1. Introduction

1.1 Research background and motivation

Queuing delay models in the manufacturing system have attracted great attention from many researchers. These models constitute a majority of the manufacturing lead time. Lead time is generally defined as the duration of job arrival and job completion, which can be divided into several manufacturing processes such as pre-production waiting time, setup time, and processing time. The manufacturer should provide the estimated lead time to customers so that they will have an idea how long it will take to complete their job order. Thus, lead time has a significant impact on manufacturing schedule because it is extensively used to determine the target completion date of a customer's order even during sales negotiation (Gordon et al. [1]).

Two measures can be used in evaluating the performance of a lead time estimate: first is accuracy and the second is precision. Accuracy is a measure of how closely the estimated lead time agrees with the actual lead time, that is to say, lead time prediction. If lead time prediction can be significantly improved, then the costs incurred between actual completion times and committed delivery dates can be reduced based on lead time prediction. These costs, reflected in various earliness- and tardiness-related measures, are considered as key indicators of performance in the negotiation of the due date with the customer (Enns [2]). Precision is a measure of the variability of the lead time, which can be used to establish appropriate safety stock levels and safety lead times as protection in the uncertainties in demand and supply (e.g., Chopra et al. [3], Wang and Hill [4], Ruiz-Torres and Mahmoodi [5], Van Kampen et al. [6]). In general, if variability in the lead time estimates is low, then better estimates can be obtained and more accurate due dates can be given.

For a single finite-capacity machine that can process several product types, the setup is necessary to adjust current machine settings in order to complete a particular job. It was reported that 20% or even as much as 50% loss of available capacity may arise from setup activities (Liu and Chang [7], Trovinger and Bohn [8]). Market demand, uncertainties in job arrival time and types of product, make the setup estimation time very complicated. Hence setting output targets may have significant errors compared with actual levels due to the possible heavy loss of capacity and the difficulty in calculating required setup time. This gap cannot be disregarded. At least three additional factors affect the magnitude of required setup time. They are as follows:

I. The total arrival rate of all types of incoming jobs.

II. The combination of the arrival rates of various types of jobs.

III. The dispatching rule applied in selecting the next job for processing by the machine.

If a lengthy setup is required in product type change, then the setup activities may exhaust the machine capacity and increase the work-in-process (WIP). An overloaded system with large WIP would cause long waiting time, resulting in extended lead time and late job deliveries.

Based on the discussion, the difficulty in calculating the required setup time leads to the challenge of developing a system capable of accurately estimating waiting time. Therefore, this research aims to develop a relatively simple yet accurate model of queuing delays with setup time.

1.2 Research goals

Setup activities may cause wastage in machine capacity and extend job lead time. Uncertainties in job arrival time and product type can complicate the calculation of required setup time and the setting of output target. Thus, this research aims to estimate the lead time for each product type using the First-In First-Out (FIFO) rule to facilitate performance evaluation from the customer's perspective.

The family-based scheduling rule (FSR), which consecutively handles jobs belonging to the same product family, and which require the same machine setting, can be used to reduce setup frequency and amount of setup time. Both expected setup time and service time are estimated by FSR analytic model to efficiently evaluate the effects on capacity saving. The effect of FSR in reducing setup time and capacity loss is explored further by comparing the results with those of the FIFO rule.

1.3 Research domain and assumptions

In order to simplify the problem, this research focuses on the production system with a single finite-capacity machine providing several different product types of services for jobs, and is built under an environment with the assumptions:

I. The inter-arrival time of a specific product type of job is distributed independently and exponentially.

II. Jobs are serviced by FIFO rule and FSR for the next job.

III. A setup time is incurred between jobs of different product.

IV. Sum of the processing time and the setup time is treated as the service time.

1.4 Research process

Figure 1-1 is the flowchart of this research.

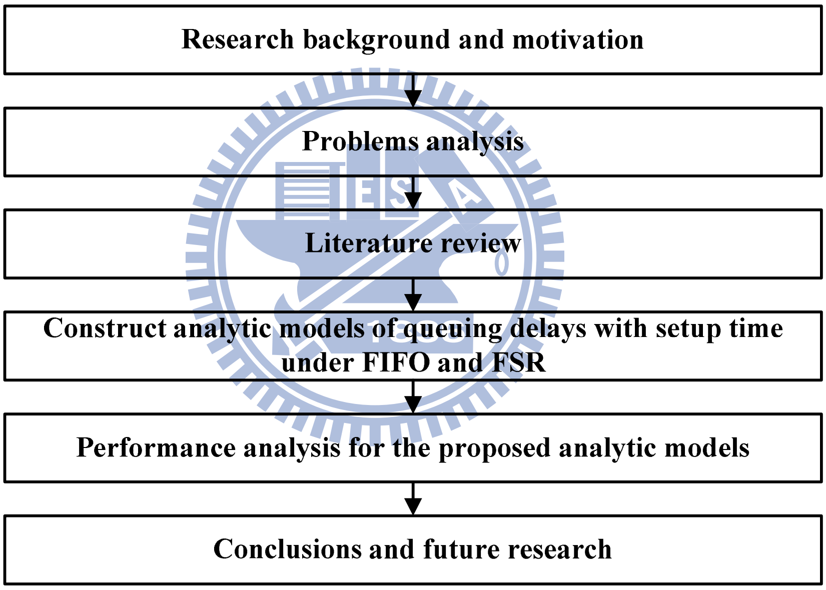

Figure 1-1 Research Process

2. Literature Review

As stated in Section 1.2, this research aims to estimate the lead time for each product type with setup time using FIFO rule and to evaluate the effects on capacity saving by FSR. Related literatures of analytic models are discussed in this chapter. Specifically, Section 2.1 discusses an analytic model under FIFO, and Section 2.2 describes an analytic model under FSR. Section 2.3 reviews other related work.

2.1 Analytic model under FIFO

The estimate of setups under the FIFO has been investigated in several studies. The FIFO rule describes the principle of processing in queue by a first-come first-served process—what job comes in first is dispatched first, what job comes in next waits until the first is finished.

2.1.1 Analytic model of setups under FIFO

Probability of setups under FIFO

Missbauer [9] defined the probability of setups for producing several types of services with a single machine underlying the FIFO rule. In Missbauer [9], each product type consists of several individual jobs and is assumed to be an independent Poisson distribution for the number of arriving jobs of the product type j with parameter λ_j, where $j=1, 2, \ldots, J$, and J is the number of product type. The probability λ_j/λ is used to represent the probability that an arriving job is product type j, and the probability $(1-\lambda_j/\lambda)$ is used to represent the probability that an arriving job is from a different product type j. For a single machine, setup occurs when each of the two consecutive jobs belongs to different product types.

According to Missbauer [9], the probability that a setup is necessarily a product type j job under FIFO rule on a single machine is equal to the probability of a product type j job arriving at the system multiplied by the probability that this arriving job has a different product type as its predecessor, as shown in Equation (2-1).

$$
\begin{aligned}
&\Pr[\text{Setup is necessary for product type } j \text{ job under FIFO rule}] \\
&= \Pr[\text{Type } j \text{ job arrives}] \times \\
&\quad \Pr[\text{Setup is necessary for an arriving job} \mid \text{an arriving job is of type } j] \\
&= \Pr[\text{Type } j \text{ job arrives}] \times \\
&\quad \Pr[\text{The type of the predecessor is different from type } j] \\
&= \frac{\lambda_j}{\lambda}\left(1 - \frac{\lambda_j}{\lambda}\right)
\end{aligned}
\tag{2-1}
$$

The calculation result only depends on the job arrival rate; hence, it is constant if the job arrival rates are known.

Chern and Liu [10] extended Missbauer's research [9] to parallel machines scheduling under FIFO. The probability of requiring setup of a job for parallel machines under FIFO must consider the number of jobs existing in the system. This can be divided into three parts: 1.) there is no job in the system; 2.) there are n jobs in the system and $n < m$; and 3.) there are n jobs in the system and $n \geqq m$, where m is the number of parallel machines. First, if there is no job in the system and a job of product type j arrived at the system, then a setup is necessary if no jobs belonging to the product type j are among the last jobs being processed on the parallel machines. Second, if there are n jobs in the system and $n < m$, setup would be necessary for this newly arrived job while no jobs belonging to the product type j are among the last jobs being processed on the idle machines. Finally, if there are n jobs in the system and $n \geqq m$, setup would not be necessary for this newly arrived job while the last job being processed on the assigned machine belongs to product type j. Thus, the probability that a setup is necessary for product type j job in parallel machines under FIFO can be shown as Equation (2-2).

$$
\begin{aligned}
&\Pr[\text{Setup is necessary for an arriving job} \mid \text{an arriving job is of type } j] \\
&= p_0\left(1 - \frac{\lambda_j}{\lambda}\right)^m + \sum_{n=1}^{m-1} p_n \left(1 - \frac{\lambda_j}{\lambda}\right)^{m-n} + \sum_{n=m}^{\infty} p_n \left(1 - \frac{\lambda_j}{\lambda}\right)
\end{aligned}
\tag{2-2}
$$

where p_0 and p_n are the probabilities that there are no jobs in the system and there are n ($n \geqq 1$) jobs in the system.

Studies on setup time estimation are quite limited. The same setup time for each product type is assumed to simplify the model by Missbauer [9].

Service time under FIFO

Karmarkar et al. [11] and Kuik and Tielemans [12][13][14] studied the service time probability distribution of batches with setup time for a single-machine system by using FIFO.

In their model, the batch service time of a specific product type is defined as the sum of the batch processing time and the setup time, which can be shown as Equation (2-3).

$$x_j = \tau_j + ST_{batch_j} \tag{2-3}$$

where x_j represents the batch service time of product type j, τ_j represents the setup time of product type j, and ST_{batch_j} represents the batch service time of product type j. The corresponding probability is given by the relative arrival rates of the batches, which can be written as Equation (2-4).

$$\Pr\left[X_j = x_j\right] = \frac{\lambda_j}{\lambda} \tag{2-4}$$

where λ_j represents the batch arrival rate of product type j. Thus, the moments of the service time of batches can be calculated based on this probability distribution.

Waiting time under FIFO

The methodology of previous studies on waiting time estimation was applied to the queuing analysis because their properties and results are relatively well understood and are available for many important system characteristics (Cheng and Jiang [15], Enns [2][16], Baykasoğlu et al. [17]). In these models, job arrivals are randomly generated by the Poisson process and processing times follow the negative exponential distribution. However, setup times are ignored or are assumed as zero. Meanwhile, literature on queue analysis with setup times studied the expected waiting time of batches with setup time for single-machine system by applying the Pollaczek–Khintchine formula for an $M/G/1$ queue in Equation (2-5) (Karmarkar et al. [11], Kuik and Tielemans [12][13][14], Koo et al. [18][19], Missbauer [20]).

$$E\left[W_q\right]_{M/G/1} = \frac{\lambda E\left[ST_{FIFO}^2\right]}{2\left(1-\lambda E\left[ST_{FIFO}\right]\right)} \tag{2-5}$$

where λ is the total arrival rate, $E[ST_{FIFO}]$ and $E[ST_{FIFO}^2]$ are the first and second moments of the service time under FIFO, respectively, and $E[W_q]_{M/G/1}$ is the expected waiting time for an $M/G/1$ queue.

Karmarker et al. [11] presented a multi-item heuristic batching aimed to minimize queuing delays, and developed the upper and lower bounds on the optimal batch size. Kuik and Tielemans [12] studied the upper bound for the setup utilization at optimal batch sizes by queuing delay batching model for multiple products. Kuik and Tielemans [13] discussed the analytical expressions for the optimal multi-item batch sizes and the minimal expected queuing delay. Kuik and Tielemans [14] also investigated the relationship between batch size and lead time at the low levels of machine utilization for multi-items. Koo et al. [18][19] presented a linear search algorithm to find the optimal throughput rate and batch size with single product and multi-products. Missbauer [20] analyzed the impact of lot sizing on the functional relationships between WIP level and flow time or capacity utilization in lot sizing models. In addition to the batch waiting time, Kuik and Tielemans [21] also investigated the variance of batch waiting time with setup time by applying the $M/G/1$ queuing system in Equation (2-6) because these are important for the safety stock or safety time determination and due date assignment.

$$Var\left[W_q\right]_{M/G/1} = \frac{\lambda E\left[ST_{FIFO}^3\right]}{3\left(1-\lambda E\left[ST_{FIFO}\right]\right)} + E\left[W_q\right]_{M/G/1}^2 \tag{2-6}$$

Among the abovementioned literature that studied batch waiting time, an analytical model for estimating waiting time was modeled by an $M/G/1$ queue because the superposition of the batch arrival processes for different products can be approximated by a Poisson process (Jönsson and Silver [22], Zipkin [23], Karmarkar [24], Tielemans and Kuik [25]). Thus, the expected value and variance of waiting time for a batch can be obtained easily if the first and second moments of the service time are given. Batch service time is defined as the sum of batch processing time and setup time. As such, an analytical expression in standard form for the expected value and variance of waiting time by queuing theory can be applied if the batch process is assumed. However, it cannot respond completely to the impact of setup time to waiting time by substituting the first and second moments of the service time into the formula for an $M/G/1$ queue. There are at least three additional factors relevant to the magnitude of the required setup time: the total arrival rate of all incoming jobs, the mix of the jobs arrival rates of various types, and the dispatching rule used in queuing jobs on the machine. The influence of these factors on setup time and the relationship between setup time, service time, and waiting time should be discussed to estimate the waiting time in the setup time.

2.2 Analytic model under FSR

2.2.1 Family-based scheduling (FSR) rule

FSR has been assessed in several studies. According to FSR, when a job arrives at the system and there already exists one or more jobs belonging to the same product type, then this arriving job is jointed with the same product type jobs and these jobs become a batch. The jobs within each batch and each batch are dispatched under FIFO rule. When a job arrives at the system and if there are no jobs belonging to the same product type, then this arriving job is dispatched under the FIFO rule.

Missbauer [9] proved that setup time could be saved using FSR for the single-machine system. Jensen et al. [26] considered the case of the semiconductor testing facility with parallel machines and dynamic job arrival; FSR has been credited for reducing setup time in batch production industries. Chern and Liu [10] proposed FSR to dispatch wafer lots in the photolithography stage of the wafer fabrication system. Kannan and Lyman [27] examined the combined effect of lot splitting and FSR in a manufacturing cell by simulation and showed that FSR can reduce the negative impact on flow time by lot splitting. Nomden et al. [28] refined the existing rules for family-based scheduling by including data on upcoming jobs and showed that flow time performance can be improved significantly. Therefore, FSR not only has an effect on saving machine setups, it also indirectly reduces job flow time.

In the foregoing investigations, except for Missbauer [9] and Chern and Liu [10], the simulation approach is applied to evaluate the effect of FSR in reducing setups and flow time. Numerous computer runs are needed to produce reliable results; thus, this method is both time-consuming and costly.

2.2.2 Analytic model of setups under FSR

Probability of setups under FSR

To simplify the setups for single machine under FSR, the system state can be divided into two parts according to the number of jobs processed in the system. First, assume that no jobs are in the system and there is a job arriving; thus, a setup is needed if the product type of the last job that has been processed completely on the machine is different from the arriving job. Second, assume that there are one or more jobs in the system and a job is coming; thus, a setup is needed if the product types of jobs among the jobs in the system are all different from the arriving job. Therefore, the probability that a setup is necessary for product type j job in a single machine under FSR by Missbauer [9] can be shown as Equation (2-7).

$$\Pr\left[\text{Setup is necessary for product type } j \text{ job for single machine under FSR}\right]$$
$$= \frac{\lambda_j}{\lambda}\left[p_0\left(1-\frac{\lambda_j}{\lambda}\right) + \sum_{n=1}^{\infty} p_n \left(1-\frac{\lambda_j}{\lambda}\right)^n \right] \tag{2-7}$$

where p_0 and p_n are the probabilities that there are no jobs in the system and there are n ($n \geqq 1$) jobs in the system, respectively.

Chern and Liu [10] extended the result by Missbauer [9] to a more complicated system with parallel machines and multiple job re-entrances under FSR. Based on FSR, the probability of requiring setup for jobs in parallel machines has to consider the number of jobs existing in the system. These can be divided into three parts: there is no job in the system, there are n jobs in the system and $n < m$, and there are n jobs in the system and $n \geqq m$, where m is the number of parallel machines. First, if there are no jobs in the system and the product type j job arrived, then a setup is necessary if no jobs belonging to the product type j are among the last jobs processed on the parallel machines. Second, if there are n jobs in the system and $n < m$, setup would be necessary for this newly arrived job while there is no job belonging to the product type j among the last jobs processed on the idle machines. Finally, if there are n jobs in the system and $n \geqq m$, setup would be necessary for this newly arrived job while all last jobs being processed do not belong to product type j. Thus, the probability that a setup is necessary for product type j job in parallel machines under FSR can be shown as Equation (2-8).

$$\Pr\left[\text{Setup is necessary for product type } j \text{ job with parallel machines under FSR}\right]$$
$$= p_0\left(1-\frac{\lambda_j}{\lambda}\right)^m + \sum_{n=1}^{m-1} p_n \left(1-\frac{\lambda_j}{\lambda}\right)^{m-n} + \sum_{n=m}^{\infty} p_n \left(1-\frac{\lambda_j}{\lambda}\right)^n \tag{2-8}$$

Results show that the setup time can be saved by FSR as compared with FIFO and the utilization rate can be increased by FSR as compared with FIFO.

Number of setups under FSR

Vieira et al. [29] considered the number of setups to be observed in a time interval for producing several types of services with a single machine, in which the FSR is used to dispatch the jobs. Each product type consists of n_j individual jobs in the time interval. Vieira et al. [29] assumed an independent Poisson arrival of jobs with product type j and arrival rate λj. The probability of product type j job arriving in the time interval P_r is equal to $\Pr[T_j \leqq P_r]$, where T_j is the inter-arrival time of product type j jobs and is an exponential distribution with parameter λ_j, and j=1, 2, …, J. They simplified the setup probability to constantly be $(1-1/J)$ without considering the effect of the product type of an arriving job. Therefore, the expected total number of setup for a single machine depends on probabilities $\Pr[T_j \leqq P_r]$ and $(1-1/J)$, which can be shown as Equation (2-9).

$$N_s = \left(1 - \frac{1}{J}\right) \sum_{j=1}^{J} \Pr\left[T_j \le P_r\right] \tag{2-9}$$

With regard to parallel machine, Vieira et al. [30] also extended the result to the parallel-machine system in case of machine failure. The expected total number of setups for parallel machine is given by Equation (2-10).

$$N_p = \left(1 - \frac{N_m}{J}\right) \sum_{j=1}^{J} \Pr\left[T_j \le P_r\right] \tag{2-10}$$

where N_m represents the number of parallel machines. Based on FSR, N_m product types will not require a setup and $(1-N_m/J)$ represents the probability of requiring setups.

According to the discussion, Vieira et al. [29][30] did not consider the possible differences in the arrival rates of various job types; instead, they simplified the setup probability by categorizing all types of arriving jobs as a constant.

2.3 Other related work

Rossetti and Stanford [31] considered the aforementioned problem on the single machine and presented a case study that examined the use of heuristics to estimate the expected number of setups. Bagherpour et al. [32] estimated the sequence-dependent setup time for the single machine using the fuzzy approach. However, their fuzzy estimation was significantly lower compared with the simulated results. Estimation error of the fuzzy setup time cannot be controlled in an acceptable range.

2.4 Summary

The primary focus of this research is to conduct an analytic methodology. New closed-form analytical expressions of queuing delays with setup time for the case of a Poisson arrival process by FIFO and FSR rules are constructed in the following sections.

3. FIFO analytic model for estimating lead time

In this chapter, an analytical expression of queuing delays with setup time for the case of a Poisson arrival process by FIFO rule is discussed to estimate lead time. Two parts are derived: one is the estimator of the lead time, and the other is the variance of the lead time

3.1 Problem Analysis

In this research, a single finite-capacity machine for processing several product types of jobs is considered. We assume that the jobs arrive at the system in a time interval (0, RT], which is a positive integer. Beginning time is labeled as 0. We also assume that the number of newly arrived jobs of product type j follows the Poisson distribution for the arrival of jobs at rate λ_j, thus there arrives $n_j = \lambda_j RT$ product type j jobs at time interval (0, RT]. In addition, the arrival time of the i^{th} arrived job of product type j, T_{ij}, is the gamma distribution with parameters i and λ_j under the given Poisson assumption. Thus, the probability of the i^{th} job of product type j arrived at the system in the time interval (0, RT] can be shown as Equation (3-1).

$$\Pr\left[T_{ij} \le RT\right] = \int_0^{RT} \frac{\left(\lambda_j\right)^i}{\Gamma(i)} \left(t_{ij}\right)^{i-1} e^{-\lambda_j t_{ij}} dt_{ij} \tag{3-1}$$

where $j = 1, 2, \ldots, J$, $i = 1, 2, \ldots, n_j$, and J represents the number of product type. The probability of i^{th} job of product type j arriving at the system but out of time interval (0, RT] is denoted by $\Pr[T_{ij} > RT] = 1 - \Pr[T_{ij} \leqq RT]$.

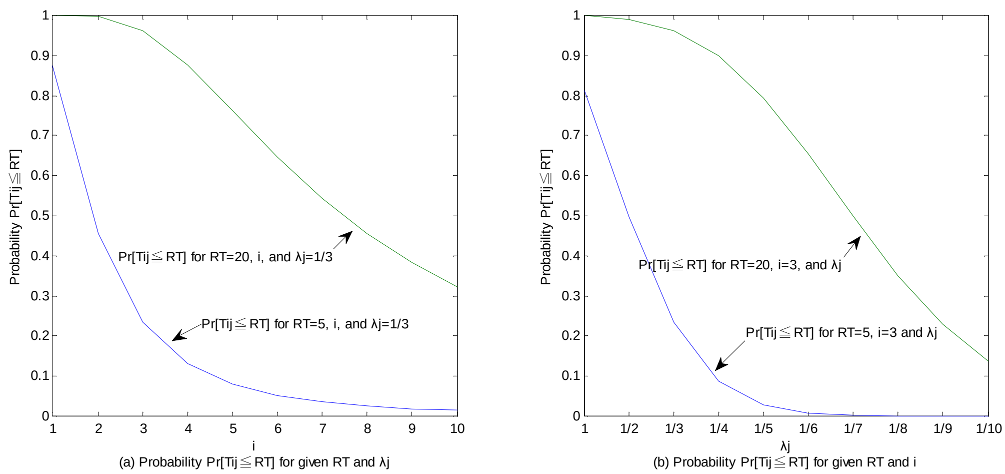

Figure 3-1 Plot of probability $\Pr[T_{ij} \leqq RT]$

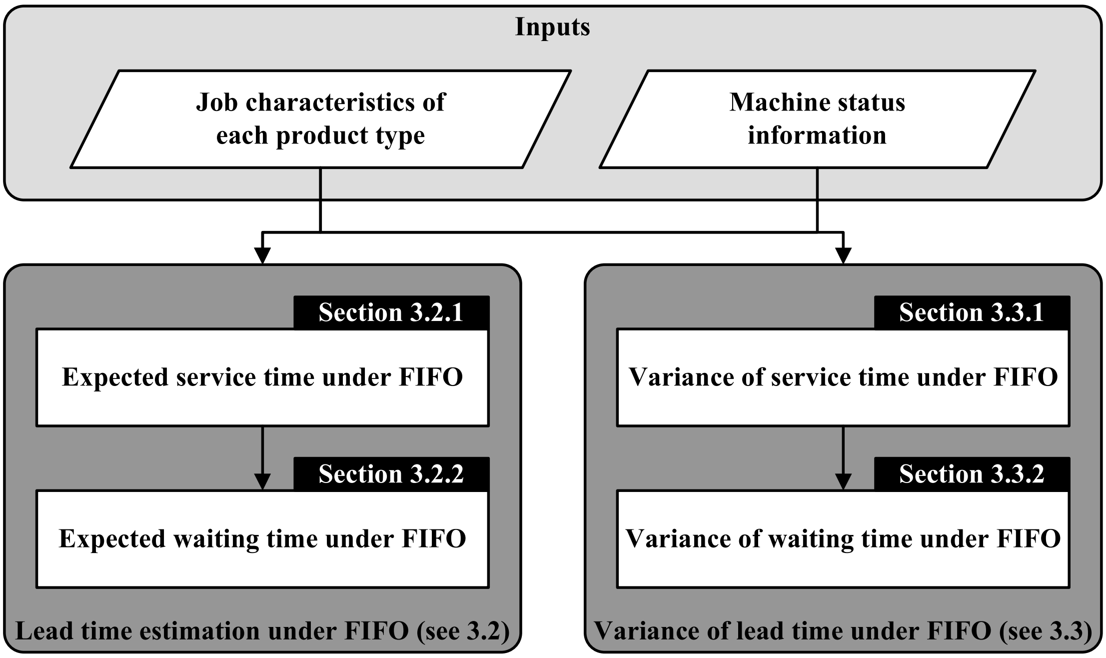

Figure 3-2 Estimator and variance of lead time under FIFO

We know that the probability $\Pr[T_{ij} \leqq RT]$ depends on the length of the time interval (0, RT], the sequence of the arrived jobs (i), and the job arrival rate (λ_j). Figure 3-1 (a) plots the probability $\Pr[T_{ij} \leqq RT]$ against the sequence of the arrived jobs (i) when $\lambda_j = 1/3$ for $RT = 5$, 20, and Figure 3-1 (b) plots the probability $\Pr[T_{ij} \leqq RT]$ against the job arrival rate (λ_j) when $i = 3$ for $RT = 5$, 20. From Figure 3-1 (a) to Figure 3-1 (b), we see that the probability $\Pr[T_{ij} \leq RT]\big|_{RT=20}$ is larger then the probability $\Pr[T_{ij} \leq RT]\big|_{RT=5}$. This means that the possibility of jobs arrived at the system is increasing when the length of the time interval is longer, that is, more jobs will arrive at the system to process in the time interval (0, RT]. In Figure 3-1 (a), the probability $\Pr[T_{ij} \leqq RT]$ is decreasing as the sequence of the arrived jobs (i) climbing up for given RT and job arrival rate, which implies that the possibility of the later jobs arrived at the system is decreasing progressively. Moreover, the probability $\Pr[T_{ij} \leqq RT]$ is also decreasing as the job arrival rate is smaller for given parameters RT and i in Figure 3-1 (b). To decrease arrival rate means increasing inter-arrival time, thus the possibility of jobs arrived at the system in the time interval (0, RT] becomes lower.

When a jobs in specific type of demand, arrives at the system, it will enter the waiting line and be serviced in a FIFO rule. A job will be processed after a period time of waiting. Before starting the processing of next job, the machine should be occupied and proceeds setup if a new setup is required. Therefore, not only job processing but also machine setup consumes the capacity of machine, which indicates that total setup time is related to the utilization rate of capacity and is the cost must be paid due to the sharing of a machine for providing various services. The lead time of jobs can be determined with the summation of the waiting time of jobs in queue and the service time of jobs on the machine. The waiting time of job relates to the expected service time of some specific number of jobs already arrived. The service time of jobs is defined by adding the estimated setup time and the processing time of jobs. Since the service time is independent of its waiting time in queue, and then the variance of lead time is expressed as the sum of variances of the service time and the waiting time in queue. Both the estimator and the variance of lead time are shown in Figure 3-2 and are presented as follows.

3.2 Estimator of lead time of jobs under FIFO

The job's lead time for each product type is defined as the time that is spent in the system from their arrival until the job completed its processing on the machine. Therefore, the lead time of jobs for each product type can be estimated with the summation of the average waiting time of jobs in queue and the average service time of jobs, where the service time of jobs includes both its setup time and its processing time. Let $LT_{j,FIFO}$, $W_{q,j,FIFO}$, and $ST_{j,FIFO}$ be the lead time, waiting time and service time of the product type j job under FIFO, hence the estimate of lead time of the product type j job under FIFO, $E[LT_{j,FIFO}]$, is given by Equation (3-2).

$$E\left[LT_{j,FIFO}\right] = E\left[W_{q,j,FIFO}\right] + E\left[ST_{j,FIFO}\right] \tag{3-2}$$

where $E[W_{q,j,FIFO}]$ and $E[ST_{j,FIFO}]$ represent the expected waiting time and expected service time of product type j jobs in queue under FIFO, and $j = 1, 2, \ldots, J$. The calculations of $E[W_{q,j,FIFO}]$ and $E[ST_{j,FIFO}]$ are presented as follows.

3.2.1 Expected service time under FIFO

The service time of jobs is defined by the time that is spent on the machine from the setup process until the jobs completed its processing. Therefore, the service time of jobs equals its setup time plus its processing time, where the processing time of jobs depends on its product type and the setup time depends on the product type change between any two consecutive jobs. Let $ST_{ij,FIFO}$ be the service time of the i^{th} job of product type j under FIFO. The probability mass function of $ST_{ij,FIFO}$ can be classified three parts according to the case of arriving time and the setup case and is explained in the below discussion.

In the first part, the service time of the i^{th} job of product type j would be zero because this job not arrived in time interval (0, RT] with the probability ($1-\Pr[T_{ij} \leqq RT]$). In the second part, the i^{th} job of product type j arrived within the time interval (0, RT] but no setup is required, then the service time of this arrived job would be equal to its processing time with the probability $\Pr[T_{ij} \leqq RT](1-P_{s,j,FIFO})$. In the last part, the i^{th} job of product type j arrived in the time interval (0, RT] and a setup is necessary, which implies that the arrived job is different from its predecessor. The different product type of the predecessor r has the fixed finite number ($J-1$) of classifications with the probabilities λ_r/λ^c and the setup time for product type j job after product type r job is defined as s_{rj}. Thus, the service time of the i^{th} job of product type j would be equal to its processing time plus its setup time with the probability $\Pr[T_{ij} \leqq RT]P_{s,j,FIFO}(\lambda_r/\lambda^c)$, where $r=1, 2, \ldots, J$, $r \neq j$, $\lambda^c = \sum_{r=1,r\neq j}^{J} \lambda_r$, and the probability $P_{s,j,FIFO}$ represents the probability that a setup is required given a job of product type j arrived at the system in time interval (0, RT], at where jobs are dispatched by the FIFO rule. It can be seen that the probability mass function of $ST_{ij,FIFO}$ can be showed as Equation (3-3).

$$P\left(ST_{ij,FIFO} = st_{ij}\right) = \begin{cases} 1-\Pr\left[T_{ij} \leq RT\right], & \text{if } st_{ij} = 0 \\ \Pr\left[T_{ij} \leq RT\right]\left(1-P_{s,j,FIFO}\right), & \text{if } st_{ij} = pt_j \\ \Pr\left[T_{ij} \leq RT\right]P_{s,j,FIFO}\dfrac{\lambda_r}{\lambda^c}, & \text{if } st_{ij} = pt_j + s_{rj}, \\ & r = 1, 2, \ldots, J,\ r \neq j \end{cases} \tag{3-3}$$

In Equation (3-3), pt_j is the processing time of product type j jobs. The expected service time of the product type j jobs is defined by $E[ST_{j,FIFO}] = E[n_j^{-1}\sum_{i=1}^{n_j} ST_{ij,FIFO}]$ and is given by Equation (3-4) according to the probability mass function of $ST_{ij,FIFO}$.

$$\begin{aligned} E\left[ST_{j,FIFO}\right] &= E\left[n_j^{-1}\sum_{i=1}^{n_j} ST_{ij,FIFO}\right] \\ &= n_j^{-1}\sum_{i=1}^{n_j}\Pr\left[T_{ij} \leq RT\right]\left[pt_j + P_{s,j,FIFO}\sum_{\substack{r=1 \\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right] \end{aligned} \tag{3-4}$$

where $n_j = \lambda_j RT$ and j=1, 2, …, J. The result in Equation (3-4) asserts that the decomposition of the expected mean service time of the product type j jobs into two parts according to the definition of the service time results in the expected mean processing time

($n_j^{-1}\sum_{i=1}^{n_j}\Pr[T_{ij}\le RT]pt_j$) and the expected mean setup time ($n_j^{-1}\sum_{i=1}^{n_j}\Pr[T_{ij}\le RT]P_{s,j,FIFO}\sum_{r=1,r\ne j}^{J}(\lambda_r/\lambda^c)s_{rj}$) of the product type j jobs, respectively. Finally, the expected service time of jobs is defined by $E[ST_{FIFO}]=E[(\sum_{j=1}^{J}n_j)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}ST_{ij,FIFO}]$ and can be shown as equation (3-5).

$$\begin{aligned} E\left[ST_{FIFO}\right] &= E\left[\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}ST_{ij,FIFO}\right] \\ &= \left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]\left[pt_j+P_{s,j,FIFO}\sum_{\substack{r=1\\ r\ne j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right] \end{aligned} \quad (3\text{-}5)$$

3.2.1.1 Probability of requiring setups under FIFO

To solve the expected service time of jobs, the probability $P_{s,j,FIFO}$ has to be calculated in advance. Let $P_{s,ij,FIFO}$ be the probability for the i^{th} job of product type j arriving at the system in time interval (0, RT] with a setup under FIFO. If jobs not arrive at the system in the time interval (0, RT], a setup is not needed for this job. If jobs arrive at the system in the time interval (0, RT], it can be classified into two categories, "setup" and "not setup". A setup of job is not needed with the single machine according to the FIFO rule if it has the same product type as its predecessor. On the contrary, a setup of job is necessary with the single machine according to FIFO if it have the different product type as its predecessor. Therefore, the probability $P_{s,ij,FIFO}$ can be calculated with Equation (3-6).

$$P_{s,ij,FIFO}=\Pr\left[T_{ij}\le RT\right]\times P_{s,j,FIFO} \quad (3\text{-}6)$$

where the probability $P_{s,j,FIFO}$ represents the probability that a setup is required given a job of product type j arriving at the system in time interval (0, RT], at where jobs are dispatched by the FIFO rule. The calculation of $P_{s,j,FIFO}$ must consider the number of jobs already in the system, which can be partitioned into two parts: when there are no jobs in the system and when there are $n \geqq 1$ jobs in the system. Therefore, the probability $P_{s,j,FIFO}$ can be calculated by Equation (3-7).

$$P_{s,j,FIFO}=p_{0,FIFO}P_{setups,FIFO}^{n=0}+\sum_{n=1}^{\infty}p_{n,FIFO}P_{setups,FIFO}^{n\ge 1} \quad (3\text{-}7)$$

where $p_{0,FIFO}$ and $p_{n,FIFO}$ are the probabilities that there are no jobs in the system and there are

$n \geqq 1$ jobs in the system under FIFO, $P^{n=0}_{setups,FIFO}$ is the probability of an arriving job of the product type j requiring a setup under FIFO with no jobs in the system, and $P^{n\geq1}_{setups,FIFO}$ is the probability that an arriving job of the product type j needs a setups under FIFO with $n \geqq 1$ jobs in the system. The calculation of $P_{s,j,FIFO}$ is presented as follows.

First, for the case if there are no jobs in the system and the i^{th} job of product type j arrives during the time interval (0, RT], then a setup is necessary if the new arriving job is different from the product type of the last job on the machine currently being idle with the probability $(1-\lambda_j/\lambda)$, that is to say, the probability of an arriving job of product type j requiring a setup with no jobs in the system ($P^{n=0}_{setups,FIFO}$) can be rewritten as the probability that the last job on the machines currently being idle is different from the product type j and is equal to $(1-\lambda_j/\lambda)$.

Second, for the case if there are $n \geqq 1$ jobs in the system and the i^{th} job of product type j arrives at the system in the time interval (0, RT], then a setup is necessary if the new arriving job has the different product type as its predecessor. Therefore, the probability $P^{n\geq1}_{setups,FIFO}$ is also equal to $(1-\lambda_j/\lambda)$, and the probability of an arriving job of product type j not requiring a setup with $n \geqq 1$ jobs in the system ($P^{n\geq1}_{not\ setups,FIFO}$) is equal to λ_j/λ.

Thus it can be seen that the probability $P_{s,j,FIFO}$ can be rewritten as $P_{s,j,FIFO} = p_{0,FIFO}(1-\lambda_j/\lambda)+\sum_{n=1}^{\infty} p_{n,FIFO}(1-\lambda_j/\lambda)$. Therefore, the probability $P_{s,ij,FIFO}$ can be updated as Equation (3-8).

$$\begin{aligned} P_{s,ij,FIFO} &= \Pr\left[T_{ij} \leq RT\right]\left[p_{0,FIFO}\left(1-\frac{\lambda_j}{\lambda}\right)+\sum_{n=1}^{\infty} p_{n,FIFO}\left(1-\frac{\lambda_j}{\lambda}\right)\right] \\ &= \Pr\left[T_{ij} \leq RT\right]\left(1-\frac{\lambda_j}{\lambda}\right) \end{aligned} \tag{3-8}$$

According to Equation (3-8), the probability that a setup does not need for the i^{th} job of product type j under the FIFO rule can be expressed as $P_{ns,ij,FIFO} = 1 - P_{s,ij,FIFO}$.

3.2.1.2. Expected number of setups under FIFO

The i^{th} job of product type j arriving at the system in the time interval (0, RT] have two possible categories and is labelled by value 1 and value 0, in which value 1 responses "setup occurs" and value 0 responses "no setup occurs".

According to Equation (3-8), a setup of the i^{th} job of product type j takes value 1 with the probability $P_{s,ij,FIFO} = \Pr[T_{ij} \le RT](1-\lambda_j/\lambda)$ and takes value 0 with the probability $1-P_{s,ij,FIFO} = \{1-\Pr[T_{ij} \le RT]+\Pr[T_{ij} \le RT]\lambda_j/\lambda\}$. Therefore, the expected number of setups for the i^{th} job of product type j equals $E[NS_{ij,FIFO}] = \Pr[T_{ij} \le RT](1-\lambda_j/\lambda)$.

Suppose that there would arrive n_j jobs in the time interval (0, RT] for the product type j and the jobs are all independent arriving at the system, then the expected number of setups for product type j can be computed as $E[NS_{j,FIFO}] = \sum_{i=1}^{n_j} E[NS_{ij,FIFO}]$, where $n_j=\lambda_j RT$, and j=1, 2, …, J. Finally, the expected number of setups for all jobs is calculated as $E[NS_{FIFO}] = \sum_{j=1}^{J} E[NS_{j,FIFO}]$.

3.2.1.3. Expected setup time under FIFO

Since an analytical model for the number of setups can be calculated, and then the analytical model for the setup time can be developed. The definition of the setup time of jobs is the time required to change a process over from one product type to the next product type. Considering the sequence-dependent setup time for all product types in dynamic single-machine problem, in which jobs are assumed to be processed according to the FIFO rule. Let s_{rj} be the setup time for the job of product type j, in which the previous processed job belongs to product type r. The calculation of the expected setup time of jobs can be divided into three parts based on the setup cases and then is presented as follows.

In the first part, considering the case that the i^{th} job of product type j does not arrive in the time interval (0, RT], then the setup time should be zero with the probability $\Pr[T_{ij} > RT]$. In the second part, considering the i^{th} job of product type j arriving in the time interval (0, RT] and the previous job processed on the machine belonging to the same product type, the setup time would be equal to $s_{jj}=0$ and the probability is $\Pr[T_{ij} \leqq RT](1-P_{s,j,FIFO})$. In the last part, the i^{th} job of product type j arriving in the time interval (0, RT] and a setup is necessary, which imply that the job with different product type as predecessor. The different product type r has the fixed finite number $(J-1)$ of classifications, with probabilities (λ_r/λ^c) for $r=1, 2, \ldots, J$, $r \neq j$, and $\lambda^c = \sum_{r=1, r\neq j}^{J} \lambda_r$. Therefore, the setup time would be equal to s_{rj} and the probability is

$\Pr[T_{ij} \leqq RT]P_{s,j,FIFO}(\lambda_r/\lambda^c)$ while the i^{th} job of product type j arrives in the time interval (0, RT] and the earlier job processed on the machine belongs to product type r, at where $r = 1, 2, \ldots, J$, and $r \neq j$.

According to these three parts, Equation (3-9) shows the calculation of the expected setup time for the i^{th} job of product type j in the time interval (0, RT].

$$\begin{aligned} E\left[S_{ij,FIFO}\right] &= \Pr\left[T_{ij} > RT\right] \times 0 + \Pr\left[T_{ij} \le RT\right]\left(1 - P_{s,j,FIFO}\right)s_{jj} + \\ &\quad \Pr\left[T_{ij} \le RT\right]P_{s,j,FIFO}\sum_{\substack{r=1 \\ r \neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \\ &= \Pr\left[T_{ij} \le RT\right]P_{s,j,FIFO}\sum_{\substack{r=1 \\ r \neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \end{aligned} \tag{3-9}$$

Suppose that there are n_j jobs of product type j arrive in the time interval (0, RT] for each product types and the jobs are all independent arriving at the system, then the expected value of the mean setup time for product type j jobs arriving in the time interval (0, RT] can be computed as Equation (3-10).

$$E\left[S_{j,FIFO}\right] = E\left[n_j^{-1}\sum_{i=1}^{n_j}S_{ij,FIFO}\right] = n_j^{-1}\sum_{i=1}^{n_j}\Pr\left[T_{ij} \le RT\right]P_{s,j,FIFO}\sum_{\substack{r=1 \\ r \neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \tag{3-10}$$

where $n_j = \lambda_j RT$, and $j = 1, 2, \ldots, J$. The expected value of the overall mean setup time of jobs arriving in the time interval (0, RT] can be derived as Equation (3-11).

$$\begin{aligned} E\left[S_{FIFO}\right] &= E\left[\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}S_{ij,FIFO}\right] \\ &= \left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij} \le RT\right]P_{s,j,FIFO}\sum_{\substack{r=1 \\ r \neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \end{aligned} \tag{3-11}$$

3.2.2 Waiting time of jobs under FIFO

When the i^{th} job of product type j arrives at the system in the time interval (0, RT], it may be required to wait in queue until its turn to be served, and the job processing on the machine would activate the rule of FIFO. The setup process is required to switch from one product type to another before starting the processing. The waiting time of jobs occurs if the machine is still busy when the jobs arrived. This can be decomposed into two components corresponding to the number of jobs in the system (see Figure 3-3): one is the residual service time of the unfinished job on the machine and the other is the total service time of all jobs in queue ahead of the newly arrived job. The former occurs if the system is busy, whereas the latter occurs if there are at least two jobs in the system. When there is one job in the system during the arrival time of jobs, the newly arrived job has to wait in queue until one job has completed its processing on the machine, which implies that the waiting time of the newly arrived job is equal to the former. Thus, when there are n ($n \geqq 2$) jobs in the system at the arrival time of jobs, the newly arrived job has to wait in queue until n ($n \geqq 2$) jobs completed their processing on the machine. This means that the waiting time of the newly arrived job is equal to the sum of the two components mentioned earlier.

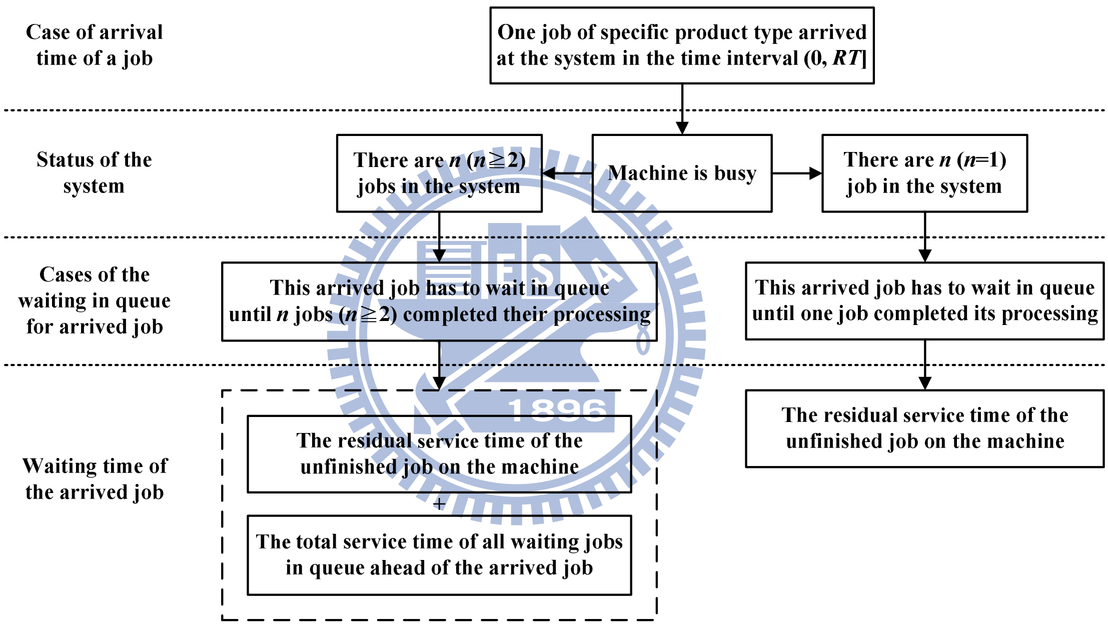

Figure 3-3 Classifications of the waiting time of the arrived job for single machine under FIFO

According to the above discussion, letting $W_{q,ij,FIFO}$ be the time of the i^{th} arrived job of product type j that has to wait in queue for the service under FIFO, then its expected value, $E[W_{q,ij,FIFO}]$, can be defined by Equation (3-12), where the notation $E[R_{ij,FIFO}]$ is used for the expected time of the i^{th} arrived job of product type j that has to wait until one unfinished job has completed its processing on the machine, while $E[TST_{ij,FIFO}]$ is used for the expected time of the i^{th} job of product type j that has to wait until all jobs waiting in queue have completed their processing ahead of the newly arrived job. Both the expected values of $R_{ij,FIFO}$ and $TST_{ij,FIFO}$ are presented as follows.

$$E\left[W_{q,ij,FIFO}\right] = E\left[R_{ij,FIFO}\right] + E\left[TST_{ij,FIFO}\right] \tag{3-12}$$

3.2.2.1 Expected residual service time of unfinished job on the machine

The remaining portion of service time for a partially served job is called the residual service time. Therefore, the residual service time of unfinished job only occurs if there are n ($n \geqq 1$) jobs in the system. If there are no jobs in the system, the residual service time is equal to zero. Supposing that there are n ($n \geqq 1$) jobs in the system during the arrival time of the i^{th} job of product type j, then the expected residual service time until one unfinished job has completed its processing on the machine can be shown as Equation (3-13), where $p_{n,FIFO}$ represents the probability that there are n ($n \geqq 1$) jobs present in the system under FIFO and this is equal to $(1-\rho_{FIFO})\rho_{FIFO}^{n}$; ρ_{FIFO} represents the utilization rate of the machine under FIFO; $R(t)$ represents the residual service time of the unfinished job on the machine; $f_{T_{ij}}(t_{ij})$ represents the probability density function of the gamma variable T_{ij} with parameters i and λ_j and $t_{ij} > 0$.

$$E\left[R_{ijn,FIFO}\right] = p_{n,FIFO}\int_0^{RT}\frac{1}{t_{ij}}\int_0^{t_{ij}} R(t)\,dt\, f_{T_{ij}}\left(t_{ij}\right)dt_{ij} \tag{3-13}$$

Considering the interval of time t_{ij} and $0 < t_{ij} \leqq RT$, the value of $\int_0^{t_{ij}} R(t)\,dt$ in Equation (3-13) can be calculated as Equation (3-14) by dividing the sum of the areas of the triangles by the length of the interval t_{ij} in the Figure 3-4. Supposing that there are n_j triangles of product type j in the interval of time t_{ij}, which is determined by the arrival rate λ_j and is equal to $\lambda_j t_{ij}$, where j=1, 2, …, J, we assume that the probability of belonging to product type j triangle is equal to λ_j/λ. The triangles in Figure 3-4 are isosceles triangles, thus their area equals the base multiplied by the height and then divided by two, where the base and the height of the triangles are the same; they have two possible categories based on the setup condition of the jobs. First, the base or the height of the triangles of product type j would be equal to its processing time (pt_j) with the probability ($1-P_{s,j,FIFO}$) if no setup is required. Second, a setup is necessary, the base or the height of the triangles of product type j would be equal to its processing time plus its setup time ($pt_j + s_{rj}$) with the probability $P_{s,j,FIFO}(\lambda_r/\lambda^c)$, where $P_{s,j,FIFO} = (1-\lambda_j/\lambda)$, $r = 1, 2, \ldots, J$, $r \neq j$ and $\lambda^c = \sum_{r=1, r\neq j}^{J}\lambda_r$. Accordingly, $E[R_{ijn,FIFO}]$ can be rewritten as Equation (3-15) by using Equation (3-14) in Equation (3-13), where $\int_0^{RT} f_{T_{ij}}\left(t_{ij}\right)dt_{ij} = \Pr[T_{ij} \leq RT]$. Using the summation of $E[R_{ijn,FIFO}]$ for all n, the expected time of the i^{th} job of product type j has to wait until one unfinished job has completed its processing on the machine under FIFO, which can be shown as $E[R_{ij,FIFO}] = \sum_{n=1}^{\infty} E[R_{ijn,FIFO}]$.

$$\int_0^{t_{ij}} R(t)\,dt = \sum_{j=1}^{J}\sum_{i=1}^{\lambda_j t_{ij}} \frac{\lambda_j}{\lambda}\left[\sum_{\substack{r=1\\ r\neq j}}^{J} P_{s,j,FIFO}\frac{\lambda_r}{\lambda^c}\frac{\left(pt_j+s_{rj}\right)^2}{2} + \left(1-P_{s,j,FIFO}\right)\frac{pt_j^2}{2}\right] \tag{3-14}$$

$$E\left[R_{ijn,FIFO}\right] = p_{n,FIFO}\left\{\sum_{j=1}^{J}\frac{\lambda_j^2}{2\lambda}\left[\sum_{\substack{r=1\\ r\neq j}}^{J} P_{s,j,FIFO}\frac{\lambda_r}{\lambda^c}\left(pt_j+s_{rj}\right)^2 + \left(1-P_{s,j,FIFO}\right)pt_j^2\right]\right\}\times \int_0^{RT} f_{T_{ij}}\left(t_{ij}\right)dt_{ij} \tag{3-15}$$

Figure 3-4 Diagram of the calculation of the residual service time of the unfinished job

3.2.2.2 Expected total service time of all waiting job in queue

The expected total service time of all waiting jobs in queue ahead of the newly arrived job would be necessary when there are at least two jobs in the system at the arrival time of the newly arrived job. Therefore, the calculation of this time is related to the number of jobs in queue and the service time of some specific product type jobs that have already arrived before the newly arrived job.

First, in case there are two jobs in the system at the arrival time of the i^{th} job of product type j, between two jobs in the system, one would stay in queue while the other is processed on the machine. Thus, when the i^{th} job of product type j arrives at the system in the time interval (0, RT], the expected total service time of one waiting job in queue ahead of the newly arrived job can be computed as Equation (3-16), where $p_{2,FIFO}$ represents the probability that there are two jobs present in the system under FIFO and λ_j/λ represents the probability that the waiting job in queue belongs to the product type j. In consideration of the setup condition of the jobs, the expected service time of the waiting job in queue can have two components: one is equal to $pt_j + \sum_{r=1,r\neq j}^{J} (\lambda_r / \lambda^c) s_{rj}$ with the probability $P_{s,j,FIFO}$ and the other is equal to pt_j with the probability $(1 - P_{s,j,FIFO})$.

$$
\begin{aligned}
ETST_{1,ij,FIFO} = {} & \Pr\left[T_{ij} \leq RT\right] p_{2,FIFO} \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} P_{s,j,FIFO} \left[pt_j + \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \right] + \\
& \Pr\left[T_{ij} \leq RT\right] p_{2,FIFO} \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \left(1 - P_{s,j,FIFO}\right) pt_j \\
= {} & \Pr\left[T_{ij} \leq RT\right] p_{2,FIFO} \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \left[pt_j + P_{s,j,FIFO} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \right] \\
= {} & \Pr\left[T_{ij} \leq RT\right] ETST_{1,FIFO}
\end{aligned}
\tag{3-16}
$$

$$
\begin{aligned}
ETST_{2,ij,FIFO} = {} & \Pr\left[T_{ij} \leq RT\right] p_{3,FIFO} \sum_{j^1=1}^{J} \sum_{j^2=1}^{J} \frac{\lambda_{j^1}}{\lambda} \frac{\lambda_{j^2}}{\lambda} \times \\
& \Bigg\{ P_{s,j^1,FIFO} P_{s,j^2,FIFO} \left[\sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} \frac{\lambda_{r^2}}{\lambda^{c^2}} \left(pt_{j^1} + pt_{j^2} + s_{r^1 j^1} + s_{r^2 j^2} \right) \right] + \\
& P_{s,j^1,FIFO} \left(1 - P_{s,j^2,FIFO}\right) \left[\sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} \left(pt_{j^1} + pt_{j^2} + s_{r^1 j^1} \right) \right] + \\
& \left(1 - P_{s,j^1,FIFO}\right) P_{s,j^2,FIFO} \left[\sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} \left(pt_{j^1} + pt_{j^2} + s_{r^2 j^2} \right) \right] + \\
& \left(1 - P_{s,j^1,FIFO}\right) \left(1 - P_{s,j^2,FIFO}\right) \left(pt_{j^1} + pt_{j^2} \right) \Bigg\} \\
= {} & \Pr\left[T_{ij} \leq RT\right] p_{3,FIFO} \times \\
& \sum_{j^1=1}^{J} \sum_{j^2=1}^{J} \frac{\lambda_{j^1}}{\lambda} \frac{\lambda_{j^2}}{\lambda} \left[pt_{j^1} + P_{s,j^1,FIFO} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1} + pt_{j^2} + P_{s,j^2,FIFO} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2} \right] \\
= {} & \Pr\left[T_{ij} \leq RT\right] ETST_{2,FIFO}
\end{aligned}
\tag{3-17}
$$

Second, for the case where there are three jobs in the system at the arrival time of the i^{th} job of product type j, among these three jobs in the system, two jobs would stay in queue while one job is processed on the machine. Thus, when the i^{th} job of product type j arrives at the system in the time interval (0, RT], the expected total service time of the two waiting jobs in queue ahead of the newly arrived job can be computed as Equation (3-17), where $p_{3,FIFO}$ is the probability that there are three jobs present in the system under FIFO, the superscript k in j, r, and c refers to the k^{th} waiting job in queue, λ_{j^k}/λ is the probability that the k^{th} waiting jobs in queue belongs to product type j, and $j=1, 2, \ldots, J$.

The setup cases for these two waiting jobs in queue list are as follows: "setup, setup," "setup, no setup," "no setup, setup," and "no setup, no setup." Thus, the expected total service time of these two waiting jobs in queue can have four components:

I. $[pt_{j^1}+s_{r^1j^1}+pt_{j^2}+s_{r^2j^2}]$ with the probability $P_{s,j^1,FIFO}P_{s,j^2,FIFO}(\lambda_{r^1}/\lambda^{c^1})(\lambda_{r^2}/\lambda^{c^2})$

II. $[pt_{j^1}+s_{r^1j^1}+pt_{j^2}]$ with the probability $P_{s,j^1,FIFO}(1-P_{s,j^2,FIFO})(\lambda_{r^1}/\lambda^{c^1})$

III. $[pt_{j^1}+pt_{j^2}+s_{r^2j^2}]$ with the probability $(1-P_{s,j^1,FIFO})P_{s,j^2,FIFO}(\lambda_{r^2}/\lambda^{c^2})$

IV. $[pt_{j^1}+pt_{j^2}]$ with the probability $(1-P_{s,j^1,FIFO})(1-P_{s,j^2,FIFO})$

where $r^k=1, 2, \ldots, J$, $r^k \neq j^k$, and $\lambda^{c^k}=\sum_{r^k=1,r^k\neq j^k}^{J}\lambda_{r^k}$. In order to simplify the calculation of Equation (3-17), $ETST_{2,FIFO}$ in Equation (3-17) can be reformulated as $ETST_{2,FIFO}=\rho_{FIFO}ETST_{1,FIFO}+\rho_{FIFO}ETST_{1,FIFO}=2\rho_{FIFO}ETST_{1,FIFO}$, which is the function of $ETST_{1,FIFO}$ (see Appendix A).

For this reason, when the i^{th} job of product type j arrives at the system in the interval $(0, RT]$ and there are n ($n \geqq 2$) jobs present in the system, the expected total service time of $(n-1)$ waiting jobs in queue ahead of the newly arrived job can be shown as Equation (3-18), which is the function of $ETST_{n-1,FIFO}$ in Equation (3-19) (see Appendix B).

$$\begin{aligned} ETST_{n-1,ij,FIFO} &= \Pr\left[T_{ij}\le RT\right]p_{n,FIFO}\times \\ &\sum_{j^1=1}^{J}\cdots\sum_{j^{n-1}=1}^{J}\left(\prod_{k=1}^{n-1}\frac{\lambda_{j^k}}{\lambda}\right)\left[\sum_{k=1}^{n-1}\left(pt_{j^k}+P_{s,j^k,FIFO}\sum_{\substack{r^k=1\\ r^k\neq j^k}}^{J}\frac{\lambda_{r^k}}{\lambda^{c^k}}s_{r^kj^k}\right)\right] \\ &= \Pr\left[T_{ij}\le RT\right]ETST_{n-1,FIFO} \end{aligned} \tag{3-18}$$

$$ETST_{n-1,FIFO} = \begin{cases} ETST_{1,FIFO}, & \text{where } n = 2 \\ \rho_{FIFO}^{n-2} ETST_{1,FIFO} + \rho_{FIFO} ETST_{n-2,FIFO}, & \text{where } n \geq 3 \end{cases} \tag{3-19}$$

$$ETST_{n-1,ij,FIFO} = \Pr\left[T_{ij} \leq RT\right] p_{n,FIFO} \left\{ (n-1) \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \left[pt_j + P_{s,j,FIFO} \sum_{\substack{r=1 \\ r \neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \right] \right\} \tag{3-20}$$

Note that $ETST_{n\text{-}2,FIFO}$ in Equation (3-19) can be expressed as $ETST_{n-2,FIFO} = \rho_{FIFO}^{n-3} ETST_{1,FIFO} + \rho_{FIFO} ETST_{n-3,FIFO}$ from Equation (3-19) and can be used in Equation (3-19). By repeating the aforementioned steps, we easily get $ETST_{n-1,FIFO} = (n-1)\rho_{FIFO}^{n-2} ETST_{1,FIFO}$ for $n \geqq 3$. Getting $ETST_{1,FIFO}$ from Equation (3-16) and then substituting it into $ETST_{n\text{-}1,FIFO}$, $ETST_{n\text{-}1,ij,FIFO}$ can be reformulated as Equation (3-20).

From Equation (3-20), $ETST_{n\text{-}1,ij,FIFO}$ represents the expected total service time of $(n-1)$ waiting jobs in queue ahead of the i^{th} arrived job of product type j. This occurs if there are n jobs in the system at the arrival time of the i^{th} job of product type j. Thus, the probability of the i^{th} job of product type j arriving at the system in the time interval (0, RT] is equal to $\Pr[T_{ij} \leqq RT]$. The probability that there are n jobs present in the system is equal to $p_{n,FIFO} = (1-\rho_{FIFO})\rho_{FIFO}^{n}$. The total service time of $(n-1)$ waiting jobs in queue ahead of the i^{th} arrived job of product type j is equal to the number of jobs in queue multiplied by the mean service time, where there are $(n-1)$ jobs in queue and the mean service time can be shown as $\sum_{j=1}^{J}(\lambda_j/\lambda)[pt_j + P_{s,j,FIFO}\sum_{r=1,r\neq j}^{J}(\lambda_r/\lambda^c)s_{rj}]$.

Thus, the expected total service time of all waiting jobs in queue ahead of the i^{th} arrived job of product type j is equal to the summation of $ETST_{n\text{-}1,ij,FIFO}$ for all n ($n \geqq 2$), which is given by $E[TST_{ij,FIFO}] = \sum_{n=2}^{\infty} ETST_{n-1,ij,FIFO}$.

3.2.2.3 Approximation of expected waiting time

Looking at the expected values $E[R_{ij,FIFO}]$ and $E[TST_{ij,FIFO}]$, the expected waiting time of product type j jobs can be determined accurately by using these two expected values and this is given by Equation (3-21), where $n_j = \lambda_j RT$ and $j = 1, 2, \ldots, J$.

$$E\left[W_{q,j,FIFO}\right] = n_j^{-1} \sum_{i=1}^{n_j} E\left[R_{ij,FIFO}\right] + E\left[TST_{ij,FIFO}\right] \tag{3-21}$$

However, an analytical model for estimating waiting time with setup time for single-machine system was modeled by an $M/G/1$ queue in many literatures. Thus, an approximation of the expected waiting time related to the formula for an $M/G/1$ queue is derived in this section, and then the relative performance of this approximation is assessed.

When RT is long enough, the expected mean waiting time of product type j jobs under FIFO in Equation (3-21) can be shown as $\lim_{RT\to\infty} E[W_{q,j,FIFO}] = \alpha_j E[W_q]_{M/G/1}$, which is proportional to the expected waiting time for the $M/G/1$ queuing theory ($E[W_q]_{M/G/1} = \lambda E[ST_{FIFO}^2] / [2(1-\rho_{FIFO})]$) (see Appendix C).

$$\alpha_j = w_j \rho_{FIFO} \left\{ (1-\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{2}{1+C_v^2} \right\} \tag{3-22}$$

$$\Delta_j'' = \sum_{\substack{r=1 \\ r\neq j}}^{J} P_{s,j,FIFO} \frac{\lambda_r}{\lambda^c} \left(pt_j + s_{rj} \right)^2 + \left(1-P_{s,j,FIFO}\right) pt_j^2 \tag{3-23}$$

In Equation (3-22), the gradient α_j corresponds to machine utilization rate (ρ_{FIFO}), w_j is the mean of all probabilities of product type j jobs arriving at the system in the time interval (0, RT] and is given by $w_j = \sum_{i=1}^{n_j} \Pr[T_{ij} \le RT] / n_j$, and C_v^2 is the squared coefficient of variation of the service time and is defined as $C_v^2 = Var[ST_{FIFO}] / E[ST_{FIFO}]^2$. The variance of the service time under FIFO can be shown as $Var[ST_{FIFO}] = E[ST_{FIFO}^2] - E[ST_{FIFO}]^2$. The first and second moments of the service time under FIFO ($E[ST_{FIFO}]$ and $E[ST_{FIFO}^2]$) are defined by $(\sum_{j=1}^{J} n_j)^{-1} \sum_{j=1}^{J} \sum_{i=1}^{n_j} E[ST_{ij,FIFO}]$ and $(\sum_{j=1}^{J} n_j)^{-1} \sum_{j=1}^{J} \sum_{i=1}^{n_j} E[ST_{ij,FIFO}^2]$, and then they can be calculated according to the probability mass function of $ST_{ij,FIFO}$ in Equation (3-3). An inequality expressed as $\alpha_j \geqq f(\rho_{FIFO})$ is derived because $2/(1+C_v^2) \ge \rho_{FIFO}$ from Equation (C.8) in Appendix C, where $f(\rho_{FIFO})$ is given by Equation (3-24).

$$f\left(\rho_{FIFO}\right) = w_j \rho_{FIFO} \left\{ (1-\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \rho_{FIFO} \right\} \tag{3-24}$$

Thus, it can be seen that $f(\rho_{FIFO}) = 0$ and $\alpha_j \geqq 0$ if $\rho_{FIFO} = 0$ and $f(\rho_{FIFO}) = 1$ and $\alpha_j \geqq 1$ if $\rho_{FIFO} =$

1 when RT is long enough, where $\lim_{RT \to \infty} w_j = 1$. Moreover, the first derivative of the gradient α_j with respect to ρ_{FIFO} is given by Equation (3-25) (see Appendix D).

$$\frac{d\alpha_j}{d\rho_{FIFO}} = w_j\left(1-2\rho_{FIFO}\right)\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{6w_j}{1+C_v^2} > 0 \tag{3-25}$$

Note that the first derivative of this gradient with respect to ρ_{FIFO} is a positive number with $0 < \rho_{FIFO} \leqq 1$ and $\lambda_j > 0$. Hence, the gradient α_j increases with the rise in the utilization rate of the machine.

According to the abovementioned results, two distinct possibilities exist. First, $0 \leqq \alpha_j < 1$ when the machine utilization rate is smaller, which implies that the expected waiting time of product type j jobs is lower than the expected waiting time for the $M/G/1$ queuing model. Second, $\alpha_j \geqq 1$ when the machine utilization rate is higher, which implies that the expected waiting time of product type j jobs is higher than the expected waiting time for the $M/G/1$ queuing model. The gap between the expected mean waiting time of product type j jobs and the expected waiting time for the $M/G/1$ queuing model grows with high levels of workload on the machine because gradient α_j is more than one.

3.3 Variance of lead time of jobs under FIFO

Suppose that the service time of jobs on the machine and the waiting time of jobs in queue are independent, the variance of lead time is equal to the sum of variances of the service time and the waiting time in queue. Both the variances of the service time and the waiting time in queue are presented as follows.

3.3.1 Variance of service time under FIFO

In Section 3.2.1, the service time is equal to the sum of the processing time and setup time. Accordingly, the variance of the service time of the product type j jobs can be given by $Var[ST_{j,FIFO}] = Var[PT_j + S_{j,FIFO}]$, where PT_j and $S_{j,FIFO}$ are the variables of processing time and setup time of the product type j jobs, and they are independent. Thus, the variance of the service time of the product type j jobs is equal to the sum of the variances of PT_j and $S_{j,FIFO}$.

If the i^{th} job of product type j does not arrive in the time interval $(0, RT]$, then its processing time should be zero with the probability $\Pr[T_{ij} > RT]$. If the i^{th} job of product type j arriving in the time interval (0, RT], then its processing time would be equal to pt_j with the probability $\Pr[T_{ij} \leqq RT]$. The first and second moments of the processing of the product type j jobs are defined by $E[PT_j] = n_j^{-1}\sum_{i=1}^{n_j}\Pr[T_{ij} \le RT]pt_j$ and $E[PT_j^2] = n_j^{-1}\sum_{i=1}^{n_j}\Pr[T_{ij} \le RT]pt_j^2$, respectively. Therefore, the variance of the processing time of product type j jobs can be shown as $Var[PT_j] = E[PT_j^2] - E[PT_j]^2$. In addition, the variance of the setup time of the product type j jobs can be expressed as Equation (3-26) according to Equation (3-9).

$$
\begin{aligned}
Var\left[S_{j,FIFO}\right] &= E\left[S_{j,FIFO}^2\right] - E\left[S_{j,FIFO}\right]^2 \\
&= n_j^{-1}\sum_{i=1}^{n_j}\Pr\left[T_{ij} \le RT\right]P_{s,j,FIFO}\sum_{\substack{r=1 \\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}^2 - \\
&\quad \left\{n_j^{-1}\sum_{i=1}^{n_j}\Pr\left[T_{ij} \le RT\right]P_{s,j,FIFO}\sum_{\substack{r=1 \\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right\}^2
\end{aligned} \tag{3-26}
$$

where the second moment of the setup time of the product type j jobs, $E[S_{j,FIFO}^2]$, is defined by the arithmetic means of the second moment of the setup time for all product type j jobs, which can be expressed by $E[S_{j,FIFO}^2] = n_j^{-1}\sum_{i=1}^{n_j}E[S_{ij,FIFO}^2]$ if there are $n_j = \lambda_j RT$ jobs of product type j arrived at the system in the time interval (0, RT].

When RT is long enough, the limit of the variance of the processing time of product type j jobs equals zero ($\lim_{RT\to\infty} Var[PT_j] = 0$) because $\lim_{RT\to\infty}\Pr[T_{ij} \le RT] = 1$. Consequently, the variance of the service time of the product type j jobs can be easy to simplify as $Var[ST_{j,FIFO}] = Var[S_{j,FIFO}]$ when RT is long enough.

3.3.2 Variance of waiting time under FIFO

The variance of the waiting time of the product type j jobs is defined as the second moment of $W_{q,j,FIFO}$ minus the first moment squared and then can be derived as $Var[W_{q,j,FIFO}] = E[W_{q,j,FIFO}^2] - E[W_{q,j,FIFO}]^2$. Suppose that there are $n_j = \lambda_j RT$ jobs of product type j arrived at the system in the time interval (0, RT], the first and the second moments of $W_{q,j,FIFO}$ are defined by the arithmetic means of the first and second of the waiting time for all product type j jobs and are given by $E[W_{q,j,FIFO}] = n_j^{-1}\sum_{i=1}^{n_j}E[W_{q,ij,FIFO}]$ and

$E[W_{q,j,FIFO}^2] = n_j^{-1}\sum_{i=1}^{n_j} E[W_{q,ij,FIFO}^2]$. The time of the i^{th} arrived job of product type j has to wait in queue for the service under FIFO, $W_{q,ij,FIFO}$, is defined by $W_{q,ij,FIFO} = R_{ij,FIFO} + TST_{ij,FIFO}$, then it is easy to establish the formula for calculating $Var[W_{q,j,FIFO}]$ in Equation (3-27).

$$Var\left[W_{q,j,FIFO}\right] = \frac{1}{n_j}\sum_{i=1}^{n_j} E\left[R_{ij,FIFO}^2\right] + \frac{1}{n_j}\sum_{i=1}^{n_j} E\left[TST_{ij,FIFO}^2\right] + \frac{2}{n_j}\sum_{i=1}^{n_j}\left\{E\left[R_{ij,FIFO}\right]E\left[TST_{ij,FIFO}\right]\right\} - E\left[\frac{\sum_{i=1}^{n_j} W_{q,ij,FIFO}}{n_j}\right]^2 \tag{3-27}$$

where $R_{ij,FIFO}$ and $TST_{ij,FIFO}$ are assumed to be independent. In Equation (3-27), the first moments of $R_{ij,FIFO}$, $TST_{ij,FIFO}$ and $W_{q,ij,FIFO}$ can be obtained in Section 3.2. As for the second moments of $R_{ij,FIFO}$ and $TST_{ij,FIFO}$, they are presented as follows.

3.3.2.1. Second moment of residual service time of unfinished job on the machine

The residual service time of unfinished job only occurs if there are n ($n \geqq 1$) jobs in the system. For the case if there are no jobs in the system, this time is equal to zero. Suppose that there are n ($n \geqq 1$) jobs present in the system at the arrival time of the i^{th} job of product type j, and then the second moment of residual service time until one unfinished job completed its processing on the machine can be shown as Equation (3-28).

$$E\left[R_{ijn,FIFO}^2\right] = p_{n,FIFO}\int_0^{RT}\frac{1}{t_{ij}}\int_0^{t_{ij}} R(t)^2\,dt\, f_{T_{ij}}\left(t_{ij}\right)dt_{ij} \tag{3-28}$$

where $R(t)$ represents the residual service time of the unfinished job on the machine, $f_{T_{ij}}(t_{ij})$ represents the probability density function of the gamma variable T_{ij} with parameters i and λ_j and $t_{ij} > 0$.

Consider the interval of time t_{ij} and $0 < t_{ij} \leqq RT$, Equation (3-29) shows the value of $\int_0^{t_{ij}} R(t)^2\,dt$ in Equation (3-28) and can be computed by dividing the sum of the areas under the parabola (Height = Base^2) by the length of the interval t_{ij} in the Figure 3-5, where the height represents the square of the residual service time $R(t)^2$, and the base represents the residual service time and ranges from zero to the service time. Suppose that there are n_j areas belonging to the product type j in the interval of time t_{ij}, which is determined by the its arrival rate λ_j and is equal to $\lambda_j t_{ij}$. The probability of belonging to the product type j is given by the relative arrival rates of the product types (λ_j/λ). In the Figure 3-5, the area under the parabola equals the cube of the base divide by 3, where the base has two possible categories depending on the setup condition of the jobs.

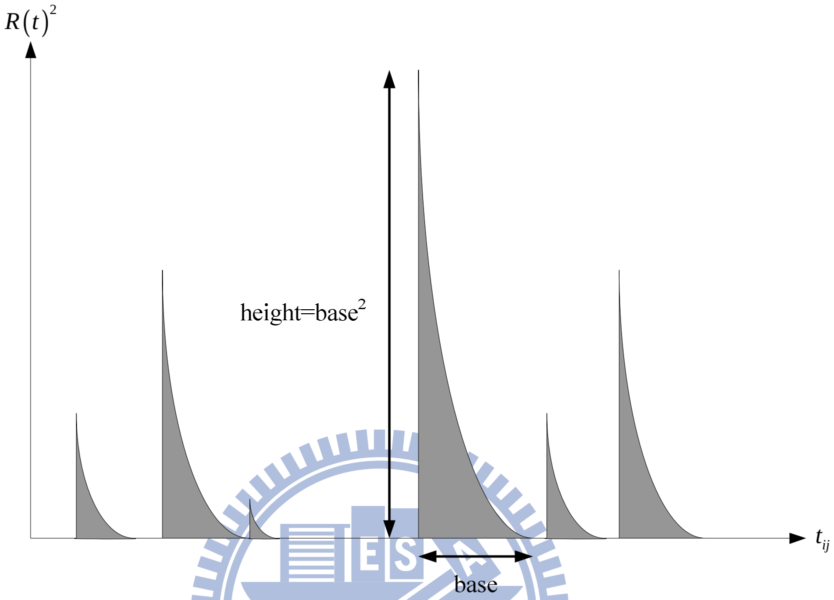

Figure 3-5 Diagram of the calculation of the square of the residual service time of the unfinished job

First, the base of the product type j would be equal to its processing time (pt_j) with the probability ($1-P_{s,j,FIFO}$) if no setup is required. Second, the base of the product type j would be equal to its processing time plus its setup time (pt_j+s_{rj}) with the probability $P_{s,j,FIFO}(\lambda_r/\lambda^c)$ if the setup is necessary, where $P_{s,j,FIFO}=(1-\lambda_j/\lambda$, $r=1, 2, \ldots, J$, $r \neq j$, $\lambda^c = \sum_{r=1, r\neq j}^{J} \lambda_r$. Accordingly, $E[R^2_{ijn,FIFO}]$ can be rewritten as Equation (3-30) by using Equation (3-28) in Equation (3-29).

$$\int_0^{t_{ij}} R(t)^2\, dt = \sum_{j=1}^{J} \sum_{i=1}^{\lambda_j t_{ij}} \frac{\lambda_j}{\lambda} \left[P_{s,j,FIFO} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} \frac{\left(pt_j + s_{rj}\right)^3}{3} + \left(1 - P_{s,j,FIFO}\right) \frac{pt_j^3}{3} \right] \tag{3-29}$$

$$E\left[R^2_{ijn,FIFO}\right] = \Pr\left[T_{ij} \le RT\right] p_{n,FIFO} \times \left\{ \sum_{j=1}^{J} \frac{\lambda_j^2}{\lambda} \left[P_{s,j,FIFO} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} \frac{\left(pt_j + s_{rj}\right)^3}{3} + \left(1 - P_{s,j,FIFO}\right) \frac{pt_j^3}{3} \right] \right\} \tag{3-30}$$

3.3.2.2. Second moment of total service time of all waiting job in queue

Total service time of all waiting job in queue ahead of the arrived job would be necessary while there are at least two jobs in the system at the arrival time of the arrived job. Therefore, the calculation of the second moment of this time relates to the number of jobs in queue and the service time of some specific product type jobs already arrived before the new arrived job.

First, for the case of there are two jobs in the system at the arrival time of the i^{th} job of product type j, this means that one is staying in queue and the other is processing present on the machine between these two jobs in the system. Thus, when the i^{th} job of product type j arrived at the system in the time interval (0, RT], the second moment of total service time of one waiting job in queue ahead of the arrived job ($ESTST_{1,ij,FIFO}$) is the expected value of the square of total service time of one waiting job in queue ahead of the arrived job, which is computed as Equation (3-31).

$$\begin{aligned} ESTST_{1,ij,FIFO} &= \Pr\left[T_{ij} \le RT\right] p_{2,FIFO} \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \left[P_{s,j,FIFO} \sum_{\substack{r=1 \\ r \ne j}}^{J} \frac{\lambda_r}{\lambda^c} \left(pt_j + s_{rj}\right)^2 + \left(1 - P_{s,j,FIFO}\right) pt_j^2 \right] \\ &= \Pr\left[T_{ij} \le RT\right] p_{2,FIFO} SST_{FIFO} \end{aligned} \tag{3-31}$$

where λ_j/λ represents the probability that the waiting job in queue belongs to the product type j. Depending on the setup condition of the jobs, the square of service time of this waiting job in queue can have two components: one is equal to pt_j^2 with the probability ($1-P_{s,j,FIFO}$) and the other is equal to $(pt_j + s_{rj})^2$ with the probability $P_{s,j,FIFO}(\lambda_r/\lambda^c)$, where $r = 1, 2, \ldots, J$, $r \ne j$, $\lambda^c = \sum_{r=1,r\ne j}^{J} \lambda_r$.

Second, for the case of there are three jobs in the system at the arrival time of the i^{th} job of product type j, this means that two jobs are staying in queue and one job is processing present on the machine among these three jobs in the system. Thus, when the i^{th} job of product type j arrived at the system in the time interval (0, RT], the second moment of total service time of two waiting jobs in queue ahead of the arrived job, $ESTST_{2,ij,FIFO}$, is equal to the expected value of the square of total service time of two waiting jobs in queue ahead of the arrived job, which can be computed as Equation (3-32).

$$
\begin{aligned}
ESTST_{2,ij,FIFO} = \Pr\left[T_{ij} \le RT\right] p_{3,FIFO} \sum_{j^1=1}^{J}\sum_{j^2=1}^{J} \frac{\lambda_{j^1}}{\lambda}\frac{\lambda_{j^2}}{\lambda} \times \\
\Bigg[P_{s,j^1,FIFO} P_{s,j^2,FIFO} \sum_{\substack{r^1=1 \\ r^1 \ne j^1}}^{J} \sum_{\substack{r^2=1 \\ r^2 \ne j^2}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}}\frac{\lambda_{r^2}}{\lambda^{c^2}} \left(pt_{j^1} + s_{r^1 j^1} + pt_{j^2} + s_{r^2 j^2}\right)^2 + \\
P_{s,j^1,FIFO}\left(1-P_{s,j^2,FIFO}\right) \sum_{\substack{r^1=1 \\ r^1 \ne j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} \left(pt_{j^1} + s_{r^1 j^1} + pt_{j^2}\right)^2 + \\
\left(1-P_{s,j^1,FIFO}\right) P_{s,j^2,FIFO} \sum_{\substack{r^2=1 \\ r^2 \ne j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} \left(pt_{j^1} + pt_{j^2} + s_{r^2 j^2}\right)^2 + \\
\left(1-P_{s,j^1,FIFO}\right)\left(1-P_{s,j^2,FIFO}\right)\left(pt_{j^1} + pt_{j^2}\right)^2 \Bigg]
\end{aligned}
\qquad (3\text{-}32)
$$

where λ_{j^k}/λ is the probability that the k^{th} waiting jobs in queue belongs to the product type j, and $j = 1, 2, \ldots, J$. The setup cases for these two waiting jobs in queue list as follows: "setup, setup", "setup, no setup", "no setup, setup", and "no setup, no setup". Thus, the square of total service time of these two waiting jobs in queue can have four components:

V. $[pt_{j^1} + s_{r^1 j^1} + pt_{j^2} + s_{r^2 j^2}]^2$ with the probability $P_{s,j^1,FIFO} P_{s,j^2,FIFO} (\lambda_{r^1}/\lambda^{c^1})(\lambda_{r^2}/\lambda^{c^2})$

VI. $[pt_{j^1} + s_{r^1 j^1} + pt_{j^2}]^2$ with the probability $P_{s,j^1,FIFO}(1-P_{s,j^2,FIFO})(\lambda_{r^1}/\lambda^{c^1})$

VII. $[pt_{j^1} + pt_{j^2} + s_{r^2 j^2}]^2$ with the probability $(1-P_{s,j^1,FIFO})P_{s,j^2,FIFO}(\lambda_{r^2}/\lambda^{c^2})$

VIII. $[pt_{j^1} + pt_{j^2}]^2$ with the probability $(1-P_{s,j^1,FIFO})(1-P_{s,j^2,FIFO})$

where $r^k = 1, 2, \ldots, J$, $r^k \ne j^k$, and $\lambda^{c^k} = \sum_{r^k=1, r^k \ne j^k}^{J} \lambda_{r^k}$. In Equation (3-32), all terms of these four components on the right hand side of the equal sign are shown in Table 3-1. In Table 3-1, the main diagonal represents the square terms of the processing times and the setup times of two waiting jobs in queue, and the elements outside the main diagonal represent the product of the processing time and setup time of two waiting jobs in queue and are symmetric.

Table 3-1 All terms on the right hand side of the equal sign in Equation (3-32)

	Processing time of first waiting job in queue	Processing time of second waiting job in queue	Setup time of first waiting job in queue	Setup time of second waiting job in queue
Processing time of first waiting job in queue	$pt_{j^1}^2$	$pt_{j^1} pt_{j^2}$	$P_{s,j^1,FIFO}\, pt_{j^1} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	$P_{s,j^2,FIFO}\, pt_{j^1} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2}$
Processing time of first waiting job in queue	$pt_{j^1} pt_{j^2}$	$pt_{j^2}^2$	$P_{s,j^1,FIFO}\, pt_{j^2} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	$P_{s,j^2,FIFO}\, pt_{j^2} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2}$
Setup time of first waiting job in queue	$P_{s,j^1,FIFO}\, pt_{j^1} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	$P_{s,j^1,FIFO}\, pt_{j^2} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	$P_{s,j^1,FIFO} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}^2$	$\left(\prod_{k=1}^{2} P_{s,j^k,FIFO}\right) \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \prod_{k=1}^{2} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k}$
Setup time of second waiting job in queue	$P_{s,j^2,FIFO}\, pt_{j^1} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2}$	$P_{s,j^2,FIFO}\, pt_{j^2} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2}$	$\left(\prod_{k=1}^{2} P_{s,j^k,FIFO}\right) \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \prod_{k=1}^{2} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k}$	$P_{s,j^2,FIFO} \sum_{\substack{r^2=1 \\ r^2\neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2}^2$

For this reason, when the i^{th} job of product type j arrives at the system in the interval (0, RT] and there are n ($n \geqq 2$) jobs present in the system, the second moment of total service time of ($n-1$) waiting jobs in queue ahead of the newly arrived job can be shown as Equation (3-33) according to Table 3-2.

$$
\begin{aligned}
& ESTST_{n-1,ij,FIFO} \\
&= \Pr\left[T_{ij} \leq RT\right] p_{n,FIFO} \times \\
&\left\{ \sum_{j^1=1}^{J} \cdots \sum_{j^{n-1}=1}^{J} \left(\prod_{k=1}^{n-1} \frac{\lambda_{j^k}}{\lambda} \right) \left[\sum_{k=1}^{n-1} \left(pt_{j^k}^2 + 2P_{s,j^k,FIFO}\, pt_{j^k} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k} + P_{s,j^k,FIFO} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k}^2 \right) \right] + \right. \\
&\quad \sum_{j^1=1}^{J} \cdots \sum_{j^{n-1}=1}^{J} \left(\prod_{k=1}^{n-1} \frac{\lambda_{j^k}}{\lambda} \right) \left[\sum_{\substack{k,k'=1 \\ k<k'}}^{n-1} \left(2\, pt_{j^k} pt_{j^{k'}} + 2P_{s,j^{k'},FIFO}\, pt_{j^k} \sum_{\substack{r^{k'}=1 \\ r^{k'} \neq j^{k'}}}^{J} \frac{\lambda_{r^{k'}}}{\lambda^{c^{k'}}} s_{r^{k'} j^{k'}} + \right. \right. \\
&\qquad 2P_{s,j^k,FIFO}\, pt_{j^{k'}} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k} + \\
&\qquad \left.\left.\left. 2P_{s,j^k,FIFO} P_{s,j^{k'},FIFO} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \sum_{\substack{r^{k'}=1 \\ r^{k'} \neq j^{k'}}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} \frac{\lambda_{r^{k'}}}{\lambda^{c^{k'}}} s_{r^k j^k} s_{r^{k'} j^{k'}} \right) \right] \right\} \\
&= \Pr\left[T_{ij} \leq RT\right] p_{n,FIFO} \left(\mathrm{SS}_{ST} + \mathrm{Prod}_{ST} \right)
\end{aligned}
\qquad (3\text{-}33)
$$

From Equation (3-33), $ESTST_{n\text{-}1,ij,FIFO}$ represents the second moment of total service time of ($n-1$) waiting jobs in queue ahead of the i^{th} arrived job of product type j, which is equal to the expected value of the square of total service time of ($n-1$) waiting jobs in queue ahead of the i^{th} arrived job of product type j. This occurs if there are n jobs in the system at the arrival time of the i^{th} job of product type j. Thus, the probability of the i^{th} job of product type j arriving at the system in the time interval (0, RT] is equal to Pr[$T_{ij} \leqq RT$]. The probability that there are n jobs present in the system is equal to $p_{n,FIFO} = (1-\rho_{FIFO})\rho_{FIFO}^{n}$. Besides, the square of total service time of ($n-1$) waiting jobs in queue ahead of the i^{th} arrived job of product type j equals the sum of the squares of each service time in ($n-1$) waiting jobs in queue (SS_{ST}) plus twice the product of all combinations of the two service times of getting exactly two jobs in ($n-1$) waiting jobs in queue (Prod_{ST}). These two terms (SS_{ST} and Prod_{ST}) are presented as follows.

I. Sum of the squares of each service time in (n −1) waiting jobs in queue (SS$_{ST}$)

In Equation (3-33), the term SS_{ST} includes the elements in Table 3-2 with gray background and can be rewritten as Equation (3-34), where $\{\sum_{j=1}^{J}(\lambda_j/\lambda)[P_{s,j,FIFO}\sum_{r=1,r\neq j}^{J}(\lambda_r/\lambda^c)(pt_j+s_{rj})^2+(1-P_{s,j,FIFO})pt_j^2]\}$ is the square of the service time. Thus, the term SS_{ST} equals the sum of the squares of each service time in ($n-1$) waiting jobs in queue.

$$\begin{aligned}\text{SS}_{ST} &= \sum_{j^1=1}^{J}\cdots\sum_{j^{n-1}=1}^{J}\left(\prod_{k=1}^{n-1}\frac{\lambda_{j^k}}{\lambda}\right)\left(\sum_{k=1}^{n-1}pt_{j^k}^2+2P_{s,j^k,FIFO}\,pt_{j^k}\sum_{\substack{r^k=1\\r^k\neq j^k}}^{J}\frac{\lambda_{r^k}}{\lambda^{c^k}}s_{r^k j^k}+\right.\\ &\qquad\left. P_{s,j^k,FIFO}\sum_{\substack{r^k=1\\r^k\neq j^k}}^{J}\frac{\lambda_{r^k}}{\lambda^{c^k}}s_{r^k j^k}^2\right)\\ &=(n-1)\sum_{j=1}^{J}\frac{\lambda_j}{\lambda}\left[pt_j^2+2P_{s,j,FIFO}\,pt_j\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}+P_{s,j,FIFO}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}^2\right]\\ &=(n-1)\sum_{j=1}^{J}\frac{\lambda_j}{\lambda}\left[P_{s,j,FIFO}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}\left(pt_j+s_{rj}\right)^2+\left(1-P_{s,j,FIFO}\right)pt_j^2\right]\end{aligned} \tag{3-34}$$

II. Twice the product of all combinations of the two service times of getting exactly two jobs in (n−1) waiting jobs in queue

In Equation (3-33), the term Prod_{ST} includes the elements in Table 2 with white background and can be rewritten as Equation (3-35) , where there are $C_2^{n-1}=(n-1)(n-2)/2$ combinations of two jobs can be selected from ($n-1$) waiting jobs in queue, and the product of the two service times is equal to the square of the mean service time of job ($\{\sum_{j=1}^{J}(\lambda_j/\lambda)[P_{s,j,FIFO}\sum_{r=1,r\neq j}^{J}(\lambda_r/\lambda^c)(pt_j+s_{rj})+(1-P_{s,j,FIFO})pt_j]\}^2$). Thus, the term Prod_{ST} is the twice the product of all combinations of the two service times of getting exactly two jobs in ($n-1$) waiting jobs in queue. Thus we have succeeded in writing the square of total service time of ($n-1$) waiting jobs in queue ahead of the i^{th} arrived job of product type j in terms of

SS_{ST}, plus $Prod_{ST}$.

The second moment of total service time of all waiting jobs in queue ahead of the i^{th} arrived job of product type j is equal to the summation of $ESTST_{n-1,ij,FIFO}$ for all n ($n \geqq 2$) according to Equation (3-33), which is given by $E[TST^2_{ij,FIFO}] = \sum_{n=2}^{\infty} ESTST_{n-1,ij,FIFO}$.

$$
\begin{aligned}
\mathrm{Prod}_{ST} &= \sum_{j^1=1}^{J} \cdots \sum_{j^{n-1}=1}^{J} \left(\prod_{k=1}^{n-1} \frac{\lambda_{j^k}}{\lambda} \right) \\
&\Bigg[\sum_{\substack{k,k'=1 \\ k<k'}}^{n-1} \Bigg(2pt_{j^k} pt_{j^{k'}} + 2P_{s,j^{k'},FIFO} pt_{j^k} \sum_{\substack{r^{k'}=1 \\ r^{k'} \neq j^{k'}}}^{J} \frac{\lambda_{r^{k'}}}{\lambda^{c^{k'}}} s_{r^{k'} j^{k'}} + \\
&\quad 2P_{s,j^k,FIFO} pt_{j^{k'}} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} s^2_{r^k j^k} + \\
&\quad 2P_{s,j^k,FIFO} P_{s,j^{k'},FIFO} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \sum_{\substack{r^{k'}=1 \\ r^{k'} \neq j^{k'}}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} \frac{\lambda_{r^{k'}}}{\lambda^{c^{k'}}} s_{r^k j^k} s_{r^{k'} j^{k'}} \Bigg) \Bigg] \\
&= 2C_2^{n-1} \left[\sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \left(pt_j + P_{s,j,FIFO} \sum_{\substack{r=1 \\ r \neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{jr} \right) \right]^2 \\
&= 2C_2^{n-1} \left\{ \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \left[P_{s,j,FIFO} \sum_{\substack{r=1 \\ r \neq j}}^{J} \frac{\lambda_r}{\lambda^c} \left(pt_j + s_{rj} \right) + \left(1 - P_{s,j,FIFO} \right) pt_j \right] \right\}^2
\end{aligned}
\tag{3-35}
$$

3.3.2.3. Approximation of variance of waiting time

From Equation (2-6), the variance of waiting time for $M/G/1$ queueing system ($Var[W_q]_{M/G/1}$) depends on the square of the expected waiting time for $M/G/1$ queueing system ($E[W_q]^2_{M/G/1}$). In this section, an approximation of the variance of waiting time related to $E[W_q]^2_{M/G/1}$ is derived when the length of time interval (RT) is long enough. In order to do this, the limit of the second moment of waiting time of product type j jobs under FIFO has to be found as RT approaches infinity, and then the relative performance of an approximation of the variance of waiting time is also assessed.

Table 3-2 All terms in the square bracket on the right hand side of the equal sign in Equation (3-34)

	Processing time of first waiting job in queue	…	Processing time of final waiting job in queue	Setup time of first waiting job in queue	…	Setup time of final waiting job in queue
Processing time of first waiting job in queue	$pt_{j^1}^2$	…	$pt_{j^1} pt_{j^{n-1}}$	$P_{s,j^1,FIFO}\, pt_{j^1} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	…	$P_{s,j^{n-1},FIFO}\, pt_{j^1} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \frac{\lambda_{r^{n-1}}}{\lambda^{c^{n-1}}} s^2_{r^{n-1} j^{n-1}}$
…	…	…	…	…	…	…
Processing time of final waiting job in queue	$pt_{j^1} pt_{j^{n-1}}$	…	$pt_{j^{n-1}}^2$	$P_{s,j^1,FIFO}\, pt_{j^{n-1}} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	…	$P_{s,j^{n-1},FIFO}\, pt_{j^{n-1}} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \frac{\lambda_{r^{n-1}}}{\lambda^{c^{n-1}}} s^2_{r^{n-1} j^{n-1}}$
Setup time of first waiting job in queue	$P_{s,j^1,FIFO}\, pt_{j^1} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	…	$P_{s,j^1,FIFO}\, pt_{j^{n-1}} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1}$	$P_{s,j^1,FIFO} \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s^2_{r^1 j^1}$	…	$\left(\prod_{k\in\{1,n-1\}} P_{s,j^k,FIFO}\right)\times \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \prod_{k=\{1,n-1\}} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k}$
…	…	…	…	…	…	…
Setup time of final waiting job in queue	$P_{s,j^{n-1},FIFO}\, pt_{j^1} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \frac{\lambda_{r^{n-1}}}{\lambda^{c^{n-1}}} s^2_{r^{n-1} j^{n-1}}$	…	$P_{s,j^{n-1},FIFO}\, pt_{j^{n-1}} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \frac{\lambda_{r^{n-1}}}{\lambda^{c^{n-1}}} s^2_{r^{n-1} j^{n-1}}$	$\left(\prod_{k\in\{1,n-1\}} P_{s,j^k,FIFO}\right)\times \sum_{\substack{r^1=1 \\ r^1\neq j^1}}^{J} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \prod_{k=\{1,n-1\}} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k}$	…	$P_{s,j^{n-1},FIFO} \sum_{\substack{r^{n-1}=1 \\ r^{n-1}\neq j^{n-1}}}^{J} \frac{\lambda_{r^{n-1}}}{\lambda^{c^{n-1}}} s^2_{r^{n-1} j^{n-1}}$

When RT is long enough, the second moment of waiting time of product type j jobs under FIFO can be shown as $\lim_{RT\to\infty} E[W_{q,j,FIFO}^2] = \beta_{1j} E[R^2]_{M/G/1} + 2\beta_{2j} E[W_q]_{M/G/1}^2$ (see Appendix E), the coefficients β_{1j} and β_{2j} are expressed as Equations (3-36) and (3-37).

$$\beta_{1j} = w_j \rho_{FIFO} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j'''}{\sum_{j=1}^{J} n_j \lambda \Delta_j'''} \leq 1 \tag{3-36}$$

$$\beta_{2j} = w_j \left(1-\rho_{FIFO}\right) \frac{2}{1+C_v^2} \left(1 + \frac{\rho_{FIFO}}{1-\rho_{FIFO}} \frac{2}{1+C_v^2} + \rho_{FIFO}^2 \frac{\sum_{j=1}^{J}\sum_{i=1}^{n_j} \lambda_j \Delta_j''}{\sum_{j=1}^{J}\sum_{i=1}^{n_j} \lambda \Delta_j''} \right) > 0 \tag{3-37}$$

Thus, it can be seen that $\beta_{2j} = 2/(1+C_v^2)$ if $\rho_{FIFO}=0$ and $\beta_{2j} = [2/(1+C_v^2)]^2$ if $\rho_{FIFO}=1$ when RT is long enough, where $\lim_{RT\to\infty} w_j = 1$. Additionally, the first derivative of the gradient β_{2j} with respect to ρ_{FIFO} is given by Equation (3-38) (see Appendix F).

$$\frac{d\beta_{2j}}{d\rho_{FIFO}} = w_j \frac{2}{1+C_v^2} \left\{ 5\left(\frac{2}{1+C_v^2} - \rho_{FIFO}^2 \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \right) + \frac{4\left(\rho_{FIFO} - \frac{3}{8}\right)^2 + \frac{23}{16}}{\rho_{FIFO}} \right\} > 0 \tag{3.38}$$

Note that the first derivative of the gradient β_{2j} with respect to ρ_{FIFO} is a positive number with $0 < \rho_{FIFO} \leqq 1$ and $\lambda_j > 0$. Hence, the gradient β_{2j} increases with the rise in the utilization rate of the machine.

Now forming an approximation of the variance of waiting time of product type j jobs under FIFO in terms of $\lim_{RT\to\infty} E[W_{q,j,FIFO}]$ and in terms of $\lim_{RT\to\infty} E[W_{q,j,FIFO}^2]$, it can be derived as Equation (3-39) as RT is increased and becomes vary large.

$$\begin{aligned} App1 &= \lim_{RT\to\infty} Var\left[W_{q,j,FIFO}\right] \\ &= \beta_{1j} E\left[R^2\right]_{M/G/1} + \left(2\beta_{2j} - \alpha_j^2\right) E\left[W_q\right]_{M/G/1}^2 \end{aligned} \tag{3-39}$$

According to Equation (3-39), an approximation of the variance of waiting time of product type j jobs under FIFO depends on the parameters α_j^2 and β_{2j}. Note that an approximation of the expected waiting time of product type j is given by $\lim_{RT\to\infty} E[W_{q,j,FIFO}] = \alpha_j E[W_q]_{M/G/1}$ when RT is long enough. From Equation (3-40), the latter on the right hand side of the equals sign ($\alpha_j^2 E[W_q]_{M/G/1}^2$) means that $\lim_{RT\to\infty} E[W_{q,j,FIFO}]$ is used to substitute the expected waiting time $E[W_q]_{M/G/1}$ as compared with Equation (2-6).

$$App2 = \beta_{1j} E\left[R^2\right]_{M/G/1} + \alpha_j^2 E\left[W_q\right]_{M/G/1}^2 \tag{3-40}$$

In order to compare App1 in Equation (3-39) with App2 in Equation (3-40), the properties of the parameters α_j^2 and β_{2j} are discussed as follows.

I. The gradients α_j^2 and β_{2j} increase with the rise in the utilization rate of the machine.

II. According to Equation (3-22) and Equation (3-37), $0 \le \alpha_j^2 \le [2/(1+C_v^2)]^2$ and $2/(1+C_v^2) \le \beta_{2j} \le [2/(1+C_v^2)]^2$ with $0 \leqq \rho_{FIFO} \leqq 1$ and $\lambda_j > 0$, where $\lim_{RT\to\infty} w_j = 1$. The minimum of α_j^2 is less than the minimum of β_{2j} and the maximums of α_j^2 and β_{2j} are the same.

III. The first derivative of the gradient β_{2j} with respect to ρ_{FIFO} is more than or equal to the first derivative of the gradient α_j^2 with respect to ρ_{FIFO} for $0 \leqq \rho_{FIFO} \leqq 1$ and $\lambda_j > 0$, where $\lim_{RT\to\infty} w_j = 1$. The difference between the first derivative of the gradient β_{2j} with respect to ρ_{FIFO} and the first derivative of the gradient α_j^2 with respect to ρ_{FIFO} is shown in Equation (3-41) (see Appendix G). The function $g_2(\rho_{FIFO})$ in Equation (3-41) depends on ρ_{FIFO} and $\pi = \sum_{j=1}^{J} n_j \lambda_j \Delta_j'' / \sum_{j=1}^{J} n_j \lambda \Delta_j''$ and is plotted against ρ_{FIFO} and π in Figure 3-6. It is seen that $g_2(\rho_{FIFO}) \geqq 0$ with $0 \leqq \rho_{FIFO} \leqq 1$ and $0 \leqq \pi \leqq 1$, and then $g_2(\rho_{FIFO}) \leqq g_3(\rho_{FIFO})$ can be derived because $0 \leqq \rho_{FIFO} \leqq 1$ and $0 \leqq \pi \leqq 1$. The proof of $g_1(\rho_{FIFO}) - g_3(\rho_{FIFO}) > 0$ with $\lambda_j \leqq (2/3)\lambda$ and $J > 1$ is shown in Appendix H, which implies that $g_1(\rho_{FIFO}) - g_2(\rho_{FIFO}) > 0$. Thus, change in β_{2j} is more than or equal to change in α_j^2 with respect to increase in ρ_{FIFO} with $\lambda_j \leqq (2/3)\lambda$ and $J > 1$.

According to the discussion above, we concluded that $\beta_{2j} \geq \alpha_j^2$ and then an inequality $(2\beta_{2j} - \alpha_j^2) \geq \alpha_j^2$ can be derived if the variation among job arrival rates is controlled in an adequate range. Consequently, an approximation of the variance of waiting time App1 is greater than or equal to an approximation of the variance of waiting time App2.

$$\frac{d\beta_{2j}}{d\rho_{FIFO}} - \frac{d\alpha_j^2}{d\rho_{FIFO}} \geq g_1(\rho_{FIFO}) - g_2(\rho_{FIFO}) \tag{3-41}$$

$$g_1(\rho_{FIFO}) = 2\rho_{FIFO}^{-1} - 2\rho_{FIFO} + 2 \tag{3-42}$$

$$\begin{aligned} g_2(\rho_{FIFO}) &= \rho_{FIFO} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \left\{ 8 - 5\rho_{FIFO} + 2(1-\rho_{FIFO})(1-2\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \right\} \\ &\leq \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \left[8 - 5\rho_{FIFO} + 2(1-\rho_{FIFO})(1-2\rho_{FIFO}) \right] = g_3(\rho_{FIFO}) \end{aligned} \tag{3-43}$$

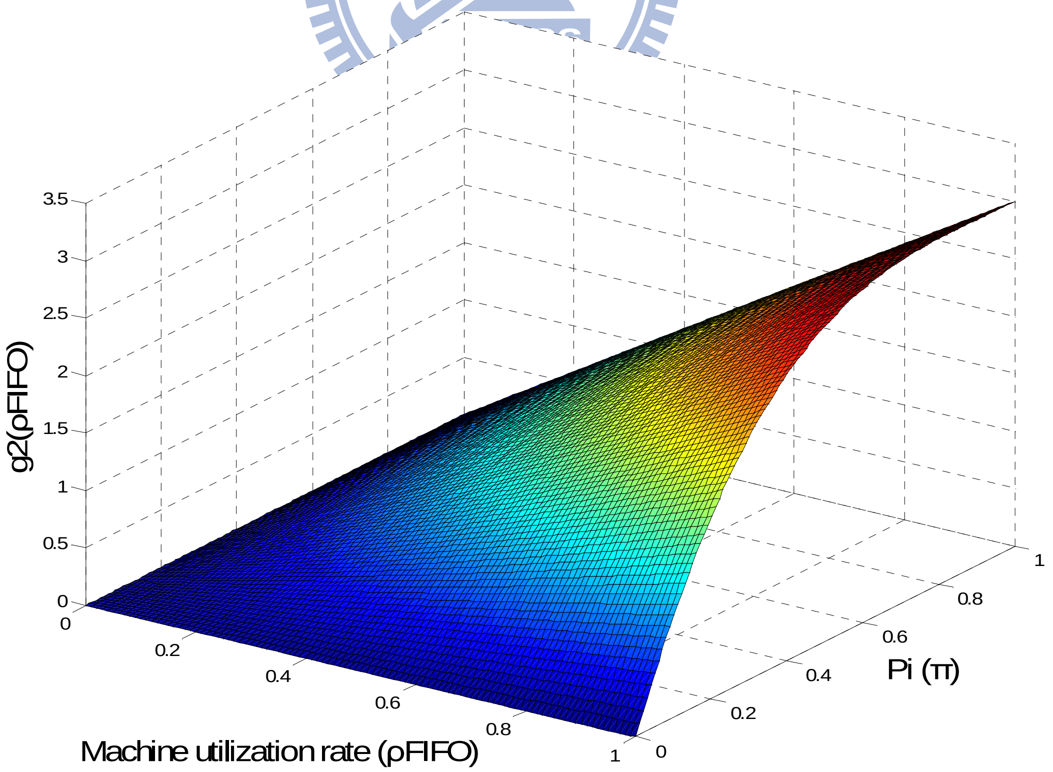

Figure 3-6 Plot of function $g_2(\rho_{FIFO})$

In general, the square of the expected waiting time is more than the second moment of the residual service time greatly; hence a large proportion of the variance of waiting time is the square of the expected waiting time. Value of α_j in Equation (3-22) has two distinct possibilities: (1) $0 \leqq \alpha_j < 1$ when the machine utilization rate is smaller and (2) $\alpha_j \geqq 1$ when the machine utilization rate is higher. This means that an approximation of the square of the expected waiting time of product type j jobs ($\{\lim_{RT \to \infty} E[W_{q,j,FIFO}]\}^2$) may be higher than the square of the expected waiting time for the $M/G/1$ queuing model ($[W_q]^2_{M/G/1}$), that is to say, the variance of waiting time of product type j jobs of this research could be greater than the variance of waiting time in Equation (2-6) by substituting the moments of the service time into the formula for an $M/G/1$ queue with high levels of workload on the machine.

3.4 Summary

In this section, a single machine system for processing several product types according to the FIFO rule is considered, and a setup of job to switch from the current setting to a different one is necessary. The expected values and the variances of the service time and the waiting time for multi-product types with setup time are determined. The advantage of using these estimates is that it can estimate important performance measure like the lead time, which contributes to the possibility of providing a due date for the order.

4. FSR analytic model for evaluating the effect of capacity-saving

FSR implies following the criterion for selecting jobs that are of the same product type and need the same machine setting, hence those that are processed consecutively. Queued jobs with the same product type as the previous job on the machine indicates higher priority for processing (Missbauer [9]).

In this section, FSR analytic model is developed to estimate the number of setups and the setup time for the single-machine problem in order to evaluate the effect of capacity-saving with the adoption of FSR. Due to the difficulty in directly solving analytical solutions for the expected setup time and service time, a numerical analysis is used. If the numerical solutions of the expected setup time and service time are solved, then the amounts of capacity wastage due to changes in the machine setting across several product types are evaluated. In Section 3, FIFO analytic model is adopted to estimate the expected setup time, and consequently, for comparison with those under FSR. After replacing FIFO with FSR, the effect of the latter on reducing setup time and capacity loss is explored further. The flowchart of FSR analytic model on the reduction of setup time and capacity loss is shown in Figure 4-1.

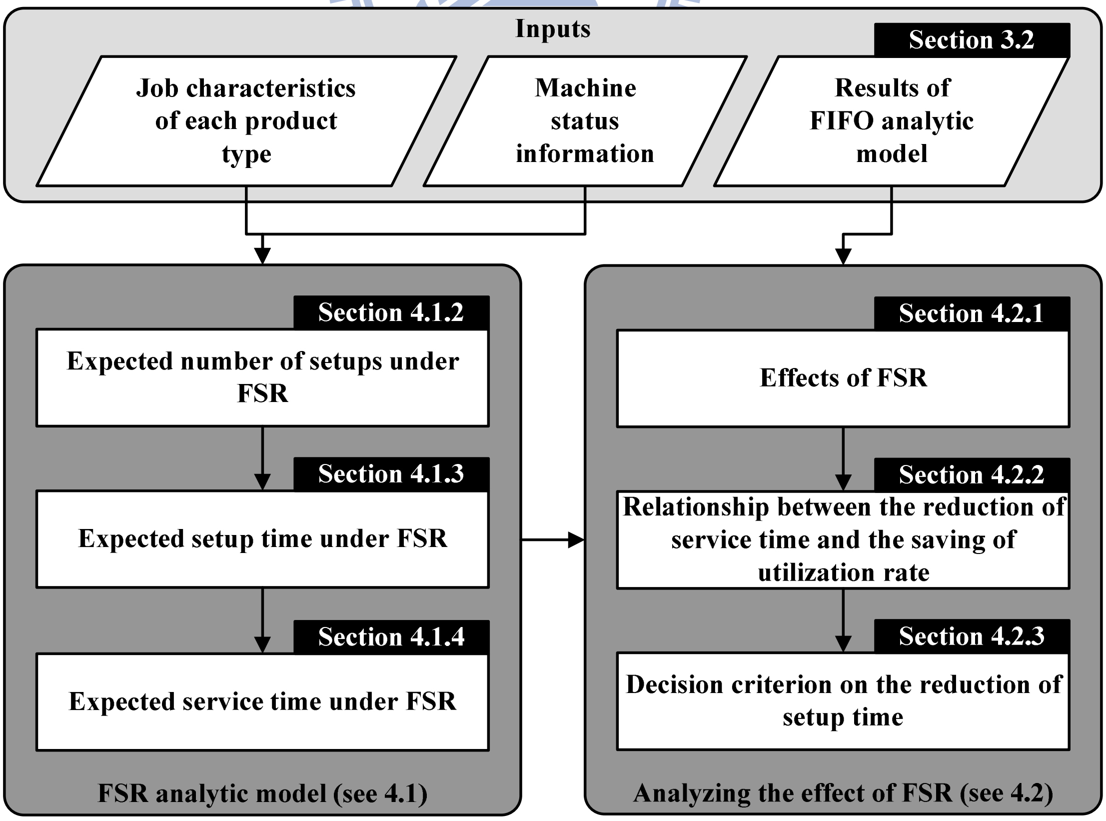

Figure 4-1 Flowchart of FSR analytic model on the reduction of setup time and capacity loss

4.1 FSR analytic model

When a job of specific product type arrives at the system, it may enter the queue of the batch (i.e., by product type) and wait for processing on machine, as required by FSR. FSR consists of two parts: (1) the assignment of a newly arrived job to a specific batch on queue based on the type of product family, which cannot be dispatched immediately on the machine, and (2) the dispatching of a next candidate job from several batches on queue that should be processed by the busy machine.

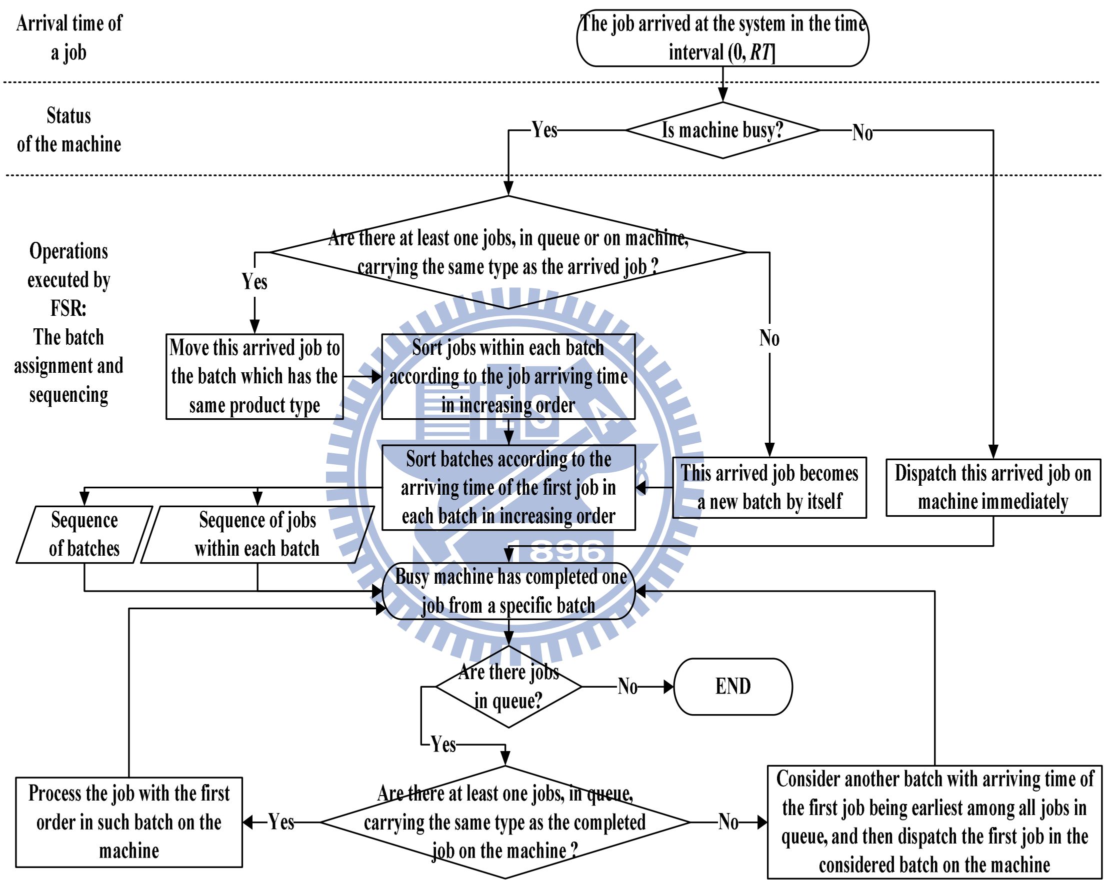

Figure 4-2 Flow chart of family-based scheduling rule

The operation executed by FSR is illustrated in Figure 4-2. When a job of a specific product type arrives at the system, if the machine is idle, FSR immediately dispatches this newly arrived job on machine. However, if the machine is busy and there is at least one job on queue or on machine, by carrying the same type as the new arrived job, FSR moves the arrived job to the batch with the same product type. If the machine is busy but there are no jobs (i.e., either on queue or on machine), by carrying the same type as the newly arrived job, FSR by itself transforms the arrived job into a new batch. When an arrived job is moved into an existing batch, jobs are sorted according to job arrival time in increasing order. Once the busy machine has completed one job on a specific batch, then the job with the first order in the same batch is processed. After all jobs in this batch are completed, another batch designated as having the earliest arrival time of the first job among all jobs on queue is picked. Then, the first job is dispatched on machine. If FSR cannot find another batch on queue for machine processing, implying that no jobs are waiting on queue, then the machine becomes idle.

4.1.1 Probability of requiring setups under FSR

Note that before starting the processing of a new job, a setup is required if the type of job is different from the last completed job on machine. Similarly, when a job of specific product type arrives at the system at a time when the machine is busy, a setup is required if there is an additional new batch generated. For this purpose, let $P_{s,ij,FSR}$ be the probability of requiring a setup under FSR, given that i^{th} job of product type j arrives at the system at time interval (0, RT]. The probability $P_{s,ij,FSR}$ is given by Equation (4-1), where $P_{s,j,FSR}$ is the probability of requiring a setup under FSR, given that product type j job arrives at the system at time interval (0, RT].

$$P_{s,ij,FSR} = \Pr\left[T_{ij} \le RT\right] \times P_{s,j,FSR} \tag{4-1}$$

The probability $P_{s,j,FSR}$ should consider the number of jobs queued in the system. This includes two cases: (1) no jobs and (2) n ($n \geqq 1$) jobs. Thus, $P_{s,j,FSR}$ is defined by Equation (4-2).

$$P_{s,j,FSR} = p_{0,FSR} P_{setups,FSR}^{n=0} + \sum_{n=1}^{\infty} p_{n,FSR} P_{setups,FSR}^{n\ge1} \tag{4-2}$$

In Equation (4-2), $p_{0,FSR}$ and $p_{n,FSR}$ are the probabilities under FSR under conditions that there are no jobs and there are n ($n \geqq 1$) jobs in the system, and $P_{setups,FSR}^{n=0}$ and $P_{setups,FSR}^{n\ge1}$ are the probabilities of requiring a setup under FSR for a job of type j arriving at a time when there are no jobs and there are n ($n \geqq 1$) jobs in the system.

The probability $P_{s,j,FSR}$ is presented as follows: For the first condition, the i^{th} job of type j arrives at time interval (0, RT] and there are no jobs in the system. A setup is necessary if this arrived job is different from the job previously completed by the current idle machine. Therefore, $P_{setups,FSR}^{n=0}$ can be expressed as $(1-\lambda_j/\lambda)$, which indicates the probability that the previously completed job on the current idle machine is different from type j. For the second condition, the i^{th} job of type j arrives at time interval (0, RT] and there are n ($n \geqq 1$) jobs in the system. A setup is necessary if there are no jobs in the system belonging to type j. Therefore, $P^{n\geq 1}_{setups,FSR}$ is equal to $(1-\lambda_j/\lambda)^n$.

By referring to Equations (4-1) and (4-2), the probability of requiring a setup for i^{th} job of product type j under FSR ($P_{s,ij,FSR}$) is rewritten as Equation (4-3). Note that $P_{ns,ij,FSR}$ is the probability of a setup that is not required by i^{th} job of product type j under FSR, which is given as $(1-P_{s,ij,FSR})$.

$$P_{s,ij,FSR} = \Pr\left[T_{ij} \leq RT\right]\left[p_{0,FSR}\left(1-\frac{\lambda_j}{\lambda}\right)+\sum_{n=1}^{\infty} p_{n,FSR}\left(1-\frac{\lambda_j}{\lambda}\right)^n\right] \tag{4-3}$$

To simplify the calculation of $P_{s,ij,FSR}$, the probabilities ($p_{0,FSR}$ and $p_{n,FSR}$) need to be defined. If $p_{0,FSR}$ and $p_{n,FSR}$ are approximated by the $M/G/1$ formula, then $p_{0,FSR}$ and $p_{n,FSR}$ are approximately set to $(1-\rho_{FSR})$ and $(1-\rho_{FSR})(\rho_{FSR})^n$, respectively, as executed in Missbauer [9] and Chern and Liu [10]. Subsequently, $P_{s,ij,FSR}$ can be reformulated as Equation (4-4), where ρ_{FSR} is the machine utilization rate under FSR for the single machine. It is equal to $\lambda E[ST_{FSR}]$, where λ is the total arrival rate and $E[ST_{FSR}]$ is the expected service time of jobs under FSR.

$$P_{s,ij,FSR} = \Pr\left[T_{ij} \leq RT\right]\left(1-\frac{\lambda_j}{\lambda}\right)\left\{1-\rho_{FSR}\left[1-\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-1}\right]\right\} \tag{4-4}$$

4.1.2 Expected number of setups under FSR

$P_{s,ij,FSR}$ represents the probability of requiring "one" setup under FSR and given by i^{th} new job of type j; $(1-P_{s,ij,FSR})$ represents the probability of requiring "no" setup under FSR and given by i^{th} new job of type j. The expected number of setups under FSR for the i^{th} arrived job of product type j can be derived as Equation (4-5).

$$E\left[NS_{ij,FSR}\right] = 1\times P_{s,ij,FSR} + 0\times\left(1-P_{s,ij,FSR}\right) = P_{s,ij,FSR} \tag{4-5}$$

Suppose there arrives n_j independent product type j jobs at time interval (0, RT]. Using the summation of $E[NS_{ij,FSR}]$ for all i, the expected number of setups of product type j under FSR is computed as $E[NS_{j,FSR}] = \sum_{i=1}^{n_j} E[NS_{ij,FSR}]$, where $n_j=\lambda_j RT$ and j=1, 2, …, J. Finally, using the summation of $E[NS_{j,FSR}]$ for all j, the expected number of setups for all jobs under FSR is calculated as $E[NS_{FSR}] = \sum_{j=1}^{J} \sum_{i=1}^{n_j} E[NS_{ij,FSR}]$.

4.1.3 Expected setup time under FSR

For this purpose, let s_{rj} be the setup time prior to the processing of a job with product type j right after the last completed job belonging to product type r, referred to as predecessor. The length of the required setup time depends on product type change between any two consecutive jobs. We consider the following three cases with the inclusion of job arrival time: (1) The i^{th} job of product type j does not arrive at time interval (0, RT]. Then, the setup time should equal 0 with the probability (1 − Pr[$T_{ij} \leqq RT$]). (2) The i^{th} job of product type j arrives at time interval (0, RT] but a setup is not needed. Thus, the setup time s_{jj} would be equal to 0 with the probability Pr[$T_{ij} \leqq RT$](1 − $P_{s,j,FSR}$). (3) The i^{th} job of product type j arrives at time interval (0, RT] and a setup is needed. This implies that the product type of the arrived job is different from the predecessor. Therefore, the setup time would be equal to s_{rj} with the probability Pr[$T_{ij} \leqq RT$]$P_{s,j,FSR}(\lambda_r/\lambda^c)$, where r = 1, 2, …, J, $r \neq j$, and $\lambda^c = \sum_{r=1, r \neq j}^{J} \lambda_r$.

Based on the abovementioned three cases, Equation (4-6) can be used to estimate the expected setup time for i^{th} job of product type j arriving at time interval (0, RT] under FSR.

$$
\begin{aligned}
E\left[S_{ij,FSR}\right] &= \left(1-\Pr\left[T_{ij} \le RT\right]\right)\times 0 + \Pr\left[T_{ij} \le RT\right]\left(1-P_{s,j,FSR}\right)\times s_{jj} + \\
&\qquad \Pr\left[T_{ij} \le RT\right] P_{s,j,FSR} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \\
&= \Pr\left[T_{ij} \le RT\right] P_{s,j,FSR} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \\
&= P_{s,ij,FSR} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj}
\end{aligned}
\qquad (4\text{-}6)
$$

Then, the expected mean setup time for product type j jobs and the expected mean setup time for a job under FSR are expressed as $E[S_{j,FSR}] = E[\sum_{i=1}^{n_j} S_{ij,FSR} / n_j]$ and $E[S_{FSR}] = E[\sum_{j=1}^{J} \sum_{i=1}^{n_j} S_{ij,FSR} / \sum_{j=1}^{J} n_j]$, respectively. Applying Equation (4-6) to $E[S_{j,FSR}]$ and $E[S_{FSR}]$ yields Equations (4-7) and (4-8), where $n_j = \lambda_j RT$ and j=1, 2, …, J.

$$E\left[S_{j,FSR}\right]=n_j^{-1}\sum_{i=1}^{n_j}P_{s,ij,FSR}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \tag{4-7}$$

$$E[S_{FSR}]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}P_{s,ij,FSR}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \tag{4-8}$$

4.1.4 Expected service time under FSR

The service time of a job is equal to the sum of its processing time and its setup time. Therefore, the expected service time for a job also relates to the three cases when estimating the setup time, as mentioned in Section 4.1.3. Moreover, the processing time of a job depends on its product type.

In this context, let $ST_{ij,FSR}$ be the random variable of service time for i^{th} job of product type j under FSR. The probability mass function of $ST_{ij,FSR}$ can then be shown as Equation (4-9). The expected mean service time for specific type j jobs and expected mean service time for a job are defined by $E[ST_{j,FSR}]=E[\sum_{i=1}^{n_j}ST_{ij,FSR}/n_j]$ and $E[ST_{FSR}]=E[\sum_{j=1}^{J}\sum_{i=1}^{n_j}ST_{ij,FSR}/\sum_{j=1}^{J}n_j]$, respectively. According to the probability mass function of $ST_{ij,FSR}$, $E[ST_{j,FSR}]$ and $E[ST_{FSR}]$ can be derived as Equations (4-10) and (4-11), where pt_j is the job processing time of product type j, $n_j=\lambda_j RT$, and j=1, 2, …, J.

$$P\left(ST_{ij,FSR}=st_{ij}\right)=\begin{cases}1-\Pr\left[T_{ij}\leq RT\right], & \text{if } st_{ij}=0\\ \Pr\left[T_{ij}\leq RT\right]\left(1-P_{s,j,FSR}\right), & \text{if } st_{ij}=pt_j\\ P_{s,ij,FSR}\dfrac{\lambda_r}{\lambda^c}, & \text{if } st_{ij}=pt_j+s_{rj},\\ & \quad r=1,2,\ldots,J,\ r\neq j\end{cases} \tag{4-9}$$

$$E\left[ST_{j,FSR}\right]=n_j^{-1}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\leq RT\right]\left[pt_j+P_{s,j,FSR}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right] \tag{4-10}$$

$$E[ST_{FSR}]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\leq RT\right]\left[pt_j+P_{s,j,FSR}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right] \tag{4-11}$$

4.2 Analyzing the effect of FSR on the reduction of setup time and capacity loss

With the analytic model developed in Section 4.1, the effect of FSR on the reduction of setup time and capacity loss is further explored by comparing the results with the FIFO rule. Relative to FSR, FIFO dispatches jobs even without batching some jobs into the same type in order to process them consecutively. This implies wastage in setup frequency. Based on the FIFO principle, a setup occurs when any two consecutive jobs in the sequence have different product types and the total setup time may take up a large part of the machine capacity. Therefore, selecting FSR instead of FIFO may contribute to a reduction in setup frequency, setup time, and machine capacity utilization rate, and consequently, lessened capacity loss. In this section, we first compare the effect of FSR with FIFO in terms of reduced setup time and machine utilization rate. Second, we provide details on how machine utilization rate is saved by FSR while dispatching jobs as a result of setup time reduction, and then demonstrate how the effect of FSR on reducing utilization rate is related to the level of total arrival rate.

4.2.1 Effects of FSR

According to Equation (4-4) and the definition of $P_{s,ij,FIFO}$ as $P_{s,ij,FIFO} = \Pr[T_{ij} \leqq RT](1 - \lambda_j/\lambda)$ in Equation (3-8), the probability of $P_{s,ij,FSR}$ can be rewritten as Equation (4-12), where $P_{s,ij,FIFO}$ is the probability of requiring a setup under FIFO, given that the i^{th} job of type j arrives at time interval $(0, RT]$.

$$P_{s,ij,FSR} = P_{s,ij,FIFO}\left\{1-\rho_{FSR}\left[1-\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-1}\right]\right\} \tag{4-12}$$

The following theorems can then be used to state the effect of FSR in relation to FIFO.

Theorem 1. $P_{s,ij,FSR} \leqq P_{s,ij,FIFO}$, if $J > 0$ and $0 \leqq \rho_{FSR} < 1$ with $\lambda_j > 0$ for all j.

Theorem 2. $P_{s,ij,FSR} < P_{s,ij,FIFO}$, if $J > 0$ and $0 < \rho_{FSR} < 1$ with $\lambda_j > 0$ for all j.

An inequality expressed as Equation (4-13) can be used to explain the above theorems. In particular, the probability of requiring a setup under FSR is always less than or equal to the probability of requiring a setup under FIFO. Therefore, FSR can be used to reduce the setup frequency by assigning jobs on queue to a specific batch according to their product type. The effect of FSR on reducing setup time, service time, and capacity loss based on Theorem 1 can be expressed as the following.

$$1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\begin{cases}=1, \text{ if } \rho_{FSR}=\sum_{n=1}^{\infty}p_{n,FSR}=0 \text{ with } \lambda_j>0,\ \forall j\\ >1, \text{ if } 0<\rho_{FSR}=\sum_{n=1}^{\infty}p_{n,FSR}<1 \text{ with } \lambda_j>0,\ \forall j\end{cases} \tag{4-13}$$

Lemma 1. $E[S_{FSR}] \leqq E[S_{FIFO}]$

The expected mean setup time under FIFO for jobs arriving at time interval (0, RT] $E[S_{FIFO}]$ in Equation (3-11) can be expressed as Equation (4-14). According to Theorem 1, the expected mean setup time under FSR in Equation (4-8), $E[S_{FSR}]$, is always less than or equal to that under FIFO.

$$E[S_{FIFO}]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}P_{s,ij,FIFO}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj} \tag{4-14}$$

Lemma 2. $E[ST_{FSR}] \leqq E[ST_{FIFO}]$

The expected mean service time of jobs under FIFO $E[ST_{FIFO}]$ in Equation (3-5) can be given by Equation (4-15) based on Equation (4-14). The expected mean service time of jobs under FSR, $E[ST_{FSR}]$, can be reformulated as Equation (4-16) based on Equations (4-8) and (4-11).

$$E[ST_{FIFO}]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\leq RT\right]pt_j+E[S_{FIFO}] \tag{4-15}$$

$$E[ST_{FSR}]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\leq RT\right]pt_j+E[S_{FSR}] \tag{4-16}$$

Then, $E[ST_{FIFO}]-E[ST_{FSR}]=E[S_{FIFO}]-E[S_{FSR}]$ is derived from Equations (4-15) and (4-16). Note that $E[ST_{FSR}] \leqq E[ST_{FIFO}]$ is the result of $E[S_{FIFO}] \geqq E[S_{FSR}]$. This means that service time can be reduced by using FSR when dispatching jobs.

Lemma 3. $\rho_{FSR} \leqq \rho_{FIFO}$

For a single machine, machine utilization rates under FIFO and FSR are shown as $\rho_{FIFO}=\lambda E[ST_{FIFO}]$ and $\rho_{FSR}=\lambda E[ST_{FSR}]$, respectively. In accordance with $E[ST_{FSR}] \leqq E[ST_{FIFO}]$, $\rho_{FSR} \leqq \rho_{FIFO}$ if the total arrival rate is given. This implies that machine utilization rate can be reduced by replacing FIFO with FSR when dispatching jobs.

Savings in machine utilization rate by replacing FIFO with FSR can be written as $\Delta\rho = \rho_{FIFO} - \rho_{FSR} = \lambda(E[S_{FIFO}] - E[S_{FSR}])$. From Equations (4-8), (4-12), and (4-14), $\Delta\rho$ can then be written as Equation (4-17) depending on the machine utilization rate under FSR (ρ_{FSR}).

Prior to the discussion of the influence of ρ_{FSR} on the savings in machine utilization rate, the first derivative of $\Delta\rho$ with respect to ρ_{FSR} is used and given by Equation (4-18). Note that $d\Delta\rho/d\rho_{FSR} \geqq 0$ with $0 \leqq \rho_{FSR} < 1$ and $\lambda_j > 0$ based on Equation (4-13). Let ρ_{FSR1} and ρ_{FSR2} be two different machine utilization rates under FSR and $\rho_{FSR1} \geqq \rho_{FSR2}$. Using ρ_{FSR1} and ρ_{FSR2} in Equation (4-17), $\Delta\rho(\rho_{FSR1})$ and $\Delta\rho(\rho_{FSR2})$ can then be computed. Next, $\Delta\rho(\rho_{FSR1}) \geqq \Delta\rho(\rho_{FSR2})$ is set in accordance with Equation (4-18), where $0 \leqq \rho_{FSR1} < 1$ and $0 \leqq \rho_{FSR2} < 1$. Thus, savings in machine utilization rate achieved by replacing FIFO with FSR increases with the rise in utilization rate of the machine. This implies that more savings in machine utilization rate is achieved with high levels of workload on machine.

$$\Delta\rho = \lambda\rho_{FSR}\left(\sum_{j=1}^{J} n_j\right)^{-1} \sum_{j=1}^{J}\sum_{i=1}^{n_j} P_{s,ij,FIFO}\left(\sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj}\right)\left[1-\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-1}\right] \tag{4-17}$$

$$\frac{d\Delta\rho}{d\rho_{FSR}} = \lambda\left(\sum_{j=1}^{J} n_j\right)^{-1} \sum_{j=1}^{J}\sum_{i=1}^{n_j} P_{s,ij,FIFO}\left(\sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj}\right) \times \left[1-\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-1} + \frac{\lambda_j}{\lambda}\frac{\rho_{FSR}}{(1-\rho_{FSR})^2}\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-2}\right] \geq 0 \tag{4-18}$$

4.2.2 Relationship between the reduction of service time and the saving of utilization rate by varying total arrival rate

In earlier discussions, we mentioned that savings in machine utilization rate ($\Delta\rho$) depends on machine utilization rate under FSR (ρ_{FSR}), which also depends on total arrival rate (λ) and reduced service time. Next, we investigate how savings in machine utilization rate can be affected by the changes in total arrival rate and reduction of service time. The result is plotted in Figure 4-3.

For a single machine system, by referring to the queuing theory, the expected service time ($E[ST]$) is proportional to the utilization rate of machine (ρ) with gradient $1/\lambda$; this denotes an inverse of total arrival rate (Ross [33]). Thus, the expected service time behaves as a function of machine utilization rate. In relation, the straight line in Figure 4-3 can be depicted, which passes through the origin with the slope equal to the inverse of total arrival rate ($1/\lambda$). In Figure 4-3(a), a line with slope $1/\lambda^*$ and intercept zero, $E[ST]=\rho/\lambda^*$ can be obtained for a given specific total arrival rate λ^* and the vector of job processing time **PT**. Therefore, the expected service time under FIFO ($E[ST_{FIFO}]\big|_{\lambda^*,\mathbf{PT}}$) is calculated by Equation (4-15) using λ^* and **PT**. The machine utilization rate under FIFO ($\rho_{FIFO}\big|_{\lambda^*,\mathbf{PT}}$) can then be computed by $\lambda^* E[ST_{FIFO}]\big|_{\lambda^*,\mathbf{PT}}$.

Similarly, a curve of expected service time under FSR ($E[ST_{FSR}]$) by varying the machine utilization rate under can be seen in Figure 4-3(a). Based on Equations (4-4), (4-8), and (4-16), the estimation of expected service time under FSR ($E[ST_{FSR}]$) is required by the machine utilization rate under FSR (ρ_{FSR}) in order to compute the expected setup time ($E[S_{FSR}]$). However, by referring to the queuing theory, the machine utilization rate under FSR (ρ_{FSR}) also depends on the expected service time under FSR ($E[ST_{FSR}]$). Therefore, it is difficult to solve an analytical solution for $E[ST_{FSR}]$. Instead, a numerical analysis can be used to compute $E[ST_{FSR}]$. The numerical solution of $E[ST_{FSR}]$ can be solved by solving the two equations, $E[ST_{FSR}]=\rho_{FSR}/\lambda$ and $E[ST_{FSR}]=E[ST_{FIFO}]-E[S_{FIFO}]+E[S_{FSR}]$, derived from Equations (4-15) and (4-16). If $E[ST_{FSR}]=\rho_{FSR}/\lambda$ is substituted in $E[ST_{FSR}]=E[ST_{FIFO}]-E[S_{FIFO}]+E[S_{FSR}]$, then a new equation can be written as $f(\rho_{FSR})=E[ST_{FIFO}]-E[S_{FIFO}]+E[S_{FSR}]-\rho_{FSR}/\lambda=0$ and then it can be rewritten as Equation (4-19) based on Equation (4-15), where $E[S_{FSR}]$ can be derived by substituting Equation (4-4) with Equation (4-8).

$$f(\rho_{FSR})=\left(\sum_{j=1}^{J} n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]pt_j+E\left[S_{FSR}\right]-\frac{\rho_{FSR}}{\lambda}=0 \tag{4-19}$$

As $f(\rho_{FSR})$ is the function of ρ_{FSR} and is differentiable, the Newton's method can be used to solve the nonlinear equation, $f(\rho_{FSR})=0$. According to $f(\rho_{FSR})$ and its derivative with respect to ρ_{FSR}, we begin with a first guess of ρ_{FSR}^0 by setting $0<\rho_{FSR}^0\le 1$. An approximate solution ρ_{FSR}^1 can be obtained by calculating $\rho_{FSR}^0 - f(\rho_{FSR}^0)/f'(\rho_{FSR}^0)$, in which ρ_{FSR}^1 should be a better approximation to the solution of $f(\rho_{FSR})=0$. Once we have ρ_{FSR}^1, the process can be repeated to obtain ρ_{FSR}^{2}. After n steps, if we have an approximate solution of ρ_{FSR}^{n}, then the next step is to calculate ρ_{FSR}^{n+1} and $\rho_{FSR}^{n+1}=\rho_{FSR}^{n}-f(\rho_{FSR}^{n})/f'(\rho_{FSR}^{n})$. Note that value of ρ_{FSR}^{n} moving closer to the value of ρ_{FSR}^{n+1} indicate that the approximate solution of $f(\rho_{FSR})=0$ after n steps has been determined.

The curve of the function $f(\rho_{FSR})=0$ for various machine utilization rates is plotted in Figure 4-3(a). The function of $f(\rho_{FSR})=0$ is the expected service time under FSR ($E[ST_{FSR}]$) that shifts down with shifts in quantum ρ_{FSR}/λ. Thus, a root of $f(\rho_{FSR})=0$; that is, ρ_{FSR}^{*} is identified using the Newton's method. By giving $\rho_{FSR}=\rho_{FSR}^{*}$ for Equation (4-16) to calculate $E[ST_{FSR}]$, then machine utilization rate under FSR is obtained; that is, $\rho_{FSR}\big|_{\lambda^*,\mathbf{PT},\rho_{FSR}^*}=\lambda^* E[ST_{FSR}]\big|_{\lambda^*,\mathbf{PT},\rho_{FSR}^*}$.

In Figure 4-3(b), by changing the total arrival rate from λ^* to λ^{**}, λ^{**} is found to be smaller compared with λ^* along with the same vector of job processing time **PT**. A line $E[ST]=\rho/\lambda^{**}$ is drawn with slope $1/\lambda^{**}$; that is, the inverse of the total arrival rate and this line is steeper because $1/\lambda^{**}$ is larger compared with $1/\lambda^*$. By repeating the aforementioned steps, the expected service time and the machine utilization under FSR can then be depicted as $E[ST_{FSR}]\big|_{\lambda^{**},\mathbf{PT},\rho_{FSR}^{**}}$ and $\lambda^{**}E[ST_{FSR}]\big|_{\lambda^{**},\mathbf{PT},\rho_{FSR}^{**}}$. The expected service time and machine utilization rate under FIFO can be computed as $E[ST_{FIFO}]\big|_{\lambda^{**},\mathbf{PT},\rho_{FSR}^{**}}$ and $\lambda^{**}E[ST_{FIFO}]\big|_{\lambda^{**},\mathbf{PT},\rho_{FSR}^{**}}$.

The varied total arrival rate from λ^* to λ^{**} with small increment is depicted by the two bold curves in Figure 4-3(c). They represent the relationships between the expected service time and the machine utilization rate for various total arrival rates under FIFO and FSR, respectively. Figure 4-3(c) also illustrates the effect of varying total arrival rates on the reduction of service time, which corresponds to the pairs of machine utilization rates under FIFO and FSR. These show that the reductions of service time and machine utilization rate become larger as total arrival rate increases. Therefore, FSR can effectively reduce service time and machine utilization rate at peak demand times.

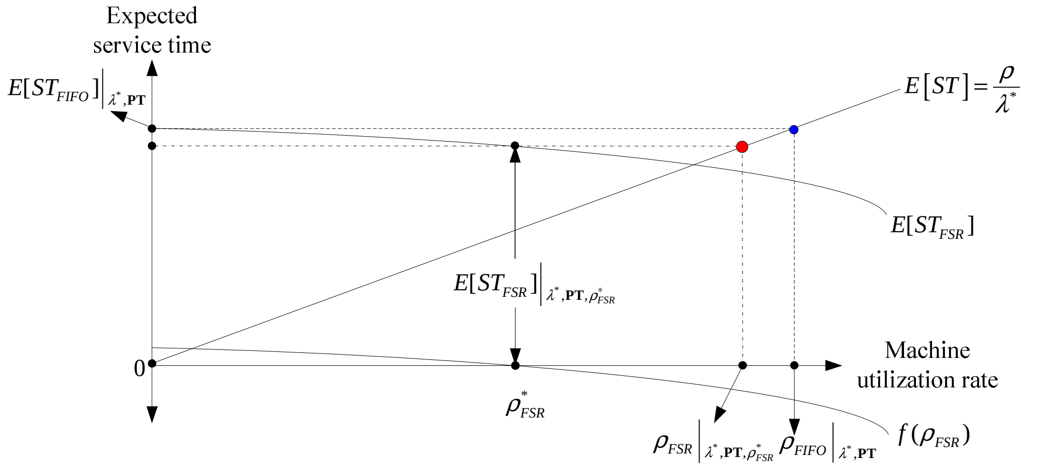

Figure 4-3(a) The expected service time under FSR for a given total arrival rate λ^*

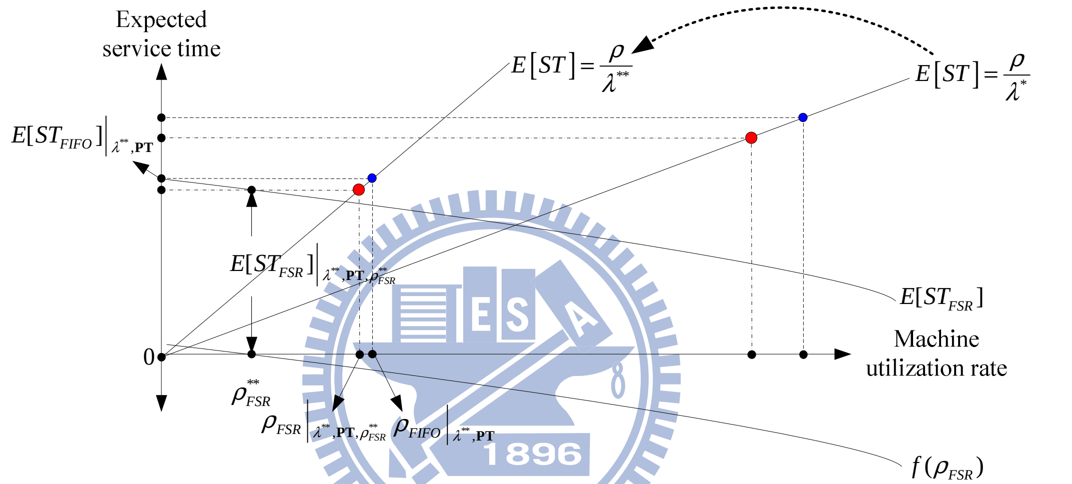

Figure 4-3(b) The expected service time by changing total arrival rate from λ^* to λ^{**}

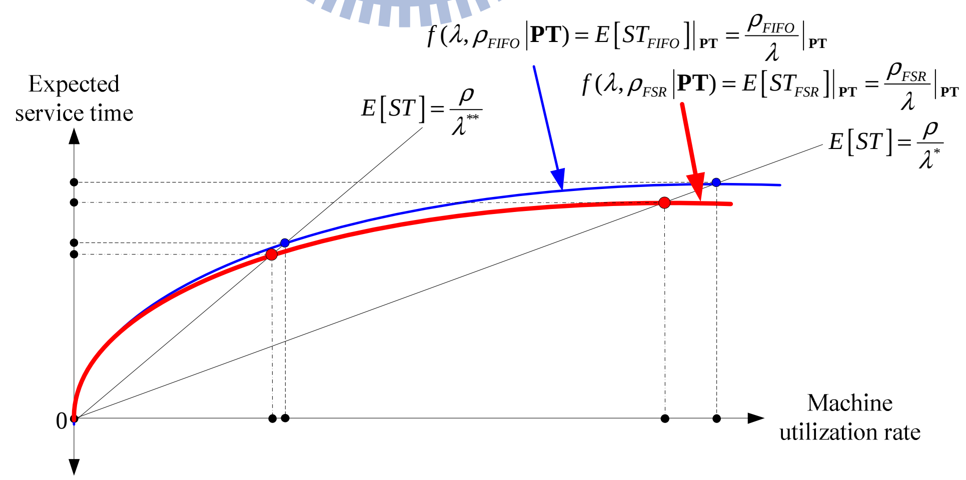

Figure 4-3(c) Two curves of expected service time for various total arrival rates under FIFO and FSR

4.2.3 Decision criterion on the reduction of setup time

In this section, we develop a decision criterion to conclude on which condition the setup time can be saved significantly by replacing FIFO with FSR. According to Lemma 1, the expected mean setup time under FSR is always less than or equal to that under FIFO. To evaluate the magnitude of setup time reduction by replacing FIFO with FSR, the difference of setup time for the i^{th} job of type j by comparing FIFO and FSR is defined as $DS_{ij,FIFO\text{-}FSR} = S_{ij,FIFO} - S_{ij,FSR}$, where $S_{ij,FIFO}$ and $S_{ij,FSR}$ represent the setup time for the i^{th} job of type j under FIFO and FSR, $i = 1, 2, \ldots, n_j$, and $j=1, 2, \ldots, J$. Therefore, the mean difference $\overline{DS}_{FIFO-FSR}$ is expressed as Equation (4-20).

$$\overline{DS}_{FIFO-FSR} = \left(\sum_{j=1}^{J} n_j\right)^{-1} \sum_{j=1}^{J}\sum_{i=1}^{n_j} DS_{ij,FIFO-FSR} = \left(\sum_{j=1}^{J} n_j\right)^{-1} \sum_{j=1}^{J}\sum_{i=1}^{n_j} \left(S_{ij,FIFO} - S_{ij,FSR}\right) \quad (4\text{-}20)$$

Let $\overline{DS}_{FIFO-FSR} > \mu_0$ represent the magnitude of setup time reduction is more than μ_0 by replacing FIFO with FSR, where $\mu_0 \geqq 0$. Therefore, by replacing FIFO with FSR, $\Pr[\overline{DS}_{FIFO-FSR} > \mu_0] \geq 1-\alpha$ indicates the probability of having the setup time reduction larger than μ_0 is more than or equal to $(1-\alpha)$, where $0<\alpha<1$.

$$\begin{aligned} &\Pr\left[\overline{DS}_{FIFO-FSR} > 0\right] = \Pr\left[Z > \frac{0 - E\left[\overline{DS}_{FIFO-FSR}\right]}{\sqrt{Var\left[\overline{DS}_{FIFO-FSR}\right]}}\right] \geq 1-\alpha \\ &\Rightarrow \frac{0 - E\left[\overline{DS}_{FIFO-FSR}\right]}{\sqrt{Var\left[\overline{DS}_{FIFO-FSR}\right]}} \leq -z_\alpha \\ &\Rightarrow E\left[\overline{DS}_{FIFO-FSR}\right] \geq z_\alpha \sqrt{Var\left[\overline{DS}_{FIFO-FSR}\right]} \end{aligned} \quad (4\text{-}21)$$

where the expected value of $\overline{DS}_{FIFO-FSR}$ and the variance of $\overline{DS}_{FIFO-FSR}$ are derived as Equation (4-22) and Equation (4-23).

$$E\left[\overline{DS}_{FIFO-FSR}\right] = E\left[\left(\sum_{j=1}^{J} n_j\right)^{-1} \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(S_{ij,FIFO} - S_{ij,FSR}\right)\right] = E\left[S_{FIFO}\right] - E\left[S_{FSR}\right] \quad (4\text{-}22)$$

$$\begin{aligned} Var\left[\overline{DS}_{FIFO-FSR}\right] = &\left(\sum_{j=1}^{J} n_j\right)^{-2} \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(P_{s,ij,FIFO} + P_{s,ij,FSR}\right)\left(\sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj}^2\right) - \\ &\left(\sum_{j=1}^{J} n_j\right)^{-2} \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(P_{s,ij,FIFO}^2 + P_{s,ij,FSR}^2\right)\left(\sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj}\right)^2 \end{aligned} \quad (4\text{-}23)$$

The further detail of mathematical proof for $Var[\overline{DS}_{FIFO-FSR}]$ is shown in Appendix I.

Thus, the probability $\Pr[\overline{DS}_{FIFO-FSR} > 0] \geq 1-\alpha$ can be computed as Equation (4-21), where z_α indicates the proportion of the area under the curve, from 0 to positive, of a standardized normal distribution is equal to $(1-\alpha)$. To calculate the probability in Equation (4-21), the random variable Z is defined as Equation (4-24), whose value is given by the difference between $\overline{DS}_{FIFO-FSR}$ and its expected value ($E[\overline{DS}_{FIFO-FSR}]$), divided by the standard error of the mean ($\sqrt{Var[\overline{DS}_{FIFO-FSR}]}$). According to the central limit theorem, the distribution of the random variable Z approaches that of a standard normal distribution as $n \to \infty$, where $n = \sum_{j=1}^{J} n_j = \lambda RT$.

$$Z = \frac{\overline{DS}_{FIFO-FSR} - E\left[\overline{DS}_{FIFO-FSR}\right]}{\sqrt{Var\left[\overline{DS}_{FIFO-FSR}\right]}} \tag{4-24}$$

Thus, if $E[\overline{DS}_{FIFO-FSR}]$ is larger than $z_\alpha \sqrt{Var[\overline{DS}_{FIFO-FSR}]}$, then the probability that the setup time of jobs can be saved by replacing FIFO with FSR is guaranteed to be more than or equal to $(1-\alpha)$.

4.3 Summary

In this section, FSR analytic model is proposed to estimate the number of setups, the setup time, and the service time for a single-machine system facing uncertain job arrival. Through this analytic model, the amount of the capacity waste can be evaluated due to the changing of machine setting among several product types and the effect of the setup time reduction by replacing FIFO with FSR can be analyzed.

5. Performance analysis for the proposed analytic models

In order to evaluate the FIFO and FSR analytical models, simulation models are built to test the performance of FIFO and FSR analytical models in several different scenarios. In the simulation models, the inter-arrival time of jobs is an exponential distribution; the dispatching of jobs for processing on a single machine depends on FIFO and FSR; and the setups occur for two different types of jobs being consecutively processed on the machine.

As stated in Section 3, the lead time under FIFO consists of two parts: the waiting time in queue and the service time on the machine. If these two parts are accurate, then the performance of the estimate of the lead time by FIFO analytical mode is guaranteed. Therefore, the accuracy of the waiting time in queue and the service time on the machine under FIFO are presented in the following sections. Next, the simulation results under FSR are collected from a fixed time period of jobs arriving with various arriving rates.

5.1 Experimental design

Missbauer [20] and Vieira et al. [29] assumed that five and ten job types can arrive to the system. Eight is simply the middle number. Therefore eight product types are processed in the simulated production system.

In FIFO simulation model, the machine is implemented to work 24 hours a day ($RT = 24$ hours $= 86{,}400$ seconds). In FSR simulation model, three levels of run time (RT) are considered: 8, 16, and 24 hours. The simulation model contains the vector of job processing time among eight product types (**PT**) and the matrix of setup time ($\mathbf{ST} = [s_{rj}]$) for switching product types on the machine, where s_{rj} is the setup time for product type j job when product type j job follows product type r job and is the element at the j^{th} row and r^{th} column of **ST**. “Second” is the unit of processing time and setup time.

According to the queuing theory, the utilization rate of the machine, which depends on the total arrival rate, has great impact on the waiting time of jobs. Thus, the total arrival rates (λ) are set to control the machine utilization rates. The arrival rates among eight product types are defined by $\lambda_j = \delta\tau_j$ and the corresponding values of τ_j are shown in Table 5-1. The total arrival rate can be calculated as $\lambda = \sum_{j=1}^{8} \lambda_j = \delta \sum_{j=1}^{8} \tau_j$ jobs in 60 seconds and is proportional to the sum of τ_j with the parameter δ. The difference between any two adjacent levels of δ is the same and equals 0.05.

In addition to the total arrival rate, the mix of the arrival rates of various types of jobs can also affect the waiting time in queue because different products have different processing time and setup frequency, as well as time consumed on the machine. Thus, the coefficient of variation among job arrival rates (symbol *CV*) is considered and is defined as the ratio of the standard deviation (s_λ) of the job arrival rate of product types to the mean ($\bar{\lambda}$) of the job arrival rate of product types multiplied by 100. Thus, it is calculated as $CV = (s_\lambda/\bar{\lambda})\times 100\% = (s_\tau/\bar{\tau})\times 100\%$, where $\bar{\tau}$ and s_τ are the mean and the standard deviation of τ_j, respectively, which are shown in Table 1. Using the information of $\bar{\tau}$ and s_τ in Table 1, the *CV*s for job arrival rates can easily be calculated as 0, 27.9753%, and 53.7234%. The higher the *CV* means the greater dispersion in the jobs arrival rates among eight product types.

Different scenarios are created by varying the total arrival rates and by varying the *CV*s for job arrival rates. All other parameters remained constant. Note that the simulated production system is always stable (i.e. the production system has enough capacity to process all jobs) because $\rho_{FIFO} = \lambda E[ST_{FIFO}]$ and $\rho_{FSR} = \lambda E[ST_{FSR}]$ are always smaller than one. Note that each scenario is simulated for five replications and the simulation results for each combination are collected after 10,000 independent simulation runs.

$$\mathbf{PT} = \begin{bmatrix} 15 & 75 & 85 & 45 & 55 & 10 & 80 & 125 \end{bmatrix}$$

$$\mathbf{ST} = \begin{bmatrix} s_{rj} \end{bmatrix} = \begin{bmatrix} 0 & 90 & 60 & 15 & 15 & 30 & 45 & 30 \\ 15 & 0 & 75 & 30 & 45 & 75 & 90 & 45 \\ 30 & 60 & 0 & 45 & 90 & 90 & 75 & 60 \\ 45 & 75 & 90 & 0 & 45 & 30 & 60 & 45 \\ 60 & 75 & 45 & 45 & 0 & 45 & 75 & 15 \\ 45 & 30 & 30 & 30 & 75 & 0 & 60 & 75 \\ 60 & 45 & 60 & 15 & 45 & 15 & 0 & 45 \\ 15 & 30 & 15 & 30 & 60 & 30 & 45 & 0 \end{bmatrix}$$

Table 5-1 Total arrival rate and coefficient of variation

Parameters among eight product types								Mean of parameters	Standard deviation of parameters	CVs (%)
τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8			
0.001250	0.001250	0.001250	0.001250	0.001250	0.001250	0.001250	0.001250	0.001250	0	0
0.001342	0.001577	0.001804	0.000917	0.001145	0.001443	0.000817	0.000955	0.001250	3.49691×10^{-4}	27.9753
0.001821	0.001255	0.000297	0.001721	0.000946	0.000363	0.001467	0.002130	0.001250	6.71543×10^{-4}	53.7234

Let $\overline{x}_{kj,VAR,RULE}$ and $s^2_{kj,VAR,RULE}$ be the sample mean and the sample variance of specific *VAR* of product type *j* jobs for the k^{th} simulation run under specific *RULE* by the simulation model. The suffix symbol of *VAR* represents the variance, where *VAR*=*S* (Setup time), *VAR*=*ST* (Service time), *VAR*=W_q (Waiting time), and *VAR*=P_s (Probability of setups), and the suffix symbol of *RULE* represents the dispatching rule, where *RULE*=*FIFO* and *RULE*=*FSR*. In simulation model, we have 10,000 sets of data containing N_{1j}, …, and N_{10000j} jobs of product type *j* with means $\overline{x}_{1j,VAR,RULE}$, …, and $\overline{x}_{10000j,VAR,RULE}$, and variances $s^2_{1j,VAR,RULE}$, …, and $s^2_{10000j,VAR,RULE}$, and then the combined mean and the combined variance of all product type *j* jobs are given by Equations (5-1) and (5-2), respectively.

$$\overline{x}_{j,VAR,RULE}=\left(\sum_{k=1}^{K}N_{kj}\right)^{-1}\sum_{k=1}^{K}N_{kj}\overline{x}_{kj,VAR,RULE} \tag{5-1}$$

$$s^2_{j,VAR,RULE}=\left(\sum_{k=1}^{K}N_{kj}\right)^{-1}\sum_{k=1}^{K}N_{kj}\left[s^2_{kj,VAR,RULE}+\left(\overline{x}_{kj,VAR,RULE}-\overline{x}_{j,VAR,RULE}\right)^2\right] \tag{5-2}$$

where N_{kj} is the sample size of product type *j* jobs for the k^{th} simulation run. The average size of product type *j* jobs can be computed as $N_j=\sum_{k=1}^{K}N_{kj}/K$ and the combined standard deviations can be calculated by taking the square roots of $s^2_{j,VAR,RULE}$.

5.2 Accuracy analysis of FIFO analytic model in estimating lead time

To analyze the accuracy of FIFO analytic model on estimating lead time, an experimental design with various arrival conditions among eight types of services, which correspond to various resource utilization rates, is conducted. Furthermore, the numerical result of the proportionality α_j in Equation (3-22) is also presented compared with $\alpha_j=1$ because an approximation of the expected waiting time for each product type is equivalent to the expected waiting time for the *M*/*G*/1 queuing theory if $\alpha_j=1$.

In terms of the accuracy of FIFO analytic model, the initial and final levels of δ are 0.75 and 0.95, respectively. For the case of the numerical analysis of the proportionality α_j, the initial and final levels of δ are 0.05 and 0.95, respectively.

5.2.1 Accuracy analysis of expected service time

By substituting the probability $P_{s,j,FIFO}=(1-\lambda_j/\lambda)$ into Equation (3-4), the expected service time of the product type j jobs can be easy to simplify as Equations (5-3), and then the limit of the expected service time of the product type j jobs can be given by Equations (5-4) as RT approaches infinity because $\lim\limits_{ER\to\infty} w_j = \lim\limits_{ER\to\infty} \sum_{i=1}^{n_j}\Pr[T_{ij}\le RT]\big/n_j = 1$.

$$E\left[ST_{j,FIFO}\right]=w_j\left(pt_j+\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda}s_{rj}\right)=w_j\left(pt_j+\sum_{r=1}^{J}\frac{\lambda_r}{\lambda}s_{rj}\right) \tag{5-3}$$

$$\lim_{RT\to\infty}E\left[ST_{j,FIFO}\right]=\left(pt_j+\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda}s_{rj}\right)=\left(pt_j+\sum_{r=1}^{J}\frac{\lambda_r}{\lambda}s_{rj}\right) \tag{5-4}$$

where w_j is the mean of all probabilities of product type j jobs arrived at the system in the time interval (0, RT]. The expected service times of single job for each product type and their limits as RT approaches infinity by varying the CVs of job arrival rate and total arrival rates are drawn in Figure 5-1.

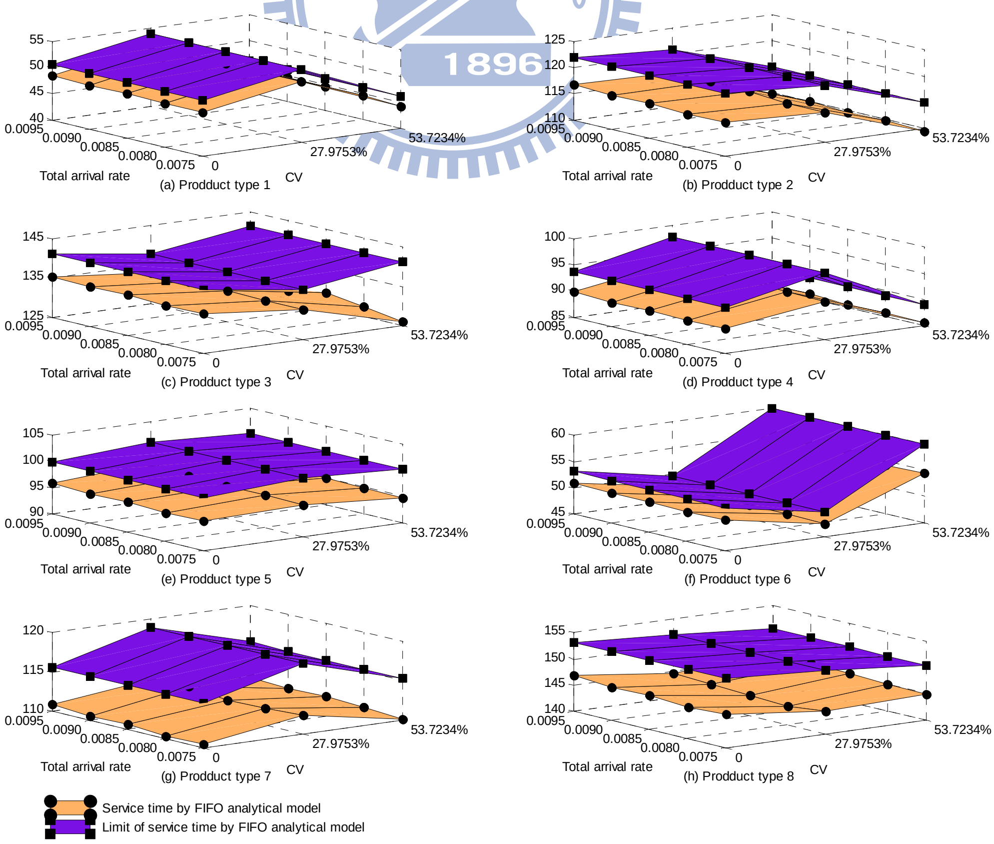

Figure 5-1 Service time and its limit by FIFO analytic model

Owing to the probability $w_j \leqq 1$, the service times by analytic model are less than their limits, which is apparent in Figure 5-1. In addition, the probabilities that the predecessor of the arrived job belongs to the specific product type ($\lambda_j / \lambda = \delta\tau_j / \delta\sum_{j=1}^{8}\tau_j = \tau_j / \sum_{j=1}^{8}\tau_j$) is unconcerned with the constant δ in the experimental design. These probabilities are fixed even the total arrival rate is changed. Thus, the change of the service time for varying total arrival rates is insignificant in Figure 5-1.

According to Equations (5-3) and (5-4), the expected service time and its limit depend on arrival rate, processing time, and setup time. When *CV* is equal to zero, the arrival rate parameters among eight product types (τ_j) are the same and then the expected service time depends on the processing time and the setup time. Thus, the expected service times for each product type by analytic model are near to their average service times as *CV*=0, where the average service time of product type *j* job is defined by the sum of the processing time and the average setup time of product type *j* job, and the average setup time of product type *j* job can be computed as the summation of s_{jr} in **ST** for all *r* divided by the number of product type.

When *CV* is equal to 27.9753% or 53.7234%, the arrival rate parameters among eight product types are unequal and then it makes setup time and service time to be changed as compared with *CV*=0. Taking product type 1 as an example, the setup times of product type 1 are the elements at the first row of **ST.** Note that the setup times of product type 1 after product type 2 and product type 3 are larger (90 and 60), but otherwise the setup times of product type 1 are smaller. When *CV* is equal to 27.9753%, the arrival rates of product type 2 and product type 3 (0.001577δ and 0.001804δ) are increasing as compared with *CV*=0. Thus, the setup time of single job of product type 1 becomes larger and then it leads to the larger service time for product type 1 when *CV* is equal to 27.9753%. When *CV* is equal to 53.7234%, the arrival rates of product type 4 and product type 8 (0.001721δ and 0.002130δ) are increasing as compared with *CV*=0. This result causes the saving in setup time and service time. It can be apparent from Figure 5-1(a).

Apart from the foregoing, there are some specific cases in Figure 5-1(c) and Figure 5-1(h). First, the expected service time of product type 3 decreases but its limit increases when *CV* increases from 27.9753% to 53.7234%. According to Equations (5-4), an increase of the limit of the expected service time is caused because of an increase of $\sum_{r=1, r\neq j}^{J}(\lambda_r/\lambda^c)s_{rj}$. As for the expected service time in Equations (5-3), it depends on w_j and $\sum_{r=1, r\neq j}^{J}(\lambda_r/\lambda^c)s_{rj}$.

The decrease by a wide margin of the arrival rate of product type 3 leads to the probability w_j being far less than one when CV increases from 27.9753% to 53.7234%. Relative to an increase of its limit, the expected service time of product type 3 decreases when CV increases from 27.9753% to 53.7234%. On the contrary, the expected service time of product type 8 increases but its limit decreases when CV increases from 27.9753% to 53.7234% because the increasing margin of the probability w_j is larger than the decreasing margin of $\sum_{r=1,r\neq j}^{J}(\lambda_r/\lambda^c)s_{rj}$. Thus, the lower and higher arrival rate resulted in the trend of the expected service time being different from its limit.

Next, to evaluate the limit of the expected service time by analytical model the simulation results of the service time are compared with the values that analytical model predicted in the experimental design. In the simulation model, the sample mean of the service time of product type j jobs for the k^{th} simulation run under FIFO ($\bar{x}_{kj,ST,FIFO}$) can be shown as Equation (5-5).

$$\bar{x}_{kj,ST,FIFO} = pt_j + \sum_{r=1}^{J} \frac{N_{krj}}{N_{kj}} s_{rj} \tag{5-5}$$

where N_{krj} represents the number of product type r job when the product type j job follows the product type r job for the k^{th} simulation run, and N_{kj} is the sample size of product type j jobs for the k^{th} simulation run and $N_{kj} = \sum_{r=1}^{J} N_{krj}$. The combined mean of the service time of product type j jobs can be calculated by Equation (5-1). Because each combination is simulated for five replications, average of five combined means of jobs service time and the limit of the expected service time by varying the CVs of job arrival rate and total arrival rates are plotted in Figure 5-2. It can be seen that the average service times of jobs by simulation model are close to the limits of the expected service time by analytic model. Meanwhile, the average service times of jobs by simulation model are smaller than the limits of the expected service time by analytic model in many cases. The further analyses are to be described as follows.

To compare the results of service time of single job generated respectively by the simulation model and analytic model, the error of estimated service time is defined by Equation (5-6).

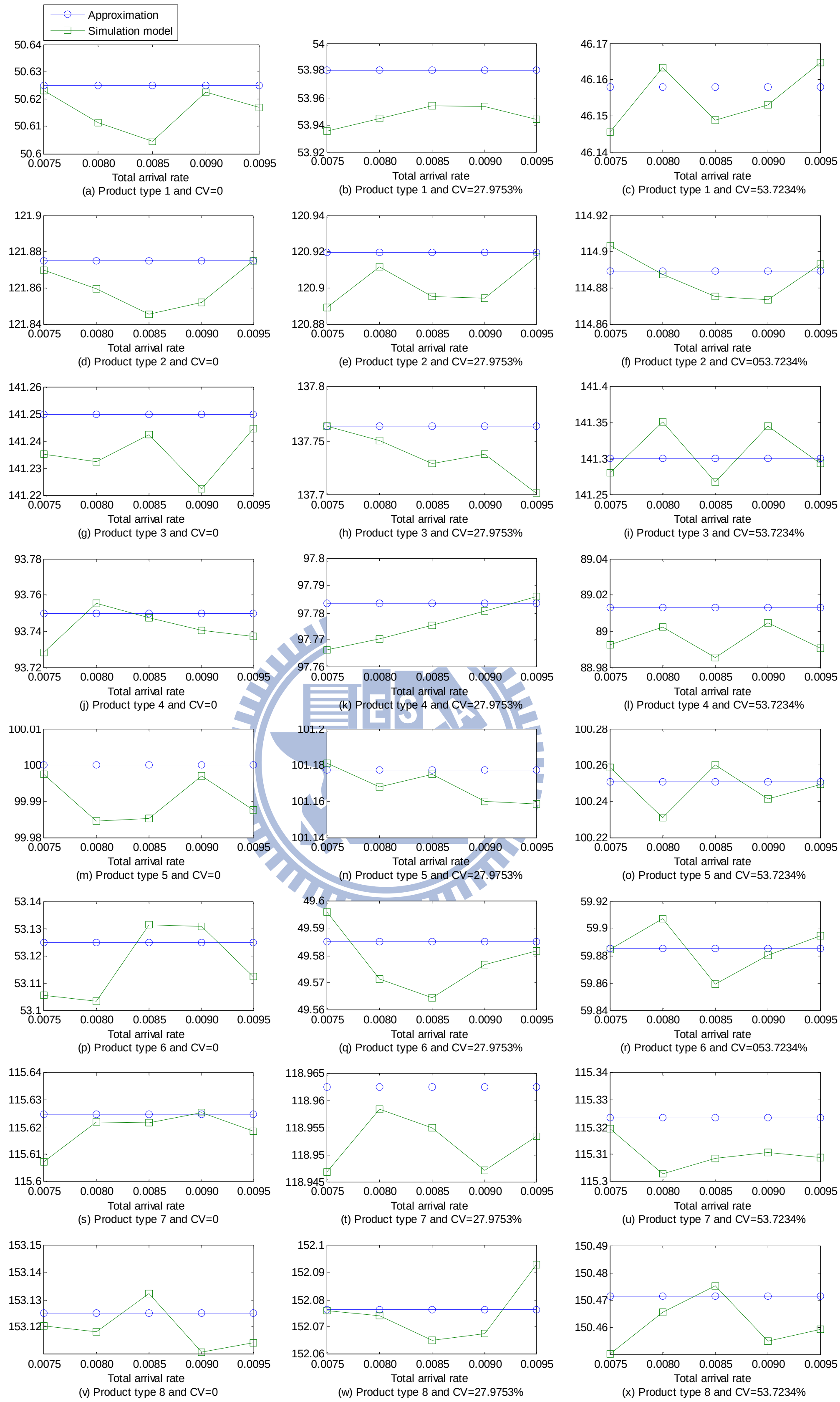

Figure 5-2 Limit of service time by analytic model and average service time by simulation model

$$Error_{j,ST,FIFO} = \lim_{RT\to\infty} E\left[ST_{j,FIFO}\right] - \overline{x}_{j,ST,FIFO} \tag{5-6}$$

where $\overline{x}_{j,ST,FIFO}$ represents the sample mean of the service time of product type j jobs by simulation model and $E[ST_{j,FIFO}]$ represents the expected service time of product type j jobs by analytic model. Figure 5-3 shows the mean errors of estimated service time by varying the *CV*s of job arrival rate and the total arrival rates. Note that these errors are almost positive. This indicates that the average service times of jobs by simulation model are smaller than the limits of the expected service time by analytic model, thus $\lambda_r/\lambda > N_{krj}/N_{kj}$ according to Equation (5-4) and Equation (5-5).

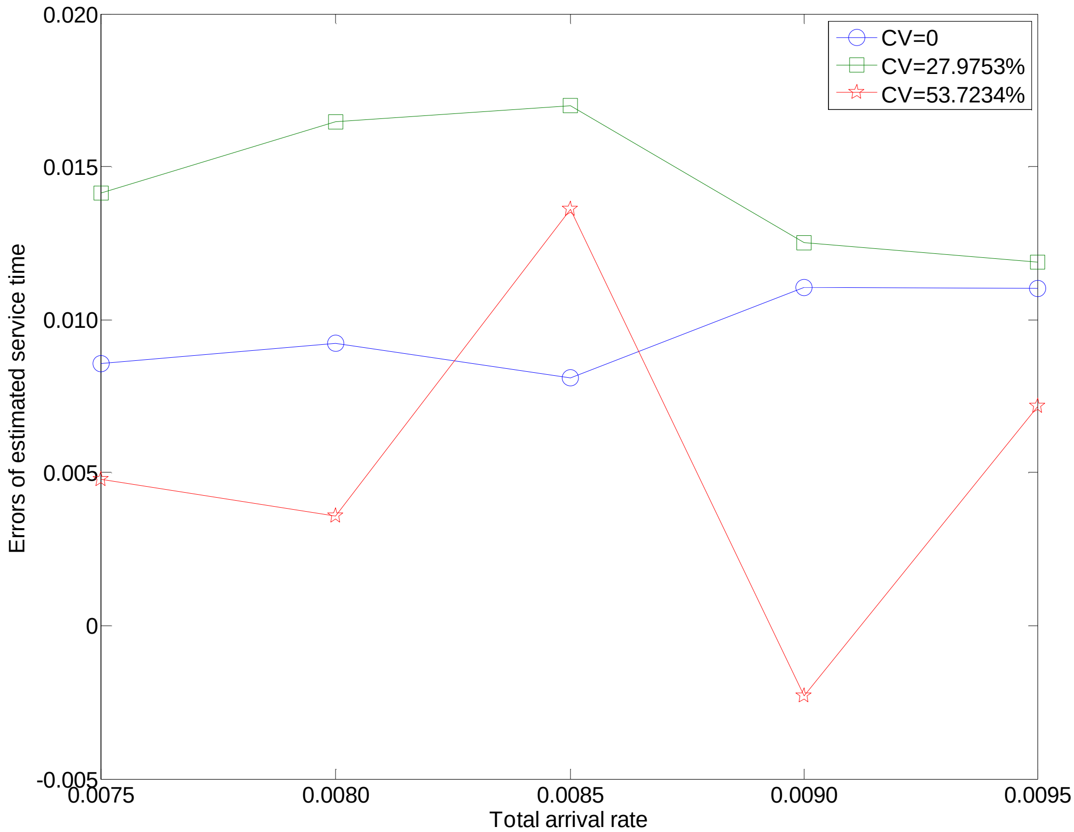

Figure 5-3 Errors of estimated service time by varying the *CV*s of job arrival rate and the total arrival rates

Figure 5-4 shows the relationship between the errors of estimated service time for each product type and the arrival rate parameters among eight product types. When *CV* equals zero, the errors are vertical because the arrival rate parameters among eight product types are the same and the errors range between 0.005716 and 0.01478. Moreover, there is a increasing relationship between the errors of the estimated service time and the arrival rate parameters among eight product types when CV=27.9753% and CV=53.7234%. When CV=27.9753%

and CV=53.7234%, the correlation coefficients are 0.6225 and 0.7456, respectively. This means that the value for the errors of the estimated service time increase as the value for the arrival rate parameters among eight product types increases. The error may be negative for the smaller arrival rate of jobs and may be positive for the larger arrival rate of jobs because there is a greater possibility of $\lambda_r/\lambda > N_{krj}/N_{kj}$ when the arrival rate of jobs is large.

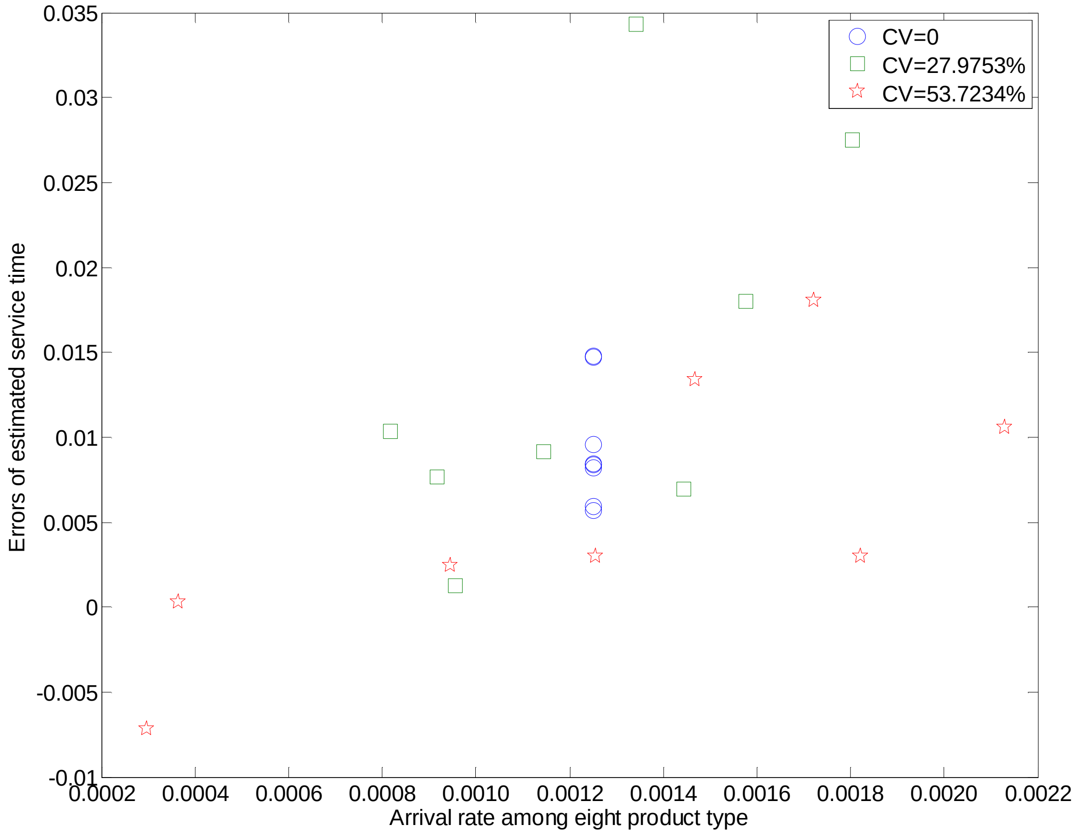

Figure 5-4 Scatter plot for errors of estimated service time under FIFO

5.2.2 Accuracy analysis of standard deviation of service time

The standard deviations of service time and their limits for each product type by varying the *CV*s of the job arrival rate and the total arrival rates by FIFO analytic model are plotted in Figure 5-5. In FIFO analytical model, the variance of the service time of the product type *j* jobs is given by $Var[ST_{j,FIFO}]=Var[PT_j]+Var[S_{j,FIFO}]$ and its limit is shown as Equation (5-7) based on Equation (3-26).

$$\lim_{RT\to\infty} Var\left[ST_{j,FIFO}\right] = \lim_{RT\to\infty} Var\left[S_{j,FIFO}\right]$$

$$= \left(1-\frac{\lambda_j}{\lambda}\right)\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}^2 - \left[\left(1-\frac{\lambda_j}{\lambda}\right)\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right]^2 \tag{5-7}$$

$$= \sum_{r=1}^{J}\frac{\lambda_r}{\lambda}s_{rj}^2 - \left(\sum_{r=1}^{J}\frac{\lambda_r}{\lambda}s_{rj}\right)^2$$

where PT_j and $S_{j,FIFO}$ are the variables of processing time and setup time of the product type j jobs.

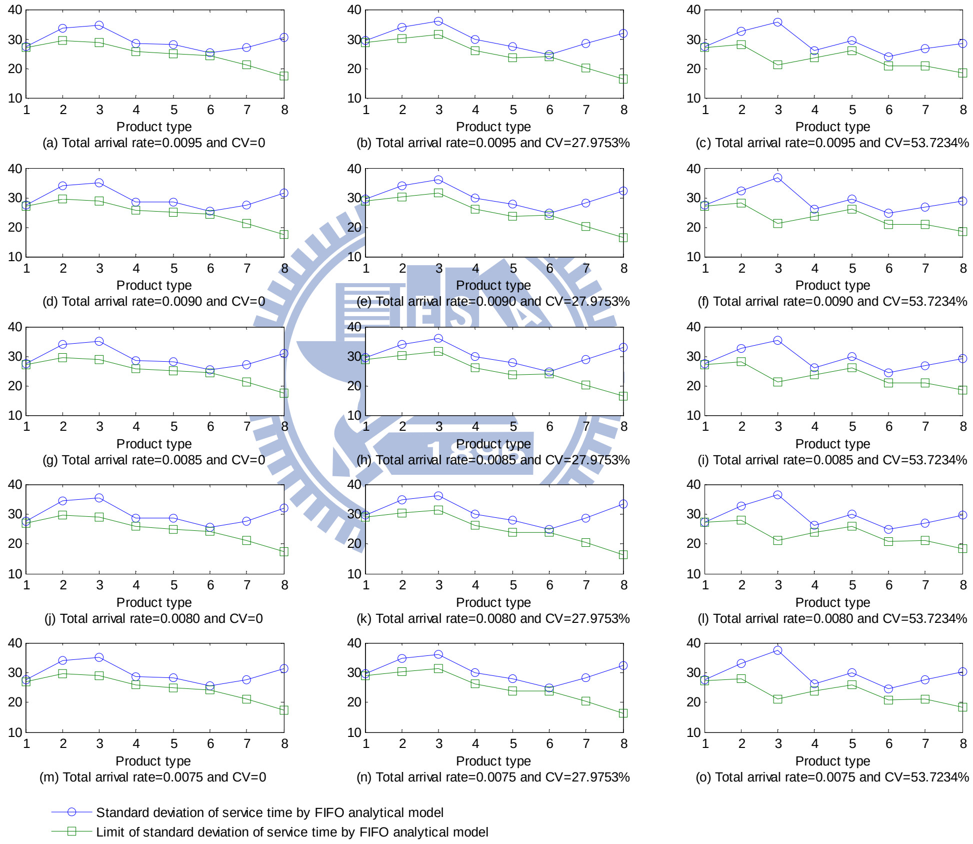

Figure 5-5 Standard deviations of service time and their limits by FIFO analytic model

In Figure 5-5, the differences between the standard deviation of service time and their limits for product type 6 and product type 8 by FIFO analytic model are minimum and maximum, respectively. The difference between the standard deviation of service time and their limits is equivalent to the standard deviation of processing time. There are two possible categories of the processing time of product type j and are labelled by value pt_j and value 0, in which value pt_j responses "job arrives in (0, RT]" and value 0 responses "job does not arrive in (0, RT]". Thus there is a greater dispersion in the jobs processing time when the jobs processing time becomes larger, and then the gaps between the standard deviations of service time and their limits are widened as the processing time of jobs increases.

Next, the simulation results of the standard deviation of service time are compared with the values that FIFO limit analytical model predicted in order to evaluate the limit of the standard deviation of service time by FIFO analytical model. In simulation model, the service time of the product type *j* jobs is equal to its processing plus its setup time ($pt_j + s_{rj}$), where the processing time of the product type *j* (pt_j) depends on its product type and the setup time (s_{rj}) depends on its product type and the product type of its predecessor. Thus, the sample variance of the service time of the product type *j* jobs in simulation model ($s^2_{j,ST,FIFO}$) can be calculated as Equation (5-8).

$$
\begin{aligned}
s^2_{j,ST,FIFO} &= \frac{\sum_{r=1}^{J} N_{krj}\left(pt_j + s_{rj}\right)^2 - N_{kj}\overline{x}^2_{j,ST,FIFO}}{N_{kj}} \\
&= \frac{\sum_{r=1}^{J} N_{krj}\left(pt_j + s_{rj}\right)^2 - N_{kj}\left(pt_j + \sum_{r=1}^{J}\frac{N_{krj}}{N_{kj}}s_{rj}\right)^2}{N_{kj}} \\
&= \sum_{r=1}^{J}\frac{N_{krj}}{N_{kj}}s^2_{rj} - \left(\sum_{r=1}^{J}\frac{N_{krj}}{N_{kj}}s_{rj}\right)^2 \\
&= \frac{\sum_{r=1}^{J} N_{krj}s^2_{rj} - N_{kj}\left(\sum_{r=1}^{J}\frac{N_{krj}}{N_{kj}}s_{rj}\right)^2}{N_{kj}} \\
&= s^2_{j,S,FIFO}
\end{aligned}
\qquad (5\text{-}8)
$$

According to Equation (5-8), the sample variance of the service time of the product type *j* jobs in simulation model is equal to the sample variance of the setup time of the product type *j* jobs in simulation model ($s^2_{j,S,FIFO}$) because the processing time of the product type *j* is fixed. In other words, a comparison of the limits of standard deviation of service time from FIFO analytical model to the values of standard deviation of service time from simulation model is equivalent to a comparison of the limits of standard deviation of setup time from FIFO analytical model to the values of standard deviation of setup time from simulation model.

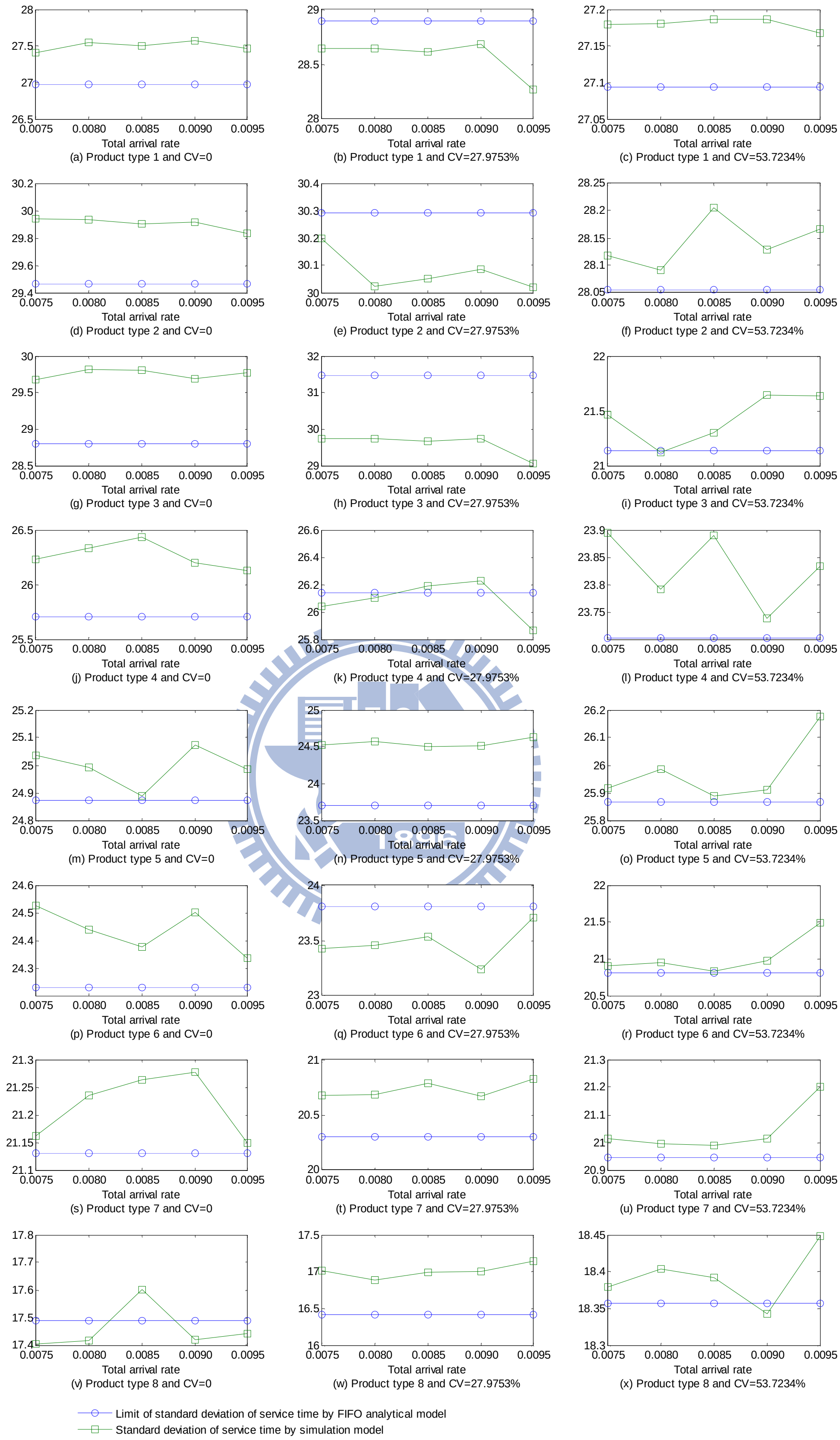

Figure 5-6 Limit of standard deviation of service time by FIFO analytic model and standard deviation of service time by simulation model

Figure 5-6 shows the limits of standard deviation of service time from FIFO analytical model and the values of standard deviation of service time from simulation model for each product type by varying the *CV*s of the job arrival rate and the total arrival rates. The error of estimated standard deviation of service time is defined by Equation (5-9).

$$Error_{j,SD_{ST},FIFO} = \sqrt{\lim_{RT \to \infty} Var\left[ST_{j,FIFO}\right]} - s_{j,ST,FIFO} \tag{5-9}$$

According to the discussion of the service time under FIFO, it is obvious that the average service times of jobs by simulation model are smaller than the limits of the expected service time by analytic model, which implies that $\lambda_r/\lambda > N_{krj}/N_{kj}$ based on Equation (5-4) and Equation (5-5). By referring to Equation (5-7) and Equation (5-8), the error of estimated standard deviation of service time can be updated as Equation (5-10).

$$\begin{aligned}
Error_{j,SD_{ST},FIFO} &= \sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)s_{rj}^2 - \left[\left(\sum_{r=1}^{J}\frac{\lambda_r}{\lambda}s_{rj}\right)^2 - \left(\sum_{r=1}^{J}\frac{N_{krj}}{N_{kj}}s_{rj}\right)^2\right] \\
&= \sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)s_{rj}^2 - \left[\sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}+\frac{N_{krj}}{N_{kj}}\right)s_{rj}\right]\left[\sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)s_{rj}\right] \\
&\le \sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)s_{rj}^2 - \left[\sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)s_{rj}\right]^2 \\
&\le \sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)^2 s_{rj}^2 - \left[\sum_{r=1}^{J}\left(\frac{\lambda_r}{\lambda}-\frac{N_{krj}}{N_{kj}}\right)s_{rj}\right]^2 \le 0
\end{aligned} \tag{5-10}$$

Note that the sum of squares is less than or equal to the square of the sum. Thus, the error $Error_{j,SD_{ST},FIFO}$ is a negative value if $\lambda_r/\lambda > N_{krj}/N_{kj}$, that is, the standard deviations of service times of jobs by simulation model are larger than the limits of the standard deviation of service time by FIFO analytic model if $\lambda_r/\lambda > N_{krj}/N_{kj}$, which are apparent in Figures 5-6.

Figure 5-7 shows the mean errors of the estimated standard deviation of service time by varying the *CV*s of job arrival rate and the total arrival rates. According to Equation (5-10), the relationship between $Error_{j,ST,FIFO}$ and $Error_{j,SD_{ST},FIFO}$ is given by $Error_{j,SD_{ST},FIFO} \le \sum_{r=1}^{J}(\lambda_r/\lambda - N_{krj}/N_{kj})^2 s_{rj}^2 - Error_{j,ST,FIFO}^2$. Thus, it is obvious that the error $Error_{j,SD_{ST},FIFO}$ is negative growth if the error $Error_{j,ST,FIFO}$ is the positive growth. In Figure 5-2, the value of $Error_{j,ST,FIFO}$ for CV = 53.7234% is minimum, the value of $Error_{j,ST,FIFO}$ for CV = 27.9753% is maximum, and the value of $Error_{j,ST,FIFO}$ for CV=0 is in between. In Figure 5-15, thus, the absolute value of $Error_{j,SD_{ST},FIFO}$ for CV = 53.7234% is minimum, the absolute value of $Error_{j,SD_{ST},FIFO}$ for CV = 27.9753% is maximum, and the absolute value of $Error_{j,SD_{ST},FIFO}$ for CV=0 is in between. From smaller to larger CV, overall means of the errors of estimated standard deviation of service time are equal to -0.2862, -0.3288, and -0.1072, respectively.

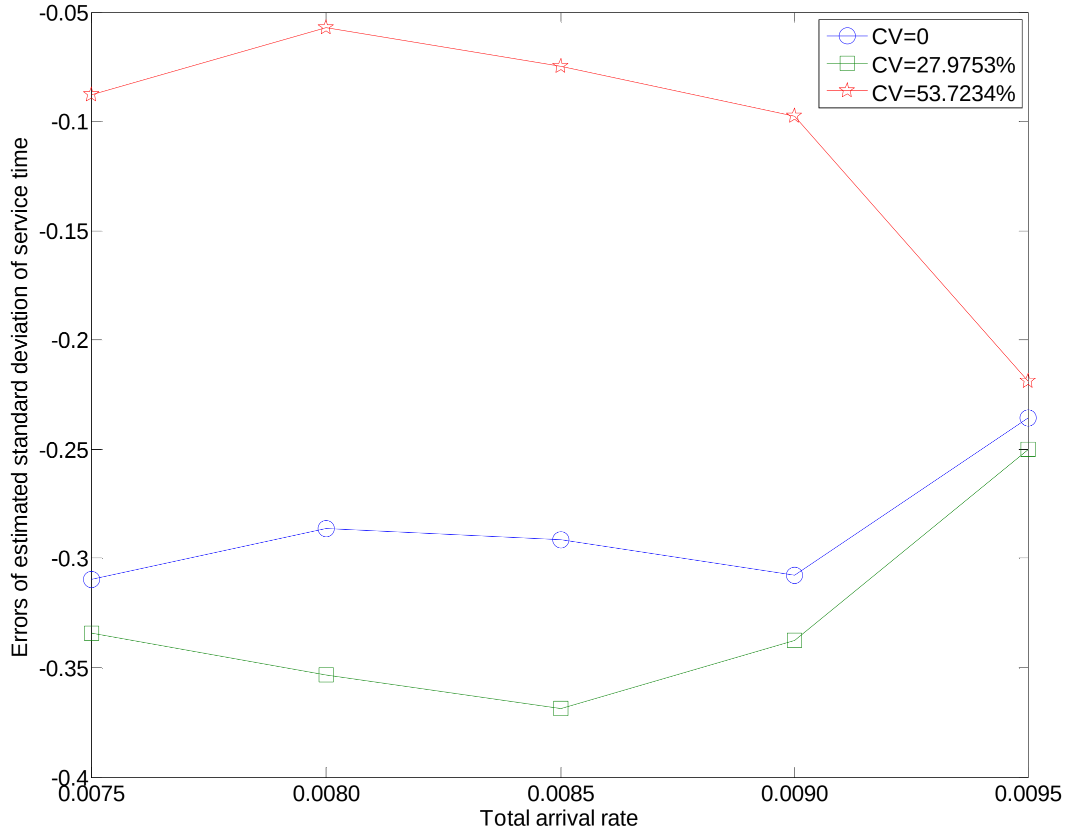

Figure 5-7 Mean errors of estimated standard deviation of service time by varying the *CV*s of job arrival rate and the total arrival rates

Then, the values of the mean errors of estimated standard deviation of service time versus the corresponding values of the arrival rate parameters are also plotted in Figure 5-8 in order to interpret the influence of the arrival rate parameters among eight product types on estimating the standard deviation of service time.

In Figure 5-8, the graph is used to show relationship between the mean errors of estimated standard deviation of service time and the arrival rate parameters among eight product types. It can be seen that the error tends to usually go down as the arrival rate of jobs goes up except CV = 0. This implies that the higher arrival rates of various job types contribute to decrease not only the error percentage of the estimated service time but also the error percentage of the estimated standard deviation of service time.

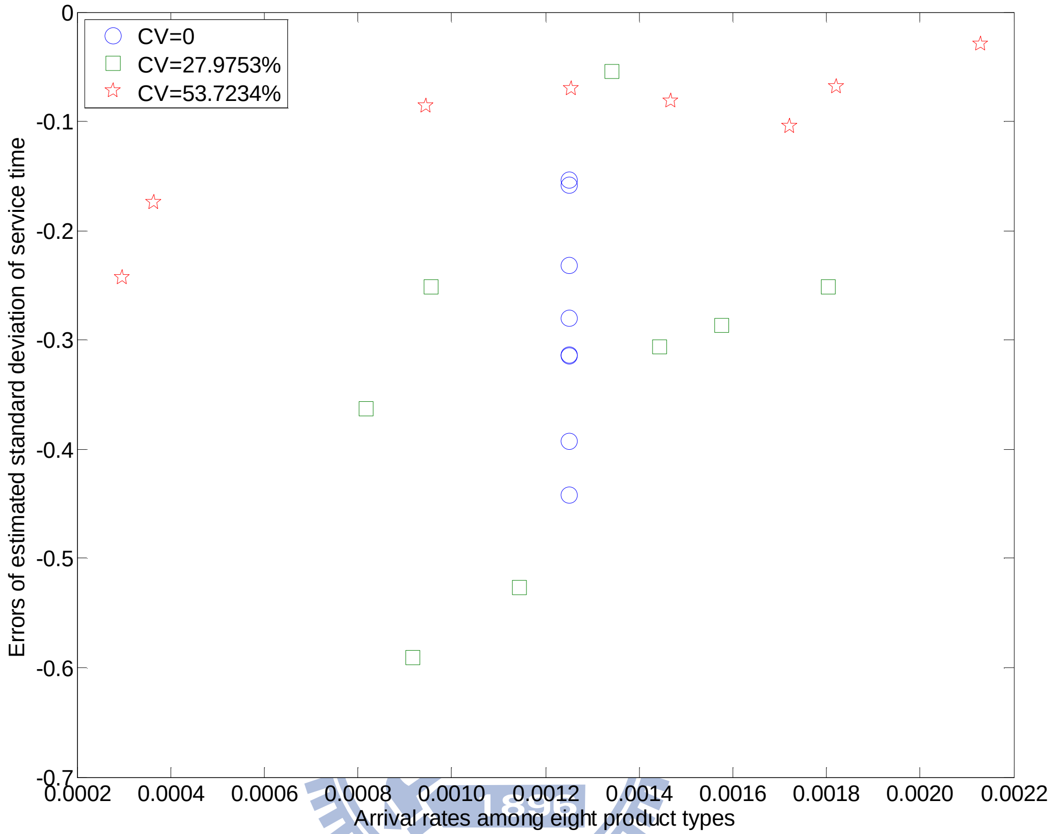

Figure 5-8 Scatter plot for errors of estimated standard deviation of service time under FIFO

5.2.3 Accuracy analysis of expected waiting time

In this research, the waiting time is decomposed into two components: one is the residual service time of the unfinished job on the machine ($R_{ij,FIFO}$) and the other is the total service time of all jobs in queue ahead of the newly arrived job ($TST_{ij,FIFO}$). Both the expected values of $R_{ij,FIFO}$ and $TST_{ij,FIFO}$ depend on the probability $\Pr[T_{ij} \leqq RT]$. Note that $\Pr[T_{ij} \geq RT] \leq \lim_{RT\to\infty} \Pr[T_{ij} \geq RT] = 1$ when RT is long enough. Thus, the limits of the expected values of $R_{ij,FIFO}$ and $TST_{ij,FIFO}$ are larger than or equal to the expected values of $R_{ij,FIFO}$ and $TST_{ij,FIFO}$ and consequently the limit of the expected waiting time is more than or equal to the expected waiting time ($\lim_{RT\to\infty} E[W_{q,ij,FIFO}] \geq E[W_{q,ij,FIFO}]$), which are apparent in Figures 9. In Figure 9, the waiting times grow steeper with the rise in the total arrival rate because of the growth of the number of jobs in the system, which is a characteristic of almost all queuing systems.

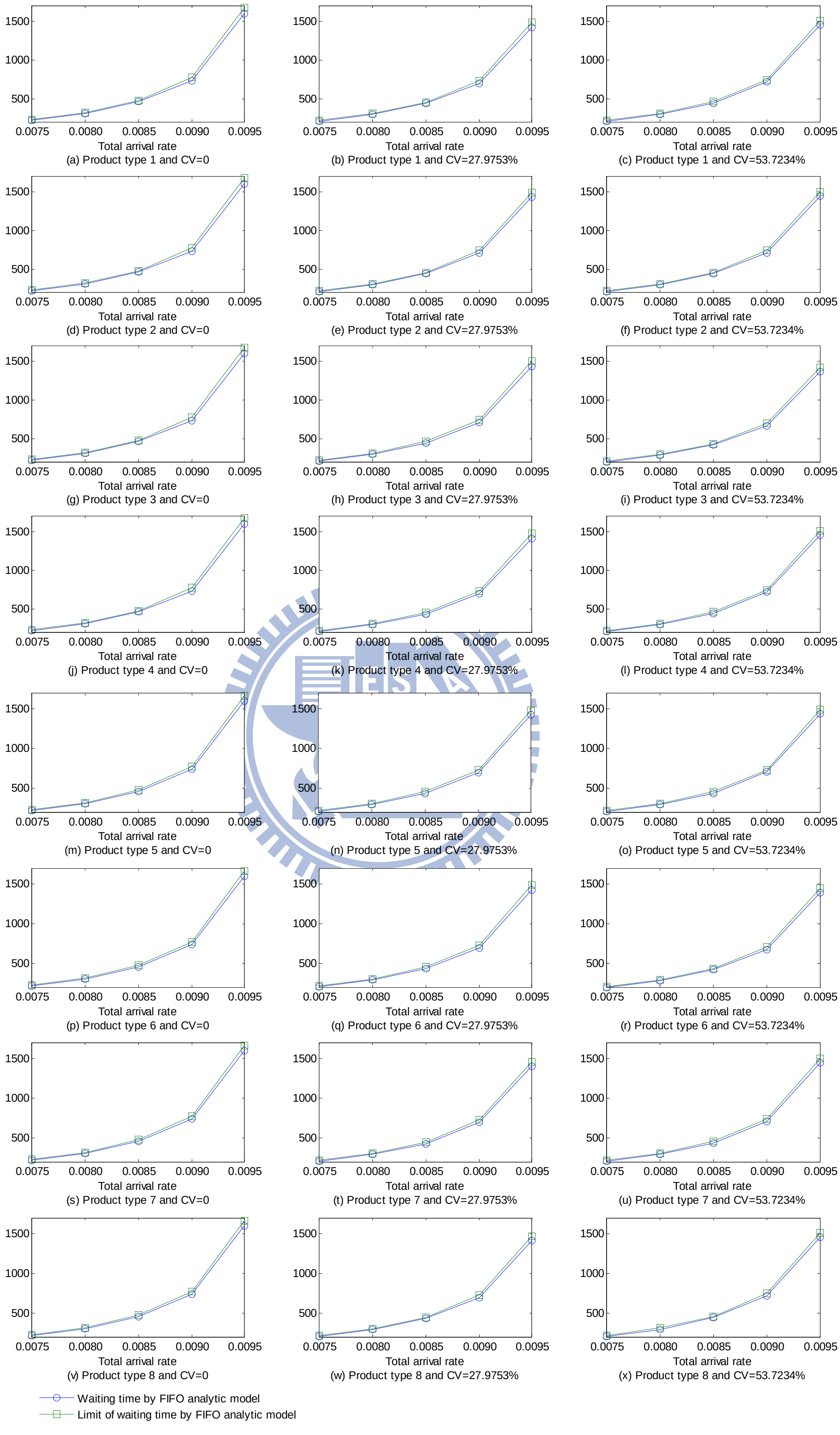

Figure 5-9 Waiting time for each product type jobs and its limit by FIFO analytic model

In order to compare the results of the waiting time of a single job generated by the simulation model and the analytic model, we have to identify the error of the estimated waiting time, which is defined by Equation (5-11).

$$EP_{waiting\ time,j} = \lim_{RT \to \infty} E\left[W_{q,j,FIFO}\right] - \overline{x}_{j,W_q,FIFO} \tag{5-11}$$

where $\overline{x}_{j,W_q,FIFO}$ represents the overall sample mean of the waiting time of a single job of product type *j* by the simulation model, $E[W_{q,j,FIFO}]$ represents the expected waiting time of a single job of product type *j* by the analytic model. Figure 5-10 shows the mean error of the estimated waiting time by varying the *CV*s of job arrival rate and the total arrival rates. Clearly, the error of estimated waiting time ascends as the level of the total arrival rate increases. The higher level of the total arrival rate leads to the high level of WIP. The high level of WIP means jobs are waiting in queue for longer. Thus, relatively few jobs have long waiting time at higher level of total arrival rate causing larger error of the estimated waiting time.

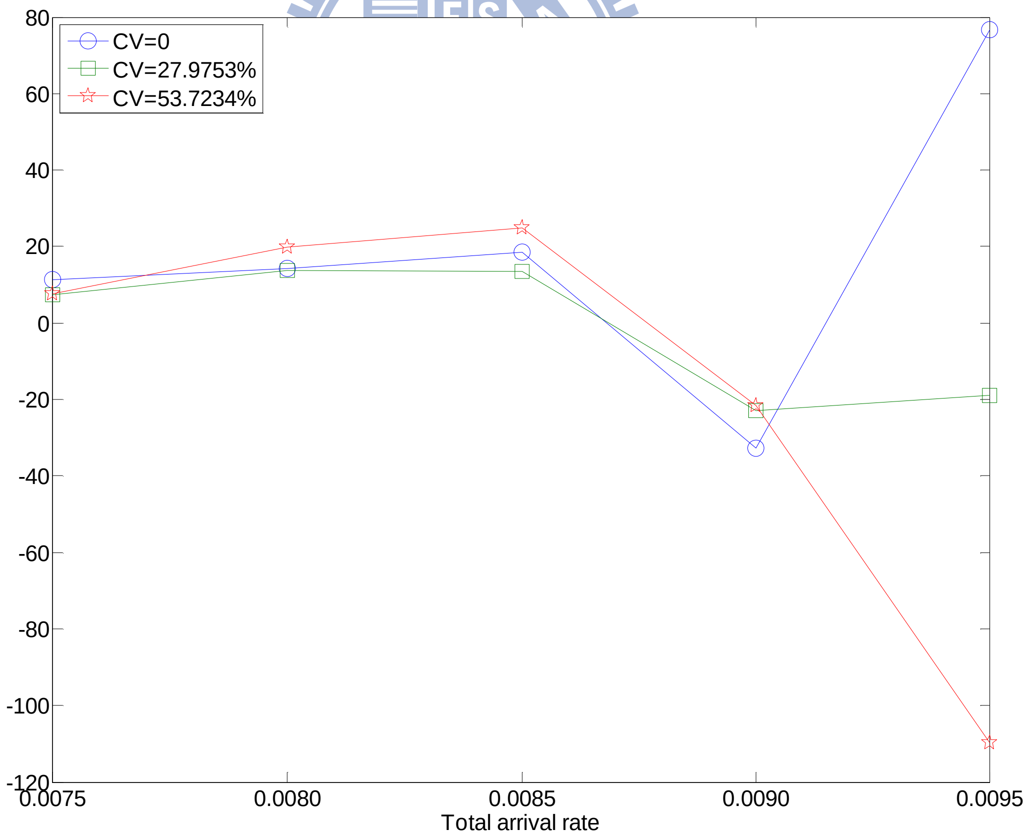

Figure 5-10 Mean error of estimated waiting time by varying the *CV*s of job arrival rate and the total arrival rates

Besides, the arrival rates among various product types affect the performance of the analytic model in estimating waiting time because the waiting time depends on the service time. The values of the mean error of estimated waiting time versus the corresponding values of the arrival rate parameters are plotted in Figure 5-11. Excluding *CV*=0, the pattern of the mean error of the estimated waiting time for each product type slopes from the lower left to the upper right part when *CV*=27.9753% and *CV*=53.7234%, respectively. The error of estimated waiting time is similar to the error of estimated service time, which is negative for the smaller arrival rate of jobs and is positive for the larger arrival rate of jobs. Thus, the extreme of the arrival rates of jobs would affect the performance of the analytic model more in estimating waiting time when the dispersion of the arrival rates of jobs increases.

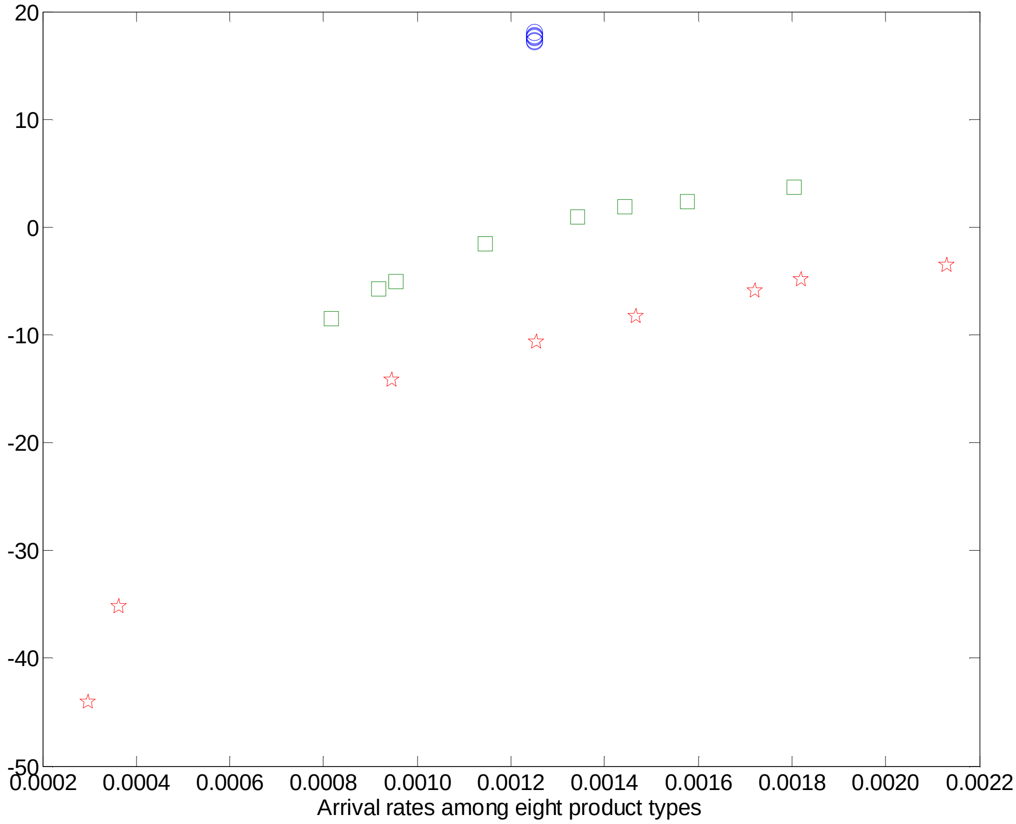

Figure 5-11 Scatter plot for errors of estimated waiting time under FIFO

5.2.4. Accuracy analysis of standard deviation of waiting time

Three standard deviations of the waiting time for each product type by varying the *CV*s of the job arrival rate and the total arrival rates in the analytic model are given graphically in Figure 5-12:

I. According to Equation (3-27), the standard deviation of waiting time by analytical model can be shown as

$$\sigma_{W_{q,j,FIFO}} = \left[\begin{array}{l} \frac{1}{n_j}\sum_{i=1}^{n_j} E\left[R_{ij,FIFO}^2\right] + \frac{1}{n_j}\sum_{i=1}^{n_j} E\left[TST_{ij,FIFO}^2\right] + \\ \frac{2}{n_j}\sum_{i=1}^{n_j}\left\{E\left[R_{ij,FIFO}\right]E\left[TST_{ij,FIFO}\right]\right\} - E\left[\frac{\sum_{i=1}^{n_j} W_{q,ij,FIFO}}{n_j}\right]^2 \end{array} \right]^{\frac{1}{2}}$$

II. According to Equation (3-39), an approximation of the standard deviation of waiting time with the parameters α_j^2 and β_{2j} is given by

$$\sigma'_{W_{q,j,FIFO}} = \left[\beta_{1j} E\left[R^2\right]_{M/G/1} + \left(2\beta_{2,j} - \alpha_j^2\right) E\left[W_q\right]_{M/G/1}^2\right]^{\frac{1}{2}}$$

III. According to Equation (3-40), an approximation of the standard deviation of waiting time with the parameter α_j^2 is given by

$$\sigma''_{W_{q,j,FIFO}} = \left[\beta_{1j} E\left[R^2\right]_{M/G/1} + \alpha_j^2 E\left[W_q\right]_{M/G/1}^2\right]^{\frac{1}{2}}$$

Note that $\sigma'_{W_{q,j,FIFO}} = [\lim_{RT\to\infty} \sigma^2_{W_{q,j,FIFO}}]^{0.5} \geq [\sigma^2_{W_{q,j,FIFO}}]^{0.5}$ because $\lim_{RT\to\infty} \Pr[T_{ij} \leq RT] = 1$. Thus, $\sigma'_{W_{q,j,FIFO}}$ is an upper bound of the standard deviation of waiting time ($\sigma_{W_{q,j,FIFO}}$). Meanwhile, $\sigma'_{W_{q,j,FIFO}} \geq \sigma''_{W_{q,j,FIFO}}$ because $(2\beta_{2j} - \alpha_j^2) > \alpha_j^2$. In Figure 5-12, the value of $\sigma''_{W_{q,j,FIFO}}$ is minimum. Thus, $\sigma''_{W_{q,j,FIFO}}$ can be treated as the lower bound of the standard deviation of waiting time ($\sigma_{W_{q,j,FIFO}}$). Moreover, approximations of the standard deviation of waiting time are the function of the square of the expected waiting time for $M/G/1$ queueing model ($E[W_q]^2_{M/G/1}$). Thus, they have a characteristic of the expected waiting time for $M/G/1$ queueing model, which grow steeper with the rise in the total arrival rate.

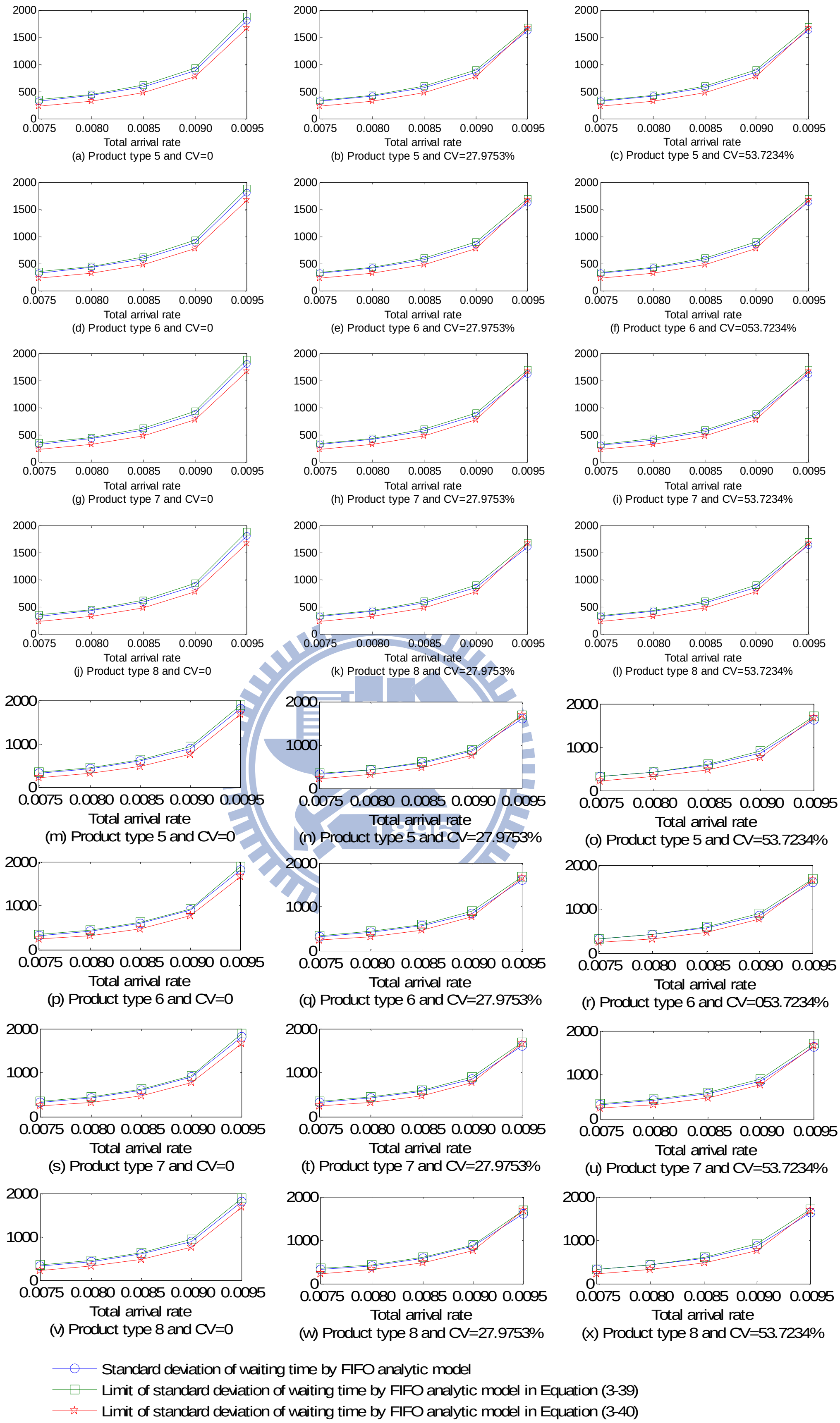

Figure 5-12 Standard deviation of waiting time and its approximation by FIFO analytic model

In Figure 5-12, it can be seen that $\sigma'_{W_{q,j,FIFO}} \ge \sigma_{W_{q,j,FIFO}} \ge \sigma''_{W_{q,j,FIFO}}$. If the larger standard deviation of the waiting time is used in the variability of the lead time, then it will lead to the due-date slackness. On the other hand, the due-date tightness will occur if the lower standard deviation of the waiting time is used in the variability of the lead time. The accuracy of due-date assignment has a profound effect on the production management. In order to obtain the better standard deviation of waiting time, a new approximation of the standard deviation of waiting time is defined the square roots of the mean of $\sigma'^2_{W_{q,j,FIFO}}$ and $\sigma''^2_{W_{q,j,FIFO}}$, which is given by Equation (5-12).

$$\begin{aligned}\sigma'''_{W_{q,j,FIFO}} &= \left[\frac{\sigma'^2_{W_{q,j,FIFO}} + \sigma''^2_{W_{q,j,FIFO}}}{2}\right]^{\frac{1}{2}} \\ &= \left[\beta_{1j} E\left[R^2\right]_{M/G/1} + \beta_{2j} E\left[W_q\right]^2_{M/G/1}\right]^{\frac{1}{2}} \\ &= \left[\beta_{1j} E\left[R^2\right]_{M/G/1} + \frac{\beta_{2j}}{\alpha_j^2}\left\{\lim_{RT\to\infty} E\left[W_{q,j,FIFO}\right]\right\}^2\right]^{\frac{1}{2}}\end{aligned} \tag{5-12}$$

The accuracy analysis of this new approximation is discussed as follows.

To compare the results of the standard deviation of the waiting time generated respectively by the simulation model and approximation form developed in Equation (5-12), we compute the error of estimated standard deviation of the waiting time defined by Equation (5-13).

$$EP_{sd\ of\ waiting\ time,j} = \sigma'''_{W_{q,j,FIFO}} - s_{j,W_q,FIFO} \tag{5-13}$$

where $s_{j,W_q,FIFO}$ is the combined standard deviation of the waiting time of product type j jobs. Figure 5-13 shows the mean error of the estimated standard deviation of waiting time by varying the CVs of job arrival rate and the total arrival rates. Note that an approximation of the standard deviation of waiting time $\sigma'''_{W_{q,j,FIFO}}$ in Equation (5-12) depends on the limit of the expected waiting time ($\lim_{RT\to\infty} E[W_{q,j,FIFO}]$) and the performance of $\lim_{RT\to\infty} E[W_{q,j,FIFO}]$ can be influence by the high level of the total arrival rate. Thus, the error of estimated standard deviation of waiting time increases as the level of the total arrival rate increases in Figure 5-13.

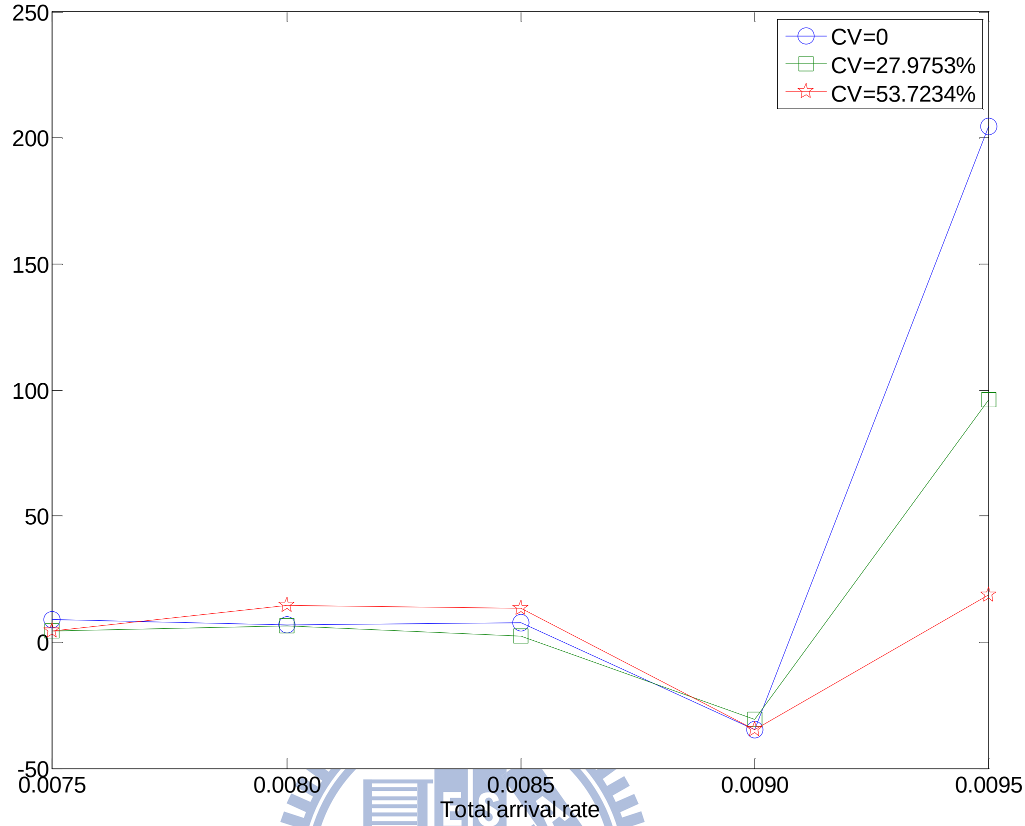

Figure 5-13 Mean errors of estimated standard deviation of waiting time by varying the *CV*s of job arrival rate and the total arrival rates

In addition to standard deviation, the coefficient of variation is also useful to describe the dispersion of variable. The coefficient of variation of waiting time is defined as the ratio of its standard deviation to its mean, which is given by Equation (5-14) if *RT* is long enough.

$$
\begin{aligned}
CV &= \frac{\sigma'''_{W_{q,j,FIFO}}}{\lim_{RT\to\infty} E\left[W_{q,j,FIFO}\right]} \\
&= \frac{\left\{\beta_{1j}E\left[R^2\right]_{M/G/1} + \beta_{2j}E\left[W_q\right]^2_{M/G/1}\right\}^{\frac{1}{2}}}{\alpha_j E\left[W_q\right]_{M/G/1}} \\
&\approx \frac{\sqrt{\beta_{2j}}}{\alpha_j}
\end{aligned}
\qquad (5\text{-}14)
$$

where $\beta_{1j}E[R^2]_{M/G/1}$ is assumed to be ignored. From Equation (5-14), the coefficient of variation of waiting time by analytic model is the function of the proportionalities α_j and β_{2j} and is plotted in Figure 5-14 as compared with the coefficient of variation of waiting time by simulation model.

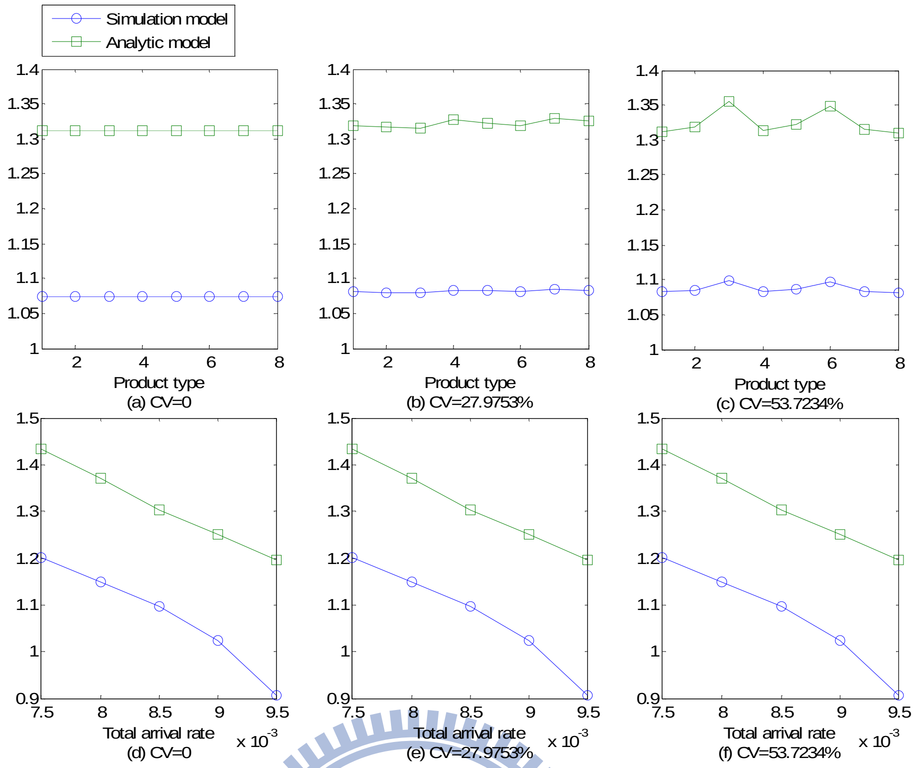

Figure 5-14 Coefficient of variation of waiting time by varying the *CV*s of job arrival rate and the total arrival rates in the simulation model and analytic model

In Figure 5-14, the coefficient of variation by analytic model is larger than the coefficient of variation by simulation model, which implies that the waiting time by analytic model has greater dispersion. Meanwhile, the coefficient of variations of waiting time in the simulation model and analytic model are decreasing with the rise in total arrival rate, indicating that the waiting time has the lower dispersion of waiting time and then the waiting time is more stable and lies close to the mean. The difference in the coefficient of variation of waiting time between analytic model and simulation model is stable except the high level of the total arrival rate. It causes the larger error of estimated standard deviation of the waiting time at higher level of total arrival rate.

Moreover, the relationship between the values of the mean error of estimated waiting time and the corresponding values of the arrival rate parameters is plotted in Figure 5-15. Excluding *CV*=0, the errors of the estimated waiting time tend to increase as the arrival rate parameters increase when *CV*=27.9753% and *CV*=53.7234%, respectively. The error of estimated standard deviation of waiting time is similar to the error of estimated waiting time, which is negative for the smaller arrival rate of jobs and is positive for the larger arrival rate of jobs. Thus, the performance of the analytic model more in estimating waiting time is affected by the extreme of the arrival rates of jobs.

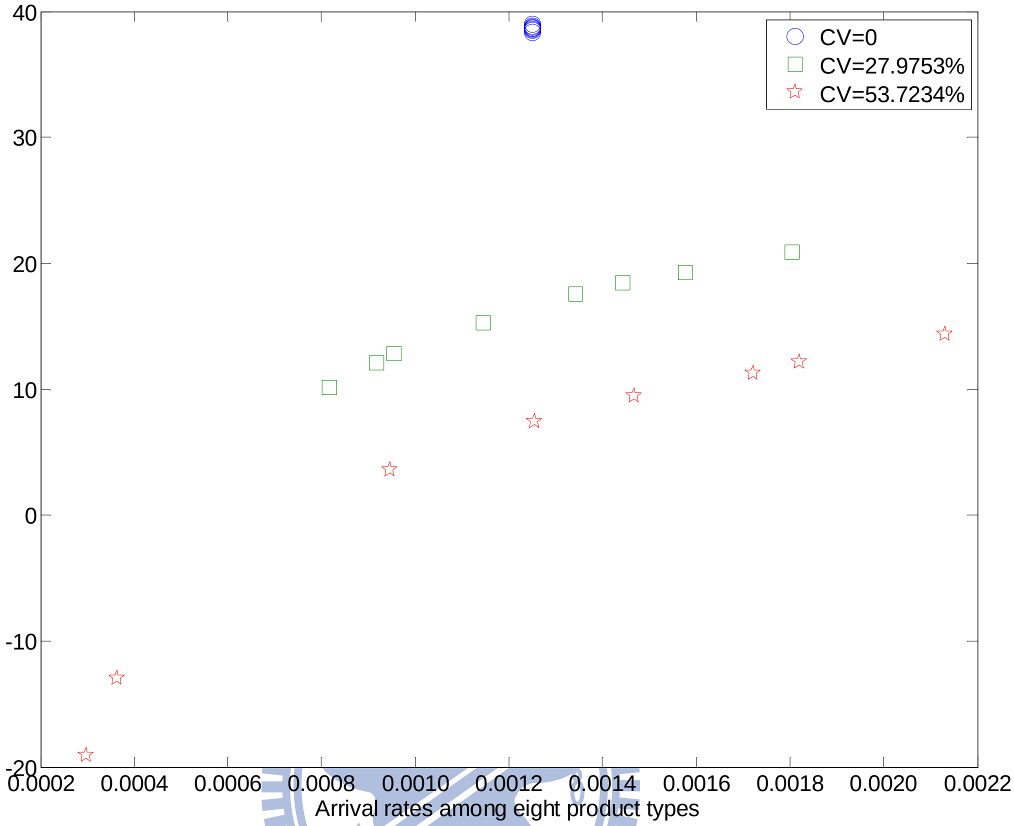

Figure 5-15 Scatter plot or errors of estimated standard deviation of waiting time under FIFO

Based on the results of test for service time and waiting time under FIFO, we know that the differences in the service time and the waiting time between the simulation model and the limit FIFO analytic model are insignificant. This shows that the service time and the waiting time can be estimated accurately using our models, thus managers can set lead time and assign customer order due date for internal control and due dates quoted to customers based on the predicted service time and waiting time.

5.2.5 Numerical analysis of approximations of expected value and variance of waiting time

An approximation of the expected waiting time is proportional to the expected waiting time for the $M/G/1$ queuing theory with α_j if RT is long enough. This indicates that an approximation of the expected waiting time and the expected waiting time for the $M/G/1$ queuing theory are the same if $\alpha_j = 1$. Figure 5-16 shows the proportionality α_j for each product type by varying the *CV*s of job arrival rate and the machine utilization rates; the machine utilization rates vary along with the total arrival rates.

In Figure 5-16, the proportionality α_j for each product type is drawn in pink and closely corresponds to a linear growth as the machine utilization rate increases. The proportionality α_j for each product type is more than one when the machine utilization rates are higher, indicating that the expected waiting time with setup time is underestimated by the formula for the queuing theory. Meanwhile, the proportionality α_j for each product type is smaller than one when the machine utilization rates are lower, showing that the expected waiting time with setup time is overestimated by the formula for the queuing theory.

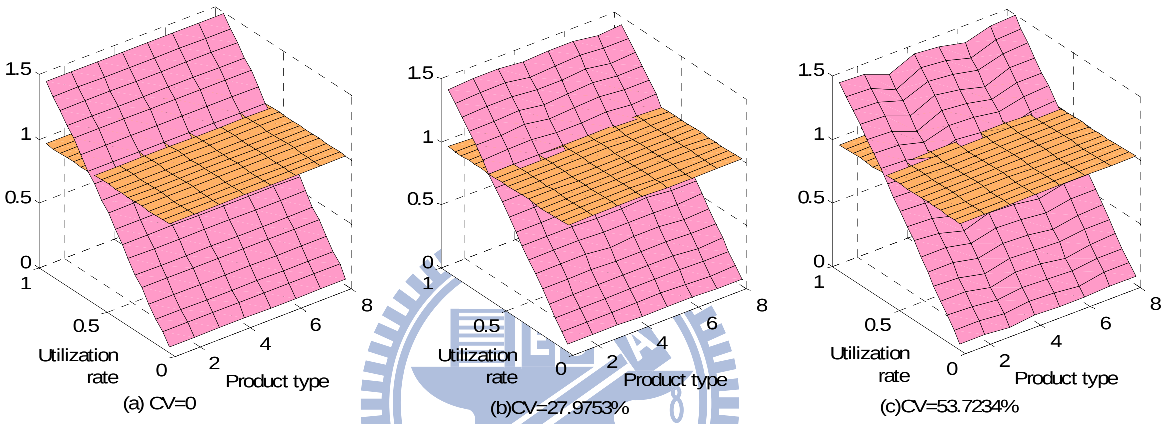

Figure 5-16 Values of the proportionality α_j

Besides, the variance of waiting time for $M/G/1$ queueing system depends on the square of the expected waiting time for $M/G/1$ queueing system in Equation (2-6). In approximations of the standard deviations App2 and App3, α_j^2 and β_{2j} are the coefficients of the square of the expected waiting time for the $M/G/1$ queuing theory. Figure 5-17 displays the proportionalities α_j^2 and β_{2j} for each product type by varying the CVs of job arrival rate and the machine utilization rates. The proportionalities α_j^2 and β_{2j} grow as the machine utilization rate increases, yet the rate of change of α_j^2 is increasing and the rate of change of β_{2j} is decreasing. Moreover, the proportionality β_{2j} is not only more than the proportionality α_j^2 but also more than one.

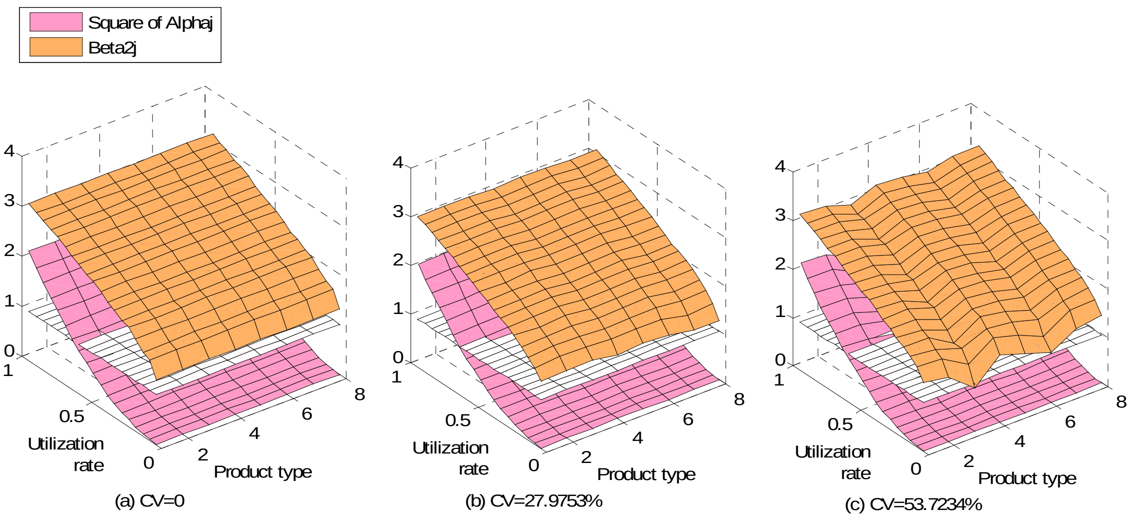

Figure 5-17 Values of the proportionalities α_j^2 and β_{2j}

Compared with the results in the sections 5.2.2 and 5.2.4, it can be seen that the estimation error of the waiting time and its variance with setup time by the formula for the queuing theory can not be restricted in an accepted range, especially at condition of high machine utilization rate. Thus, if the formula for $M/G/1$ queue is used to calculate the expected value and variance of the waiting time including setup time for a single finite-capacity machine, then the proportionalities α_j and β_{2j} should be established to be able to obtain more accurate and precise waiting time and its variance.

5.3 Accuracy analysis for FSR analytic model on the reduction of setup time and capacity loss

Simulation results under FSR for the number of setups and setup time are compared with those calculated by FSR analytic model. Then, the numerical results of the sensitivity analysis on the reductions of the expected setups and expected setup time for each product type are conducted by replacing FIFO with FSR analytic models. Finally, an analysis on the reduction of setup time by replacing FIFO with FSR is conducted.

5.3.1. Accuracy analysis for FSR analytic model in estimating number of setups

Figure 5-18 shows that the probability of setups and the limit of the probability of setups for each product type in the FSR analytic model by varying the *CV*s of job arrival rate, total arrival rates, and run times. For FSR analytic model, the expected number of setups of product type *j* under FSR can be shown as Equation (5-15).

$$E\left[NS_{j,FSR}\right]=\sum_{i=1}^{n_j}E\left[NS_{ij,FSR}\right]=\sum_{i=1}^{n_j}P_{s,ij,FSR} \tag{5-15}$$

It is seen that $E[NS_{j,FSR}]$ depends on the probability of requiring a setup for i^{th} job of product type j under FSR ($P_{s,ij,FSR}$), which is given by Equations (4-4). When RT is long enough, the limit of the probability can be expressed as Equation (5-16).

$$\begin{aligned}\lim_{RT\to\infty}P_{s,ij,FSR} &= \lim_{RT\to\infty}\Pr\left[T_{ij}\le RT\right]P_{s,j,FSR}\\ &= \lim_{RT\to\infty}P_{s,j,FSR}\\ &=\left(1-\frac{\lambda_j}{\lambda}\right)\left\{1-\rho_{FSR}\left[1-\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-1}\right]\right\}\end{aligned} \tag{5-16}$$

This means that the probabilities of setups under FSR are the same for all product types when RT is long enough. Therefore, $\lim_{RT\to\infty}P_{s,ij,FSR}$ is always larger than $P_{s,ij,FSR}$ because $\Pr[T_{ij}\leqq RT]\leqq 1$, which is apparent in Figure 5-20.

Meanwhile, the first derivative of $\lim_{RT\to\infty}P_{s,ij,FSR}$ with respect to ρ_{FSR} is given by Equation (5-17) and is negative with $0\leqq\rho_{FSR}<1$ and $\lambda_j>0$ based on Equation (4-13).

$$\frac{d\lim_{RT\to\infty}P_{s,ij,FSR}}{d\rho_{FSR}}=\left(1-\frac{\lambda_j}{\lambda}\right)\left\{-\left[1-\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-1}\right]-\frac{\lambda_j}{\lambda}\frac{\rho_{FSR}}{\left(1-\rho_{FSR}\right)^2}\left[\left(1+\frac{\rho_{FSR}}{1-\rho_{FSR}}\frac{\lambda_j}{\lambda}\right)^{-2}\right]\right\}\le 0 \tag{5-17}$$

Hence, the probability $\lim_{RT\to\infty}P_{s,ij,FSR}$ decreases with the rise in the utilization rate of the machine, which implies that the probability of setups of product type j under FSR is lower and then it lead to lower setup frequency when the machine utilization rate is higher. The dispersions of the probability of setups for each product type in the FSR analytic model increase with the rise in *CV*. This implies that the extreme values of the arrival rate parameters among various product type increase and can lead to higher and lower setup frequencies.

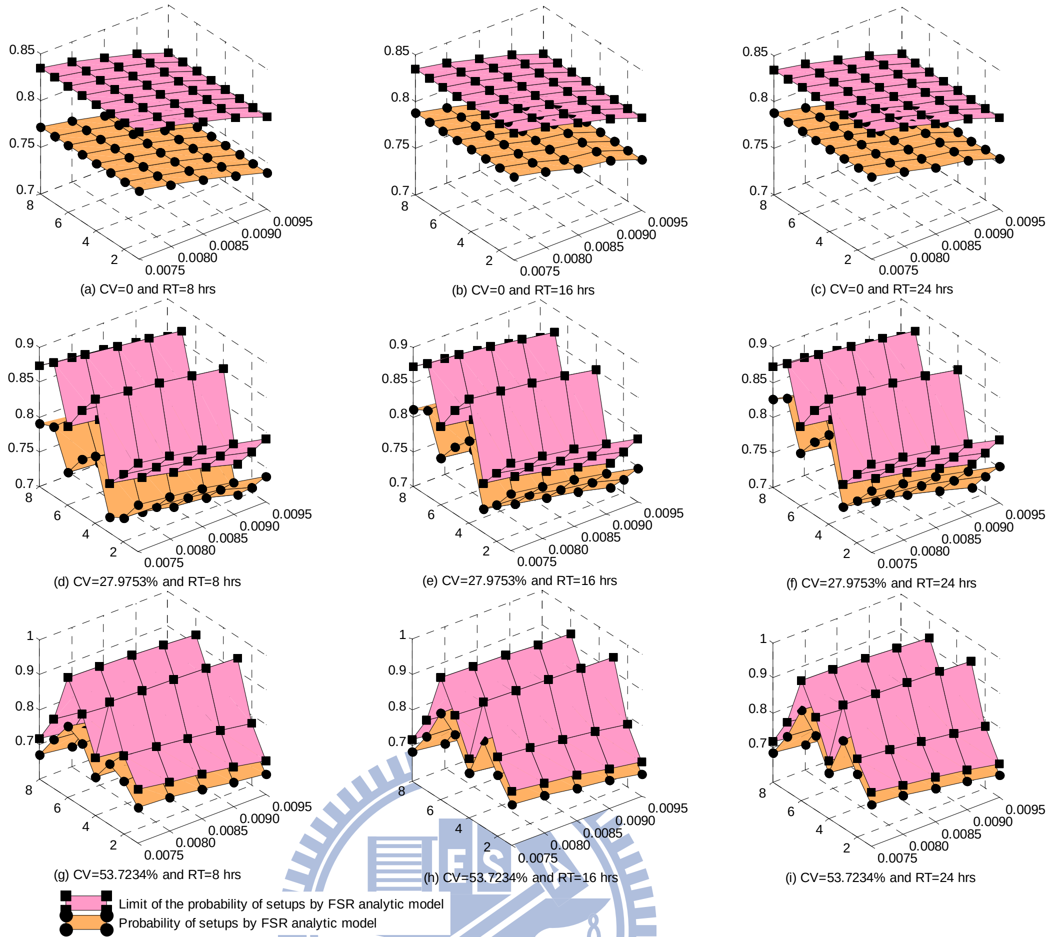

Figure 5-18 Probability of setups and its limit in FSR analytic model

For FSR simulation model, let $NS_{,kj,FSR}$ be the number of setups of the product type j jobs for the k^{th} simulation run. The probability of setups of the product type j jobs for the k^{th} simulation run is defined by the number of setups of the product type j jobs for the k^{th} simulation run divided by the number of the product type j jobs for the k^{th} simulation run ($\bar{x}_{kj,Ps,FSR} = NS_{kj,FSR} / N_{kj}$). To compare the result of the probability of setups generated by the FSR simulation model and the FSR limit analytic model, the error of estimated probability of setups is given by Equation (5-18).

$$Error_{j,Ps,FSR} = \lim_{RT \to \infty} P_{s,j,FSR} - \bar{x}_{kj,Ps,FSR} \tag{5-18}$$

Figures 5-19 illustrates the mean errors of estimated probability of setups under FSR by varying the CVs of job arrival rate, total arrival rates, and run times.

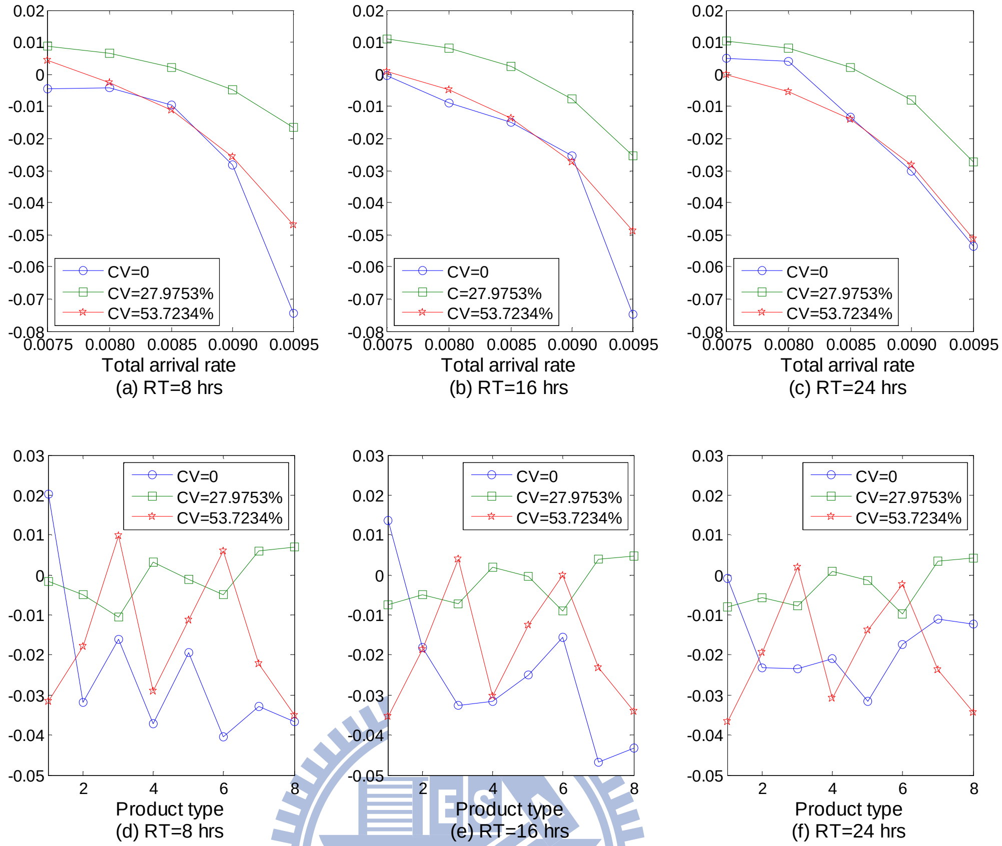

Figure 5-19 Mean errors of estimated the probability of setups between FSR limit analytic model and simulation model

When the total arrival rate increases, the mean error percentage increases correspondingly, as shown in Figures 5-19(a) to 5-19(c). In particular, the larger error occurs at a higher level of machine utilization rate. In Figures 5-19(d) to 5-19(f), the value and the dispersion of the error for each *CV* decrease with a lengthened *RT*. The larger error occurs at *RT*=8 hours because of the few setups. When the run time becomes longer, the error decreases as a result of the larger setups. The overall means of the errors are equal to -0.01383, -0.01530, and -0.01348, as *RT* changes from 8 hours to 24 hours.

In Figure 5-20, the scatter plot provides a graphical display of the relationship between the errors of estimated probability of setups and the arrival rates among eight product types. When *CV* equal to zero, the errors are vertical because the values for the arrival rates are the same. When *CV* equal to 27.9753% and 53.7234%, the values of the errors are negative growth as the values of the arrival rates increase. The extreme values of the arrival rate parameters among various product types increase with larger *CV*. This can lead to an increase in extreme values in the number of setups and can make the error of estimated probability of setups rise as well. Thus, the moderate dispersion of job arrival rate among various types is related to the accuracy of the proposed FSR analytic model in estimating the number of setups.

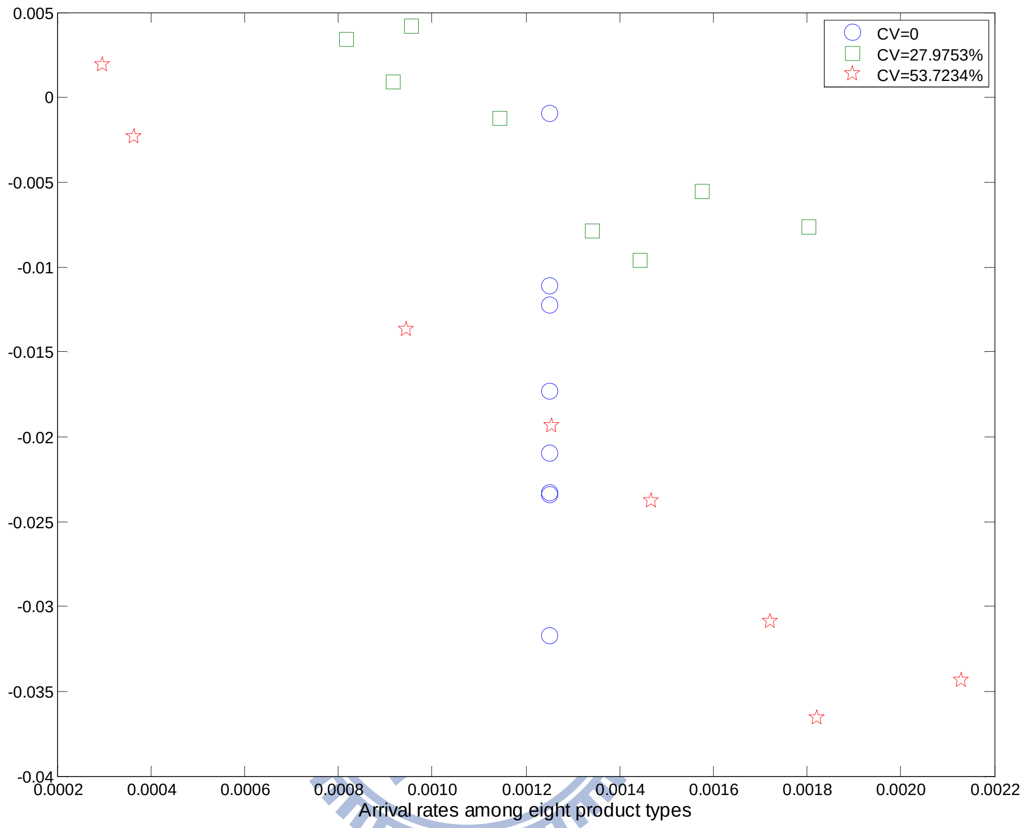

Figure 5-20 Scatter plot for errors of estimated probability of setups under FSR

5.3.2. Accuracy analysis for FSR analytic model in estimating setup time

Figure 5-21 shows the setup times of single job and their limits in FSR analytic model by varying the *CV*s of job arrival rate, run times, and total arrival rates. The setup time of a single job is defined by the total setup time of all jobs with the same product type at a time interval divided by the total number of jobs arrival specific for that. According to Equation (4-7), the limit of the setup time of the product type *j* job can be shown as Equation (5-19).

$$\lim_{RT\to\infty} E\left[S_{j,FSR}\right] = \lim_{RT\to\infty} w_j P_{s,j,FSR} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} = P_{s,j,FSR} \sum_{\substack{r=1 \\ r\neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \quad (5\text{-}19)$$

It can be seen that the limit of the setup time depends on the probability $P_{s,j,FRS}$, the arrival rates, and setup time matrix. Because the probability $w_j = n_j^{-1}\sum_{i=1}^{n_j}\Pr[T_{ij} \le RT] \le 1$, thus $\lim_{RT\to\infty} E[S_{j,FSR}] \ge E[S_{j,FSR}]$, which is depicted on the diagram of Figure 5-21.

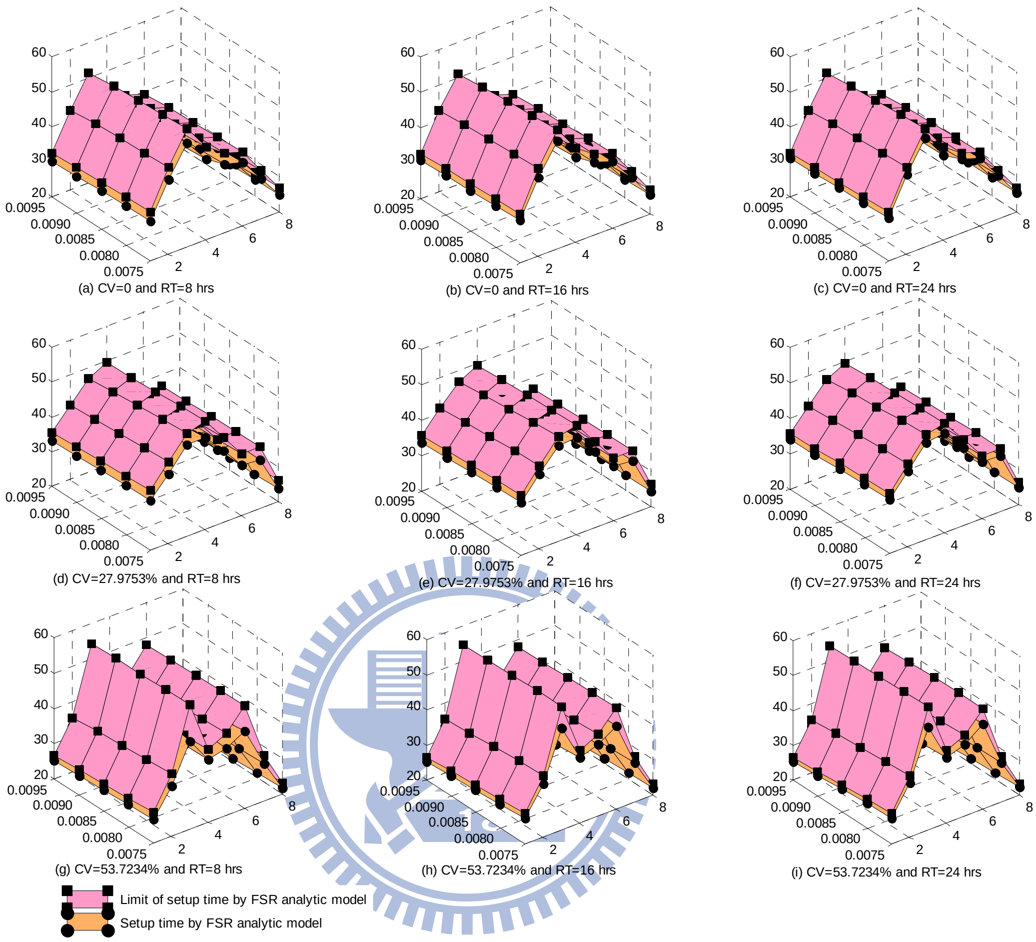

Figure 5-21 Setup times of single job and their limits in FSR analytic model

When *CV* is equal to zero (i.e., the arrival rates among eight product types are the same), the setup time of a single job only depends on the setup time matrix. The average setup time of product type *j* job ($\bar{s}_j$) can be calculated as the summation of s_{rj} in **ST** for all *r* divided by the number of product type. Product types 3 has the largest average setup time ($\bar{s}_3 = 56.2500$), and product type 8 has the minimum value of average setup time ($\bar{s}_8 = 28.1250$). The setup times of a single job for each product type and their limits in FSR analytic model, as shown in Figures 5-21(a) to 5-21(c), are near their average setup times as *CV*=0. When *CV* is equal to 27.9753%, product type 3 obtains the larger arrival rate, whereas product type 4 achieves the smaller arrival rate. In Figures 5-21(d) to 5-21(f), thus, a vast amount of setup time of product type 3 is saved because its larger arrival rate as compared with *CV*=0. In contrast, product type 3 has the smaller arrival rate while product type 4 has the larger arrival rate with

CV=53.7234%. Thus, the wider reduction in setup time for product type 4 leads to the larger gap between the setup times of product type 3 and product type 4 as compared with CV=0, which is apparent in Figures 5-21(g) to 5-21(i). Product type 8 also has the smaller arrival rate as CV=27.9753% and the larger arrival rate as CV=53.7234%. However, its setup time reduction is limited as CV=53.7234% because of the shorter average setup time. Thus, the setup times of product type 8 upon varying the CVs are nearly equal, whether to adopt the FSR analytic model or to implement the FSR limit analytic model.

A comparison of the results of setup time for single jobs generated by the simulation model and the analytic model suggests that the error percentage of estimated setup time is defined by Equation (5-20).

$$Error_{j,S,FSR} = \lim_{RT \to \infty} E\left[S_{j,FSR} \right] - \bar{x}_{kj,S,FSR} \qquad (5\text{-}20)$$

Figures 5-22(a) to 5-22(c) show the mean errors of the estimated setup time between FSR limit analytic model and the simulation model by varying the CVs of job arrival rate, total arrival rates, and run times. Meanwhile, the mean errors of estimated setup time for each product type between FSR limit analytic model and simulation model by varying the CVs of job arrival rate and the run times are shown by Figures 5-22(d) to 5-22(f).

As setup time depends on the probability of setups, the behavior of the error of estimated setup time in Figure 5-22 is similar to that in Figure 5-19. From shorter to longer run time, the error of estimated setup time decreases and the lower error of estimated setup time is attained at longer run time, regardless of the CVs of job arrival rate and total arrival rates. The overall means of the error of estimated setup time range from -0.856 to -0.634, as RT changes from 8 hours to 24 hours. Meanwhile, when CV equals 27.9753%, the lowest error of estimated setup time is obtained. Finally, when total arrival rate increases, the error of estimated setup time increases correspondingly; that is, lower error of estimated setup time occurs at lower levels of machine utilization rate.

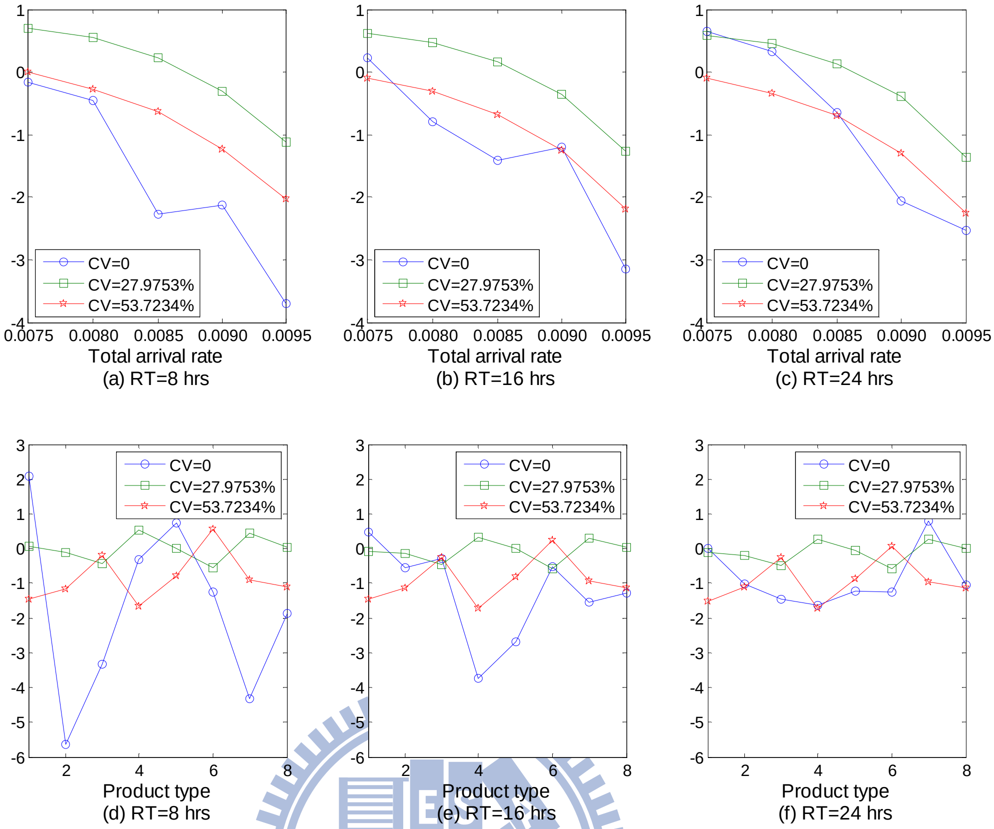

Figure 5-22 Mean errors of estimated setup time between FSR analytic model and simulation model

In Figure 5-23, the scatter plot is used to investigate the relationship between the errors of estimated setup time under FSR and the arrival rates among eight product types. Owing to the same values for the arrival rates, the errors are vertical when *CV* equal to zero. Moreover, the values of the errors are negative growth as the values of the arrival rates increase when *CV* equal to 27.9753% and 53.7234% because the behavior of the error of estimated setup time in Figure 5-22 is similar to the behavior of the error of estimated probability of setups. Thus, the moderate dispersion of job arrival rate among various types is related to the accuracy of the proposed FSR analytic model in estimating the setup time.

In general, the number of setups and the setup time can be estimated accurately using our models to a certain extent. Based on the analysis, better accuracy of the proposed FSR analytic models in estimating the number of setups and setup time can be obtained for longer run times, smaller total arrival rates, and moderate dispersion of job arrival rates among various types. This result can be offered to managers as reference for evaluating capacity loss and others.

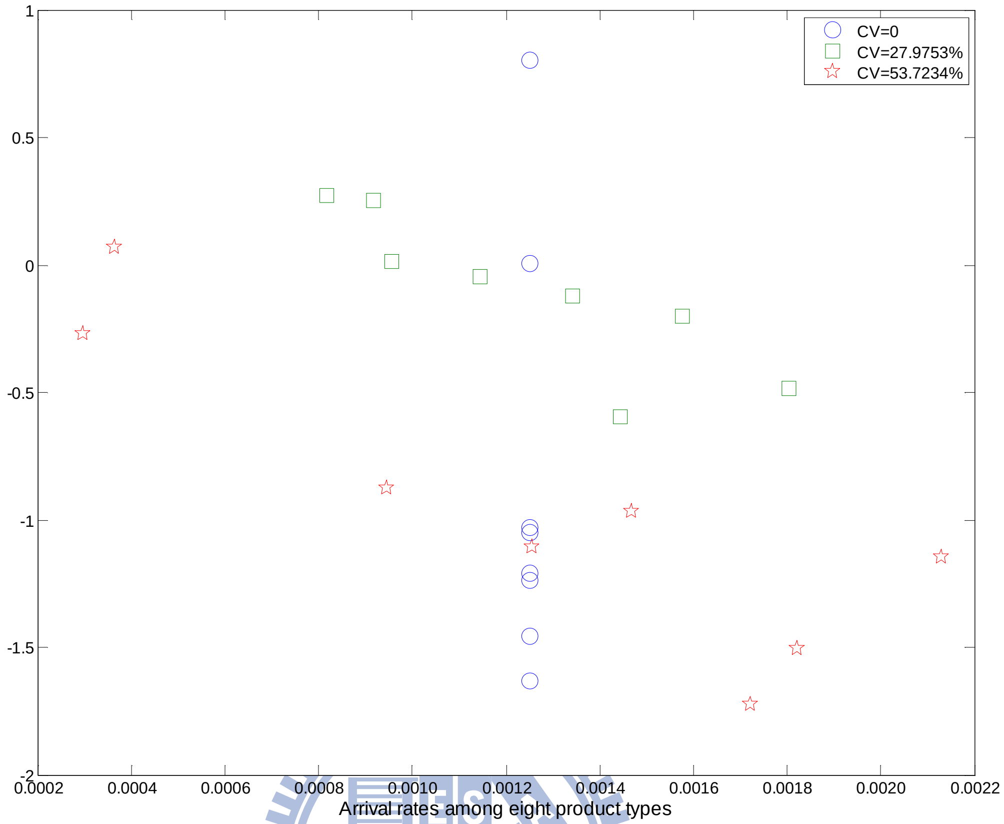

Figure 5-23 Scatter plot for errors of estimated setup time under FSR

5.3.3 Sensitivity analysis of the reduction of number of setups for each product type

The differences of the expected number of setups between FIFO and FSR are defined by the expected number of setups under FIFO minus the expected number of setups under FSR. The mean of the difference of the expected number of setups between FIFO and FSR for each product type by varying the *CV*s of job arrival rate is illustrated in Figure 5-24(a).

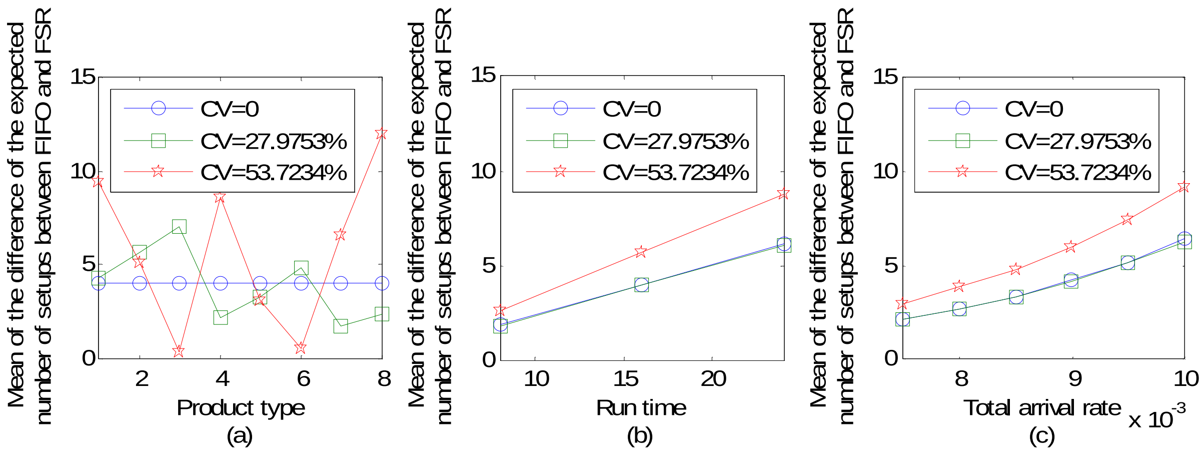

Figure 5-24 Mean of the difference of the expected number of setups between FIFO and FSR

The mean of the difference of the expected number of setups between FIFO and FSR for each product type is constant when *CV* equals zero. Moreover, the dispersion of the mean of the difference of the expected number of setups between FIFO and FSR increases with *CV*, which implies that the extreme value of arrival rate parameters among various product type increases and can influence the performance of the FSR analytic model in reducing setup frequency. The positive correlation coefficients are calculated as 0.998 and 0.991 when the *CV*s equal 27.9753% and 53.7234%, respectively. These positive correlation coefficients indicate a relationship between the mean of the difference of the expected number of setups between FIFO and FSR and the arrival rate parameters among eight product types. As values for the arrival rate parameters among eight product types increase, the values for reducing setup frequency also increase. Therefore, by replacing FIFO with FSR, the largest reduction of the number of setups occurs at *CV* =53.7234%, which is apparent in Figures 5-24(b) and 5-24(c).

5.3.4 Sensitivity analysis of the reduction of expected setup time for each product type

The differences of the expected setup time between FIFO and FSR are defined by the expected setup time under FIFO minus the expected setup time under FSR, and the mean of the difference of the expected setup time between FIFO and FSR for each product type by varying the *CV*s of job arrival rate is displayed in Figure 5-25(a). The dispersion of the mean of the difference of the expected setup time between FIFO and FSR increases with *CV*. The correlation coefficients are positive and are calculated as 0.906 and 0.845 when the *CV*s equal 27.9753 and 53.7234%, respectively. Therefore, the arrival rate parameters among eight product types and the mean of the difference of the expected setup time tend to increase and decrease, respectively, along with each other.

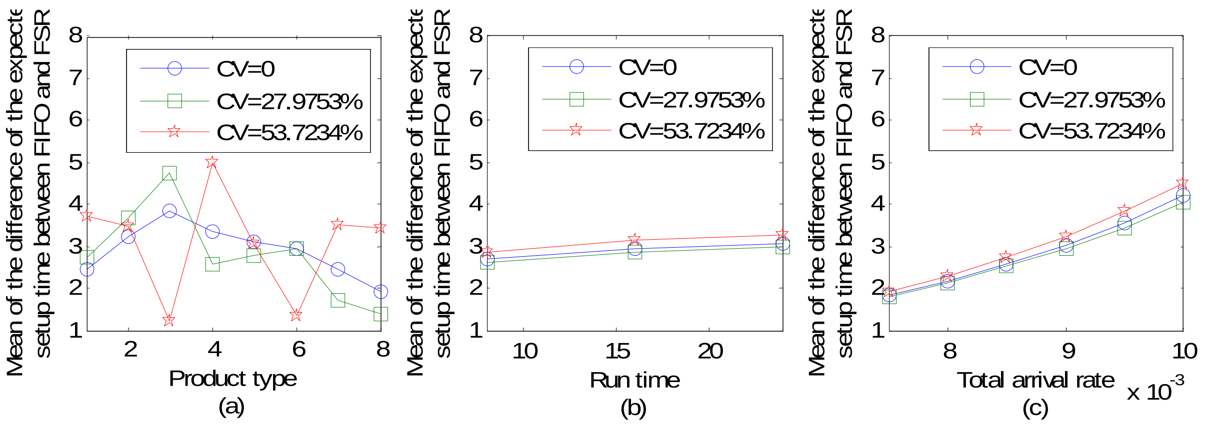

Figure 5-25 Mean of the difference of the expected setup time between FIFO and FSR

Job arrivals tend to concentrate on fewer product types as CV increases. The types obtaining high possibilities of setup reduction leading to the largest reductions of the setup time occur at CV =53.7234%, which are showed in Figures 5-25(b) and 5-25(c).

5.3.5 Evaluation of the effect on the setup time reduction

A set of numerical data are used to evaluate the effect of the decision criterion developed in Section 4.2.3, by which we can find the conditions that the setup time can be saved significantly by replacing FIFO with FSR. According to Equation (4-24), letting DS represent $E[\overline{DS}_{FIFO-FSR}]$ and CP represent $z_{\alpha}\sqrt{Var[\overline{DS}_{FIFO-FSR}]}$, and then the setup time can be reduced significantly by applying FSR instead of FIFO if DS is larger than CP.

There are two control factors included, the total arrival rate (λ) and the coefficient of variation among job arrival rates (CV). The total arrival rate λ (jobs in 60 seconds) is given by $\lambda = \delta\sum_{j=1}^{8}\lambda_j$, where $0.88 \leqq \delta \leqq 1.00$. The λ_j parameters and CV are identical to those in Table 5-1. The vector of job processing time (**PT**) among eight product types and the matrix of sequence-dependent setup time (**ST**) are consistent with the experimental design in Section 5.1. Set the run time (RT) to 8 hours and α to 0.05, Thus, two lines, DS and CP, are depicted in Figure 5-26(a) ~ 5-26(c) by varying the total arrival rates from 0.0088 to 0.0100 and for three levels of CV. We note that the intersection of the two lines, DS and CP, is the condition identified by the decision criterion developed in Section 4.2.3.

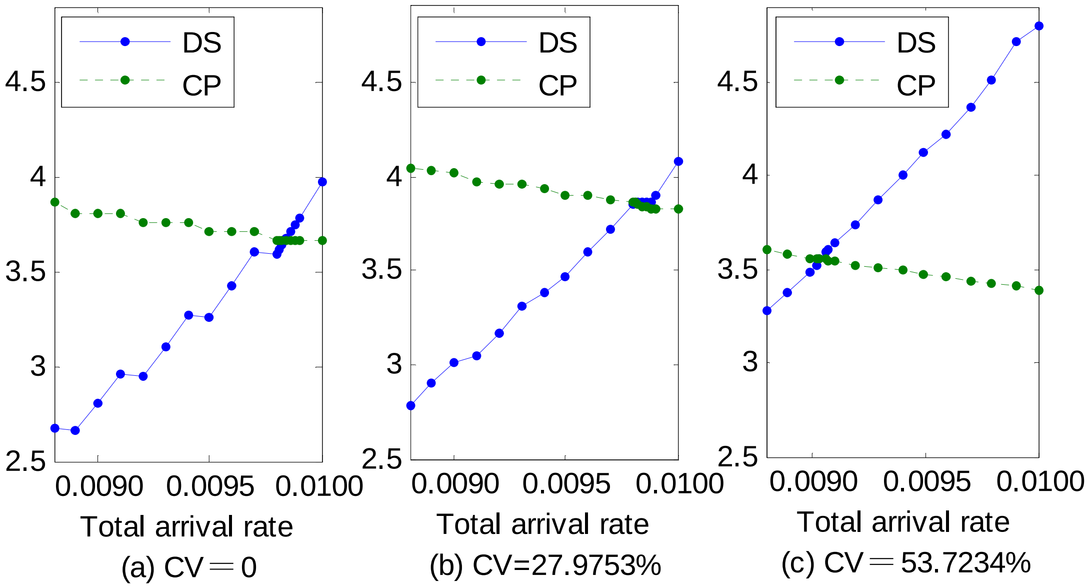

Figure 5-26 Magnitudes of DS and CP by varying the total arrival rate and the CVs of job arrival rate

In Figure 5-26(a) ~ 5-26(c), according to the levels of *CV*, the intersections of the two lines, DS and CP, occur at various levels of total arrival rate λ. After the identification of the intersection point of CP and DS, the setup time reduction would be increasing as the raising of total arrival rate. Therefore, this implies that the total arrival rate of various types of incoming jobs and the mix of the arriving rates of various types of job both affect the effect in reducing the setup time by applying FSR to replace FIFO.

6. Conclusions and future research

6.1 Conclusions

This research considers a system with finite capacity to process several types of jobs; a setup process is necessary before the machine is switched from the current setting to a different one, and this cannot be regarded as a part of the job processing time. With uncertainties in job arrival time and types of demand, setting an output target may be significantly different from actual scenarios due to possible heavy capacity loss and difficulty in calculating the required setup time. Thus, a relatively simply yet accurate analytic method is established to estimate the lead time for each product type by FIFO rule in order to facilitate the performance evaluation from the customer's perspective. Besides, FSR analytic model is developed to estimate expected setup time and service time. The effect on capacity wastage due to changes in machine setting among several product types can then be evaluated. Due to the difficulty in obtaining analytical solutions for the expected setup time and service time, the numerical solutions of expected setup time and service time are provided in this research.

The lead time for each product type is estimated with the summation of the expected waiting time in queue and the expected service time for each product type, where the service time of jobs includes both its setup time and its processing time. Results of the proposed FIFO analytic model in estimating service time and waiting time in queue are compared with simulation results. Computational results show that overall means of the error percentages of estimated service time and waiting time in queue are equal to 4.7551% and 4.4548%, respectively. Meanwhile, the dispersions of error percentages of estimated service time, waiting time in queue, and lead time for each product type are increasing as the coefficient of variation among job arrival rates (*CV*) becomes larger. Generally speaking, these error percentages can be controlled in an acceptable range.

As regards the results of the proposed FSR analytic model, it is also compared with simulation results. Computational results show that error percentages of estimated setups and setup time are larger when *CV* and total arrival rate increase, but they are reduced when run time is lengthened. Generally speaking, the smaller error percentage of estimated setups and setup time can be obtained with longer run time, smaller total arrival rate, and moderate dispersion of job arrival rate among various types. In this paper, we also provide the sensitivity analyses to discuss how the reductions of the setup frequency and the setup time can be affected by the changes of three factors (*CV*, λ, and *RT*). Compared with FIFO, FSR can be used to reduce the frequency of setups and the length of the setup time, hence leading to a reduction in machine utilization rate, especially at conditions of high total arrival rate and high dispersion of arrival rates among several types of job.

The FIFO analytic model can estimate accurately the service time and waiting time, and then evaluate efficiently the lead time based on the estimations of the service time and waiting time. Managers can utilize the lead time prediction to set the due date of customer orders. The FSR models can, to some extent, estimate accurately the setup time and evaluate efficiently the capacity of wastage arising from switching the machine setting responding to uncertainties in job arrivals. Managers can utilize the expected setup time as threshold and tolerance during production planning.

6.2 Future research

This research proposes the lead time estimation of single bottleneck machine to represent a production system. In the future, we can extend the result to the parallel-machine or flexible flow shop environment. These types of problems are applicable in process industry. Then, the extreme error percentages affect the performance of the analytic model in estimating lead time when the mix of the arrival rates of various job types is unbalanced. Thus, we can divide the mix of the arrival rates of various job types into several subgroups for balancing the dispersion within subgroups to reduce the influence of the extreme error percentages on the performance of the analytic model in estimating lead time.

Moreover, the sequence of batches by FSR is sorted according to arrival time of the first jobs in each batch in increasing order. In the future, the rule of sorting batches may change to using the setup time for any two batches in increasing order in order to minimize total setup time.

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Appendixes

Appendix A:

To prove the following statement:

$$ETST_{2,FIFO} = \rho_{FIFO} ETST_{1,FIFO} + \rho_{FIFO} ETST_{1,FIFO} = 2\rho_{FIFO} ETST_{1,FIFO} \tag{A.1}$$

Proof:

According to Equation (6), $ETST_{2,FIFO}$ can be shown as

$$\begin{aligned} ETST_{2,FIFO} = p_{3,FIFO} \sum_{j^1=1}^{J} \frac{\lambda_{j^1}}{\lambda} \Bigg\{ \Bigg[pt_{j^1} + P_{s,j^1,FIFO} \sum_{\substack{r^1=1 \\ r^1 \neq j^1}}^{J} \frac{\lambda_{r^1}}{\lambda^{c^1}} s_{r^1 j^1} \Bigg] \sum_{j^2=1}^{J} \frac{\lambda_{j^2}}{\lambda} + \\ \sum_{j^2=1}^{J} \frac{\lambda_{j^2}}{\lambda} \Bigg[pt_{j^2} + P_{s,j^2,FIFO} \sum_{\substack{r^2=1 \\ r^2 \neq j^2}}^{J} \frac{\lambda_{r^2}}{\lambda^{c^2}} s_{r^2 j^2} \Bigg] \Bigg\} \end{aligned} \tag{A.2}$$

where

$$\sum_{j^k=1}^{J} \frac{\lambda_{j^k}}{\lambda} = 1 \tag{A.3}$$

$$\sum_{j^k=1}^{J} \frac{\lambda_{j^k}}{\lambda} \Bigg[pt_{j^k} + P_{s,j^k,FIFO} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k} \Bigg] = \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \Bigg[pt_j + P_{s,j,FIFO} \sum_{\substack{r=1 \\ r \neq j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \Bigg] = \frac{ETST_{1,FIFO}}{p_{2,FIFO}} \tag{A.4}$$

according to Equation (5). From Equation (A.3) and Equation (A.4), $ETST_{2,FIFO}$ can be rewritten as

$$\begin{aligned} ETST_{2,FIFO} &= p_{3,FIFO} \left\{ \frac{ETST_{1,FIFO}}{p_{2,FIFO}} + \sum_{j^1=1}^{J} \frac{\lambda_{j^1}}{\lambda} \frac{ETST_{1,FIFO}}{p_{2,FIFO}} \right\} \\ &= \frac{p_{3,FIFO}}{p_{2,FIFO}} ETST_{1,FIFO} + \frac{p_{3,FIFO}}{p_{2,FIFO}} \sum_{j^1=1}^{J} \frac{\lambda_{j^1}}{\lambda} ETST_{1,FIFO} \\ &= \rho_{FIFO} ETST_{1,FIFO} + \rho_{FIFO} ETST_{1,FIFO} \\ &= 2\rho_{FIFO} ETST_{1,FIFO} \end{aligned} \tag{A.5}$$

where $p_{2,FIFO} = (1-\rho_{FIFO})\rho_{FIFO}^2$ and $p_{3,FIFO} = (1-\rho_{FIFO})\rho_{FIFO}^3$. It has now been proven that

$ETST_{2,FIFO}$ can be reformulated as the function of $ETST_{1,FIFO}$.

Appendix B:

Mathematical induction can be used to prove that the statement

$$ETST_{n-1,FIFO} = \rho_{FIFO}^{n-2} ETST_{1,FIFO} + \rho_{FIFO} ETST_{n-2,FIFO} \tag{B.1}$$

holds for $n \geqq 3$.

Proof:

(1) Base case: Show that the statement holds for $n = 3$

According to Equation (A.1) in Appendix A, we obtain the following:

$$ETST_{2,FIFO} = \rho_{FIFO} ETST_{1,FIFO} + \rho_{FIFO} ETST_{1,FIFO} \tag{B.2}$$

From Equation (B.1), we have this equation:

$$ETST_{2,FIFO} = \rho_{FIFO}^{3-2} ETST_{1,FIFO} + \rho_{FIFO} ETST_{3-2,FIFO} \tag{B.3}$$

Thus, they are equal.

(2) Inductive step: Show that if $ETST_{n\text{-}1,FIFO}$ holds, then $ETST_{n,FIFO}$ also holds. This can be done as follows.

Suppose $ETST_{n\text{-}1,FIFO}$ holds, such that

$$\begin{aligned} ETST_{n-1,FIFO} &= p_{n,FIFO} \sum_{j^1=1}^{J} \cdots \sum_{j^{n-1}=1}^{J} \left(\prod_{k=1}^{n-1} \frac{\lambda_{j^k}}{\lambda} \right) \left[\sum_{k=1}^{n-1} \left(pt_{j^k} + P_{s,j^k,FIFO} \sum_{\substack{r^k=1 \\ r^k \neq j^k}}^{J} \frac{\lambda_{r^k}}{\lambda^{c^k}} s_{r^k j^k} \right) \right] \\ &= \rho_{FIFO}^{n-2} ETST_{1,FIFO} + \rho_{FIFO} ETST_{n-2,FIFO} \end{aligned} \tag{B.4}$$

It must then be shown that $ETST_{n,FIFO}$ holds, such that

$$
\begin{aligned}
&ETST_{n,FIFO} \\
&= p_{n+1,FIFO}\sum_{j^1=1}^{J}\cdots\sum_{j^n=1}^{J}\left(\prod_{k=1}^{n}\frac{\lambda_{j^k}}{\lambda}\right)\left[\sum_{k=1}^{n}\left(pt_{j^k}+P_{s,j^k,FIFO}\sum_{\substack{r^k=1\\ r^k\neq j^k}}^{J}\frac{\lambda_{r^k}}{\lambda^{c^k}}s_{r^k j^k}\right)\right] \\
&= p_{n+1,FIFO}\sum_{j^1=1}^{J}\frac{\lambda_{j^1}}{\lambda}\left\{\left(pt_{j^1}+P_{s,j^1,FIFO}\sum_{\substack{r^1=1\\ r^1\neq j^1}}^{J}\frac{\lambda_{r^1}}{\lambda^{c^1}}s_{r^1 j^1}\right)\left[\sum_{j^2=1}^{J}\cdots\sum_{j^n=1}^{J}\left(\prod_{k=2}^{n}\frac{\lambda_{j^k}}{\lambda}\right)\right]+\right. \\
&\qquad\qquad \left.\sum_{j^2=1}^{J}\cdots\sum_{j^n=1}^{J}\left(\prod_{k=2}^{n}\frac{\lambda_{j^k}}{\lambda}\right)\left[\sum_{k=2}^{n}\left(pt_{j^k}+P_{s,j^k,FIFO}\sum_{\substack{r^k=1\\ r^k\neq j^k}}^{J}\frac{\lambda_{r^k}}{\lambda^{c^k}}s_{r^k j^k}\right)\right]\right\}
\end{aligned} \tag{B.5}
$$

Note that we have the following relations according to Equation (B.4):

$$
\sum_{j^2=1}^{J}\cdots\sum_{j^n=1}^{J}\left(\prod_{k=2}^{n}\frac{\lambda_{j^k}}{\lambda}\right)=1 \tag{B.6}
$$

$$
\sum_{j^2=1}^{J}\cdots\sum_{j^n=1}^{J}\left(\prod_{k=2}^{n}\frac{\lambda_{j^k}}{\lambda}\right)\left[\sum_{k=2}^{n}\left(pt_{j^k}+P_{s,j^k,FIFO}\sum_{\substack{r^k=1\\ r^k\neq j^k}}^{J}\frac{\lambda_{r^k}}{\lambda^{c^k}}s_{r^k j^k}\right)\right]=\frac{ETST_{n-1,FIFO}}{p_{n,FIFO}} \tag{B.7}
$$

From Equation (A.3), Equation (A.4), Equation (B.6), and Equation (B.7), $ETST_{n,FIFO}$ can be rewritten as follows:

$$
\begin{aligned}
ETST_{n,FIFO} &= p_{n+1,FIFO}\left\{\frac{ETST_{1,FIFO}}{p_{2,FIFO}}+\sum_{j^1=1}^{J}\frac{\lambda_{j^1}}{\lambda}\frac{ETST_{n-1,FIFO}}{p_{n,FIFO}}\right\} \\
&= \frac{p_{n+1,FIFO}}{p_{2,FIFO}}ETST_{1,FIFO}+\frac{p_{n+1,FIFO}}{p_{n,FIFO}}\sum_{j^1=1}^{J}\frac{\lambda_{j^1}}{\lambda}ETST_{n-1,FIFO} \\
&= \rho_{FIFO}^{n-1}ETST_{1,FIFO}+\rho_{FIFO}ETST_{n-1,FIFO}\sum_{j^1=1}^{J}\frac{\lambda_{j^1}}{\lambda} \\
&= \rho_{FIFO}^{n-1}ETST_{1,FIFO}+\rho_{FIFO}ETST_{n-1,FIFO}
\end{aligned} \tag{B.8}
$$

where $p_{n,FIFO}=(1-\rho_{FIFO})\rho_{FIFO}^{n}$. This shows that $ETST_{n,FIFO}$ indeed holds. Since both the base case and the inductive step have been proven, it has now also been proven by mathematical induction that $ETST_{n-1,FIFO}$ in Equation (7-2) holds for $n \geqq 3$.

Appendix C:

To prove the following statement:

$$\lim_{RT\to\infty} E\left[W_{q,j,FIFO}\right] = w_j \rho_{FIFO} \left\{ (1-\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{2}{1+C_v^2} \right\} E\left[W_q\right]_{M/G/1} \tag{C.1}$$

Proof:

The ratio $n_j^{-1}\sum_{i=1}^{n_j} E[TST_{ij,FIFO}] \big/ E[TST]_{M/G/1}$ can be shown as

$$\frac{n_j^{-1}\sum_{i=1}^{n_j} E\left[TST_{ij,FIFO}\right]}{E\left[TST\right]_{M/G/1}} = \frac{w_j \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left[pt_j + P_{s,j,FIFO}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{jr}\right]}{\frac{1+C_v^2}{2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]\left[pt_j + P_{s,j,FIFO}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{jr}\right]} \tag{C.2}$$

where $E[TST]_{M/G/1}$ represents the expected total service time of all waiting jobs ahead in queue for the $M/G/1$ queuing model and is shown as $E[TST]_{M/G/1} = E[N_q]E[ST_{FIFO}]$, where $E[N_q]$ represents the expected number of jobs in queue and can be computed as $[(1+C_v^2)/2][\rho_{FIFO}^2/(1-\rho_{FIFO})]$ and $E[ST_{FIFO}]$ can be calculated by the probability mass function of $ST_{ij,FIFO}$ in Equation (3-3). If RT is large enough, then the following holds:

$$\lim_{RT\to\infty} n_j^{-1}\sum_{i=1}^{n_j} E\left[TST_{ij,FIFO}\right] = \frac{2w_j}{1+C_v^2} E\left[TST\right]_{M/G/1} \tag{C.3}$$

where $\lim_{RT\to\infty} \Pr[T_{ij} \le RT] = 1$.

Next, the ratio $n_j^{-1}\sum_{i=1}^{n_j} E[R_{ij,FIFO}] \big/ E[R]_{M/G/1}$ can be given by

$$\frac{n_j^{-1}\sum_{i=1}^{n_j} E\left[R_{ij,FIFO}\right]}{E\left[R\right]_{M/G/1}} = \frac{w_j \rho_{FIFO}\left\{\frac{1}{2}\left(\sum_{j=1}^{J} n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\lambda_j\Delta_j''\right\}}{\frac{1}{2}\left(\sum_{j=1}^{J} n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\lambda\Pr\left[T_{ij}\le RT\right]\Delta_j''} = \frac{w_j\rho_{FIFO}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\lambda_j\Delta_j''}{\lambda\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]\Delta_j''} \tag{C.4}$$

where $E[R]_{M/G/1}$ represents the expected residual service time for the $M/G/1$ queuing model and is shown as $E[R]_{M/G/1} = \lambda E[ST_{FIFO}^2]/2$, where the second moment of service time ($E[ST_{FIFO}^2]$) can be calculated by the probability mass function of $ST_{ij,FIFO}$ in Equation (3-3). If RT is large enough, then the following holds:

$$\lim_{RT\to\infty} n_j^{-1}\sum_{i=1}^{n_j} E\left[R_{ij,FIFO}\right] = w_j \rho_{FIFO} \frac{\sum_{j=1}^{J}\sum_{i=1}^{n_j}\lambda_j \Delta_j''}{\lambda \sum_{j=1}^{J}\sum_{i=1}^{n_j}\Delta_j''} E[R]_{M/G/1}$$
$$= w_j \rho_{FIFO} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} E[R]_{M/G/1} \tag{C.5}$$

where $\lim_{RT\to\infty} \Pr[T_{ij} \le RT] = 1$.

According to Equation (C.3) and Equation (C.5), $\lim_{RT\to\infty} E[W_{q,j,FIFO}]$ can be shown as

$$\lim_{RT\to\infty} E\left[W_{q,j,FIFO}\right] = \lim_{RT\to\infty} n_j^{-1}\sum_{i=1}^{n_j}\left\{E\left[R_{ij,FIFO}\right] + E\left[TST_{ij,FIFO}\right]\right\}$$
$$= w_j \rho_{FIFO} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} E[R]_{M/G/1} + \frac{2w_j}{1+C_v^2} E[TST]_{M/G/1}$$
$$= w_j \left\{ \rho_{FIFO} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} E[R]_{M/G/1} + \frac{\rho_{FIFO}^2}{1-\rho_{FIFO}} E[ST_{FIFO}] \right\} \tag{C.6}$$
$$= w_j \rho_{FIFO} E[R]_{M/G/1} \left\{ \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{\rho_{FIFO}}{1-\rho_{FIFO}} \frac{E[ST_{FIFO}]}{E[R]_{M/G/1}} \right\}$$

where

$$E[R]_{M/G/1} = \frac{\lambda E\left[ST_{FIFO}^2\right]}{2} = (1-\rho_{FIFO}) E\left[W_q\right]_{M/G/1} \tag{C.7}$$

$$\rho_{FIFO} \frac{E[ST_{FIFO}]}{E[R]_{M/G/1}} = \frac{2\rho_{FIFO} E[ST_{FIFO}]}{\lambda E\left[ST_{FIFO}^2\right]} = \frac{2E[ST_{FIFO}]^2}{E\left[ST_{FIFO}^2\right]} = \frac{2}{1+C_v^2} \tag{C.8}$$

Then, $\lim_{RT\to\infty} E\left[WT_{j,FIFO}\right]$ can be rewritten as follows:

$$\lim_{RT\to\infty} E\left[W_{q,j,FIFO}\right] = w_j \rho_{FIFO} \left\{ (1-\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{2}{1+C_v^2} \right\} E\left[W_q\right]_{M/G/1} \tag{C.9}$$

It has now been proven that the expected mean waiting time of the product type j jobs is proportional to the expected waiting time for the $M/G/1$ queuing theory when RT is long enough.

Appendix D:

To prove the following statement:

$$\frac{d}{d\rho_{FIFO}} w_j \rho_{FIFO} \left\{ (1-\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{2}{1+C_v^2} \right\} > 0 \tag{D.1}$$

Proof:

The gradient in Equation (10-1) can be reformulated as

$$\alpha_j = w_j \rho_{FIFO} \left\{ (1-\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + \frac{2\rho_{FIFO}^2}{\lambda^2 E\left[ST_{FIFO}^2\right]} \right\} \tag{D.2}$$

because $E[ST_{FIFO}^2] = (1+C_v^2)E[ST_{FIFO}]^2$ and $\rho_{FIFO} = \lambda E[ST_{FIFO}]$. The first derivative of this gradient with respect to ρ_{FIFO} is then given by the following:

$$\begin{aligned} \frac{d\alpha_j}{d\rho_{FIFO}} &= w_j (1-2\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + w_j \frac{6\rho_{FIFO}^2}{\lambda^2 E\left[ST_{FIFO}^2\right]} \\ &= w_j (1-2\rho_{FIFO}) \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} + w_j \frac{6}{1+C_v^2} \end{aligned} \tag{D.3}$$

According to Equation (C.8), $2/(1+C_v^2) \geq \rho_{FIFO}$ because $E[ST_{FIFO}] \geq E[R]_{M/G/1}$. Thus, we get

$$\begin{aligned}\frac{d\alpha_j}{d\rho_{FIFO}} &\geq w_j\left(1-2\rho_{FIFO}\right)\frac{\sum_{j=1}^{J} n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J} n_j\lambda\Delta_j''}+3w_j\rho_{FIFO} \\ &\geq w_j\rho_{FIFO}\left(3-2\frac{\sum_{j=1}^{J} n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J} n_j\lambda\Delta_j''}\right)+w_j\frac{\sum_{j=1}^{J} n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J} n_j\lambda\Delta_j''}>0\end{aligned} \tag{D.4}$$

where $\sum_{j=1}^{J} n_j\lambda\Delta_j'' \geq \sum_{j=1}^{J} n_j\lambda_j\Delta_j''$, and $J>0$ and $0<\rho_{FIFO}\leqq 1$ with $\lambda_j>0$ for all j. It has now been proven that the first derivative of the gradient α_j with respect to ρ_{FIFO} is a positive number.

Appendix E:

To prove the following statement:

$$\lim_{RT\to\infty} E\left[W^2_{q,j,FIFO}\right]=\beta_{1j}E\left[R^2\right]_{M/G/1}+2\beta_{2j}E\left[W_q\right]^2_{M/G/1} \tag{E.1}$$

Proof:

When RT is long enough, the second moment of waiting time of product type j jobs under FIFO in Equation (3-27) can be shown as

$$\begin{aligned}\lim_{RT\to\infty} E\left[W^2_{q,j,FIFO}\right]=&\lim_{RT\to\infty} n_j^{-1}\sum_{i=1}^{n_j} E\left[R^2_{ij,FIFO}\right]+\lim_{RT\to\infty} n_j^{-1}\sum_{i=1}^{n_j} E\left[TST^2_{ij,FIFO}\right]+ \\ &\lim_{RT\to\infty} 2n_j^{-1}\sum_{i=1}^{n_j}\left\{E\left[R_{ij,FIFO}\right]E\left[TST_{ij,FIFO}\right]\right\}\end{aligned} \tag{E.2}$$

First, the ratio $\lim_{RT\to\infty} n_j^{-1}\sum_{i=1}^{n_j} E[R^2_{ij,FIFO}]\big/E[R^2]_{M/G/1}$ can be shown as

$$\lim_{RT\to\infty}\frac{n_j^{-1}\sum_{i=1}^{n_j}E\left[R_{ij,FIFO}^2\right]}{E\left[R^2\right]_{M/G/1}}=\lim_{RT\to\infty}\frac{w_j\rho_{FIFO}\left\{\frac{1}{3}\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\lambda_j\Delta_j'''\right\}}{\frac{\lambda}{3}\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]\Delta_j'''}=w_j\rho_{FIFO}\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j'''}{\sum_{j=1}^{J}n_j\lambda\Delta_j'''} \tag{E.3}$$

$$\Delta_j'''=P_{s,j,FIFO}\sum_{\substack{r=1\\r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}\left(pt_j+s_{rj}\right)^3+\left(1-P_{s,j,FIFO}\right)pt_j^3 \tag{E.4}$$

where the second moment of the residual service time for $M/G/1$ queueing system is given by $E[R^2]_{M/G/1}=\lambda E[ST_{FIFO}^3]/3$, $E[ST_{FIFO}^3]$ is the third moment of the service time under FIFO and can be computed according to the probability mass function of $ST_{ij,FIFO}$ in Equation (3-3), and $\lim_{RT\to\infty}\Pr[T_{ij}\le RT]=1$. Thus, the limit of $n_j^{-1}\sum_{i=1}^{n_j}E[R_{ij,FIFO}^2]$ is the function of $E[R^2]_{M/G/1}$ when RT approaches infinity.

Second, the second moment of total service time of all waiting jobs in queue ahead of the product type j jobs can be expressed as Equation (E.5).

$$\begin{aligned}n_j^{-1}\sum_{i=1}^{n_j}E\left[TST_{ij,FIFO}^2\right]&=n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}ESTST_{n-1,ij,FIFO}\\&=n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr\left[T_{ij}\le RT\right]p_{n,FIFO}\mathrm{SS}_{ST}+\\&\quad n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr\left[T_{ij}\le RT\right]p_{n,FIFO}\mathrm{Prod}_{ST}\end{aligned} \tag{E.5}$$

The limits of $n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr[T_{ij}\le RT]p_{n,FIFO}\mathrm{SS}_{ST}$ and $n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr[T_{ij}\le RT]p_{n,FIFO}\mathrm{Prod}_{ST}$ divide by $E[N_q]_{M/G/1}E[ST_{FIFO}^2]$ as RT approaches infinity can be shown in Equation (E.6) and Equation (E.7). where, $E[N_q]_{M/G/1}$ represents the expected number of jobs in queue for the $M/G/1$ queuing model and can be computed as $[(1+C_v^2)/2][\rho_{FIFO}^2/(1-\rho_{FIFO})]$, and $E[ST_{FIFO}]$ and $E[ST_{FIFO}^2]$ represent the first and second moments of the service time under FIFO and can be computed by the probability mass function of $ST_{ij,FIFO}$ in Equation (3-3) and are expressed as Equation (E.8) and Equation (E.9).

$$\lim_{RT\to\infty}\frac{n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr\left[T_{ij}\le RT\right]p_{n,FIFO}\mathrm{SS}_{ST}}{E\left[N_q\right]_{M/G/1}E\left[ST_{FIFO}^2\right]}=w_j\frac{2}{1+c_v^2} \tag{E.6}$$

$$\begin{aligned}\lim_{RT\to\infty}\frac{n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr\left[T_{ij}\le RT\right]p_{n,FIFO}\mathrm{Prod}_{ST}}{E\left[N_q\right]_{M/G/1}E\left[ST_{FIFO}^2\right]}&=\lim_{RT\to\infty}\frac{n_j^{-1}\sum_{i=1}^{n_j}\sum_{n=2}^{\infty}\Pr\left[T_{ij}\le RT\right]p_{n,FIFO}\mathrm{Prod}_{ST}}{E\left[N_q\right]_{M/G/1}\dfrac{E\left[ST_{FIFO}^2\right]}{E\left[ST_{FIFO}\right]^2}E\left[ST_{FIFO}\right]^2}\\&=w_j\frac{2\rho_{FIFO}}{1-\rho_{FIFO}}\frac{2}{\left(1+C_v^2\right)^2}\end{aligned} \tag{E.7}$$

where

$$\begin{aligned}E\left[ST_{FIFO}\right]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]\times\\\left\{P_{s,j,FIFO}\sum_{\substack{r=1\\r\ne j}}^{J}\frac{\lambda_r}{\lambda^c}\left(pt_j+s_{rj}\right)+\left(1-P_{s,j,FIFO}\right)pt_j\right\}\end{aligned} \tag{E.8}$$

$$\begin{aligned}E\left[ST_{FIFO}^2\right]=\left(\sum_{j=1}^{J}n_j\right)^{-1}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\Pr\left[T_{ij}\le RT\right]\times\\\left\{P_{s,j,FIFO}\sum_{\substack{r=1\\r\ne j}}^{J}\frac{\lambda_r}{\lambda^c}\left(pt_j+s_{rj}\right)^2+\left(1-P_{s,j,FIFO}\right)pt_j^2\right\}\end{aligned} \tag{E.9}$$

Thus, the limit of $n_j^{-1}\sum_{i=1}^{n_j}E[TST_{ij,FIFO}^2]$ can be derived as the function of the square of the expected waiting time for $M/G/1$ queueing system ($E[W_q]_{M/G/1}^2$) as RT approaches infinity and is given by Equation (E.10).

$$\begin{aligned}&\lim_{RT\to\infty}n_j^{-1}\sum_{i=1}^{n_j}E\left[TST_{ij,FIFO}^2\right]\\&=\left[w_j\frac{2}{1+C_v^2}+w_j\frac{2\rho_{FIFO}}{1-\rho_{FIFO}}\frac{2}{\left(1+C_v^2\right)^2}\right]E\left[N_q\right]_{M/G/1}\frac{E\left[ST_{FIFO}^2\right]}{E\left[ST_{FIFO}\right]^2}E\left[ST_{FIFO}\right]^2\\&=2w_j\left(1-\rho_{FIFO}\right)\frac{2}{1+C_v^2}\left[1+\frac{\rho_{FIFO}}{1-\rho_{FIFO}}\frac{2}{1+C_v^2}\right]E\left[W_q\right]_{M/G/1}^2\end{aligned} \tag{E.10}$$

where

$$\frac{E\left[ST_{FIFO}^2\right]}{E\left[ST_{FIFO}\right]^2}=1+C_v^2 \tag{E.11}$$

$$E\left[W_q\right]_{M/G/1}=\frac{\lambda E\left[ST_{FIFO}^2\right]}{2\left(1-\rho_{FIFO}\right)}=\frac{1+C_v^2}{2}\frac{\rho_{FIFO}}{1-\rho_{FIFO}}E\left[ST_{FIFO}\right] \tag{E.12}$$

Finally, the limit of $2n_j^{-1}\sum_{i=1}^{n_j}E[R_{ij,FIFO}]E[TST_{ij,FIFO}]$ divide by $E[R]_{M/G/1}E[TST]_{M/G/1}$ as RT approaches infinity is considered and is given by Equation (E.13), where $E[TST]_{M/G/1}$ represents the expected total service time of all waiting jobs ahead in queue for the $M/G/1$ queuing model and is shown as $E[TST]_{M/G/1}=E[N_q]E[ST_{FIFO}]$.

$$\lim_{RT\to\infty}\frac{2n_j^{-1}\sum_{i=1}^{n_j}\left\{E\left[R_{ij,FIFO}\right]E\left[TST_{ij,FIFO}\right]\right\}}{E\left[R\right]_{M/G/1}E\left[TST\right]_{M/G/1}}=2w_j\rho_{FIFO}\frac{2}{1+C_v^2}\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''} \tag{E.13}$$

$$\Delta_j''=P_{s,j,FIFO}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}\left(pt_j+s_{rj}\right)^2+\left(1-P_{s,j,FIFO}\right)pt_j^2 \tag{E.14}$$

From Equation (E.13), the limit of $2n_j^{-1}\sum_{i=1}^{n_j}E[R_{ij,FIFO}]E[TST_{ij,FIFO}]$ can be also derived as the function of the square of the expected waiting time for $M/G/1$ queueing system ($E[W_q]_{M/G/1}^2$) as RT approaches infinity and is given by Equation (E.15).

$$\begin{aligned}&\lim_{RT\to\infty}2n_j^{-1}\sum_{i=1}^{n_j}\left\{E\left[R_{ij,FIFO}\right]E\left[TST_{ij,FIFO}\right]\right\}\\&=2w_j\rho_{FIFO}\frac{2}{1+C_v^2}\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}E\left[R\right]_{M/G/1}^2E\left[N_q\right]_{M/G/1}\frac{E\left[ST_{FIFO}\right]}{E\left[R\right]_{M/G/1}}\\&=2w_j\rho_{FIFO}^2\left(1-\rho_{FIFO}\right)\frac{2}{1+C_v^2}\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}E\left[W_q\right]_{M/G/1}^2\end{aligned} \tag{E.15}$$

where

$$E\left[R\right]_{M/G/1}=\left(1-\rho_{FIFO}\right)E\left[W_q\right]_{M/G/1} \tag{E.16}$$

$$\frac{E[ST_{FIFO}]}{E[R]_{M/G/1}}=\frac{E[ST_{FIFO}]}{\lambda E\left[ST_{FIFO}^2\right]/2}=\frac{2E[ST_{FIFO}]^2}{\rho_{FIFO}E\left[ST_{FIFO}^2\right]}=\rho_{FIFO}^{-1}\frac{2}{1+C_v^2} \tag{E.17}$$

Referring Equation (E.3), Equation (E.10) and Equation (E.15), the limit of $E[W_{q,j,FIFO}^2]$ can be obtained as Equation (E.18).

$$\lim_{RT\to\infty}E\left[W_{q,j,FIFO}^2\right]=\beta_{1j}E\left[R^2\right]_{M/G/1}+2\beta_{2j}E\left[W_q\right]_{M/G/1}^2 \tag{E.18}$$

where

$$\beta_{1j}=w_j\rho_{FIFO}\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j'''}{\sum_{j=1}^{J}n_j\lambda\Delta_j'''}\le 1 \tag{E.19}$$

$$\beta_{2j}=w_j\left(1-\rho_{FIFO}\right)\frac{2}{1+C_v^2}\left(1+\frac{\rho_{FIFO}}{1-\rho_{FIFO}}\frac{2}{1+C_v^2}+\rho_{FIFO}^2\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}\right)>0 \tag{E.20}$$

It has now been proven that the second moment of the waiting time of the product type j jobs is proportional to the second moment of the residual service time and the square of the expected waiting time for the $M/G/1$ queuing theory when RT is long enough.

Appendix F:

To prove the following statement:

$$\beta_{2j}'=\frac{d}{d\rho_{FIFO}}w_j\left(1-\rho_{FIFO}\right)\frac{2}{1+C_v^2}\left(1+\frac{\rho_{FIFO}}{1-\rho_{FIFO}}\frac{2}{1+C_v^2}+\rho_{FIFO}^2\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}\right)>0 \tag{F.1}$$

Proof:

The gradient β_{2j} in Equation (3-37) can be reformulated as

$$\beta_{2j} = w_j \left[\frac{2\rho_{FIFO}^2 - 2\rho_{FIFO}^3}{\lambda^2 E\left[ST_{FIFO}^2\right]} + \frac{4\rho_{FIFO}^5}{\lambda^4 E\left[ST_{FIFO}^2\right]^4} + \frac{2\rho_{FIFO}^4 - 2\rho_{FIFO}^5}{\lambda^2 E\left[ST_{FIFO}^2\right]} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \right] \tag{F.2}$$

because $[2/(1+c_v^2)] = 2\rho_{FIFO}^2 / \lambda^2 E[ST_{FIFO}^2]$. The first derivative of the gradient β_{2j} with respect to ρ_{FIFO} is then given by the following:

$$\begin{aligned}
\frac{d\beta_{2j}}{d\rho_{FIFO}} &= w_j \left[\frac{4\rho_{FIFO} - 6\rho_{FIFO}^2}{\lambda^2 E\left[ST_{FIFO}^2\right]} + \frac{20\rho_{FIFO}^4}{\lambda^4 E\left[ST_{FIFO}^2\right]^4} + \frac{8\rho_{FIFO}^3 - 10\rho_{FIFO}^4}{\lambda^2 E\left[ST_{FIFO}^2\right]} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \right] \\
&= w_j \left[\frac{4\rho_{FIFO}}{\lambda^2 E\left[ST_{FIFO}^2\right]} - 3\frac{2}{1+C_v^2} + 5\left(\frac{2}{1+C_v^2}\right)^2 + \right. \\
&\qquad \left. \left(4\rho_{FIFO} - 5\rho_{FIFO}^2\right) \frac{2}{1+C_v^2} \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \right] \\
&= w_j \frac{2}{1+C_v^2} \left\{ 5\left(\frac{2}{1+C_v^2} - \rho_{FIFO}^2 \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} \right) + \frac{1}{\rho_{FIFO}} \left[4\left(\rho_{FIFO} - \frac{3}{8}\right)^2 + \frac{23}{16} \right] \right\}
\end{aligned} \tag{F.3}$$

In Equation (3.22), the gradient $\alpha_j = 2/(1+C_v^2) \geq 1$ if $\rho_{FIFO}=1$ when RT is long enough, where $\lim_{RT \to \infty} w_j = 1$. Thus, we get $d\beta_{2j}/d\rho_{FIFO} > 0$, where $\sum_{j=1}^{J} n_j \lambda \Delta_j'' \geq \sum_{j=1}^{J} n_j \lambda_j \Delta_j''$, and $J > 0$ and $0 < \rho_{FIFO} \leqq 1$ with $\lambda_j > 0$ for all j. It has now been proven that the first derivative of the gradient β_{2j} with respect to ρ_{FIFO} is a positive number.

Appendix G:

To prove the following statement:

$$\frac{d\beta_{2j}}{d\rho_{FIFO}} - \frac{d\alpha_j^2}{d\rho_{FIFO}} \geq g_1\left(\rho_{FIFO}\right) - g_2\left(\rho_{FIFO}\right) \tag{G.1}$$

Proof:

According to Equation (3.22) and Equation (3.25), the first derivatives of α_j^2 and β_{2j}

with respect to ρ_{FIFO} can be reformulated as Equation (G.2) and Equation (G.3).

$$
\begin{aligned}
\frac{d\alpha_j^2}{d\rho_{FIFO}} &= 2\alpha_j \frac{d\alpha_j}{d\rho_{FIFO}} \\
&= 2\rho_{FIFO}\left(1-\rho_{FIFO}\right)\left(1-2\rho_{FIFO}\right)\left[\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j}\right]^2 + \\
&\quad 2\rho_{FIFO}\left(4-5\rho_{FIFO}\right)\frac{2}{1+C_v^2}\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j} + 6\rho_{FIFO}\left(\frac{2}{1+C_v^2}\right)^2
\end{aligned}
\tag{G.2}
$$

$$
\frac{d\beta_{2j}}{d\rho_{FIFO}} = 5\left(\frac{2}{1+C_v^2}\right)^2 - 5\rho_{FIFO}^2 \frac{2}{1+C_v^2}\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j} + \frac{2}{1+C_v^2}\left(\frac{2}{\rho_{FIFO}} + 4\rho_{FIFO} - 3\right)
\tag{G.3}
$$

where $\lim_{RT \to \infty} w_j = 1$. The difference between $d\beta_{2j}/d\rho_{FIFO}$ and $d\alpha_j^2/d\rho_{FIFO}$ is given by Equation (G.4).

$$
\begin{aligned}
\frac{d\beta_{2j}}{d\rho_{FIFO}} - \frac{d\alpha_j^2}{d\rho_{FIFO}} &= \left(5-6\rho_{FIFO}\right)\left(\frac{2}{1+C_v^2}\right)^2 + \frac{2}{1+C_v^2}\left(\frac{2}{\rho_{FIFO}} + 4\rho_{FIFO} - 3\right) - \\
&\quad 2\rho_{FIFO}\left(1-\rho_{FIFO}\right)\left(1-2\rho_{FIFO}\right)\left[\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j}\right]^2 - \\
&\quad \left(8\rho_{FIFO} - 5\rho_{FIFO}^2\right)\frac{2}{1+C_v^2}\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j} \\
&\geq \left(5-6\rho_{FIFO}\right) + \left(\frac{2}{\rho_{FIFO}} + 4\rho_{FIFO} - 3\right) - \\
&\quad \frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j}\left\{2\rho_{FIFO}\left(1-\rho_{FIFO}\right)\left(1-2\rho_{FIFO}\right)\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta''_j}{\sum_{j=1}^{J} n_j \lambda \Delta''_j} + 8\rho_{FIFO} - 5\rho_{FIFO}^2\right\}
\end{aligned}
\tag{G.4}
$$

where $2/(1+C_v^2) \geq 1$, and

$$g_1\left(\rho_{FIFO}\right)=\left(5-6\rho_{FIFO}\right)+\left(\frac{2}{\rho_{FIFO}}+4\rho_{FIFO}-3\right) \tag{G.5}$$

$$g_2\left(\rho_{FIFO}\right)=\rho_{FIFO}\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}\left\{8-5\rho_{FIFO}+2\left(1-\rho_{FIFO}\right)\left(1-2\rho_{FIFO}\right)\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}\right\} \tag{G.6}$$

Thus, an inequality expressed as $d\beta_{2j}/d\rho_{FIFO}-d\alpha_j^2/d\rho_{FIFO}\geq g_1\left(\rho_{FIFO}\right)-g_2\left(\rho_{FIFO}\right)$ can be derived.

Appendix H:

To prove the following statement:

$$g_1(\rho_{FIFO})-g_3(\rho_{FIFO})>0 \tag{H.1}$$

where $g_1(\rho_{FIFO})$ and $g_3(\rho_{FIFO})$ are shown as Equation (3-42) and Equation (3-43).

Proof:

The law of trichotomy reminds us that there exist three distinct possibilities, exactly one of the following holds:

Case I. $g_1(\rho_{FIFO})-g_3(\rho_{FIFO})<0$ (H.2)

Case II. $g_1(\rho_{FIFO})-g_3(\rho_{FIFO})=0$ (H.3)

Case III. $g_1(\rho_{FIFO})-g_3(\rho_{FIFO})>0$ (H.3)

For Case I and Case II, an inequality expressed as Equation (H.4) can be derived. Moreover, another inequality expressed as Equation (H.5) can be got under condition that $g_1(\rho_{FIFO})-g_3(\rho_{FIFO})>0$.

$$\frac{\sum_{j=1}^{J}n_j\lambda_j\Delta_j''}{\sum_{j=1}^{J}n_j\lambda\Delta_j''}\geq\frac{\frac{2}{\rho_{FIFO}}-2\rho_{FIFO}+2}{8-5\rho_{FIFO}+2\left(1-\rho_{FIFO}\right)\left(1-2\rho_{FIFO}\right)} \tag{H.4}$$

$$\frac{\sum_{j=1}^{J} n_j \lambda_j \Delta_j''}{\sum_{j=1}^{J} n_j \lambda \Delta_j''} < \frac{\frac{2}{\rho_{FIFO}} - 2\rho_{FIFO} + 2}{8 - 5\rho_{FIFO} + 2(1-\rho_{FIFO})(1-2\rho_{FIFO})} \tag{H.5}$$

where $0 < \rho_{FIFO} \leqq 1$ and then

$$\frac{\frac{2}{\rho_{FIFO}} - 2\rho_{FIFO} + 2}{8 - 5\rho_{FIFO} + 2(1-\rho_{FIFO})(1-2\rho_{FIFO})} \geq \frac{2}{3}$$

In Case I and Case II, the solution set of λ_j can be shown as Equation (H.6) and may be more than λ. This is illegitimate because $\sum_{j=1}^{J} \lambda_j = \lambda$. In Case III, the solution set of λ_j can be shown as Equation (H.6) and is legitimate.

$$\sum_{j=1}^{J} (3\lambda_j) n_j \Delta_j'' \geq \sum_{j=1}^{J} (2\lambda) n_j \Delta_j'' \Rightarrow \lambda_j \geq \frac{2}{3}\lambda \tag{H.6}$$

$$\sum_{j=1}^{J} (3\lambda_j) n_j \Delta_j'' < \sum_{j=1}^{J} (2\lambda) n_j \Delta_j'' \Rightarrow \lambda_j < \frac{2}{3}\lambda \tag{H.7}$$

Therefore an inequality expressed as $g_1(\rho_{FIFO}) - g_3(\rho_{FIFO}) > 0$ can be derived, where $J > 1$.

Appendix I:

The variance of $\overline{DS}_{FIFO-FSR}$ ($Var[\overline{DS}_{FIFO-FSR}]$) is determined as the follows:

The variance of $\overline{DS}_{FIFO-FSR}$ is given by using the definition of covariance and is defined as the equation below.

$$\begin{aligned} Var\left[\overline{DS}_{FIFO-FSR}\right] &= Var\left[\left(\sum_{j=1}^{J} n_j\right)^{-1} \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(S_{ij,FIFO} - S_{ij,FSR}\right)\right] \\ &= \left(\sum_{j=1}^{J} n_j\right)^{-2} \sum_{j=1}^{J}\sum_{i=1}^{n_j} Var\left[S_{ij,FIFO} - S_{ij,FSR}\right] \\ &= \left(\sum_{j=1}^{J} n_j\right)^{-2} \sum_{j=1}^{J}\sum_{i=1}^{n_j}\left\{Var\left[S_{ij,FIFO}\right] + Var\left[S_{ij,FSR}\right] - \right. \\ &\qquad \left. 2\operatorname{cov}\left(S_{ij,FIFO}, S_{ij,FSR}\right)\right\} \end{aligned} \tag{I.1}$$

where $Var[S_{ij,FIFO}] = E[S_{ij,FIFO}^2] - E[S_{ij,FIFO}]^2$ and $Var[S_{ij,FSR}] = E[S_{ij,FSR}^2] - E[S_{ij,FSR}]^2$.

According to the probability mass functions of $S_{ij,FIFO}$ and $S_{ij,FSR}$, then $E[S^2_{ij,FIFO}]$ and $E[S^2_{ij,FSR}]$ are respectively expressed as Equations (I.2) and (I.3). For associated reference of the probability mass functions of $S_{ij,FIFO}$, refer to Equation (3-3).

$$
\begin{aligned}
E\left[S^2_{ij,FIFO}\right] &= \Pr\left[T_{ij} > RT\right] \times 0^2 + \Pr\left[T_{ij} \le RT\right]\left(1 - P_{s,j,FIFO}\right) s^2_{jj} + P_{s,ij,FIFO} \sum_{\substack{r=1 \\ r \ne j}}^{J} \frac{\lambda_r}{\lambda^c} s^2_{rj} \\
&= P_{s,ij,FIFO} \sum_{\substack{r=1 \\ r \ne j}}^{J} \frac{\lambda_r}{\lambda^c} s^2_{rj}
\end{aligned} \tag{I.2}
$$

$$
\begin{aligned}
E\left[S^2_{ij,FSR}\right] &= \Pr\left[T_{ij} > RT\right] \times 0^2 + \Pr\left[T_{ij} \le RT\right]\left(1 - P_{s,j,FSR}\right) s^2_{jj} + P_{s,ij,FSR} \sum_{\substack{r=1 \\ r \ne j}}^{J} \frac{\lambda_r}{\lambda^c} s^2_{rj} \\
&= P_{s,ij,FSR} \sum_{\substack{r=1 \\ r \ne j}}^{J} \frac{\lambda_r}{\lambda^c} s^2_{rj}
\end{aligned} \tag{I.3}
$$

The covariance for two random variables $S_{ij,FIFO}$ and $S_{ij,FSR}$ is defined by $\operatorname{cov}(S_{ij,FIFO}, S_{ij,FSR}) = E[S_{ij,FIFO} S_{ij,FSR}] - E[S_{ij,FIFO}]E[S_{ij,FSR}]$. Let Y be equal to $Y = S_{ij,FIFO} S_{ij,FSR}$, where Y is a discrete distribution having two possible values labelled by $Y = 0$ and $Y = s_{jr} s_{jr'}$, $r = 1, 2, \ldots, J$ and $r \ne j$, and $r' = 1, 2, \ldots, J$ and $r' \ne j$. According to the probability mass functions of $S_{ij,FIFO}$ and $S_{ij,FSR}$, the probability mass function of the random variable Y is shown as Equation (I.4).

$$
\Pr\left[Y = y\right] = \begin{cases} 1 - \displaystyle\sum_{\substack{r=1 \\ r \ne j}}^{J} \sum_{\substack{r'=1 \\ r' \ne j}}^{J} P_{s,ij,FIFO} P_{s,ij,FSR} \frac{\lambda_r}{\lambda^c} \frac{\lambda_{r'}}{\lambda^c}, & \text{if } y = 0 \\ P_{s,ij,FIFO} P_{s,ij,FSR} \dfrac{\lambda_r}{\lambda^c} \dfrac{\lambda_{r'}}{\lambda^c}, & \text{if } y = s_{rj} s_{r'j} \end{cases} \tag{I.4}
$$

Then the expected value of Y can be calculated as Equation (I.5).

$$
\begin{aligned}
E\left[Y\right] &= E\left[S_{ij,FIFO} S_{ij,FSR}\right] \\
&= \left[1 - \sum_{\substack{r=1 \\ r \ne j}}^{J} \sum_{\substack{r'=1 \\ r' \ne j}}^{J} P_{s,ij,FIFO} P_{s,ij,FSR} \frac{\lambda_r}{\lambda^c} \frac{\lambda_{r'}}{\lambda^c}\right] \times 0 + \sum_{\substack{r=1 \\ r \ne j}}^{J} \sum_{\substack{r'=1 \\ r' \ne j}}^{J} P_{s,ij,FIFO} P_{s,ij,FSR} \frac{\lambda_r}{\lambda^c} \frac{\lambda_{r'}}{\lambda^c} s_{jr} s_{r'j} \\
&= P_{s,ij,FIFO} P_{s,ij,FSR} \left(\sum_{\substack{r=1 \\ r \ne j}}^{J} \frac{\lambda_r}{\lambda^c} s_{rj} \right)^2
\end{aligned} \tag{I.5}
$$

Substituting $E[S_{ij,FIFO}S_{ij,FSR}]$, $E[S_{ij,FIFO}]$, and $E[S_{ij,FSR}]$ into $\operatorname{cov}(S_{ij,FIFO}, S_{ij,FSR})$, the covariance for two random variables $S_{ij,FIFO}$ and $S_{ij,FSR}$ is rewritten as Equation (I.6).

$$
\begin{aligned}
\operatorname{cov}\left(S_{ij,FIFO}, S_{ij,FSR}\right) &= E\left[S_{ij,FIFO}S_{ij,FSR}\right] - E\left[S_{ij,FIFO}\right]E\left[S_{ij,FSR}\right] \\
&= P_{s,ij,FIFO}P_{s,ij,FSR}\left(\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right)^2 - \\
&\quad P_{s,ij,FIFO}\left(\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right)P_{s,ij,FSR}\left(\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right) \\
&= 0
\end{aligned}
\tag{I.6}
$$

Then the variance of $\overline{DS}_{FIFO-FSR}$ is given Equation (I.7).

$$
\begin{aligned}
Var\left[\overline{DS}_{FIFO-FSR}\right] &= \left(\sum_{j=1}^{J}n_j\right)^{-2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\left\{E\left[S_{ij,FIFO}^2\right] + E\left[S_{ij,FSR}^2\right] - \right. \\
&\quad \left. E\left[S_{ij,FIFO}\right]^2 - E\left[S_{ij,FSR}\right]^2\right\} \\
&= \left(\sum_{j=1}^{J}n_j\right)^{-2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\left\{P_{s,ij,FIFO}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}^2 + P_{s,ij,FSR}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}^2\right\} - \\
&\quad \left(\sum_{j=1}^{J}n_j\right)^{-2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\left\{\left[P_{s,ij,FIFO}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right]^2 + \left[P_{s,ij,FSR}\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right]^2\right\} \\
&= \left(\sum_{j=1}^{J}n_j\right)^{-2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}^2\right)\left(P_{s,ij,FIFO} + P_{s,ij,FSR}\right) - \\
&\quad \left(\sum_{j=1}^{J}n_j\right)^{-2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\left(\sum_{\substack{r=1\\ r\neq j}}^{J}\frac{\lambda_r}{\lambda^c}s_{rj}\right)^2\left(P_{s,ij,FIFO}^2 + P_{s,ij,FSR}^2\right)
\end{aligned}
\tag{I.7}
$$