國立交通大學

資訊科學與工程研究所

碩 士 論 文

在 PMC 模式下超立方體之 *g*-good-neighbor 條件式診斷能力

The *g*-good-neighbor Conditional Diagnosability of Hypercube under PMC Model

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摘要

在多處理器系統當中，為維持處理器在計算上的可靠度，處理器偵錯一直是很重要的議題。對於許多著名的連結網路，已經有相關的處理器偵錯之診斷能力的研究結果。 舉例而言，n 維的超立方體(hypercubes)、n 維的交叉立方體(crossed cubes)、n 維的梅氏立方體(möbius cubes)、n 維的雙扭立方體(twisted cubes)之處理器偵錯之診斷能力皆為 n。而 n 維的超立方體在 PMC 模式下條件式處理器偵錯之診斷能力為 $4(n\text{-}2)+1$。在本文中我們將探討 n 維的超立方體在 PMC 模式下的 g-good-neighbor 條件式處理器偵錯之診斷能力，並証明其為 $2^g(n\text{-}g)+2^g\text{-}1$，其中 $0 \le g \le n-3$。 在 g-good-neighbor 條件式下處理器偵錯之診斷能力為傳統的處理器偵錯之診斷能力的數倍。

關鍵字：超立方體、PMC 診斷模式、t-可診斷性、診斷能力、g-good-neighbor 條件式診斷能力。

The *g*-good-neighbor Conditional Diagnosability of Hypercube under PMC Model

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Abstract

Processor fault diagnosis plays an important role in multiprocessor systems for reliable computing, and the diagnosability of many well-known networks has been explored. For example, hypercubes, crossed cubes, möbius cubes, and twisted cubes of dimension n all have diagnosability n. The conditional diagnosability of n-dimensional hypercube Q_n is proved to be $4(n - 2) + 1$ under the PMC model.
In this thesis, we study the g-good-neighbor conditional diagnosability of Q_n under the PMC model and show that it is $2^g(n - g) + 2^g - 1$ for $0 \leq g \leq n - 3$. The g-good-neighbor conditional diagnosability of Q_n is several times larger than the classical diagnosability of Q_n.

Keywords: hypercube, PMC diagnosis model, t-diagnosable, diagnosability, g-good-neighbor conditional diagnosability.

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1 Introduction

With the rapid development of technology, the need for high-performance large multiprocessor systems has been continuously increasing day by day. Since all the processors run in parallel, the reliability of each processor in multiprocessor systems becomes an important issue for parallel computing. In order to maintain the reliability of such multiprocessor systems, whenever a processor (node or vertex) is found faulty, it should be replaced by a fault-free processor.

The hypercube [25, 32] is a well-known interconnection network for multiprocessor systems. Fault-tolerant computing for the hypercube has been of interest to many researchers. The process of identifying faulty vertices is called the *diagnosis* of the system. System diagnosis can be done in two different approaches, that is, circuit-level diagnosis and system-level diagnosis. In circuit-level diagnosis, the processors must be tested one after one by the human labor, which induces diagnosis complicated and possibly inaccurate. On the other hand, system-level diagnosis could be done automatically by the system itself. Thus, system-level diagnosis appears to be an alternative to circuit-level testing in a large multiprocessor system. Many terms for system-level diagnosis have been defined and various models have been proposed in literature [1, 11, 28, 31]. If all allowable fault sets can be diagnosed correctly and completely based on a single syndrome, then the diagnosis is referred to as *one-step diagnosis* or *diagnosis without repairs*.

In this study, we use the widely adopted PMC model [31] as the fault diagnosis model. In [15], Hakimi and Amin proved that a multiprocessor system is *t-diagnosable* if it is t-connected with at least $2t + 1$ vertices. Besides, they gave a necessary and sufficient condition for verifying if a system is t-diagnosable under the PMC model. Reviewing the previous papers, there are several variations of the hypercube [19], for example, the crossed cube [9], the möbius cube [10], and the twisted cube [16]. For each of these cubes, an n-dimensional cube can be constructed from two copies of $(n-1)$-dimensional cubes by adding a perfect matching between them. One of the common property among them is that all these variations have diagnosability n under the PMC model.

In classical measures of system-level diagnosability for multiprocessor systems, it has generally been assumed that any subset of processors can potentially fail at the same time. If there is a vertex v whose neighboring vertices are faulty simultaneously, there is no way of knowing the faulty or faut-free status of v. As a consequence, the diagnosability of a system is upper bounded by its minimum degree. Motivated by the deficiency of the classical measurement of diagnosability, Lai et al. [21] introduced a measure of *conditional diagnosability* by claiming the property that any faulty set cannot contain all neighbors of any processor. Under this condition, they showed that the conditional diagnosability of the n-dimensional hypercube Q_n is $4(n-2)+1$. We are then led to the following question: how large the maximum value t can be such that a graph G remains t-diagnosable under the condition that every vertex v has at least g fault-free neighboring vertices. More precisely, we assume the faulty set F satisfies the condition that each vertex v in $G-F$ has at least g good neighbors. We notice that, considering the situation that all the neighbors of each vertex cannot fail simultaneously, many properties of the network would be much better, including the connectivity and diagnosability. The aim of this thesis is to study more of these better properties.

In this thesis, we extend the concept of conditional diagnosis and propose a new measure of diagnosability. We define *g-good-neighbor conditional diagnosability* as the maximum number of faulty vertices that the system can guarantee to identify under the condition that every fault-free vertex has at least g fault-free neighbors. We show that the g-good-neighbor conditional diagnosability of Q_n is $2^g(n-g)+2^g-1$ under the PMC model, which is several times larger than the classical diagnosability of Q_n.

The rest of this thesis is organized as follows: Section 2 provides terminology and preliminaries for a multiprocessor system. In Section 3, we introduce the n-dimensional hypercube. We introduce system diagnosis and propose the concept of the g-good-neighbor conditional diagnosis in Section 4. In Section 5, we show the proof of the g-good-neighbor conditional diagnosability of Q_n. Finally, our conclusions are given in Section 6.

2 Terminology and Notations

A multiprocessor system or a network is usually represented as an undirected graph where vertices represent processors and edges represent communication links. Throughout this thesis, we follow [33] for the graph definitions and notations, and we focus on the undirected graph without loops (simply abbreviated as graph).

Let $G = (V, E)$ be a *graph* where V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. We use $n(G) = |V|$ to denote the cardinality of V. The *degree* of a vertex v in a graph G, written as $deg_G(v)$ or $deg(v)$, is the number of edges incident to v. The *maximum degree* of graph G is denoted by $\Delta(G) = max\{deg_G(v) \mid v \in V(G)\}$, and the *minimum degree* of graph G is $\delta(G) = min\{deg_G(v) \mid v \in V(G)\}$. The graph G is *regular* if $\Delta(G)$=$\delta(G)$. It is *k-regular* if the common degree is k. The *neighborhood* of a vertex v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v. We use $N(A) = \{x \mid y \in A$, $x \in G - A$, and $(x, y) \in E(G)\}$ to denote the neighborhood of a vertex subset A of G.

Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, in which all the vertices $v_0, v_1, \ldots, v_k$ are distinct except possibly $v_0 = v_k$. A graph G is *connected* if for any two vertices, there is a path joining them, otherwise it is *disconnected*. For a set S of V, the notation $G - S$ represents the graph obtained by removing the vertices in S from G and deleting those edges with at least one end vertex in S. If $G - S$ is disconnected, then S is called a *separating set* (or a *vertex cut*). A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *component* of a graph G is its maximal connected subgraph. A component is *trivial* if it has no edges; otherwise, it is *nontrivial*. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left. A graph G is *k-connected* if its connectivity is at least k.

3 The n-dimensional Hypercube

An n-dimensional hypercube, Q_n, is an undirected n-regular graph containing 2^n vertices and $n2^{n-1}$ edges (See Figure1). Let $\mathbf{u} = u_{n-1}u_{n-2}\ldots u_1u_0$ be an n-bit binary string. In a sense, the hypercube Q_n consists of all n-bit binary strings as its vertices. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if their binary string representations differ in exactly one bit position. For $0 \le i \le n-1$, we use $\mathbf{u}^i$ to denote the *i-th neighbor* of $\mathbf{u}$, i.e., the binary string $v_{n-1}v_{n-2}\ldots v_1v_0$ where $v_i = 1 - u_i$ and $v_k = u_k$ if $k \ne i$.

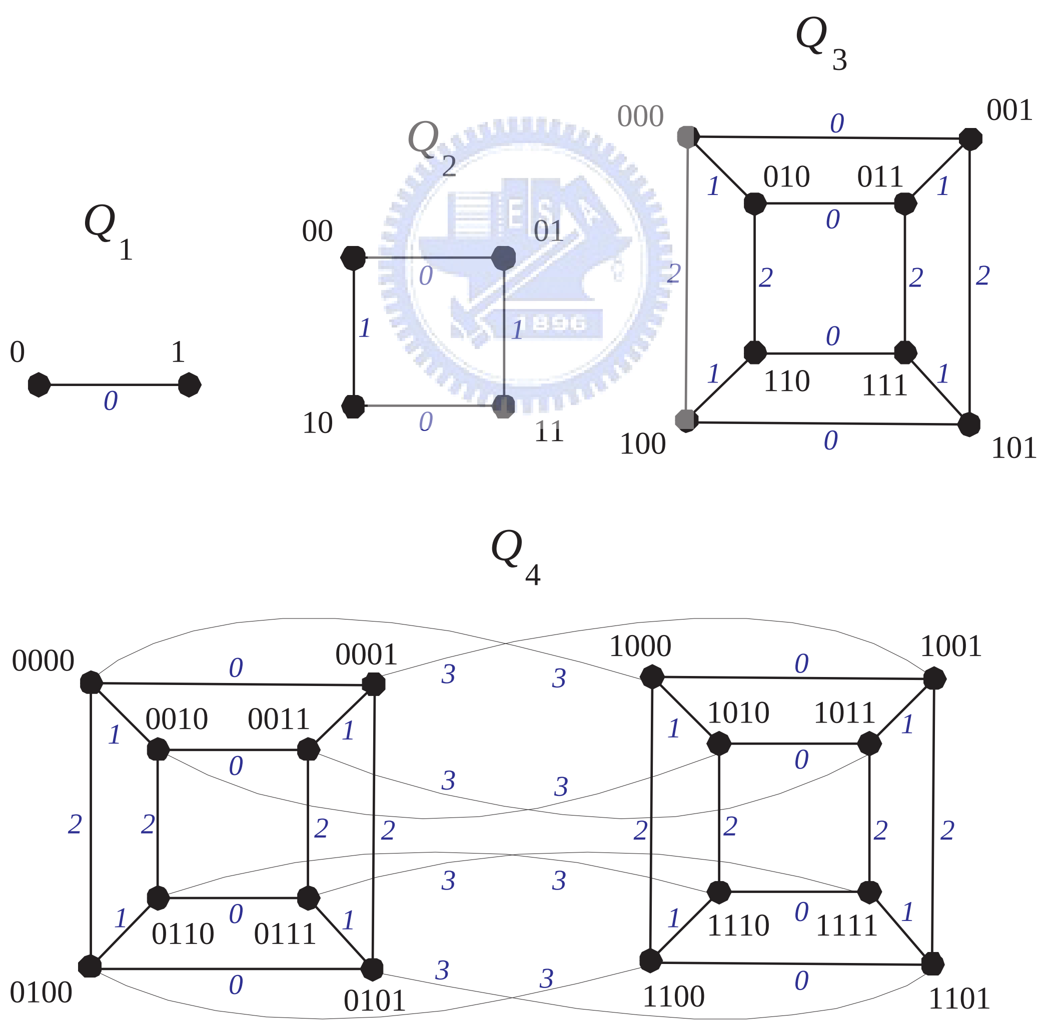

Figure 1: Illustration of the hypercube of dimension 1, 2, 3 and 4.

The *Hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is the number of i such that $u_i = 1$. The hypercube Q_n is a bipartite graph with bipartition $\{\mathbf{u} \mid w(\mathbf{u}) \text{ is odd}\}$ and $\{\mathbf{u} \mid w(\mathbf{u}) \text{ is even}\}$. We use black vertices to denote those vertices of odd weight and white vertices to denote those vertices of even weight. For $i \in \{0, 1\}$, we set Q_n^i to be the subgraph of Q_n which is induced by $\{\mathbf{u} \in V(Q_n) \mid u_{n-1} = i\}$. The hypercube of n-dimension Q_n is consisted of two Q_{n-1}, and Q_n^i is isomorphic to Q_{n-1} for $i = 0, 1$, as shown in Figure 2. It is well known that Q_n is vertex transitive and edge transitive [14, 25]. Furthermore, the permutation on the coordinate of Q_n and the componentwise complement operations are graph isomorphisms.

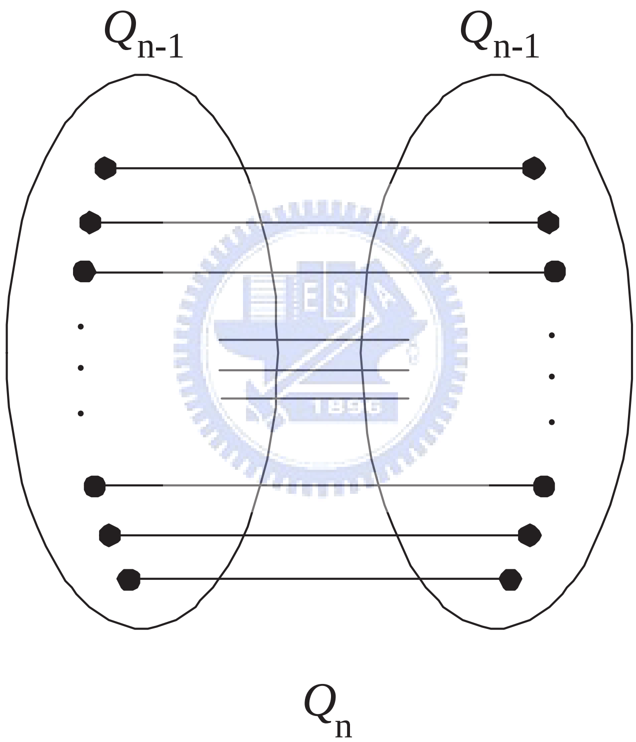

Figure 2: Illustration of hypercube of dimension n consisted of two Q_{n-1}.

The topological properties of Q_n has been studied extensively in recent years. Readers can refer [25] for a survey on the properties of hypercubes. For further study of the hypercube Q_n, we need to use the following property of hypercube.

Lemma 1. *For every two distinct vertices* $\mathbf{u}$ *and* $\mathbf{v}$ *in hypercube, they have at most* 2 *common neighbors. That is* $|N(\mathbf{u}) \cap N(\mathbf{v})| \leq 2$.

4 Diagnosability

Under the classical PMC model [31], adjacent processors are capable of performing tests on each other. For two adjacent vertices u and v in V, the ordered pair (u, v) represents the test performed by u on v. In this situation, u is called the *tester* and v is called the *tested vertex*. The outcome of a test (u, v) is either 1 or 0 with the assumption that the testing result is regarded as reliable if the tester u is fault-free. However, the outcome of a test (u, v) is unreliable, provided that the tester u itself is originally a faulty processor. Suppose that the tester u is fault-free, then the result would be 0 (respectively, 1) if v is fault-free (respectively, faulty). In this thesis, for each pair of adjacent vertices (u, v), u and v can perform the test to each other. All the possible results of v tested by u are listed in the following table.

u	v	(u, v)
fault-free	fault-free	0
fault-free	faulty	1
faulty	fault-free	0/1
faulty	faulty	0/1

Table 1: All possible results of v tested by u

A *test assignment* T for a system G is a collection of tests for every adjacent pairs of vertices. It can be modeled as a directed testing graph $T = (V, L)$ where $(u, v) \in L$ implies that u and v are adjacent in G. See Figure 3 for an illustration of T produced from G. Throughout this thesis, we assume that each vertex tests the other whenever there is an edge between them and all these tests are gathered in the test assignment. The collection of all test results for a test assignment T is called a *syndrome*. Formally, a syndrome is a function $\sigma : L \to \{0, 1\}$. The set of all faulty processors in the system is called a faulty set. This can be any subset of V. The process of identifying all the faulty vertices is called the diagnosis of the system. The maximum number of faulty vertices that the system G

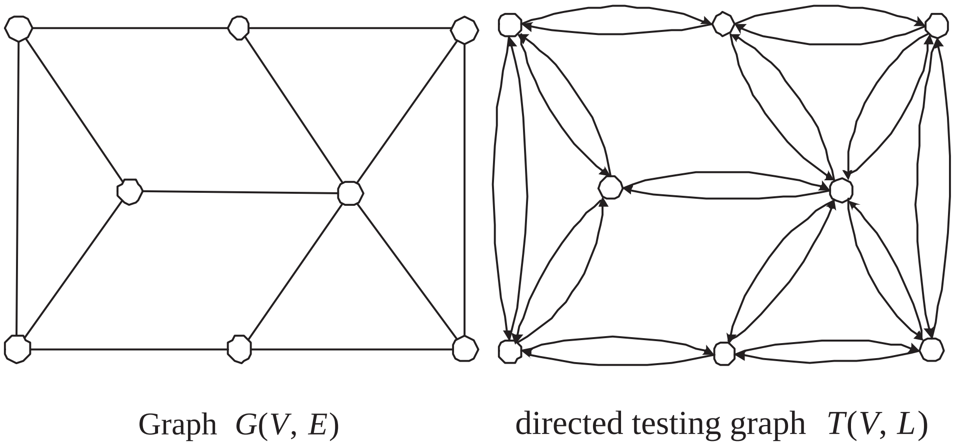

Figure 3: Illustration of the directed testing graph T produced from graph G.

can guarantee to identify is called the diagnosability of G, written as $t(G)$. For a given syndrome σ, a subset of vertices $F \subseteq V$ is said to be consistent with σ if syndrome σ can be produced from the situation that, for any $(u, v) \in L$ such that $u \in V - F, \sigma(u, v) = 1$ if and only if $v \in F$. Because a faulty tester can lead to an unreliable result, a given set F of faulty vertices may produce different syndromes. We use notation $\sigma(F)$ to represent the set of all syndromes which could be produced if F is the set of faulty vertices. Two distinct sets F_1 and F_2 in V are said to be *indistinguishable* if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$, otherwise, F_1 and F_2 are said to be *distinguishable*. Besides, we say (F_1, F_2) is an *indistinguishable pair* if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$, else (F_1, F_2) is a *distinguishable pair*. Figure 4 illustrates an example of a faulty vertex embedded in Q_2. In this example, $F = \{x_1\}$ and $\sigma(F) = \{(x_1, x_2), (x_2, x_1), (x_1, x_4), (x_4, x_1), (x_2, x_3), (x_3, x_2), (x_3, x_4), (x_4, x_3)\} = \{\{0, 1, 0, 1, 0, 0, 0, 0\}, \{0, 1, 1, 1, 0, 0, 0, 0\}, \{1, 1, 0, 1, 0, 0, 0, 0\}, \{1, 1, 1, 1, 0, 0, 0, 0\}\}$ with 4 possible syndromes.

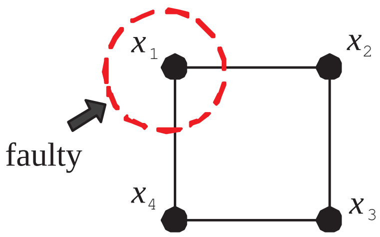

Figure 4: An example of a faulty vertex x_1 embedded in Q_2.

Some known results about the t-diagnosable systems and related concepts are listed as follows. Some of these previous results are on directed graphs and others are on undirected graphs.

Definition 1. *[31] A system of n units is t-diagnosable if all faulty units can be identified without replacement, provided that the number of faults presented does not exceed t.*

Let F_1 and F_2 be two distinct subsets of V, and let the symmetric difference $F_1 \Delta F_2 = (F_1 - F_2) \cup (F_2 - F_1)$. DahBura and Masson [5] proposed a polynomial time algorithm to check whether a system is t-diagnosable.

Theorem 1. *[5] A system $G = (V, E)$ is t-diagnosable if and only if, for any two distinct subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$, there is at least one test from $V - (F_1 \cup F_2)$ to $F_1 \Delta F_2$.*

The following two results related to t-diagnosable systems are due to Hakimi and Amin [15], and Preparata et al. [31], respectively.

Theorem 2. *[31] Let $G(V, E)$ be the graph representation of a system G with V representing the processors and E the interconnection among them. Let $|V| = n$. The following two conditions are necessary for G to be t-diagnosable under PMC model:*

1. *$n \geq 2t + 1$, and*

2. *each processor is tested by at least t other processors.*

Theorem 3. *[15] The following two conditions are sufficient for a system G of n processors to be t-diagnosable under PMC model:*

1. *$n \geq 2t + 1$, and*

2. *$\kappa(G) \geq t$.*

For a directed graph G and a vertex $v \in V(G)$, the notation $d_{in}(v)$ is used to denote the number of edges directed toward v in G. Let $\Gamma(v) = \{v_i | (v, v_i) \in E\}$ and $\Gamma(X) = \cup_{v \in X} \Gamma(v) - X$, where $X \subset V$. Hakimi and Amin [15] presented a necessary and sufficient condition for a system G to be t-diagnosable as follows:

Theorem 4. *[15] Let $G(V,E)$ be the directed graph of a system G with n vertices. Then G is t-diagnosable under PMC model if and only if (i) $n \geq 2t+1$, (ii) $d_{in}(v) \geq t$ for all $v \in V$, and (iii) $|\Gamma(X)| > p$ for each integer p with $0 \leq p \leq t-1$ and for each $X \subset V$ with $|X| = n - 2t + p$.*

In this thesis, we propose some new viewpoints on diagnosis, and we will focus on undirected graph (simply abbreviated as graph). Let $G(V,E)$ be an undirected graph of a system G. The following result follows directly from Theorem 1.

Theorem 5. *For any two distinct subsets F_1 and F_2 of V, (F_1, F_2) is a distinguishable-pair under PMC model if and only if there is a vertex $u \in V - (F_1 \cup F_2)$ and there is another vertex $v \in F_1 \Delta F_2$ such that $(u,v) \in E$. (See Figure 5.)*

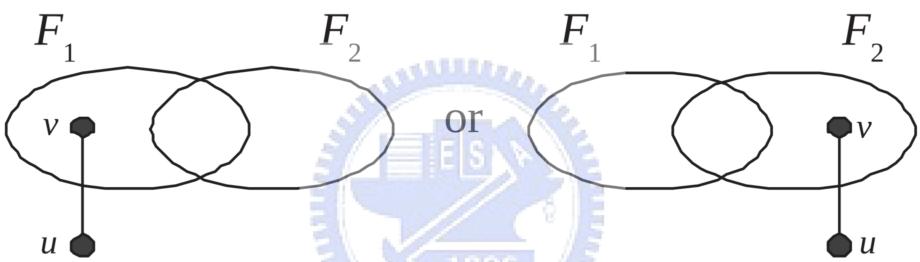

Figure 5: Illustration of a distinguishable pair (F_1, F_2).

Duo to Definition 1, the following lemma holds.

Lemma 2. *A system is t-diagnosable under PMC model if and only if for each distinct pair of subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$, F_1 and F_2 are distinguishable.*

An equivalent way of stating the above lemma is the following:

Lemma 3. *A system is t-diagnosable under PMC model if and only if for each indistinguishable pair of sets F_1 and F_2 of V, it implies that $|F_1| > t$ or $|F_2| > t$.*

By Lemma 2, a similar result for undirected graph is stated as follows.

Corollary 1. *[31] Let $G(V,E)$ be an undirected graph with n vertices. The following two conditions are necessary for G to be t-diagnosable under PMC model:*

1. $n \geq 2t+1$, *and*

2. $\delta(G) \geq t$.

For classical measurement of diagnosability, it is usually assumed that processor failures are statically independent. It does not reflect the total number of processors in the system and the probabilities of processor failures. In [29], Najjar and Gaudiot proposed fault resilience as the maximum number of failures that can be sustained while the network remains connected with a reasonably high probability. For hypercube, the fault resilience is shown as 25 percent for Q_4 and it increases to 33 percent for Q_{10}. More particularly, for the 10-dimensional hypercube Q_{10}, 33 percent of processors can fail and the network still remains connected with a probability of 99 percent. They also gave a conclusion that large-scale systems with a constant degree are more susceptible to failures by disconnection than smaller networks. With the observation of Theorem 5, a connected network gives higher probability to diagnose faulty processors and has better ability to distinguish any two sets of processors.

In an n-dimensional hypercube, Q_n has $\binom{2^n}{n}$ vertex subsets of size n, among which there are only 2^n vertex subsets which contains all the neighbors of some vertex. Since the ratio $2^n/\binom{2^n}{n}$ is very small for large n, the probability of a faulty set with size n containing all the neighbors of any vertex is very low. For this reason, Lai et al. [21] introduced a new restricted diagnosability of multiprocessor systems called *conditional diagnosability*. They consider the situation that any faulty set cannot contain all the neighbors of any vertex in a system.

Motivated by this concept [21], we extend this idea about conditional diagnosis. In this thesis, we introduce g-good-neighbor condition by claiming that for every fault-free vertex in a system, it has at least g fault-free neighbors. We now give some formal terms related to the g-good-neighbor conditional diagnosis in the following. A faulty set $F \subset V$ is called a *g-good-neighbor conditional faulty set* if $|N(v) \cap (V-F)| \geq g$ for every vertex v in $V-F$. A system G is g-good-neighbor conditional t-diagnosable if F_1 and F_2 are distinguishable, for each distinct pair of g-good-neighbor conditional faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$. Let H be a subgraph of graph G, we say that the

g-good-neighbor property of H, $P_g(H)$, holds for H if and only if every vertex in H has at least g neighbors in H.

Definition 2. *The g-good-neighbor conditional diagnosability $t_g(G)$ of a graph G is the maximum value of t such that G is g-good-neighbor conditional t-diagnosable*

Follow from this definition, we observe that under the stronger g-good-neighbor condition, the diagnosability of G is greater as well.

Lemma 4. *For any given graph G, $t_g(G) \leq t_{g'}(G)$ if $g \leq g'$.*

The g-good-neighbor conditional diagnosability is a new concept, so there are not many known results. Finding the exact value of $t_g(G)$ for general graphs would be a difficult problem. So we study a popular topology, the hypercube Q_n, as an example. In the next section, we will derive the the g-good-neighbor conditional diagnosability $t_g(Q_n)$ of the hypercube.

5 The g-good-neighbor Conditional Diagnosability of Hypercube

Before discussing the g-good-neighbor conditional diagnosability of hypercube, we have some useful observations as follows:

Theorem 6. *[30] Let $n \geq 3$ and $1 < p \leq n$. Suppose that F is a minimum cardinality cut of Q_n such that $|N_{Q_n}(\mathbf{x}) \cap F| \leq p$ for all $\mathbf{x} \in V(Q_n) - F$. Then $|F| = p2^{n-p}$.*

In the above theorem, we note that F is a g-good-neighbor conditional faulty set if $p = n - g$.

Theorem 7. *[24] Let H be a subgraph of Q_n satisfying $P_g(H)$. Then $|V(H)| \geq 2^g$, for $0 < g \leq n$.*

To find the g-good-neighbor conditional diagnosability of the hypercube Q_n, we first give an example to show that $t_g(Q_n)$ is no more than $2^g(n-g) + 2^g - 1$. We are going to show that there exist two g-good-neighbor conditional faulty sets F_1 and F_2 of $V(Q_n)$ with $|F_1| \leq 2^g(n-g) + 2^g$ and $|F_2| \leq 2^g(n-g) + 2^g$, but F_1 and F_2 are indistinguishable. Thus, we know Q_n is not g-good-neighbor conditional $(2^g(n-g) + 2^g)$-diagnosable.

We set $A = \{\mathbf{y} = y_{n-1}y_{n-2}\ldots y_0 \mid y_i = 0$ for $i \in \{g, g+1, \ldots, n-1\}$ and $y_j \in \{0,1\}$ for $j \in \{0, 1, \ldots, g-1\}\}$ and $V_k = \{\mathbf{y}^{n-k} \mid \mathbf{y} \in A$ for every $1 \leq k \leq n-g\}$. Then we set $F_1 = \cup_{i=1}^{n-g} V_i$ and $F_2 = A \cup F_1$. Since $|A| = 2^g$ and $|V_i| = 2^g$ for every $1 \leq i \leq n-g$, we obtain that $|F_1| = 2^g(n-g)$ and $|F_2| = 2^g + 2^g(n-g)$. By Theorem 5, we conclude that (F_1, F_2) is an indistinguishable pair because $A = F_1 \Delta F_2$ and $N(A) = F_1$. See Figure 6.

Now we verify that both F_1 and F_2 are g-good-neighbor conditional faulty sets. Let X be the set $V(Q_n) - (F_1 \cup F_2)$. Since F_1 is the subset of F_2, $X = V(Q_n) - F_2$. Therefore, it is sufficient to verify both $P_g(A)$ and $P_g(X)$ are satisfied. For every vertex $\mathbf{u}$ in A, it is easy to see that $\mathbf{u}^i$ in A for every $i \in \{0, 1, \ldots, g-1\}$. Thus, $P_g(A)$ holds. Now we

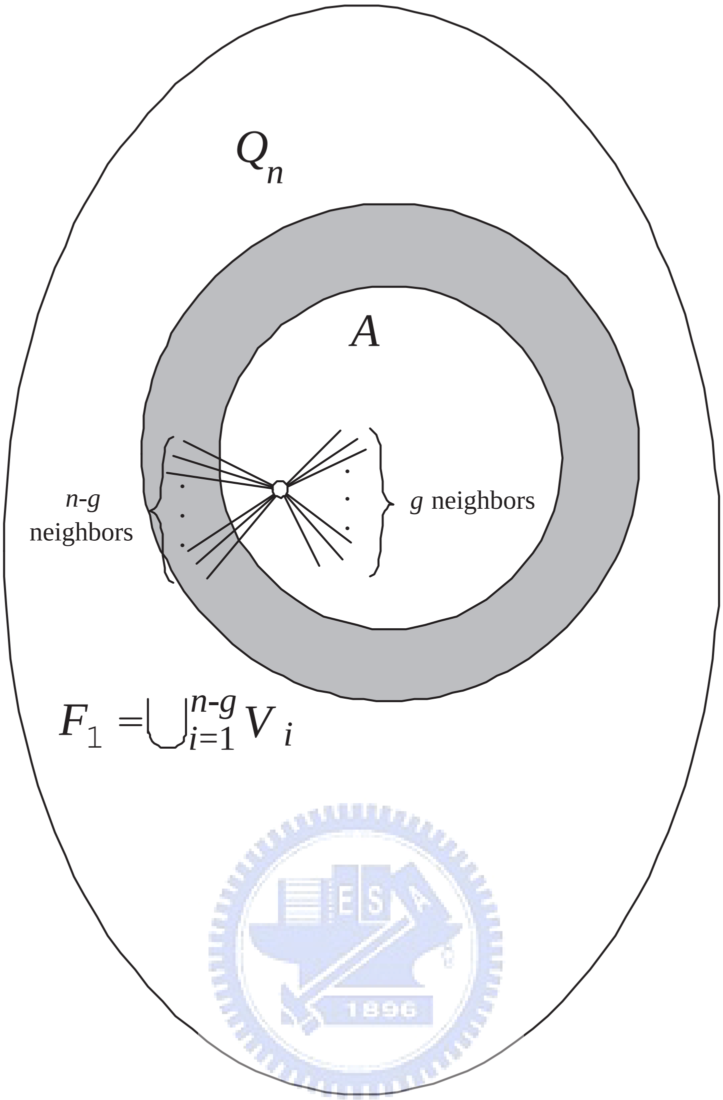

Figure 6: Illustration of F_1 and F_2.

consider the vertices in X. By the definition of X, we know that for every $\mathbf{x}$ in X, $\mathbf{x}^i \in X$ for every $0 \leq i \leq g-1$. Thus, the property $P_g(X)$ also holds. Therefore, both F_1 and F_2 are g-good-neighbor conditional faulty sets of Q_n.

Since (F_1, F_2) is an indistinguishable pair with $|F_1| = 2^g(n-g)$ and $|F_2| = 2^g(n-g)+2^g$, we conclude that the g-good-neighbor conditional diagnosability of Q_n is less than $2^g(n-g)+2^g$. The following lemma states the fact.

Lemma 5. *For $0 \leq g \leq n-3$, $t_g(Q_n) \leq 2^g(n-g)+2^g-1$.*

The following theorem shows that the g-good-neighbor conditional diagnosability of hypercube $t_g(Q_n)$ is exactly $2^g(n-g)+2^g-1$.

Theorem 8. *For $0 \leq g \leq n-3$, $t_g(Q_n) = 2^g(n-g) + 2^g - 1$.*

Proof:

By Lemma 5, we already know that $t_g(Q_n) \leq 2^g(n-g) + 2^g - 1$. Therefore, we only need to verify that $t_g(Q_n) \geq 2^g(n-g) + 2^g - 1$. To prove Q_n is g-good-neighbor conditional $(2^g(n-g)+2^g-1)$-diagnosable, it is equivalent to prove that F_1 and F_2 must be distinguishable for every two distinct g-good-neighbor conditional faulty sets F_1 and F_2 of Q_n, provided that both the cardinality of F_1 and cardinality of F_2 are no more than $2^g(n-g) + 2^g - 1$.

We prove this theorem by contradiction. Suppose that there are two distinct g-good-neighbor conditional faulty sets F_1 and F_2, which are indistinguishable with $|F_1| \leq 2^g(n-g)+2^g-1$ and $|F_2| \leq 2^g(n-g)+2^g-1$. Now we consider all the possible cases such that F_1 and F_2 are indistinguishable. By Theorem 5, there are two cases such that F_1 and F_2 are indistinguishable: $V(Q_n) = F_1 \cup F_2$ or $V(Q_n) \neq (F_1 \cup F_2)$ but there is no test from $V(Q_n) - (F_1 \cup F_2)$ to $F_1 \Delta F_2$. Without loss of generality, we assume that $F_2 - F_1 \neq \emptyset$. We show that each case has contradiction with our assumption.

Case 1: $V(Q_n) = F_1 \cup F_2$.

Since $g \leq n-3$ and all the vertices of Q_n are in $F_1 \cup F_2$, we obtain the following equation with contradiction:

$$\begin{aligned}
2^n &= |V(Q_n)| \\
&= |F_1| + |F_2| - |F_1 \cap F_2| \\
&\leq |F_1| + |F_2| \\
&\leq 2(2^g(n-g) + 2^g - 1) \\
&\leq 2(2^{n-3}(n-(n-3)+1)) - 2 \\
&= 2^n - 2
\end{aligned}$$

which is a contradiction.

Case 2: $V(Q_n) \neq (F_1 \cup F_2)$.

In this case, we show $|F_2| \geq 2^g + (n-g)2^g$, which is a contradiction with our assumption, regardless $F_1 \subset F_2$ or not. Since F_1 and F_2 are indistinguishable, there are no edges between $V(Q_n) - (F_1 \cup F_2)$ and $F_1 \Delta F_2$. By the assumption that $F_2 - F_1 \neq \emptyset$ and F_1 is a g-good-neighbor conditional faulty set, any vertex in $F_2 - F_1$ has at least g good neighbors in subgraph $F_2 - F_1$. By Theorem 7, the size of $F_2 - F_1$ is characterized, and thus we have $|F_2 - F_1| \geq 2^g$. In addition, since $V(Q_n) - (F_1 \cup F_2)$ and $F_1 \Delta F_2$ are disconnected, Q_n would be divided into several components by deleting $F_1 \cap F_2$. Besides, since F_1 and F_2 are both g-good-neighbor conditional faulty sets, $F_1 \cap F_2$ is also a g-good-neighbor conditional faulty set. By Theorem 6, the minimum cardinality cut of Q_n with g-good-neighbor condition is $p2^{n-p} = (n-g)2^g$. Thus, we obtain that $|F_2 \cap F_1| \geq (n-g)2^g$. As a result, $|F_2| = |F_2 - F_1| + |F_2 \cap F_1| \geq 2^g + (n-g)2^g$ which contradicts with that $|F_2| \leq 2^g + (n-g)2^g - 1$.

Based on these two cases above, we conclude that $t_g(Q_n) \geq 2^g(n-g) + 2^g - 1$. Thus, the g-good-neighbor conditional diagnosability $t_g(Q_n) = 2^g(n-g) + 2^g - 1$. This completes the proof of this theorem. $\square$

The following table shows the g-good-neighbor conditional diagnosability of n-dimensional hypercube $t_g(Q_n)$ of small n where $0 \leq g \leq n-3$.

n	g	$\lvert V(Q_n)\rvert$	$t_g(Q_n)$	ratio
3	0	8	3	0.375
4	0	16	4	0.25
4	1	16	7	0.4375
5	0	32	5	0.15625
5	1	32	9	0.28125
5	2	32	15	0.46875
6	0	64	6	0.09375
6	1	64	11	0.171875
6	2	64	19	0.296875
6	3	64	31	0.484375
7	0	128	7	0.0546875
7	1	128	13	0.1015625
7	2	128	23	0.1796875
7	3	128	39	0.3046875
7	4	128	63	0.4921875

Table 2: $t_g(Q_n)$ of small n

6 Conclusions

In probabilistic models of multiprocessor systems, processors fail independently, but with different probabilities. The probability that all faulty processors are neighbors of one processor is very small. In this thesis, we propose the concept of g-good-neighbor conditional diagnosis with any fault-free vertex has at least g neighboring fault-free vertices. To grant more accurate measurement of diagnosability for a large-scale processing system, we introduce the g-good-neighbor conditional diagnosability of a system under the PMC model. The g-good-neighbor conditional diagnosability of the hypercube Q_n is demonstrated to be $2^g(n-g)+2^g-1$.

Observing that when $g=0$, there is no restriction on the faulty sets and we have the traditional diagnosability on the hypercube as n. In addition, in the special case of $g=1$, our result is slightly different from the measure of diagnosability given by Lai et al. [21]. The difference between these two measures is that we only consider the condition of the fault-free vertices in the network. A thorough investigation of the diagnosability with the requirement of having at least g good neighbors for all vertices would be an interesting problem to study in the future.

In the area of diagnosability, the comparison model is another well-known and widely chosen fault diagnosis model. Hence, for further discussion, it is worthy to determining the g-good-neighbor conditional diagnosability of a system under comparison model.

The classical diagnosability of a system is small owing to the assumption that all neighbors of each processor can potentially fail at the same time regardless of the probability. If there are exactly n faulty processors in a system of minimum degree n, however, the probability of the faulty set containing all the neighbors of any vertex is statistically low for large multiprocessor systems. Therefore, it is an attractive work to develop more different measures of g-good-neighbor conditional diagnosability based on application environment, network topology, network reliability, and statistics related to fault patterns.

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