# Wilson loop and contour shapes

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By computing the Wilson loop expectation value $W[C]$ in the two-dimensional Schwinger model on $R^1\times S^1$, we show that nonleading terms depend on the shape of the contours both on $R^1\times R^1$ and $R^1\times S^1$. We also find that the rhombic contour and triangular contour lead to the same static potentials both on $R^1\times R^1$ and $R^1\times S^1$. The binding energy is also affected by the choice of contour shapes on the $R^1\times S^1$ model. This indicates that the model on $R^1\times S^1$ deserves more study.

## I. INTRODUCTION

It is known that the gauge-invariant Wilson loop expectation value

$$W[C]=\left\langle \mathrm{Tr}P\exp\left[\oint_C iQA_\mu dx^\mu\right]\right\rangle \tag{1.1}$$

is related to the binding energy of a quark-antiquark pair.[1] Therefore, $W[C]$ has been employed by a number of authors[1,2] to test the idea of quark confinement. Moreover, the gauge-invariant dynamical variable[3] $W[C]$ is useful as a toy model in discussing hadronic physics.[4]

Furthermore, many articles[5] have argued that by averaging over different contours $C$, the Green's function of the quark currents is derivable from the gauge-invariant Wilson loop expectation value $W[C]$ in the large-$N_c$ limit. Accordingly, a study of the equations for the loop averages was first proposed by Polyakov and Nambu in order to make transparent the relevant relation between QCD and the dual resonance model.[6]

It was shown in Ref. 2 that $W[C]$ depends nontrivially on the shapes of the contour $C$. This property was also discussed by several authors to analyze the effect of the long-wavelength string fluctuations[7] on $W[C]$. Moreover, by considering a smoothly shaped contour one can avoid singularities[8] usually plagued with a rectangular contour. Indeed, it was shown in Ref. 2 that nonleading terms depend sensitively on the contour shapes $C$.

Also, the Wilson loop expectation value $W[C]$ is expected to be a linearly increasing quark-antiquark pair static potential of the form

$$V(d)=\frac{Q^2d}{2} . \tag{1.2}$$

Here $d$ ($=\rho\theta$ on a circle) is the distance between the quark and antiquark. Here $\rho$ and $\theta$ denote, respectively, the radial and polar coordinates on $S^1$.

For simplicity, we will consider a two-dimensional Schwinger model[9] on $R^1\times S^1$ with massless fermions. Extending our result from two-dimensional QED ($QED_2$) to diagonal $QCD_2$ ($DQCD_2$) is straightforward and will only bring in an additional group-theoretical factor[2]

$$\ln W_{\mathrm{DQCD}_2}[C]=\frac{N-1}{2N}\ln W_{\mathrm{QED}_2}[C] . \tag{1.3}$$

Here $N$ stands for the dimension of the fundamental representation of the symmetry group SU($N$).

There are several advantages[10] for considering $R^1\times S^1$ instead of $R^1\times R^1$. First of all, it is much easier to consider the model on $R^1\times S^1$ since most of the relevant physics remains unaffected. Secondarily, the annoying infrared divergence usually plagued with most two-dimensional gauge theories can be shown to be absent on $R^1\times S^1$. Moreover, the flat-space limit can be easily reproduced by letting $\rho$, the radial coordinate, go to infinity. Also the fermions are introduced (in contrast with a pure gauge theory) in order to study not only the gauge-invariant dynamical variable $W[C]$ in the loop space more directly but also the mechanism for quark trapping.[9,11] We reported part of the results concerning the binding energy in $QED_2$ in a previous paper.[12] In this paper, we will present complete and consistent details and make transparent some technical tricks in showing the decoupling theorem.

Therefore, we will study the shape dependence of $W[C]$ in $QED_2$. In Sec. II, we will briefly review the loop calculations of $W[C]$ by the path-integral method. In Sec. III, we will review the computation of the anomalous mass $\mu^2$ on the $R^1\times S^1$ model using Fujikawa's method. In Sec. IV, we present the details of the calculations for four different contours $C$: namely, the rectangular, rhombic, triangular, and elliptic contours. We also demonstrate a special trick in taking limits. Finally, several comments and discussions are in order in Sec. V.

## II. THE THEORY ON $R^1\times S^1$

We will give a brief review of the derivation of the chiral anomaly[12,13] using Fuijkawa's path-integral method[14] before we get into the details of computing $W[C]$. The Lagrangian for the two-dimensional (Euclidean) Schwinger model is given by

$$\mathcal{L}=-i\bar\psi\gamma^\mu D_\mu\psi+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_{\text{gauge fixing}} . \tag{2.1}$$

Here $D_\mu=\partial_\mu+eA_\mu$. Also the $\gamma$ matrices satisfy

 

$$\{\gamma_\mu,\gamma_\nu\}=-2g_{\mu\nu} . \tag{2.2}$$

In this paper we will work on the Euclidean (Riemannian) base manifold after a Wick rotation. Writing $\gamma^2=i\gamma^0$, we find that $\gamma^\mu D_\mu$ becomes a Hermitian operator after a Wick rotation $x^0\to -ix^2$ and $A_0\to iA_2$. Note that $\gamma^{\mu\dagger}=-\gamma^\mu$ and $g_{\mu\nu}=\delta_{\mu\nu}$ with $\mu=1,2$ in Euclidean space. Furthermore, we can show that $\gamma_5^\dagger=-\gamma_5$ and $\gamma_\mu\gamma_5=\epsilon_{\mu\nu}\gamma^\nu$ by defining $\gamma_5=-\gamma_1\gamma_2$ and $\epsilon_{12}=1$. Note that we have followed the notation of Fujikawa[14] (1980). Also note that $\gamma_5^\dagger=-\gamma_5$ in two dimensions (in contrast with $\gamma_5^\dagger=\gamma_5$ in four dimensions) is a general feature in $4k+2$ dimensions. Moreover, the gauge-fixing term is

$$\mathcal{L}_{\text{gauge fixing}}=\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \tag{2.3}$$

in the Lorentz gauge (i.e., $\partial_\mu A^\mu=0$). On the spatial circle $S^1$, we need to specify appropriate boundary conditions in order to define the theory properly. Therefore, by requiring that $\mathcal{L}$ and $F_{\mu\nu}$ (hence $J^\mu=\bar\psi\gamma^\mu\psi$) be single valued, one obtains the following constraints on the photon and fermion fields:

$$A_\mu(t,x+2\pi\rho)=A_\mu(t,x) , \tag{2.4}$$

$$\psi(t,x+2\pi\rho)=e^{i\varphi}\psi(t,x) . \tag{2.5}$$

Note that it was argued[9] that one should take $e^{i\varphi}=-1$ due to the stability of the vacuum. Hence (2.4) and (2.5) are sufficient to restrict ourselves to $R^1\times S^1$. Note also that all the algebraic and differential operations are not affected by this restriction since the connection on $R^1\times S^1$ is trivial, namely, $\Gamma^\lambda_{\mu\nu}=0$. In fact, $g_{\mu\nu}=\text{diag}(1,\rho^2)$ on $R^1\times S^1$. Hence the loop average is

$$W[C]=\left\langle \exp\left[\oint_C iQA_\mu dx^\mu\right]\right\rangle_{A,\psi,\bar\psi} \tag{2.6}$$

$$=N\int \mathcal{D}A_\mu\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left[-\int\sqrt{g}\,\mathcal{L}\,d^2x\right]\times\exp\left[\oint_C iQA_\mu dx^\mu\right] . \tag{2.7}$$

Here $C$ denotes closed contours on $R^1\times S^1$ defined by $x_\mu=x_\mu(t)$, and $N$ denotes all irrelevant normalization factors.

## III. THE ANOMALY ON $R^1\times S^1$

Note that the gauge field $A_\mu$ can be written as $A_\mu=A_\mu^L+A_\mu^T\;[=\partial_\mu a-(1/e)\epsilon_{\mu\nu}\partial^\nu\phi(x)]$ in two dimensions. Therefore, by the chiral $\gamma_5$ transformation

$$\psi(x)=e^{\gamma_5\phi(x)}\chi(x) , \tag{3.1}$$

$$\bar\psi(x)=\bar\chi(x)e^{\gamma_5\phi(x)} , \tag{3.2}$$

the fermions $\chi$ decouple from $A_\mu^L$, the longitudinal component of the gauge fields $A_\mu$, in the model defined by (2.1). Here $\phi$ is related to $A_\mu^T$, the transverse component of $A_\mu$, by

$$A_\mu^T=-\frac{1}{e}\epsilon_{\mu\nu}\partial^\nu\phi(x) . \tag{3.3}$$

The nontrivial dynamics of the model is actually hidden in the Jacobian factor of the path-integral measure[14,15] for the above chiral $\gamma^5$ transformation. In fact, this Jacobian factor will contribute as a dynamical photon mass.

Following Ref. 14, the Jacobian factor ($\Delta$) of the path-integral measure for the above chiral transformation is

$$\mathcal{D}\psi=\mathcal{D}\chi\exp\left[-\int d^2x\,\phi(x)\Delta(x)\right] , \tag{3.4}$$

with

$$\Delta(x)=\lim_{M\to\infty}\text{Tr}M\int\frac{dk_2}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\gamma_5\exp\left[-k_\mu k^\mu-\frac{1}{4M^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right]$$

$$=\lim_{M\to\infty}\text{Tr}M\int\frac{dk}{(2\pi)^2}\gamma_5\exp\left[-k^2-\frac{1}{4M^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right]\sum_{n=-\infty}^{\infty}e^{-(n/M)^2} . \tag{3.5}$$

Note that we have scaled the momentum $k_2$ according to $k_2\to Mk$. Moreover, we have suppressed the trivial radial factor $\rho$ since $\rho$ will not appear in the final expression (3.12).

In order to carry on our computations, we need to know the asymptotic properties of the Jacobi $\theta$ function[16] of the form

$$\theta(y)=\sum_{n=-\infty}^{\infty}e^{-n^2\pi y} . \tag{3.6}$$

In fact, we want to know the large-$M$ behavior of

$$\theta(1/\pi M^2)=\sum_{n=-\infty}^{\infty}e^{-(n/M)^2} . \tag{3.7}$$

If $\theta(y)$ is in the Schwartz space (the set of $C^\infty$ functions on $R^n$ which, along with their partial derivatives of all orders, tend to vanish rapidly at infinity), one can obtain the following functional equation of the Jacobi $\theta$ function: namely,

$$\theta(y^{-1})=\sqrt{y}\,\theta(y) . \tag{3.8}$$

Note that Gaussian functions (3.6) are certainly in the Schwartz space. Hence, using the functional equation of the Jacobi $\theta$ function (3.8), one obtains

$$\theta(1/\pi M^2)=M\sqrt{\pi}\theta(\pi M^2)$$

$$=M\sqrt{\pi}\sum_n e^{-n^2\pi^2M^2} . \tag{3.9}$$

Therefore,

$$\lim_{M\to\infty}\theta(1/\pi M^2)=M\sqrt{\pi}\ . \tag{3.10}$$

Hence the Jacobian factor can be shown to be

$$\Delta(x)=\lim_{M\to\infty}\mathrm{Tr}M^2\sqrt{\pi}\int\frac{dk}{(2\pi)^2}\gamma_5 \times\exp\left[-k^2-\frac{1}{4M^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right]$$
$$=\sqrt{\pi}\,\mathrm{Tr}\gamma_5\frac{-1}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\int\frac{dk}{(2\pi)^2}e^{-k^2}$$
$$=\frac{1}{4\pi}F^{\mu\nu}\epsilon_{\mu\nu}\ . \tag{3.11}$$

Note that (3.11) is exactly the same as the anomalous contribution in the flat-space Schwinger model. Therefore we have the same anomalous contribution: namely,

$$\mathcal{D}\psi\mathcal{D}\bar{\psi}=\mathcal{D}\chi\mathcal{D}\bar{\chi}\exp\left[-\frac{\mu^2}{2}\int A^{\mu T}A^T_\mu d^2x\right], \tag{3.12}$$

with $\mu^2$ given by

$$\mu^2=\frac{e^2}{\pi}\ . \tag{3.13}$$

Accordingly, the Wilson loop expectation value becomes

$$W[C]$$
$$=N\int\mathcal{D}A_\mu\mathcal{D}\bar{\chi}\mathcal{D}\chi\exp(-S_{\mathrm{eff}})\exp\left[\oint_C iQA_\mu dx^\mu\right]. \tag{3.14}$$

Here

$$S_{\mathrm{eff}}=\int d^2x\left[-i\bar{\chi}\gamma^\mu\partial_\mu\chi+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} -\frac{\mu^2}{2}A^T_\mu A^{T\mu}+\mathcal{L}'(A^L_\mu,\chi)\right]. \tag{3.15}$$

Note that the conserved current $J^\mu$ (i.e., $\partial\cdot J=0$) will not couple to $A^L_\mu$. Therefore, we can collect all irrelevant $A^L_\mu$ and $\chi$ terms in $\mathcal{L}'$. Thus, by integrating out the trivial $A^L_\mu$ and $\chi$ degrees of freedom and introducing the current

$$J_\mu(\mathbf{x})=iQ\oint_C\delta^{(2)}(\mathbf{x}-\mathbf{x}(t))\frac{dx_\mu(t)}{dt}dt\ , \tag{3.16}$$

we have

$$W[C]=N\int\mathcal{D}A^T_\mu\exp[-S_{\mathrm{eff}}(A^T_\mu)+\int d^2x\,J\cdot A] \tag{3.17}$$
$$=N\exp[\tfrac{1}{2}\int d^2x\,d^2yJ^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)]\ . \tag{3.18}$$

Note that the propagator for $A^T_\mu$ is

$$\Delta_{\mu\nu}(x-y)=\int d^2k\frac{e^{-ik\cdot(x-y)}}{(2\pi)^2}\frac{\delta_{\mu\nu}-k\mu k\nu}{k^2+\mu^2}\ .$$

Transforming into momentum space by

$$J_\mu(\mathbf{k})=\int d^2x\,J_\mu(x)e^{-ik\cdot x}\ , \tag{3.19}$$

we have, from (3.16),

$$J_\mu(\mathbf{k})=iQ\oint_C e^{-ik\cdot x(t)}dx_\mu(t)\ . \tag{3.20}$$

Therefore,

$$\ln W[C]=\frac{1}{8\pi^2}\int d^2k\,J_\mu(-\mathbf{k})\frac{1}{k^2+\mu^2}J^\mu(\mathbf{k})\ . \tag{3.21}$$

Note that the spatial momentum $k_1$ becomes discretized, namely,

$$k_1=\frac{n}{\rho},\quad n\in Z\ , \tag{3.22}$$

due to the periodic boundary condition (2.4) and (2.5) in the spatial direction. Accordingly, the integration on $k_1$ becomes a discrete sum,

$$\int_{-\infty}^{\infty}dk_1\to\frac{1}{\rho}\sum_{n=-\infty}^{\infty}\ . \tag{3.23}$$

## IV. CONTOUR SHAPE DEPENDENCE

Let us first consider a rectangular contour given by the oriented boundary of the rectangular disk

$$\{(x_1,x_2)|\ |x_1|\le d/2,\ |x_2|\le T/2\}\ .$$

The current can be shown to be

$$J_\mu(\mathbf{k})=iQ\oint_C e^{-i(n/\rho)x_1-ik_2x_2}dx_\mu \tag{4.1}$$
$$=\frac{\epsilon_{\mu\nu}k_\nu 4Q\rho}{k_2 n}\sin\frac{nd}{2\rho}\sin\frac{k_2T}{2}\ . \tag{4.2}$$

Here $k_\mu=(n/\rho,k_2)$. Note that the above current is indeed conserved. Therefore,

$$\ln W[C]=\frac{1}{8\pi^2\rho}\sum_{n=-\infty}^{\infty}\int dk_2\frac{1}{k_2^2+(n/\rho)^2+\mu^2}(4Q)^2\left[-\frac{1}{k_2^2}\sin^2\frac{k_2T}{2}\sin^2\frac{nd}{2\rho}-\left[\frac{\rho}{n}\right]^2\sin^2\frac{nd}{2\rho}\sin^2\frac{k_2T}{2}\right]^2 \tag{4.3}$$

$$=-\frac{Q^2}{\pi\rho}\sum_{n=-\infty}^{\infty}\frac{\sin^2(nd/2\rho)}{(n/\rho)^2}\left[T+\mu^2\frac{1-\exp\{-T[(n/\rho)^2+\mu^2]^{1/2}\}}{(n/\rho)^2[(n/\rho)^2+\mu^2]^{1/2}}\right]. \tag{4.4}$$

The effective static potential of charge $\pm Q$ separated by a mean distance $d$ can be defined as

$$V(d)=-\lim_{T\to\infty}\left[\frac{1}{T}\ln W[C]\right]. \tag{4.5}$$

Hence this effective static potential becomes

$$V_\Box(d)=\frac{2Q^2}{\pi\rho^2}\sum_{n=-\infty}^{\infty}\frac{\sin^2(nd/2\rho)}{(n/\rho)^2+\mu^2} \tag{4.6}$$

$$=\frac{Q^2}{2\mu}[\coth\mu\rho\pi-\coth\mu\rho\pi\cosh\mu d+\sinh\mu d], \tag{4.7}$$

in the case of the rectangular contour.

Next, we will consider a rhombic contour defined by the equation $|x_1|/a+|x_2|/b=1$. The current can be shown to be

$$J_\mu(\mathbf{k})=4abQ\epsilon_{\mu\nu}k_\nu\frac{\cos k_2 b-\cos(na/\rho)}{(k_2b)^2-(na/\rho)^2}. \tag{4.8}$$

Also note that the current conserved is apparent. Therefore,

$$\ln W_\Diamond(a,b)=-\frac{2a^2bQ^2}{\pi^2\rho}\int dk\sum_{n\in Z}\frac{k^2+(n^2/\rho^2)b^2}{k^2+(n^2/\rho^2+\mu^2)b^2}\times\left[\frac{\cos k-\cos(na/\rho)}{k^2-n^2a^2/\rho^2}\right]^2. \tag{4.9}$$

Note that in (38) we performed a change of integration variable $k_2\to k/b$, in order to extract the $b$ dependence in the integrand. After doing this, it is straightforward to compute the effective static potential

$$V_\Diamond(d)=-\lim_{T\to\infty}\frac{1}{T}\ln W(d/\rho,T/2) \tag{4.10}$$

$$=-\lim_{b\to\infty}\frac{1}{2b}\ln W(d/\rho,b). \tag{4.11}$$

Here we have set $a=d/\rho$ and $b=T/2$ to ensure that the average separation distance remains $d$. This can be done by requiring $Td=$area of the contour. Therefore the effective potential becomes

$$V_\Diamond(d)=-\frac{a^2Q}{\pi^2\rho}\int dk\sum_{n\in Z}\frac{n^2/\rho^2}{n^2/\rho^2+\mu^2}\times\left[\frac{\cos k-\cos(nd/\rho)}{k^2-n^2d^2/\rho^2}\right]^2. \tag{4.12}$$

This expression can be simplified to the form

$$V_\Diamond(d)=\frac{Q^2}{4\mu}\left[2\coth\mu\rho\pi+\frac{\sinh\mu(\rho\pi-2d)-\sinh\mu\rho\pi}{\mu d\sinh\mu\rho\pi}\right]. \tag{4.13}$$

Also, we can consider a slightly different contour, namely the triangular contour[2] following the following contour path: i.e., starting from $(x_1,x_2)=(0,-T/2)\to(d,T/2)\to(-d,T/2)\to(-d,T/2)$ and back to $(0,-T/2)$. The current can be shown to be

$$J_\mu(\mathbf{k})=-i2Q\epsilon_{\mu\nu}k_\nu\frac{\rho T}{n}\left[e^{-ind/2\rho}\frac{\sin\frac{1}{2}k_+}{k_+}-e^{ind/2\rho}\frac{\sin\frac{1}{2}k_-}{k_-}\right]. \tag{4.14}$$

Here $k_+=k_2T+nd/\rho$ and $k_-=k_2T-nd/\rho$. Note that the current is indeed conserved. Hence one can derive

$$\ln W_\nabla(d,T)=-\frac{T^2Q^2}{2\pi^2\rho}\int dk_2\sum_{n\in Z}\frac{1}{k^2+(n^2/\rho^2+\mu^2)}\frac{n^2+k_2^2\rho^2}{n^2}\left[\frac{\sin^2\frac{1}{2}k_+}{k_+^2}+\frac{\sin^2\frac{1}{2}k_-}{k_-^2}-2\frac{\sin\frac{1}{2}k_+\sin\frac{1}{2}k_-\cos(nd/\rho)}{k_+k_-}\right]. \tag{4.15}$$

Replacing $k_2$ by $k/T$, we have

$$V_\nabla(d,T)=\frac{Q^2}{2\pi^2\rho}\sum_{n\in Z}\frac{1}{(n^2/\rho^2+\mu^2)}\int dk\left[\frac{\sin^2\frac{1}{2}k_+}{k_+^2}+\frac{\sin^2\frac{1}{2}k_-}{k_-^2}-2\frac{\sin\frac{1}{2}k_+\sin\frac{1}{2}k_-\cos(nd/\rho)}{k_+k_-}\right]. \tag{4.16}$$

Here $k_+$ and $k_-$ become $k+nd/\rho$ and $k-nd/\rho$, respectively. After some algebra, one has

$$V_\nabla(d)=\frac{Q^2}{2\mu}\left[\coth\mu\rho\pi+\frac{\sinh\mu(\rho\pi-2d)-\sinh\mu\rho\pi}{2\mu d\sinh\mu\rho\pi}\right]. \tag{4.17}$$

Note that $V_\nabla=V_\Diamond$ although their contour shapes are not the same. Note also that $V_\triangle$ (from a reversed triangular contour) equals to $V_\nabla$ due to time-reversal invariance.

Finally, let us consider an elliptical contour sketched by the equation $x_1^2/a^2+x_2^2/b^2=1$. The current can be shown to be

$$J_\mu(\mathbf{k})=2\pi Qab\epsilon_{\mu\nu}k_\nu\frac{J_1(\sqrt{k_1^2a^2+k_2^2b^2})}{\sqrt{k_1^2a^2+k_2^2b^2}}. \tag{4.18}$$

Here $J_1(y)$ is the Bessel's function.[17] Again, the current is apparently conserved. Now, by inserting (4.18) into (3.21), one obtains

$$\ln W_{\text{ellip}}(a,b)=-\frac{a^2b^2Q^2}{2\rho}\sum_{n\in Z}\int dk_2\frac{k_1^2+k_2^2}{k_1^2+k_2^2+\mu^2}\times\frac{J_1^2(\sqrt{k_1^2a^2+k_2^2b^2})}{k_1^2a^2+k_2^2b^2}\ . \tag{4.19}$$

Here $k_1=n/\rho$. By scaling $k_2\to k/b$, one obtains

$$\ln W_{\text{ellip}}(a,b)=-\frac{a^2b^2Q^2}{2\rho}\sum_{n\in Z}\int dk\frac{k^2+k_1^2b^2}{k^2+(k_1^2+\mu^2)b^2}\times\frac{J_1^2(\sqrt{k^2+k_1^2a^2})}{k^2+k_1^2a^2}\ . \tag{4.20}$$

The effective potential should be defined as

$$V_{\text{ellip}}(d)=-\lim_{T\to\infty}\frac{1}{T}\ln W_{\text{ellip}}(T/2,2d/\pi)=-\lim_{b\to\infty}\frac{1}{2b}\ln W_{\text{ellip}}(a,b)\ . \tag{4.21}$$

Note that the area enclosed by the elliptical contour is set to be $\pi ab=Td$ in order to make $d$ equal to the average separation. Therefore,

$$V_{\text{ellip}}(d)=-\frac{a^2Q^2}{4}\sum_{n\in Z}\int dk\frac{k_1^2}{k_1^2+\mu^2}\frac{J_1^2(\sqrt{k^2+k_1^2a^2})}{k^2+k_1^2a^2}\ . \tag{4.22}$$

After some algebra, one can then derive

$$V_{\text{ellip}}(d)=\frac{Q^2}{4\mu}\pi\left[I_1\left[\frac{4\mu d}{\pi}\right]-\coth\mu\rho\pi L_1\left[\frac{4\mu d}{\pi}\right]\right]\ . \tag{4.23}$$

Here $I_1$ and $L_1$ are, respectively, the associated Bessel function and associated Struve's function[17] of order 1. Note that in deriving Eq. (4.23) we have made use of the integral representation of $I_m$ and $L_m$: i.e.,

$$I_m(x)=\frac{2(x/2)^m}{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1(1-y^2)^{m-1/2}\cosh xy\,dy\ , \tag{4.24}$$

$$L_m(x)=\frac{2(x/2)^m}{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1(1-y^2)^{m-1/2}\sinh xy\,dy\ . \tag{4.25}$$

## V. CONCLUSION

In summary, we have

$$V_{\Box}(d)=\frac{Q^2}{2\mu}(\coth\mu\rho\pi-\coth\mu\rho\pi\cosh\mu d+\sinh\mu d)\ , \tag{5.1}$$

$$V_{\Diamond}(d)=V_{\nabla}(d)=\frac{Q^2}{2\mu}\left[\coth\mu\rho\pi+\frac{\sinh\mu(\rho\pi-2d)-\sinh\mu\rho\pi}{2\mu d\sinh\mu\rho\pi}\right], \tag{5.2}$$

$$V_{\text{ellip}}(d)=\frac{Q^2}{4\mu}\pi\left[I_1\left[\frac{4\mu d}{\pi}\right]-\coth\mu\rho\pi L_1\left[\frac{4\mu d}{\pi}\right]\right]\ . \tag{5.3}$$

Note that $\theta$ $(=d/\rho)\in[0,\pi]$ for rectangular contour and rhombic contour, $\theta\in[0,\pi/2]$ for triangular contour, and $\theta\in[0,\pi^2/4]$ for elliptical contour since we are living in a finite-size system, namely, $\theta\in[0,2\pi]$. Note also that the equivalence between rhombic contour and triangular contour is not quite a surprise. For instance, imagine $\theta=\pi$ in the case of triangular contour: then the triangular contour becomes the $\theta=\pi/2$ rhombic contour except for some differences in the order and direction in the contour.

It is also straightforward to show that

$$V_{\Box}(d)\to\frac{Q^2}{2}d\left[1-\frac{\mu d}{2}\coth\mu\rho\pi\right]\ , \tag{5.4}$$

$$V_{\Diamond}(d)=V_{\nabla}(d)\to\frac{Q^2}{2}d\left[1-\frac{2\mu d}{3}\coth\mu\rho\pi\right]\ , \tag{5.5}$$

$$V_{\text{ellip}}(d)\to\frac{Q^2}{2}d\left[1-\frac{16\mu d}{3\pi^2}\coth\mu\rho\pi\right]\quad\text{as } \mu d\ll 1\ . \tag{5.6}$$

By keeping $d=\rho\theta$ fixed and letting $\rho\to\infty$, one reproduces immediately the $R^1\times R^1$ flat-space static potentials of the forms

$$V_{\Box}(d)=\frac{Q^2}{2\mu}(1-e^{-\mu d})\ , \tag{5.7}$$

$$V_{\Diamond}(d)=V_{\nabla}(d)=\frac{Q^2}{2\mu}\left[1-\frac{1-e^{-2\mu d}}{2\mu d}\right]\ , \tag{5.8}$$

$$V_{\text{ellip}}(d)=\frac{Q^2}{4\mu}\pi\left[I_1\left[\frac{4\mu d}{\pi}\right]-L_1\left[\frac{4\mu d}{\pi}\right]\right]\ , \tag{5.9}$$

which is expected to agree with Ref. 2. Also,

$$V_{\Box}(d)\to\frac{Q^2}{2}d\left[1-\frac{\mu d}{2}\right]\ , \tag{5.10}$$

$$V_{\Diamond}(d)=V_{\nabla}(d)\to\frac{Q^2}{2}d\left[1-\frac{2\mu d}{3}\right]\ , \tag{5.11}$$

$$V_{\text{ellip}}(d) \to \frac{Q^2}{2} d \left[ 1 - \frac{16\mu d}{3\pi^2} \right] \quad \text{as } \mu d \ll 1 \ , \tag{5.12}$$

in the flat-space limit. Note that the higher-order contributions of the binding energy differ from one another and depend on both the contour shapes and the radial function $\rho$ in the model. This indicates that the model on $R^1 \times S^1$ deserves more study.

Furthermore, in the long-distance limit $\mu d \gg 1$, one has

$$V_{\Box}(d) \to \frac{Q^2}{2\mu} \ , \tag{5.13}$$

$$V_{\Diamond}(d) \to \frac{Q^2}{2\mu} \left[ 1 - \frac{1}{2\mu d} \right] , \tag{5.14}$$

$$V_{\text{ellip}}(d) \to \frac{Q^2}{2\mu} \left[ 1 - \frac{\pi^2}{16} \frac{1}{(\mu d)^2} \right] . \tag{5.15}$$

In particular, we discover the van der Wall force for $V_{\Diamond}$, $V_{\nabla}$, and $V_{\text{ellip}}$ on $R^1 \times R^1$.

It is, however, not easy to discuss similar long-distance behavior on $R^1 \times S^1$ since $\theta$ space is compact. Nonetheless, we can show that $V_{\Box}(d)$ and $V_{\Diamond}(d)$ are both monotonically increasing functions in $d$. For example, we can show that

$$\frac{2\mu}{Q^2 \rho} \frac{\partial V_{\Box}}{\partial \theta} = \sinh\mu d (\coth\mu d - \coth\mu\rho\pi) > 0 \ .$$

Hence, $V_{\Box}(d)$ is a monotonically increasing function in $d$. Moreover,

$$\frac{2\mu}{Q^2 \rho} \frac{\partial V_{\Diamond}}{\partial \theta} = \frac{Y}{2\mu^2 d^2 \sinh\mu\rho\pi}$$

with

$$Y = \sinh\mu\rho\pi - \sinh\mu(\rho\pi - 2d) - 2\mu d \cosh\mu(\rho\pi - 2d) \ .$$

Note that $Y(\theta=0)=0$, $Y(\theta=\pi/2)>0$, and $\partial Y/\partial\theta > 0$ for all $\theta \in [0, \pi/2]$. Therefore, $Y$ (hence $\partial V_{\Diamond}/\partial\theta) > 0$ for all $\theta \in [0, \pi/2]$. Hence $V_{\Diamond}(d)$ is also a monotonically increasing function in $d$. This agrees with the large-distance limit on $R^1 \times R^1$ shown above.

In this paper, we study the two-dimensional Schwinger model and its dependence on different contour shapes on $R^1 \times S^1$. By letting $\rho \to \infty$, we reproduce immediately the results on $R^1 \times R^1$. In this limit, we also discover the long-range van der Wall force form. We consider a rhombic shape contour instead of the ambiguous circular contour considered in Ref. 2. It is hard to imagine separating a $q\bar{q}$ pair by an infinite distance on the one hand, it is also difficult to analyze the circular contour within a presumed finite-sized $R^1 \times S^1$ model on the other hand. Hence we consider rhombic and triangular shape contours instead of the ambiguous circular contour considered in Ref. 2.

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