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Wide diameters of de Bruijn graphs

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Abstract The wide diameter of a graph is an important parameter to measure fault-tolerance of interconnection network. This paper proves that for any two vertices in de Bruijn undirected graph $UB(d,n)$, there are $2d-2$ internally disjoint paths of length at most $2n+1$. Therefore, the $(2d-2)$-wide diameter of $UB(d,n)$ is not greater than $2n+1$.

Keywords de Bruijn · Internally disjoint path · Wide diameter

1 Introduction and preliminaries

Reliability and efficiency are important criteria in the design of interconnection networks. In graph theory and the study of fault-tolerance and transmission delay of networks, connectivity and diameter are two very important parameters and have been studied by many researchers. Wide diameter of a graph, which combines connectivity with diameter, is a parameter that measures simultaneously the fault-tolerance and efficiency of parallel processing computer networks. Diameter with width w of a graph G is defined as the minimum integer l for which between any two distinct vertices in G there exist at least w internally vertex disjoint paths of length at most l. Throughout this paper, "disjoint paths" always means "internally vertex disjoint paths".

The connectivity $\kappa(G)$ of a network G is the minimum number of vertices whose removal results in a disconnected or trivial network. According to Menger's theorem, there are κ disjoint paths between any two vertices in a network of connectivity κ.

Dedicated to Professor Frank K. Hwang on the occasion of his sixty fifth birthday.

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The w-*wide diameter* $d_w(G)$ of a network G is the minimum l such that for any two vertices $\mathbf{x}$ and $\mathbf{y}$ there exists w disjoint paths of length at most l from x to y. The notation of w-wide diameter was introduced by Hsu (1994) to unify the concept of diameter and connectivity (Liaw and Chang 1999).

The well-known de Bruijn digraph is a class of important digraphs, which have been widely used in the design and analysis of interconnection networks. We recall the definition of de Bruijn digraph $B(d,n)$ for $n \geq 1$ and $d \geq 2$. The de Bruijn diagraph $B(d,n)$ has vertex-set $V = \{x_1x_2\ldots x_n : x_i \in Z_d = \{0,1,\ldots,d-1\}, i = 1,2,\ldots,n\}$ and directed edge-set E, where for $\mathbf{x} = x_1x_2\ldots x_n$, $\mathbf{y} = y_1y_2\ldots y_n \in V$, $\mathbf{xy} \in E$ if and only if $y_i = x_{i+1}$ for $i = 1,2,\ldots,n-1$. It has been shown that the digraph $B(d,n)$ is d-regular and has connectivity $\kappa = d-1$ (Sridhar 1988). In (Sridhar 1988) Sridhar also showed that for any two vertices $\mathbf{x}$ and $\mathbf{y}$ in $B(d,n)$, there are at least $d-1$ disjoint paths of length at most $n+1 = \lfloor \log N \rfloor + 1$, where $N = |V|$.

The de Bruijn undirected graph, denoted by $UB(d,n)$, is obtained from $B(d,n)$ by omitting the orientation of all directed edges and omitting multiple edges and loops. In $UB(d,n)$, there are d vertices with degree $2d-2$, $d(d-1)$ vertices with degree $2d-1$, and the others with degree $2d$. Moreover, $UB(d,n)$ has diameter n, vertex-connectivity $\kappa = 2d-2$, and edge-connectivity $\lambda = 2d-2$ (Esfahnian and Hakimi 1985). This implies that, for any two vertices $\mathbf{x}$ and $\mathbf{y}$ in $UB(d,n)$, there are at least $2d-2$ disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. But, the length of these $2d-2$ paths may be very large compared to n. It is interesting and important to know the $(2d-2)$-wide diameter of $UB(d,n)$. In this paper, we will show that $d_{2d-2}(UB(d,n)) \leq 2n+1$, i.e. for any two vertices in $UB(d,n)$ there are $2d-2$ disjoint paths of length at most $2n+1$.

2 The main result

We use $P[x_1x_2\ldots x_m]$, $m > n$, to denote the following path by deleting the repeated vertices.

$$x_1\ldots x_n \to x_2\ldots x_{n+1} \to \cdots \to x_{m-n}\ldots x_{m-1} \to x_{m-n+1}\ldots x_m.$$

For example: for $n=5$ and $d=3$, $P[00100200] = 00100 \to 01002 \to 10020 \to 00200$ and $P[00000101010] = 00000 \to 00001 \to 00010 \to 00101 \to 01010$.

Suppose that $\mathbf{x} = x_1x_2\ldots x_n$, $\mathbf{y} = y_1y_2\ldots y_n$. In what follows, for each $a \in Z_d$, we use $P[\mathbf{x}a\mathbf{y}]$ to denote the path

$$\mathbf{x} \to x_2\ldots x_na \to x_3\ldots x_nay_1 \to x_4\ldots x_nay_1y_2 \to \cdots \to ay_1\ldots y_{n-1} \to \mathbf{y}.$$

$P(\mathbf{x}a\mathbf{y}]$ is the path of vertices including $\mathbf{y}$ and excluding $\mathbf{x}$ (and similarly for $P[\mathbf{x}a\mathbf{y})$ and $P(\mathbf{x}a\mathbf{y})$) (Esfahnian and Hakimi 1985). So we can also use $\mathbf{x} \to P(\mathbf{x}a\mathbf{y}]$ or $P[\mathbf{x}a\mathbf{y}) \to \mathbf{y}$, or $\mathbf{x} \to P(\mathbf{x}a\mathbf{y}) \to \mathbf{y}$ to mean $P[\mathbf{x}a\mathbf{y}]$. If necessary,

$$x_1x_2\ldots x_n \to P(x_1x_2\ldots x_nay_1y_2\ldots y_n) \to y_1y_2\ldots y_n \quad \text{equals } P[\mathbf{x}a\mathbf{y}].$$

We add superscripts for long string for reading easily. For example:

$$P[x_1x_2\ldots x_nay_1y_2\ldots y_n] = P[x_1^1x_2^2\ldots x_n^na^{n+1}y_1^{n+2}y_2^{n+3}\ldots y_n^{2n+1}].$$

The boldface of a character is one vertex of $UB(d,n)$, for example $\mathbf{x}$, $\mathbf{y}$.

The following lemmas and corollary are essential to the proof of the main result Theorem 2.4.

Lemma 2.1 *For any vertex* $\mathbf{y} = y_1 y_2 \dots y_n \in UB(d,n)$, *there are exactly* $2d-2$ *disjoint paths from* $\mathbf{y}$ *to* $\mathbf{x} = 00\dots 0$ *of length at most* $2n-2$.

Proof Define $A_1 = \{00\dots 0a : a \in Z_d{}^* = Z_d \setminus \{0\}\}$, $B_1 = \{b0\dots 0 : b \in Z_d{}^*\}$,

$$A_i = \{0^1 \dots 0^{n-i} a y_1 \dots y_{i-1} : a \in Z_d{}^*\}, \quad i = 2, 3, \dots, n, \quad \text{and}$$
$$B_j = \{y_{n-j+2} \dots y_n b 0^{j+1} \dots 0^n : b \in Z_d{}^*\}, \quad j = 2, 3, \dots, n.$$

For any $i, j = 1, 2, \dots, n$, and $i \neq j$, $A_i \cap A_j = \phi$ and $B_i \cap B_j = \phi$, since the number of 0's before a and after b are different respectively. If $A_i \cap B_j = \phi$ for any $i, j = 1, 2, \dots, n$, then we have found $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x}$ with length $n+1$, namely $WD = \{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}] : a, b \in Z_d{}^*\}$.

Now, suppose $A_i \cap B_j = \{0\dots 0a\dots b0\dots 0\} \neq \phi$, for some $a, b \in Z_d{}^*$, where $t = i + j - n > 0$ is the length of overlap between A_i and B_j by deleting the consecutive 0's from head and from tail respectively. Then the length of $a\dots b$ in $A_i \cap B_j$ is $t (\leq n)$. According to i and j, we have the following cases.

Case 1: $n - j + 2 > i - 1 \Rightarrow t < 3$.

Case 1.1: $t = 1 \Rightarrow \mathbf{y} = 0^1 \dots 0^{i-1} y_i 0^{n-j+2} \dots 0^n$.

Note that $y_i > 0$ and $y_j = 0$, for $j \neq i$.

- $i = 1$:
  Let $y_1 = w > 0$. Then,

$$\begin{aligned} WD = &\{\mathbf{x} \to 0\dots 0a \to \mathbf{y} : a \in Z_d{}^*\} \cup \{\mathbf{x} \to \mathbf{y}\} \\ &\cup \{\mathbf{y} \to bw0\dots 0 \to P[w^1 0\dots 0 b^n 0^{n+1} \dots 0^{2n}) \to \mathbf{x} : b \in Z_d{}^* \backslash \{w\}\}. \end{aligned}$$

- $i = n$:
  Let $y_n = w > 0$. Then,

$$\begin{aligned} WD = &\{\mathbf{y} \to a0\dots 0 \to \mathbf{x} : a \in Z_d{}^*\} \cup \{\mathbf{x} \to \mathbf{y}\} \\ &\cup \{\mathbf{x} \to P(0^1 \dots 0^n b^{n+1} 0^{n+2} \dots 0^{2n-1} w^{2n} b^{2n+1}] \to \mathbf{y} : b \in Z_d{}^* \backslash \{w\}\}. \end{aligned}$$

- $1 < i < n$:
  Let $y_i = w > 0$. Then,

$$\begin{aligned} WD = \{&\mathbf{x} \to P(0^1 \dots 0^n a^{n+1} \dots a^{n+i}) \to 0^1 \dots 0^{n-i} a^{n-i+1} \dots a^n \\ &\text{connect } \mathbf{y} \to P(0^1 \dots w^i \dots 0^n a^{n+1} \dots a^{n+i}) \\ &\to 0^1 \dots 0^{n-i} a^{n-i+1} \dots a^n : a \in Z_d{}^*\} \\ &\cup \{b^1 \dots b^i 0^{i+1} \dots 0^n \to P(b^1 \dots b^i 0^{i+1} \dots 0^{n+i}) \to \mathbf{x} \\ &\quad \text{connect } b^1 \dots b^i 0^{i+1} \dots 0^n \to P(b^1 \dots b^i 0^{i+1} \dots w^{2i} \dots 0^{n+i}) \\ &\quad \to \mathbf{y} : b \in Z_d{}^*\}. \end{aligned}$$

  This case has the longest path of length $2n-2$.

Case 1.2: $t=2 \Rightarrow \mathbf{y}=b0^2\dots0^{i-1}0^{n-j+2}\dots0^{n-1}a$.
Let $\mathbf{y}=w0\dots0u$, $w,u>0$. Then,

$$\begin{aligned}WD &= \{P[\mathbf{x}a\mathbf{y}]: a\in {Z_d}^*\}\\ &\quad \cup\{P[\mathbf{y}b\mathbf{x}]: b\in {Z_d}^*\setminus\{w\}\}\cup\{\mathbf{x}\to w0\dots0\to 0w0\dots0\to w0\dots0u=\mathbf{y}\}.\end{aligned}$$

Case 2: $n-j+2\le i-1\Rightarrow t\ge 3$.
Case 2.1: $n-t+2\le t-1\Rightarrow n+3\le 2t\Rightarrow \frac{n+3}{2}\le t\le n$.

$$\mathbf{y}=y_1\dots y_{n-j+1}0^{n-j+2}\dots0^{n-t+1}a^{n-t+2}y_{n-t+3}\dots y_{t-2}b^{t-1}0^t\dots0^{i-1}y_i\dots y_n,$$

with $y_1\dots y_{t-2}=y_{n-t+3}\dots y_n$. Let $y_{t-1}=b=w>0$. Then,

$$WD=\{P[\mathbf{x}a\mathbf{y}]: a\in {Z_d}^*\}\cup\{P[\mathbf{y}b\mathbf{x}]: b\in {Z_d}^*\setminus\{w\}\}\cup\{P[\mathbf{y}ww\mathbf{x}]\}.$$

Example 1: $n+3=2t$.

$$y=0^1\dots0^{n-j}y_{n-j+1}0^{n-j+2}\dots0^{n-t+1}a^{n-t+2=t-1}0^t\dots0^{i-1}y_{n-j+1}0^{i+1}\dots0^n.$$

Example 2: $n+4=2t$.

$$\mathbf{y}=b^10^2\dots0^{n-t+1=t-3}a^{t-2}b^{t-1}0^t\dots0^{n-1}a^n.$$

Case 2.2: $n-t+2>t-1\Rightarrow n+3>2t$.
Define $\text{Min}=\min\{t,n-j+2\}$ and $\text{Max}=\max\{i-1,n-t+1\}$. Then

$$\mathbf{y}=y_1\dots y_{\text{Min}-1}0^{\text{Min}}\dots0^{\text{Max}}y_{\text{Max}+1}\dots y_n$$

with $y_1\dots y_{t-2}=y_{n-t+3}\dots y_n$.
Case 2.2.1: $\text{Min}=t$ and $\text{Max}=n-t+1$.

$$t\le n-j+2\quad\text{and}\quad i-1\le n-t+1\Rightarrow 3\le t\le\frac{n+4}{3},$$
$$\mathbf{y}=y_1\dots y_{t-2}b^{t-1}0^t\dots0^{n-t+1}a^{n-t+2}y_{n-t+3}\dots y_n.$$

Let $y_{t-1}=b=w>0$. Then,

$$WD=\{P[\mathbf{x}a\mathbf{y}]: a\in {Z_d}^*\}\cup\{P[\mathbf{y}b\mathbf{x}]: b\in {Z_d}^*\setminus\{w\}\}\cup\{P[\mathbf{y}ww\mathbf{x}]\}.$$

Case 2.2.2: $\text{Min}=t$ and $\text{Max}=i-1$.

$$\mathbf{y}=0^1\dots0^{i+t-n-1}y_{i+t-n}\dots y_{t-2}b^{t-1}0^t\dots0^{i-1}y_i\dots y_n\Rightarrow a=y_{n-t+2}=0,$$

this is impossible.
Case 2.2.3: $\text{Min}=n-j+2$ and $\text{Max}=n-t+1$.

$$\mathbf{y}=y_1\dots y_{n-j+1}0^{n-j+2}\dots0^{n-t+1}a^{n-t+2}y_{n-t+3}\dots y_{2n-t-j+3}0^{2n-t-j+4}\dots0^n,$$

and $b=y_{t-1}=0$, this is impossible.

Case 2.2.4: Min $= n - j + 2$ and Max $= i - 1$.

$$\mathbf{y} = 0^1 \dots 0^{i+t-n-1} y_{i+t-n} \dots y_{n-j+1} 0^{n-j+2} \dots 0^{i-1} y_i \dots y_{2n-t-j+3} \times 0^{2n-t-j+4} \dots 0^n,$$

$a = y_{n-t+2} = 0$, and $b = y_{t-1} = 0$. This is impossible.

Since cases 2.2.2, 2.2.3, and 2.2.4 are impossible, we get exactly $2d - 2$ disjoint paths. This completes the Lemma. □

Lemma 2.2 *For any vertex* $\mathbf{y} = y_1 y_2 \dots y_n \in UB(d, n)$, *there are at least* $2d - 2$ *disjoint paths from* $\mathbf{y}$ *to* $\mathbf{x} = 100\dots0$ *of length at most* $2n$.

Proof First, we consider the case when $\mathbf{y} = u0\dots0$, $u \neq 1$. If $u = 0$, then by Lemma 2.1, we are done. Now, suppose $u > 1$. Then, the $2d - 2$ disjoint paths are

$$WD = \{\mathbf{x} \to 0\dots0a \to \mathbf{y} : a \in Z_d{}^*\} \cup \{\mathbf{y} \to P[b^1 u^2 0^3 \dots 0^n b^{n+1} 1^{n+2} 0^{n+3} \dots 0^{2n}] \to \mathbf{x} : b \in Z_d{}^*\}.$$

In what follows, we suppose $\mathbf{y} \neq u0\dots0$.

Define $A_1 = \{00\dots0a : a \in Z_d{}^*\}$, $B_1 = \{b10\dots0 : b \in Z_d{}^*\}$,

$$A_i = \{0^1 \dots 0^{n-i} a y_1 \dots y_{i-1} : a \in Z_d{}^*\}, \quad i = 2, 3, \dots, n, \quad \text{and}$$
$$B_j = \{y_{n-j+2} \dots y_n b 1^{j+1} 0^{j+2} \dots 0^n : b \in Z_d{}^*\}, \quad j = 2, 3, \dots, n.$$

For $i, j = 1, 2, \dots, n$, $i \neq j$, $A_i \cap A_j = \phi$ and $B_i \cap B_j = \phi$, since the number of 0's before a and after b are different respectively. If $A_i \cap B_j = \phi$ for any $i, j = 1, 2, \dots, n$, then we have found $2d - 2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x}$ with length $n + 1$, namely $WD = \{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}] : a, b \in Z_d{}^*\}$.

Now, suppose $A_i \cap B_j = \{0\dots0a\dots b10\dots0\} \neq \phi$, for some $a, b \in Z_d{}^*$, where $t = i + j - n > 0$ is the length of overlap between A_i and B_j by deleting the consecutive 0's from head and from tail respectively and the special 1 following after b. In other words, $a\dots b1$ has length t. According to i and j, we have the following cases.

Case 1: $n - j + 2 > i - 1 \Rightarrow t < 3$.

Case 1.1: $t = 1 \Rightarrow \mathbf{y} = 10\dots0y_i0\dots0$.

Note that $y_i > 0$ and $i > 1$.

- $i = n$

$$WD = \{\mathbf{x} \to a10\dots0 \to \mathbf{y} : a \in Z_d{}^*\} \cup \{\mathbf{x} \to P(\mathbf{x}b^{n+1} \dots b^{2n}) \to b\dots b \text{ connect } \mathbf{y} \to P(\mathbf{y}b^{n+1} \dots b^{2n}) \to b\dots b : b \in Z_d{}^*\}.$$

This case has the longest length $2n$.

- $1 < i < n$

$$WD = \{\mathbf{x} \to P(\mathbf{x}a^{n+1}\dots a^{n+i+1}) \to 0^1 \dots 0^{n-i-1}a^{n-i}\dots a^n$$
$$\text{connect } \mathbf{y} \to P(\mathbf{y}a^{n+1}\dots a^{n+i+1}) \to 0^1 \dots 0^{n-i-1}a^{n-i}\dots a^n, : a \in Z_d{}^*\}$$
$$\cup \{b^1 \dots b^n \to P(b^1 \dots b^n\mathbf{x}) \to \mathbf{x} \text{ connect } b^1 \dots b^n$$
$$\to P(b^1 \dots b^n\mathbf{y}) \to \mathbf{y} : b \in Z_d{}^*\}.$$

This case also has the longest length $2n$.

Case 1.2: $t = 2 \Rightarrow \mathbf{y} = b\delta 0^3 \dots 0^{i-1}0^{n-j+2}\dots 0^{n-1}a$, where $\delta = 0$ or 1. Let $y = w\delta 0\dots 0u$, $\delta = 0$ or 1, $w, u > 0$.

- $w = 1$ and $\delta = 0$

$$WD = \{\mathbf{x} \to a10\dots 0 \to \mathbf{y} : a \in Z_d{}^*\}$$
$$\cup \{\mathbf{x} \to P(\mathbf{x}b^{n+1}\dots b^{2n}) \to b\dots b \text{ connect } \mathbf{y} \to P(\mathbf{y}b^{n+1}\dots b^{2n})$$
$$\to b\dots b : b \in Z_d{}^*\}.$$

- otherwise

$$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d \setminus \{w\}\}.$$

Case 2: $n - j + 2 \le i - 1 \Rightarrow t \ge 3$.
Case 2.1: $n - t + 2 \le t - 1 \Rightarrow n + 3 \le 2t \Rightarrow \frac{n+3}{2} \le t \le n$

$$\mathbf{y} = y_1 \dots y_{n-j+1}0^{n-j+2}\dots 0^{n-t+1}a^{n-t+2}y_{n-t+3}\dots y_{t-2}b^{t-1}\delta^t$$
$$\times 0^{t+1}\dots 0^{i-1}y_i \dots y_n,$$

with $y_1 \dots y_{t-2} = y_{n-t+3}\dots y_n$, and $\delta = 0$ or 1. Let $y_{t-1} = b = w > 0$. Then,

$$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d{}^* \setminus \{w\}\} \cup \{P[\mathbf{y}ww\mathbf{x}]\}.$$

Case 2.2: $n - t + 2 > t - 1 \Rightarrow n + 3 > 2t$.
Define $\text{Min} = \min\{t, n - j + 2\}$ and $\text{Max} = \max\{i - 1, n - t + 1\}$. Then

$$\mathbf{y} = y_1 \dots y_{\text{Min}-1}0^{\text{Min}}\dots 0^{\text{Max}}y_{\text{Max}+1}\dots y_n$$

with $y_1 \dots y_{t-2} = y_{n-t+3}\dots y_n$, and $y_t = \delta = 0$ or 1.
Case 2.2.1: $\text{Min} = t$ and $\text{Max} = n - t + 1$.

$$t \le n - j + 2 \quad \text{and} \quad i - 1 \le n - t + 1 \Rightarrow 3 \le t \le \frac{n+4}{3},$$
$$\mathbf{y} = y_1 \dots y_{t-2}b^{t-1}\delta^t 0^{t+1}\dots 0^{n-t+1}a^{n-t+2}y_{n-t+3}\dots y_n.$$

Let $y_{t-1} = b = w > 0$. Then,

$$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d{}^* \setminus \{w\}\} \cup \{P[\mathbf{y}ww\mathbf{x}]\}.$$

Case 2.2.2: Min $= t$ and Max $= i-1$.

$$\mathbf{y} = 0^1 \dots 0^{i+t-n-1} y_{i+t-n} \dots y_{t-2} b^{t-1} \delta^t 0^{t+1} \dots 0^{i-1} y_i \dots y_n,$$

and $a = y_{n-t+2} = 0$, this is impossible.

Case 2.2.3: Min $= n-j+2$ and Max $= n-t+1$.

$$\mathbf{y} = y_1 \dots y_{n-j+1} 0^{n-j+2} \dots 0^{n-t+1} a^{n-t+2} y_{n-t+3} \dots y_{2n-t-j+3} 0^{2n-t-j+4} \dots 0^n,$$

and $b = y_{t-1} = 0$, this is impossible.

Case 2.2.4: Min $= n-j+2$ and Max $= i-1$.

$$\mathbf{y} = 0^1 \dots 0^{i+t-n-1} y_{i+t-n} \dots y_{n-j+1} 0^{n-j+2} \dots 0^{i-1} y_i \dots y_{2n-t-j+3} \times 0^{2n-t-j+4} \dots 0^n,$$

and $a = b = 0$, this is impossible.

Combining all the cases, we have at least $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = 100\dots0$ of length at most $2n$. □

Corollary 2.3 *For any vertex* $\mathbf{y} \in UB(d,n)$, *there are at least* $2d-2$ *disjoint paths from* $\mathbf{y}$ *to distinct vertex* $\mathbf{x}$ *of length at most* $2n$ *when* $\mathbf{x} = cc\dots c$, *or* $ce\dots e$, *or* $c\dots ce$, *where* $c, e \in Z_d$, *and* $c \neq e$.

Proof The proof follows directly from Lemmas 2.1 and 2.2. □

Theorem 2.4 *For any two distinct vertices* $\mathbf{y} = y_1 y_2 \dots y_n$, $\mathbf{z} = z_1 z_2 \dots z_n \in UB(d,n)$, *there are* $2d-2$ *disjoint paths from* $\mathbf{y}$ *to* $\mathbf{z}$ *with length at most* $2n+1$.

Proof If $\mathbf{y}$ or $\mathbf{z}$ is in the forms of $\mathbf{x}$ in Corollary 2.3 then we are done. Therefore, we suppose that both of $\mathbf{y}$ and $\mathbf{z}$ are not in the forms of $\mathbf{x}$ in Corollary 2.3.

By Lemma 2.1, we can get $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = 0\dots0$ and from $\mathbf{z}$ to $\mathbf{x}$, respectively. Now, let the sets A_i and B_j are defined as in Lemma 2.1. Similar to Lemma 2.1, define $C_1 = A_1$, $D_1 = B_1$,

$$C_i = \{0^1 \dots 0^{n-i} a z_1 \dots z_{i-1} : a \in Z_d{}^*\}, \quad i = 2, 3, \dots, n, \quad \text{and}$$

$$D_j = \{z_{n-j+2} \dots z_n b 0^{j+1} \dots 0^n : b \in Z_d{}^*\}, \quad j = 2, 3, \dots, n.$$

We use $A_i(a')(\in A_i)$ to denote the vertex $0^1 \dots 0^{n-i} a' y_1 \dots y_{i-1}$ in the set A_i. Then $A_i \cap A_j = B_i \cap B_j = C_i \cap C_j = D_i \cap D_j = \phi$, for $i \neq j$, and $i, j = 1, 2, \dots, n$. This immediately implies $C_j(a') = B_i(b') = C_{j'}(a'')$ which is impossible.

Since $\mathbf{y} \neq \mathbf{z}$, we let $p = \min\{i : y_i \neq z_i\}$ and $q = \max\{i : y_i \neq z_i\}$, then $1 \leq p \leq q \leq n$. Now, $A_i \cap C_j = \phi$ for $i, j > p$, and $B_i \cap D_j = \phi$, for $i, j > n-q+1$.

If $A_i(a') = B_j(b')$ (or $C_{i'}(a'') = D_{j'}(b'')$), then we choose an arbitrary β from $Z_d{}^* \setminus \{b', b''\}$, and replace $\mathbf{x}$ with $\beta\beta\dots\beta$. This implies that there are $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = \beta\beta\dots\beta$, $\{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}], a, b \in Z_d \setminus \{\beta\}\}$, with $A_i \cap B_j = \phi$ (respectively from $\mathbf{z}$ to $\mathbf{x}$ with $C_i \cap D_j = \phi$).

If $\mathbf{y} = a'b'a'b' \dots a'b'$ (or $a'b'a'b' \dots a'b'a'$), then $\mathbf{y}$ has degree $2d-1$. The case $a = b'$ and $b = a'$ (or $a = b'$, $b = b'$) may cause the vertex $b'a'b'a' \dots b'a'$

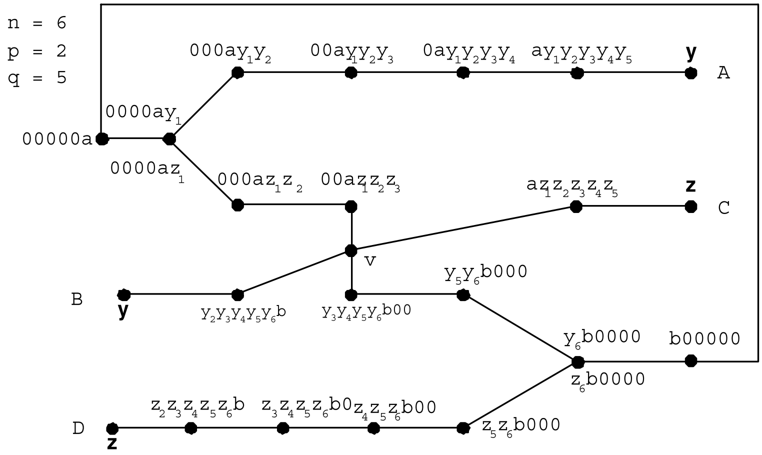

Fig. 1 Main process

(or $b'a'b' \ldots b'a'b'$) repeated. Then we replace $\mathbf{x}$ with $\beta\beta \ldots \beta$, where $\beta = a'$. Now we have $2d - 2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x}$ and $WD = \{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}], a, b \in Z_d \setminus \{a'\}\}$ with $A_i \cap B_j = \phi$. As to $\mathbf{z}$, the process is similar, we omit the details.

Now we have case $A_i \cap D_j \neq \phi$ or $B_i \cap C_j \neq \phi$ left to consider. Suppose not. Then, $A_i \cap D_j = B_i \cap C_j = \phi$, we can find $2d - 2$ disjoint paths from $\mathbf{y}$ to $\mathbf{z}$ as follows:

$$
\begin{aligned}
WD = \{&0 \ldots 0ay_1 \ldots y_{p-1} \to P(0^1 \ldots 0^{n-p} a^{n-p+1} {y_1}^{n-p+2} \ldots {y_n}^{2n-p}) \\
&\to \mathbf{y} \text{ connect } 0 \ldots 0az_1 \ldots z_{p-1} \to P(0^1 \ldots 0^{n-p} a^{n-p+1} {z_1}^{n-p+2} \ldots {z_n}^{2n-p}) \\
&\to \mathbf{z} : a \in {Z_d}^*\} \\
&\cup \{\mathbf{y} \to P({y_1}^1 \ldots {y_n}^n b^{n+1} 0^{n+2} \ldots 0^{n+q}) \to y_{q+1} \ldots y_n b0 \ldots 0 \\
&\quad \text{connect } \mathbf{z} \to P({z_1}^1 \ldots {z_n}^n b^{n+1} 0^{n+2} \ldots 0^{n+q}) \\
&\quad \to z_{q+1} \ldots z_n b0 \ldots 0 : b \in {Z_d}^*\}.
\end{aligned}
$$

Note that $0 \ldots 0ay_1 \ldots y_{p-1} = 0 \ldots 0az_1 \ldots z_{p-1}$ and $y_{q+1} \ldots y_n b0 \ldots 0 = z_{q+1} \ldots z_n b0 \ldots 0$. Moreover, 0 may be replaced with β if it is necessary in the previous cases. We call this "normal process".

Suppose $A_{j'} \cap D_{i'} \neq \phi$ or $B_i \cap C_j \neq \phi$:

- $A_{j'} \cap D_{i'} = \phi$ and $B_i(b') = v = C_j(a')$, for some $i > n - q + 1$, $j > p$, a' and b'. We fix the two disjoint paths as:

$$
\mathbf{y} \to B_n(b') \to B_{n-1}(b') \to \cdots \to v \to C_{j+1}(a') \to \cdots \to C_n(a') \to \mathbf{z},
$$

$$
\mathbf{z} \to P({z_1}^1 \ldots {z_n}^n b' 0^{n+2} \ldots 0^{2n}] \to P[0^1 \ldots 0^{n-1} a' {y_1}^{n+1} \ldots {y_n}^{2n}) \to \mathbf{y}.
$$

We use Fig. 1 as an example to explain our idea.

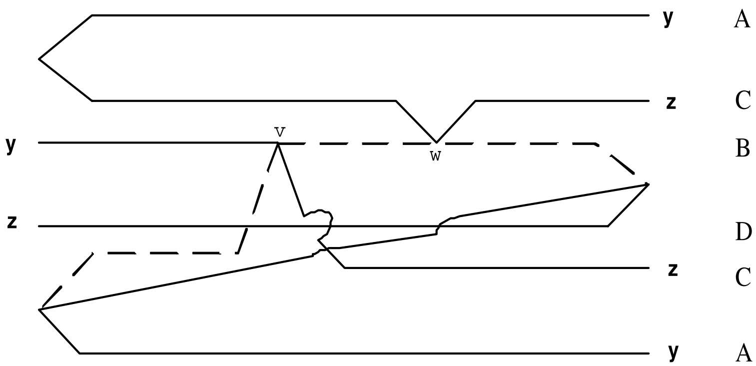

Fig. 2 Special process

- $A_{j'}(a') = w = D_{i'}(b')$ and $B_i \cap C_j = \phi$, for some i', j', a' and b'.

$$\mathbf{z} \to D_n(b') \to D_{n-1}(b') \to \cdots \to w \to A_{j'+1}(a') \to \cdots \to A_n(a') \to \mathbf{y},$$

$$\mathbf{y} \to P(y_1{}^1 \ldots y_n{}^n b' 0^{n+2} \ldots 0^{2n}] \to P[0^1 \ldots 0^{n-1} a' z_1{}^{n+1} \ldots z_n{}^{2n}) \to \mathbf{z}.$$

- $A_{j'}(a') = w = D_{i'}(b')$ and $B_i(b') = v = C_j(a')$ for some i, j, i', j', a' and b'.

$$\mathbf{y} \to B_n(b') \to B_{n-1}(b') \to \cdots \to v \to C_{j+1}(a') \to \cdots \to C_n(a') \to \mathbf{z},$$

$$\mathbf{z} \to D_n(b') \to D_{n-1}(b') \to \cdots \to w \to A_{j'+1}(a') \to \cdots \to A_n(a') \to \mathbf{y}.$$

We call the previous three cases "main process".

Finally, $B_i(b') = v = C_j(a')$ and $B_{i'}(b') = w = C_k(a'')$ happen at the same time for i, i', j, k, a', a'', b' and $i > i'$. In this case, we have $\mathbf{y} \to B_n(b') \cdots \to B_i(b') = v = C_j(a') \to \cdots \to C_n(a') \to \mathbf{z}$ first, and the others are decided by Fig. 2. We call this "special process".

We give an algorithm to end this proof.

Algorithm

Step 1: According to $\mathbf{y}$ and $\mathbf{z}$, define A_i, B_j, C_i, and D_j. If $A_i(a') = B_j(b')$ or $C_i(a'') = D_j(b'')$, then choose an arbitrary β from $Z_d^* \setminus \{b', b''\}$, and let $\mathbf{x} = \beta \ldots \beta$.

Step 2: Routing by "normal process".

Step 3: repeat if $B_i(b') = C_j(a')$ and $B_{i'}(b') = C_k(a'')$
routing "special process"
end-repeat.

Step 4: repeat if $A_i \cap D_j \neq \emptyset$ or $B_i \cap C_j \neq \emptyset$
routing "main process"
end-repeat.

end

Note here that $\mathbf{y}$ and $\mathbf{z}$ are not equal to either $ce \ldots e$ or $c \ldots ce$ (in Corollary 2.3), so we always have $a0 \ldots 0$ and $0 \ldots 0b$ (complete bipartite graph) to use. Since the main process and the special process do not occur usually, the "normal process" always work. □

3 Concluding remark

For any two distinct vertices **y** and **z**, indeed we have at least $2d - 2$ paths from **y** to **z** with length $n + 1$: $\{P[\mathbf{y}a\mathbf{z}]$ and $P[\mathbf{z}b\mathbf{y}]$ $a, b \in Z_d\}$. But they may not be internally disjoint. In this paper, we manage to find $2d - 2$ disjoint paths in $UB(d, n)$ for any two vertices and these paths have length at most $2n + 1$. Therefore, the $(2d - 2)$-wide diameter is not greater than $2n + 1$. It is interesting to know the exact $(2d - 2)$-wide diameter of a de Bruijn graph.

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