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Note

An Upper Bound for the Transversal Numbers of 4-Uniform Hypergraphs[1]

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The main purpose of this paper is to prove that if H is a 4-uniform hypergraph with n vertices and m edges, then the transversal number $\tau(H) \leqslant 2(m+n)/9$.

All standard terminology of hypergraphs is from [1]. Suppose $H=(V, E)$ is a k-uniform hypergraph with n vertices and m edges. Tuza [2] proposed the problem of finding an upper bound for the transversal number $\tau(H)$, of the form $\tau(H) \leqslant c_k(n+m)$, where c_k depends only on k. More precisely, we want to determine $c_k \equiv \sup \tau(H)/(m+n)$, where H runs over all k-uniform hypergraphs of n vertices and m edges. It is easy to see that $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{3}$. Tuza [2] proved that $c_3 = \frac{1}{4}$ and asked if c_k is $O(1/k)$.

For any positive integer p we can construct a k-uniform hypergraph H of $n = k + p$ vertices $x_1, ..., x_n$ and $m = \lceil n/p \rceil$ edges $e_1, ..., e_m$, where

$$e_i = \{x_1, ..., x_n\} - \{x_{ip-p+1}, ..., x_{ip}\} \qquad \text{for} \quad 1 \leqslant i \leqslant m-1$$

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and

$$e_m = \{x_1, ..., x_k\}.$$

Then $\tau(H) = 2$. To make $\tau(H)/(n+m)$ large, we only have to find p such that $n+m = k+1+p+\lceil k/p \rceil$ is minimum. It is an easy exercise to check that $n+m$ achieves minimum at $p = \lfloor \sqrt{k} \rfloor$. So $c_k \geqslant b_k \equiv 2/(k+1+\lfloor \sqrt{k} \rfloor + \lceil k/\lfloor k \rfloor \rceil)$. Note that $c_k = b_k = 1/(k+1)$ for $k \leqslant 3$. The main purpose of this paper is to prove that $c_4 = b_4 = \frac{2}{9}$. In fact we prove a more general theorem.

THEOREM. *Suppose $H = (V, E)$ is a 4-uniform hypergraph of n vertices and m edges. If H has t end edges, which are edges containing vertices of degree one, then $\tau(H) \leqslant (2n+2m-t)/9$.*

Proof. Without loss of generality we may assume that H is connected. We shall prove the theorem by induction on m. The theorem is trivial for $m = 0$ since $\tau(H) = t = 0$.

Suppose $X = \{v_1, ..., v_k\} \subseteq V$ is a set of k vertices. Denote by H' the hypergraph obtained from H by deleting all edges $e_1, ..., e_p$ that meet X and all vertices only in $e_1 \cup \cdots \cup e_p$. Then

$$\tau(H) \leqslant k + \tau(H') \leqslant (2n' + 2m' - t' + 9k)/9,$$

$$n' = n - r,$$

$$m' = m - p,$$

$$t' = t - q + s,$$

where r is the number of vertices only in $e_1 \cup \cdots \cup e_p$, q is the number of end edges in $e_1, ..., e_p$, and s is the number of end edges in H' which are not end edges in H. Consequently,

$$\tau(H) \leqslant (2n+2m-t)/9 - (2p+2r+s-q-9k)/9.$$

So the theorem holds when $\Delta^* \equiv 2p + 2r + s - q \geqslant 9|X|$. In the case when each vertex in X is of degree $\geqslant 2$, $r \geqslant q + k$ and the theorem holds when $\Delta \equiv 2p + r + s \geqslant 8|X|$.

Case 1. If there is a vertex x of degree $p \geqslant 4$, then choose $X = \{x\}$. $p \geqslant 4$ implies $\Delta \geqslant 8|X|$.

Case 2. Suppose all vertices of H are of degree 3. Choose a k-cycle $C = (x_1, e_1, x_2, e_2, ..., x_k, e_k, x_1)$ of minimum length. Denote by e_1' the edge other than e_{i-1} and e_i that contains x_i.

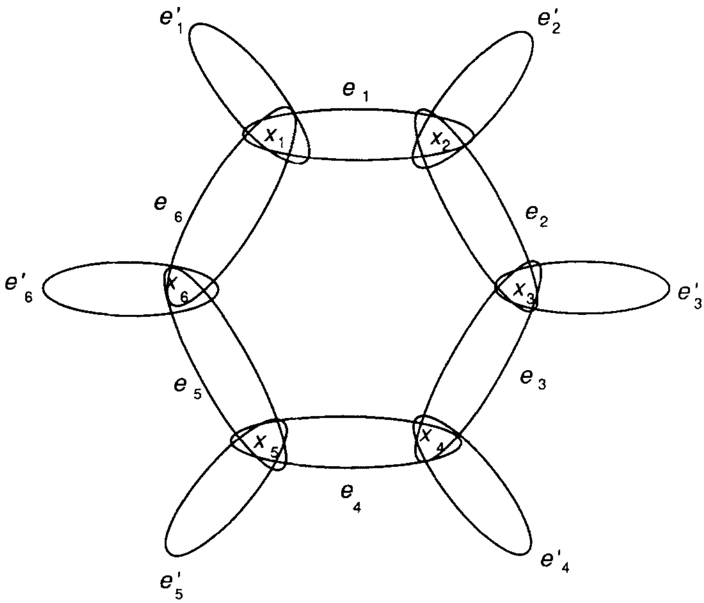

FIGURE 1

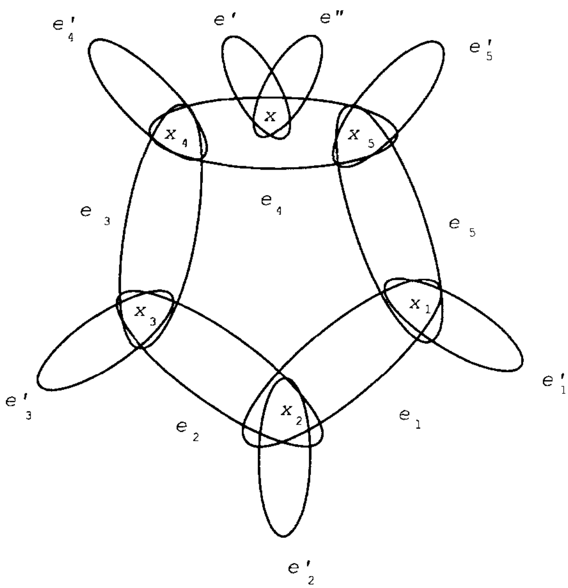

FIGURE 2

In the case $k=2$, let $X=\{x_1\}$. Then $p=3$; $r\geqslant 2$ when $e'_1=e'_2$; $r\geqslant 1$ and $s\geqslant 1$ when $e'_1\neq e'_2$. Thus $\Delta\geqslant 8|X|$.

Now assume $k\geqslant 3$. Choose a vertex $x\in e_{k-1}-\{x_{k-1}, x_k\}$; denote by e' and e'' the edges other than e_{k-1} that contain x. By the minimality of the length of C, the edges $e_1, ..., e_k, e'_1, ..., e'_k, e', e''$ are distinct. Now consider $X=\{x_1, x_3, x_5, ..., x_{k-1}\}$ when k is even and $X=\{x_1, x_3, ..., x_{k-2}, x\}$ when k is odd. Then $p=3k/2$ and $r, s\geqslant |X|=k/2$ when k is even (see Fig. 1 for $k=6$); and $p=3(k+1)/2$ and $r, s\geqslant |X|=(k+1)/2$ when k is odd (see Fig. 2 for $k=5$). In any case $\Delta\geqslant 8|X|$.

Case 3. Suppose all vertices are of degree at most 3 and there is a vertex x of degree 3 contained in e_1, e_2, e_3; but e_1 has at least one vertex of degree $\leqslant 2$.

If e_1 is an end edge, then consider $X=\{x\}$. $p=3$ and $r\geqslant 2$ imply $\Delta\geqslant 8|X|$. So we can assume that e_1 is not an end edge. Suppose y is a vertex of degree 2 in e_1, say $y\in e_1\cap e$. If $e=e_2$ or e_3, then consider $X=\{x\}$. Again $p=3$ and $r\geqslant 2$ imply $\Delta\geqslant 8|X|$. So assume $e\neq e_2$ and $e\neq e_3$. If e is not an end edge, then consider $X=\{x\}$. $p=3$, $r\geqslant 1$ and $s\geqslant 1$ imply $\Delta\geqslant 8|X|$.

Now we can assume $e=\{y, y_1, y_2, y_3\}$ with $1=\deg(y_1)\leqslant\deg(y_2)\leqslant\deg(y_3)\leqslant 3$. If $\deg(y_3)=3$, then choose $X=\{y_3\}$ to get $\Delta\geqslant 8|X|$. So assume $\deg(y_3)=2$, say $y_3\in e\cap e'$. In the case of $\deg(y_2)=1$ or e' is an end edge we consider $X=\{y_3\}$, then $p=2$, $r\geqslant 3$, and $s\geqslant 1$; in the case of $\deg(y_2)=2$ and e' is not an end edge we consider $X=\{y_2\}$, then $p=2$, $r\geqslant 2$, and $s\geqslant 2$. In any case $\Delta\geqslant 8|X|$.

Case 4. Suppose all vertices of H are of degree at most 2. Suppose there is an end edge e containing at least two vertices of degree one. Choose $X=\{x\}$, where x is a vertex in e of maximum degree p. Then either $p=q=1$ and $r=4$, which imply $\Delta^*\geqslant 9=9|X|$; or else $p=2$ and $r\geqslant q+2\geqslant 3$ which imply $\Delta^*=2p+2r-q+s\geqslant 2p+2q+4-q+s\geqslant 9=9|X|$. So we can assume that every and edge has exactly one vertex of degree one. Count the degrees of all vertices. There are t vertices of degree one and $n-t$ of degree 2; then we have $4m=t+2(n-t)=2n-t$.

Suppose there are two distinct edges e and f such that $|e\cap f|=3$, say $e=\{x_1, x_2, x_3, x_4\}$ and $f=\{x_1, x_2, x_3, x_5\}$. If $\deg(x_4)=1$ or $\deg(x_5)=1$, then we choose $X=\{x_1\}$. In this case $p=2$ and $r\geqslant 4$, which imply $\Delta\geqslant 8|X|$. So we may assume $\deg(x_4)=\deg(x_5)=2$, say $x_4\in e\cap e'$ and $x_5\in f\cap f'$. If e' or f' is not an edge, then we choose $X=\{x_1\}$. In this case $p=2$, $r=3$, and $s\geqslant 1$, which imply $\Delta\geqslant 8|X|$. So we may assume that e' and f' are end edges, say $e'=\{x_4, y_1, y_2, y_3\}$ with $\deg(y_3)=1$ and $\deg(y_1)=\deg(y_2)=2$ (since every end edge has exactly one vertex of degree one). Let $y_1\in e'\cap e_1$ and $y_2\in e'\cap e_2$. If e_1 or e_2 is an end edge, we can without loss of generality assume that e_1 is an end edge. Now choose $X=\{y_1\}$. Then either $p=2$, $r=3$, and $s \geqslant 1$ (when e_1 is an end edge), or else $p=2$, $r=2$, and $s \geqslant 2$ (when e_1 is not an end edge). In any case $\Delta \geqslant 8$. So we can assume that $|e \cap f| \leqslant 2$ for any two distinct edges e and f.

Choose maximum number of vertices $x_1, ..., x_k$ of degree 2, say $x_i \in e_i \cap f_i$ for $1 \leqslant i \leqslant k$, such that $e_1, f_1, e_2, f_2, ..., e_k, f_k$ are distinct. Let $g_1, ..., g_{m-2k}$ be the remaining edges of E. By the maximality of k, the edges g_j are pairwise disjoint. Each edge g_j has at least three vertices of degree 2, which are also in some e_i or f_i; call such vertices *common vertices*. Note that there are at least $3(m-2k)$ common vertices. On the other hand, $e_i \cup f_i$ has at most three common vertices for any i. Otherwise the fact that any two distinct edges intersect at no more than two vertices implies that either $|e_i \cap f_i| = |e_i \cap g_j| = |f_i \cap g_j| = 2$ for some g_j or else there are common vertices $x_i^* \in e_i \cap g_j$ and $x_i^{**} \in f_i \cap g_{j'}$ for distinct j and j'. For the former case, H has exactly 3 edges and 6 vertices and $\tau(H)=2$, so the theorem holds. For the latter case, we can replace x_i by x_i^* and x_i^{**} to get $k+1$ vertices whose containing edges are distinct, in contradiction to the maximality of k. Hence $3(m-2k) \leqslant 3k$, i.e., $m \leqslant 3k$. Since $x_1, ..., x_k$ together with a vertex in g_j for $1 \leqslant j \leqslant m-2k$ form a transversal,

$$\begin{aligned} \tau(H) &\leqslant k+m-2k \\ &\leqslant 2m/3 && (\text{since } m \leqslant 3k) \\ &= (2m+2n-t)/9 && (\text{since } 4m = 2n-t). \end{aligned}$$

Q.E.D.

References

1. C. Berge, "Graphs and Hypergraphs", North-Holland, Amsterdam, 1973.
2. Z. Tuza, Covering all cliques of a graph, preprint.