國立交通大學

資訊科學與工程研究所

碩 士 論 文

增強立方體之最大區域連通性質

Maximally Local Connectivity on Augmented Cubes

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摘要

連通性在連結網路中是很重要的問題。Choudum 和 Sunitha 在 2002 年時提出了增強立方體 AQ_n 並指出增強立方體具有以下性質: (1)當 $n \geq 4$ 時，對 AQ_n 中任意壞點所成的集合 F 而言，若壞點個數總和不超過 $2n - 2$，則 $AQ_n - F$ 爲一包含 $2^n - |F|$個點的連通單元。 (2)當 $n \geq 4$ 時，AQ_n 中任意兩點 u 和 v 之間，存在 $2n - 1$ 條點不相交路徑。在本篇論文中，我們提供了更進一步的結果，包括 (1)當 $n \geq 4$ 時，對 AQ_n 中任意壞點所成的集合 F 而言，若壞點個數總和不超過 $4n - 9$，則 $AQ_n - F$ 中的最大連通單元至少有 $2^n - |F| - 1$ 個點。 (2)當 $n \geq 4$ 時，對 AQ_n 中任意壞點所成的集合 F 而言，若壞點個數總和不超過 $2n - 7$ 時，則對於 AQ_n 中任意兩個非壞點 u 和 v 之間，存在 $\min\{\deg_f(u), \deg_f(v)\}$條點不相交且無壞點路徑。

關鍵字：連通性，最大連通單元，點不相交路徑，增強立方體。

Maximally Local Connectivity on Augmented Cubes

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Abstract

Connectivity is an important issue in interconnection networks. In 2002, Choudum and Sunitha proposed the augmented cube AQ_n and indicated that it has the following properties: (1)for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 2n-2$ for $n \geq 4$, $AQ_n - F$ is a connected component with $2^n - |F|$ vertices; and (2)for any two vertices u and v of AQ_n with $n \geq 4$, there are $2n-1$ vertex-disjoint paths joining u and v. In this paper, we show some further results about (1)for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 4n-9$ for $n \geq 4$, the maximal connected component of $AQ_n - F$ has at least $2^n - |F| - 1$ vertices; and (2)for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 2n-7$ for $n \geq 4$, each pair of non-faulty vertices u and v in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths.

Keywords: connectivity, maximal connected component, vertex-disjoint path, augmented cube.

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中華民國九十七年六月

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Chapter 1

Introduction

For the graph definitions and notations, we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(a, b)|(a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. For the interconnection network topology, it is usually represented by a graph $G = (V, E)$, while vertices represent processors and edges represent links between processors. The *neighborhood* of vertex v, denoted by $N(v)$, is $\{x|(v, x) \in E\}$. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of vertices in $N(v)$. A graph G is *k-regular* if $\deg(v) = k$ for every vertex $v \in V$. For the purpose of connecting hundreds or thousands of processing elements, many interconnection network topologies have been proposed in the literature. Graph theory can be used to analyze the network reliability. We use the terminology of *graphs* and *networks* synonymously.

The *fault tolerance* of a network with respect to processor failures is directly related to the connectivity of the corresponding graph. The connectivity of a graph is an important issue related to the reliability and fault tolerance of a network in graph theory. The *connectivity* of G, denoted by $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. The traditional connectivity only consider that under how many faulty vertices, the network may be disconnected. We know that $\kappa(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. However, while the number of faulty vertices is greater than the connectivity of a network, what will happen? Many measures on fault tolerance of networks are related to the maximal size of connected component of networks with faulty vertices. To estimate the maximal connected component of the network with the faulty vertices is essential [1], Yang et al. [11, 12, 13] have continually proposed results on the maximal connected component of the *n-dimensional hypercube*.

A useful distributed system offers the advantage of improved connectivity. Menger's Theorem [7] is a famous result on connectivity, it shows that if a network G is k-connected, then every pair of vertices in G is connected by k vertex-disjoint (parallel) paths. An efficient routing can be achieved by vertex-disjoint paths. A routing using vertex-disjoint paths provides parallel routing and high fault tolerance, increases the efficiency of data transmission, and decreases transmission time. Saad and Schultz [9] have shown the n vertex-disjoint parallel paths of an n-dimensional hypercube Q_n. Day and Tripathi [5] also have shown the $n-1$ vertex-disjoint parallel paths of an $(n-1)$-dimensional star graph S_n for any two vertices of S_n.

There are many useful topologies proposed to balance the performance and some cost parameters. Among them, the binary hypercube Q_n is one of the most popular topologies, and has been studied for parallel networks. The augmented cubes are derivatives of the hypercubes with good geometric nature and retain all the favorable properties of the hypercubes, such as vertex symmetry, maximum connectivity, best possible wide diameter, routing, and broadcasting procedures with linear time complexity. The augmented cube of dimension n is a Cayley graph, $(2n-1)$-regular, $(2n-1)$-connected, and has diameter $\lceil n/2 \rceil$ [4]. In this paper, we consider the maximal connected component of the augmented cube with faulty vertices. We shall show that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 4n-9$ for $n \geq 4$, the maximal connected component of $AQ_n - F$ has at least $2^n - |F| - 1$ vertices. In addition, we show that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 2n-7$ for $n \geq 4$, each pair of non-faulty vertices u and v in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $AQ_n - F$, respectively.

In the next chapter, we give the definition of the augmented cube AQ_n for $n \geq 1$. Chapter 3 deals with the maximal connected component of $AQ_n - F$ with $|F| \leq 4n-9$ for $n \geq 4$. Chapter 4 studies the vertex-disjoint fault-free paths in $AQ_n - F$ with $|F| \leq 2n-7$ for $n \geq 4$. Finally, we draw the conclusion in Chapter 5.

Chapter 2

The Augmented Cube AQ_n

The definition of the n-dimensional augmented cube is stated as follows. Let $n \geq 1$ be a positive integer. The *n-dimensional augmented cube* [4], denoted by AQ_n, is a vertex transitive and $(2n-1)$-regular graph with 2^n vertices. Each vertex is labeled by an n-bit binary string and $V(AQ_n) = \{u_n u_{n-1} ... u_1 | u_i \in \{0,1\}\}$. AQ_1 is the complete graph K_2 with vertex set $\{0,1\}$ and edge set $\{(0,1)\}$. As for $n \geq 2$, AQ_n consists of (1)two copies of $(n-1)$-dimensional augmented cubes, denoted by AQ^0_{n-1} and AQ^1_{n-1}; and (2)2^n edges (two perfect matchings of AQ_n) between AQ^0_{n-1} and AQ^1_{n-1}. We may write AQ_n as $AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ for $n \geq 2$. $V(AQ^0_{n-1}) = \{0u_{n-1}u_{n-2}...u_1 \mid u_i \in \{0,1\}\}$ and $V(AQ^1_{n-1}) = \{1v_{n-1}v_{n-2}...v_1 \mid v_i \in \{0,1\}\}$. Vertex $u = 0u_{n-1}u_{n-2}...u_1$ of AQ^0_{n-1} is joined to vertex $v = 1v_{n-1}v_{n-2}...v_1$ of AQ^1_{n-1} if and only if either

(i) $u_i = v_i$ for $1 \leq i \leq n-1$; in this case, (u,v) is called a *hypercube edge* and we set $v = u^h$, or

(ii) $u_i = \bar{v}_i$ for $1 \leq i \leq n-1$; in this case, (u,v) is called a *complement edge* and we set $v = u^c$.

The augmented cubes AQ_1, AQ_2, and AQ_3 are illustrated in Figure 2.1.

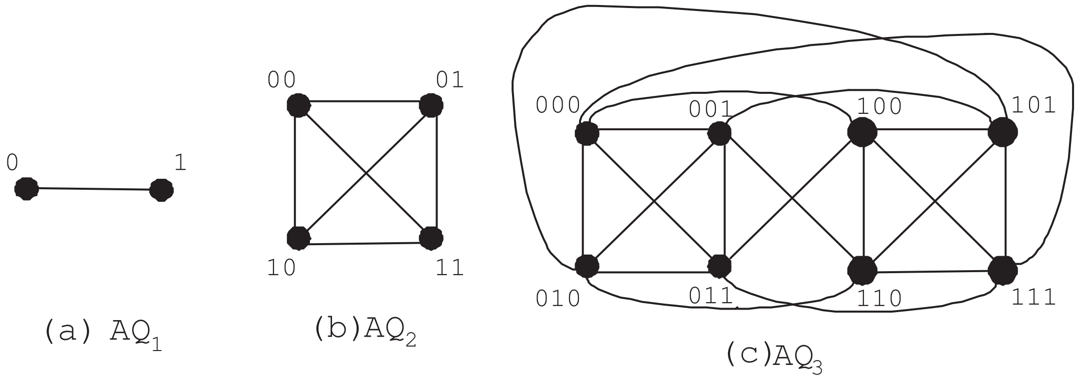

Figure 2.1: The augmented cubes AQ_1, AQ_2, and AQ_3.

Let the hypercube edge set of AQ_n be E_n^h and the complete edge set of AQ_n be E_n^c. Thus, $E_n^h = \{(u, u^h)|\ u \in V(AQ_{n-1}^0)\}$ and $E_n^c = \{(u, u^c)|\ u \in V(AQ_{n-1}^0)\}$. Obviously, each of E_n^h and E_n^c is a perfect matching between the vertices of AQ_{n-1}^0 and AQ_{n-1}^1. Then, both $|E_n^h|$ and $|E_n^c|$ are equal to 2^{n-1}.

Chapter 3

Maximally Connected Component

Lemma 1 *Assume n is an integer with $n \geq 3$. Let $AQ_n = AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ be an n-dimensional augmented cube, and let u and v be any two vertices in AQ^0_{n-1}. Then, the vertices u and v have totally two distinct neighborhoods in AQ^1_{n-1} if $u = 0a_{n-1}\dots a_1$ and $v = 0\overline{a_{n-1}\dots a_1}$ with $a_i \in \{0,1\}$ for $1 \leq i \leq n-1$. Otherwise, the vertices u and v have totally four distinct neighborhoods in AQ^1_{n-1}. That is,*

$$|(N(u) \cup N(v)) \cap V(AQ^1_{n-1})| = \begin{cases} 2 & \text{if } u = 0a_{n-1}\dots a_1 \text{ and } v = 0\overline{a_{n-1}\dots a_1}, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. We first suppose that $u = 0a_{n-1}\dots a_1$ and $v = 0\overline{a_{n-1}\dots a_1}$ with $a_i \in \{0,1\}$ for $1 \leq i \leq n-1$. Then, vertices u and v have two distinct neighborhoods $1a_{n-1}\dots a_1$ and $1\overline{a_{n-1}\dots a_1}$ in AQ^1_{n-1}. Otherwise, suppose that $u = 0a_{n-1}\dots a_1$ and $v = 0b_{n-1}\dots b_1$, where $a_{n-1}\dots a_1 \neq \overline{b_{n-1}\dots b_1}$. Then, vertices u and v have four distinct neighborhoods $1a_{n-1}\dots a_1$, $1\overline{a_{n-1}\dots a_1}$, $1b_{n-1}\dots b_1$, and $1\overline{b_{n-1}\dots b_1}$ in AQ^1_{n-1}. As a result, this lemma follows. □

Lemma 2 *For $n \geq 3$, let AQ_n be an n-dimensional augmented cube, any two vertices u and v of AQ_n have at most four common neighborhoods. That is, $|N(u) \cap N(v)| \leq 4$.*

Proof. We shall prove it by induction. For $n = 3$, it is clear that $|N(u) \cap N(v)| \leq 4$ by Table 3.1. As for the inductive hypothesis, we assume that the result is true for AQ_{n-1}. Now we consider AQ_n and show that any two vertices u and v of AQ_n have at most four common neighborhoods. Without loss of generality, we may divide the proof into the following two cases.

Case 1: $u \in AQ^0_{n-1}$ and $v \in AQ^1_{n-1}$. Because u has two neighborhoods in AQ^1_{n-1} and v has two neighborhoods in AQ^0_{n-1}, $|N(u) \cap N(v) \cap AQ^0_{n-1}| \leq 2$ and $|N(v) \cap N(u) \cap AQ^1_{n-1}| \leq 2$. Therefore, $|N(u) \cap N(v)| \leq 4$.

Case 2: Both u and v are in AQ^0_{n-1}. By Lemma 1, u and v have no common neighborhood in AQ^1_{n-1} except for the case that $u = 0a_{n-1}\ldots a_1$ and $v = 0\overline{a_{n-1}\ldots a_1}$ with $a_i \in \{0,1\}$ for $1 \leq i \leq n-1$. Now, consider that $u = 0a_{n-1}\ldots a_1$ and $v = 0\overline{a_{n-1}\ldots a_1}$, vertices u and v have two common neighborhoods in AQ^1_{n-1} and two common neighborhoods in AQ^0_{n-1}. That is, $N(u)\cap N(v)\cap V(AQ^1_{n-1}) = \{1a_{n-1}a_{n-2}\ldots a_1, 1\overline{a_{n-1}a_{n-2}\ldots a_1}\}$ and $N(u)\cap N(v)\cap V(AQ^0_{n-1}) = \{0\overline{a_{n-1}}a_{n-2}\ldots a_1, 0a_{n-1}\overline{a_{n-2}\ldots a_1}\}$. Hence, u and v have at most four common neighborhoods in AQ_n and this lemma follows. □

u and v	common neighbors of u and v
$u = 000, v = 001$	$010, 011$
$u = 000, v = 010$	$001, 011$
$u = 000, v = 011$	$001, 010, 100, 111$
$u = 000, v = 100$	$011, 111$
$u = 000, v = 101$	$001, 010, 100, 111$
$u = 000, v = 110$	$001, 010, 100, 111$
$u = 000, v = 111$	$011, 100$

Table 3.1: Common neighbors of two vertices in AQ_3.

Lemma 3 [4] $\kappa(AQ_n) = 2n - 1$ *except* $\kappa(AQ_3) = 4$.

Lemma 4 *Assume that n is an integer with $n \geq 2$. Let $AQ_n = AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ be an n-dimensional augmented cube, $F \subset V(AQ_n)$ be a set of vertices of AQ_n, and $F_1 = F \cap V(AQ^1_{n-1})$ with $|F_1| \leq 1$. Then $AQ_n - F$ is still a connected graph containing $2^n - |F|$ vertices.*

Proof. According to Lemma 3, $AQ^1_{n-1} - F_1$ is a connected component with $2^{n-1} - |F_1|$ vertices. For each vertex $v \in V(AQ^0_{n-1})$, at least one of its two neighborhoods located in $V(AQ^1_{n-1})$ is fault-free since $|F_1| \leq 1$. Therefore, $AQ_n - F$ is connected, and its cardinality of the fault-free vertex set is $2^n - |F|$. This lemma is completed. $\square$

Lemma 5 *For a 4-dimensional augmented cube AQ_4, let $F \subset V(AQ_4)$ be a faulty vertex set with $|F| = 4n - 9 = 7$. Then, $AQ_4 - F$ has a connected component containing at least $2^n - |F| - 1 = 8$ vertices.*

Proof. Let $F_0 = F \cap V(AQ^0_3)$ and $F_1 = F \cap V(AQ^1_3)$, thus $F = F_0 \cup F_1$. Without loss of generality, we may assume that $|F_0| \geq |F_1|$. Thus, $|F_0| \geq 4$, $|F_1| \leq 3$, and $AQ^1_3 - F_1$ is connected by Lemma 3. In the following, we divide the proof according to the cardinality of F_0.

Case 1: $|F_0| \geq 6$.

Since $|F_0| \geq 6$, $|F_1| \leq 1$. By Lemma 4, $AQ_4 - F$ is a connected component containing 9 vertices, and this case follows.

Case 2: $4 \leq |F_0| \leq 5$.

Let C be the connected component with minimal cardinality in $AQ_3^0 - F_0$. First, suppose C consists of only one vertex, say vertex u. Then, $N_{AQ_3^0}(u) \subset F$ and $|F_0| = 5$. Then, $AQ_3^0 - (F_0 \cup \{u\})$ is a connected component with 2 vertices, and thus $AQ_4 - F$ has a connected component containing at least 8 vertices.

Second, suppose C consists of two vertices, then either $F_0 = \{0000, 0011, 0101, 0110\}$ or $F_0 = \{0001, 0010, 0100, 0111\}$. Thus, $AQ_3^0 - F_0$ is composed of two connected components with two vertices respectively. The vertex set of $AQ_3^0 - F_0$ is either $\{0001, 0010, 0100, 0111\}$ or $\{0000, 0011, 0101, 0110\}$, and both are illustrated in Figure 3.1. In addition, $AQ_3^1 - F_1$ is connected. For each of the two connected components in $AQ_3^0 - F_0$, it has four distinct neighborhoods in AQ_3^1, and it is connected to $AQ_3^1 - F_1$. Therefore, $AQ_4 - F$ is a connected component containing 9 vertices.

Now, suppose C consists of three or four vertices, it is easy to see that $AQ_3^0 - F_0$ is connected. Let u,v be two vertices in C such that $u = 0a_3a_2a_1$ and $v \neq 0\overline{a_3a_2a_1}$. Hence, by Lemma 1, u or v has at least one fault-free neighborhood in AQ_3^1. Therefore, $AQ_4 - F$ is a connected component containing 9 vertices, and this lemma follows. □

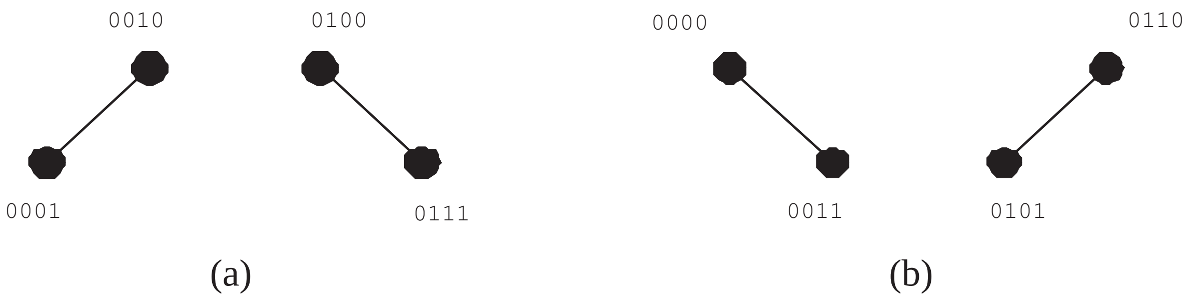

Figure 3.1: The graph of $AQ_3^0 - F_0$.

Theorem 1 *Let AQ_n be an n-dimensional augmented cube with $n \geq 4$ and let $F \subset V(AQ_n)$ be a faulty vertex set with $|F| = 4n-9$. Then, $AQ_n - F$ has a connected component containing at least $2^n - |F| - 1$ vertices.*

Proof. We prove this theorem by inducting on n. For $n = 4$, it is already proved by Lemma 5 that $AQ_4 - F$ has a connected component containing at least 8 vertices. As the inductive hypothesis, we assume that the result is true for AQ_{n-1} with $|F| = 4 \times (n-1) - 9 = 4n - 13$. Now we consider AQ_n with $|F| = 4n - 9$ and show that $AQ_n - F$ has a connected component containing at least $2^n - |F| - 1$ vertices.

Let $F_0 = F \cap V(AQ^0_{n-1})$ and $F_1 = F \cap V(AQ^1_{n-1})$. Without loss of generality, we may assume that $|F_0| \geq |F_1|$. Thus $|F_0| \geq 2n-4$, $|F_1| \leq 2n-5$, and $AQ^1_{n-1} - F_1$ is connected according to Lemma 3. In the following, we divide the proof into three cases according to the cardinality of F_0.

Case 1: $|F_0| \geq 4n - 10$.

Since $|F_0| \geq 4n-10$, $|F_1| \leq 1$. According to Lemma 4, $AQ_n - F$ is a connected component containing $2^n - |F|$ vertices, and this case follows.

Case 2: $4n - 11 \geq |F_0| \geq 4n - 12$. Let $AQ^0_{n-1} - F_0$ be composed of connected components $C_1, C_2, \ldots, C_x$, and let $|V(C_1)| \leq |V(C_2)| \leq \ldots \leq |V(C_x)|$ with $x \geq 1$. Now, we shall show that **(1)** $|V(C_i)| \geq 2$ for $2 \leq i \leq x$; and **(2)** For each $|V(C_i)| \geq 2$ where $1 \leq i \leq x$, C_i is connected to $AQ^1_{n-1} - F_1$. With **(1)** and **(2)** holds, $AQ_n - F$ contains a connected component containing at least $2^n - |F| - 1$ vertices, and this case follows.

Proof of (1): Suppose (1) is incorrect, then $|V(C_1)| = |V(C_2)| = 1$, we denote that $V(C_1) = \{u\}$, $V(C_2) = \{v\}$, and $(u, v) \notin E(AQ_n)$. Because any two vertices have at most four common neighborhoods by Lemma 2, $|F_0| \geq |N_{AQ^0_{n-1}}(u) \cup N_{AQ^0_{n-1}}(v)| \geq (2(n-1)-1) \times 2 - 4 = 4n - 10$, which is a contradiction to our assumption that $4n - 11 \geq |F_0| \geq 4n - 12$.

Proof of (2): First, suppose $|V(C_i)| = 2$. Let (u, v) be the edge of C_i. By Lemma 1, $|(N(u) \cup N(v)) \cap V(AQ^1_{n-1})|$ is either 2 or 4. Suppose $|(N(u) \cup N(v)) \cap V(AQ^1_{n-1})| = 2$, u and v will have at most two common neighborhoods in AQ^0_{n-1} according to Lemma 2. Thus, $|F_0| \geq |(N_{AQ^0_{n-1}}(u) \cup N_{AQ^0_{n-1}}(v)) - \{u, v\}| \geq (2(n-1)-2) \times 2 - 2 = 4n - 10$, which is a contradiction to our assumption that $4n - 11 \geq |F_0| \geq 4n - 12$. Suppose $|(N(u) \cup N(v)) \cap V(AQ^1_{n-1})| = 4$. Because $|F_1| \leq 3$, there is at least one fault-free edge between C_i and $AQ^1_{n-1} - F_1$. Therefore, C_i is connected to $AQ^1_{n-1} - F_1$.

Now, suppose $|V(C_i)| \geq 3$. Each C_i in AQ^0_{n-1} has two vertices u and v such that $u = 0a_{n-1} \ldots a_1$ and $v \neq 0\overline{a_{n-1} \ldots a_1}$. Note that $|F_1| \leq 3$. Hence, according to Lemma 1, u or v has at least one fault-free neighborhood in AQ^1_{n-1}. As a result, C_i is connected to $AQ^1_{n-1} - F_1$.

Case 3: $|F_0| \leq 4n - 13$**.** By inductive hypothesis, $AQ^0_{n-1} - F_0$ has a connected component containing at least $2^{n-1} - |F_0| - 1$ vertices. In addition, $AQ^1_{n-1} - F_1$ is a connected component with $2^{n-1} - |F_1|$ vertices. Suppose $n = 4$, $|F| = 4n - 9 = 7$ and $|F_0| \leq 4n - 13 = 3$, which contradicts to our assumption that $|F_0| \geq |F_1|$. Hence, without loss of generality, we may assume that $n \geq 5$. Now, $AQ^0_{n-1} - F_0$ is connected to $AQ^1_{n-1} - F_1$ since $(2^{n-1} - |F_0| - 1) + (2^{n-1} - |F_1|) > |E^h_n| = |E^c_n| = 2^{n-1}$, where $n \geq 5$. Therefore, $AQ_n - F$ has a connected component containing at least $(2^{n-1} - |F_0| - 1) + (2^{n-1} - |F_1|) = 2^n - |F| - 1$ vertices and the theorem is established. □

Corollary 1 *For an n-dimensional augmented cube AQ_n with $n \geq 4$, let $F \subset V(AQ_n)$ be any vertex set with $|F| \leq 4n - 9$. Then, $AQ_n - F$ has a connected component containing at least $2^n - |F| - 1$ vertices.*

Chapter 4

Vertex-Disjoint Paths

Menger's Theorem [7] is a classical result on connectivity and it states that if a network G is k-connected, then every pair of vertices in G is connected by k vertex-disjoint paths. A k-regular graph G is *strongly Menger-connected* if for any copy $G - F$ of G with at most $k - 2$ vertices removed, each pair u and v of $G - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths in $G - F$, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $G - F$, respectively [8]. It is proved in [8] that the *star graphs* are strongly Menger-connected.

However, the augmented cubes are not strongly Menger-connected according to their structures. Figure 4.1 provides an example. For an n-dimensional augmented cube AQ_n with $n \geq 4$, AQ_n is $(2n-1)$-regular. Let (u, x) be an edge of AQ_n such that $|N(u) \cap N(x)| = 4$, F be a faulty vertex set such that $F = N(x) - (N(u) \cup \{u\})$, and v be a vertex of AQ_n such that $v \in V(AQ_n) - (N(u) \cup N(x))$. Note that $|F| = 2n - 6$. As a result, the vertices u and v are **not** connected by $\min\{\deg_f(u), \deg_f(v)\} = 2n - 1$ vertex-disjoint fault-free paths in $AQ_n - F$.

Now, we give the definition of *maximally local connectivity*. Given a graph G and a vertex set $F \subset V(G)$, (G, F) is said to be maximally local connected if and only if for each pair

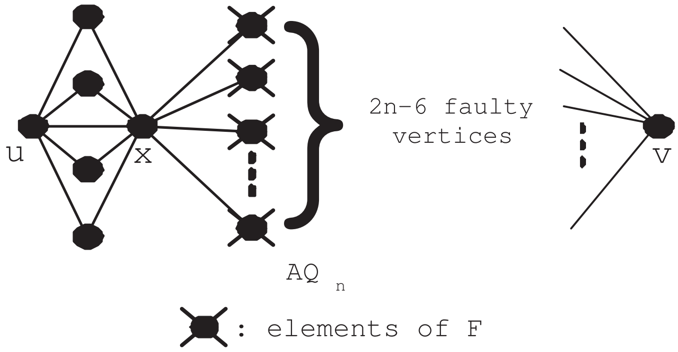

Figure 4.1: An example for $|F| = 2n - 6$.

of vertices, denoted by u and v, of $G - F$, u and v are connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths in $G - F$, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $G - F$, respectively. For the vertex-disjoint fault-free paths of AQ_n under a set of faulty vertices with $|F| \leq 2n - 7$, we have the following result.

Theorem 2 *For an n-dimensional augmented cube AQ_n with $n \geq 4$, let $F \subset V(AQ_n)$ be a set of faulty vertices with $|F| \leq 2n - 7$. Then, each pair of vertices u and v in $AQ_n - F$ is connected by* $\min\{\deg_f(u), \deg_f(v)\}$ *vertex-disjoint fault-free paths in $AQ_n - F$.*

Proof. We shall prove this theorem by using contradiction. Let u and v be two distinct vertices in $AQ_n - F$ and let $m = \min\{\deg_f(u), \deg_f(v)\}$. Suppose that there do not exist m vertex-disjoint fault-free paths connecting u and v in $AQ_n - F$. By Menger Theorem, u and v will be separated in $(AQ_n - F) - V_f$ for some faulty vertex sets $V_f \subset V(AQ_n - F)$ and $|V_f| = m-1$. Thus, the total number of faulty vertices in AQ_n is $|F|+|V_f| \leq (2n-7)+(m-1) \leq (2n - 7) + (2n - 1 - 1) = 4n - 9$. By Corollary 1, $(AQ_n - F) - V_f$ has a connected component containing at least $2^n - (4n-9) - 1$ vertices. That is, if $(AQ_n - F) - V_f$ is disconnected, it consists of two connected components and one of which is an isolated vertex, denoted by s. We may let $s = u$ or $s = v$. Hence, V_f must contain all the neighbors of s, so $|V_f| \geq m$, which is a contradiction to our assumption that $|V_f| = m - 1$. Consequently, this theorem is proved. □

Corollary 2 *For an n-dimensional augmented cube AQ_n with $n \geq 4$, let $F \subset V(AQ_n)$ be any vertex set with $|F| \leq 2n - 7$. Then, (AQ_n, F) is maximally local connected.*

Chapter 5

Conclusion

The hypercube, which is discussed extensively, plays an important role among many topologies. Many graphs relevant to the hypercube are brought up in succession, and the augmented cube is one of them. Both the augmented cubes and the hypercube have many good qualities and are appropriate for parallel routing. In this paper, we discuss the maximal connected component of the augmented cube with faulty vertices. We show that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 4n-9$ for $n \geq 4$, the maximal connected component of $AQ_n - F$ has at least $2^n - |F| - 1$ vertices. In addition, we show that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leq 2n-7$ for $n \geq 4$, each pair of non-faulty vertices u and v in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths.

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