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Near automorphisms of trees with small total relative displacements

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Abstract For a permutation f of the vertex set $V(G)$ of a connected graph G, let $\delta_f(x,y) = |d(x,y) - d(f(x), f(y))|$. Define the displacement $\delta_f(G)$ of G with respect to f to be the sum of $\delta_f(x,y)$ over all unordered pairs $\{x,y\}$ of distinct vertices of G. Let $\pi(G)$ denote the smallest positive value of $\delta_f(G)$ among the $n!$ permutations f of $V(G)$. In this note, we determine all trees T with $\pi(T) = 2$ or 4.

Keywords Near automorphism · Tree · Total relative displacement

Suppose G is a connected graph. The distance between two distinct vertices x and y in G is denoted by $d_G(x,y)$ or $d(x,y)$ for short. For a permutation f of the vertex set $V(G)$ of G, let $\delta_f(x,y) = |d(x,y) - d(f(x), f(y))|$ and $\delta_f(x) = \sum_{y \in V(G)} \delta_f(x,y)$. The *displacement* $\delta_f(G)$ of G with respect to f is the sum of $\delta_f(x,y)$ over all the $\binom{n}{2}$ unordered pairs $\{x,y\}$ of distinct vertices of G. It is easy to see that $\sum_{x \in V(G)} \delta_f(x) = 2\delta_f(G)$. Clearly, a permutation f of $V(G)$ is an automorphism of G if and only if $\delta_f(G) = 0$. Let $\pi(G)$ denote the smallest positive value of $\delta_f(G)$ among the $n!$ permutations f of $V(G)$. A permutation f for which $\delta_f(G) = \pi(G)$ is called a *near automorphism* of G, and $\pi(G)$ is the value of the near automorphism. It is not difficult to see that both $\delta_f(G)$ and $\pi(G)$ are even and if G is not a complete graph, then $2 \le \pi(G) \le 2|V(G)| - 4$. Chartrand et al. (1999) conjectured that $\pi(P_n) = 2n - 4$. The conjecture was soon verified by Aitken (1999). In fact he also determined all near automorphisms of P_n. Recently, Chang et al. (submitted) proved that $\pi(C_n) = 4\lfloor \frac{n}{2} \rfloor - 4$ and determined all near automorphisms of C_n. As for the complete t-partite graphs, Reid (2002) obtained the following result.

Dedicated to Professor Frank K. Hwang on the occasion of his 65th birthday.

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Theorem 1 (Reid 2002) *For positive integers*, $n_1 \le n_2 \le \cdots \le n_t$, *where* $t \ge 2$ *and* $n_t \ge 2$,

$$\pi(K_{n_1,n_2,\ldots,n_t}) = \begin{cases} 2n_{h+1}-2 & \text{if } 1 = n_1 = \cdots = n_h < n_{h+1} \le \cdots \le n_t, \\ & \quad \text{and } t \ge (h+1), \text{ for some } h \ge 2, \\ 2n_{k_0} & \text{if } 1 = n_1 < n_2 \text{ or } n_1 \ge 2, \\ & \quad n_{k+1} = n_k + 1 \text{ for some } k, 1 \le k \le t-1, \\ & \quad \text{and } 2 + n_{k_0} \le n_1 + n_2, \\ 2(n_1+n_2-2) & \text{otherwise}, \end{cases}$$

where k_0 *is the smallest index for which* $n_{k_0+1} = n_{k_0} + 1$.

For the terms we use in this note, the readers may refer to the book by West (2001). First, we need several lemmas. Since they are easy to be checked, we omit their proofs.

Lemma 2 *If* f *is a permutation of* $V(G)$, *then* $\delta_f(G) = \delta_{f^{-1}}(G)$.

Lemma 3 *If* G *is a graph and* f *is a permutation of* $V(G)$ *which is not an automorphism, then there is an edge* (u, v) *of* G *such that* $\delta_f(u, v) \ge 1$.

Lemma 4 *If* u, v, w *are three vertices in a tree* T, *then* $d(u, v) \equiv d(w, u) + d(w, v) \pmod 2$.

Before we prove the main results, we also need a notion called displacement graph.

Definition 5 Suppose G is a graph and f is a permutation of $V(G)$. The displacement graph of G with respect to f is the directed multigraph $G[f]$ whose vertex set $V(G[f]) = \{a_1, a_2, \ldots, a_t\}$, where $t = \mathrm{diam}(G)$, and arc set $A(G[f]) = \{\langle a_i, a_j\rangle : i \ne j,$ there is a pair of vertices u and v such that $d(u, v) = i$ and $d(f(u), f(v)) = j\}$.

Note that if there are exactly s pairs of vertices u and v such that $d(u, v) = i$ and $d(f(u), f(v)) = j$, then $\langle a_i, a_j\rangle$ occurs in $G[f]$ exactly s times, i.e., $\langle a_i, a_j\rangle$ is of multiplicity s. For each unordered pair $\{u, v\}$ of distinct vertices of G, let $\alpha(u, v)$ denote $\langle a_i, a_j\rangle$, where $d(u, v) = i$ and $d(f(u), f(v)) = j$.

Now, we have a couple of results about the structure of $G[f]$.

Lemma 6 *For each vertex* $a_i \in V(G[f])$, $1 \le i \le \mathrm{diam}(G)$, $\deg^+(a_i) = \deg^-(a_i)$.

Proof This is a direct consequence of the fact that $|\{\{u, v\} : u, v \in V(G), d(u, v) = i\}| = |\{\{z, w\} : z, w \in V(G), d(f(z), f(w)) = i\}|$ where $1 \le i \le \mathrm{diam}(G)$. □

Lemma 7 $\delta_f(G) = \sum_{\langle a_i, a_j\rangle \in A(G[f])} |i - j|$.

Proof The lemma follows from Definition 5 easily. □

We are now ready to determine bipartite graphs G with $\pi(G)=2$ and trees T with $\pi(T)=4$.

Theorem 8 *Suppose G is a connected graph without 3-cycles and 5-cycles, and $|V(G)|\ge 3$. Then, $\pi(G)=2$ if and only if $G\simeq P_3$.*

Proof It is clear that $\pi(P_3)=2$. On the other hand, suppose $\pi(G)=2$ but $G\not\simeq P_3$. In this case, $|V(G)|\ge 4$. Choose a near automorphism f of G such that $\delta_f(G)=2$. By Lemma 3, there exists an edge $(u,v)\in E(G)$ such that $\delta_f(u,v)\ge 1$. Also, by Lemma 6 $\deg^+(a_1)=\deg^-(a_1)\ge 1$ and so in fact $A(G[f])=\{\langle a_1,a_2\rangle,\langle a_2,a_1\rangle\}$ as $\pi(G)=2$.

Let $\alpha(u,v)=\langle a_1,a_2\rangle$. Therefore, $d(f(u),f(v))=2$. Let w be the vertex in G such that $(f(u),f(w),f(v))$ is a path in G. Clearly, $d(f(u),f(w))=d(f(w),f(v))=1$. Thus, $d(u,w)\le 2$ and $d(w,v)\le 2$, and at most one of them equals to 2. Furthermore, w is not adjacent to both u and v, for otherwise there is a C_3 which is not possible. By symmetric we may assume that $d(w,v)=1$ and $d(u,w)=2$, and so (u,v,w) is a path in G.

Since $|V(G)|\ge 4$ and G is connected, there exists a vertex z adjacent to some vertex of $\{u,v,w\}$. If z is adjacent to u, then z is not adjacent to v and so $d(z,v)=d(f(z),f(w))=2$ implying that $d(z,w)=2$ and so there is a C_5, a contradiction. The case z is adjacent to w can be treated similarly. If z is adjacent to v, then $d(z,u)=d(z,w)=2$ imply that $d(f(z),f(u))=d(f(z),f(w))=2$ and so there is a C_5, again a contradiction. The theorem hence is true. □

If $f(u)=v$ and there exists an automorphism g such that $g(v)=u$, then we can replace f by $g\circ f$ and say that u is a fixed vertex. Hence, for stars, the only non-automorphism is a transposition of the center vertex and one leaf. Thus we have the value of $\pi(S_{n-1})$ where S_{n-1} is a star with $n-1$ edges.

Lemma 9 $\pi(S_{n-1})=2n-4$.

Theorem 10 *If T is a tree of order at least 4. Then $\pi(T)=4$ if and only if there exists a vertex x such that $T-x$ contains an isolated vertex and a component K_2, with a only exception that $\pi(S_3)=4$.*

Proof By Theorem 8, we have $\pi(T)\ge 4$ if $|V(T)|\ge 4$. It is easy to see that trees with order 4 are P_4 and S_3, and $\pi(P_4)=4$ and $\pi(S_3)=4$. Assume that u is an isolated vertex in $T-x$, and v belongs to a component of K_2 of $T-x$ and is adjacent to x. Let the transposition $f=(uv)$. Then it is easy to check that $\delta_f(T)=4$. Thus, $\pi(T)=4$.

Conversely, suppose that $\pi(T)=4$. Then $A(T[f])$ is equal to $\{\langle a_1,a_2\rangle,\langle a_2,a_1\rangle,\langle a_i,a_{i+1}\rangle,\langle a_{i+1},a_i\rangle : i\ge 1\}$, $\{\langle a_1,a_2\rangle,\langle a_2,a_3\rangle,\langle a_3,a_1\rangle\}$, $\{\langle a_1,a_3\rangle,\langle a_2,a_1\rangle,\langle a_3,a_2\rangle\}$ or $\{\langle a_1,a_3\rangle,\langle a_3,a_1\rangle\}$.

Case 1. $A(T[f])=\{\langle a_1,a_2\rangle,\langle a_2,a_1\rangle,\langle a_i,a_{i+1}\rangle,\langle a_{i+1},a_i\rangle : i\ge 1\}$. For $i=1$, we claim that $T\simeq S_3$. By the same argument as in Theorem 8, there are three vertices u,v,w of $V(T)$ such that (u,v,w) and $(f(u),f(w),f(v))$ are two paths in T, and $|V(T)|\ge 4$. Since $d(u,w)=2$, $d(f(u),f(w))=d(f(v),f(w))=1$ and $\langle a_l,a_k\rangle\notin A(T[f])$ for $l\ge 3$ or $k\ge 3$. Hence, none of the vertices of $V(T)\setminus\{u,v,w\}$

is adjacent to u or w in T. With the same argument, since $d(f(u), f(v)) = 2$, $d(u, v) = d(v, w) = 1$ and $\langle a_l, a_k\rangle \notin A(T[f])$ for $l \geq 3$ or $k \geq 3$, we get that for each vertex y of $V(T) \setminus \{u, v, w\}$, $f(y)$ must be adjacent to $f(w)$. It's easy to count that each vertex of $V(T) \setminus \{u, v, w\}$ contribute 2 to $\pi(T)$. Hence $|V(T)| = 4$, and $T \simeq S_3$.

Otherwise, for $i \geq 2$, we have $|V(T)| \geq 5$. Let $\alpha(x_0, x_i) = \langle a_i, a_{i+1}\rangle$ and the path from x_0 to x_i in T be $(x_0, x_1, \ldots, x_i)$ and y be a vertex of $V(T)$ such that $f(y)$ lies on the $f(x_0) \sim f(x_i)$ path. Without loss of generality, we let $d(x_0, y) \geq d(x_i, y)$. Since $d(x_0, x_i) \neq d(f(x_0), f(x_i))$, we have that $d(x_0, x_1) \neq d(f(x_0), f(x_1))$ or $d(x_1, x_i) \neq d(f(x_1), f(x_i))$, i.e. $\alpha(x_0, x_1)$ or $\alpha(x_1, x_i) \in A(T[f])$.

If $\alpha(x_0, x_1) \in A(T[f])$, then $\alpha(x_0, x_1) = \langle a_1, a_2\rangle$. Furthermore, if $i \geq 3$, then $\alpha(x_0, x_2) = \langle a_2, a_1\rangle$ or $\alpha(x_2, x_i) = \langle a_2, a_1\rangle$. But, when $\alpha(x_0, x_2) = \langle a_2, a_1\rangle$, $d(f(x_0), f(x_i)) \leq d(f(x_0), f(x_2)) + d(f(x_2), f(x_i)) = 1 + (i - 2) = i - 1$, a contradiction. On the other hand, if $\alpha(x_2, x_i) = \langle a_2, a_1\rangle$, $i = 4$, $d(f(x_0), f(x_4)) \leq d(f(x_0), f(x_2)) + d(f(x_2), f(x_4)) = 2 + 1 = 3$, also a contradiction. Hence the possible case left is $i = 2$. Then, $\alpha(x_0, x_1) = \langle a_1, a_2\rangle$ and $\delta_f(x_1, x_2) = 0$ imply that $d(f(x_0), f(x_2)) = d(f(x_0), f(x_1)) + d(f(x_1), f(x_2))$, $f(x_1)$ is on $f(x_0) \sim f(x_2)$ path and the path is $(f(x_0), f(y), f(x_1), f(x_2))$. Since $d(x_0, y) \geq d(x_2, y)$, $\alpha(x_0, y) \in A(T[f])$, in fact $\alpha(x_0, y) = \langle a_2, a_1\rangle$. The induced subgraph of $\{x_0, x_1, x_2, y\}$ in T is a star with center x_1. Since $|V(T)| \geq 5$, there exists another vertex z which is adjacent to one of $\{x_0, x_1, x_2, y\}$, and no matter which one is adjacent to z, $\delta_f(z) \geq 2$, a contradiction.

Now, suppose that $\alpha(x_1, x_i) \in A(T[f])$ and $\delta_f(x_0, x_1) = 0$. Then $\alpha(x_1, x_i)$ is equal to $\langle a_1, a_2\rangle$ or $\langle a_2, a_1\rangle$, $i = 2$ or $i = 3$. First, for $i = 3$, we have $d(f(x_0), f(x_3)) \leq d(f(x_0), f(x_1)) + d(f(x_1), f(x_3)) = 1 + 1 = 2$, a contradiction. Hence, $i = 2$ and the $f(x_0) \sim f(x_2)$ path is $(f(x_0), f(x_1), f(y), f(x_2))$. Since $d(f(y), f(x_1)) = d(f(y), f(x_2)) = 1$ and $d(x_1, x_2) = 1$, by Lemma 4, we have $\alpha(y, x_1) = \langle a_2, a_1\rangle$ or $\alpha(y, x_2) = \langle a_2, a_1\rangle$ in the tree T. If $\alpha(y, x_1) = \langle a_2, a_1\rangle$, then we have $\delta_f(y, x_2) = 0$ and the induced subgraph of $\{x_0, x_1, x_2, y\}$ is a path (x_0, x_1, x_2, y); if $\alpha(y, x_2) = \langle a_2, a_1\rangle$, then we have $\delta_f(y, x_1) = 0$ and the induced subgraph of $\{x_0, x_1, x_2, y\}$ is a star with the center x_1. Since $|V(T)| \geq 5$, there exists a vertex z' which is adjacent to one of the vertices in $\{x_0, x_1, x_2, y\}$. Clearly, no matter which one is adjacent to z', we also have $\delta_f(z') \geq 2$. Thus, this case is not possible.

Case 2. $A(T[f]) = \{\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \langle a_3, a_1\rangle\}$. Let $\alpha(u, v) = \langle a_1, a_2\rangle$ and $(f(u), f(w), f(v))$ be the path in T for some vertex w. Then $d(f(w), f(u)) = d(f(w), f(v)) = 1$ implies that one of the elements in $\{d(w, u), d(w, v)\}$ is 1 and the other one is 3. By Lemma 4, this case is impossible.

Case 3. $A(T[f]) = \{\langle a_1, a_3\rangle, \langle a_2, a_1\rangle, \langle a_3, a_2\rangle\}$. By Lemma 2, if f is a near automorphism, then f^{-1} is also a near automorphism. Moreover, $A(T[f^{-1}]) = \{\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \langle a_3, a_1\rangle\}$. Thus, by Case 2, Case 3 is not possible either.

Case 4. $A(T[f]) = \{\langle a_1, a_3\rangle, \langle a_3, a_1\rangle\}$. Let $\alpha(u, v) = \langle a_1, a_3\rangle$ and $\{x, y\}$ be a pair of vertices of T such that $(f(u), f(x), f(y), f(v))$ is a path in T. Then, $d(f(u), f(y)) = d(f(x), f(v)) = 2$ implies that $d(u, y) = d(x, v) = 2$, and $d(f(u), f(x)) = d(f(x), f(y)) = d(f(y), f(v)) = 1$ implies that one of the elements in $\{d(u, x), d(x, y), d(y, v)\}$ is 3 and the other two are 1 in tree T.

If $\alpha(x, y) = \langle a_3, a_1\rangle$, then in T the graph induced by the vertex set $\{u, v, x, y\}$ is the path (x, u, v, y). If $|V(T)| = 4$, then $T \simeq P_4$. If $|V(T)| \geq 5$, then there is a vertex w which is adjacent to one vertex in $\{u, v, x, y\}$ and keeps the condition that $\delta_f(w) = 0$. This is impossible, since $\delta_f(w) \geq \delta_f(w, x) + \delta_f(w, y) > 0$.

In addition, since $\alpha(u, x) = \langle a_3, a_1 \rangle$ and $\alpha(y, v) = \langle a_3, a_1 \rangle$ are similar cases, we consider the case $\alpha(u, x) = \langle a_3, a_1 \rangle$. Then the graph induced by $\{u, v, x, y\}$ is (u, v, y, x), obviously, it's an exchange of x and v. If $|V(T)| \geq 5$, then for each vertex w in $V(T) \setminus \{u, v, x, y\}$, $\delta_f(w) = 0$. Moreover, in order to maintain $\delta_f(w) = 0$, all the paths from $\{u, v, x, y\}$ to $V(T) \setminus \{u, v, x, y\}$ must pass through y. This implies that $T - y$ contains an isolated vertex x and K_2 induced by $\{u, v\}$. Furthermore, the near automorphism is the transposition (vx). This concludes the proof. □

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