

## Sums of Idempotent Matrices

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### ABSTRACT

We show that any complex square matrix  $T$  is a sum of finitely many idempotent matrices if and only if  $\text{tr } T$  is an integer and  $\text{tr } T \geq \text{rank } T$ . Moreover, in this case the idempotents may be chosen such that each has rank one and has range contained in that of  $T$ . We also consider the problem of the minimum number of idempotents needed to sum to  $T$  and obtain some partial results.

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A complex square matrix  $T$  is *idempotent* if  $T^2 = T$ . In this paper, we characterize matrices which can be expressed as a sum of finitely many idempotent matrices and consider the minimum number of idempotents needed in such expressions.

In the following,  $\text{tr } T$  denotes the *trace* of a matrix  $T$ ,  $\text{ran } T$  denotes its *range*,  $\text{rank } T$  the dimension of  $\text{ran } T$ , and  $\text{ker } T$  the *kernel* of  $T$ . The  $n \times n$  identity matrix is denoted by  $I_n$ , or  $I$  if the size is not emphasized. Similarly for the zero matrix:  $0_n$  or  $0$ . Two matrices  $T$  and  $S$  are *similar*, denoted  $T \approx S$ , if  $XT = SX$  for some nonsingular matrix  $X$ ; they are *unitarily equivalent*,  $T \cong S$ , if the above  $X$  can be chosen to be unitary. If  $T$  and  $S$  act on spaces  $H$  and  $K$ , respectively, then

$$T \oplus S = \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$$

acts on  $H \oplus K$ , the orthogonal direct sum of  $H$  and  $K$ .

We start with the characterization of sums of idempotents.

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Now we prove the converse. Let  $t = \text{tr } T$  and  $r = \text{rank } T$ , and assume that  $T$  acts on the space  $H$ . Then  $T$  is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix}$$

with respect to the decomposition  $H = \text{ran } T^* \oplus \text{ker } T$ . Here  $T_1$  is of size  $r \times r$ . We have two cases to consider:

(1)  $T_1$  is not a scalar matrix. We choose positive integers  $t_1, \dots, t_r$  with  $\sum_j t_j = t$ . Note that this is always possible, since  $t$  is an integer and  $t \geq r$ . By a result of Fillmore's [2, Theorem 2],  $T_1$  is similar to a matrix of the form

$$\begin{bmatrix} t_1 & & * \\ & \ddots & \\ * & & t_r \end{bmatrix}.$$

Hence  $T$  is similar to

$$T' \equiv \left[ \begin{array}{cccc|c} t_1 & & & * & 0 \\ * & & \ddots & & \\ & & & t_r & \\ \hline & & & * & 0 \end{array} \right].$$

If  $x_1, \dots, x_r$  denote the first  $r$  column vectors of  $T'$ , then  $T'$  can be expressed as

$$t_1 \begin{bmatrix} \frac{1}{t_1} x_1 & 0 & \cdots & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & \frac{1}{t_2} x_2 & 0 & \cdots & 0 \end{bmatrix} \\ + \cdots + t_r \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{t_r} x_r & 0 & \cdots & 0 \end{bmatrix}.$$

Note that each of these latter matrices is idempotent. Hence we have expressed  $T'$  as a sum of  $t$  rank-1 idempotents, each with range contained in that of  $T'$ . Since these properties are all preserved under similarity of matrices, they also hold for  $T$ .

(2)  $T_1$  is a scalar matrix. Say  $T_1 = aI_r$ . Since  $\text{tr } T = \text{tr } T_1 = ra \geq r$ , we have  $a \geq 1$ . If  $a = 1$ , that is,  $t = r$ , then

$$T \cong \begin{bmatrix} I_r & 0 \\ T_2 & 0 \end{bmatrix} = [y_1 \quad 0 \quad \cdots \quad 0] \\ + \cdots + [0 \quad \cdots \quad 0 \quad y_r \quad 0 \quad \cdots \quad 0],$$

where  $y_1, \dots, y_r$  are the first  $r$  column vectors of the preceding matrix. Thus  $T$  is a sum of  $t$  idempotents with all the required properties. Otherwise  $a > 1$  or, equivalently,  $t > r$ . In this case,

$$T \cong \begin{bmatrix} aI_r & 0 \\ T_2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{a}y_1 & 0 & \cdots & 0 \end{bmatrix} \\ + \left[ \left(1 - \frac{1}{a}\right)y_1 \quad y_2 \quad \cdots \quad y_r \quad 0 \quad \cdots \quad 0 \right] \equiv S_1 + S_2,$$

where  $S_1$  is idempotent, the upper left corner

$$\begin{bmatrix} a-1 & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{bmatrix}$$

of  $S_2$  is nonscalar, and  $\text{tr } S_2 = t - 1 \geq r = \text{rank } S_2$ . Arguing as in (1), we obtain that  $S_2$  is expressible as a sum of  $t - 1$  idempotents with the desired properties. Thus  $T$  is a sum of  $t$  idempotents as asserted. ■

**COROLLARY 2.** *The following statements are equivalent for a matrix  $T$ :*

- (1)  $T$  is a sum of finitely many idempotents;
- (2)  $T$  is a sum of infinitely many idempotents;
- (3)  $T$  is the limit of a sequence of matrices each of which is a sum of finitely many idempotents.

*Proof.* Since the implications (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are trivial, we need only prove (3)  $\Rightarrow$  (1).

Let  $T = \lim_n T_n$ , where each  $T_n$  is a sum of finitely many idempotents. By Theorem 1,  $\text{tr } T_n$  is an integer and  $\text{tr } T_n \geq \text{rank } T_n$  for all  $n$ . The continuity of trace and the lower semicontinuity of rank [4, Appendix] implies that the same hold for  $T$ . Thus  $T$  is a sum of idempotents. ■

Note that similar arguments to the above show that the analogue of Corollary 2 for projections also holds.

Recall that a matrix  $T$  is an *involution* if  $T^2 = I$ . Idempotents and involutions are closely related through the one-to-one correspondence:  $T$  is idempotent if and only if  $2T - I$  is an involution. Using Theorem 1, we may easily characterize sums of involutions.

**COROLLARY 3.** *An  $n \times n$  matrix  $T$  is a sum of finitely many involutions if and only if  $\text{tr } T$  is an integer and (for  $n$  even) even.*

*Proof.* The necessity is easy: the trace of any involution is an integer and even if  $n$  is. The same hold for sums of involutions.

To prove the sufficiency, we assume that  $n \geq 2$ . Let  $m$  be an integer such that  $m \geq 2 - (1/n)\text{tr } T$  and  $m$  and  $\text{tr } T$  are even or odd at the same time. Consider  $S = \frac{1}{2}(T + mI)$ . Since  $\text{tr } S = \frac{1}{2}\text{tr } T + \frac{1}{2}mn$  is an integer and  $\text{tr } S \geq n \geq \text{rank } S$ , Theorem 1 implies that  $S$  is a sum of idempotents:  $S = \sum_{j=1}^k E_j$ . Thus  $T = \sum_{j=1}^k (2E_j - I) + (k - m)I$  is a sum of involutions. ■

We remark in passing that using the result of Percy and Topping [8], we may easily show that every operator on an infinite-dimensional Hilbert space is a sum of at most five involutions.

In the proof of Theorem 1, we actually showed that any  $T$  with  $\text{tr } T$  integer and  $\text{tr } T \geq \text{rank } T$  is a sum of  $\text{tr } T$  many idempotents. We may suspect that  $\text{tr } T$  is the minimum number of idempotents needed to sum to  $T$ . For sums of projections, this is indeed the case: any positive semidefinite  $T$  with  $\text{tr } T$  integer and  $\text{tr } T \geq \text{rank } T$  is a sum of  $\text{tr } T$  many projections, and the number  $\text{tr } T$  is sharp in the sense that for any integers  $n, t$ , and  $r$  satisfying  $n, t \geq r \geq 1$ , there exists an  $n \times n$  positive semidefinite matrix  $T$  with  $\text{tr } T = t$  and  $\text{rank } T = r$  which is not a sum of less than  $t$  projections. The first assertion is proved in [1]; the second follows from the following simple arguments. Let  $t_1, \dots, t_r$  be real numbers satisfying  $0 < t_r \leq \dots \leq t_2 < 1 < t_1$  and  $\sum_j t_j = t$ , and let

$$T = \begin{bmatrix} t_1 & & & & & & & & & 0 \\ & \ddots & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & t_r & & & & & & \\ & & & & 0 & & & & & \\ & & & & & \ddots & & & & \\ 0 & & & & & & & & & 0 \end{bmatrix}.$$

Then  $T$  is positive semidefinite with  $\text{tr } T = t$  and  $\text{rank } T = r$ . If  $T = \sum_{j=1}^m P_j$

is a sum of  $m$  projections, we need to check that  $m \geq t$ . Indeed, in this case, we must have  $\text{rank } P_j \leq 1$  for all  $j$ , for otherwise, if  $\text{rank } P_j \geq 2$  for some  $j$ , then  $P_j \leq T$  implies that  $1 \leq t_2$  by the well-known Courant-Fisher min-max theorem (cf. [7, p. 182, Corollary 4.3.3]), contradicting our choice of  $t_2$ . Thus  $t = \text{tr } T = \sum_j \text{tr } P_j = \sum_j \text{rank } P_j \leq m$  as asserted.

The situation is quite different for sums of idempotents: the number  $\text{tr } T$  is not always sharp. This is quite plausible considering the fact that the class of idempotents is much larger than that of projections. Put more precisely, the problem asks, for any integers  $n, t$ , and  $r$  satisfying  $n, t \geq r \geq 1$ , what the minimum number  $m$  is for which every  $n \times n$  matrix  $T$  with  $\text{tr } T = t$  and  $\text{rank } T = r$  is a sum of  $m$  idempotents. We have not been able to solve this problem completely. Only some partial results are obtained. In the following, we will show that  $m = t$  if  $t - r = 0$  or  $1$  and completely determine the number  $m$  for  $t \leq 4$ .

**THEOREM 4.** *For any integers  $n$  and  $r$  satisfying  $n \geq r \geq 1$ , there exists an  $n \times n$  matrix  $T$  ( $S$ ) with  $\text{tr } T = \text{rank } T = r$  ( $\text{tr } S = r + 1$  and  $\text{rank } S = r$ ) which is not a sum of less than  $r(r + 1)$  idempotents.*

*Proof.* The assertion is trivial for  $n = 1$ . Hence from now on we assume that  $n \geq 2$ .

Let  $0 < a < 1$  and  $T = T' \oplus 0_{n-r}$ , where  $T'$  is the  $r \times r$  matrix

$$\begin{bmatrix} r - (r - 1)a & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{bmatrix}.$$

Trivially,  $\text{tr } T = \text{rank } T = r$ . Assume that  $T = \sum_{k=1}^m E_k$  is a sum of  $m$  idempotents, where  $m < r$ . Let  $t_k = \text{tr } E_k = \text{rank } E_k$  for each  $k$ . Since  $r = \text{tr } T = \sum_{k=1}^m t_k > m$ , at least one of the  $t_k$ 's is not less than 2, say,  $t_1 \geq 2$ . Note that each  $E_k$  is similar to a rank- $t_k$  projection:  $E_k = X_k^{-1}(0_{s_{k-1}} \oplus I_{t_k} \oplus 0_{n-s_k})X_k$ , where  $s_k = \sum_{l=1}^k t_l$ . Let  $x_1^{(k)}, \dots, x_n^{(k)}$  ( $y_1^{(k)}, \dots, y_n^{(k)}$ ) denote the row (column) vectors of  $X_k$  ( $X_k^{-1}$ ). Then the  $x$ 's and  $y$ 's satisfy  $x_i^{(k)} y_j^{(k)} = \delta_{ij}$  for all  $i, j$ , and  $k$ . On the other hand, from the above we deduce that  $E_k = y_{s_{k-1}+1}^{(k)} x_{s_{k-1}+1}^{(k)} + \dots + y_{s_k}^{(k)} x_{s_k}^{(k)}$ . Therefore,

$$T = \sum_{k=1}^m E_k = \sum_{k=1}^m \sum_{j=s_{k-1}+1}^{s_k} y_j^{(k)} x_j^{(k)} = YX,$$

where  $Y$  is the  $n \times r$  matrix

$$\left[ y_1^{(1)} \quad \cdots \quad y_{s_1}^{(1)} \quad y_{s_1+1}^{(2)} \quad \cdots \quad y_{s_2}^{(2)} \quad \cdots \quad y_{s_{m-1}+1}^{(m)} \quad \cdots \quad y_{s_m}^{(m)} \right]$$

and  $X$  is the  $r \times n$  matrix

$$\left[ x_1^{(1)t} \quad \cdots \quad x_{s_1}^{(1)t} \quad x_{s_1+1}^{(2)t} \quad \cdots \quad x_{s_2}^{(2)t} \quad \cdots \quad x_{s_{m-1}+1}^{(m)t} \quad \cdots \quad x_{s_m}^{(m)t} \right]^t,$$

$x^t$  being the transpose of  $x$ . A result of Flanders's [3] implies that  $XY \approx T'$ . However,  $XY$  is of the form

$$\begin{bmatrix} I_{t_1} & & * \\ & \ddots & \\ * & & I_{t_m} \end{bmatrix}.$$

Thus we have

$$T' \approx R \equiv \begin{bmatrix} I_{t_1} & A \\ B & C \end{bmatrix},$$

and it follows that

$$T' - aI \approx R - aI = \begin{bmatrix} (1-a)I_{t_1} & A \\ B & C - aI \end{bmatrix}.$$

Assume that

$$R = \begin{bmatrix} I_{t_1} & A \\ B & C \end{bmatrix}$$

acts on  $H = K \oplus L$ . We claim that  $(R - aI)|K$  is one-to-one. Indeed, if  $x \in K$  and  $(R - aI)x = 0$ , then  $(R - I)x = (a - 1)x$ . Since

$$(R - I)x = \begin{bmatrix} 0 & A \\ B & C - I \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ Bx \end{bmatrix}$$

belongs to  $L$ , and  $(a - 1)x$  belongs to  $K$ , we may conclude that  $(a - 1)x = 0$ ,

or  $x = 0$ , as asserted. Hence

$$t_1 = \dim K = \dim(R - aI)K \leq \text{rank}(R - aI) = \text{rank}(T' - aI) = 1,$$

contradicting our assumption that  $t_1 \geq 2$ . This proves that  $T$  is not a sum of less than  $r$  idempotents.

Next we construct  $S$ : let  $S = S' \oplus 0_{n-r}$ , where  $S'$  is the  $r \times r$  matrix  $(1 + 1/r)I_r$ . Trivially,  $\text{tr} S = r + 1$  and  $\text{rank} S = r$ . If  $S$  is a sum of less than  $r + 1$  idempotents, then, as before, we deduce that  $S' \oplus 0_1$  is similar to a matrix of the form

$$R \equiv \begin{bmatrix} I_{t_1} & A \\ B & C \end{bmatrix}$$

acting on  $K \oplus L$ , where  $t_1 \geq 2$ . Again, we can show that  $(R - (1 + 1/r)I_{r+1})|_K$  is one-to-one. Hence

$$\begin{aligned} t_1 = \dim K &= \dim \left[ R - \left( 1 + \frac{1}{r} \right) I_{r+1} \right] K \leq \text{rank} \left[ R - \left( 1 + \frac{1}{r} \right) I_{r+1} \right] \\ &= \text{rank} \left[ 0_r \oplus \left( -1 - \frac{1}{r} \right) I_1 \right] = 1, \end{aligned}$$

contradicting the fact that  $t_1 \geq 2$ . This shows that  $S$  is not a sum of less than  $r + 1$  idempotents, completing the proof.  $\blacksquare$

That, in general,  $\text{tr} T$  is not sharp can be seen from the next proposition.

**PROPOSITION 5.** *Any  $n \times n$  matrix  $T$  ( $n \geq 2$ ) with  $\text{tr} T = 4$  and  $\text{rank} T = 1$  or  $2$  is a sum of three idempotents, and the number three is sharp.*

For the proof, we need a recent result of Hartwig and Putcha's [5] which characterizes sums (or differences) of two idempotents. In particular, the following simple lemma will be repeatedly used.



LEMMA 6. For any matrix  $T$ ,  $(I + T) \oplus (I - T)$  is a sum of two idempotents.

*Proof.* Let

$$E = \begin{bmatrix} \frac{1}{2}(I + T) & I \\ \frac{1}{4}(I - T^2) & \frac{1}{2}(I - T) \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \frac{1}{2}(I + T) & -I \\ \frac{1}{4}(T^2 - I) & \frac{1}{2}(I - T) \end{bmatrix}.$$

It is easily seen that  $E$  and  $F$  are idempotent and  $E + F = (I + T) \oplus (I - T)$ . ■

*Proof of Proposition 5.* If  $T$  has trace 4 and rank 1, then it must be similar to

$$\begin{bmatrix} 4 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 4 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \ddots \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 3 & & & 0 \\ & -1 & & \\ & & 0 & \ddots \\ 0 & & & 0 \end{bmatrix},$$

where the first matrix is idempotent and the second one is a sum of two

idempotents by Lemma 6. Hence  $T$  is a sum of three. That

$$\begin{bmatrix} 4 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

is not a sum of two idempotents follows from the characterization of such matrices given by Hartwig and Putcha [5, Theorem 1b].

Next we assume that  $T$  has trace 4 and rank 2. By the Jordan form,  $T$  is similar to one of the following matrices:

$$\begin{bmatrix} 2 & 1 & & 0 \\ & 2 & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & & & 0 \\ & 0 & 1 & \\ & 0 & 0 & \\ & & & 0 & \\ 0 & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

$$\text{and } \begin{bmatrix} a & & & 0 \\ & b & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix},$$

where  $a, b \neq 0$  and  $a + b = 4$ . Then

$$\begin{bmatrix} 2 & 1 & & 0 \\ & 2 & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & & 0 \\ 0 & 1 & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

expresses the first matrix as a sum of three idempotents. As for the second one, we have

$$\begin{bmatrix} 4 & & & & 0 \\ & 0 & 1 & & \\ & 0 & 0 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & & 0 \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix} + \begin{bmatrix} 3 & & & & 0 \\ & 0 & 1 & & \\ & 0 & -1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 3 & & & & 0 \\ & 0 & 1 & & \\ & 0 & -1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix} \approx \begin{bmatrix} 3 & & & & 0 \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix}$$

and the latter is a sum of two idempotents by Lemma 6, this shows that, in this case,  $T$  is a sum of three idempotents. Finally, for the last one, we have

$$\begin{bmatrix} a & & & & 0 \\ & b & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} + \begin{bmatrix} a-1 & & & & 0 \\ & b-1 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix},$$

the second of which is a sum of two idempotents by Lemma 6. That three is sharp in this case can be observed from the matrix

$$\begin{bmatrix} 3 & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$$

using [5, Theorem 1b]. ■

The preceding proposition, together with Theorem 4, disposes of the problem of the minimum number of idempotents for matrices with trace 4. The cases with trace less than 4 can be similarly handled. We leave the details to the reader.

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