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Profile minimization on compositions of graphs

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Abstract The profile minimization problem arose from the study of sparse matrix technique. In terms of graphs, the problem is to determine the profile of a graph G which is defined as

$$P(G)=\min_{f}\sum_{v\in V(G)}\max_{x\in N[v]}(f(v)-f(x)),$$

where f runs over all bijections from $V(G)$ to $\{1,2,\ldots,|V(G)|\}$ and $N[v]=\{v\}\cup\{x\in V(G):xv\in E(G)\}$. This is equivalent to the interval graph completion problem, which is to find a super-graph of a graph G with as few number of edges as possible. The purpose of this paper is to study the profiles of compositions of two graphs.

Keywords Profile · Composition · Interval graph · Chordal graph · Simplicial vertex · Join · Cycle

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Dedicated to Professor Frank K. Hwang on the occasion of his 65th birthday.

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1 Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless and without multiple edges. For a graph G, we use $V(G)$ to denote the set of vertices of G and $E(G)$ the set of edges. The profile minimization problem arose from the study of sparse matrix technique. It can be defined in terms of graphs as follows.

In a graph G, the *neighborhood* of a vertex v is $N_G(v) = \{x \in V(G) : xv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N(v)$. If there is no ambiguity, we often use $N(v)$ for $N_G(v)$ and $N[v]$ for $N_G[v]$.

A *proper numbering* of a graph G of n vertices is a 1–1 mapping $f : V(G) \to \{1, 2, \ldots, n\}$. Given a proper numbering f, the *profile width* of a vertex v in G is

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x)).$$

The *profile* of a proper numbering f of G is $P_f(G) = \sum_{v \in V(G)} w_f(v)$, and the *profile* of G is

$$P(G) = \min\{P_f(G) : f \text{ is a proper numbering of } G\}.$$

A *profile numbering* of G is a proper numbering f such that $P_f(G) = P(G)$. The *profile minimization problem* is to determine the profile of a graph. It is equivalent to the interval graph completion problem described as below.

Recall that an *interval graph* is a graph whose vertices correspond to closed intervals in the real line, and two distinct vertices are adjacent if and only if their corresponding intervals intersect. It is well-known that a graph G of n vertices is an interval graph if and only if it has an *interval ordering* which is an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ such that

$$i < j < k \quad \text{and} \quad v_i v_k \in E(G) \quad \text{imply} \quad v_j v_k \in E(G). \tag{1}$$

This property can be re-stated as: A graph G of n vertices is an interval graph if and only if it has an *interval numbering* which is a proper numbering f such that

$$f(x) < f(y) < f(z) \quad \text{and} \quad xz \in E(G) \quad \text{imply} \quad yz \in E(G). \tag{2}$$

Interval orderings or interval numberings are used frequently in this paper.

Having the concept of interval numbering (2) in mind, it is then easy to see that for any proper numbering f of G, the graph G_f defined by the following is an interval super-graph of G with $|E(G_f)| = P_f(G)$:

$$V(G_f) = V(G) \quad \text{and} \quad E(G_f) = \{yz : f(x) \le f(y) < f(z), xz \in E(G)\}.$$

In other words, we have

Proposition 1 (Lin and Yuan 1994) *The profile minimization problem is the same as the interval graph completion problem. Namely,*

$$P(G) = \min\{|E(\widehat{G})| : \widehat{G} \text{ is an interval super-graph of } G\}.$$

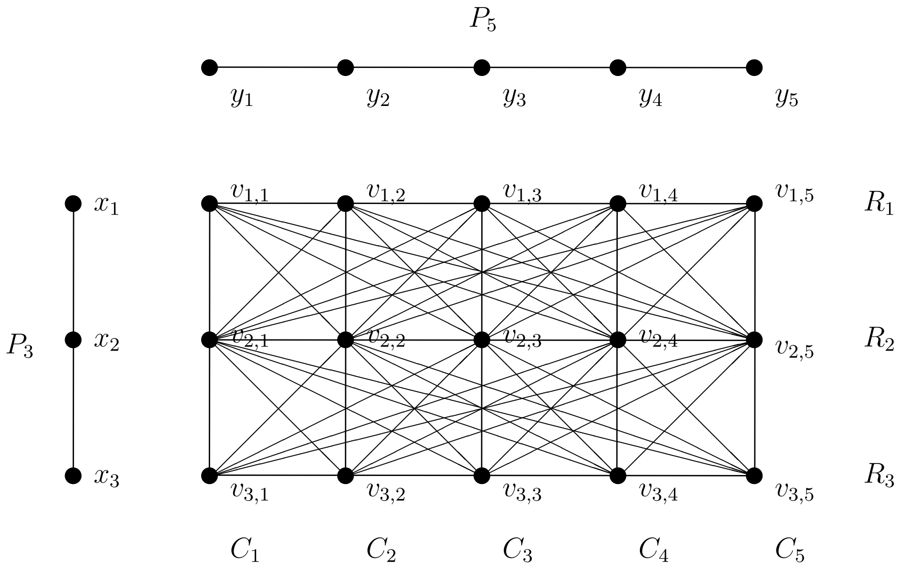

Fig. 1 The graph $P_3[P_5]$

An interval super-graph $\widehat{G}$ of a graph G with $|E(\widehat{G})| = P(G)$ is called an *interval completion* of G (see Fomin and Golovach 2000).

The profile minimization problem has been extensively studied in the literature (e.g. Guan and Williams 1998; Lai and Williams 1997; Odlyzko and Wilf 1987; Snay 1976; Wiegers and Monien 1988), for a good survey see (Lai and Williams 1999). From an algorithmic point of view, the problem is known to be NP-complete (see Garey and Johnson 1979). While many approximation algorithms for profiles of various graphs have been developed (see Gibbs et al. 1976; Koo and Lee 1992; Luo 1992; Smyth 1985), Kuo and Chang (1994) gave a polynomial-time algorithm for finding profiles of trees. Among the non-algorithmic results for profiles, we are most interested in those graphs obtained from graph operations. The classes of graphs in this line include Cartesian product of certain graphs (Lin and Yuan 1994; Mai 1996), sum of two graphs (Lin and Yuan 1994), composition of certain graphs (Lai 1997, 2002), and Coronas of certain graphs (Lai 1997).

The purpose of this paper is to study the profiles of compositions of graphs. The *composition* of two graphs G and H is the graph $G[H]$ vertex set $V(G) \times V(H)$ such that (x, y) is adjacent to (x', y') in $G[H]$ if $xx' \in E(G)$ or $x = x'$ with $yy' \in E(H)$. Notice that $G[H]$ has $|V(G)||V(H)|$ vertices and $|E(G)||V(H)|^2 + |V(G)||E(H)|$ edges.

For convenience, suppose $V(G) = \{x_i : 1 \le i \le |V(G)|\}$ and $V(H) = \{y_j : 1 \le j \le |V(H)|\}$. We may write (x_i, y_j) as $v_{i,j}$. Let $R_i = \{v_{i,j} : 1 \le j \le |V(H)|\}$ represents the ith row (a copy of H) of $G[H]$ and $C_j = \{v_{i,j} : 1 \le i \le |V(G)|\}$ the jth column (a copy of G). See Fig. 1 for the example $P_3[P_5]$.

In this paper we establish bounds for profiles $P(G[H])$ of compositions of graphs G and H. Also, exact value is determined when G is an interval graph as well as certain graphs.

2 Preliminary

A close related class of graphs to interval graphs are chordal graphs. A graph is *chordal* if every cycle of length greater than three has a chord. A vertex v of a graph G is *simplicial* if neighborhood $N(v)$ is a clique. It is well-known that a graph G of n vertices is chordal if and only if it has a *perfect elimination ordering* which is an ordering $v_1, v_2, \dots, v_n$ of $V(G)$ such that

$$i < j < k, \quad v_i v_j \in E(G) \quad \text{and} \quad v_i v_k \in E(G) \quad \text{imply} \quad v_j v_k \in E(G). \tag{3}$$

It is clear that an interval ordering is a perfect elimination ordering. Consequently, interval graphs are chordal. Notice that v_i is a simplicial vertex of the induced subgraph $G_{\{v_i, v_{i+1}, \dots, v_n\}}$ for $1 \le i \le n$.

Denote by $S(G)$ the set of all simplicial vertices of a graph G. It is clear by the definition that $S(G)$ induces a subgraph $G_{S(G)}$ in which every component is a complete graph. It is then the case that the number of components of $G_{S(G)}$ equals to the maximum number of an independent set in $G_{S(G)}$. We use $s(G)$ to denote this number.

Suppose now G is an interval graph, and $v_1, v_2, \dots, v_n$ is an interval ordering of G. For $1 \le i \le n$ and $x \in V(G)$, let

$$N_i(x) = \{v_j \in N(x) : j \ge i\}, \qquad N_i[x] = \{v_j \in N[x] : j \ge i\},$$
$$N^-(v_i) = \{v_j \in N(v_i) : j < i\}.$$

If necessary, we use $N^-(v_i; v_1, v_2, \dots, v_n)$ for $N^-(v_i)$ to emphasize the ordering. We use $\sigma(G; v_1, v_2, \dots, v_n)$ to denote the number of vertices v_i with $N^-(v_i) = \emptyset$. And let $\sigma(G) = \max \sigma(G; v_1, v_2, \dots, v_n)$, where the maximum is taken over all interval orderings of G.

Lemma 2 *Suppose $v_1, v_2, \dots, v_n$ is an interval ordering of an interval graph G. If $v_q \in N^-(v_p)$ and $N_q[v_p] \subseteq N_q[v_q]$, then the ordering $u_1, u_2, \dots, u_n$ resulted from $v_1, v_2, \dots, v_n$ by moving v_q to the position just after v_p is also an interval ordering of G.*

Proof For $i < j < k$ with $u_i u_k \in E(G)$, we shall verify that $u_j u_k \in E(G)$ by considering three cases. Let $u_i = v_{i'}$, $u_j = v_{j'}$ and $u_k = v_{k'}$.

Case 1. $i' < j' < k'$. In this case, $v_{i'} v_{k'} = u_i u_k \in E(G)$ implies $v_{j'} v_{k'} \in E(G)$ and so $u_j u_k \in E(G)$.

Case 2. $q = k' < j' \le p$. In this case, $v_p v_q \in E(G)$ implies $v_{j'} \in N_q[v_p] \subseteq N_q[v_q]$ and so $u_j u_k = v_{j'} v_q \in E(G)$.

Case 3. $q = j' < i' \le p < k'$. In this case, $v_{i'} v_{k'} = u_i u_k \in E(G)$ implies $v_{k'} \in N_q[v_p] \subseteq N_q[v_q]$ and so $u_j u_k = v_q v_{k'} \in E(G)$. □

Proposition 3 *For any interval graph G, we have $\sigma(G) = s(G)$.*

Proof Suppose $v_1, v_2, \dots, v_n$ is an interval ordering of G with $\sigma(G; v_1, v_2, \dots, v_n) = \sigma(G)$. By the definition of an interval ordering, any vertex v_i with $N^-(v_i) = \emptyset$ is simplicial. Also, $N^-(v_i) = N^-(v_j) = \emptyset$ imply that v_i and v_j are not adjacent. So, $\sigma(G) \le s(G)$.

Suppose $\sigma(G) < s(G)$. Then, by the definitions of $\sigma(G)$ and $s(G)$, the graph $G_{S(G)}$ has a component C containing no vertex v_i with $N^-(v_i) = \emptyset$. Let v_p be an arbitrarily vertex in C. For $v_{p-1} \in N^-(v_p)$, since v_p is simplicial, $N_{p-1}[v_p] \subseteq N_{p-1}[v_{p-1}]$. According to Lemma 2, we can move v_{p-1} to the position just after v_p to get a new interval ordering of G. Continue this process we shall get an interval ordering $u_1, u_2, \ldots, u_n$ with $N^-(v_p) = \emptyset$. More precisely, if $N^-(v_p; v_1, v_2, \ldots, v_n) = \{v_q, v_{q+1}, \ldots, v_{p-1}\}$, then in fact $u_1, u_2, \ldots, u_n$ is obtained from $v_1, v_2, \ldots, v_n$ by moving v_p into the position between v_{q-1} and v_q. So, $N^-(v_i; u_1, u_2, \ldots, u_n) = N^-(v_i; v_1, v_2, \ldots, v_n)$ for $i < q$ or $i > p$. Notice that by the definition of C and v_p, we have $N^-(v_i; v_1, v_2, \ldots, v_n) \ne \emptyset$ for $q \le i \le p$. Hence, $N^-(v_p; u_1, u_2, \ldots, u_n) = \emptyset$ implies that $\sigma(G; v_1, v_2, \ldots, v_n) < \sigma(G; u_1, u_2, \ldots, u_n)$, a contradiction. This proves the proposition. □

For a graph G, define $\hat{s}(G) = \max\{s(\widehat{G}) : \widehat{G}$ is an interval completion of $G\}$ and $\hat{\sigma}(G) = \max\{\sigma(\widehat{G}) : \widehat{G}$ is an interval completion of $G\}$.

Proposition 4 *If x is a simplicial vertex of a graph G, then x is also simplicial in any interval completion $\widehat{G}$ of G.*

Proof Suppose to the contrary that x is not simplicial in $\widehat{G}$. Choose an interval ordering $v_1, v_2, \ldots, v_n$ of $\widehat{G}$ with $x = v_p$. We may assume that the interval ordering is chosen such that p is as small as possible. Then, there are $v_q, v_r \in N_{\widehat{G}}(v_p)$ such that $q < r$ and $v_q v_r \notin E(\widehat{G})$. We may assume that q is chosen as large as possible. It is the case that $q < p$ by the interval ordering property. In fact, $q = p - 1$ for otherwise we have $N_{p-1}[v_p] \subseteq N_{p-1}[v_{p-1}]$. In this case, by Lemma 2, we may switch v_{p-1} and v_p to get a new interval ordering of G in which x has a smaller index than p, a contradiction.

Let s be the least index with $v_s \in N^-(v_p)$. It is easy to see that $v_1, v_2, \ldots, v_n$ is an interval ordering of $\widehat{G} - v_s v_p$. If $v_s v_p \notin E(G)$, then $\widehat{G} - v_s v_p$ is an interval super-graph of G with fewer edges than $\widehat{G}$, a contradiction. So, $v_s v_p \in E(G)$.

Since $v_r \in N_{\widehat{G}}(v_p) - N_{\widehat{G}}(v_{p-1})$, the least index t with $v_t \in N^-(v_r)$ is p. Again, $v_p v_r \in E(G)$ for otherwise $v_1, v_2, \ldots, v_n$ is an interval ordering of $\widehat{G} - v_p v_r$ which is an interval super-graph of G with fewer edges than $\widehat{G}$.

Since v_p is simplicial in G, both $v_s v_p, v_p v_r \in E(G)$ imply that $v_r v_s \in E(G) \subseteq E(\widehat{G})$. As $s \le q < r$, by the interval ordering property, $v_q v_r \in E(\widehat{G})$, a contradiction. □

Proposition 5 *If I is an independent set of a graph G and $I \subseteq S(\widehat{G})$ for an interval completion $\widehat{G}$ of G, then I is also independent in $\widehat{G}$ and so $|I| \le \hat{\sigma}(G)$.*

Proof Suppose to the contrary that $x, y \in I$ are such that $xy \notin E(G)$ but $xy \in E(\widehat{G})$. Choose an interval ordering $v_1, v_2, \ldots, v_n$ of $\widehat{G}$. Let $x = v_p$ and $y = v_{p'}$. Without loss of generality, we may assume that $p < p'$ and the interval ordering is chosen so that p is as small as possible. We then have $N^-(v_p) = \emptyset$, for otherwise there is some vertex $v_q \in N^-(v_p)$. Since v_p is simplicial in $\widehat{G}$, we have $N_q[v_p] \subseteq N_q[v_q]$. According to Lemma 2, we can move v_q to the position just after v_p to get a new interval ordering of $\widehat{G}$ in which x has a smaller index than p, a contradiction.

As $x = v_p$ and $y = v_{p'}$ are two adjacent simplicial vertices in $\widehat{G}$, we have $N_{\widehat{G}}[v_p] = N_{\widehat{G}}[v_{p'}]$. The fact that $N^-(v_p) = \emptyset$ then implies that the least index t with $v_t \in N^-(v_{p'})$ is p. It is then easy to see that $\widehat{G} - v_p v_{p'}$ is an interval supper-graph of G, a contradiction. This proves the proposition. □

3 Bounds for profiles of compositions of graphs

This section establishes upper and lower bounds for the profiles $P(G[H])$ of compositions of graphs G and H. Exact value is also determined when G is an interval graph.

First, an upper bound.

Theorem 6 *If $\widehat{G}$ is an interval supper-graph of a graph G of order m and H is a graph of order n, then*

$$P(G[H]) \leq |E(\widehat{G})|n^2 + (m - \sigma(\widehat{G}))\binom{n}{2} + \sigma(\widehat{G})P(H).$$

Proof Choose an interval completion $\widehat{H}$ of H. Then, $G[H]$ is a subgraph of $\widehat{G}[\widehat{H}]$ and so $P(G[H]) \leq P(\widehat{G}[\widehat{H}])$. Choose an interval ordering $x_1, x_2, \ldots, x_m$ of $\widehat{G}$ such that there are exactly $\sigma(\widehat{G})$ vertices x_i with $N^-(x_i) = \emptyset$. Also, choose an interval ordering $y_1, y_2, \ldots, y_n$ of $\widehat{H}$. Consider the ordering

$$v_{1,1}, v_{1,2}, \ldots, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,n}$$

using the lexicographical ordering. That is, $(i, j) < (i', j')$ if and only if $i < i'$ or $i = i'$ with $j < j'$. We shall check below that this is an interval ordering for the supper-graph Θ of $\widehat{G}[\widehat{H}]$ with $V(\Theta) = V(\widehat{G}[\widehat{H}])$ and $E(\Theta) = E(\widehat{G}[\widehat{H}]) \cup \{v_{i,j}v_{i,j'} : N^-(x_i) \neq \emptyset, 1 \leq j \neq j' \leq n\}$. Suppose $(i_1, j_1) < (i_2, j_2) < (i_3, j_3)$ with $v_{i_1,j_1}v_{i_3,j_3} \in E(\Theta)$.

Case 1. $i_1 \leq i_2 < i_3$. In this case, $v_{i_1,j_1}v_{i_3,j_3} \in E(\Theta)$ implies that $x_{i_1}x_{i_3} \in E(\widehat{G})$. By the interval ordering property, $x_{i_2}x_{i_3} \in E(\widehat{G})$ and so $v_{i_2,j_2}v_{i_3,j_3} \in E(\widehat{G}[\widehat{H}]) \subseteq E(\Theta)$.

Case 2. $i_1 < i_2 = i_3$. In this case, $v_{i_1,j_1}v_{i_3,j_3} \in E(\Theta)$ implies that $x_{i_1}x_{i_3} \in E(\widehat{G})$ and so $N^-(x_{i_3}) \neq \emptyset$. By the definition of Θ, we have $v_{i_2,j_2}v_{i_3,j_3} \in E(\Theta)$ since $i_2 = i_3$ and $j_2 \neq j_3$.

Case 3. $i_1 = i_2 = i_3$. In this case, $j_1 < j_2 < j_3$. Suppose $v_{i_2,j_2}v_{i_3,j_3} \notin E(\Theta)$. By the definition of Θ, we have $N^-(x_{i_3}) = \emptyset$ and so $v_{i_1,j_1}v_{i_3,j_3} \in E(\widehat{G}[\widehat{H}])$. Then, $y_{j_1}y_{j_3} \in E(\widehat{H})$ and so $y_{j_2}y_{j_3} \in E(\widehat{H})$ which in turn implies that $v_{i_2,j_2}v_{i_3,j_3} \in E(\widehat{G}[\widehat{H}]) \subseteq E(\Theta)$.

Therefore, Θ is an interval super-graph of $\widehat{G}[\widehat{H}]$ with $|E(\widehat{G})|n^2 + (m - \sigma(\widehat{G}))\binom{n}{2} + \sigma(\widehat{G})P(H)$ edges. The theorem then follows. □

Corollary 7 *If G is a graph of order m and H is a graph of order n, then*

$$P(G[H]) \leq P(G)n^2 + (m - \hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H).$$

Proof The corollary follows from Theorem 6 by choosing an interval completion $\widehat{G}$ of G with $\hat{\sigma}(G) = \sigma(\widehat{G})$. □

Next, we consider a lower bound.

Theorem 8 *If G is a graph of order m without $K_{2,3}$ as an induced subgraph and H is a graph of order n, then*

$$P(G[H]) \geq |E(G)|n^2 + (m - \hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H).$$

Proof Suppose K is an interval completion of $G[H]$. Notice that K is chordal. Let

$$V(G) = \{x_i : 1 \leq i \leq m\},$$
$$V(H) = \{y_j : 1 \leq j \leq n\},$$
$$V(K) = \{v_{i,j} = (x_i, y_j) : 1 \leq i \leq m, 1 \leq j \leq n\},$$
$$R(K) = \{x_i \in V(G) : K_{R_i} \text{ is not a clique in } K\} \text{ and } \eta = |R(K)|,$$
$$R'(K) = \{x \in R(K) : x \text{ is not simplicial in } G\}.$$

Claim 1 *$R(K)$ is an independent set in G.*

Proof of Claim 1 Suppose to the contrary that $x_p x_q \in E(G)$ for some $x_p, x_q \in R(K)$. By the definition of $R(K)$, there are four vertices $v_{p,a}, v_{p,b}, v_{q,c}, v_{q,d}$ in K such that $v_{p,a}v_{p,b} \notin E(K)$ and $v_{q,c}v_{q,d} \notin E(K)$. Since $x_p x_q \in E(G)$, we have $\{v_{p,a}v_{q,c}, v_{p,a}v_{q,d}, v_{p,b}v_{q,c}, v_{p,b}v_{q,d}\} \subseteq E(K)$ and hence $v_{p,a}v_{q,c}v_{p,b}v_{q,d}v_{p,a}$ is a chordless 4-cycle, a contradiction to the fact that K is chordal. □

Claim 2 *If $x_i \in R(K)$ and $x_p \neq x_q$ are in $N_G(x_i)$, then $v_{p,a}v_{q,b} \in E(K)$ for $1 \leq a,b \leq n$.*

Proof of Claim 2 By the definition of $R(K)$, $v_{i,j}v_{i,k} \notin E(K)$ for two distinct vertices $v_{i,j}$ and $v_{i,k}$. For $1 \leq a, b \leq n$, in the 4-cycle $v_{p,a}v_{i,j}v_{q,b}v_{i,k}v_{p,a}$, since $v_{i,j}v_{i,k} \notin E(K)$ we have $v_{p,a}v_{q,b} \in E(K)$. □

Claim 3 *If $\hat{\sigma}(G) < \eta$, then K has at least $(|E(G)| + \lceil \frac{\eta - \hat{\sigma}(G)}{2} \rceil)n^2$ non-horizontal edges.*

Proof of Claim 3 According to Claim 1, $R(K)$ is independent in G. Since $\hat{\sigma}(G) < \eta$, by Proposition 5, in each interval completion $\widehat{G}$ of G, there are at least $r = \eta - \hat{\sigma}(G) = \eta - \hat{s}(G)$ vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ of $R(K)$ which are not simplicial in $\widehat{G}$.

By Proposition 4, they are not simplicial in G and so are in $R'(K)$. For each x_{i_j} choose two neighbors $x_{p_j} \neq x_{q_j}$ with $x_{p_j}x_{q_j} \notin E(G)$. By Claim 2, there are n^2 non-horizontal edges $v_{p_j,a}v_{q_j,b}$ in K, where $1 \le a, b \le n$. As G contains no $K_{2,3}$ as an induced subgraph, each $\{x_{p_j}, x_{q_j}\}$ may equal to at most one $\{x_{p_{j'}}, x_{q_{j'}}\}$ with $j \neq j'$. Therefore, there are at least $\lceil \frac{\eta-\hat{\sigma}(G)}{2} \rceil n^2$ non-horizontal edges other than those already in $G[H]$. □

We are now ready to prove the theorem. First, by the definition of $R(K)$, there are at least $(m-\eta)\binom{n}{2} + \eta P(H)$ horizontal edges in K. If $\hat{\sigma}(G) \ge \eta$, then

$$\begin{aligned} P(G[H]) &\ge |E(G)|n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\ge |E(G)|n^2 + (m-\hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H), \end{aligned}$$

since $P(H) \le \binom{n}{2}$. If $\hat{\sigma}(G) < \eta$, then by Claim 3 we have

$$\begin{aligned} P(G[H]) &\ge \left(|E(G)| + \left\lceil \frac{\eta-\hat{\sigma}(G)}{2} \right\rceil\right)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\ge |E(G)|n^2 + (m-\hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H), \end{aligned}$$

since $\frac{n^2}{2} > \binom{n}{2}$ and $\eta > \hat{\sigma}(G)$. The theorem then follows. □

Corollary 9 *If G is a chordal graph of order m and H is a graph of order n, then*

$$P(G[H]) \ge |E(G)|n^2 + (m-\hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H).$$

Proof The corollary follows from that any chordal graph does not contain $K_{2,3}$ as an induced subgraph. □

Notice that the difference between the upper bound in Corollary 7 and the lower bound in Corollary 9 is at their first terms $P(G)n^2$ and $|E(G)|n^2$. For the case when the graph is interval, we have $P(G) = |E(G)|$ and so

Corollary 10 *If G is an interval graph of order m and H is a graph of order n, then*

$$P(G[H]) = P(G)n^2 + (m-\hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H). \tag{4}$$

It is our interest to know for which graph G equality (4) holds for any graph H. For this purpose, let

$$\begin{aligned} \Omega = \{G : P(G[H]) = P(G)|V(H)|^2 + (|V(G)| - \hat{\sigma}(G))\binom{|V(H)|}{2} \\ + \hat{\sigma}(G)P(H) \text{ for any graph } H\}. \end{aligned}$$

So, we have that Ω contains all interval graphs.

A slightly different lower bound is as follows.

Theorem 11 *If G is a graph of order m and H is a graph of order n, then either $G \in \Omega$ or*

$$P(G[H]) \geq (P(G)+1)n^2 + (m-\eta)\binom{n}{2} + \eta P(H)$$
$$\geq (P(G)+1)n^2 + (m-\alpha(G))\binom{n}{2} + \alpha(G)P(H)$$

for some nonnegative integer $\eta \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G.

Proof We use precisely the same notation K, $V(G)$, $V(H)$, $V(K)$, $R(K)$, η, $R'(K)$ as in the proof of Theorem 8. Notice that Claims 1 and 2 are still valid.

Case 1. $\eta \leq \hat{\sigma}(G)$.

For $j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, n\}$, The subgraph $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ is an interval supergraph of G and so has at least $P(G)$ edges. For each non-horizontal edge $v_{i',j'}v_{i'',j''}$ in K, there are n^{m-2} subgraphs $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ contain this edge. Since there are n^m subgraphs $K_{\{v_{i,j_i}:1\leq i\leq m\}}$, there are at least $n^m P(G)/n^{m-2} = P(G)n^2$ non-horizontal edges in K. By the definition of η, we have

$$P(G[H]) \geq P(G)n^2 + (m-\eta)\binom{n}{2} + \eta P(H)$$
$$\geq P(G)n^2 + (m-\hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H),$$

This together with Corollary 7 gives that $G \in \Omega$.

Case 2. $\eta > \hat{\sigma}(G)$.

In this case, we claim that each $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ has at least $P(G)+1$ edges and hence the desired inequalities hold. Suppose to the contrary that there is some $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ having just $P(G)$ edges. We may view v_{i,j_i} as x_i and then $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ is an interval completion of G. By Claim 1, $R(K)$ is independent in G. By Claim 2, $R(K) \subseteq S(K_{\{v_{i,j_i}:1\leq i\leq m\}})$. Hence, by Proposition 5, $R(K)$ is also independent in $K_{\{v_{i,j_i}:1\leq i\leq m\}}$. And then $\eta = |R(K)| \leq \hat{\sigma}(G)$, a contradiction. □

Corollary 12 *If $\alpha(G) - \hat{\sigma}(G) \leq 2$, then $G \in \Omega$.*

Proof Suppose to the contrary that $G \notin \Omega$. According to Corollary 7 and Theorem 11,

$$(P(G)+1)n^2 + (m-\alpha(G))\binom{n}{2} + \alpha(G)P(H)$$
$$\leq P(G)n^2 + (m-\hat{\sigma}(G))\binom{n}{2} + \hat{\sigma}(G)P(H)$$

for some graph H of n vertices. This gives $n^2 \leq (\alpha(G) - \hat{\sigma}(G))(\binom{n}{2} - P(H)) \leq n(n-1) - 2P(H)$, which is impossible. Therefore, $G \in \Omega$. □

4 Gap between the upper and the lower bounds

There is a gap between the upper bound in Corollary 7 and the lower bound in Theorem 11. This section gives examples for which the upper or the lower bound are attainable. We also give conditions for which the upper bound attains.

The *join* of graphs G_1 and G_2 is the graph $G_1 \vee G_2$ with $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1) \text{ and } y \in V(G_2)\}$.

Theorem 13 *If G_1 is a graph of m_1 vertices and G_2 a graph of m_2 vertices, then*

$$P(G_1 \vee G_2) = \min\left\{P(G_1) + m_1 m_2 + \binom{m_2}{2}, P(G_2) + m_1 m_2 + \binom{m_1}{2}\right\}.$$

Furthermore, if $P(G_1 \vee G_2) = P(G_i) + m_1 m_2 + \binom{m_j}{2}$ for $i \neq j$, then $\hat{\sigma}(G_1 \vee G_2) = \hat{\sigma}(G_i)$.

Proof The theorem follows from the fact that for any interval super-graph K of $G_1 \vee G_2$, either $V(G_1)$ or $V(G_2)$ is a clique in K. This is because if $x_i y_i \notin E(K)$ for $x_i, y_i \in V(G_i)$ $(i = 1, 2)$, then $x_1 x_2 y_1 y_2 x_1$ is a chordless 4-cycle in K which is impossible. □

Theorem 14 *If G_1 is a graph of m_1 vertices, G_2 a graph of m_2 vertices and H a graph of n vertices, then $(G_1 \vee G_2)[H] = G_1[H] \vee G_2[H]$ and so*

$$\begin{aligned} &P((G_1 \vee G_2)[H]) \\ &\quad = \min\left\{P(G_1[H]) + m_1 m_2 n^2 + \binom{m_2 n}{2}, P(G_2[H]) + m_1 m_2 n^2 + \binom{m_1 n}{2}\right\}. \end{aligned}$$

Proof The first equality follows from definition. The second equality then follows from Theorem 13. □

Now, let G_1 be the path P_7 of 7 vertices and G_2 the graph obtained from $K_{1,6}$ by adding a new edge. Notice that both G_1 and G_2 are interval graphs of 7 vertices; and G_1 has 6 edges while G_2 has 7 edges. Also, $\sigma(G_1) = 2$ and $\sigma(G_2) = 5$. Then, for any graph H of n vertices, we have

$$\begin{aligned} P(G_1 \vee G_2) &= 6 + 7 \cdot 7 + \binom{7}{2} = 76, \\ P(G_1[H]) &= 6n^2 + (7-2)\binom{n}{2} + 2P(H) = 8.5n^2 - 2.5n + 2P(H), \\ P(G_2[H]) &= 7n^2 + (7-5)\binom{n}{2} + 5P(H) = 8n^2 - n + 5P(H), \\ P((G_1 \vee G_2)[H]) &= \min\{P(G_1[H]), P(G_2[H])\} + \binom{7n}{2} + 7n \cdot 7n \\ &= \min\{P(G_1[H]), P(G_2[H])\} + 73.5n^2 - 3.5n, \end{aligned}$$

$$\hat{\sigma}(G_1 \vee G_2) = 2,$$
$$\alpha(G_1 \vee G_2) = 5,$$
$$\text{upper bound} = 76n^2 + (14-2)\binom{n}{2} + 2P(H) = 82n^2 - 6n + 2P(H),$$
$$\text{lower bound} = (76+1)n^2 + (14-5)\binom{n}{2} + 5P(H)$$
$$= 81.5n^2 - 4.5n + 5P(H).$$

Depending on H, it is possible that $P(G_1[H]) < P(G_2[H])$ or $P(G_1[H]) \geq P(G_2[H])$. For the former case, $P((G_1 \vee G_2)[H])$ is equal to the upper bound; for the later case, $P((G_1 \vee G_2)[H])$ is equal to the lower bound.

Theorem 15 *Suppose G_1, G_2 and H are graphs of order m_1, m_2 and n, respectively. If $G_1 \in \Omega$, $G_2 \notin \Omega$, $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and $\alpha(G_2) \leq \hat{\sigma}(G_1) + 2$, then $G_1 \vee G_2 \in \Omega$ and*

$$P((G_1 \vee G_2)[H])$$
$$= \left(P(G_1) + m_1m_2 + \binom{m_2}{2}\right)n^2 + (m_1 + m_2 - \hat{\sigma}(G_1))\binom{n}{2} + \hat{\sigma}(G_1)P(H).$$

Proof By the assumption $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and Theorem 13, we have

$$P(G_1 \vee G_2) = P(G_1) + m_1m_2 + \binom{m_2}{2} \quad \text{and} \quad \hat{\sigma}(G_1 \vee G_2) = \hat{\sigma}(G_1).$$

Now

$$P(G_1[H]) + m_1m_2n^2 + \binom{m_2n}{2}$$
$$= P(G_1)n^2 + (m_1 - \hat{\sigma}(G_1))\binom{n}{2} + \hat{\sigma}(G_1)P(H) + m_1m_2n^2 + \binom{m_2n}{2}$$
$$= \left(P(G_1) + m_1m_2 + \binom{m_2}{2}\right)n^2 + (m_1 + m_2 - \hat{\sigma}(G_1))\binom{n}{2} + \hat{\sigma}(G_1)P(H)$$
$$\leq \left(P(G_2) + m_1m_2 + \binom{m_1}{2}\right)n^2 + (m_1 + m_2 - \alpha(G_2))\binom{n}{2}$$
$$+ \alpha(G_2)P(H) + 2\binom{n}{2}$$
$$\leq (P(G_2) + 1)n^2 + (m_2 - \alpha(G_2))\binom{n}{2} + \alpha(G_2)P(H) + m_1m_2n^2 + \binom{m_1n}{2}$$
$$\leq P(G_2[H]) + m_1m_2n^2 + \binom{m_1n}{2}.$$

Notice that in the above formulas, the first equality follows from that $G_1 \in \Omega$, the second equality from that $\binom{m_2 n}{2} = \binom{m_2}{2}n^2 + m_2\binom{n}{2}$, the third inequality from that $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and $\alpha(G_2) \leq \hat{\sigma}(G_1) + 2$, the forth inequality from that $\binom{m_1 n}{2} = \binom{m_1}{2}n^2 + m_1\binom{n}{2}$ and $2\binom{n}{2} \leq n^2$, and the fifth inequality from Theorem 11.

The theorem then follows from Theorem 14. □

Theorem 16 *Suppose* G_1, G_2 *and* H *are graphs of order* m_1, m_2 *and* n, *respectively. If* $G_1 \in \Omega$, $G_2 \in \Omega$, $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ *and* $\hat{\sigma}(G_2) \leq \hat{\sigma}(G_1)$, *then* $G_1 \vee G_2 \in \Omega$ *and*

$$\begin{aligned} &P((G_1 \vee G_2)[H]) \\ &\quad = \left(P(G_1) + m_1 m_2 + \binom{m_2}{2}\right)n^2 + (m_1 + m_2 - \hat{\sigma}(G_1))\binom{n}{2} + \hat{\sigma}(G_1)P(H). \end{aligned}$$

Proof The arguments are similar to those for the proof of Theorem 15. □

5 Exact value

By using the theorems in the previous sections, we are able to get exact values for many $P(G[H])$ when G are given precisely. At the end of this paper we only consider one of the case that can not be deduced directly by the previous properties, namely for the case when $G = C_m$ with $m \geq 4$.

Lemma 17 *If* $m \geq 4$ *and* C *is a non-complete interval super-graph of* C_m, *then* $|E(C)| \geq 2m - 5 + s(C)$.

Proof Since C is chordal, C contains at least $m - 3$ chords of C_m and so $|E(C)| \geq 2m - 3$. The lemma is clearly true for $s(C) \leq 2$. We may now assume that $s(C) \geq 3$. It is then the case that $m \geq 6$. Choose an interval ordering $v_1, v_2, \ldots, v_m$ of C. Let $i < j < k$ and v_i, v_j, v_k are independent simplicial vertices of C. Choose an v_i–v_k path P in C_m not passing v_j. As $i < j < k$, in this path there are adjacent vertices $v_{i'}$ and $v_{k'}$ with $i' < j < k'$. By the interval ordering property, we have $v_j v_{k'} \in E(C)$. Let $v_{j'}, v_{j''}$ be the two neighbors of v_j in C_m. Then $v_{j'} v_{j''} \in E(C)$ as v_j is simplicial in C. So $C' = C - v_j$ is an interval super-graph of C_{m-1} with $s(C') \geq s(C) - 1 \geq 2$, which implies that C' is not a complete graph. By the induction hypothesis, $|E(C')| \geq 2(m-1) - 5 + s(C) - 1$. As the path P does not pass v_j, we have $v_{k'} \notin \{v_{j'}, v_{j''}\}$ and so $|E(C)| \geq |E(C')| + 3 \geq 2m - 5 + s(C)$. □

Theorem 18 *If* $m \geq 4$ *and* H *is a graph of order* n, *then*

$$P(C_m[H]) = (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H).$$

Consequently, $C_m \in \Omega$.

Proof Let $G = C_m$ and we use the same notation K, $V(G)$, $V(H)$, $V(K)$, $R(K)$, η, $R'(K)$ as in the proof of Theorem 8. Notice that Claims 1 and 2 are still valid.

Consider the interval super-graph C' obtained from C_m by adding $m-3$ chords passing a fixed vertex. Then $|E(C')| = 2m-3$ and $\hat{\sigma}(C') = s(C') = 2$. Suppose C'' is an interval completion of C_m with $\sigma(C'') = \hat{\sigma}(C_m)$. It is clear that C'' is not a complete graph, and so $\hat{\sigma}(C'') \geq 2$. By Lemma 17, $2m-3 = |E(C')| \geq |E(C'')| \geq 2m-5+\sigma(C'') \geq 2m-3$ and so in fact $P(C_m) = 2m-3$ and $\hat{\sigma}(C_m) = 2$.

By Corollary 7, $P(C_m[H]) \leq (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H)$. To see the other inequality, we consider two cases.

Case 1. $\eta \leq 2$.

For $j_i, j_2, \ldots, j_m \in \{1, 2, \ldots, n\}$, the subgraph $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ is an interval super-graph of C_m and so has at least $P(C_m) = 2m-3$ edges. For each non-horizontal edge $v_{i',j'}v_{i'',j''}$ in K, there are n^{m-2} subgraphs $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ contain this edge. Since there are n^m subgraphs $K_{\{v_{i,j_i}:1\leq i\leq m\}}$, there are at least $n^m(2m-3)/n^{m-2} = (2m-3)n^2$ non-horizontal edges in K. By the definition of η, we have

$$\begin{aligned} P(C_m[H]) &\geq (2m-3)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\geq (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H). \end{aligned}$$

Case 2. $\eta > 2$.

In this case, we may view v_{i,j_i} as x_i and then $C = K_{\{v_{i,j_i}:1\leq i\leq m\}}$ is an interval super-graph of C_m. By Claim 1, $R(K)$ is independent in C_m. By Claim 2, $R(K) \subseteq S(C)$. Hence, $s(C) \geq \eta$ and so $|E(C)| \geq 2m-5+s(C) \geq 2m-5+\eta$ by Lemma 17. As in the proof of case 1, there are at least $(2m-5)+\eta$ non-horizontal edges. By the definition of η, we have

$$\begin{aligned} P(C_m[H]) &\geq (2m-5+\eta)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\geq (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H). \end{aligned}$$

□

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