

## A UNIFYING APPROACH TO THE STRUCTURES OF THE STABLE MATCHING PROBLEMS

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(Received July 1989)

**Abstract**—It is well-known that the structure of the set of stable marriages of a stable marriage instance can be represented as a finite distributive lattice and, conversely, every finite distributive lattice is a set of stable marriages for some stable marriage instance. Recently, Irving [12] and Gusfield [9] propose some representations of the set of all stable assignments for a given solvable instance of the stable roommates problem. In this paper, we will give a unifying approach to the structures of the stable marriage problem and the stable roommates problem. To achieve this purpose, we first study the duality in the structure of a stable marriage instance, then transform every stable roommates instance into a corresponding stable marriage instance and obtain the structure of the stable roommates instance directly from that of the corresponding stable marriage instance. The main results of this paper are: (1) There is a one-one correspondence between the set of stable marriages for a stable marriage instance and the set of feasible words of some Faigle geometry; (2) There is a one-one correspondence between the set of stable assignments for a stable roommates instance and the set of basic words of some Faigle geometry.

### 1. INTRODUCTION

An instance of size  $n$  of the *stable marriage problem* consists of  $n$  men and  $n$  women, where each of the  $n$  men and the  $n$  women ranks the members of the opposite sex in order of preference. A complete matching of the men and women is called a *marriage*. A marriage  $M$  is *unstable*, if there is a man and a woman who are not married to each other in  $M$ , but who both prefer each other to their partners in  $M$ . A marriage that is not unstable is called *stable*. It is well-known that there is a stable marriage for any instance of the stable marriage problem [1]. It is also well-known that the structure of the set of stable marriages can be represented as a finite distributive lattice. Conversely, it is shown [2,3] that every finite distributive lattice is a set of stable marriages for some instance of the stable marriage problem.

There is a closely related problem to the stable marriage problem, called the *stable roommates problem*. An instance of size  $n$  of the stable roommates problem consists of a set of  $2n$  people, where each person in the set ranks the  $2n - 1$  others in order of preference. A pairing of the  $2n$  people into  $n$  disjoint pairs is called an *assignment*. An assignment  $\alpha$  is called *unstable* if there are two persons who are not paired together in  $\alpha$ , but they prefer each other to their respective mates in  $\alpha$ . A *stable assignment* is one which is not unstable. An instance of the stable roommates problem is called *solvable* if there is at least one stable assignment. Contrary to the case of the stable marriage problem, there are unsolvable instances of the stable roommates problem.

Recently, Irving [4] and Gusfield [5] give some "small," "implicit" representations of the set of all stable assignments for a given solvable instance of the stable roommates problem: the poset  $\Pi^*$  on the set of "rotations," the poset  $\Pi$  and the undirected graph  $G$  on the set of "nonsingleton rotations." An interesting result (Theorem 5.3., Gusfield [9]) says that there is a one-one correspondence between the maximal independent sets in  $G$  and the set of stable assignments. Furthermore, every maximal independent set in  $G$  has the same cardinality (an alternative version of Lemma 5.6. in [5]). If we let

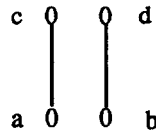
$$\mathcal{M} = (G, \mathcal{I})$$

be the system such that  $\mathcal{I}$  is the collection of all independent sets in  $G$ , including the empty set, then the system  $\mathcal{M}$  is indeed a “matroid” [6] like structure, referred to as “Faigle geometry” [7,8], on the poset  $\Pi$ . This observation motivates our study on the combinatorial structures of the stable matching problems. In addition, we observe that the poset  $\Pi$  is a member of the class of “self-dual posets.”

The *dual poset* of a poset  $\mathcal{P} = (S; \leq)$  is the poset  $\mathcal{P}^d = (S; \geq)$ . If  $\mathcal{P}$  is order-isomorphic to  $\mathcal{P}^d$ , i.e., if there is a bijective function  $\delta : x \mapsto x^\delta$  from  $S$  into itself such that for all elements  $x, y \in S$

- (1)  $x \leq y$ , if and only if,  $y^\delta \leq x^\delta$ ,
- (2)  $(x^\delta)^\delta = x$ ,

then  $\mathcal{P}$  is called a *self-dual poset*. Such a function  $\delta$  is called a *dual assignment* on  $\mathcal{P}$ . Note that there may be many non-isomorphic dual assignments on a given self-dual poset. For example, consider the poset  $\mathcal{P}$  whose diagram is given by



Let

$$a^\delta = c, \quad b^\delta = d, \quad c^\delta = a, \quad d^\delta = b,$$

and

$$a^{\delta'} = d, \quad b^{\delta'} = c, \quad c^{\delta'} = b, \quad d^{\delta'} = a.$$

Then it is easy to see that both  $\delta$  and  $\delta'$  are dual assignments on  $\mathcal{P}$ . However, since  $a \leq a^\delta$  and  $a \parallel a^{\delta'}$  ( $a$  is incomparable to  $a^{\delta'}$ ), we have that  $\delta$  and  $\delta'$  are non-isomorphic. This motivates the following definition:

**DEFINITION 1.1.** A *self-dualized poset*  $\mathcal{P}^\delta = (S; \leq, \delta)$  is a structure consisting of a set  $S$ , a partial order  $\leq$  on  $S$ , and a dual assignment  $\delta$  on the poset  $(S; \leq)$ . For each  $x \in S$ , the element  $x^\delta$  is called the *dual element* of  $x$  in  $\mathcal{P}^\delta$ .

The purpose of this paper is to obtain a unifying combinatorial structure, called *Faigle geometry* (we follow the terminology used in Korte and Lovász [8]), for both the stable marriage and the stable roommates problems. To achieve this purpose, we first study the duality in the structure of the stable marriage problem, then transform every instance of the stable roommates problem into a corresponding instance of the stable marriage problem. The structure of the stable roommates problem can be obtained from that of the stable marriage problem directly by duality.

Given a stable roommates instance  $RI$  of size  $n$ . Let  $S$  be the set of the given  $2n$  preference lists. Then the instance  $RI$  can be transformed into an instance of size  $2n$  of the stable marriage problem by the following:

- (1) Add person  $i$  to the end of the list of himself,  $i = 1, \dots, 2n$ . Let  $S'$  be the resulting set of lists.
- (2) Let  $MS$  and  $WS$  be two identical copies of  $S'$ .
- (3) Let  $MI$  be the instance of the stable marriage problem with  $MS$  and  $WS$  as the sets of male and female preference lists, respectively.

**DEFINITION 1.2.** The instance  $MI$  is called the *instance of the stable marriage problem corresponding to  $RI$* . The *rotation poset* of  $MI$  is also called the *rotation poset* of  $RI$ .

It should be noted that Definition 1.2. is valid for both solvable and unsolvable instances of the stable roommates problem, and the rotation poset of a stable roommates instance under this definition is different from that given in Irving [4] or Gusfield [5].

In this paper, we will show that the rotation poset of a given instance of the stable roommates problem, together with some dual assignment on it, is a self-dualized poset and, conversely, every finite self-dualized poset is an instance of the stable roommates problem. Moreover, we will show

that there is a one-one correspondence between the stable marriages of an instance  $I$  of the stable marriage problem and the feasible words of a Faigle geometry on the rotation poset of  $I$ , and there is a one-one correspondence between the stable assignments of an instance  $RI$  of the stable roommates problem and the basic words of a Faigle geometry on the rotation poset of  $RI$ .

## 2. DEFINITIONS AND ALGORITHMS FOR THE STABLE MARRIAGE PROBLEM

Given an instance of size  $n$  of the stable marriage problem, there is a fundamental "proposal-rejection" algorithm [1] which finds a stable marriage of the given instance, called the *male optimal marriage* (or the *female pessimal marriage*). Recall that a marriage is a complete matching of the  $n$  men and  $n$  women. Henceforth, we will denote a marriage by the notation

$$\{\text{man } i/\text{woman } j_i; i = 1, 2, \dots, n\}.$$

**DEFINITION 2.1.** We say that woman  $j$  accepts the proposal from man  $i$  if she removes from her list each man  $k$  ranked below man  $i$  on her list and, at the same time, is removed from man  $k$ 's list.

### GALE-SHAPLEY ALGORITHM.

*Input:* A set of  $n$  male-preference lists and  $n$  female-preference lists.

*Step 1.* Every man proposes to the first woman in his current list.

*Step 2.* Every woman who receives proposals accepts the best proposal.

*Step 3.* If every woman has a proposal, then STOP; otherwise, GO TO Step 1.

The output of this algorithm is a set of  $2n$  sublists of the original preference lists, and the male optimal marriage is the set of the pairs

$$\{\text{man } i/\text{woman } j_i; i = 1, \dots, n\},$$

where woman  $j_i$  is the first on man  $i$ 's list. The set of all lists of this output possesses several interesting properties [9,10]:

(T1) Every list is nonempty.

(T2) Woman  $j$  is first on man  $i$ 's list, if and only if man  $i$  is last on woman  $j$ 's list.

(T3) Man  $i$  is on woman  $j$ 's list, if and only if woman  $j$  prefers man  $i$  to the last man on her list.

(T4) Woman  $j$  is on man  $i$ 's list, if and only if man  $i$  is on hers.

**DEFINITION 2.2.** Given an instance of size  $n$  of the stable marriage problem. A table is a set of  $2n$  lists, each of which is a sublist of the original preference list, such that the above properties (T1) ~ (T4) are satisfied.

Obviously, the output of the Gale-Shapley algorithm is a table. We will call this table the *male optimal table* (or the *female pessimal table*).

**LEMMA 2.3.** For any table  $T$  of an instance of size  $n$  of the stable marriage problem, if we pair each man with the first woman on his list in  $T$ , then the resulting matching is a stable marriage.

**PROOF.** Let woman  $j_i$  be the first on man  $i$ 's list,  $i = 1, 2, \dots, n$ . For any pair (man  $i$ , woman  $j$ ), where  $j \neq j_i$ , let man  $i_j$  be the last on woman  $j$ 's list. If woman  $j$  prefers man  $i$  to man  $i_j$ , then, by properties (T3) and (T4), man  $i$  is on woman  $j$ 's list and vice versa. However, since woman  $j_i$  is first on man  $i$ 's list, man  $i$  does not prefer woman  $j$  to woman  $j_i$ . This concludes that the matching, by pairing each man  $i$  with woman  $j_i$ , is a stable marriage. ■

**DEFINITION 2.4.** Given a table  $T$ . The stable marriage obtained as in Lemma 2.3 is called the *stable marriage corresponding to  $T$* .

**LEMMA 2.5.** Given a stable marriage  $M$  of an instance of size  $n$  of the stable marriage problem. There exists a table  $T$  such that  $M$  is the stable marriage corresponding to  $T$ .

**PROOF.** Let  $M = \{\text{man } i/\text{woman } j_i; i = 1, 2, \dots, n\}$  and let  $T$  be the set of lists obtained from the original preference lists by letting woman  $j_i$  accept the proposal from man  $i$ . We claim that  $T$  is a table and, hence, is the desired table.

After woman  $j_i$  accepts the proposal from man  $i$ , properties (T1), (T3), (T4) and the property that man  $i$  is on the last of woman  $j_i$ 's list follow directly from Definition 2.1. It remains for us to show that woman  $j_i$  is the first on man  $i$ 's list. Suppose woman  $j_k$  is the first on man  $i$ 's list and  $k \neq i$ . Again, by Definition 2.1, woman  $j_k$  must prefer man  $i$  to man  $k$ . Since man  $i$  prefers woman  $j_k$  to woman  $j_i$ , the given marriage  $M$  is unstable. This yields a contradiction. Hence,  $T$  is a table. ■

**DEFINITION 2.6.** Given an instance of the stable marriage problem. A cyclic sequence  $R$  of man/woman pairs

$$(\text{man } i_k / \text{woman } j_k; k = 0, 1, \dots, r-1)$$

is called a rotation if there exists a table  $T$  such that woman  $j_k$  is the first and woman  $j_{k+1 \bmod r}$  is the second on man  $i_k$ 's list in  $T$  for  $k = 0, 1, \dots, r-1$ . The rotation  $R$  is said to be exposed in  $T$ .

The notion of rotations for the stable marriage problem has been studied in detail in Irving and Leather [10]. In Gusfield [9], a rotation-elimination algorithm is proposed to find all rotations of a stable marriage instance of size  $n$  in  $O(n^2)$  time. To help understand this algorithm, we need the following definition and results.

**DEFINITION 2.7.** Let  $R = (\text{man } i_k / \text{woman } j_k; k = 0, 1, \dots, r-1)$  be a rotation exposed in a table  $T$ . If each woman  $j_k$  accepts the proposal from man  $i_{k-1 \bmod r}$ , where  $k = 0, 1, \dots, r-1$ , then the rotation  $R$  is said to be eliminated from  $T$ .

**LEMMA 2.8.** Let  $R$  be a rotation exposed in a table  $T$ . Let  $T'$  be the set of lists obtained by eliminating  $R$  from  $T$ . Then,  $T'$  is a table.

**PROOF.** It is sufficient to show that  $T'$  possesses property (T2). Observe that man  $i$  is removed from a list in  $T$ , if and only if he is ranked below man  $i_{k-1 \bmod r}$  on woman  $j_k$ 's list in  $T$  for some  $k$ . Moreover, woman  $j$  is removed from a list in  $T$  if  $j = j_k$  for some  $k$ . Let man  $i$  be such that  $i \neq i_k$  for any  $k = 0, 1, \dots, r-1$ , and let woman  $j$  be the first on man  $i$ 's list in  $T$ . Since the matching, by pairing each man with the first woman on his list in  $T$ , is a stable marriage, we have that  $j \neq j_k$  for any  $k$ . Thus  $T'$  inherits the property that man  $i$  is the last on woman  $j$ 's list and woman  $j$  is the first on man  $i$ 's list. As for man  $i_k$ ,  $k = 0, 1, \dots, r-1$ , by Definition 2.1., it is clear that woman  $j_k$  is the first on the list of man  $i_{k-1 \bmod r}$  and he is the last on woman  $j_k$ 's list in  $T'$ . ■

In notation, the table  $T'$  will be denoted as  $T \setminus R$ . Observe that the proof of Lemma 2.8. also implies the following result.

**COROLLARY 2.9.** Let  $R$  and  $R'$  be two distinct rotations exposed in a table  $T$ . Then  $R'$  is also a rotation exposed in the table  $T \setminus R$ .

It is shown (Lemma 4.6. in [10]) that each table can be obtained from the male optimal table by a sequence of zero or more rotation eliminations. We are now in a position to describe the rotation-elimination algorithm.

**ROTATION-ELIMINATION ALGORITHM.**

*Input:* The male optimal table.

*Step 1.* Let  $T$  be the current table.

*Step 2.* If there are no rotations exposed in  $T$ , then STOP; otherwise, GO TO next step.

*Step 3.* Find a rotation  $R$  exposed in  $T$ .

*Step 4.* Eliminate  $R$  from  $T$ ; GO TO Step 1.

This algorithm outputs all rotations of a given stable marriage instance. Gusfield [9] also uses this algorithm to find all stable pairs, which are the pairs appearing in at least one stable marriage. An earlier version of this algorithm is proposed in McVitie and Wilson [11] and is used to find all stable marriages.

### 3. THE LATTICE OF STABLE MARRIAGES AND THE ROTATION POSET

Let  $\mathcal{S}$  be the set of stable marriages of a given stable marriage instance of size  $n$ . For any two stable marriages

$$M_1 = \{\text{man } i / \text{woman } j_i; i = 1, 2, \dots, n\}$$

and

$$M_2 = \{\text{man } i/\text{woman } j'_i; i = 1, 2, \dots, n\},$$

let

$$k_i = \begin{cases} j_i & \text{if man } i \text{ prefers woman } j_i \text{ to woman } j'_i \\ j'_i & \text{otherwise,} \end{cases}$$

and

$$k'_i = \begin{cases} j_i & \text{if } k_i = j'_i, \\ j'_i & \text{if } k_i = j_i. \end{cases}$$

Then the two sets of man/woman pairs

$$M_3 = \{\text{man } i/\text{woman } k_i; i = 1, 2, \dots, n\}$$

and

$$M_4 = \{\text{man } i/\text{woman } k'_i; i = 1, 2, \dots, n\}$$

are stable marriages [15]. Define

$$M_1 \wedge M_2 = M_3 \quad \text{and} \quad M_1 \vee M_2 = M_4.$$

Then, the algebra

$$\mathcal{L} = (\mathcal{S}; \wedge, \vee, O, I)$$

is a distributive lattice [15], where the least element  $O$  is the male optimal marriage and the greatest element  $I$  is the *female optimal marriage*. Note that the female optimal marriage can be obtained from the Gale-Shapley algorithm by reversing the roles of men and women. Dually, if we define

$$M_1 \wedge' M_2 = M_4 \quad \text{and} \quad M_1 \vee' M_2 = M_3,$$

then we have the dual lattice

$$\mathcal{L}^d = (\mathcal{S}; \wedge', \vee', O', I') = (\mathcal{S}; \vee, \wedge, I, O).$$

Let  $\mathcal{E}$  the set of all rotations of a given stable marriage instance. Let

$$R_1 = (\text{man } i_k/\text{woman } j_k; k = 0, 1, \dots, r-1)$$

and

$$R_2 = (\text{man } i'_h/\text{woman } j'_h; h = 0, 1, \dots, s-1)$$

be two distinct rotations in  $\mathcal{E}$ .

**DEFINITION 3.1.** *Rotation  $R_1$  is said to explicitly precede  $R_2$ , if and only if there exist  $k$  and  $h$  ( $0 \leq k \leq r-1$ ,  $1 \leq h \leq s-1$ ), such that woman  $j_{k+1 \bmod r}$  prefers man  $i_k$  to man  $i'_h$  and man  $i'_h$  prefers woman  $j_{k+1 \bmod r}$  to woman  $j'_h$ .*

Now, define a binary relation " $\leq$ " on  $\mathcal{E}$  as below:

$$R \leq R' \text{ if and only if there are rotations } R_1 = R, R_2, \dots, R_k = R' \\ \text{such that } R_{i-1} \text{ explicitly precedes } R_i \text{ for each } i = 2, \dots, k.$$

It is easy to see that the binary relation  $\leq$  is a partial order on  $\mathcal{E}$  and the structure  $\mathcal{B} = (\mathcal{E}, \leq)$  is a poset.

**DEFINITION 3.2.** *The poset  $\mathcal{B} = (\mathcal{E}; \leq)$  is called the rotation poset of the given instance.*

**DEFINITION 3.3.** *Let  $\mathcal{P} = (\mathcal{S}; \leq)$  be a poset. A subset  $H$  of  $\mathcal{S}$  is called hereditary if  $h \in H$  and  $x \leq h$  imply  $x \in H$  for all  $x \in \mathcal{S}$ .*

For any finite poset  $\mathcal{P} = (S; \leq)$ , let  $\text{Hered}(\mathcal{P})$  denote the set of all hereditary subsets of  $S$ . It is a fundamental theorem in lattice theory [12–14] that the lattice

$$\mathcal{L}(\text{Hered}(\mathcal{P})) = (\text{Hered}(\mathcal{P}); \cap, \cup, \phi, S)$$

is a distributive lattice. Conversely, let  $\mathcal{J}$  be the poset of all nontrivial join-irreducible elements of a finite distributive lattice  $\mathcal{L}$  under the partial order from  $\mathcal{L}$ , then

$$\mathcal{L} \cong \mathcal{L}(\text{Hered}(\mathcal{J})).$$

In other words, if  $\mathcal{L}$  is the distributive lattice of stable marriages and  $\mathcal{B}$  is the rotation poset of a given instance, then

$$\mathcal{L} \cong \mathcal{L}(\text{Hered}(\mathcal{B})).$$

The explicit meaning of this isomorphism [10] can be rephrased as:

If  $M$  is the corresponding stable marriage of the hereditary subset  $\mathcal{H}$  of  $\mathcal{E}$ , then  $M$

is the stable marriage corresponding to the table obtained by eliminating all rotations in  $\mathcal{H}$ .

Given a rotation  $R$  in  $\mathcal{E}$ , let  $\mathcal{H}(R)$  denote the *hereditary subset* and  $\mathcal{H}^d(R)$  denote the *dual hereditary subset* of  $\mathcal{E}$  generated by  $R$ . That is,

$$\mathcal{H}(R) = \{Q \in \mathcal{E} : Q \leq R\}$$

and

$$\mathcal{H}^d(R) = \{Q \in \mathcal{E} : R \leq Q\}.$$

**LEMMA 3.4.** *For any  $R \in \mathcal{E}$ , the difference subset  $\mathcal{E} - \mathcal{H}^d(R)$  is hereditary.*

**PROOF.** Let  $Q \in \mathcal{E} - \mathcal{H}^d(R)$  and  $Q' \leq Q$ . If  $Q' \in \mathcal{H}^d(R)$ , then  $R \leq Q' \leq Q$ , a contradiction. Hence,  $Q' \in \mathcal{E} - \mathcal{H}^d(R)$  and  $\mathcal{E} - \mathcal{H}^d(R)$  is hereditary. ■

**DEFINITION 3.5.** *The hereditary subset  $\mathcal{E} - \mathcal{H}^d$  is called the *dual-exclusive hereditary subset* of  $\mathcal{E}$  generated by  $R$  and is denoted by  $\mathcal{H}^e(R)$ .*

**REMARK 3.6.** Given a rotation  $R$  in  $\mathcal{E}$ . Let  $T$  be the resulting table by eliminating all rotations in  $\mathcal{H}^e(R)$  starting from the male optimal table. Since  $R \leq Q$  for any rotation  $Q$  not in  $\mathcal{H}^e(R)$ , we have that  $R$  is the only rotation exposed in  $T$ .

**LEMMA 3.7.** *For any  $R \in \mathcal{E}$ , the subset  $\mathcal{H}(R)$  is join-irreducible in  $\mathcal{L}(\text{Hered}(\mathcal{B}))$ . Conversely, if  $\mathcal{H}$  is a nontrivial join-irreducible element in  $\mathcal{L}(\text{Hered}(\mathcal{B}))$ , then  $\mathcal{H} = \mathcal{H}(R)$  for some  $R \in \mathcal{E}$ .*

**PROOF.** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two hereditary subsets of  $\mathcal{E}$ , such that  $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}(R)$ . Then  $R \in \mathcal{H}_1$  or  $R \in \mathcal{H}_2$ . If  $R \in \mathcal{H}_1$ , then  $\mathcal{H}_1 = \mathcal{H}(R)$ ; if  $R \in \mathcal{H}_2$ , then  $\mathcal{H}_2 = \mathcal{H}(R)$ . Hence,  $\mathcal{H}(R)$  is join-irreducible. Conversely, let  $\mathcal{H} \neq \phi$  be join-irreducible in  $\mathcal{L}(\text{Hered}(\mathcal{B}))$ . We claim that  $\mathcal{H}$  has a greatest element  $R \in \mathcal{E}$  and hence  $\mathcal{H} = \mathcal{H}(R)$ .

Suppose  $\mathcal{H}$  does not have a greatest element. Let  $R_1, R_2, \dots, R_k$  be all the maximal elements of  $\mathcal{H}$ . Let  $\mathcal{H}_1 = \mathcal{H}(R_1)$  and  $\mathcal{H}_2 = \mathcal{H}(R_2) \cup \dots \cup \mathcal{H}(R_k)$ . It is clear that  $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$ . Since  $\mathcal{H}$  is join-irreducible, either  $\mathcal{H}_1 = \mathcal{H}$  or  $\mathcal{H}_2 = \mathcal{H}$ . If  $\mathcal{H}_1 = \mathcal{H}$ , then  $R_i \leq R$  for all  $i \in \{2, \dots, k\}$ ; if  $\mathcal{H}_2 = \mathcal{H}$ , then  $R \leq R_i$  for some  $i \in \{2, \dots, k\}$ . Either case will result in a contradiction. ■

Similarly, we have:

**LEMMA 3.8.** *For any  $R$  in  $\mathcal{E}$ , the subset  $\mathcal{H}^e(R)$  is meet-irreducible in  $\mathcal{L}(\text{Hered}(\mathcal{B}))$ . Conversely, if  $\mathcal{H}$  is a nontrivial meet-irreducible element in  $\mathcal{L}(\text{Hered}(\mathcal{B}))$ , then  $\mathcal{H} = \mathcal{H}^e(R)$  for some  $R$  in  $\mathcal{E}$ .*

## 4. DUALITY IN THE STABLE MARRIAGE PROBLEM

Note that the previous definitions for tables and rotations in Section 2 are male oriented. Henceforth, we will call them *male-oriented tables* and *male-oriented rotations*, respectively. If we reverse the roles of men and women in those definitions, we have the definitions for *female-oriented tables* and *female-oriented rotations*. Moreover, after this reversal, Gusfield's algorithm finds the set of all female-oriented rotations. As for stable marriages, there is no such distinction. That is, the marriage  $\{\text{man } i/\text{woman } j_i; i = 1, \dots, n\}$  is exactly the same as the marriage  $\{\text{woman } j_i/\text{man } i; i = 1, \dots, n\}$ .

Let  $\mathcal{F}$  be the set of female-oriented rotations. Applying the same reversal to the definition of the partial order on  $\mathcal{E}$ , we have a partial  $\leq'$  on  $\mathcal{F}$ . We call the poset  $\mathcal{B}' = (\mathcal{F}; \leq')$  the *female-oriented rotation poset* of the given instance. It is known [15] that  $\mathcal{L}(\text{Hered}(\mathcal{B}')) \cong \mathcal{L}^d$ .

LEMMA 4.1. *Let  $\mathcal{B} = (\mathcal{E}; \leq)$  be the male-oriented rotation poset and  $\mathcal{B}' = (\mathcal{F}; \leq')$  the female-oriented rotation poset of a given stable marriage instance. Then  $\mathcal{B}' \cong \mathcal{B}^d$ . In particular, there is a one-one correspondence between male-oriented rotations in  $\mathcal{E}$  and female-oriented rotations in  $\mathcal{F}$ .*

PROOF. Let  $\mathcal{B}^d = (\mathcal{E}; \geq)$  be the dual poset of the male-oriented rotation poset  $\mathcal{B} = (\mathcal{E}; \leq)$ . Since  $\mathcal{L}^d \cong \mathcal{L}(\text{Hered}(\mathcal{B}^d))$  and a finite distributive lattice is uniquely characterized by the poset of its nontrivial joint-irreducible elements up to isomorphism [14], we have

$$\mathcal{B}' = (\mathcal{F}, \leq') \cong \mathcal{B}^d = (\mathcal{E}, \geq). \quad \blacksquare$$

Given any rotation  $Q$  (male- or female-oriented), let  $\mathcal{H}(Q)$  be the hereditary subset and  $\mathcal{H}^e(Q)$  the dual-exclusive hereditary subset generated by  $Q$ . Then, let  $M(Q)$  and  $M^e(Q)$  be the respective stable marriages corresponding to  $\mathcal{H}(Q)$  and  $\mathcal{H}^e(Q)$ .

For any male-oriented rotation  $R$  in  $\mathcal{E}$ , since  $M(R)$  is join-irreducible in  $\mathcal{L}$ ,  $M(R)$  is meet-irreducible in  $\mathcal{L}^d$ . Hence,  $R$  corresponds to a female-oriented rotation  $R'$  in  $\mathcal{F}$  such that  $M(R) = M^e(R')$ . The following lemma gives an explicit form of such correspondence.

LEMMA 4.2. *Let  $R = (\text{man } i_k/\text{woman } j_k; k = 0, 1, \dots, r-1)$  be a male-oriented rotation in  $\mathcal{E}$ . If  $R'$  is a female-oriented rotation in  $\mathcal{F}$  such that  $M(R) = M^e(R')$ , then*

$$R' = (\text{woman } j_k/\text{man } i_{k-1 \bmod r}; k = 0, 1, \dots, r-1).$$

PROOF. Let  $T'$  be the female-oriented table corresponding to  $M^e(R')$ . We claim that  $(\text{woman } j_k/\text{man } i_{k-1 \bmod r}; k = 0, 1, \dots, r-1)$  is a rotation exposed in  $T'$ .

First, since the marriage  $M(R)$  contains the pairs  $\text{man } i_{k-1 \bmod r}/\text{woman } j_k$ , where  $k = 0, 1, \dots, r-1$ , we have that, in  $T'$ ,  $\text{man } i_{k-1 \bmod r}$  is the first on  $\text{woman } j_k$ 's list and  $\text{woman } j_k$  is the last on  $\text{man } i_{k-1 \bmod r}$ 's list for each  $k$ . Next, since  $\text{man } i_k$  prefers  $\text{woman } j_k$  to  $\text{woman } j_{k+1 \bmod r}$ , by properties (T3) and (T4),  $\text{man } i_k$  is on  $\text{woman } j_k$ 's list and vice versa for each  $k$ . If there is an  $h, 0 \leq h \leq r-1$ , such that  $\text{man } i_h$  is not the second on  $\text{woman } j_h$ 's list, let  $\text{man } i$  be the second on  $\text{woman } j_h$ 's list, then  $\text{woman } j_h$  prefers  $\text{man } i$  to  $\text{man } i_h$ . Let  $\text{woman } j$  be the partner of  $\text{man } i$  in  $M^e(R')$ . Since, in  $T'$ ,  $\text{woman } j$  is the last on  $\text{man } i$ 's list and, by property (T4),  $\text{woman } j_h$  is also on  $\text{man } i$ 's list, we must have that  $\text{man } i$  prefers  $\text{woman } j_h$  to  $\text{woman } j$ . In summary,  $\text{man } i$  and  $\text{woman } j_h$  are not married to each other in  $M^e(R')$  but they prefer each other to their partners in  $M^e(R')$ . That is, the marriage  $M^e(R')$  is unstable, a contradiction. Thus,  $\text{man } i_k$  is the second on  $\text{woman } j_k$ 's list for each  $k$  and then  $(\text{woman } j_k/\text{man } i_{k-1 \bmod r}; k = 0, 1, \dots, r-1)$  is a rotation exposed in  $T'$ .

Since, by Remark 3.6.,  $R'$  is the only rotation exposed in  $T'$ , we conclude that  $R' = (\text{woman } j_k/\text{man } i_{k-1 \bmod r}; k = 0, 1, \dots, r-1)$ .  $\blacksquare$

Dually, we have:

LEMMA 4.3. *Let  $R' = (\text{woman } j_k/\text{man } i_k; k = 0, 1, \dots, r-1)$  be a female-oriented rotation in  $\mathcal{F}$ . If  $R$  is a male-oriented rotation in  $\mathcal{E}$  such that  $M(R') = M^e(R)$ , then*

$$R = (\text{man } i_k/\text{woman } j_{k-1 \bmod r}; k = 0, 1, \dots, r-1).$$

COROLLARY 4.4. For any  $R$  in  $\mathcal{E}$  and  $R'$  in  $\mathcal{F}$ ,

$$M(R) = M^e(R') \text{ if and only if } M(R') = M^e(R).$$

PROOF. Assume  $R = (\text{man } i_k/\text{woman } j_k; k = 0, 1, \dots, r-1)$  and  $M(R) = M^e(R')$ . By Lemma 4.2., we have

$$R' = (\text{woman } j_k/\text{man } i_{k-1 \bmod r}; k = 0, 1, \dots, r-1).$$

If  $Q$  is a male-oriented rotation in  $\mathcal{E}$  such that  $M(R') = M^e(Q)$ , then, by Lemma 4.3.,

$$\begin{aligned} Q &= (\text{man } i_{k-1 \bmod r}/\text{woman } j_{k-1 \bmod r}; k = 0, 1, \dots, r-1) \\ &= (\text{man } i_k/\text{woman } j_k; k = 0, 1, \dots, r-1) \\ &= R \end{aligned}$$

Hence,  $M(R') = M^e(R)$ . The "if" part then follows by duality.  $\blacksquare$

DEFINITION 4.5. A pair  $(R, R')$  of male- and female-oriented rotations with  $M(R) = M^e(R')$  is called a dual pair of rotations. In notation, we write

$$R^d = R' \text{ and } (R')^d = R.$$

For any rotation  $Q$ , the rotation  $Q^d$  is called the dual rotation of  $Q$  in opposite sex orientation.

REMARK 4.6. From Corollary 4.4., it follows that  $(Q^d)^d = Q$  for any rotation  $Q$ .

LEMMA 4.7. For any two rotations  $R_1$  and  $R_2$  in  $\mathcal{E}$ ,

$$R_1 \leq R_2 \text{ in } \mathcal{B} \text{ if and only if } R_2^d \leq R_1^d \text{ in } \mathcal{B}'.$$

PROOF. If  $R_1 \leq R_2$  in  $\mathcal{B}$ , then  $M(R_1) \leq M(R_2)$  in  $\mathcal{L}$ . Since  $M(R_1) = M^e(R_1^d)$  and  $M(R_2) = M^e(R_2^d)$ , we have  $M^e(R_2^d) \leq M^e(R_1^d)$  in  $\mathcal{L}^d$ . Hence,  $R_2^d \leq R_1^d$  in  $\mathcal{B}'$ . The proof of the "if" part is similar.  $\blacksquare$

The duality in the stable marriage problem plays a central role in studying the structure of the rotation poset of a stable roommates instance. This is the main subject of the next section.

## 5. ROTATION POSETS OF THE STABLE ROOMMATES PROBLEM

In Section 1, we transform a stable roommates problem instance  $RI$  into a stable marriage problem instance  $MI$  and call  $MI$  the instance of the stable marriage problem corresponding to  $RI$ . The rotation posets (male- and female-oriented) of  $MI$  are also called the rotation posets of  $RI$ . To start exploiting the structure of these rotation posets, we make the following observation.

OBSERVATION 5.1. Let  $\mathcal{E}$  and  $\mathcal{F}$  be the sets of male- and female-oriented rotations of  $RI$ , respectively. Since the two sets  $MS$  and  $WS$  of preference lists are identical up to sex reversal, these two sets  $\mathcal{E}$  and  $\mathcal{F}$  are also identical in the following sense:

If  $R = (\text{man } i_k/\text{woman } j_k, k = 0, 1, \dots, r-1)$  is a male-oriented rotation in  $\mathcal{E}$ , then  $R' = (\text{woman } i_k/\text{man } j_k, k = 0, 1, \dots, r-1)$  is a female-oriented rotation in  $\mathcal{F}$ , and vice versa.

Such property will be called the *equal right property* of rotations.

Henceforth, for the sake of convenience, we will make use of the following notations and terminology:

- (1) A male-oriented rotation  $R = (\text{man } i_k/\text{woman } j_k, k = 0, 1, \dots, r-1)$  in  $\mathcal{E}$  will be simply written as

$$R = (i_k/j_k, k = 0, 1, \dots, r-1).$$

- (2) An *assignment*  $\alpha$  of the instance  $RI$  is denoted as a permutation

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & 2n \\ j_1 & j_2 & & j_{2n} \end{pmatrix}$$

such that  $(ij_i)$  is a transposition in  $\alpha$  for each  $i = 1, \dots, 2n$ .



(3) Let  $M$  be a stable marriage of the instance  $MI$ . If  $M$  is of the form

$$M = \{\text{man } i/\text{woman } j_i; i = 1, \dots, 2n\},$$

then the mapping  $\sigma_M: i \mapsto j_i$  is a permutation of  $\{1, \dots, 2n\}$ , called the *permutation corresponding to  $M$* .

(4) A permutation  $\sigma$  of  $\{1, \dots, 2n\}$  is called a *feasible permutation* of the instance  $RI$  if  $\sigma$  is the permutation corresponding to a stable marriage of  $MI$ . If, in addition,  $\sigma$  is an assignment of  $RI$ , then it is called a *feasible assignment*.

**LEMMA 5.2.** *An assignment  $\sigma$  of an instance  $RI$  of the stable roommates problem is stable if and only if it is feasible.*

**PROOF.** It is trivial that  $\alpha$  is feasible if it is stable. Conversely, assume  $\alpha$  is feasible but is not stable. Then there exist two persons, say person  $i$  and person  $j$ , such that person  $i$  prefers person  $j$  to person  $\alpha(i)$  and person  $j$  prefers person  $i$  to person  $\alpha(j)$ . Then, in the corresponding stable marriage problem instance  $MI$ , there exist man  $i$  and woman  $j$  such that man  $i$  prefers woman  $j$  to woman  $\alpha(i)$  and woman  $j$  prefers man  $i$  to man  $\alpha(j)$ . That is, the marriage  $\{\text{man } k/\text{woman } \alpha(k); k = 1, \dots, 2n\}$  is unstable, a contradiction. Hence,  $\alpha$  is stable if it is feasible. ■

Let  $R = (i_k/j_k; k = 0, 1, \dots, r-1)$  be a rotation in  $\mathcal{E}$ . Recall that the dual rotation  $R^d$  of  $R$  in the opposite sex orientation is a female-oriented rotation in  $\mathcal{F}$  of the form  $R^d = (\text{woman } j_k/\text{man } i_{k-1 \bmod r}; k = 0, 1, \dots, r-1)$ . By the equal right property, the male-oriented rotation  $(j/i_{k-1 \bmod r}; k = 0, 1, \dots, r-1)$  is also in  $\mathcal{E}$ . Now, we reach a position to establish the following theorem.

**THEOREM 5.3.** *Given a stable roommates instance  $RI$ . Let  $\delta$  be the function from  $\mathcal{E}$  into itself defined by: for any  $R = (i_k/j_k; k = 0, 1, \dots, r-1)$ ,*

$$\delta(R) = (j_k/i_{k-1 \bmod r}; k = 0, 1, \dots, r-1).$$

*Then, the structure  $\mathcal{B}^\delta = (\mathcal{E}; \leq, \delta)$  is a self-dualized poset.*

**PROOF.** From Remark 4.6., Lemma 4.7. and the equal right property, it is easy to see that the function  $\delta: R \mapsto \delta(R)$  is a dual assignment on  $(\mathcal{E}; \leq)$ . Hence,  $\mathcal{B}^\delta$  is self-dualized. ■

**DEFINITION 5.4.** *For any rotation  $R$  in  $\mathcal{E}$ , the rotation  $\delta(R)$  given in Theorem 5.3. is called the dual rotation of  $R$  in same sex orientation.*

**REMARK 5.5.** If there is no danger of confusion, the rotation  $\delta(R)$  will be simply called the *dual rotation* of  $R$ , and will be written as  $R^\delta$ . Also, the male-oriented rotation poset  $\mathcal{B} = (\mathcal{E}; \leq)$  will be simply called the *rotation poset* of a given stable roommates instance.

**LEMMA 5.6.** *Let  $\mathcal{H}$  be a hereditary subset of  $\mathcal{E}$  and  $M$  be the stable marriage corresponding to  $\mathcal{H}$ . If for some rotation  $R$ , both  $R$  and  $R^\delta$  are in  $\mathcal{H}$ , then the permutation  $\sigma_M$  is not a feasible assignment.*

**PROOF.** Assume  $R = (i_k/j_k, k = 0, 1, \dots, r-1)$  and  $R^\delta$  be both in  $\mathcal{H}$ . Let  $T$  be the male-oriented table corresponding to  $\mathcal{H}$ . If man  $i_0$  / woman  $j$  is a pair in  $M$ , since  $R$  has been eliminated, woman  $j$  is ranked below woman  $j_0$  in man  $i_0$ 's list. Similarly, since  $R^\delta$  has been eliminated, woman  $i_0$  cannot be paired with any man ranked below man  $j_0$ . Thus, man  $j$ /woman  $i_0$  is not in  $M$ . ■

Dually, we have:

**LEMMA 5.7.** *Let  $\mathcal{H}$  be a hereditary subset of  $\mathcal{E}$ , and let  $M$  be its corresponding stable marriage. For any rotation  $R$  in  $\mathcal{E}$ , if  $\mathcal{H}$  does not contain  $R$  and  $R^\delta$ , then the permutation  $\sigma_M$  is not a feasible assignment.*

We summarize the above results as in the next theorem.

**THEOREM 5.8.** *Given an instance  $RI$  of the stable roommates problem. Let  $\mathcal{E}$  be the set of all male-oriented rotations of  $RI$ . Let  $\mathcal{H}$  be a hereditary subset of  $\mathcal{E}$  and  $M$  be the stable marriage*

corresponding to  $\mathcal{H}$ . Then the permutation  $\sigma_M$  is a feasible assignment of RI if and only if for each  $R$  in  $\mathcal{E}$ , either  $R$  or  $R^\delta$  is in  $\mathcal{H}$  but not both.

DEFINITION 5.9. A rotation  $R$  in  $\mathcal{E}$  is called *self-dual* if  $R = R^\delta$ .

COROLLARY 5.10. An instance of the stable roommates problem is unsolvable if and only if there is a self-dual rotation.

LEMMA 5.11. Any self-dual rotation is of odd length.

PROOF. Let  $R = (i_k/j_k; k = 0, 1, \dots, r-1)$  be a self-dual rotation of length  $r$ . Since  $R$  is self-dual, that is,  $(j_k/i_{k-1 \bmod r}; k = 0, 1, \dots, r-1) = (i_k/j_k; k = 0, 1, \dots, r-1)$ , we have  $\{i_k; k = 0, 1, \dots, r-1\} = \{j_k; k = 0, 1, \dots, r-1\}$ . Let  $j_0 = i_m$  for some  $m > 0$ . Then, observe that  $j_1 = i_{m+1}, \dots, j_{r-m-1} = i_{r-1}$ . On the other hand, since  $i_m/j_m = j_0/i_{r-1}$ , we have  $i_{r-1} = j_m$ . Therefore,  $m = r - m - 1$  and  $r = 2m + 1$ . ■

For the reason of completeness, we establish the converse of Theorem 5.3. at the end of this section.

THEOREM 5.12. Let  $\mathcal{P}^\delta = (S; \leq, \delta)$  be a finite self-dualized poset. Then the poset  $P = (S; \leq)$  is the rotation poset of a stable roommates instance.

PROOF. See Appendix. ■

## 6. COMBINATORIAL STRUCTURES OF THE STABLE MATCHING PROBLEMS

“Greedoids” are combinatorial structures introduced by Korte and Lovász [7,8] as a structural framework for the greedy algorithm. These structures generalize the well-known combinatorial structures “matroids” by extending the independence axioms of matroids from set systems to languages. There are other combinatorial structures, called *Faigle geometries*, which extend the concept of matroids on finite sets to posets [16,17]. We briefly introduce them in the following.

DEFINITION 6.1. Let  $S$  be a finite set. A word  $\alpha$  on  $S$  is a finite sequence of elements of  $S$ , and is shortly written as the form  $\alpha = x_1x_2\dots x_r$ , where  $x_i$ 's are elements of  $S$ . The number  $r$  is called the *length* of  $\alpha$  and is usually denoted as  $|\alpha|$ . The collection of all possible words on  $S$  is denoted by  $S^*$ .

DEFINITION 6.2. Let  $L$  be a subset of  $S^*$ . The pair  $(S; L)$  is called a *language* on  $S$ .

DEFINITION 6.3. A word  $\alpha$  is called *simple* if no element in  $\alpha$  is repeated. A language  $(S; L)$  is called *simple* if any word in  $L$  is simple.

DEFINITION 6.4. A language  $(S; L)$  is called *hereditary* if it satisfies:

- (H1)  $\phi \in L$ ;
- (H2) if  $\alpha \in L$  and  $\alpha = \beta\gamma$  then  $\beta \in L$ .

DEFINITION 6.5. Let  $(S; L)$  be a simple hereditary language on  $S$ . Any word in  $L$  is called a *feasible word*. Maximal feasible words are called *basic words*. An element  $x$  in  $S$  is called an *isthmus* of  $L$  if it belongs to every basic word.

DEFINITION 6.6. A simple hereditary language  $\mathcal{G} = (S; L)$  is called a *greedoid*, if in addition it satisfies:

- (G3) if  $\alpha, \beta \in L$  and  $|\alpha| > |\beta|$ , then there is an element  $x \in \alpha$  such that  $\beta x$  is in  $L$ .

Note that property (G3) means that every feasible word can be extended to a basic word and every basic word is of the same length.

DEFINITION 6.7. The length of any basic word of a greedoid  $\mathcal{G}$  is called the *rank* of  $\mathcal{G}$ .

Let  $\mathcal{G} = (S; L)$  be a greedoid. For any word  $\alpha$ , let  $\tilde{\alpha}$  denote the underlying set of elements in  $\alpha$ . Then define a binary relation  $\cong$  on  $L$  by:

$$\alpha \cong \beta \quad \text{if and only if} \quad \tilde{\alpha} = \tilde{\beta}.$$

It is easy to see that  $\cong$  is an equivalence relation on  $L$ . Let  $\bar{L}$  be the set of equivalence classes induced by  $\cong$ . Obviously, the structure  $\bar{\mathcal{G}} = (S; \bar{L})$  is a greedoid.

**DEFINITION 6.8.** The greedoid  $\bar{\mathcal{G}} = (S; \bar{L})$  is called the *quotient greedoid of  $\mathcal{G}$  relative to the equivalence relation  $\cong$* .

**DEFINITION 6.9.** Let  $\mathcal{P} = (S; \leq)$  be a finite poset. For any subset  $A$  of  $S$ , a simple word  $\alpha = x_1 x_2 \dots x_r$  is called a *linear extension of  $A$*  if  $\bar{\alpha} = A$  and  $x_i \leq x_j$  implies  $i \leq j$  for any  $1 \leq i, j \leq r$ .

**LEMMA 6.10.** Let  $\mathcal{P} = (S; \leq)$  be a finite poset and  $\alpha, \beta, \gamma$  be words on  $S$  such that  $\alpha = \beta\gamma$ . If  $\alpha$  is a linear extension of  $\tilde{\alpha}$ , then  $\beta$  is a linear extension of  $\tilde{\beta}$ . If, in addition,  $\tilde{\alpha}$  is a hereditary subset of  $S$ , then so is  $\tilde{\beta}$ .

**PROOF.** Let  $\beta = x_1 x_2 \dots x_r$  and  $\gamma = x_{r+1} \dots x_{r+s}$ . If  $\alpha$  is a linear extension of  $\tilde{\alpha}$ , then, in particular,  $x_i \leq x_j$  implies  $i \leq j$  for  $1 \leq i, j \leq r$ . Hence,  $\beta$  is a linear extension of  $\tilde{\beta}$ . Moreover, if  $\tilde{\alpha}$  is hereditary, then for any  $x_j \in \tilde{\beta} \subset \tilde{\alpha}$  and  $y \leq x_j$ , we have  $y \in \tilde{\alpha}$ . That is,  $y = x_k$  for some  $1 \leq k \leq r + s$ . Since  $\alpha$  is a linear extension,  $y = x_k \leq x_j$  implies  $k \leq j$ . Hence,  $y \in \tilde{\beta}$  and  $\tilde{\beta}$  is hereditary. ■

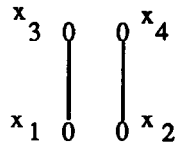
**DEFINITION 6.11.** Let  $\mathcal{P} = (S; \leq)$  be a finite poset and  $(S; L)$  be a greedoid on  $S$ . Then the structure  $\mathcal{F} = (S; \leq, L)$  is a *Faigle geometry* if:

(F4) if  $\alpha \in L$ , then  $\alpha$  is a linear extension of  $\tilde{\alpha}$ ;

(F5) for all hereditary subsets  $H_1$  and  $H_2$  with  $H_1 \subset H_2$ , if  $x \in H_1$  is an isthmus of  $L \cap H_2^*$  then  $x$  is an isthmus of  $L \cap H_1^*$ .

**REMARK 6.12.** If  $\mathcal{F} = (S; \leq, L)$  is a Faigle geometry and  $(S; \bar{L})$  is the quotient greedoid of  $(S; L)$ , then it is easy to see that  $\bar{\mathcal{F}} = (S; \leq, \bar{L})$  is also a Faigle geometry on  $(S; \leq)$ . This geometry  $\bar{\mathcal{F}}$  will be called the *quotient geometry of  $\mathcal{F}$* .

**EXAMPLE 6.13.** Let  $(S; \leq)$  be a poset with the following diagram



Let  $L_1 = \{\phi, x_2, x_2 x_1\}$ . Then  $(S; L_1)$  is a greedoid. However, consider the hereditary subsets  $H_1 = \{x_1\}$  and  $H_2 = \{x_1, x_2\}$ , since  $x_1$  is an isthmus of  $L_1 \cap H_2^* = L_1$  but is not an isthmus of  $L_1 \cap H_1^* = \{\phi\}$ , the structure  $(S; \leq, L_1)$  is not a Faigle geometry.

If we let  $L_2 = \{\phi, x_1, x_2, x_1 x_2, x_2 x_1\}$ , then it is easy to see that the structure  $(S; \leq, L_2)$  is a Faigle geometry.

**THEOREM 6.14.** Let  $\mathcal{P} = (S; \leq)$  be a finite poset and let  $(S; L)$  be a simple hereditary language on  $S$  such that the basic words are the linear extensions of  $S$ . Then  $(S; \leq, L)$  is a Faigle geometry on  $\mathcal{P}$ .

**PROOF.** From Lemma 6.11., it is easy to verify that properties (H1), (H2) and (F4) hold. Observe that any element in a hereditary subset  $H$  is in all linear extensions of  $H$ . That is, every element in  $H$  is an isthmus of  $L \cap H^*$ . Particularly, property (F5) also holds. It remains for us to show property (G3) is satisfied. Let  $\alpha, \beta \in L$  with  $|\alpha| > |\beta|$ . Let  $x$  be a minimal element of  $\tilde{\alpha} - \tilde{\beta}$ . We claim that  $\beta x$  is in  $L$ . First, since  $\beta$  is hereditary and  $x$  is not in  $\beta$ , we have that  $x$  is not  $\leq y$  for any  $y$  in  $\beta$ . Hence,  $\beta x$  is a linear extension of  $\tilde{\beta} \cup \tilde{x}$ . Next, let  $z \in (\tilde{\beta} \cup \tilde{x})$  and  $y \leq z$ .

Case 1.  $z \in \tilde{\beta}$ .

Since  $\tilde{\beta}$  is hereditary, we have  $y \in \tilde{\beta} \subset (\tilde{\beta} \cup \tilde{x})$ .

Case 2.  $z = x$ .

Since  $\tilde{\alpha}$  is hereditary,  $y \in \tilde{\alpha}$ . If  $y \notin \tilde{\beta}$ , since  $x$  is minimal in  $\tilde{\alpha} - \tilde{\beta}$ , we have  $y = x$ .

Hence, either  $y \in \tilde{\beta}$  or  $y = x$ .

Combining Case 1 and Case 2, we have that  $\tilde{\beta} \cup \tilde{x}$  is hereditary and, hence, is in  $L$ . ■

**DEFINITION 6.15.** Let  $\mathcal{P} = (S; \leq)$  be a finite poset. The Faigle geometry  $(S; \leq, L)$  on  $\mathcal{P}$  obtained as in Theorem 6.14. is called the complete Faigle geometry on  $\mathcal{P}$  (is called the poset greedoid on  $S$  in [7,8]).

**COROLLARY 6.16.** Let  $(S; \leq, L)$  be the complete Faigle geometry on the finite poset  $\mathcal{P} = (S; \leq)$ . Then there is one-one correspondence between the feasible words of  $\bar{L}$  and the hereditary subsets of  $S$ .

**PROOF.** The mapping by sending  $\bar{\alpha}$  to  $\tilde{\alpha}$  is one to one from  $\bar{L}$  onto  $\text{Hered}(\mathcal{P})$ . ■

Therefore, the quotient geometry  $(S; \leq, \bar{L})$  is a combinatorial aspect of the distributive lattice

$$\mathcal{L} = (\text{Hered}(\mathcal{P}); \cap, \cup, \phi, S).$$

Corresponding to the stable marriage problem, we have the following theorem.

**THEOREM 6.17.** Given an instance  $I$  of the stable marriage problem. Let  $\mathcal{B} = (\mathcal{E}; \leq)$  be the rotation poset of  $I$ , and let  $\mathcal{F} = (\mathcal{E}; \leq, L)$  be the complete Faigle geometry on  $\mathcal{B}$ . Then there is a one-to-one correspondence between the stable marriages of  $I$  and the feasible words of the quotient geometry  $\bar{\mathcal{F}} = (\mathcal{E}; \leq, \bar{L})$  on  $\mathcal{B}$ .

**DEFINITION 6.18.** Given a self-dualized poset  $\mathcal{P}^\delta = (S; \leq, \delta)$ . Let  $\mathcal{F} = (S; \leq, L)$  be the complete Faigle geometry on  $(S; \leq)$ . A word  $\alpha$  in  $L$  is called dual-exclusive if for any  $x$  in  $S$ ,  $x$  and  $x^\delta$  cannot appear in  $\alpha$  at the same time.

It should be clear that a dual-exclusive word does not contain any self-dual element. Let  $L^\epsilon = \{\alpha \in L : \alpha \text{ is dual-exclusive}\}$ .

**LEMMA 6.19.** The structure  $\mathcal{F}^\epsilon = (S; \leq, L^\epsilon)$  is a Faigle geometry.

**PROOF.** It is enough for us to verify that property (G3) holds. Let  $\alpha, \beta$  be in  $L^\epsilon$  with  $|\alpha| > |\beta|$ . Let  $\tilde{\beta}^\delta = \{x^\delta : x \in \tilde{\beta}\}$  and assume  $\tilde{\alpha} \subset (\tilde{\beta} \cup \tilde{\beta}^\delta)$ . Since  $\alpha$  is dual-exclusive,

$$|\tilde{\alpha}| \leq (|\tilde{\beta}| + |\tilde{\beta}^\delta|)/2 = |\tilde{\beta}|,$$

we have a contradiction. Hence, the set  $A = \tilde{\alpha} - (\tilde{\beta} \cup \tilde{\beta}^\delta)$  is nonempty. Choose a minimal element  $x$  in  $A$ . Obviously,  $\beta x$  is dual-exclusive and is a linear extension of  $\tilde{\beta} \cup \tilde{x}$ . To complete the proof, we have to show that  $\tilde{\beta} \cup \tilde{x}$  is hereditary. Let  $z \in (\tilde{\beta} \cup \tilde{x})$  and  $y \leq z$ .

Case 1.  $z \in \tilde{\beta}$ .

Since  $\tilde{\beta}$  is hereditary,  $z \in \tilde{\beta}$  implies  $y \in \tilde{\beta}$ .

Case 2.  $z = x$ .

Since  $\tilde{\alpha}$  is hereditary, we have  $y \in \tilde{\alpha}$ . If  $y \notin (\tilde{\beta} \cup \tilde{\beta}^\delta)$ , then the minimality of  $x$  implies  $y = x$ . If  $y \in (\tilde{\beta} \cup \tilde{\beta}^\delta)$ , then we must have  $y \in \tilde{\beta}$ ; otherwise,  $y^\delta \in \tilde{\beta}$  and  $x^\delta \leq y^\delta$  imply  $x^\delta \in \tilde{\beta}$  and  $x \in \tilde{\beta}^\delta$ .

Therefore, the set  $\tilde{\beta} \cup \tilde{x}$  is hereditary. ■

**REMARK 6.20.** The rank of  $\mathcal{F}^\epsilon$  is less than or equal to  $|S|/2$ . If the rank of  $\mathcal{F}^\epsilon$  is less than  $|S|/2$ , then there must be an element  $x \in S$  with  $x = x^\delta$ , i.e.,  $x$  is a self-dual element.

**DEFINITION 6.21.** The Faigle geometry  $\mathcal{F}^\epsilon$  is called the dual-exclusive geometry on  $(S; \leq)$ .

Corresponding to the stable roommates problem, we have:

**THEOREM 6.22.** Given an instance  $RI$  of the stable roommates problem. Let  $\mathcal{B} = (\mathcal{E}; \leq)$  be the rotation poset of  $RI$  and  $\mathcal{F}^\epsilon = (\mathcal{E}; \leq, L^\epsilon)$  be the dual-exclusive geometry on  $\mathcal{B}$ . Then:

- (1) The instance  $RI$  is solvable if and only the rank of  $\mathcal{F}^\epsilon$  is equal to  $|\mathcal{E}|/2$ .
- (2) If  $RI$  is solvable, then there is a one-to-one correspondence between the stable assignments of  $RI$  and the basic words of the quotient geometry  $\bar{\mathcal{F}}^\epsilon = (\mathcal{E}; \leq, \bar{L}^\epsilon)$  of  $\mathcal{F}^\epsilon$ .

## 7. CONCLUSION

Given a stable marriage instance of size  $n$ . Its rotation poset can be constructed [9] in  $O(n^2)$  time. Hence, the rotation poset of a stable roommates instance of size  $n$  can be constructed in  $O(n^2)$  time as well. It should be noted that the "singleton rotations" mentioned in Irving [4,18] are the rotations  $R$  with  $R \leq R^\delta$ , and the "nonsingleton rotations" are the rotations  $R$  with  $R \parallel R^\delta$ . Moreover, a path from the root to a leaf in the execution tree  $D$  defined in [5] is a basic word in the Faigle geometry  $\mathcal{F}^e$  and a path set is a basic word in the quotient geometry  $\bar{\mathcal{F}}^e$ . Therefore, the Faigle geometry  $\mathcal{F}^e$  can be served as the universal structure on the rotation-elimination algorithm of the stable roommates problem.

Finally, since the greedy algorithm for some structure on finite poset works if and only if this structure is a Faigle geometry [16], the greedy algorithm might work for some optimization problems of the stable matching problems.

## REFERENCES

1. D. Gale and L. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* **69** (1962).
2. C. Blair, Every finite distributive lattice is a set of stable matchings, *J. Comb. Theory, Series A* **37**, 353-356 (1984).
3. D. Gusfield, R. Irving, P. Leather and M. Saks, Every finite distributive lattice is a set of stable matchings for a small stable marriage, *J. Comb. Theory, Series A* **44**, 304-309 (1987).
4. R. Irving, On the stable room-mates problem, Res. Report CSC/86/R5, Department of Computing Science, University of Glasgow, (1986).
5. D. Gusfield, The structure of the stable roommate problem: efficient representation and enumeration of all stable assignments, *SIAM J. Comput.* **17**, 742-769 (1988).
6. D. Welsh, *Matroid Theory*, A.P., (1976).
7. B. Korte and L. Lovász, Posets, matroids, and greedoids, in matroid theory, *Coll. Math. Soc. J. Bolyai* **40**, 239-265 (1982).
8. B. Korte and L. Lovász, Greedoids—A structural framework for the greedy algorithm, *Progress in Combinatorial Optimization*, A.P., pp. 221-243, (1984).
9. D. Gusfield, Three fast algorithms for four problems in stable marriage, *SIAM J. Comput.* **16**, 111-128 (1987).
10. R. Irving and P. Leather, The complexity of counting stable marriages, *SIAM J. Comput.* **15**, 655-667 (1986).
11. D. McVitie and L.B. Wilson, The stable marriage problem, *CACM* **14**, 486-492 (1971).
12. G. Birkhoff, *Lattice Theory*, AMS, Providence, R.I., (1979).
13. P. Crawley and R. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall Inc., Englewood Cliffs, N.J., (1973).
14. G. Grätzer, *General Lattice Theory*, A.P. (1978).
15. D. Knuth, *Mariages Stables*, Les Presses de L'Université de Montreal, (1976).
16. U. Faigle, The greedy algorithm for partially ordered sets, *Discrete Math.* **28**, 153-159 (1979).
17. U. Faigle, Geometries on partially ordered sets, *J. Comb. Theory, Series B* **28**, 26-51 (1980).
18. R. Irving, An efficient algorithm for the stable room-mates problem, *J. Algorithms* **6**, 577-595 (1985).

## APPENDIX

EVERY FINITE SELF-DUALIZED POSET IS THE  
ROTATION POSET OF AN INSTANCE OF THE STABLE ROOMMATES PROBLEM

Let  $\mathcal{P}^\delta = (S; \leq, \delta)$  be a finite self-dualized poset. Label the elements of  $S$  so that

$$S = \{x_1, x_1^\delta, x_2, x_2^\delta, \dots, x_k, x_k^\delta, x_{k+1}, x_{k+1}^\delta, \dots, x_{k+s}, x_{k+s}^\delta\}$$

and

$$\text{either } x_i \leq x_i^\delta \text{ or } x_i \parallel x_i^\delta \text{ in } P, \text{ for each } i = 1, \dots, k.$$

For each  $i = 1, \dots, k$ , we associate it with four persons, person  $4(i-1) + j$ ,  $j = 1, 2, 3, 4$ , and construct a portion of the preference lists as follows:

		first position	current-last position
person	4(i-1)+1	4(i-1)+3	4(i-1)+4
	4(i-1)+2	4(i-1)+4	4(i-1)+3
	4(i-1)+3	4(i-1)+2	4(i-1)+1
	4(i-1)+4	4(i-1)+1	4(i-1)+2

For each  $t = 1, \dots, s$ , we associate it with three persons, person  $3(t-1) + 4k + j$ ,  $j = 1, 2, 3$  and construct a portion of the preference lists as follows:

		first position	...	current-last position
person	$3(t-1)+4k+1$	$3(t-1)+4k+2$	...	$3(t-1)+4k+3$
	$3(t-1)+4k+2$	$3(t-1)+4k+3$	...	$3(t-1)+4k+1$
	$3(t-1)+4k+3$	$3(t-1)+4k+1$	...	$3(t-1)+4k+2$

Then consider the following cases:

Case 1.  $x_i \leq x_j^6$  for some  $i$ ,  $1 \leq i \leq k$ .

Place  $4(i-1)+4$  in any position between the first and current-last positions on the list of person  $4(i-1)+3$  and place  $4(i-1)+3$  in any position between the first and current-last positions on the list of person  $4(i-1)+4$ .

Case 2.  $x_i \leq x_j$  for some  $i \neq j$ ,  $1 \leq i, j \leq k$ .

Place  $4(i-1)+3$  in any position between the first and current-last positions on the list of person  $4(j-1)+1$  and place  $4(j-1)+1$  in any position between the first and current-last positions on the list of person  $4(i-1)+3$ .

Case 3.  $x_i \leq x_j^6$  for some  $i \neq j$ ,  $1 \leq i, j \leq k$ .

Place  $4(i-1)+3$  in any position between the first and current-last positions on the list of person  $4(j-1)+3$  and place  $4(j-1)+3$  in any position between the first and current-last positions on the list of person  $4(i-1)+3$ .

Case 4.  $x_i \leq x_{k+t}$  for some  $i$  and  $t$ ,  $1 \leq i \leq k$  and  $1 \leq t \leq s$ .

Place  $4(i-1)+3$  in any place between the first and current-last positions on the list of person  $3(t-1)+4k+1$  and place  $3(t-1)+4k+1$  in any position between the first and current-last positions on the list of person  $4(i-1)+3$ .

Case 5.  $x_{k+t} \leq x_i$  for some  $i$  and  $t$ ,  $1 \leq i \leq k$  and  $1 \leq t \leq s$ .

Place  $3(t-1)+4k+1$  in any position between the first and current-last positions on the list of person  $4(i-1)+1$  and place  $4(i-1)+1$  in any position between the first and current-last positions on the list of person  $3(t-1)+4k+1$ .

To complete the preference lists, place any missing entries after the current-last position on each list in any order. If  $s$  is odd, then we join person  $3s+4k+1$  to the group, place  $3s+4k+1$  in the last position on the list of person  $j$  for each  $j = 1, \dots, 3s+4k$ , and fill up the list of person  $3s+4k+1$  arbitrarily.

Let  $RI$  denote the constructed instance of the roommates problem. We claim that the rotation poset of  $RI$  is isomorphic to the self-dualized poset  $\mathcal{P}^6$ .

Applying the Gale-Shapley algorithm to the corresponding stable marriage instance  $MI$ , we obtain the following male-optimal table:

		first position	...	current-last position
male lists:	man 1	$f_1$	...	$l_1$
	$\vdots$			
	$3s+4k$	$f_{3s+4k}$	...	$l_{3s+4k}$
	$3s+4k+1$	$3s+4k+1$		(if $s$ is odd)

		first position	...	current-last position
female lists:	woman 1	$f_1$	...	$l_1$
	$\vdots$			
	$3s+4k$	$f_{3s+4k}$	...	$l_{3s+4k}$
	$3s+4k+1$	$3s+4k+1$		(if $s$ is odd)

where

- (1) if  $j = 4(i-1)+1$ ,  $1 \leq i \leq k$ , then  $f_j = 4(i-1)+3$  and  $l_j = 4(i-1)+4$ ;
- (2) if  $j = 4(i-1)+2$ ,  $1 \leq i \leq k$ , then  $f_j = 4(i-1)+4$  and  $l_j = 4(i-1)+3$ ;
- (3) if  $j = 4(i-1)+3$ ,  $1 \leq i \leq k$ , then  $f_j = 4(i-1)+2$  and  $l_j = 4(i-1)+1$ ;
- (4) if  $j = 4(i-1)+4$ ,  $1 \leq i \leq k$ , then  $f_j = 4(i-1)+1$  and  $l_j = 4(i-1)+2$ ;
- (5) if  $j = 3(t-1)+4k+1$ ,  $1 \leq t \leq s$ , then  $f_j = 3(t-1)+4k+2$  and  $l_j = 3(t-1)+4k+3$ ;
- (6) if  $j = 3(t-1)+4k+2$ ,  $1 \leq t \leq s$ , then  $f_j = 3(t-1)+4k+3$  and  $l_j = 3(t-1)+4k+1$ ;
- (7) if  $j = 3(t-1)+4k+3$ ,  $1 \leq t \leq s$ , then  $f_j = 3(t-1)+4k+1$  and  $l_j = 3(t-1)+4k+2$ .

Observe that the rotations of the instance  $MI$  are:

$$R_i = (4(i-1) + 1/4(i-1) + 3, 4(i-1) + 2/4(i-1) + 4)$$

$$R_i^\delta = (4(i-1) + 3/4(i-1) + 2, 4(i-1) + 4/4(i-1) + 1), \quad i = 1, \dots, k,$$

and

$$R_t = (3(t-1) + 4k + 1/3(t-1) + 4k + 2, 3(t-1) + 4k + 2/3(t-1) + 4k + 3, 3(t-1) + 4k + 3/3(t-1) + 4k + 1,$$

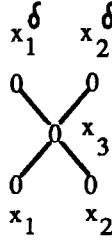
$$= R_t^\delta \quad t = 1, \dots, s.$$

Furthermore, observe that the mapping  $x_i \mapsto R_i$  is an isomorphism from the poset  $\mathcal{P}^\delta = (S; \leq, \delta)$  into the rotation poset  $\mathcal{B}^\delta = (\mathcal{E}; \leq, \delta)$  of  $MI$ . For instance, if  $x_i \leq x_j$ ,  $1 \leq i \neq j \leq k$ , then on the list of person  $4(j-1) + 1$  before the elimination of  $R_i$ ,  $4(i-1) + 3$  is sitting between  $4(j-1) + 3$  and  $4(j-1) + 4$ . Thus,  $R_i \leq R_j$ . The other cases are similar. We summarize the above result as the following theorem.

**THEOREM.** *Let  $\mathcal{P}^\delta$  be a self-dualized poset with  $2k$  non-self-dual elements and  $s$  self-dual elements. Then there is an instance  $RI$  of size  $4k + 3s$  if  $s$  is even and  $4k + 3s + 1$  if  $s$  is odd of the stable roommates problem such that  $\mathcal{P}^\delta$  is order-isomorphic to the rotation poset of  $RI$ .*

**EXAMPLE.**

Let



Then a corresponding instance of the stable roommates problem is one with the following preference lists.

person	1	3	4	12
	2	4	3	12
	3	2	4	9
	4	1	3	2
	5	7	8	
	6	8	7	
	7	6	8	9
	8	5	7	6
	9	10	3	7
	10	11	9	
	11	9	10	
	12			

(fill up the rest of the preference lists arbitrarily)

The rotations of the above instance are:

$$R_1 = (1/3, 2/4), \quad R_1^\delta = (3/2, 4/1),$$

$$R_2 = (5/7, 6/8), \quad R_2^\delta = (7/6, 8/5),$$

and

$$R_3 = R_3^\delta = (9/10, 10/11, 11/9).$$