On the Nature of the Boxer–Thaler and Madwed Integrators and Their Applications in Digitizing a Continuous-Time System

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Abstract—**The nature of Boxer–Thaler and Madwed integrators is explored in this note. A consistent derivation of the Madwed integrator from the well-known derivation of the Boxer–Thaler integrator is first proposed. A new general computerized algorithm is also proposed for the kth-order Boxer–Thaler and Madwed integrators. These two discrete integrators are used in this note to replace the Tustin integrator for digitizing a continuous-time system. A more systematic and precise formulation of the Q-matrix is presented for the s-domain to z-domain transformation via Boxer–Thaler and Madwed integrators. Due to the more accurate nature of these two discrete integrators, better results can be obtained. A set of MATLAB programs is written to implement the proposed algorithms in this note.**

I. Introduction

The Boxer–Thaler and Madwed discrete integrators were proposed in 1951 [1] and 1956 [2], [3], respectively. These two integrators are more precise than the Tustin integrator in digitizing a continuous-system [4]. Also, the Boxer–Thaler integrator is claimed to be more accurate than the Madwed integrator [4]. But no proof has been seen regarding this fact. In this note, a consistent manner of deriving these two discrete integrators is presented. It is by this consistent manner of derivations that we can clearly prove the more accurate property of the Boxer–Thaler integrator over that of the Madwed integrator. Furthermore, we use these two discrete integrators to replace the Tustin integrator in digitizing a continuous-time system. In comparison to the Q-matrix implementation of the Boxer–Thaler integrator [5] and the Tustin integrator [6], our method is more precise and therefore, more suitable for computer programming. Excellent results are obtained which are better than those obtained by using the Tustin method. A set of MATLAB programs is written to implement the derived algorithms in this note.

II. Derivations of the Boxer–Thaler and Madwed Integrators

In this section, the Boxer–Thaler and Madwed discrete integrators are derived in a consistent manner. It is found in this section that these two integrators can be derived from the same starting point, which is contained in the well-known derivation presented by Boxer and Thaler [2], [3].

A. Derivation of the Boxer–Thaler Integrator

First, starting from the general substitution formula between the s-plane and z-plane

$$s = \ln(z)/t_s \Rightarrow s^{-1} = \frac{t_s}{\ln(z)}$$

then we expand the natural logarithm of z as a Laurent series. Let

$$z = \frac{1+u}{1-u}.$$

Then, from Taylor's theorem, it follows that

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - + \cdots \tag{1}$$

$$-\ln(1-u) = \ln\frac{1}{1-u} = u + \frac{u^2}{2} + \frac{u^3}{3} + \cdots. \tag{2}$$

By adding (1) and (2), we obtain

$$\ln\frac{1+u}{1-u} = 2\left[u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots\right].$$

Now, replacing $\frac{1+u}{1-u}$ by z, we get

$$\ln(z) = 2\left[u + \frac{1}{3}u^3 + \frac{1}{5}u^5 + \cdots\right].$$

Therefore,

$$\begin{aligned} s^{-1} &= \frac{t_s}{\ln(z)} = \frac{t_s}{2}\frac{1}{u + \frac{1}{3}u^3 + \frac{1}{5}u^5 + \cdots} \\ &\simeq \frac{t_s}{2}\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right]. \end{aligned} \tag{3}$$

The Boxer–Thaler integrator of power 1 retains the principal part of (3). Higher order expansions of (3) can be obtained by raising both sides to the desired power, and retaining only the principal part and constant term of the expanded series. For example

$$s^{-1} \simeq \frac{t_s}{2}\left[\frac{1}{u}\right] = \frac{t_s}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

$$s^{-2} \simeq \left[\frac{t_s}{2}\right]^2\left[\frac{1}{u^2} - \frac{2}{3}\right] = \left[\frac{t_s}{2}\right]^2\frac{\frac{1}{3} + \frac{10}{3}z^{-1} + \frac{1}{3}z^{-2}}{(1-z^{-1})^2}$$

$$s^{-3} \simeq \left[\frac{t_s}{2}\right]^3\left[\frac{1}{u^3} - \frac{1}{u}\right] = \left[\frac{t_s}{2}\right]^3\frac{4z^{-1} + 4z^{-2}}{(1-z^{-1})^3}.$$

The general expression is as follows:

$$s^{-k} = \left[\frac{t_s}{2}\right]^k\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right]^k \tag{4}$$

$$\simeq \left[\frac{t_s}{2}\right]^k\left[v_0 + \frac{v_1}{u} + \frac{v_2}{u^2} + \cdots + \frac{v_k}{u^k}\right] \tag{5}$$

$$\begin{aligned} &= \left[\frac{t_s}{2}\right]^k\left[v_0 + v_1\frac{1+z^{-1}}{1-z^{-1}} + \cdots + v_k\frac{(1+z^{-1})^k}{(1-z^{-1})^k}\right] = \left[\frac{t_s}{2}\right]^k \\ &\quad \frac{v_0(1-z^{-1})^k + v_1(1+z^{-1})(1-z^{-1})^{k-1} + \cdots + v_k(1+z^{-1})^k}{(1-z^{-1})^k} \\ &= \left[\frac{t_s}{2}\right]^k\frac{\sum_{i=0}^{k} v_i(1-z^{-1})^{k-i}(1+z^{-1})^i}{(1-z^{-1})^k} \\ &= \left[\frac{t_s}{2}\right]\frac{P_k(z^{-1})}{(1-z^{-1})^k}. \end{aligned} \tag{6}$$

The transition from (4) to (5) can be accomplished easily via computer programming. For instance, the DECONV and CONV commands in MATLAB [7] can be used to implement the polynomial inversion in (3) and polynomial multiplication in (4), respectively. By using the binomial theorem and discrete convolution, we get the numerator $P_k(z^{-1})$ of the

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kth power Boxer–Thaler integrator as follows:

$$
\begin{aligned}
P_k(z^{-1}) &= \sum_{i=0}^{k} v_i (1-z^{-1})^{k-i}(1+z^{-1})^i \\
&= \sum_{i=0}^{k} v_i \left[\sum_{q=0}^{k-i} (-1)^q \binom{k-i}{q} z^{-q}\right]\left[\sum_{r=0}^{i}\binom{i}{r} z^{-r}\right] \\
&= \sum_{i=0}^{k} v_i \left[\binom{k-i}{0} - \binom{k-i}{1} z^{-1} \right. \\
&\qquad \left. + \cdots + (-1)^{k-i}\binom{k-i}{k-i} z^{-(k-i)}\right] \\
&\qquad \cdot \left[\binom{i}{0} + \binom{i}{1} z^{-1} + \cdots + \binom{i}{i} z^{-i}\right] \\
&= \sum_{i=0}^{k} v_i [h_{i0} + h_{i1} z^{-1} + h_{i2} z^{-2} + \cdots + h_{ik} z^{-k}] \\
&= [v_0 \quad v_1 \quad v_2 \quad \cdots \quad v_k] \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0k} \\ h_{10} & h_{11} & \cdots & h_{1k} \\ \vdots & \vdots & & \vdots \\ h_{k0} & h_{k1} & \cdots & h_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-k} \end{bmatrix} \\
&= V^t \mathcal{H} Z
\end{aligned} \tag{7}
$$

where "t" denotes matrix transpose and

$$
\begin{aligned}
V^t &= [v_0 \quad v_1 \quad v_2 \quad \cdots \quad v_k] \\
Z^t &= [1 \quad z^{-1} \quad z^{-2} \quad \cdots \quad z^{-k}] \\
\mathcal{H} &= [h_{ij}]_{\{i,j=0,1,\cdots,k\}} \\
h_{ij} &= \sum_{q=0}^{j} (-1)^{j-q} \binom{k-i}{j-q}\binom{i}{q}.
\end{aligned} \tag{8}
$$

B. Derivation of the Madwed Integrator

The numerators of Madwed integrators can also be derived by a procedure similar to the derivation of Boxer–Thaler integrators. The only difference is that the higher order Madwed integrators are obtained by multiplying the lower order Madwed integrator and (3) together and retaining only the principal part and constant term of the expanded series. From this phenomenon, we can realize that the Boxer–Thaler integrator is indeed more accurate than the Madwed integrator. For example

$$
\begin{aligned}
s^{-1} &\simeq \frac{t_s}{2}\left[\frac{1}{u}\right] = \frac{t_s}{2}\frac{1+z^{-1}}{1-z^{-1}} \\
s^{-2} &\simeq \frac{t_s}{2}\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right] \\
&\quad * \frac{t_s}{2}\left[\frac{1}{u}\right] \\
&\simeq \left[\frac{t_s}{2}\right]^2 \left[\frac{1}{u^2} - \frac{1}{3}\right] = \frac{t_s^2}{3!}\frac{1+4z^{-1}+z^{-2}}{(1-z^{-1})^2} \\
s^{-3} &\simeq \frac{t_s}{2}\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 + \cdots\right] \\
&\quad * \left[\frac{t_s}{2}\right]^2 \left[\frac{1}{u^2} - \frac{1}{3}\right] \\
&\simeq \left[\frac{t_s}{2}\right]^3 \left[\frac{1}{u^3} - \frac{2}{3u}\right] = \frac{t_s^3}{4!}\frac{1+11z^{-1}11z^{-2}+z^{-3}}{(1-z^{-1})^3}.
\end{aligned} \tag{9}
$$

Following the previous manner, and assuming that the Madwed integrator for s^{-k-1} is

$$s^{-k-1} \simeq F_{k-1}(u) = F_{k-1}(z) \tag{10}$$

we have the Madwed integrator for s^{-k} as follows:

$$s^{-k} \simeq \left[\frac{t_s}{2}\right]\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right] * F_{k-1}(u) \tag{11}$$

$$\simeq \left[\frac{t_s}{2}\right]^k \left[v_0' + \frac{v_1'}{u} + \frac{v_2'}{u^2} + \cdots + \frac{v_k'}{u^k}\right]. \tag{12}$$

However, the above equation is identical to (5) except for the $[v_0' \quad v_1' \quad \cdots \quad v_k']$ vector. Therefore, we can get the following equation immediately:

$$s^{-k} \simeq \left[\frac{t_s}{2}\right]^k \frac{N_k(z^{-1})}{(1-z^{-1})^k} \tag{13}$$

and the numerator $N_k(z^{-1})$ of the Madwed integrator can also be obtained as in (8), i.e.,

$$
\begin{aligned}
N_k(z^{-1}) &= \sum_{i=0}^{k} v_i'(1-z^{-1})^{k-i}(1+z^{-1})^i \\
&= \sum_{i=0}^{k} v_i'[h_{i0} + h_{i1}z^{-1} + h_{i2}z^{-2} + \cdots + h_{ik}z^{-k}] \\
&= V'^t \mathcal{H} Z
\end{aligned} \tag{14}
$$

where $\mathcal{H}$ and Z are defined in (7) and

$$V'^t = [v_0' \quad v_1' \quad v_2' \quad \cdots \quad v_k']. \tag{15}$$

The transition from (11) to (12) can also be accomplished easily via CONV command in MATLAB. Note that (7) is identical to (14). Therefore, the only difference between (7) and (14) is the variation between V^t and V'^t.

Example 1: Derive the Boxer–Thaler and Madwed integrators for s^{-4}.

First, the $\mathcal{H}$ matrix defined in (7) and (8) must be computed as follows:

$$\mathcal{H} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

1) The Boxer–Thaler Integrator: From (4) we have

$$
\begin{aligned}
s^{-4} &= \left[\frac{t_s}{2}\right]^4 \left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right]^4 \\
&\simeq \left[\frac{t_s}{2}\right]^4 \left[\frac{1}{u^4} - \frac{4}{3}\frac{1}{u^2} + \frac{14}{45}\right] \\
&= \left[\frac{t_s}{2}\right]^4 \frac{P_4(z^{-1})}{(1-z^{-1})^4}.
\end{aligned}
$$

We can get $P_4(z^{-1})$ from (7) as follows:

$$
\begin{aligned}
P_4(z^{-1}) &= \left[\frac{14}{45} \quad 0 \quad -\frac{4}{3} \quad 0 \quad 1\right] \\
&\quad * \mathcal{H} * [1 \quad z^{-1} \quad z^{-2} \quad z^{-3} \quad z^{-4}]^t \\
&= \left[-\frac{1}{45} \quad \frac{124}{45} \quad \frac{474}{45} \quad \frac{124}{45} \quad -\frac{1}{45}\right] \\
&\quad * [1 \quad z^{-1} \quad z^{-2} \quad z^{-3} \quad z^{-4}]^t \\
&= -\frac{1}{45} + \frac{124}{45}z^{-1} + \frac{474}{45}z^{-2} + \frac{124}{45}z^{-3} - \frac{1}{45}z^{-4}.
\end{aligned}
$$

Finally, the Boxer–Thaler integrator for s^{-4} is

$$s^{-4} \simeq \left[\frac{t_s}{2}\right]^4 \cdot \left[\frac{-1/45 + 124/45z^{-1} + 474/45z^{-2} + 124/45z^{-3} - 1/45z^{-4}}{(1-z^{-1})^4}\right] = \frac{t_s^4}{6}\frac{z^{-1}+4z^{-2}+z^{-3}}{(1-z^{-1})^4} - \frac{t_s^4}{720}.$$

2) The Madwed Integrator: From (9) and (11), we have

$$s^{-4} \simeq \left[\frac{t_s}{2}\right]\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right] * F_3(u)$$

$$= \left[\frac{t_s}{2}\right]^4\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 - \cdots\right] * \left[\frac{1}{u^3} - \frac{2}{3u}\right]$$

$$= \left[\frac{t_s}{2}\right]^4\left[\frac{1}{u^4} - \frac{1}{u^2} + \frac{2}{15}\right] = \left[\frac{t_s}{2}\right]^4 \frac{N_4(z^{-1})}{(1-z^{-1})^4}.$$

We can get $N_4(z^{-1})$ from (14) as follows:

$$N_4(z^{-1}) = [2/15 \quad 0 \quad -1 \quad 0 \quad 1] * \mathscr{H} * [1 \quad z^{-1} \quad z^{-2} \quad z^{-3} \quad z^{-4}]^t$$
$$= 2/15 + 52/15z^{-1} + 132/15z^{-2} + 52/15z^{-3} + 2/15z^{-4}.$$

Finally, the Madwed integrator for s^{-4} is

$$s^{-4} \simeq \left[\frac{t_s}{2}\right]^4\left[\frac{2/15 + 52/15z^{-1} + 132/15z^{-2} + 52/15z^{-3} + 2/15z^{-4}}{(1-z^{-1})^4}\right] = \frac{t_s^4}{120}\frac{1 + 26z^{-1} + 66z^{-2} + 26z^{-3} + z^{-4}}{(1-z^{-1})^4}.$$

The previous two discrete integrators are in accordance with those in [1], [4].

III. Digitizing A Continuous-Time System Via Boxer–Thaler and Madwed Integrators

The Boxer–Thaler and Madwed integrators derived previously are basically similar to the Tustin integrator. They are all derived from the Taylor's expansion of $z = e^{sT}$. However, the more accurate properties of the Boxer–Thaler and Madwed integrators [4] can enable us to yield a more accurate result in digitizing a continuous-time system. The Q-matrix implementation for the Boxer–Thaler integrator has already been shown in [6]. However, a more systematic and precise formulation is proposed in this section. Actually, (7) and (14) play a major role in the following formulation. Let us consider a continuous-time transfer function as follows:

$$G(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_{m-1}s + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n} = \frac{a_0 s^{m-n} + a_1 s^{m-n-1} + \cdots + a_{m-1}s^{-n-1} + a_m s^{-n}}{b_0 + b_1 s^{-1} + b_2 s^{-2} + \cdots + b_{n-1}s^{-n+1} + b_n s^{-n}} \quad n \geq m. \tag{16}$$

The first objective is to obtain an equivalent discrete-time form of the continuous-time transfer function $G(s)$. Let $T = t_s/2$ and

$$R_k(z^{-1}) = P_k(z^{-1}) \quad \text{of the Boxer–Thaler integrator,}$$
$$= N_k(z^{-1}) \quad \text{of the Madwed integrator.}$$

Then we obtain the corresponding discrete-time transfer function as follows:

$$\begin{aligned}
G(z^{-1}) &= \frac{a_0 T^{n-m}\dfrac{R_{n-m}(z^{-1})}{(1-z^{-1})^{n-m}} + \cdots + a_m T^n \dfrac{R_n(z^{-1})}{(1-z^{-1})^n}}{b_0 + b_1 T\dfrac{R_1(z^{-1})}{(1-z^{-1})} + b_2 T^2 \dfrac{R_2(z^{-1})}{(1-z^{-1})^2} + \cdots + b_n T^n \dfrac{R_n(z^{-1})}{(1-z^{-1})^n}} \\
&= \frac{a_0 T^{n-m} R_{n-m}(z^{-1})(1-z^{-1})^m + a_1 T^{n-m+1} R_{n-m+1}(z^{-1})(1-z^{-1})^{m-1}}{b_0(1-z^{-1})^n + b_1 T R_1(z^{-1})(1-z^{-1})^{n-1} + b_2 T^2 R_2(z^{-1})(1-z^{-1})^{n-2}} \\
&\quad \frac{+ \cdots + a_{m-1}T^{n-1}R_{n-1}(z^{-1})(1-z^{-1}) + a_m T^n R_n(z^{-1})}{+ \cdots + b_{n-1}T^{n-1}R_{n-1}(z^{-1})(1-z^{-1}) + b_n T^n R_n(z^{-1})} \\
&= \frac{\sum_{j=0}^{m} a_j T^{n-m+j} R_{n-m+j}(z^{-1})(1-z^{-1})^{m-j}}{\sum_{i=0}^{n} b_i T^i R_i(z^{-1})(1-z^{-1})^{n-i}} \qquad (\text{since } R_0(z^{-1}) = 1) \\
&= \frac{\sum_{j=m-n}^{m} a_j T^{n-m+j} R_{n-m+j}(z^{-1})(1-z^{-1})^{m-j}}{\sum_{i=0}^{n} b_i T^i R_i(z^{-1})(1-z^{-1})^{n-i}} \qquad (\text{since } a_j = 0 \text{ for } j < 0) \\
&= \frac{\sum_{i=0}^{n} a_{m-n+i} T^i R_i(z^{-1})(1-z^{-1})^{n-i}}{\sum_{i=0}^{n} b_i T^i R_i(z^{-1})(1-z^{-1})^{n-i}}.
\end{aligned} \tag{17}$$

Therefore, the numerator and denominator are in the same form in (17). It is shown from (7) and (14) that $R_k(z^{-1})$ can be represented as follows:

$$R_k(z^{-1}) = e_k(0) + e_k(1)z^{-1} + \cdots + e_k(k)z^{-k} \tag{18}$$

where $e_k(j) = \sum_{i=0}^{k} v_i h_{ij}$ (or $\sum_{i=0}^{k} v_i' h_{ij}$). Therefore, we can get the numerator and denominator of the z-transfer function in matrix form as follows:

$$\sum_{i=0}^{n} b_i T^i R_i(z^{-1})(1-z^{-1})^{n-i}$$

$$= \sum_{i=0}^{n} b_i T^i R_i(z^{-1}) \sum_{j=0}^{n-j} (-1)^j \binom{n-i}{j} z^{-j}$$

$$= \sum_{i=0}^{n} b_i T^i [e_i(0) + e_i(1)z^{-1} + \cdots + e_i(i)z^{-i}] \cdot \left[\binom{n-i}{0} - \binom{n-i}{1} z^{-1} + \cdots + (-1)^{n-i}\binom{n-i}{n-i} z^{-(n-i)}\right]$$

$$= \sum_{i=0}^{n} b_i [q_{i0} + q_{i1}z^{-1} + q_{i2}z^{-2} + \cdots + q_{i,n-1}z^{-(n-1)} + q_{i,n}z^{-n}]$$

$$= [b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_n] \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0n} \\ q_{10} & q_{11} & \cdots & q_{1n} \\ \vdots & \vdots & & \vdots \\ q_{n0} & q_{n1} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix}$$

$$= [b_0 \quad \cdots \quad b_n] \boldsymbol{Q} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix}$$

$$= d_0 + d_1 z^{-1} + \cdots + d_{n-1}z^{-(n-1)} + d_n z^{-n}$$

$$= \sum_{i=0}^{n} d_i z^{-i} \tag{19}$$

where

$$d_i = \sum_{j=0}^{n} b_j q_{ji} \qquad \boldsymbol{Q} = [q_{ij}]_{\{i,j=0,1,\cdots,n\}} \tag{20}$$

$$q_{ij} = T^i \sum_{r=0}^{j} (-1)^{j-r} e_i(r) \binom{n-i}{j-r} \tag{21}$$

and by the same procedure, the numerator is

$$\sum_{i=0}^{n} a_{m-n+1} T^i R_i(z^{-1})(1-z^{-1})^{n-i}$$

$$= \sum_{i=0}^{n} a_{m-n+i} T^i R_i(z^{-1}) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} z^{-j}$$

$$= \sum_{i=0}^{n} a_{m-n+i} T^i [e_i(0) + e_i(1)z^{-1} + \cdots + e_i(i)z^{-i}] \cdot \left[\binom{n-i}{0} - \binom{n-i}{1} z^{-1} + \cdots + (-1)^{n-i}\binom{n-i}{n-i} z^{-(n-i)}\right]$$

$$= \sum_{i=0}^{n} a_{m-n+i} [q_{i0} + q_{i1}z^{-1} + \cdots + q_{i,n-1}z^{-(n-1)} + q_{i,n}z^{-n}]$$

$$= [\underbrace{0 \ \cdots \ 0}_{n-m \text{ terms}} \quad a_0 \quad a_1 \cdots a_m] \boldsymbol{Q} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix}$$

$$= c_0 + c_1 z^{-1} + \cdots + c_{n-1}z^{-(n-1)} + c_n z^{-n}$$

$$= \sum_{i=0}^{n} c_i z^{-i} \tag{22}$$

where

$$c_i = \sum_{j=0}^{n} a_j q_{ji}. \tag{23}$$

Also, the $\boldsymbol{Q}$ and q_{ij} are defined in (20) and (21), respectively.

The aforementioned derivations are basically the s-domain to z-domain transformation, i.e.,

$$G(s) = \frac{a(s)}{b(s)} \Rightarrow G(z) = \frac{c(z)}{d(z)} \tag{24}$$

and

$$c^t = a^t \boldsymbol{Q} \quad d^t = b^t \boldsymbol{Q} \tag{25}$$

where Q is defined in (20) and

$$a^t = [\underbrace{0 \ \cdots \ 0}_{n-m \text{ terms}} \quad a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_m] \tag{26}$$

$$b^t = [b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_n]$$

$$c^t = [c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_n]$$

$$d^t = [d_0 \quad d_1 \quad d_2 \quad \cdots \quad d_n].$$

Note that the Q-matrix defined in (20) is a function of $T(= t_s/2)$, n (order of s-domain transfer function), and integrator type (Boxer–Thaler or Madwed).

Example 2: Consider the following fifth-order system in [5]:

$$G(s) = \frac{s^2 + 2s + 0.75}{s^5 + 27.5s^4 + 261.6s^3 + 1039s^2 + 1668s + 864}.$$

The sampling time is also set to $t_s = 0.01$ as in [5]. We must first compute the Q-matrices from (21) for the Boxer–Thaler and Madwed cases. Then, the numerator and denominator coefficients can be obtained from (25) and (26). The following MATLAB program performs all of the three discrete approximations (i.e., Tustin, Madwed, and Boxer–Thaler) of $G(s)$:

```
clear
format long e
num=[1 2 0.75];                          % Numerator of G(s)
den=[1 27.5 261.5 1039 1668 864];        % Denominator of G(s)
ts=0.01;                                 % Sampling time=0.01 sec
n=500;
t=0:ts:ts*n;                             % Time=0.0 sec→5.0 sec
y=step(num,den,t);                       % Continuous-Time Step
                                           Response
y0=y(2:n+1);
[Tnum,Tden]=Tustin(num,den,ts);          % Tustin Approximation
y1=dstep(Tnum,Tden,n);                   % Discrete-Time Step
                                           Response
```

```
[Mnum,Mden] = Madwed(num,den,ts);  % Madwed Approximation
y2 = dstep(Mnum,Mden,n);           % Discrete-Time Step
                                     Response
[Bnum,Bden] = Boxer(num,den,ts);   % Boxer-Thaler
                                     Approximation
y3 = dstep(Bnum,Bden,n);           % Discrete-Time Step
                                     Response
plot([y0 y1 y2 y3])
Tus = lsqr(y0,y1)                  % Sum-squared errors
                                     between
Mad = lsqr(y0,y2)                  % three approximations and
                                     continuous
Box = lsqr(y0,y3)                  % -time system.
```

The results are as follows:

$$G(z) = \frac{c_0 z^5 + c_1 z^4 + c_2 z^3 + c_3 z^2 + c_4 z + c_5}{d_0 z^5 + d_1 z^4 + d_2 z^3 + d_3 z^2 + d_4 z + d_5}.$$

For the Boxer–Thaler case

$$\begin{aligned}[c_0 \; c_1 \; c_2 \; c_3 \; c_4 \; c_5] &= [-2.437333168392555e-011 \quad 4.417693787709666e-007 \\ &\quad -4.301591422233286e-007 \quad -4.472204744020766e-007 \\ &\quad 4.356760458499852e-007 \quad 2.437333168392555e-011]\end{aligned}$$

$$\begin{aligned}[d_0 \; d_1 \; d_2 \; d_3 \; d_4 \; d_5] &= [1.000000000000000e+000 \quad -4.735300689390917e+000 \\ &\quad 8.965532424521376e+000 \quad -8.483852047784712e+000 \\ &\quad 4.012324404794199e+000 \quad -7.587040163291362e-001].\end{aligned}$$

For the Madwed case:

$$\begin{aligned}[c_0 \; c_1 \; c_2 \; c_3 \; c_4 \; c_5] &= [3.663488297720488e-008 \quad 3.320536109854195e-007 \\ &\quad -3.590226011977542e-007 \quad -3.706990288487865e-007 \\ &\quad 3.247558437035243e-007 \quad 3.634297228592907e-008]\end{aligned}$$

$$\begin{aligned}[d_0 \; d_1 \; d_2 \; d_3 \; d_4 \; d_5] &= [1.000000000000000e+000 \quad -4.735738861034265e+000 \\ &\quad 8.967238982065885e+000 \quad -8.486344516008330e+000 \\ &\quad 4.013942326325186e+000 \quad -7.590978556852241e-001].\end{aligned}$$

For the Tustin case

$$\begin{aligned}[c_0 \; c_1 \; c_2 \; c_3 \; c_4 \; c_5] &= [1.103441954183032e-007 \quad 1.125373821517400e-007 \\ &\quad -2.162938236459465e-007 \quad -2.206638096652478e-007 \\ &\quad 1.059824031227881e-007 \quad 1.081592024086525e-007]\end{aligned}$$

$$\begin{aligned}[d_0 \; d_1 \; d_2 \; d_3 \; d_4 \; d_5] &= [1.000000000000000e+000 \quad -4.736107367001773e+000 \\ &\quad 8.968668581331992e+000 \quad -8.488424113560411e+000 \\ &\quad 4.015286718188694e+000 \quad -7.594237434451435e-001].\end{aligned}$$

And the sum-squared errors between the step responses of $G(s)$ and three discrete approximations are

$$\text{Tus} = 1.118476206688459e-008$$

$$\text{Mad} = 1.101086722161684e-008$$

$$\text{Box} = 1.092731169729238e-008.$$

It is obvious that the Boxer–Thaler discrete approximation yield the smallest sum-squared errors. All the above data are computed using PC-MATLAB and displayed here using long exponential format. The Tustin.m, Madwed.m, and Boxer.m in the previous MATLAB program can be obtained from the authors upon request.

V. Conclusions

In this note, we examine the nature and applications of the Boxer–Thaler and Madwed discrete integration operators. A general computerized algorithm is devised to derive the Boxer–Thaler and Madwed integrators consistently. Further, the digitizations of a continuous-time system via Boxer–Thaler and Madwed integrators are proposed in a more systematic and precise way. Instead of using the Tustin integrator, the Boxer–Thaler and Madwed integrators can be used as the substitutes to reduce the accumulated truncation errors [4]. This is especially true for a system with a longer sampling period or for higher order systems.

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Directional Interpolation Via All-Pass Transfer Function Matrices and Its Application in Hankel-Norm Approximations

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Abstract—**A new approach to the problem of multivariable interpolation via all-pass transfer function matrices that are not necessarily stable is presented. It applies both state-space and classical function theoretic arguments and it obtains a very simple expression for the all-pass matrix that satisfies the interpolation requirement. Unlike the solution that is obtained by the generalized Nevanlinna–Pick algorithm, this expression is derived in closed form explicitly in terms of the interpolation parameters. It allows a detailed investigation of the structure of the all-pass solution and it is readily used in Hankel-norm approximations of linear multivariable systems.**

I. Introduction

The problem of interpolation via inner matrices gained much attention in the last few years. It has been used in the various fields of system and control such as H^∞-optimization [1], [2], robust stabilization [3], [4], and circuit theory [5], [6]. This problem was to derive a stable transfer function matrix U of dimensions that are appropriate to the system in question, that satisfies the following all-pass property:

$$U^H U = I \tag{1}$$

where $(\cdot)^H$ denotes the Hermitian transpose.

The matrix U has to also satisfy interpolation requirements that are either given in terms of matrix values, as a direct extension of the scalar

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