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Maximum cyclic 4-cycle packings of the complete multipartite graph

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Abstract A graph G is said to be m-sufficient if m is not exceeding the order of G, each vertex of G is of even degree, and the number of edges in G is a multiple of m. A complete multipartite graph is balanced if each of its partite sets has the same size. In this paper it is proved that the complete multipartite graph G can be decomposed into 4-cycles cyclically if and only if G is balanced and 4-sufficient. Moreover, the problem of finding a maximum cyclic packing of the complete multipartite graph with 4-cycles are also presented.

Keywords Complete multipartite graph · Cyclic · Cycle system · Cycle packing · 4-cycle

1 Introduction

An m-cycle, written $(c_0, c_1, \ldots, c_{m-1})$, consists of m distinct vertices $c_0, c_1, \ldots, c_{m-1}$, and m edges $\{c_i, c_{i+1}\}$, $0 \le i \le m-2$, and $\{c_0, c_{m-1}\}$. An *m-cycle system* of a simple graph G is a set $\boldsymbol{C}$ of edge disjoint m-cycles which partition the edge set of G. If G is a complete graph on v vertices, it is known as an m-cycle system of order v.

The obvious necessary conditions for the existence of an m-cycle system of a graph G are that the value of m is not exceeding the order of G, m divides the number of edges in G, and the degree of each vertex in G is even. A graph G is called *m-sufficient* if the necessary conditions are met.

Dedicated to Professor Frank K. Hwang on the occasion of his 65th birthday.

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A graph G is said to be a *complete r-partite graph* ($r > 1$) if its vertex set V can be partitioned into r disjoint non-empty sets $V_1, \ldots, V_r$ (called *partite sets*) such that there exists exactly one edge between each pair of vertices from different partite sets. If $|V_i| = n_i$ for $1 \le i \le r$, the complete r-partite graph is denoted by $K_{n_1,\ldots,n_r}$. In particular, if $n_1 = \cdots = n_r = k$ (>1), it is called *balanced* and the graph will be simply denoted by $K_{r(k)}$.

The graph decomposition problem has attracted many researchers, and it serves as useful models for a range of applications such as: serology (Ree 1967), synchronous optical network ring (Colbourn and Wan 2001; Wan 1999), and DNA library screening (Mutoh et al. 2003).

The study of m-cycle systems of the complete graph has been one of the most interesting problems in graph decomposition. The existence question for m-cycle systems of the complete graph has been completely settled by Alspach and Gavlas (2001) in the case of m odd and by Šajna (2002) in the even case.

The problem of finding the existence of m-cycle systems of the complete r-partite graph has also been considered by a number of researchers. The case when $r = 2$ and m is even was completely solved by Sotteau (1981). Cavenagh (1998) proved that there exists a k-cycle system of $K_{3(m)}$ if and only if $k \le 3m$ and k divides $3m^2$. Billington (1999) gave the necessary and sufficient conditions for the existence of a decomposition of any complete tripartite graph into specific numbers of 3-cycles and 4-cycles. Hoffman et al. (1989) proved that if both r and m are odd then there exists an m-cycle system of $K_{r(m)}$. The necessary and sufficient conditions to partition the same graph into Hamiltonian cycles are given by Laskar (1978). The existence for 5-cycle system of the complete tripartite graph has been considered by Mahmoodian and Mirzakhani (1995), Cavenagh and Billington (2002), and Cavenagh (2002). Moreover, necessary and sufficient conditions are also given (Cavenagh and Billington 2000) for the existence of m-cycle systems of the complete r-partite graph with $m = 4, 6$, and 8.

An m-cycle *packing* of a graph G is a set $\boldsymbol{P}$ of edge disjoint m-cycles in G. The *leave* of an m-cycle packing of G is the set of edges in G that occur in no m-cycle in $\boldsymbol{P}$. An m-cycle packing $\boldsymbol{P}$ of G is *maximum* if $|\boldsymbol{P}| \ge |\boldsymbol{P}'|$ for all other m-cycle packings $\boldsymbol{P}'$ of G. Obviously, a maximum packing will have a minimum leave, and an m-cycle system of G is an m-cycle packing of G for which the leave is empty.

Not much work has been done on packing complete r-partite graphs with cycles. For 3- and 6-cycles, maximum packings in $K_{r(k)}$ are respectively dealt with in (Billington and Lindner 1996; Fu and Huang 2004). In (Billington et al. 2001), the problem of finding a maximum packing of the complete r-partite graph with 4-cycles is completely solved. A natural generalization to determine a maximum packing of the λ-fold complete r-partite graph appears in (Billington et al. 2005).

Let $C = (c_0, c_1, \ldots, c_{m-1})$ be an m-cycle. An m-cycle system (packing) of a graph G, $\boldsymbol{C}(\boldsymbol{P})$, is said to be *cyclic* if $V(G) = Z_v$ and we have $(c_0 + 1, c_1 + 1, \ldots, c_{m-1} + 1)$ (mod v) $\in \boldsymbol{C}(\boldsymbol{P})$ whenever $(c_0, c_1, \ldots, c_{m-1}) \in \boldsymbol{C}(\boldsymbol{P})$.

The existence question for cyclic m-cycle systems of order v has been completely solved for $m = 3$ (Peltesohn 1938), 5 and 7 (Rosa 1966b). For m even and $v \equiv 1$ (mod $2m$), cyclic m-cycle systems of order v are proved for $m \equiv 0$ (mod 4) (Kotzig 1965) and for $m \equiv 2$ (mod 4) (Rosa 1966a). Recently, it is shown in (Buratti and Del Fra 2003; Bryant et al. 2003; Fu and Wu 2004) that for each pair of integers (m, n), there

Table 1 Best possible leaves of a maximum cyclic packing of $\boldsymbol{K}_{r(k)}$ with 4-cycles

r	k							
	0	1	2	3	4	5	6	7
0	-	F	-	F	-	F	-	F
1	-	-	-	-	-	-	-	-
2	-	F	-	$F \cup H$	-	$F \cup 2H$	-	$F \cup C^*$
3	-	H	-	$3H$	-	H	-	$3H$
4	-	F	-	$F \cup C$	-	F	-	$F \cup C$
5	-	$2H$	-	$2H$	-	$2H$	-	$2H$
6	-	F	-	$F \cup H$	-	$F \cup 2H$	-	$F \cup C$
7	-	$3H$	-	H	-	$3H$	-	H

* When $r = 2$ and $k \equiv 7 \pmod 8$, the leave is the union of a 1-factor and 3 Hamiltonian cycles

exists a cyclic m-cycle system of order $2mn + 1$, and in particular, for each odd prime p, there exists a cyclic p-cycle system (Buratti and Del Fra 2003; Fu and Wu 2004). For $v \equiv m \pmod{2m}$, cyclic m-cycle systems of order v are presented for $m \notin \mathrm{M}$ (Buratti and Del Fra 2004), where $\mathrm{M} = \{p^\alpha \mid p \text{ is prime}, \alpha > 1\} \cup \{15\}$, and in (Vietri 2004) for $m \in \mathrm{M}$. More recently, the present authors (Wu and Fu 2006) prove that for $m = 3, 4, \ldots, 32$, there exists a cyclic m-cycle system and for p a prime power, there exists a cyclic $2p$-cycle system.

In this paper, we shall focus on maximum cyclic 4-cycle packings of $K_{r(k)}$ with leave and the main result is listed in Table 1, where the values of r and k are reduced modulo 8 and the symbols -, iH, C, and F denote respectively the empty set, i Hamiltonian cycles, 2 $(rk/2)$-cycles, and a 1-factor.

In Sect. 2, we will give the essential definitions and preliminaries. In Sect. 3, a cyclic 4-cycle system of $K_{r(k)}$ will be presented, and in Sects. 4 and 5, maximum cyclic 4-cycle packings of $K_{r(k)}$ with leave and with rk odd or even will be respectively given.

2 Definitions and preliminaries

Assume $\{a, b\}$ to be any edge of G with $V(G) \in Z_v$. We shall use $\pm|a - b|$ to denote the *difference* of the edge $\{a, b\}$ in G. The number of distinct differences in a graph G defined on Z_v is called the *weight* of G, denoted by $W(G)$.

Let $C = (c_0, c_1, \ldots, c_{m-1})$ be an m-cycle of G and let $C + i = (c_0 + i, c_1 + i, \ldots, c_{m-1} + i) \pmod v$, where $i \in Z_v$. A *cycle orbit* $\boldsymbol{O}$ of C is a collection of distinct m-cycles in $\{C + i \mid i \in Z_v\}$. The *length* of a cycle orbit is its cardinality, i.e., the minimum positive integer k such that $C + k = C$. A *base cycle* of a cycle orbit $\boldsymbol{O}$ is a cycle $C \in \boldsymbol{O}$ that is chosen arbitrarily. For the convenience of notation, we write a cycle k-orbit for a cycle orbit of length k. A cycle v-orbit of C on G is said to be *full* and otherwise *short*.

Given a subset Ω of $Z_v - \{0\}$ with $\Omega = -\Omega$, the *circulant graph* $X(Z_v, \Omega)$ of order v is the Cayley graph $Cay[Z_v; \Omega]$, that is, the graph with vertex set Z_v and all possible edges of the form $\{x, x+w\}$ with $w \in \Omega$. The set Ω is called the *connection set* and its size is the degree of $X(Z_v, \Omega)$.

We first introduce a necessary condition for the existence of a cyclic m-cycle system of a graph.

Lemma 2.1 *If there is a cyclic m-cycle system of a graph G, then G is $2r$-regular for some positive integer r.*

Proof For $i = 1, \dots, p$ with $p \geq 1$, let $\boldsymbol{O_i}$ be a cycle k_i-orbit of C_i in the cyclic m-cycle system and let C_i be the base cycle of $\boldsymbol{O_i}$ with weight w_i. Note that the graph induced by the edges having the same difference is a spanning 2-regular subgraph of G. Thus, the union of the cycles $C_i, C_i+1, \dots, C_i+(k_i-1)$ forms a spanning $2w_i$-regular subgraph of G. This means that each cycle k_i-orbit $\boldsymbol{O_i}$ $(1 \leq i \leq p)$ is exactly a spanning $2w_i$-regular subgraph of G. It follows that the graph G is $(2\sum_{i=1}^{p} w_i)$-regular. □

Remark that the graph G in Lemma 2.1 is precisely a circulant graph. It is clear from Lemma 2.1 that if there exists a cyclic m-cycle system of the complete r-partite graph $K_{n_1,\dots,n_r}$, then $K_{n_1,\dots,n_r}$ is balanced, namely, $n_1 = \cdots = n_r = k$ for some integer k (>1).

A necessary condition for the existence of a cyclic m-cycle system of $K_{r(k)}$ is that any partite set in $K_{r(k)}$ is the subgroup

$$rZ_k = \{0, r, \dots, (k-1)r\}$$

of Z_{rk} or its coset. For $i = 0, \dots, r-1$, let V_i denote the ith partite set of $K_{r(k)}$. Throughout this paper we will assume the ith partite set of $K_{r(k)}$ to be $V_i = \{i, i+r, \dots, i+(k-1)r\}$ for $i = 0, \dots, r-1$. Note that the set of distinct differences of edges in $K_{r(k)}$ is $Z_{rk} \backslash \pm \{0, r, \dots, \lfloor k/2 \rfloor r\}$.

For an m-cycle C with $V(C) \in Z_v$, the necessary condition for the sum of absolute differences of edges in C is given as follows:

Lemma 2.2 *Let $C = (c_0, c_1, \dots, c_{m-1})$ be an m-cycle with $c_i \in Z_v$ where $0 \leq i \leq m-1$ and v is any positive integer. Then the sum of absolute differences of edges in C is even.*

Proof The proof follows immediately from the fact that

$$\sum_{i=1}^{m} |c_i - c_{i-1}| \equiv \sum_{i=1}^{m} (c_i - c_{i-1}) \equiv 0 \pmod 2.$$ □

The following consequences can be obtained by simple observations.

Lemma 2.3 *If C is an m-cycle with weight p in a cyclic m-cycle system of $K_{r(k)}$, then m is a multiple of p. Consequently, if $m = pq$, then the value of q is a common divisor of m and rk.*

Lemma 2.4 *Suppose $\Omega = \pm\{b\}$ with $b \in Z_{\lfloor v/2 \rfloor}$ and let $k = v/\gcd(v, b)$. Then the circulant graph $X(Z_v, \Omega)$ is the union of v/k edge-disjoint k-cycles.*

If $\gcd(v, b) = 1$, then $X(Z_v, \Omega)$ is exactly a Hamiltonian cycle in K_v and if $b = v/m$, then $X(Z_v, \Omega)$ is the union of b edge-disjoint m-cycles.

Lemma 2.5 *Let a_i $(1 \le i \le 4)$ be distinct elements in $Z^*_{\lfloor v/2 \rfloor} = Z_{\lfloor v/2 \rfloor} \setminus \{0\}$. If $\Omega = \pm\{a_1, a_2, a_3, a_4\}$ with $a_1 + a_2 = a_3 + a_4$, then there exists a cyclic 4-cycle system of $X(Z_v, \Omega)$.*

Proof The base cycle is $(0, a_1, a_1 + a_2, a_3)$. □

Lemma 2.6 *If $\Omega = \pm\{a_1, a_2\}$ with $a_1 \ne a_2$ and $a_1 + a_2 = rk/2$, then there exists a cyclic 4-cycle system of $X(Z_{rk}, \Omega)$.*

Proof Choose $(0, a_1, rk/2, rk/2 + a_1)$ as the base cycle. □

Given a positive integer $m = pq$, an m-cycle C in $K_{r(k)}$ with weight p has *index* rk/q if for each edge $\{s, t\}$ in C, the edges $\{s + i \cdot rk/q, t + i \cdot rk/q\}$ (mod rk) with $i \in Z_q$ are also in C.

For instance, the 15-cycle $C = (0, 1, 5, 7, 12, 25, 26, 30, 32, 37, 50, 51, 55, 57, 62)$ in $K_{5(15)}$ with weight 5 (differences ± 1, ± 2, ± 4, ± 5, and ± 13) has index 25.

The following consequence will be the crucial tool for constructing a cycle orbit in a cyclic m-cycle system of $K_{r(k)}$. The similar results about 1-rotational m-cycle system of the complete graph can also be found in (Buratti 2003, 2004) and so we omit the details.

Proposition 2.7 *Let $m = pq$. Then there exists an m-cycle $C = (c_0, c_1, \dots, c_{m-1})$ in $K_{r(k)}$ with weight p and index rk/q if and only if each of the following conditions is satisfied*:

(1) *For* $0 \le i \ne j \le p - 1$, $c_i \not\equiv c_j \pmod{rk/q}$;
(2) *The differences of the edges* $\{c_i, c_{i-1}\}$ $(1 \le i \le p)$ *are all distinct*;
(3) $c_p - c_0 = t \cdot rk/q$, *where* $\gcd(t, q) = 1$; *and*
(4) $c_{ip+j} = c_j + i \cdot t \cdot rk/q$ *where* $0 \le j \le p - 1$ *and* $0 \le i \le q - 1$.

It should be noticed that in Proposition 2.7, the m-cycle C can be viewed as a base cycle and the set $\{C + i \mid i \in Z_{rk/q}\}$ forms a cycle (rk/q)-orbit of C in $K_{r(k)}$. To simplify, C will be denoted by $C = [c_0 = 0, c_1, \dots, c_{p-1}]_{t \cdot rk/q}$, and we denote the set of partial differences $\pm\{(c_i - c_{i-1}) \mid 1 \le i \le p\}$ of C by ∂C.

Consider, for instance, the 8-cycle $C = (0, 15, 14, 29, 28, 43, 42, 1) = [0, 15]_{14}$ in $K_{7(8)}$ with weight 2 (i.e., $\partial C = \pm\{1, 15\}$) and index 14, and the set $\{C, C + 1, \dots, C + 13\}$ forms a cycle 14-orbit of C in $K_{7(8)}$.

Given a set $D = \{C_1, \dots, C_t\}$ of m-cycles, the list of differences from D is defined as the union of the multisets $\partial C_1, \dots, \partial C_t$, i.e., $\partial D = \bigcup_{i=1}^{t} \partial C_i$.

The next result is simple but important and will be used later.

Theorem 2.8 *A set D of m-cycles with vertices in Z_{rk} is a set of base cycles of a cyclic m-cycle system of $K_{r(k)}$ if and only if $\partial D = Z_{rk} \setminus \pm\{0, r, \dots, \lfloor k/2 \rfloor r\}$.*

3 Cyclic 4-cycle systems

Theorem 3.1 *The complete multipartite graph G can be decomposed into* 4*-cycles cyclically if and only if G is balanced and* 4*-sufficient.*

Proof (Necessity) Since G can be decomposed cyclically, it follows from Lemma 2.1 that G must be a regular graph. Hence, G is a balanced complete multipartite graph $K_{r(k)}$ for some positive integers r and k. Now, if k is even, then clearly the degree of every vertex of G is even and $4||E(G)|$. On the other hand, if k is odd, then r must be odd in order that each vertex of G is of even degree. Moreover, $4||E(G)|$ implies that $r \equiv 1 \pmod 8$. Therefore, we have that G is 4-sufficient.

(Sufficiency) By virtue of Theorem 2.8, it suffices to prove that there is a set D of base cycles in $K_{r(k)}$ so that $\partial D = Z_{rk} \setminus \pm\{0, r, \ldots, \lfloor k/2 \rfloor r\}$. We break the proof into two cases depending on whether k is even or odd.

Case 1. k is even.

(1) $k \equiv 0 \pmod 4$, say $k = 4p$.

For $i \in Z_p$ and $j \in Z_r^*$, let $C_{i,j} = [0, j + ir]_{2pr}$. Clearly, $\partial C_{i,j} = \pm\{j + ir, (2p - i)r - j\}$. Therefore, $\{C_{i,j}\}$ is a set of base cycles we need.

(2) $k \equiv 2 \pmod 4$ and $r \equiv 0 \pmod 2$, say $k = 4p + 2$.

Again, for $i \in Z_p$ and $j \in Z_r^*$, let $C_{i,j} = [0, j + ir]_{(2p+1)r}$. Moreover, let $C = (0, (2p + 1)r/2, (2p + 1)r, 3(2p + 1)r/2)$, and $C_t = [0, t + pr]_{(2p+1)r}$ for $t \in Z_{r/2}^*$. Then $\partial C \cup \{\partial C_t\} = \pm\{1 + pr, 2 + pr, \ldots, r - 1 + pr\}$. Hence, $\{C_{i,j}\} \cup \{C\} \cup \{C_t\}$ consists of a set of base cycles.

(3) $k \equiv 2 \pmod 4$ and $r \equiv 1 \pmod 2$, say $k = 4p + 2$.

For $i \in Z_p$ and $j \in Z_r^*$, let $C_{i,j} = [0, j + ir]_{(2p+1)r}$ and $C_t = [0, t + pr]_{(2p+1)r}$ for $t \in Z_{(r+1)/2}^*$. Since $(\cup \partial C_{i,j}) \cup (\cup \partial C_t) = Z_{rk} \setminus \pm\{0, r, \ldots, (2p + 1)r\}$, $\{C_{i,j}\} \cup \{C_t\}$ forms a set of base cycles.

Case 2. k is odd and $r \equiv 1 \pmod 8$, say $k = 2h + 1$ and $r = 8q + 1$.

For $i \in Z_h$ and $j \in Z_{2q}$, let $C_{i,j} = (0, 4j + 1 + ir, 8j + 5 + 2ir, 4j + 2 + ir)$, and let $C_t = (0, 4t + 1 + hr, 8t + 5 + 2hr, 4t + 2 + hr)$ for $t \in Z_q$. Since $\partial C_{i,j} = \pm\{4j + 1 + ir, 4j + 2 + ir, 4j + 3 + ir, 4j + 4 + ir\}$ and $\partial C_t = \pm\{4t + 1 + hr, 4t + 2 + hr, 4t + 3 + hr, 4t + 4 + hr\}$, we have a set $\{C_{i,j}\} \cup \{C_t\}$ of base cycles for the cycle system. □

Now, we are ready for the packings with cyclic 4-cycles. We shall classify the maximum cyclic m-cycle packings of $K_{r(k)}$ with leave into two cases: Odd and Even according as the value of order of $K_{r(k)}$ is odd or even.

4 Maximum cyclic 4-cycle packings of $K_{r(k)}$ of odd order

Since there exists a cyclic 4-cycle system of $K_{r(k)}$ whenever k is odd and $r \equiv 1$ (mod 8), here we consider the remaining cases. That is, when k is odd and $r \equiv 3$, 5, or 7 (mod 8), no cyclic 4-cycle system of $K_{r(k)}$ exists.

The following consequence indicates the possible leave of a maximum cyclic 4-cycle packing of $K_{r(k)}$ and will be utilized repeatedly in this section. Given a maximum cyclic 4-cycle packing of $K_{r(k)}$, $\boldsymbol{P}$, let $D(\boldsymbol{P})$ be the set of distinct differences in $\boldsymbol{P}$.

Lemma 4.1 *Suppose that* $rk \equiv 1 \pmod 2$ *and* $W(K_{r(k)}) \equiv i \pmod 4$ *with* $i \in Z_4^*$ *and let* $\boldsymbol{P}$ *be a maximum cyclic* 4*-cycle packing of*$K_{r(k)}$. *Then the leave of a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *is the circulant graph* $X(Z_{rk}, \Omega)$ *with* $\Omega = Z_{rk} \backslash \pm\{0, r, \dots, \lfloor k/2 \rfloor r\} \backslash D(\boldsymbol{P})$.

Proof Since the value of rk is odd, each cycle orbit in the maximum cyclic 4-cycle packing of $K_{r(k)}$, $\boldsymbol{P}$, must be full, and since $W(K_{r(k)}) \equiv i \pmod 4$ with $i \in Z_4^*$, it implies that there are exactly i distinct differences not occurring in $\boldsymbol{P}$. It follows that the leave is precisely the circulant graph $X(Z_{rk}, \Omega)$ with $\Omega = Z_{rk} \backslash \pm\{0, r, \dots, \lfloor k/2 \rfloor r\} \backslash D(\boldsymbol{P})$. □

Throughout this paper whenever we say that a circulant graph $X(Z_{rk}, \pm\{a\})$ is a Hamiltonian cycle of $K_{r(k)}$, it implies that $\gcd(rk, a) = 1$. Given a connection set $\Omega = \pm\{a_1, \dots, a_t\}$, let $\Omega \oplus i = \pm\{a_1 + i, \dots, a_t + i\}$.

We are now in a position to prove our main result with odd order, which is divided into the following five propositions.

Proposition 4.2 *If* $r \equiv 3 \pmod 8$ *and* $k \equiv 3 \pmod 4$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave* 3 *Hamiltonian cycles.*

Proof Since $W(K_{r(k)}) \equiv 3 \pmod 4$, by Lemma 4.1, the leave is a circulant graph $X(Z_{rk}, \Omega)$ with $|\Omega| = 3$. Let $\Omega_1^* = \pm\{1\}$, $\Omega_2^* = \pm\{2\}$, and $\Omega_3^* = \pm\{(rk-1)/2\}$. Then the circulant graph $X(Z_{rk}, \Omega)$ is the union of $X(Z_{rk}, \Omega_i^*)$ for $i = 1, 2, 3$. Note that by Lemma 2.4, the circulant graphs $X(Z_{rk}, \Omega_i^*)$ $(1 \le i \le 3)$ are all Hamiltonian cycles in $K_{r(k)}$. The remaining proof are split into two cases according to whether $r = 3$ or $r > 3$. Let $r = 8t + 3$ and $k = 4s + 3$.

Case 1. $r = 3$.

Let $\Omega_i = \pm\{4, 5, 7, 8\} \oplus 6i$ for $i = 0, \dots, s-1$. Note that by Lemma 2.5, there exists a cyclic 4-cycle system of $X(Z_{rk}, \Omega_i)$ for each i. It is easy to check that the union of the circulant graphs $X(Z_{rk}, \Omega_i)$ $(0 \le i \le s-1)$ consists of a maximum cyclic 4-cycle packing of $K_{r(k)}$.

Case 2. $r > 3$.

The connection sets are given as the following:

$$\Omega_i = \pm\{r+1, r+2, 2r+1, 2r+2\} \oplus 2ir, \quad i = 0, \dots, s-1;$$

$$\Omega_{i,j} = \pm\{3, 4, 5, 6\} \oplus 4i \oplus rj, \quad i = 0, \dots, 2t-1 \text{ and } j = 0, \dots, 2s;$$

$$\Omega_i' = \pm\{2sr + r + 1, 2sr + r + 2, 2sr + r + 3, 2sr + r + 4\} \oplus 4i,$$
$$i = 0, \dots, t-1.$$

Again, a routine verification shows that the union of the circulant graphs $X(Z_{rk}, \Omega_i)$, $X(Z_v, \Omega_{i,j})$, and $X(Z_v, \Omega_i')$ forms a maximum cyclic 4-cycle packing of $K_{r(k)}$. □

Proposition 4.3 *If* $r \equiv 3 \pmod 8$ *and* $k \equiv 1 \pmod 4$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave a Hamiltonian cycle.*

Proof Analogously, the leave is a Hamiltonian cycle, i.e., the circulant graph $X(Z_{rk}, \pm\{(rk-1)/2\})$. Also, we divide the proof into two cases according to whether $r = 3$ or $r > 3$. Let $r = 8t + 3$ and $k = 4s + 1$.

Case 1. $r = 3$.

Let $\Omega_i = \pm\{1, 2, 4, 5\} \oplus 6i$ for $i = 0, \dots, s-1$ and the union of the circulant graphs $X(Z_{rk}, \Omega_i)$ $(0 \le i \le s-1)$ is a maximum cyclic 4-cycle packing of $K_{r(k)}$.

Case 2. $r > 3$.

The connection sets are defined by

$$\Omega_i = \pm\{1, 2, r+1, r+2\} \oplus 2ir, \quad i = 0, \dots, s-1;$$

$$\Omega_{i,j} = \pm\{3, 4, 5, 6\} \oplus 4i \oplus rj, \quad i = 0, \dots, 2t-1 \text{ and } j = 0, \dots, 2s-1;$$

$$\Omega'_i = \pm\{2sr+1, 2sr+2, 2sr+3, 2sr+4\} \oplus 4i, \quad i = 0, \dots, t-1.$$

An easy computation shows that the union of the circulant graphs $X(Z_{rk}, \Omega_i)$, $X(Z_v, \Omega_{i,j})$, and $X(Z_v, \Omega'_i)$ forms a maximum cyclic 4-cycle packing of $K_{r(k)}$. □

Proposition 4.4 *If* $r \equiv 5 \pmod 8$ *and* $k \equiv 1 \pmod 2$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave* 2 *Hamiltonian cycles.*

Proof The Hamiltonian cycles are the circulant graphs $X(Z_{rk}, \Omega^*_i = \pm\{i\})$ for $i = 1, 2$. Let $r = 8t + 5$. Then, by a similar argument, it suffices to provide the connection sets which are the following:

$$\Omega = \pm\{3, 4, \lfloor k/2 \rfloor r + 1, \lfloor k/2 \rfloor r + 2\};$$

$$\Omega_i = \pm\{5, 6, 7, 8\} \oplus 4i, \quad i = 0, \dots, 2t-1;$$

$$\Omega_{i,j} = \pm\{r+1, r+2, r+3, r+4\} \oplus 4i \oplus rj,$$
$$i = 0, \dots, 2t \text{ and } j = 0, \dots, (k-5)/2;$$

$$\Omega'_i = \pm\{\lfloor k/2 \rfloor r + 3, \lfloor k/2 \rfloor r + 4, \lfloor k/2 \rfloor r + 5, \lfloor k/2 \rfloor r + 6\} \oplus 4i,$$
$$i = 0, \dots, t-1.$$
□

Proposition 4.5 *If* $r \equiv 7 \pmod 8$ *and* $k \equiv 1 \pmod 4$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave* 3 *Hamiltonian cycles.*

Proof For $i = 1, 2, 3$, the circulant graphs $X(Z_{rk}, \Omega^*_i)$ with $\Omega^*_1 = \pm\{1\}$, $\Omega^*_2 = \pm\{2\}$, and $\Omega^*_3 = \pm\{\lfloor rk/2 \rfloor\}$ are the Hamiltonian cycles. Let $r = 8t + 7$ and $k = 4s + 1$.

Then, with the connection sets defined below, we have the proof.

$$\Omega_i = \pm\{r+1, r+2, 2r+1, 2r+2\} \oplus 2ir, \quad i = 0, \dots, s-1;$$

$$\Omega_{i,j} = \pm\{3, 4, 5, 6\} \oplus 4i \oplus rj, \quad i = 0, \dots, 2t \text{ and } j = 0, \dots, 2s-1;$$

$$\Omega'_i = \pm\{2sr+3, 2sr+4, 2sr+5, 2sr+6\} \oplus 4i, \quad i = 0, \dots, t-1.$$
□

Proposition 4.6 *If* $r \equiv 7 \pmod 8$ *and* $k \equiv 3 \pmod 4$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave a Hamiltonian cycle.*

Proof The Hamiltonian cycle is the circulant graph $X(Z_{rk}, \pm\{\lfloor rk/2\rfloor\})$. Let $r = 8t+7$ and $k = 4s+3$.

The connection sets are given by

$$\Omega_i = \pm\{1, 2, r+1, r+2\} \oplus 2ir, \quad i = 0, \ldots, s;$$
$$\Omega_{i,j} = \pm\{3, 4, 5, 6\} \oplus 4i \oplus rj, \quad i = 0, \ldots, 2t \text{ and } j = 0, \ldots, 2s;$$
$$\Omega'_i = \pm\{(2s+1)r+3, (2s+1)r+4, (2s+1)r+5, (2s+1)r+6\} \oplus 4i,$$
$$i = 0, \ldots, t-1.$$ □

5 Maximum cyclic 4-cycle packings of $K_{r(k)}$ of even order

By Theorem 3.1, it suffices to consider the cases when r is even and k is odd. This implies that the leave of a maximum cyclic 4-cycle packing of $K_{r(k)}$ must include a 1-factor of $K_{r(k)}$ since the degree of each vertex in $K_{r(k)}$ is odd. It is clear that the 1-factor must be the circulant graph $X(Z_{rk}, \pm\{rk/2\})$.

Lemma 5.1

(1) *If* $r \equiv 4 \pmod 8$ *and* $k \equiv 3 \pmod 4$ *or* $r \equiv 2 \pmod 4$ $(r > 2)$ *and* $k \equiv 7 \pmod 8$, *then the leave of a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *is the union of a* 1*-factor and the circulant graph* $X(Z_{rk}, \pm\{a\})$ *with a even.*

(2) *If* $r \equiv 2 \pmod 4$ *and* $k \equiv 3 \pmod 8$, *then the leave of a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *is the union of a* 1*-factor and the circulant graph* $X(Z_{rk}, \pm\{a\})$ *with a odd.*

(3) *If* $r \equiv 2 \pmod 4$ *and* $k \equiv 5 \pmod 8$, *then the leave of a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *is the union of a* 1*-factor and the circulant graph* $X(Z_{rk}, \pm\{a, b\})$ *with* a, b *odd.*

Proof We consider only the case when $r \equiv 4 \pmod 8$ and $k \equiv 3 \pmod 4$ and leave the remainder to the reader. An easy computation shows that the numbers of odd and even differences in $K_{r(k)} \backslash X(Z_{rk}, \pm\{rk/2\})$ are both odd, say α and β, and $\alpha - \beta \equiv 2 \pmod 4$. Set $\alpha - \beta = 4p + 2$, $p \geq 0$. By virtue of Lemma 2.3, the weight of any 4-cycle C is a divisor of 4, i.e., $W(C) = 1, 2$, or 4. Note that if $W(C) = 2$, then two distinct differences in C must have the same parity since its index $rk/2$ is even.

In order to obtain a maximum cyclic 4-cycle packing of $K_{r(k)}$, it is necessary to use $\beta - 1$ odd differences and $\beta - 1$ even differences to construct 4-cycles having weight 4, and then construct p 4-cycles each having weight 4 and all odd differences.

Next, consider the remaining graph, that is, the circulant graph $X(Z_{rk}, \Omega = \pm\{a, b, c, d\})$, where exactly one of elements in Ω, say a, is even and the rest is all odd. The proof then follows from Lemmas 2.4 and 2.6 by constructing the circulant graphs $X(Z_{rk}, \pm\{b\})$ with $b = rk/4$ and $X(Z_{rk}, \pm\{c, d\})$ with $c + d = rk/2$. □

Remark that by Lemma 2.4, the circulant graph $X(Z_{rk}, \pm\{a\})$ with rk and a both even is not a Hamiltonian cycle. It is not difficult to see that if $r = 2$ and $k \equiv 7 \pmod 8$, then the leave of a maximum cyclic 4-cycle packing of $K_{2(k)}$ is the union of a 1-factor and the circular graph $X(Z_{rk}, \pm\{a, b, c\})$ with a, b, c odd. Moreover, the leave of a maximum cyclic 4-cycle packing of $K_{r(k)}$ is a 1-factor whenever $r \equiv 0 \pmod 8$ and $k \equiv 1 \pmod 2$, $r \equiv 2 \pmod 4$ and $k \equiv 1 \pmod 8$, or $r \equiv 4 \pmod 8$ and $k \equiv 1 \pmod 4$. Since the technique of proofs is analogous, in what follows, we shall list the connection sets without the details of verification. Furthermore, since the consequences in Lemma 5.1 will be repeatedly used later, for simplicity, we will not mention these again.

Proposition 5.2 *If* $r \equiv 0 \pmod 8$ *and* $k \equiv 1 \pmod 2$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave a* 1*-factor.*

Proof Let $r = 8t$. The proof is divided into 4 cases according to whether $k \equiv 3, 5, 7$, or $1 \pmod 8$.

Case 1. $k \equiv 3 \pmod 8$, say $k = 8s + 3$.

$$\Omega_{a,i} = \pm\{1 + i, rk/2 - 1 - i\}, \quad i = 0, 1, 2;$$

$$\Omega_{b,i} = \pm\{4, 5, 6, 7\} \oplus 4i, \quad i = 0, \ldots, 2t - 2;$$

$$\Omega_{c,i} = \pm\{(4s + 1)r + 1, (4s + 1)r + 2, (4s + 1)r + 3, (4s + 1)r + 4\} \oplus 4i,$$
$$i = 0, \ldots, t - 2;$$

$$\Omega_{d,i} = \pm\{r + 1, 2r + 1, 3r + 1, 4r + 1\} \oplus 4ir, \quad i = 0, \ldots, s - 1;$$

$$\Omega_{i,j} = \pm\{r + 2, r + 3, 2r + 2, 2r + 3\} \oplus 2i \oplus 2jr,$$
$$i = 0, \ldots, 4t - 2 \text{ and } j = 0, \ldots, 2s - 1.$$

Case 2. $k \equiv 5 \pmod 8$, say $k = 8s + 5$.

$$\Omega_a = \pm\{rk/4\};$$

$$\Omega_b = \pm\{rk/4 - 1, rk/4 + 1\};$$

$$\Omega_{c,i} = \pm\{(2s + 1)r + 1, (2s + 1)r + 2, rk/4 + 2, rk/4 + 3\} \oplus 2i,$$
$$i = 0, \ldots, t - 2;$$

$$\Omega_{d,i} = \pm\{rk/4 + 2t, rk/4 + 2t + 1, rk/4 + 2t + 2, rk/4 + 2t + 3\} \oplus 4i,$$
$$i = 0, \ldots, t - 1;$$

$$\Omega_e = \pm\{1, rk/2 - 1\};$$

$$\Omega_f = \pm\{2, 3, (4s + 2)r + 1, (4s + 2)r + 2\};$$

$$\Omega_{g,i} = \pm\{4, 5, 6, 7\} \oplus 4i, \quad i = 0, \ldots, 2t - 2;$$

$$\Omega_h = \pm\{(4s + 2)r + 3, (4s + 2)r + 4, (4s + 2)r + 5, (4s + 2)r + 6\} \oplus 4i,$$
$$i = 0, \ldots, t - 2;$$

$$\Omega'_i = \pm\{r + 1, 2r + 1, (2s + 2)r + 1, (2s + 3)r + 1\} \oplus 2ir, \quad i = 0, \ldots, s - 1;$$

$$\Omega_{i,j} = \pm\{r + 2, r + 3, (2s + 2)r + 2, (2s + 2)r + 3\} \oplus 2i \oplus rj,$$
$$i = 0, \ldots, (r - 4)/2 \text{ and } j = 0, \ldots, 2s - 1.$$

Case 3. $k \equiv 7 \pmod 8$, say $k = 8s + 7$.

$$\Omega_i = \pm\{1, r+1, 2r+1, 3r+1\} \oplus 4ir, \quad i = 0, \ldots, s;$$
$$\Omega'_i = \pm\{2, 3, r+2, r+3\} \oplus 2ir, \quad i = 0, \ldots, 2s+1;$$
$$\Omega_{i,j} = \pm\{4, 5, 6, 7\} \oplus 4i \oplus rj, \quad i = 0, \ldots, 2t-2 \text{ and } j = 0, \ldots, 4s+2;$$
$$\Omega''_i = \pm\{(4s+3)r+4, (4s+3)r+5, (4s+3)r+6, (4s+3)r+7\} \oplus 4i,$$
$$i = 0, \ldots, t-2.$$

Case 4. $k \equiv 1 \pmod 8$, say $k = 8s + 9$.

$$\Omega_a = \pm\{rk/4\};$$
$$\Omega_b = \pm\{rk/4 - 1, rk/4 + 1\};$$
$$\Omega_{c,i} = \pm\{(2s+2)r+1, (2s+2)r+2, rk/4+2, rk/4+3\} \oplus 2i,$$
$$i = 0, \ldots, t-2;$$
$$\Omega_{d,i} = \pm\{rk/4+2t, rk/4+2t+1, rk/4+2t+2, rk/4+2t+3\} \oplus 4i,$$
$$i = 0, \ldots, t-1;$$
$$\Omega_{e,i} = \pm\{1, r+1, (2s+3)r+1, (2s+4)r+1\} \oplus 2ir, \quad i = 0, \ldots, s;$$
$$\Omega_f = \pm\{(2s+1)r+2, (2s+1)r+3, 4(s+1)r+2, 4(s+1)r+3\};$$
$$\Omega_{g,i} = \pm\{(2s+1)r+4, (2s+1)r+5, (2s+1)r+6, (2s+1)r+7\} \oplus 4i,$$
$$i = 0, \ldots, 2t-2;$$
$$\Omega_h = \pm\{4(s+1)r+4, 4(s+1)r+5, 4(s+1)r+6, 4(s+1)r+7\} \oplus 4i,$$
$$i = 0, \ldots, t-2;$$
$$\Omega_{i,j} = \pm\{2, 3, (2s+3)r+2, (2s+3)r+3\} \oplus 2i \oplus rj,$$
$$i = 0, \ldots, 4t-2 \text{ and } j = 0, \ldots, 2s.$$

□

When $r \equiv 2 \pmod 8$ and $k \equiv 1, 3, 5$, or $7 \pmod 8$, the proof will be split into two cases according to whether $r = 2$ or $r > 2$.

Proposition 5.3 *If* $r \equiv 2 \pmod 8$ *and* $k \equiv 1 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave a* 1*-factor.*

Proof Let $k = 8s + 9$.

Case 1. $r = 2$.

$$\Omega_i = \pm\{1, 3, 5, 7\} \oplus 8i, \quad i = 0, \ldots, s.$$

Case 2. $r > 2$, say $r = 8t + 10$.

$$\Omega_i = \pm\{1, r+1, 2r+1, 3r+1\} \oplus 4ir, \quad i = 0, \ldots, s;$$
$$\Omega_{i,j} = \pm\{2, 3, 4, 5\} \oplus 4i \oplus rj, \quad i = 0, \ldots, 2t+1 \text{ and } j = 0, \ldots, 4s+3;$$

$$\Omega'_i = \pm\{4(s+1)r+1, 4(s+1)r+2, 4(s+1)r+3, 4(s+1)r+4\} \oplus 4i,$$
$$i = 0, \ldots, t. \qquad \square$$

Proposition 5.4 *If* $r \equiv 2 \pmod 8$ *and* $k \equiv 3 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and a Hamiltonian cycle.*

Proof The Hamiltonian cycle is the circulant graph $X(Z_{rk}, \pm\{1\})$. Let $k = 8s+3$.

Case 1. $r = 2$.

$$\Omega_i = \pm\{3, 5, 7, 9\} \oplus 8i, \quad i = 0, \ldots, s-1.$$

Case 2. $r > 2$, say $r = 8t + 10$.

$$\Omega_{a,i} = \pm\{2, 3, 4, 5\} \oplus 4i, \quad i = 0, \ldots, 2t+1;$$
$$\Omega_{b,i} = \pm\{r+1, 2r+1, 3r+1, 4r+1\} \oplus 4ir, \quad i = 0, \ldots, s-1;$$
$$\Omega_{i,j} = \pm\{r+2, r+3, r+4, r+5\} \oplus 4i \oplus rj,$$
$$i = 0, \ldots, 2t+1 \text{ and } j = 0, \ldots, 4s-1;$$
$$\Omega_{c,i} = \pm\{(4s+1)r+1, (4s+1)r+2, (4s+1)r+3, (4s+1)r+4\} \oplus 4i,$$
$$i = 0, \ldots, t. \qquad \square$$

Proposition 5.5 *If* $r \equiv 2 \pmod 8$ *and* $k \equiv 5 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and* 2 *Hamiltonian cycles.*

Proof The 2 Hamiltonian cycles are respectively the circulant graphs $X(Z_{rk}, \pm\{1\})$ and $X(Z_{rk}, \pm\{rk/2-2\})$. Let $k = 8s+5$.

Case 1. $r = 2$.

$$\Omega_i = \pm\{3, 5, 7, 9\} \oplus 8i, \quad i = 0, \ldots, s-1.$$

Case 2. $r > 2$, say $r = 8t + 10$.

$$\Omega_{a,i} = \pm\{2, 3, 4, 5\} \oplus 4i, \quad i = 0, \ldots, 2t+1;$$
$$\Omega_{b,i} = \pm\{r+1, 2r+1, 3r+1, 4r+1\} \oplus 4ir, \quad i = 0, \ldots, s-1;$$
$$\Omega_{i,j} = \pm\{r+2, r+3, r+4, r+5\} \oplus 4i \oplus rj,$$
$$i = 0, \ldots, 2t+1 \text{ and } j = 0, \ldots, 4s-1;$$
$$\Omega_{c,i} = \pm\{(4s+1)r+1, (4s+1)r+2, (4s+1)r+3, (4s+1)r+4\} \oplus 4i,$$
$$i = 0, \ldots, 2t+1;$$
$$\Omega = \pm\{(4s+2)r-1, (4s+2)r+1, rk/2-3, rk/2-1\};$$
$$\Omega_{d,j} = \pm\{(4s+2)r+2, (4s+2)r+3, (4s+2)r+4, (4s+2)r+5\} \oplus 4i,$$
$$i = 0, \ldots, t-1. \qquad \square$$

Proposition 5.6

(1) *If* $r=2$ *and* $k\equiv 7 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and* 3 *Hamiltonian cycles.*
(2) *If* $r\equiv 2 \pmod 8$ $(r>2)$ *and* $k\equiv 7 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and* 2 $(rk/2)$*-cycles.*

Proof Let $k=8s+7$.

(1) The 3 Hamiltonian cycles are respectively the circulant graphs $X(Z_{rk},\pm\{1\})$, $X(Z_{rk},\pm\{4s+3\})$, and $X(Z_{rk},\pm\{8s+5\})$.

$$\Omega_i=\pm\{3,5,4s+5,4s+7\}\oplus 4i,\quad i=0,\dots,s-1.$$

(2) By virtue of Lemma 2.4, the circulant graph $X(Z_{rk},\pm\{2\})$ is the union of 2 $(rk/2)$-cycles. Set $r=8t+10$.

$$\Omega_a=\pm\{3,rk/2-3\};$$
$$\Omega_b=\pm\{4,5,rk/2-2,rk/2-1\};$$
$$\Omega_{c,i}=\pm\{6,7,8,9\}\oplus 4i,\quad i=0,\dots,2t;$$
$$\Omega_{d,i}=\pm\{(4s+3)r+2,(4s+3)r+3,(4s+3)r+4,(4s+3)r+5\}\oplus 4i,$$
$$i=0,\dots,t-1;$$
$$\Omega_{e,i}=\pm\{1,r+1,2r+1,3r+1\}\oplus 4ir,\quad i=0,\dots,s;$$
$$\Omega_{i,j}=\pm\{r+2,r+3,r+4,r+5\}\oplus 4i\oplus rj,$$
$$i=0,\dots,2t+1 \text{ and } j=0,\dots,4s+1.$$
□

Proposition 5.7 *If* $r\equiv 4 \pmod 8$ *and* $k\equiv 1$ *or* $5 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave a* 1*-factor.*

Proof We break the proof into two cases according to whether $k\equiv 1$ or $5 \pmod 8$. Let $r=8t+4$.

Case 1. $k\equiv 1 \pmod 8$, say $k=8s+9$.

$$\Omega=\pm\{rk/4\};$$
$$\Omega_{a,i}=\pm\{1,r+1,(2s+3)r+1,(2s+4)r+1\}\oplus 2ir,\quad i=0,\dots,s;$$
$$\Omega_{b,i}=\pm\{2,3,(2s+2)r+1,(2s+2)r+2\}\oplus 2i,\quad i=0,\dots,t-1;$$
$$\Omega_{c,i}=\pm\{2t+2,2t+3,(2s+2)r+2t+2,(2s+2)r+2t+3\}\oplus 2i,$$
$$i=0,\dots,3t;$$
$$\Omega_{i,j}=\pm\{r+2,r+3,(2s+3)r+2,(2s+3)r+3\}\oplus 2i\oplus rj,$$
$$i=0,\dots,4t \text{ and } j=0,\dots,2s;$$
$$\Omega_{d,i}=\pm\{(4s+4)r+2,(4s+4)r+3,(4s+4)r+4,(4s+4)r+5\}\oplus 4i,$$
$$i=0,\dots,t-1.$$

Case 2. $k \equiv 5 \pmod 8$, say $k = 8s + 5$.

$$\Omega_a = \pm\{1, rk/2 - 1\};$$
$$\Omega_b = \pm\{rk/4\};$$
$$\Omega_{c,i} = \pm\{2, 3, (2s+1)r + 1, (2s+1)r + 2\} \oplus 2i, \quad i = 0, \ldots, t-1;$$
$$\Omega_{d,i} = \pm\{2t+2, 2t+3, (2s+1)r + 2t + 2, (2s+1)r + 2t + 3\} \oplus 2i,$$
$$i = 0, \ldots, 3t;$$
$$\Omega_{e,i} = \pm\{r+1, 2r+1, (2s+2)r + 1, (2s+3)r + 1\} \oplus 2ir, \quad i = 0, \ldots, s-1;$$
$$\Omega_{i,j} = \pm\{r+2, r+3, (2s+2)r + 2, (2s+2)r + 3\} \oplus 2i \oplus rj,$$
$$i = 0, \ldots, 4t \text{ and } j = 0, \ldots, 2s-1;$$
$$\Omega_{f,i} = \pm\{(4s+2)r + 1, (4s+2)r + 2, (4s+2)r + 3, (4s+2)r + 4\} \oplus 4i,$$
$$i = 0, \ldots, t-1.$$
□

Proposition 5.8 *If* $r \equiv 4 \pmod 8$ *and* $k \equiv 3$ *or* $7 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and* 2 $(rk/2)$*-cycles.*

Proof The circulant graph $X(Z_{rk}, \pm\{2\})$ is the union of 2 $(rk/2)$-cycles. The proof is split into two cases depending on whether $k \equiv 3$ or $7 \pmod 8$. Let $r = 8t + 4$.

Case 1. $k \equiv 3 \pmod 8$.

Subcase 1.1. $k = 3$.

$$\Omega = \pm\{1, 3r/2 - 1\};$$
$$\Omega' = \pm\{3r/4\};$$
$$\Omega_{a,i} = \pm\{3, 4, 3r/4 + 1, 3r/4 + 2\} \oplus 2i, \quad i = 0, \ldots, t-1;$$
$$\Omega_{b,i} = \pm\{2t+3, 2t+4, 2t+5, 2t+6\} \oplus 4i, \quad i = 0, \ldots, t-1;$$
$$\Omega_{c,i} = \pm\{r+1, r+2, r+3, r+4\} \oplus 4i, \quad i = 0, \ldots, t-1.$$

Subcase 1.2. $k > 3$, say $k = 8s + 3$.

$$\Omega = \pm\{1, rk/2 - 1\};$$
$$\Omega' = \pm\{rk/4\};$$
$$\Omega_{a,i} = \pm\{r-1, 2r-1, (2s+2)r - 1, (2s+3)r - 1\} \oplus 2ir, \quad i = 0, \ldots, s-1;$$
$$\Omega_{b,i} = \pm\{3, 4, 5, 6\} \oplus 4i, \quad i = 0, \ldots, 2t-1;$$
$$\Omega_{c,i} = \pm\{(4s+1)r + 1, (4s+1)r + 2, (4s+1)r + 3, (4s+1)r + 4\} \oplus 4i,$$
$$i = 0, \ldots, t-1;$$
$$\Omega_{i,j} = \pm\{r+1, r+2, (2s+1)r + 1, (2s+1)r + 2\} \oplus 2i \oplus rj,$$
$$i = 0, \ldots, 4t \text{ and } j = 0, \ldots, 2s-2;$$

$\Omega_{d,i} = \pm\{2sr+1, 2sr+2, 4sr+1, 4sr+2\} \oplus 2i, \quad i = 0, \ldots, 3t;$

$\Omega_{e,i} = \pm\{2sr+6t+4, 2sr+6t+5, 4sr+6t+3, 4sr+6t+4\} \oplus 2i,$
$i = 0, \ldots, t-1.$

Case 2. $k \equiv 7 \pmod 8$, say $k = 8s+7$.

$\Omega_a = \pm\{1, 3, (4s+3)r-1, (4s+3)r+1\};$

$\Omega_b = \pm\{rk/4\};$

$\Omega_{c,i} = \pm\{4, 5, 6, 7\} \oplus 4i, \quad i = 0, \ldots, 2t-1;$

$\Omega_{d,i} = \pm\{(4s+3)r+2, (4s+3)r+3, (4s+3)r+4, (4s+3)r+5\} \oplus 4i,$
$i = 0, \ldots, t-1;$

$\Omega_{e,i} = \pm\{r+1, 2r+1, (2s+2)r+1, (2s+3)r+1\} \oplus 2ir, \quad i = 0, \ldots, s-1;$

$\Omega_{i,j} = \pm\{r+2, r+3, (2s+2)r+2, (2s+2)r+3\} \oplus 2i \oplus rj,$
$i = 0, \ldots, 4t$ and $j = 0, \ldots, 2s-1;$

$\Omega_{f,i} = \pm\{(2s+1)r+1, (2s+1)r+2, (4s+2)r+1, (4s+2)r+2\} \oplus 2i,$
$i = 0, \ldots, 3t;$

$\Omega_{g,i} = \pm\{rk/4+1, rk/4+2, (4s+2)r+6t+3, (4s+2)r+6t+4\} \oplus 2i,$
$i = 0, \ldots, t-1.$ □

Proposition 5.9 *If* $r \equiv 6 \pmod 8$ *and* $k \equiv 1 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave a* 1*-factor.*

Proof Let $r = 8t+6$ and $k = 8s+9$.

$\Omega = \pm\{(rk-2)/4, (rk+2)/4\};$

$\Omega_{a,i} = \pm\{r-1, 2r-1, 3r-1, 4r-1\} \oplus 4ir, \quad i = 0, \ldots, s;$

$\Omega_{i,j} = \pm\{1, 2, 3, 4\} \oplus 4i \oplus rj, \quad i = 0, \ldots, 2t$ and $j = 0, \ldots, 2s+1;$

$\Omega'_{i,j} = \pm\{(2s+3)r+1, (2s+3)r+2, (2s+3)r+3, (2s+3)r+4\} \oplus 4i \oplus rj,$
$i = 0, \ldots, 2t$ and $j = 0, \ldots, 2s;$

$\Omega_{b,i} = \pm\{(2s+2)r+1, (2s+2)r+2, (rk+2)/4+1, (rk+2)/4+2\} \oplus 2i,$
$i = 0, \ldots, t-1;$

$\Omega_{c,i} = \pm\{(rk+2)/4+2t+1, (rk+2)/4+2t+2, (4s+4)r+1,$
$(4s+4)r+2\} \oplus 2i, \quad i = 0, \ldots, 2t.$ □

Proposition 5.10 *If* $r \equiv 6 \pmod 8$ *and* $k \equiv 3 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and a Hamiltonian cycle.*

Proof The Hamiltonian cycle is the circulant graph $X(Z_{rk}, \pm\{1\})$. Let $r = 8t + 6$ and $k = 8s + 3$.

$$\Omega = \pm\{(rk-2)/4, (rk+2)/4\};$$

$$\Omega_{a,i} = \pm\{r+1, 2r+1, (2s+1)r+1, (2s+2)r+1\} \oplus 2ir, \quad i = 0, \dots, s-1;$$

$$\Omega_{i,j} = \pm\{2, 3, (2s+1)r+2, (2s+1)r+3\} \oplus 2i \oplus rj, \quad i = 0, \dots, 4t+1 \text{ and } j = 0, \dots, 2s-1;$$

$$\Omega_{b,i} = \pm\{2sr+2, 2sr+3, (rk+2)/4+1, (rk+2)/4+2\} \oplus 2i, \quad i = 0, \dots, t-1;$$

$$\Omega_{c,i} = \pm\{2sr+2t+2, 2sr+2t+3, (4s+1)r+1, (4s+1)r+2\} \oplus 2i, \quad i = 0, \dots, 2t.$$

□

Proposition 5.11 *If* $r \equiv 6 \pmod 8$ *and* $k \equiv 5 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and* 2 *Hamiltonian cycles.*

Proof The Hamiltonian cycles are the circulant graphs $X(Z_{rk}, \pm\{rk/2-2\})$ and $X(Z_{rk}, \pm\{(rk-2)/4\})$. Let $r = 8t+6$ and $k = 8s+5$.

$$\Omega = \pm\{1, rk/2-1\};$$

$$\Omega_{a,i} = \pm\{r+1, 2r+1, (2s+2)r+1, (2s+3)r+1\} \oplus 2ir, \quad i = 0, \dots, s-1;$$

$$\Omega_{i,j} = \pm\{2, 3, 4, 5\} \oplus 4i \oplus rj, \quad i = 0, \dots, 2t \text{ and } j = 0, \dots, 2s;$$

$$\Omega'_{i,j} = \pm\{(2s+2)r+2, (2s+2)r+3, (2s+2)r+4, (2s+2)r+5\} \oplus 4i \oplus rj, \quad i = 0, \dots, 2t \text{ and } j = 0, \dots, 2s-1;$$

$$\Omega_{b,i} = \pm\{(2s+1)r+1, (2s+1)r+2, (rk-2)/4+1, (rk-2)/4+2\} \oplus 2i, \quad i = 0, \dots, t-1;$$

$$\Omega_{c,i} = \pm\{(rk-2)/4+2t+1, (rk-2)/4+2t+2, (4s+2)r+1, (4s+2)r+2\} \oplus 2i, \quad i = 0, \dots, 2t-1;$$

$$\Omega_d = \pm\{(rk-2)/4+6t+1, (rk-2)/4+6t+2, (rk-2)/4+6t+3, (rk-2)/4+6t+4\}.$$

□

Proposition 5.12 *If* $r \equiv 6 \pmod 8$ *and* $k \equiv 7 \pmod 8$, *then there exists a maximum cyclic* 4*-cycle packing of* $K_{r(k)}$ *with leave the union of a* 1*-factor and* 2 $(rk/2)$*-cycles.*

Proof The circulant graph $X(Z_{rk}, \pm\{rk/2-1\})$ is the union of 2 $(rk/2)$-cycles. Let $r = 8t+6$ and $k = 8s+7$.

$$\Omega_{a,i} = \pm\{1, r+1, 2r+1, 3r+1\} \oplus 4ir, \quad i = 0, \dots, s;$$

$$\Omega_{i,j} = \pm\{2, 3, 4, 5\} \oplus 4i \oplus rj, \quad i = 0, \dots, 2t \text{ and } j = 0, \dots, 4s+2;$$

$$\Omega_{b,i} = \pm\{(4s+3)r+2, (4s+3)r+3, (4s+3)r+4, (4s+3)r+5\} \oplus 4i,$$
$$i = 0, \ldots, t-1. \qquad \square$$

6 Conclusion

Combining Lemma 2.1, Theorem 3.1, and Propositions 4.2 to 4.6 and 5.2 to 5.12, we have the following main result.

Theorem 6.1 *There exists a maximum cyclic* 4*-cycle packing of the balanced complete multipartite graph* $K_{r(k)}$ *with leave* L *where* L *is obtained as follows*:

(1) L *is the empty set if* k *is even or* k *is odd and* $r \equiv 1 \pmod 8$;
(2) L *is* 3 *Hamiltonian cycles if* $r \equiv 3 \pmod 8$ *and* $k \equiv 3 \pmod 4$ *or* $r \equiv 7 \pmod 8$ *and* $k \equiv 1 \pmod 4$;
(3) L *is* 2 *Hamiltonian cycles if* $r \equiv 5 \pmod 8$ *and* $k \equiv 1 \pmod 2$;
(4) L *is a Hamiltonian cycle if* $r \equiv 3 \pmod 8$ *and* $k \equiv 1 \pmod 4$ *or* $r \equiv 7 \pmod 8$ *and* $k \equiv 3 \pmod 4$;
(5) L *is a* 1*-factor if* $r \equiv 0 \pmod 8$ *and* $k \equiv 1 \pmod 2$, $r \equiv 2 \pmod 4$ *and* $k \equiv 1 \pmod 8$, *or* $r \equiv 4 \pmod 8$ *and* $k \equiv 1 \pmod 4$;
(6) L *is the union of a* 1*-factor and* 3 *Hamiltonian cycles if* $r = 2$ *and* $k \equiv 7 \pmod 8$;
(7) L *is the union of a* 1*-factor and* 2 *Hamiltonian cycles if* $r \equiv 2 \pmod 4$ *and* $k \equiv 5 \pmod 8$;
(8) L *is the union of a* 1*-factor and a Hamiltonian cycle if* $r \equiv 2 \pmod 4$ *and* $k \equiv 3 \pmod 8$; *and*
(9) L *is the union of a* 1*-factor and* 2 $(rk/2)$*-cycles if* $r \equiv 2 \pmod 4$ (>2) *and* $k \equiv 7 \pmod 8$ *or* $r \equiv 4 \pmod 8$ *and* $k \equiv 3 \pmod 4$.

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