{c}_{rk} = c_{rk} - \delta, & c_{rk} \ge \delta \ge \text{[illegible]} \\ \hat{c}_{rk} = c_{rk}, & \end{cases} \tag{4.2}$$

Hence, the varied input and out[illegible]ncluding project k is expressed as:

$$\begin{cases} \hat{x}_{iP} = x_{iP} + \pi, & \pi \ge 0, \quad i \in \boldsymbol{I} \\ \hat{x}_{iP} = x_{iP}, & \quad i \notin \boldsymbol{I} \end{cases} \tag{4.3}$$

and

$$\begin{cases} \hat{y}_{rP} = y_{rP} - \delta, & c_{rk} \ge \delta \ge 0, \quad r \in \boldsymbol{O} \\ \hat{y}_{rP} = y_{rP}, & \quad r \notin \boldsymbol{O} \end{cases} \tag{4.4}$$

The given type of data perturbation discussed in this paper is inconsistent with other sensitivity analyses, that inputs and outputs of the remaining portfolios are unchanged. There are a half of feasible portfolios will change their inputs and/or outputs, when we perturb a_{ik}'s and/or c_{rk}'s associated with a particular project k. Let Ψ_0 and Ψ_1 be the sets of portfolios with project k is not performed and performed, respectively. Where:

$$\Psi_0=\{P=(w_1, \dots, w_K)\in\Omega \mid w_k = 0\} \tag{4.5}$$

and

$$\Psi_1=\{P=(w_1, \dots, w_K)\in\Omega \mid w_k = 1\} \tag{4.6}$$

The inputs and outputs of DMU_P, $P \in \Psi_0$ is unchanged while the inputs and outputs of DMU_P, $P \in \Psi_1$ is changed, if the inputs and/or outputs associated with the perturbed project k are changed.

4.1 Models for stability evaluation

According to Charnes et al. (1991), the set of all DMUs can be partitioned into four classes, ***E***, ***E'***, ***F***, and ***N***. Where class ***N*** is located inner the frontier, class ***F*** is on the frontier but is also inefficient, and the first two classes are efficient. Zhu & Shen (1995) show that DMUs in class ***E'*** can be expressed as the linear combinations of the DMUs in class ***E***, and each of them will become inefficient if any increase of input and/or any decrease of output occurs. Thus, the literatures of DEA sensitivity analysis only focused on measuring stability of extremely efficient DMUs.

4.1.1 Stabilities of input coefficients

Based on the given abs investigated the stability of efficient DMUs in the additive model. A ient and data are changed in inputs as (4.3). We first consider the foll model to study the stability of DMU_T for perturbing the DMUs in Ψ_1 ject k (Liu & Lai, 2005b).

$$
\begin{aligned}
\pi^* &= \text{Min } \pi \\
s.t. \quad & \sum_{P \in \Psi_0} \lambda_P x_{iP} + \sum_{P \in \Psi_1,\, P \neq T} \lambda_P (x_{iP} + \pi) \le x_{iT} + \pi, \quad i \in \boldsymbol{I}, \\
& \sum_{P \in \Omega,\, P \neq T} \lambda_P x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I}, \\
& \sum_{P \in \Omega,\, P \neq T} \lambda_P y_{rP} \ge y_{rT}, \quad r = 1, 2, \ldots, s, \\
& \sum_{P \in \Omega,\, P \neq T} \lambda_P = 1, \\
& \pi \ge 0; \quad \lambda_P \ge 0, \quad P \in \Omega \text{ and } P \neq T.
\end{aligned}
\qquad \text{(M12)}
$$

Suppose the model is feasible for a given efficient DMU_T. This minimization is completed for indices $i \in \boldsymbol{I}$, and the optimal value is denoted by π^*. The properties of inputs stability region of DMU_T are shown below:

Theorem 4.1 Given data varied in the inputs as (4.3), an efficient DMU_T remains on the efficient frontier if and only if $\pi \in [0, \pi^*]$, where π^* is the optimal value to model (M12).

Proof: We first consider the following DEA model to evaluate DMU_T with DMU_P change their inputs by the value $x_{iP}+\pi^*$ for all $P\in\Psi_1$.

$$
\begin{aligned}
\theta^* = &\text{Min } \theta \\
s.t. \ &\sum_{P\in\Psi_0}\lambda_P x_{iP} + \sum_{P\in\Psi_1,\,P\neq T}\lambda_P(x_{iP}+\pi^*) + \lambda_T(x_{iT}+\pi^*) \le \theta(x_{iT}+\pi^*), \quad i\in\boldsymbol{I}, \\
&\sum_{P\in\Omega,\,P\neq T}\lambda_P x_{iP} \qquad + \lambda_T x_{iT} \qquad \le \theta\, x_{iT}, \qquad i\notin\boldsymbol{I}, \\
&\sum_{P\in\Omega,\,P\neq T}\lambda_P y_{rP} \qquad + \lambda_T y_{rT} \qquad \ge y_{rT}, \qquad r=1,2,\ldots,s, \\
&\sum_{P\in\Omega,\,P\neq T}\lambda_P \qquad + \lambda_T \qquad = 1, \\
&\theta_T \ge 0; \quad \lambda_P \ge 0, \ \ P\in\Omega.
\end{aligned}
\tag{M13}
$$

Let the optimal solution to model (M13) be ($\lambda_P^*, \lambda_T^*, \theta^*$). Assume DMU_T is located inner the frontier, we have θ^*<1 and λ_T^*=0. By setting all variables with the optimal solution to model (M13), the constraints of (M13) have the following results:

$$\sum_{P\in\Psi_0}\lambda_P^* x_{iP} + \sum_{P\in\Psi_1,\,P\neq T}\lambda_P^*(x_{iP} + \text{[illegible]} \lambda_P^*(x_{iP}+\pi^*)$$ [illegible] for $i\in\boldsymbol{I}$

and

$$\sum_{P\in\Omega,\,P\neq T}\lambda_{\mathrm{p}}^* x_{iP} \le \theta^* x_{iT} \le x_{iT},$$

It means that (λ_j, π)=(λ_P^*, $\theta^*\pi^*$) is a feasible solution to (M12). Hence, $\theta^*\pi^* \ge \pi^*$, i.e., $\theta^*\ge 1$. It leads to a contradiction. So, DMU_T remains on the efficient frontier if $\pi=\pi^*$.

Conversely, we assume that DMU_T remains on the efficient frontier if inputs are increased as (4.3) with π units, and $\pi>\pi^*$. Model (M12) is rewritten as following:

$$
\begin{aligned}
\rho^* = &\text{Min } \rho \\
s.t. \ &\sum_{P\in\Psi_0}\lambda_P x_{iP} + \sum_{P\in\Psi_1,\,P\neq T}\lambda_P(x_{iP}+\pi+\rho) \le (x_{iT}+\pi)+\rho, \quad i\in\boldsymbol{I}, \\
&\sum_{P\in\Omega,\,P\neq T}\lambda_P x_{iP} \qquad \le x_{iT}, \qquad i\notin\boldsymbol{I}, \\
&\sum_{P\in\Omega,\,P\neq T}\lambda_P y_{rP} \qquad \ge y_{rT}, \qquad r=1,2,\ldots,s, \\
&\sum_{P\in\Omega,\,P\neq T}\lambda_P \qquad = 1, \\
&\rho\ge 0; \quad \lambda_P\ge 0, \ \ P\in\Omega \text{ and } P\neq T.
\end{aligned}
\tag{M14}
$$

Since DMU_T is located on the frontier, we must have $\rho^* \geq 0$. It implies that $\rho^* + \pi \geq \pi > \pi^*$. But according to model (M12), its optimal value must be π^*. Hence, $\rho^* + \pi = \pi^*$. This also leads to a contradiction. So, DMU_T remains efficient only if $\pi \leq \pi^*$. ■

This Theorem illustrates that the minimization of model (M12) provides the possible maximum increment of inputs as (4.3) to all DMUs in Ψ_1 for keeping DMU_T remain on the efficient frontier while the other inputs are held at constants.

4.1.2 Stabilities of output coefficients

Now, turning to consider the case of changing data in outputs. Assume that DMU_T is efficient and data are changed in the outputs as (4.4). We utilize the following DEA like model in which the test DMU_T is not included in the reference set to find the stability regions of outputs.

$$\begin{aligned}
\delta^* = \text{Min} \;\; & \delta \\
\text{s.t.} \;\; & \sum_{P\in\Psi_0} \lambda_P y_{rP} + \sum_{P\in\Psi_1, P\neq T} \lambda_P \text{ [illegible]}, \\
& \sum_{P\in\Omega, P\neq T} \lambda_P y_{rP} \text{ [illegible]}, \\
& \sum_{P\in\Omega, P\neq T} \lambda_P x_{iP} \text{ [illegible]}, 2, \ldots, m, \\
& \sum_{P\in\Omega, P\neq T} \lambda_P \text{ [illegible]} \\
& \delta \geq 0; \quad \lambda_P \geq 0, \;\; P \in \Omega \text{ and } P \neq T.
\end{aligned} \qquad \text{(M15)}$$

We first show that the model is translation invariant.

Lemma 4.1 Model (M15) is translation invariant.

Proof: Since $\sum_{P\in\Omega, P\neq T} \lambda_P = 1,$, the result follows. ■

Suppose model (M15) is also feasible for an efficient DMU_T. The sufficient and necessary conditions for preserving DMU_T remain on the frontier are shown as follows.

Theorem 4.2 Given data varied in the outputs as (4.4), the efficient DMU_T remains on the efficient frontier if and only if $\delta \in [0, \delta^*]$, where δ^* is the optimal value to model (M15).

Proof: We first show that DMU_T remains on the frontier if $\delta = \delta^*$. By Lemma 4.1, we may adjust data of outputs so that $y_{rT} > 2\delta^*$ and it follows that $\delta^*/(y_{kT} - \delta^*) < 1$ for all $r \in \boldsymbol{O}$. Then, we consider the following DEA model when DMU_T is under evaluation and DMU_P change their outputs by the value $y_{rP}-\delta^*$ for all $P\in\Psi_1$.

$$
\begin{aligned}
\phi^* = \text{Max} \;\; & \phi \\
\text{s.t.} \;\; & \sum_{P\in\Psi_0} \lambda_P y_{rP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (y_{rP}-\delta^*) + \lambda_T (y_{rT}-\delta^*) \geq \phi (y_{rT}-\delta^*), \quad r\in \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \quad + \lambda_T y_{rT} \quad \geq \phi\, y_{rT}, \quad r\notin \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \quad + \lambda_T x_{iT} \quad \leq x_{iT}, \quad i=1,2,\ldots,m, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P \quad + \lambda_T \quad = 1, \\
& \phi \geq 0; \quad \lambda_P \geq 0, \quad P\in\Omega.
\end{aligned}
\tag{M16}
$$

Let the optimal solution to model (M16) be ($\lambda_P^*, \lambda_T^*, \phi^*$). Assume DMU_T is located inside the frontier, that is ϕ^*>1 and λ_T^*=0. It follows that:

$$\phi^* > 1 > \delta^*/(y_{rT}-\delta^*) \;\; \Rightarrow \;\; \phi^* y_{rT} - \delta^*\phi^* - \delta^* > 0 \;\; \text{for all } r\in\boldsymbol{O}.$$

By setting all variables with th[illegible]del (M16), constraints of (M16) yield the following results:

$$
\begin{aligned}
\sum_{P\in\Psi_0} \lambda_P^* y_{rP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P^* (y_{rP} \text{[illegible]} \sum_{\text{[illegible]}_1,\, P\neq T} \lambda_P^* (y_{rP}-\delta^*) \\
\text{[illegible]} y_{rT} - \text{[illegible]} + (\phi^* y_{rT} - \delta^*\phi^* - \delta^*)(1-1/\phi^*) \\
\geq y_{rT} - \delta^*/\phi^*, \qquad \text{for all } r\in\boldsymbol{O}
\end{aligned}
$$

and

$$\sum_{P\in\Omega,\, P\neq T} \lambda_P^* y_{rP} \geq \phi^* y_{rT} \geq y_{rT}, \qquad \text{for all } r\notin\boldsymbol{O}.$$

It means that (λ_P, δ)=(λ_P^*, δ^*/ϕ^*) is a feasible solution to model (M15). Hence $\delta^*/\phi^* \geq \delta^*$, i.e., $\phi^*\leq 1$. It leads to a contradiction. So, DMU_T remains on the efficient frontier if $\delta=\delta^*$.

Conversely, we assume that DMU_T remains on the efficient frontier if outputs are decreased as (4.4) with δ units, and $\delta>\delta^*$. Model (M15) is rewritten as following:

$$
\begin{aligned}
\tau^* = \text{Min} \;\; & \tau \\
\text{s.t.} \;\; & \sum_{P\in\Psi_0} \lambda_P y_{rP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (y_{rP} - \delta - \tau) \geq (y_{rT} - \delta) - \tau, \quad r \in \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \geq y_{rT}, \quad r \notin \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \leq x_{iT}, \quad i = 1, 2, \ldots, m, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P = 1, \\
& \lambda_P \geq 0, \;\; P \in \Omega \text{ and } P \neq T; \quad \tau : \text{free}.
\end{aligned}
\tag{M17}
$$

Since DMU_T is located on the frontier, we must have $\tau^* \geq 0$. It implies that $\tau^* + \delta \geq \delta > \delta^*$. But according to model (M15), it must be $\tau^* + \delta = \delta^*$. This also leads to a contradiction. So, DMU_T remains efficient only if $\rho \leq \rho^*$. ■

This Theorem illustrates that the minimization of model (M15) provides the possible maximum decrement for each output to keep DMU_T to remain on the efficient frontier when the other outputs are held at cor

4.1.3 Stability for change input ... ously

Moreover, if we change ... the same time, the stability region is obtained by solving the followi

$$
\begin{aligned}
\Gamma^* = \text{Min} \;\; & \Gamma \\
\text{s.t.} \;\; & \sum_{P\in\Psi_0} \lambda_P x_{iP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (x_{iP} + \Gamma) \leq x_{iT} + \Gamma, \quad i \in \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \leq x_{iT}, \quad i \notin \boldsymbol{I}, \\
& \sum_{P\in\Psi_0} \lambda_P y_{rP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (y_{rP} - \Gamma) \geq y_{rT} - \Gamma, \quad r \in \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \geq y_{rT}, \quad r \notin \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P = 1, \\
& \Gamma \geq 0; \quad \lambda_P \geq 0, \;\; P \in \Omega.
\end{aligned}
\tag{M18}
$$

If we assume the problem is also feasible, the following result is derived.

Theorem 4.3 The efficient DMU_T remains on the frontier after the data change as (4.3) and (4.4) with $\pi = \delta = \Gamma$, if and only if $\Gamma \in [0, \Gamma^*]$, where Γ^* is the optimal value to model (M18).

Proof: The proof is analogous to the proof of Theorem 4.1 and 4.2 and is omitted. ■

We have derived the sufficient and necessary conditions for the models to preserve the efficiency of an efficient DMU under the given data change type. The following section presents an example to illustrate this proposed analysis.

4.1.4 Examples 4: stability analysis

The simulated data set consists of 8 portfolios, P1~P8, with two inputs (x_1 and x_2) and one output (y) is listed in Table 8. Portfolios of P1~P4 are VRS efficient while portfolios of P5~P8 are inefficient. We consider the case of increasing inputs of P2, P3, and P6 simultaneously while the other portfolios are held fixed. By solving model (M12), the maximum increment of input x_1 in P2, P3, and P6 to keep P2 remains on the frontier is 4/3. Figure 3 presents the stability of P2 and the frontiers before and after the change in input x_1. Under the maximum increment, P2 locates in ***E***' and can be expressed as the linear combination of P3 and P5.

The last column of Tabl [illegible] ty regions of P2 for changing input x_1 and x_2, and simultaneously cl [illegible] e same value are 4/3, 2, and 0.8, respectively. Similarly, the stab [illegible] 3, and 1. It reveals that P3 has larger stability regions than P2 under [illegible] ype. It implies that P3 is more stable than P2 while data uncertainty [illegible] 5 simultaneously.

Table 8. [illegible] for Example 4.

Portfolio	y_1	x_1	x_2	Efficiency[a]	Ψ_0 or Ψ_1[b]	Stability regions[c]
P1	1	1	12	***E***	Ψ_0	
P2	1	2	6	***E***	Ψ_1	π_1= 4/3, π_2= 2.0, π= 0.8
P3	1	4	3	***E***	Ψ_1	π_1= 10/3, π_2= 7/3, π= 1.0
P4	1	12	1	***E***	Ψ_0	
P5	1	2	8	***N***	Ψ_0	
P6	1	7	4	***N***	Ψ_1	
P7	1	6	7	***N***	Ψ_0	
P8	1	5	4	***N***	Ψ_0	

a: ***E*** means VRS efficient while ***N*** means inefficient.

b: Ψ_1 indicates the perturbed set of portfolios while Ψ_0 is the unperturbed set of portfolios.

c: π_1, π_2, and π are the stability regions corresponding to change value in input x_1, x_2, and all inputs simultaneously, respectively.

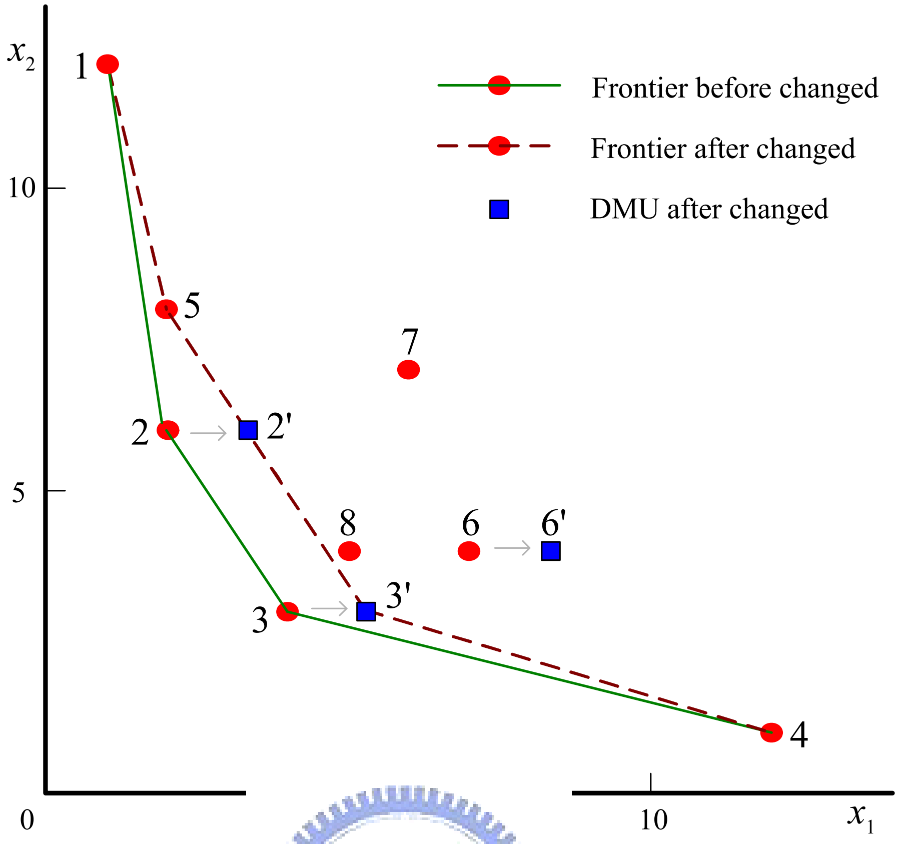

Figure 3. Stability using x_1 in P2, P3 and P6.

4.2 Properties for stability

The proposed stability Some properties related to the non-linear models are investigated in this section. Without a loss of generality, we use model (M12) to illustrate the infeasibility for all proposed stability models.

4.2.1 Infeasible and unbounded properties

When portfolio T is under evaluation, Let us employ the following super-efficiency model to assess DMU_T based on the subsets of performance indices, $i \notin \boldsymbol{I}$ and r=1, 2, … , s.

$$
\begin{aligned}
\theta^* &= \text{Min} \;\; \theta \\
\text{s.t.} \quad & \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \le \theta\, x_{iT}, \qquad i \notin \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \ge y_{rT}, \qquad r = 1, 2, \ldots, s, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P \quad = 1, \\
& \theta \ge 0; \quad \lambda_P \ge 0, \;\; P \in \Omega \text{ and } P \neq T.
\end{aligned}
\qquad \text{(M19)}
$$

In the case of $\theta^{*}>1$, it provides that DMU_T is also extremely efficient as the performance indices are augmented by set $\boldsymbol{I}$ (Chen & Ali, 2002). Now, if θ is substituted by 1 to model (M19). We have:

$$\sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I}, \tag{4.7}$$

$$\sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \ge y_{rT}, \quad r = 1, 2, \ldots, s. \tag{4.8}$$

It follows that data of DMU_T are 'infeasible' to the above constraints. One may observe that (4.7) and (4.8) are identical to the second and third constraints of model (M12). It means that DMU_T would result in an infeasible solution to model (M12) by the structure of constraints, if it remains efficient by deleting the performance indices of $i\in\boldsymbol{I}$. The infeasibility indicates DMU_T is not impacted by the data changes in indices of $i\in\boldsymbol{I}$, and states that it would always be stable under the perturbations.

In the event of $\theta^{*}\le 1$, D[illegible] n the indices of $i\notin\boldsymbol{I}$ and r=1, 2, … , s, and is a convex combination of [illegible] a feasible solution should be obtained by model (M12). It indicates t[illegible] acted by changing data of input $i\in\boldsymbol{I}$. Hence, we can use (M19) to d[illegible] DMU is impacted by the variation or not. If the impact is confirme[illegible] the stability by using model (M12). Otherwise, this DMU is always [illegible]

The BCC super-efficiency model may also result in an unbounded solution when DMU_T has the maximum value on any output since the existing constraint summed all λ_P's to one. The models proposed in this research may also have an unbounded solution. For instance, in model (M12), it first constraint can be rewritten as:

$$\sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} - x_{iT} \le \pi \sum_{P\in\Psi_0,\, P\neq T} \lambda_P, \qquad i \in \boldsymbol{I}. \tag{4.9}$$

The optimal of π would be unbounded if $\sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} - x_{iT} > 0$ and $\sum_{P\in\Psi_0,\, P\neq T} \lambda_P = 0$ for any input $i\in\boldsymbol{I}$. That is, DMU_T is super-efficiency with respect to the indices $\boldsymbol{I}$, and there is no DMU in Ψ_0 with input less than DMU_T. In this situation, as the performance worsens through increasing data of indices in $\boldsymbol{I}$, all DMUs in Ψ_1 are moved toward the interior of the frontier simultaneously. At the same time, the new frontier constructed by excluding test DMU_T is also moved in the same direction. If the above two conditions hold, DMU_T would not stop movement as the part of the frontier is simultaneously moved at the same distance. The occurrence of an unbounded solution indicates that the DMU_T possesses a vast stability on the altered indices. Note, in the case that set Ψ_1 only has element DMU_T, as discussed in Zhu (1996), an unbounded solution also exist.

4.2.2 Global optimal solution

The optimal solution of the non-linear model (M12) is a global optimal solution and can be shown here. Let us consider the case that if the data are altered as (4.3), the non-linear constraint of model (M12) is written as follows:

$$g_i(\pi, \lambda_P, \lambda_H) = \sum_{P \in \Psi_0} \lambda_P x_{iP} + \sum_{H \in \Psi_1,\, H \neq T} \lambda_H (x_{iH} + \pi) - x_{iT} - \pi \leq 0, \text{ for all } i \in \boldsymbol{I}. \quad (4.10)$$

For any point $\mathbf{z} = (\pi, \lambda_P, \lambda_H)$ on the null space of $g_i(\pi, \lambda_P, \lambda_H)$ we have a positive semi-definite Hessian matrix.

$$\mathbf{z}[\nabla^2 g_i(\pi, \lambda_P, \lambda_H)]\mathbf{z}^T = \text{[illegible]} \ \boldsymbol{I}. \quad (4.11)$$

It indicates that $g_i(\pi, \lambda_P, \lambda_H)$ i[illegible], the following set:

$$\{(\pi, \lambda_P, \lambda_H) \mid g_i(\pi, \lambda_P, \lambda\text{[illegible]} \quad (4.12)$$

is convex. Together with the other linear constraints, the feasible region of model (M12) is also convex. The same conclusion could be derived for model (M15) that changes data as (4.4), and for model (M18) that changes data as (4.3) and (4.4) simultaneously. Each model has a linear objective function subject to the convex feasible region. It implies that there is at most one local optimum. Hence, the local optimum must also be a global optimum. So, the global optimum is obtainable for all stability models proposed in this study.

4.2.3 Model extensions

Now we consider other modified DEA models by removing the constraint on the sum of the λ_P's variables in models (M12), (M15), and (M18). For instance, model (M12) can be modified by removing the following constraint:

$$\sum_{P \in \Omega,\, P \neq T} \lambda_P = 1. \quad (4.13)$$

This can be regarded as modified constants returns to scale (CRS) model (Banker et al. 1984) for finding the stability region of efficient DMU_T through changing inputs as (4.3).

$$
\begin{aligned}
\pi^* = &\ \text{Min} \ \ \pi \\
s.t. \ & \sum_{P\in\Psi_0} \lambda_P x_{iP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (x_{iP} + \pi) \le x_{iT} + \pi, \quad i \in \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \ge y_{rT}, \quad r = 1, 2, \ldots, s, \\
& \pi \ge 0; \quad \lambda_P \ge 0, \ \ P \in \Omega \text{ and } P \neq T.
\end{aligned}
\qquad \text{(M20)}
$$

Now, let us consider the following model:

$$
\begin{aligned}
\pi^* = &\ \text{Min} \ \ \pi \\
s.t. \ & \sum_{P\in\Psi_0} \lambda_P (x_{iP} - \pi) + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (x_{iP} + \pi) \le x_{iT} + \pi, \quad i \in \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \quad [illegible] \quad r = 1, 2, \ldots, s, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P \quad [illegible] \\
& \pi \ge 0; \quad \lambda_P \ge 0, \ \ P \in \ [illegible]
\end{aligned}
\qquad \text{(M21)}
$$

The minimization of (M21) pr[illegible]num increments for inputs of DMUs in Ψ_1, and the maximum decre[illegible]Js in Ψ_0, to allow an efficient DMU_T remaining on the frontier when the outputs and other inputs are held constant.

4.3 Method for Solving stability models

The stability models (M12), (M15), and (M18) proposed in the current paper are not linear programming. However, the non-linear programming model is more difficult to solve than the linear model. For simplicity, we investigate the method for solving the input-based stability model (M12). We will derive some properties that enable us to use the linear programming technique to approximate the optimal value π^* of model (M12). First, we consider the LP model given as below.

$$
\begin{aligned}
\pi(t) = \text{Min} \quad & \pi \\
s.t. \quad & \sum_{P\in\Psi_0}\lambda_P x_{iP} + \sum_{P\in\Psi_1,\, P\neq T}\lambda_P (x_{iP}+t) \le x_{iT}+\pi, && i\in \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T}\lambda_P x_{iP} \le x_{iT}, && i\notin \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T}\lambda_P y_{rP} \ge y_{rT}, && r=1,2,\dots,s, \\
& \sum_{P\in\Omega,\, P\neq T}\lambda_P = 1, \\
& \pi \ge 0; \quad \lambda_P \ge 0, \quad P\in\Omega \text{ and } P\neq T.
\end{aligned}
\tag{M22}
$$

Given positive constant t, the optimal solution to (M22) is denoted by ($\lambda_P(t)$, $\pi(t)$). Obviously, we have $\pi(t) = \pi^*$ if $t = \pi^*$. Some properties will be derived in the following.

4.3.1 Properties for stability of inputs

We will show that $\pi(t)$ is a non-decreasing function for $t \ge 0$.

Theorem 4.4 Let $\pi(t)$ be the op ... en, $\pi(t)$ is non-decreasing in t.

Proof: We will show that $\pi(t_1)$ ... positive constants t_1 and t_2 with $t_1 > t_2$. Suppose the optimal solutions t ... $t = t_1$ and $t = t_2$ are ($\lambda_P(t_1)$, $\pi(t_1)$) and ($\lambda_P(t_2)$, $\pi(t_2)$). It follows:

$$\sum_{P\in\Psi_0}\lambda_P(t_1)x_{iP} + \sum_{P\in\Psi_1,\, P\neq T}\lambda_P(t \quad \dots \quad + \sum_{P\in\Psi_1,\, P\neq T}\lambda_P(t_1)(x_{iP}+t_1)$$

$$\le x_{iT} + \pi(t_1), \qquad i\in \boldsymbol{I},$$

$$\sum_{P\in\Omega,\, P\neq T}\lambda_P(t_1)x_{iP} \le x_{iT}, \qquad i\notin \boldsymbol{I},$$

$$\sum_{P\in\Omega,\, P\neq T}\lambda_P(t_1)y_{rP} \ge y_{rT}, \qquad r=1,2,\dots,s,$$

and

$$\sum_{P\in\Omega,\, P\neq T}\lambda_P(t_1) = 1.$$

It implies ($\lambda_P(t_1)$, $\pi(t_1)$) is feasible to (M22) for $t = t_2$. Therefore, we have $\pi(t_1) \ge \pi(t_2)$. ■

Since $\pi(\pi^*) = \pi^*$, it is easy to show that $\pi(t) \le \pi^*$ for all $t < \pi^*$, and $\pi(t) \ge \pi^*$ for all $t > \pi^*$ by following the results of Theorem 4.4. Further, the following theorems will help us to approximate the optimal value to (M12).

Theorem 4.5. If $t < \pi^*$. Then $t < \pi(t) \le \pi^*$.

Proof: Since $t < \pi^*$, we have $\pi(t) \le \pi(\pi^*) = \pi^*$. Now, let us suppose $t \ge \pi(t)$. Since, $\pi(t)$ is optimal to (M22), we have

$$\sum_{P\in\Psi_0}\lambda_P(t)x_{iP} + \sum_{P\in\Psi_1,\,P\ne T}\lambda_P(t)(x_{iP}+t) \le x_{iT} + \pi(t) \le x_{iT} + t, \quad i \in \boldsymbol{I},$$

$$\sum_{P\in\Omega,\,P\ne T}\lambda_P(t)x_{iP} \le x_{iT}, \qquad i \notin \boldsymbol{I},$$

$$\sum_{P\in\Omega,\,P\ne T}\lambda_P(t)y_{rP} \ge y_{rT}, \qquad r = 1, 2, \ldots, s,$$

and

$$\sum_{P\in\Omega,\,P\ne T}\lambda_P(t) = 1 .$$

It implies that ($\lambda_P(t)$, t) is feasible to model (M12), i.e., $t \ge \pi^*$. This leads to a contradiction. So, we have $t < \pi(t) \le \pi^*$. ■

Theorem 4.6. If $t > \pi^*$. Then, $t \ge \pi(t)$.

Proof: Let (λ_P^*, π^*) be the optir[illegible]e have

$$\sum_{P\in\Psi_0}\lambda_P^* x_{iP} + \sum_{P\in\Psi_1,\,P\ne T}\lambda_P^*(x_{iP} \text{[illegible]} \quad i \in \boldsymbol{I},$$

It follows:

$$\sum_{P\in\Psi_0}\lambda_P^* x_{iP} + \sum_{P\in\Psi_1,\,P\ne T}\lambda_P^*(x_{iP} + \text{[illegible]} \sum_{P\in\Psi_0} \text{[illegible]} \sum_{P\in\Psi_1,\,P\ne T} \text{[illegible]} x_{iP} + \pi^*) + \sum_{P\in\Psi_1,\,P\ne T}\lambda_P^*(t-\pi^*)$$

$$\le x_{iT} + \pi^* + \sum_{P\in\Psi_1,\,P\ne T}\lambda_P^*(t-\pi^*)$$

$$\le x_{iT} + \pi^* + (t-\pi^*)$$

$$\le x_{iT} + t, \qquad \text{for all } i \in \boldsymbol{I},$$

It implies that (λ_P^*, t) is feasible to (M22). Thus, we have $t \ge \pi(t)$. ■

Suppose model (M12) is feasible, Theorem 4.5 and 4.6 show that if t_1 and t_2 are the lower and upper bounds of π^* respectively. One can obtain $\pi(t_1)$ and $\pi(t_2)$ from solving (M22) by setting $t = t_1$ and $t = t_2$, and identify that $t_1 < \pi(t_1) \le \pi^* \le \pi(t_2) \le t_2$. That is, the lower bound is moved upward from t_1 to $\pi(t_1)$ and the upper bound is moved downward from t_2 to $\pi(t_2)$. Conversely, if one obtains $\pi(t)$ by solving (M22) for any t, then, we have $\pi(t) \le \pi^*$ for $t < \pi(t)$ and $\pi(t) \ge \pi^*$ for $t \ge \pi(t)$. The graph of $\pi(t)$ is shown in Figure 4.

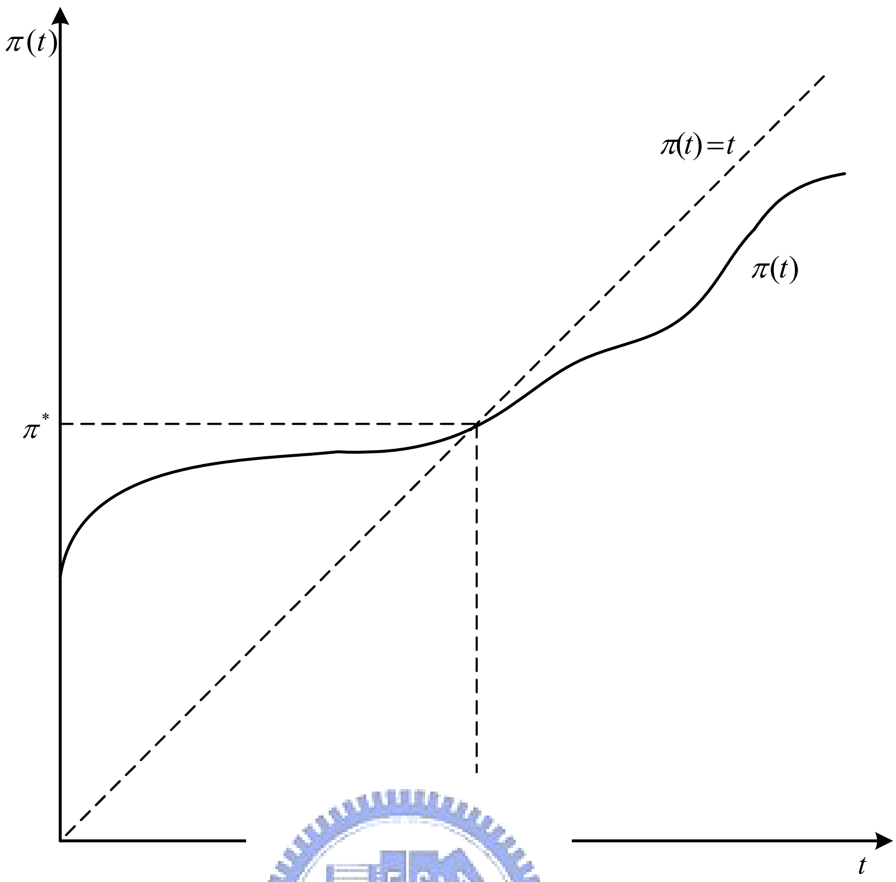

Figure 4. The ... (*t*) to model (M22).

But model (M12) could ... ed. We will state a rule to prevent the unbounded situations from sol ... r the following linear programming model modified from (M22) by excluding all DMU_P, $P \in \Psi_1$ from the reference set.

$$\begin{aligned}
\pi^{\#} = \text{Min} \;\; & \pi \\
s.t. \;\; & \sum_{P\in\Psi_0} \lambda_P x_{iP} \le x_{iT} + \pi, \quad i \in \boldsymbol{I}, \\
& \sum_{P\in\Psi_0} \lambda_P x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I}, \\
& \sum_{P\in\Psi_0} \lambda_P y_{rP} \ge y_{rT}, \quad r = 1, 2, \ldots, s, \\
& \sum_{P\in\Psi_0} \lambda_P = 1, \\
& \pi \ge 0; \quad \lambda_P \ge 0, \;\; P \in \Psi_0.
\end{aligned} \tag{M23}$$

Suppose (M23) is feasible. We have $\pi^{\#}$ is an upper bound of π^{*}.

Theorem 4.7. $\pi^{\#} \ge \pi^{*}$.

Proof: Suppose the optimal solutions to (M23) is ($\lambda_P^{\#}$, $\pi^{\#}$), where $\pi^{\#}$ is finite. It follows:

$$\sum_{P\in\Psi_0}\lambda_P^{\#}x_{iP} \le x_{iT} + \pi^{\#}, \quad i \in \boldsymbol{I},$$
$$\sum_{P\in\Psi_0}\lambda_P^{\#}x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I},$$

and

$$\sum_{P\in\Psi_0}\lambda_P^{\#}y_{rP} \ge y_{rT}, \quad r = 1, 2, \ldots, s.$$

Take $\lambda_P^{\#}$=0 for all $P\in\Psi_1$. We obtain

$$\sum_{P\in\Psi_0}\lambda_P^{\#}x_{iP} + \sum_{P\in\Psi_1,\, P\ne T}\lambda_P^{\#}(x_{iP} + \pi^{\#}) \le x_{iT} + \pi^{\#}, \quad i \in \boldsymbol{I},$$
$$\sum_{P\in\Omega,\, P\ne T}\lambda_P^{\#}x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I},$$

and

$$\sum_{P\in\Omega,\, P\ne T}\lambda_P^{\#}y_{rP} \ge y_{rT}, \quad r = 1, 2, \ldots, s.$$

It implies ($\lambda_P^{\#}$, 0, $\pi^{\#}$) is a feasib refore, $\pi^* \le \pi^{\#} < \infty$. ■

The result of Theorem finite if $\pi^{\#}$ is finite. Morevver, the following theorem will help us exact value of π^* is obtained or not.

Theorem 4.8. If $\lambda_P(0) = 0$ for a .

Proof: If $\lambda_P(0)$=0 for all $P\in\Psi_1$, we have ($\lambda_P(0)$, $\pi(0)$) is also feasible to (M23). Following the results of Theorem 4.4–4.7, we have $\pi(0) \ge \pi^{\#} \ge \pi^*$. Conversely, $\pi(t)$ is non-decreasing in t, and $\pi^* > 0$. It follows $\pi^* = \pi(\pi^*) \ge \pi(0)$. Therefore, $\pi(0) = \pi^*$. ■

However, Theorem 4.8 can be extended as: if given any $t\in[0, \pi^*]$ with $\lambda_P(t)$=0 for all $P\in\Psi_1$. Then, we have $\pi(t) = \pi^*$.

4.3.2 Approximating stability regions

Following the results of Theorem 4.4–4.8, π^* can be obtained or approximated by solving linear programming models (M22) and (M23) only, but it does not need to employ the non-linear programming model (M12) directly. An algorithm used to approximate π^* is developed as the following:

Step 0. (Initialized) Solve (M23) to obtain $\pi^{\#}$

Step 0.1. If $\pi^{\#}$ is bounded, set upper bound $U=\pi^{\#}$.

Otherwise, let $U=M$, where M is a given sufficient large number.

Step 0.2. Let lower bound $L=0$ and ε be the error tolerance for estimating π^{*}.

Step 1. Solve (M22) with $t=(U+L)/2$ to obtain $(\lambda_P(t), \pi(t))$.

Step 1.1. If $\lambda_P(t)=0$ for all $P\in\Psi_1$ then set $\pi^{*}=\pi(t)$ and stop.

Step 1.2. If $t<\pi(t)$ then set $L=\pi(t)$. Otherwise, set $U=\pi(t)$.

Step 2. If $|U-L| < 2\varepsilon$ then set $\pi^{*}=(U+L)/2$ and stop. Otherwise, go to Step 1.

A bisection procedure is applied in the algorithm for convergence. If $\pi^{\#}$ is feasible in Step 0, π^{*} must be feasible and its approximation could be obtained. However, π^{*} may occur infeasible or its value exceeds a large number such that the test portfolio tends to be stable while data is changed in a sufficient large scale. So, the upper bound U is set sufficient large value if $\pi^{\#}$ is infeasible in Step 0. In the real-world applications, one may identity that a test portfolio is always stable if the sible or large enough relatively to the data range of the perturbed proj

4.3.3 Method for solving other

For the case of changin, the optimal solution, δ^{*}, to the non-linear model (M15) could be ap ig the following LP model.

$$
\begin{aligned}
\delta(t) = \text{Min } & \delta \\
\text{s.t. } & \sum_{P\in\Psi_0} \lambda_P y_{rP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (y_{rP} - t) \geq y_{rT} - \delta, && r \in \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \geq y_{rT}, && r \notin \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \leq x_{iT}, && i = 1, 2, \ldots, m, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P = 1, \\
& \delta \geq 0; \quad \lambda_P \geq 0, \quad P \in \Omega \text{ and } P \neq T.
\end{aligned}
\tag{M24}
$$

For arbitrary positive constant t, we have $\delta(t) = \delta^{*}$ if we take $t = \delta^{*}$. However, if we use model (M15) to consider the data changed as (4.3) and (4.4), the stability is approximated by considering the following LP model.

$$
\begin{aligned}
\Gamma(t) = \text{Min } & \Gamma \\
\text{s.t. } & \sum_{P\in\Psi_0} \lambda_P x_{iP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (x_{iP}+t) \le x_{iT} + \Gamma, \quad i \in \boldsymbol{I}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P x_{iP} \le x_{iT}, \quad i \notin \boldsymbol{I}, \\
& \sum_{P\in\Psi_0} \lambda_P y_{rP} + \sum_{P\in\Psi_1,\, P\neq T} \lambda_P (y_{rP}-t) \ge y_{rT} - \Gamma, \quad r \in \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P y_{rP} \ge y_{rT}, \quad r \notin \boldsymbol{O}, \\
& \sum_{P\in\Omega,\, P\neq T} \lambda_P = 1, \\
& \Gamma \ge 0; \quad \lambda_P \ge 0, \quad P \in \Omega.
\end{aligned}
\qquad \text{(M25)}
$$

Some properties related to δ^* and Γ^* are analogous to the properties of π^*, which enable us to approximate the exact values of δ^* and Γ^*.

4.3.4 Example 1 (continued)

Let's use Example 1 as [illegible] sitivity analysis. In case of the input coefficient, a_3, of project 3 is g[illegible] In Table 9, portfolios with w_3=1 and w_3=0 are listed in the upper an[illegible] ng order of input values. We want to investigate the stability of effi[illegible] spect to the data variation. One is to find the maximum value of π s[illegible] cient portfolio 7 is remains efficient. That is, portfolio 7 remains eff[illegible] ue of project 3 is changed from a_3 to $a_3+\pi$. Model (M9) is rewritten as follows after changed.

$$
\begin{aligned}
\text{Maximize } & y = 6\,w_1 + 4.0\,w_2 + 7.2\,w_3 + 8\,w_4 + w_5 \\
\text{Minimize } & x = 4\,w_1 + 2.8\,w_2 + (5.6+\pi)\,w_3 + 9\,w_4 + 2\,w_5 \\
\text{Subject to } & w_k \in \{0,1\},\; k=1, 2,\ldots, 5.
\end{aligned}
\qquad \text{(M26)}
$$

Sets of index for classifying changed and unchanged portfolios is as follows.

Ψ_0={0, 1, 2, 3, 8, 9, 10, 11, 16, 17, 18, 19, 24, 25, 26, 27},

and

Ψ_1={4, 5, 6, 7, 12, 13, 14, 15, 20, 21, 22, 23, 28, 29, 30, 31}.

Portfolio P are shifted right if $P\in\Psi_1$ and unchanged if $P\in\Psi_0$. Changed and unchanged portfolios and their corresponding DMUs are listed in Table 9. Figure 5 presents all DMUs while they are before change. Figure 6 presents all DMUs while they are after change. The stability measure, π^*, is solved by the above algorithm and as follows:

Table 9. Changed and unchanged portfolios by perturbing project 3.

Portfolio						DMU_P	
P	w_1	w_2	w_3	w_4	w_5	x_P	y_P
4	0	0	1	0	0	$5.6+\pi$	7.2
20	0	0	1	0	1	$7.6+\pi$	8.2
6	0	1	1	0	0	$8.4+\pi$	11.2
5	1	0	1	0	0	$9.6+\pi$	13.2
22	0	1	1	0	1	$10.4+\pi$	12.2
21	1	0	1	0	1	$11.6+\pi$	14.2
7^E	1	1	1	0	0	$12.4+\pi$	17.2
23	1	1	1	0	1	$14.4+\pi$	18.2
12	0	0	1	1	0	$14.6+\pi$	15.2
28	0	0	1	1	1	$16.6+\pi$	16.2
14	0	1	1	1	0	$17.4+\pi$	19.2
13	1	0	1	1	0	$18.6+\pi$	21.2
30	0	1	[illegible]	[illegible]	[illegible]	$19.4+\pi$	20.2
29	1	0	[illegible]	[illegible]	[illegible]	$20.6+\pi$	22.2
15^E	1	1	[illegible]	[illegible]	[illegible]	$21.4+\pi$	25.2
31^E	1	1	[illegible]	[illegible]	[illegible]	$23.4+\pi$	26.2
0^E	0	0	[illegible]	[illegible]	[illegible]	0.0	0.0
16	0	0	[illegible]	[illegible]	[illegible]	2.0	1.0
2	0	1	0	0	0	2.8	4.0
1^E	1	0	0	0	0	4.0	6.0
18	0	1	0	0	1	4.8	5.0
17	1	0	0	0	1	6.0	7.0
3^E	1	1	0	0	0	6.8	10.0
19	1	1	0	0	1	8.8	11.0
8	0	0	0	1	0	9.0	8.0
24	0	0	0	1	1	11.0	9.0
10	0	1	0	1	0	11.8	12.0
9	1	0	0	1	0	13.0	14.0
26	0	1	0	1	1	13.8	13.0
25	1	0	0	1	1	15.0	15.0
11	1	1	0	1	0	15.8	18.0
27	1	1	0	1	1	17.8	19.0

E: indicates the efficient portfolio.

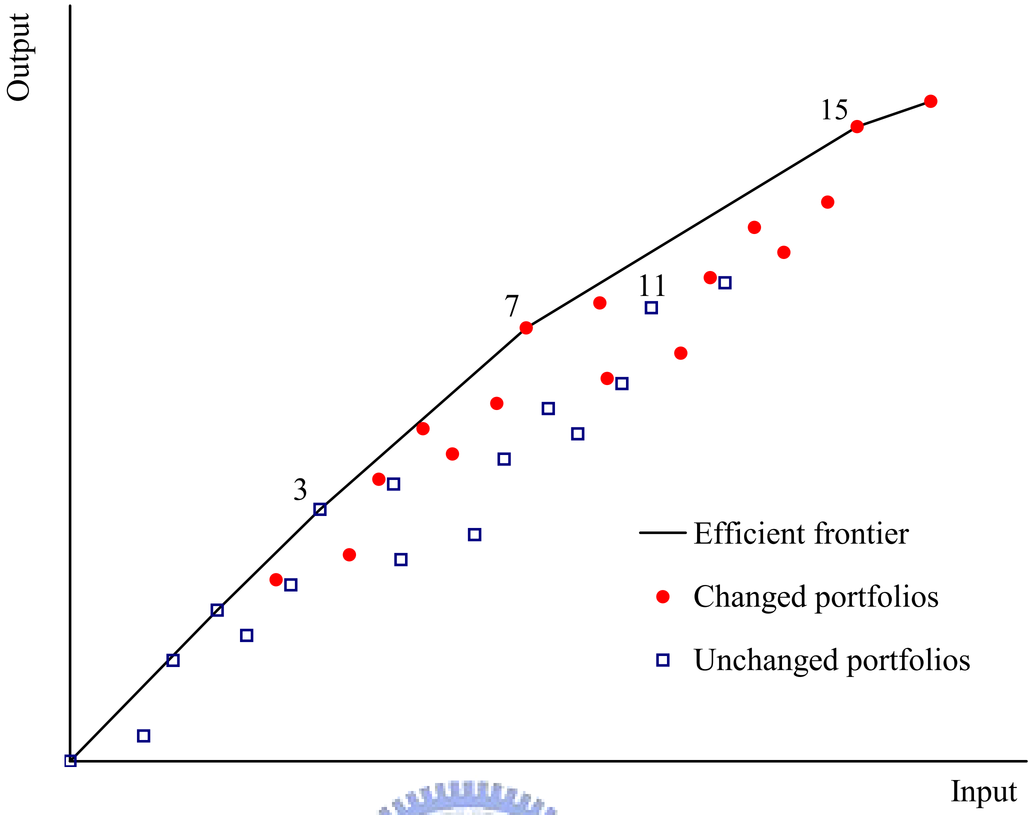

Figure 5. All [illegible]ontier before change.

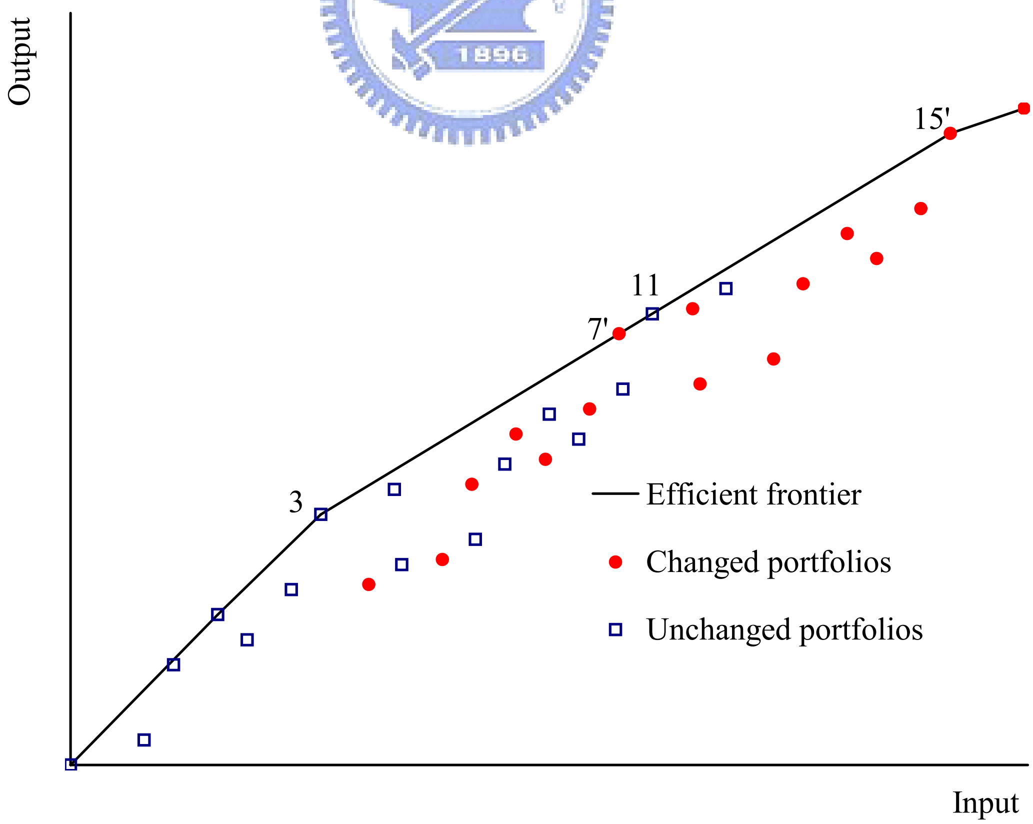

Figure 6. All portfolios and efficient frontier after change.

Step 0. Solve (M23) to obtain $\pi^{\#}$=2.5.

Step 0.1. $\pi^{\#}$=2.5 is bounded, set upper bound $U=\pi^{\#}$=2.5.

Step 0.2. Set lower bound L=0 and error tolerance for estimating π^{*} be ε=0.001.

Step 1. Solve (M22) by setting $t=(U+L)/2$=1.25.

We obtain $\lambda_3(t)$= 0.5263, $\lambda_{15}(t)$= 0.4737, and $\pi(t)$=1.9079.

Step 1.1 Since $\lambda_{15}(t)>0$ and $15\in\Psi_1$, go to step 1.2.

Step 1.2 Since $t<\pi(t)$, we set $L=\pi(t)$=1.9079.

Step 2. Since $|U–L| > 2\varepsilon$, go to Step 1.

Iteration 2: $t=(U+L)/2$=2.2040, $\lambda_3(t)$=0.5263, $\lambda_{15}(t)$=0.4737, and $\pi(t)$=2.3597.

Set $L=\pi(t)$=2.3597 and remain U= 2.5.

Iteration 3: $t=(U+L)/2$=2.4299, $\lambda_3(t)$=0.5263, $\lambda_{15}(t)$=0.4737, and $\pi(t)$=2.4668.

Set $L=\pi(t)$=2.4668 and remain U= 2.5.

Iteration 4: $t=(U+L)/2$=2.4834, $\lambda_3(t)$=0.5263, $\lambda_{15}(t)$=0.4737, and $\pi(t)$=2.4921.

Set $L=\pi(t)$=[illegible] 5.

Iteration 5: $t=(U+L)/2$=[illegible])=0.4737, and $\pi(t)$=2.4982.

Set $L=\pi(t)$=[illegible] 5.

Now, $|U–L|$=0.0018 < 2[illegible]4991 and stop the process.

The exact solution to (M12) is [illegible] not attachable if $\lambda_P(t)\neq 0$ for all $P\in\Psi_1$ occurs in Step 1. However, π^{*} could be approximated by the proposed algorithm.

5. Conclusion and Discussion

The problem of evaluation and selection of collective projects is modeled as a MOBILP. Instead of evaluating projects individually, it enables the evaluation of projects in their combination forms. In the public sector and government project selection problems, many environmental factors may be included as the objective of resources. We focused on the best use of relative resources, but not the best use of available resources. In this paper, we developed the filtering algorithm to circumvent the computational difficulties of DEA programs, to identify all efficient portfolios, and to rank them according to the stability measures of model (M7).

The simulated results show that a major portion of the inefficient portfolios and some efficient portfolios (SEC) are identified prior to the calculation of the DEA programs. The remaining portfolios are then evaluated with respect to the ECG by using DEA case-based classification model (M7). The problem size of each LP and the number of solving LP are reduced significantly. The simu [illegible] following:

1. The PDR filter is the most [illegible] sed filters. It identifies about 80% to 90% of portfolios in our sam [illegible]
2. Phase II of filtering algorith [illegible] f inefficient portfolios. It shows that the DEA programs just need [illegible] ng 1% of candidate portfolios.
3. The computing time is data dependent and its expected computing time is exponentially increased as the number of projects is increased.

DEA methodology is computationally intensive when required to solve a large number of LP. This problem has 2^K portfolios, and the number is doubled as one more project is added for evaluation. The program *MOBILP+* seems efficient for solving the problem at this moment in time. One may potentially discover new methods of determining inefficient portfolios prior to the solution of the DEA programs, further reducing the number in solving LP. However, in the real-world applications, some projects could be eliminated prior to the collective selections if their stability measures were less than a threshold value. Then, the number of projects could be reduced and also dos reduce the computational effort for solving the problem.

Table 10. Data set consists of 37 R&D projects (Oral et al., 1991).

R&D Project	Indirect Economic	Direct Economic	Technical Contribution	Social Contribution	Scientific Contribution	Resource Usage
1	67.53	70.82	62.64	44.91	46.28	84.20
2	58.94	62.86	57.47	42.84	45.64	90.00
3	22.27	19.68	6.73	10.99	5.92	50.20
4	47.32	47.05	21.75	20.82	19.64	67.50
5	48.96	48.48	34.90	32.73	26.21	75.40
6	58.88	77.16	35.42	29.11	26.08	90.00
7	50.10	58.20	36.12	32.46	18.90	87.40
8	47.46	49.54	46.89	24.54	36.35	88.80
9	55.26	61.09	38.93	47.71	29.47	95.90
10	52.40	55.09	53.45	19.52	46.57	77.50
11	55.13	55.54	55.13	23.36	46.31	76.50
12	32.09	34.04	33.57	10.60	29.36	47.50
13	27.49	39.00	34.51	21.25	25.74	58.50
14	77.17	[illegible]	[illegible]	41.37	51.91	95.00
15	72.00	[illegible]	[illegible]	36.64	25.84	83.80
16	39.74	[illegible]	[illegible]	15.79	33.06	35.40
17	38.50	[illegible]	[illegible]	59.59	48.82	32.10
18	41.23	[illegible]	[illegible]	10.18	38.86	46.70
19	53.02	[illegible]	[illegible]	17.42	46.30	78.60
20	19.91	[illegible]	[illegible]	8.66	27.04	54.10
21	50.96	[illegible]	[illegible]	30.23	54.72	74.40
22	53.36	46.47	49.72	36.53	50.44	82.10
23	61.60	66.59	64.54	39.10	51.12	75.60
24	52.56	55.11	57.58	39.69	56.49	92.30
25	31.22	29.84	33.08	13.27	36.75	68.50
26	54.64	58.05	60.03	31.16	46.71	69.30
27	50.40	53.58	53.06	26.68	48.85	57.10
28	30.76	32.45	36.63	25.45	34.79	80.00
29	48.97	54.97	51.52	23.02	45.75	72.00
30	59.68	63.78	54.80	15.94	44.04	82.90
31	48.28	55.58	53.30	7.61	36.74	44.60
32	39.78	51.69	35.10	5.30	29.57	54.50
33	24.93	29.72	28.72	8.38	23.45	52.70
34	22.32	33.12	18.94	4.03	9.58	28.00
35	48.83	53.41	40.82	10.45	33.72	36.00
36	61.45	70.22	58.26	19.53	49.33	64.10
37	57.78	72.10	43.83	16.14	31.32	66.40

It is interesting that in using output-input ratios (Theorem 3.2), the identified efficient portfolios, SEC, have higher stability measures with respect to the whole set of efficient portfolios. To illustrate the fact, we consider the data set of 37 R&D projects as listed in Table 11 (Oral et al., 1991), and evaluate all possible collectives of these projects. There are exactly 3298 VRS efficient portfolios, and 167 of them belonged to SEC. We observe that the order ranks, based on stability measures, of the memberships in SEC are significantly higher than the others. The distribution of the order ranks of portfolios in SEC is listed in Table 11. The second row of Table 11 shows there are 9 SEC portfolios in the top 10. It indicates that SEC contained superior portfolios. Therefore, one may not need to solve the collective evaluation problems by using DEA models or our proposed filtering algorithm, since SEC provides many quality portfolios for selection. Output-input ratios could be very easily obtained, even by hand calculation.

Table 11. The distribution of the order ranks of the 167 SECs.

Order ranks	Number	Percent	Cumulative number	Cumulative percent (%)
1-10	9	[illegible]	9	5.4%
11-20	6	[illegible]	15	9.0%
21-100	[illegible]	[illegible]	36	21.6%
101-500	[illegible]	[illegible]	92	55.1%
501-1000	[illegible]	[illegible]	122	73.1%
1001-2000	[illegible]	[illegible]	155	92.8%
>2000	12	7.2%	167	100.0%

The paper presented a new DEA sensitivity approach referring to the non-linear models that may be considered as the extension of super-efficiency models (Seiford & Zhu, 1998b; 1999). The new sensitivity technique provides the stability of efficient portfolios by giving the data variations on a specific project. It cause that a subset of portfolios are perturbed in the same value simultaneously. Fortunately, our proposed stability models can be applied to the case of measuring the stability of efficient DMUs by giving the data variations on a subset of perturbing DMUs simultaneously.

In contrast to the usual DEA sensitivity approaches whose data variations are considered either on the test DMU or on the allover DMUs, this approach proposed the generalized consideration that the uncertainty only affects a subset of DMUs. Sufficient and necessary conditions of stability measures are provided for upward variations of inputs and/or downward variations of outputs on a subset of DMUs simultaneously so that a test efficient DMU remains on the efficient frontier. Sensitivity analysis enhances the fine quality of the final decision. Also, one can have the insight for the comparison between DMUs. Thus, the type of data variation in our analysis is more flexible than the usual approaches.

Although the stability regions of a test efficient DMU for absolute changes in the data is identified, the data change with the same distances are not necessarily true for the real-world applications. However, rescaling all inputs and outputs suitably could be used to prevent this shortcoming. The possible future extensions of our research include: a determination of the whole stability region of a test DMU, change of different scales in different input/output, the stability of efficiency in other DEA models, and proportional data variations.

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