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Positive Radial Solutions and Non-radial Bifurcation for Semilinear Elliptic Equations in Annular Domains

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We discuss the existence and multiplicity of positive radial solutions and the non-radial bifurcation of $\Delta u + \lambda f(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$, where Ω is an annular domain of $\mathbb{R}^n$, $n \geqslant 2$. We prove that if $f(u) > 0$ for $u \geqslant 0$ and $\lim_{u \to \infty} f(u)/u = \infty$, then there exists $\lambda^* > 0$ such that there are at least two positive radial solutions for each $\lambda \in (0, \lambda^*)$, at least one for $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$. If $f(0) = 0$, $\lim_{u \to 0} f(u)/u = 1$, and $uf'(u) > (1+\varepsilon) f(u)$ for $u > 0$, $\varepsilon > 0$, then there exists a variational solution for $\lambda \in (0, \lambda_1)$, where λ_1 is the least eigenvalue of $-\Delta$. If $f(0) = 0$, $\lim_{u \to 0} f(u)/u = 0$, and $\lim_{u \to \infty} f(u)/u = \infty$, then there exists at least one positive radial solution for any $\lambda > 0$. We obtain some precise multiplicity results for narrow annulus and show that the non-radial bifurcation occurs if the growth of $f(u)$ is rapid enough as $u \to \infty$.

1. INTRODUCTION

In this paper we consider the existence and multiplicity of positive radially symmetric solutions and non-radial bifurcations (symmetry breaking) of the equation

$$\Delta u + \lambda f(u) = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\Omega = \{x \in \mathbb{R}^n : A < |x| < B\}$ is an annular domain of $\mathbb{R}^n$, $n \geqslant 2$, $\lambda > 0$, $f \in C^2$, and $f(u) > 0$ for $u > 0$.

Equation (1.1), (1.2) arises from many branches of mathematics and applied mathematics. It was studied by many authors, for example, Gelfand [11], Keller and Cohen [13], Amann [1], Crandall and Rabinowitz [8], Sattinger [22], an Lions [17].

We shall study the problems according to $f(0) > 0$ or $f(0) = 0$.

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If Ω is a bounded, smooth domain in $\mathbb{R}^n$, $f(0)>0$, f is strictly increasing and strictly convex, then there exists $\lambda^*>0$ such that for any $\lambda \in (0, \lambda^*)$, (1.1), (1.2) has a minimum solution which can be obtained by a monotone iteration scheme (see, e.g., [13,22]). In [8], Crandall and Rabinowitz showed that if the growth of $f(u)$ as $u \to \infty$ is less than the Sobolev critical exponent then (1.1), (1.2) has at least two positive solutions. Recently, Suzuki and Nagasaki [25] obtained a similar result for positive radial solutions of (1.1), (1.2) without assuming the growth conditions of $f(u)$ when Ω is an annulus. In this paper, we shall prove that the similar result holds if $f(u)$ is superlinear at $u=\infty$, i.e., $\lim_{u \to \infty} (f(u)/u)=\infty$, without assuming $f(u)$ is convex and increasing.

If Ω is an annulus and $f(0)=0$, $f(u)>0$ for $u>0$, and $f(u)/u^{1+\varepsilon}$ is strictly increasing in $u>0$ for some $\varepsilon>0$, Nehari [18] proved that there is a variational solution for (1.1), (1.2). In fact, he considered the equations

$$y''+yF(y^2, x)=0, \tag{1.3}$$

$$y(a)=0=y(b), \tag{1.4}$$

where $F(t, x)$ is continuous and positive for $t>0$ and $x>0$, and $F(t, x)/t^{\varepsilon}$ is strictly increasing in $t>0$ for some positive number ε, the functional

$$J(y)=\int_a^b [y'^2-G(y^2, x)]\, dx, \tag{1.5}$$

where

$$G(\eta, x)=\int_0^{\eta} F(t, x)\, dt,$$

and the set

$$M=\{y \text{ is absolutely continuous on } [a, b] \text{ such that } y(a)=0=y(b), y\neq 0, \text{ and } I(y)=0\},$$

where

$$I(y)=\int_a^b \{y'^2-y^2F(y^2, x)\}\, dx. \tag{1.6}$$

He showed that the minimizer of J over M is achieved which is positive and also satisfies (1.3), (1.4).

We shall prove that if f satisfies the following conditions

(H-1) $f(0)=0$, $f(u)>0$ for $u>0$, $\lim_{u \to 0^+} (f(u)/u)=1$,

(H-2) $uf'(u)>f(u)$ for $u>0$,

(H-3) $uf(u)\geqslant 2(1+\varepsilon)\int_0^u f(t)\, dt$ for u large and positive number ε,

then (1.1), (1.2) has a variational solution for any $\lambda \in (0, \lambda_1)$, where λ_1 is the least eigenvalue of Lapacian $-\Delta$ with Dirichlet boundary conditions on annulus Ω.

The existence of positive radial solutions of (1.1), (1.2) under the assumption $\lim_{u\to 0^+}(f(u)/u)=0$ has been studied by Bandle, Coffman, and Marcus [2], Garaizar [10], and Lin [15]. In [15], it was proved that (1.1), (1.2) has a positive radial solution on any annulus provided that f satisfies the following conditions:

(H-1)′ $\lim_{u\to 0^+}(f(u)/u)=0$,

(H-2)′ $\lim_{u\to\infty}(f(u)/u)=\infty$.

Since the set of positive radially symmetric solutions of (1.1), (1.2) can be very complicated, it is difficult to study the non-radial bifurcation problem in such a situation. Therefore, we shall study the problem on the narrow annulus. In fact, if the aspect ratio $B/A\leqslant(n-1)^{1/(n-2)}$ for $n\geqslant 3$ and $B/A\leqslant e$ for $n=2$, $uf'(u)>f(u)$ for $u>0$, Ni and Nussbaum [20] proved that (1.1), (1.2) has at most one positive radially symmetric solution. For such a domain, if $f(0)>0$ and f is strictly increasing and convex then we can show that there exists an unique non-minimum positive radial solution for any $\lambda\in(0,\lambda^*)$.

For such an annulus, we shall prove that the non-radial bifurcation occurs if the growth of $f(u)$ is rapid enough as $u\to+\infty$.

The problems of non-radial bifurcation from radial solutions on balls were studied by Dancer [9] and Smoller and Wasserman [23, 24], on an annulus by Suzuki and Nagasaki [26] and Lin [14], and on sectorial domains by Berestycki and Pacella [4] and Lin [16].

For simplicity, in this paper we only consider the problem of the form (1.1). With a slight modification of the arguments, we can also obtain similar results for an equation of the form

$$\Delta u+\lambda f(r,u)=0 \qquad \text{in} \quad \Omega, \tag{1.1$'$}$$

when $f(r,u)$ satisfies some appropriate conditions.

This paper is organized as follows: In Section 2, we study the existence of the second (non-minimum) positive radial solution for (1.1), (1.2) when $f(0)>0$. In Section 3, we prove the existence of a variational solution for (1.1), (1.2) when $f(0)=0$ and $\lim_{u\to 0^+}(f(u)/u)=1$. In Section 4, we study the non-radial bifurcation problems on a narrow annulus.

2. The Second Solution

In this section we shall study the existence of the second positive radial solution of (1.1), (1.2). Since we are interested in the radial solutions, we write (1.1), (1.2) as

$$u''(r)+\frac{n-1}{r}u'(r)+\lambda f(u(r))=0, \qquad r\in(A,B), \tag{2.1}$$

$$u(A)=0=u(B), \tag{2.2}$$

where $\lambda>0$ and $n\geqslant 2$.

We assume that f satisfies the following conditions:

(A-1) $f\in C^2(\mathbb{R}^1)$ and $f(u)>0$ for $u\geqslant 0$,

(A-2) $\lim_{u\to\infty}(f(u)/u)=\infty$.

For $n\geqslant 3$, set

$$s=r^{2-n} \qquad \text{and} \qquad u(s)=u(r),$$

then (2.1), (2.2) can be rewritten as

$$u''(s)+\lambda\rho(s)f(u(s))=0 \qquad \text{in} \quad (s_0,s_1) \tag{2.3}$$

$$u(s_0)=0=u(s_1), \tag{2.4}$$

where $\rho(s)=(n-2)^{-2}s^{-k}$, $k=2+2/(n-2)$, $s_0=B^{2-n}$, and $s_1=A^{2-n}$.

For $n=2$, set

$$s=-\log r \qquad \text{and} \qquad u(s)=u(r),$$

then equations (2.1), (2.2) can also be rewritten as (2.3), (2.4) with $\rho(s)=e^{-2s}$, $s_0=-\log B$, and $s_1=-\log A$.

Using the backward shooting, we consider the family of solutions of the initial value problem

$$u''(s)+\lambda\rho(s)f(u(s))=0 \qquad \text{for} \quad s<s_1 \tag{2.5}$$

$$u(s_1)=0, \qquad u'(s_1)=-b, \tag{2.6}$$

where $b>0$ is the shooting parameter.

For every $b>0$, problem (2.5), (2.6) has an unique solution $u(\cdot)\equiv u(\cdot,b,\lambda)$ with the maximal domain of existence $(\bar{s}(b,\lambda),s_1)$. It is easy to check that (2.5), (2.6) is equivalent to the following integral equation

$$u(s)=b(s_1-s)-\lambda\int_s^{s_1}(t-s)\,\rho(t)f(u(t))\,dt, \qquad \text{for} \quad s<s_1, \tag{2.7}$$

and solution u also satisfies

$$u(s)=u(\bar{s})+u'(\bar{s})(s-\bar{s})+\lambda\int_{\bar{s}}^{s}(t-s)\,\rho(t)f(u(t))\,dt \tag{2.8}$$

for $s, \bar{s} \in (\tilde{s}(b, \lambda), s_1)$. From (2.7), if u is positive in some interval (α, s_1), then

$$u(s) \leqslant b(s_1 - s) \qquad \text{in} \quad (\alpha, s_1). \tag{2.9}$$

If u has a zero in $(\tilde{s}(b, \lambda), s_1)$, denote

$$s_0(b, \lambda) = \inf\{s_0 : u(s, b, \lambda) > 0 \text{ in } (s_0, s_1)\}$$

and $\tau(b, \lambda) \in (s_0(b, \lambda), s_1)$ such that $u'(\tau(b, \lambda), b, \lambda) = 0$.

By standard results in o.d.e., the functions $u(s, b, \lambda)$ and $u'(s, b, \lambda)$ are continuously differentiable in (s, b, λ). Since $u''(s, b, \lambda) < 0$ in $(s_0(b, \lambda), s_1)$ and $u'(s_0(b, \lambda), b, \lambda) > 0$, by the implicit function theorem, $s_0(b, \lambda)$ and $\tau(b, \lambda)$ are also C^1 in (b, λ).

In this section, we only discuss the case $n \geqslant 3$; the case $n = 2$ can also be treated analogously. We first prove the following lemma.

LEMMA 2.1. *Assume conditions* (A-1) *and* (A-2) *are satisfied. Then for any* $b > 0$ *and* $\lambda > 0$, $s_0(b, \lambda) > 0$. *Furthermore, we have*

(i) $\lim_{b \to \infty} s_0(b, \lambda) = \lim_{b \to \infty} \tau(b, \lambda) = s_1$.

(ii) $\lim_{b \to 0^+} s_0(b, \lambda) = \lim_{b \to 0^+} \tau(b, \lambda) = s_1$.

Proof. Set $c_0 = \lim\{f(u) : u \geqslant 0\} > 0$. If $u(s) > 0$ in (α, s_1), then by (2.7) we have

$$\lambda c_0 \int_\alpha^{s_1} (t - \alpha)\, \rho(t)\, dt \leqslant b(s_1 - \alpha).$$

Since

$$(n-2)^2 \int_\alpha^{s_1} (t - \alpha)\, \rho(t)\, dt = \frac{1}{(k-1)(k-2)} \alpha^{2-k} + \frac{s_1^{1-k}}{k-1} \alpha - \frac{s_1^{2-k}}{k-2},$$

we have

$$\lambda c_0 \{c_1 \alpha^{2-k} - c_2\} \leqslant b s_1$$

for some positive constants $c_1 = c_1(n)$ and $c_2 = c_2(n, s_1)$. This implies that $s_0(b, \lambda) > 0$ for any $b > 0$ and $\lambda > 0$.

(i) This can be proved by an argument similar to proving Lemmas 2.1 and 2.2 of [15]; the details are omitted.

(ii) It suffices to show that $\lim_{b \to 0^+} s_0(b, \lambda) = s_1$. Suppose this were false then there would be a $\lambda > 0$, a positive number ε, and a sequence

$b_k \to 0$ such that $s_1 - 2\varepsilon \leqslant s'_k \equiv s_0(b_k, \lambda) \leqslant s_1 - \varepsilon$. By (2.7) and (2.9), for k sufficiently large, we have

$$2\varepsilon b_k \geqslant b_k(s_1 - s'_k) \geqslant \frac{1}{2} \lambda f(0) \int_{s'_k}^{s_1} (t - s'_k)\, \rho(t)\, dt \geqslant c(\varepsilon, \lambda) > 0,$$

a contradiction. This completes the proof.

Set

$$s_0^* = s_0^*(\lambda) = \min\{s_0(b, \lambda) : b > 0\}.$$

An immediate consequence of Lemma 2.1 is the following result.

COROLLARY 2.2. *If* $s_0 \in (s_0^*(\lambda), s_1)$, *then* (2.3), (2.4) *has at least two positive solutions.*

The following lemma plays the key role in this section.

LEMMA 2.3. $s_0^*(\lambda)$ *is continuous and strictly increasing in* $\lambda > 0$.

Proof. We first prove that $s_0^*(\lambda)$ is strictly increasing in $\lambda > 0$. In fact, if $0 < \lambda_1 < \lambda_2$ and u_2 is a solution of (2.3), (2.4) at $\lambda = \lambda_2$ on $(s_0^*(\lambda_2), s_1)$ and set

$$v(x) = cu_2(t) \qquad \text{and} \qquad t = \frac{x}{c} + s_1\left(1 - \frac{1}{c}\right),$$

where $c > 1$ and close to 1, then it is easy to verify that $v(x) > 0$ in $(s_0^*(\lambda_2) - \varepsilon, s_1)$ and $v(s_0^*(\lambda_2) - \varepsilon) = 0 = v(s_1)$, where

$$\varepsilon - (c-1)(s_1 - s_0^*(\lambda_2)) > 0,$$

since

$$\begin{aligned} &v''(x) + \lambda_1 \rho(x) f(v(x)) \\ &\quad = \frac{1}{c} u_2''(t) + \lambda_1 \rho(x) f(cu_2(t)) \\ &\quad = -\frac{1}{c} \{\lambda_2 \rho(t) f(u_2(t)) - c\lambda_1 \rho(x) f(cu_2(t))\}. \end{aligned}$$

If c is sufficiently close to 1, then

$$\lambda_2 \rho(t) f(u_2(t)) \geqslant c\lambda_1 \rho(x) f(cu_2(t))$$

for all $t \in (s_0^*(\lambda_2, s_1))$. Therefore, v is a supersolution of (2.3), (2.4) on the interval $(s_0^*(\lambda_2) - \varepsilon, s_1)$, which implies $s_0^*(\lambda_1) \leqslant s_0^*(\lambda_2) - \varepsilon$. Hence $s_0^*(\lambda)$ is strictly increasing in $\lambda > 0$.

Since $s_0(b, \lambda)$ is continuous in λ and $s_0^*(\lambda)$ is increasing, $\lim_{\lambda \to \lambda_0} s_0(b^*(\lambda_0), \lambda) = s_0(b^*(\lambda_0), \lambda_0) = s_0^*(\lambda_0)$ implies that $\lim_{\lambda \to \lambda_0^+} s_0^*(\lambda) = s_0^*(\lambda_0)$ immediately. On the other hand, if λ close to $\lambda_0 > 0$, then it is easy to check that $b^*(\lambda)$ is bounded and bounded away from 0 by an argument similar to proving Lemma 2.1(i), (ii). Choosing a sequence $\lambda_k \to \lambda_0$ and $\lambda_k < \lambda_0$ such that $b^*(\lambda_k) \to b^* > 0$ as $k \to \infty$, we have $\lim_{k \to \infty} s_0^*(\lambda_k) = \lim_{k \to \infty} s_0(b^*(\lambda_k), \lambda_k) = s_0(b^*, \lambda_0) \geqslant s_0^*(\lambda_0)$. Hence $\lim_{\lambda \to \lambda_0^-} s_0^*(\lambda) = s_0^*(\lambda_0)$. This proves that $s_0^*(\lambda)$ is continuous in λ. The proof is complete.

LEMMA 2.4. *Assume conditions* (A-1) *and* (A-2) *are satisfied. Then, we have*

(i) $\lim_{\lambda \to \infty} s_0^*(\lambda) = s_1$,

(ii) $\lim_{\lambda \to 0^+} s_0^*(\lambda) = 0$.

Proof. (i) Let $s_0 = s_0(b, \lambda)$, by (2.7) we have

$$b(s_1 - s_0) = \lambda \int_{s_0}^{s_1} (t - s_0)\, \rho(t) f(u(t))\, dt \geqslant \frac{1}{2} \lambda c_0 \rho(s_1)(s_1 - s_0)^2,$$

where $c_0 = \min\{f(u) : u \geqslant 0\} > 0$. This implies

$$b^*(\lambda) \geqslant \frac{\lambda}{2} c_0 \rho(s_1)(s_1 - s_0^*(\lambda)). \tag{2.10}$$

If the result were false, then there would be a number $\bar{s}_0 < s_1$ and a sequence $\lambda_k \to \infty$ such that $\lim_{k \to \infty} s_0^*(\lambda_k) = \bar{s}_0$. By (2.10), we have

$$b_k \equiv b^*(\lambda_k) \geqslant c\lambda_k, \tag{2.11}$$

for large k, where $c = \frac{1}{4} c_0 \rho(s_1)(s_1 - \bar{s}_0)$. We shall prove that

$$u_k(\tau_k) \to \infty \qquad \text{as} \quad k \to \infty, \tag{2.12}$$

where $u_k(\cdot) = u(\cdot, b_k, \lambda_k)$ and $\tau_k = \tau(b_k, \lambda_k)$.

Set

$$F(u) = \lambda \int_0^u f(s)\, ds,$$

and define

$$V(s) \equiv V(s, b, \lambda) \equiv \frac{1}{2} u'^2(s, b, \lambda) + \rho(s)\, F(u(s, b, \lambda)).$$

Since $V'(s)=\rho'(s)\,F(u(s))$, we have

$$V(s_1)=V(\tau)+\int_\tau^{s_1}\rho'(t)\,F(u(t))\,dt,$$

where $\tau=\tau(b,\lambda)$. Therefore, we have

$$\frac{1}{2}b_k^2=\rho(\tau_k)\,F(u_k(\tau_k))+\int_{\tau_k}^{s_1}\rho'(t)\,F(u_k(t))\,dt.$$

By (2.11), (2.12) holds.

By the result of Gidas, Ni, and Nirenberg [12], $\tau(b,\lambda)\leqslant\frac{1}{2}(s_0(b,\lambda)+s_1)$. Hence, we have

$$\tau_k\leqslant\frac{1}{3}\bar{s}_0+\frac{2}{3}s_1\equiv\bar{s}_1$$

for k large.

Since u is concave, the straight line l_k connecting $(s_1,0)$ and $(\tau_k,u_k(\tau_k))$ lies below the graph of u_k. Therefore, (2.12) implies that the slope m_k of l_k will tend to $-\infty$ as $k\to\infty$. Hence, $u_k(s)\to\infty$ uniformly on $[\bar{s}_1,\bar{s}_2]$ as $k\to\infty$, where $\bar{s}_2=\frac{1}{4}\bar{s}_0+\frac{3}{4}s_1$. Since u_k satisfies

$$u_{k''}+\lambda_k\rho(s)\frac{f(u_k)}{u_k}u_k=0 \qquad \text{in} \quad [\bar{s}_1,\bar{s}_2],$$

by (A-2) and the Sturm comparison principle, u_k has zeros in $[\bar{s}_1,\bar{s}_2]$ for large k, a contradiction. This proves (i).

(ii) If the result were false, then $\lim_{\lambda\to 0^+}s_0^*(\lambda)=\bar{s}_0>0$. Therefore, (2.3), (2.4) has no positive solution in $(\bar{s}_0,s_1)$ for any $\lambda>0$, which contradicts the fact that (2.3), (2.4) has a minimum solution for $\lambda>0$ and sufficiently small. The proof is complete.

Now, we can prove our main result.

THEOREM 2.5. *Assume conditions* (A-1) *and* (A-2) *are satisfied. Then there exists* $\lambda^*=\lambda^*(A,B)>0$ *such that* (2.1), (2.2) *has at least two positive solutions for all* $\lambda\in(0,\lambda^*)$ *and at least one for* $\lambda=\lambda^*$ *and none for* $\lambda>\lambda^*$.

Proof. By Lemmas 2.3 and 2.4, there exists a unique $\lambda^*>0$ such that $s_0^*(\lambda^*)=s_0$. By Corollary 2.2, (2.1), (2.2) has at least two positive solutions for any $\lambda\in(0,\lambda^*)$. It is clear that there exist at least one positive solution for $\lambda=\lambda^*$ and none for $\lambda>\lambda^*$. The proof is complete.

We give some properties concerning the solution set of (2.1), (2.2) as follows.

THEOREM 2.6. *Assume conditions* (A-1) *and* (A-2) *are satisfied. Let* $\lambda^* > 0$ *be given as in Theorem* 2.5. *Then there exists a continuous function* $M(\lambda)\colon (0, \lambda^*) \to \mathbb{R}^+$, *such that if* $u(\lambda)$ *is a solution of* (2.1), (2.2), *then* $\|u(\lambda)\|_\infty \leqslant M(\lambda)$, *where* $\|u\|_\infty = \sup\{|u(x)| : x \in \bar{\Omega}\}$. *Moreover, if* $\tilde{u}(\lambda)$ *is a non-minimum solution, then* $\|\tilde{u}(\lambda)\|_\infty \to \infty$ *as* $\lambda \to 0^+$.

Proof. The existence of $M(\lambda)$ follows from Lemma 2.1(i).

To prove the last part of the theorem, we shall show that if solution $u(\lambda)$ is not large enough as $\lambda \to 0^+$, then $u(\lambda)$ is the minimum solution $\underline{u}(\lambda)$. It is easy to verify that $\|\underline{u}(\lambda)\|_\infty \to 0$ as $\lambda \to 0^+$. Let $w(s, \lambda) = u(s, \lambda) - \underline{u}(s, \lambda)$. Then w satisfies

$$w'' + \lambda \rho g w = 0 \quad \text{in} \quad (s_0, s_1),$$
$$w(s_0) = 0 = w(s_1),$$

where

$$g(s, \lambda) = \begin{cases} \dfrac{f(u(s, \lambda)) - f(\underline{u}(s, \lambda))}{u(s, \lambda) - \underline{u}(s, \lambda)} & \text{if} \quad u(s, \lambda) > \underline{u}(s, \lambda), \\ f'(\underline{u}(s, \lambda)) & \text{if} \quad u(s, \lambda) = \underline{u}(s, \lambda), \end{cases}$$

which is continuous in $[s_0, s_1]$ for any $\lambda \in (0, \lambda^*)$. Set $G(\lambda) = \|g(\cdot, \lambda)\|_\infty$. Let $v_1 > 0$ and $\varphi_1 > 0$ in (s_0, s_1) be the least eigenvalue and an associated eigenfunction of

$$\varphi'' + v \rho \varphi = 0 \quad \text{in} \quad (s_0, s_1),$$
$$\varphi(s_0) = 0 = \varphi(s_1).$$

If $\lambda G(\lambda) < v_1$, then $w \equiv 0$ in (s_0, s_1). Otherwise, by the Sturm comparison principle, φ_1 has a zero in (s_0, s_1), a contradiction. Hence, if $u(\lambda) \neq \underline{u}(\lambda)$, then $G(\lambda) \geqslant v_1 \lambda^{-1}$. The proof is complete.

In the remaining part of the section, in addition to conditions (A-1) and (A-2), we also assume that f is strictly increasing and convex; i.e., f satisfies

(A-3) $f'(u) > 0$ and $f''(u) > 0$ for all $u \geqslant 0$.

Instead of using backward shooting, as in [20], we shall use forward shooting to study (2.3), (2.4); i.e., consider the family of solutions of the initial value problem

$$u''(s) + \lambda \rho(s) f(u(s)) = 0 \quad \text{for} \quad s > s_0, \tag{2.13}$$
$$u(s_0) = 0, \qquad u'(s_0) = d > 0. \tag{2.14}$$

If u has a zero in (s_0, ∞), let $s_1(d, \lambda)$ be the first zero of it. If u is positive in (s_0, ∞), let $s_1(d, \lambda) = \infty$. Then, it is easy to see that $s_1(d, \lambda)$ is C^1 in (d, λ) in the set $\{(d, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ : s_1(d, \lambda) < \infty\}$. Set $\varphi(s, d, \lambda) = (\partial u/\partial d)(s, d, \lambda)$, then φ satisfies

$$\varphi'' + \lambda \rho f'(u) \varphi = 0 \quad \text{in} \quad (s_0, s_1(d, \lambda)), \tag{2.15}$$

$$\varphi(s_0) = 0 \quad \text{and} \quad \varphi'(s_0) = 1. \tag{2.16}$$

DEFINITION 2.7. $u(\cdot, d, \lambda)$ is called minimum if $\varphi(\cdot, d, \lambda)$ is positive in $(s_0, s_1(d, \lambda))$ and is called non-minimum if $\varphi(\cdot, d, \lambda)$ changes signs in $(s_0, s_1(d, \lambda))$.

LEMMA 2.8. *Assume that* $s_1(d, \lambda) < \infty$. *Let* $\mu_1 = \mu_1(d, \lambda)$ *be the least eigenvalue of the linearized eigenvalue problem*

$$w'' + \lambda \rho f'(u) w = -\mu \rho w \quad \textit{in} \quad (s_0, s_1(d, \lambda)), \tag{2.17}$$

$$w(s_0) = 0 = w(s_1(d, \lambda)), \tag{2.18}$$

of (2.3), (2.4) *at* $u(\cdot, d, \lambda)$. *Then* $u(\cdot, d, \lambda)$ *is minimum if and only if* $\mu_1(d, \lambda) \geq 0$.

Proof. Let $w_1 > 0$ in $(s_0, (\cdot, d, \lambda))$ be an associated eigenfunction; i.e., w_1 satisfies

$$w_1'' + (\lambda f'(u) + \mu_1) \rho w_1 = 0 \quad \text{in} \quad (s_0, s_1(d, \lambda)),$$

$$w_1(s_0) = 0 = w_1(s_1(d, \lambda)).$$

If $\mu_1 < 0$, then by the Sturm comparison principle, φ has a zero in $(s_0, s_1(d, \lambda))$; i.e., $u(\cdot, d, \lambda)$ is non-minimum. If $\mu_1 = 0$, then $\varphi = cw_1 > 0$ in $(s_0, s_1(d, \lambda))$; i.e., $u(\cdot, d, \lambda)$ is minimum. If $\mu_1 > 0$, then φ is positive in $(s_0, s_1(d, \lambda)]$. Otherwise, by the Sturm comparison principle again, w_1 has a zero in $(s_0, s_1(d, \lambda))$, a contradiction. The proof is complete.

We need the following result which is a special case of Lemma 2.17 of Crandall and Rabinowitz [8].

PROPOSITION 2.9. *Assume conditions* (A-1)–(A-3) *are satisfied. If* $s_1(d_2, \lambda) = s_1(d_1, \lambda) = s_1$, *then if* $\mu_1(d_1, \lambda) > 0$ *we have* $u(\cdot, d_2, \lambda) \geq u(\cdot, d_1, \lambda)$ *in* (s_0, s_1), *while if* $\mu_1(d_1, \lambda) = 0$, *then* $d_2 = d_1$.

Note that, by Lemma 2.8 and Proposition 2.9, $u(\cdot, d, \lambda)$ is minimum if and only if $u(\cdot, d, \lambda)$ is the minimum solution of (2.3), (2.4) on $(s_0, s_1(d, \lambda))$.

LEMMA 2.10. *Assume conditions* (A-1)–(A-3) *are satisfied. Then there exists* $d_0(\lambda)>0$ *such that* $u(\cdot, d, \lambda)$ *is minimum if* $d \in (0, d_0(\lambda))$ *and non-minimum if* $d \in (d_0(\lambda), \infty)$ *with* $s_1(d, \lambda)<\infty$.

Proof. By an argument similar to proving Lemma 2.1, we can prove

$$\lim_{d \to \infty} s_1(d, \lambda) = s_0, \tag{2.19}$$

and

$$\lim_{d \to 0^+} s_1(d, \lambda) = s_0. \tag{2.20}$$

Using (2.20), (A-3), and the Sturm comparison principle, we can also prove that $u(\cdot, d, \lambda)$ is minimum if d is sufficiently small; i.e., the set

$$D = \{d>0 : \varphi(\cdot, d, \lambda)>0 \text{ in } (s_0, s_1(d, \lambda)] \subset (s_0, \infty)\} \supset (0, \tilde{d})$$

for some $\tilde{d}>0$. It is easy to see that D is an open set and $\mu_1(d, \lambda)>0$ for all $d \in D$. For $s_1(d, \lambda)<\infty$, we have

$$\frac{\partial s_1}{\partial d}(d, \lambda) = -\varphi(s_1(d, \lambda), d, \lambda)/u'(s_1(d, \lambda), d, \lambda). \tag{2.21}$$

By (2.19) and (2.20), there are $d \in D$ and $\bar{d}>0$ such that $s_1(\bar{d}, \lambda) = s_1(d, \lambda)$. Since $\mu_1(d, \lambda)>0$ by Lemma 2.8 and $\mu_1(\bar{d}, \lambda)<0$ by Proposition 2.9, we have $D \neq (0, \infty)$.

We shall prove that

$$D = (0, d_0(\lambda)), \tag{2.22}$$

where $d_0(\lambda) = \sup\{d \in D\} < \infty$.

Set $s_1^*(\lambda) = \sup\{s_1(d, \lambda) : d>0\}$, then there are two cases to be discussed according to $s_1^*(\lambda)<\infty$ or $s_1^*(\lambda)=\infty$.

Case 1. $s_1^*(\lambda)<\infty$.

Let $d_1>0$ such that $(0, d_1) \subset D$ and $d_1 \notin D$. Then $s_1^*(\lambda)<\infty$ implies $s_1(d_1, \lambda)<\infty$. It is clear that $\varphi(\cdot, d_1, \lambda)>0$ in $(s_0, s_1(d_1, \lambda))$ and $\varphi(s_1(d_1, \lambda), d_1, \lambda)=0$. By Lemma 2.8, $\mu_1(d_1, \lambda)=0$. Since $(\partial s_1/\partial d)(d, \lambda)>0$ for all $d \in (0, d_1)$, $s_1(d_1, \lambda)>s_1(d, \lambda)$ for all $d \in (0, d_1)$. If there were $d_2>d_1$ such that $d_2 \in D$, then by Proposition 2.9 we must have $s_1(d_2, \lambda)>s_1(d_1, \lambda)$. Now $u(\cdot, d_2, \lambda)$ is a supersolution and $u(\cdot, d_1, \lambda)$ is the minimum solution of (2.3), (2.4) on $[s_0, s_1(d, \lambda)]$, and we have $u(s, d_2, \lambda)>u(s, d_1, \lambda)$ on $(s_0, s_1(d, \lambda)]$. By (A-3), we have $f'(u(s, d_2, \lambda))>f'(u(s, d_1, \lambda))$ on $(s_0, s_1(d, \lambda))$. By the Sturm comparison principle, $\varphi((\cdot, d_2, \lambda)$ has a zero in $(s_0, s_1(d, \lambda))$, a contradiction to $d_2 \in D$. This proves that $D=(0, d_1)$.

Case 2. $s_1^*(\lambda)=\infty$.

Let $d_1>0$ such that $(0,d_1)\subset D$ and $d_1\notin D$. Then we claim that $s_1(d_1,\lambda)=\infty$. Otherwise, if $s_1(d_1,\lambda)<\infty$, then the argument in Case 1 implies that $D=(0,d_1)$. Since for any $s_1>s_1(d_1,\lambda)$, (2.3), (2.4) has the minimum solution u_1 on $[s_0,s_1]$ with $u_1=u(\cdot,d_2,\lambda)$ and $s_1(d_2,\lambda)=s_1$ for some $d_2>d_1$. By the result of Sattinger [22] we have $\mu_1(d_2,\lambda)\geqslant 0$, which implies $\varphi(\cdot,d_2,\lambda)>0$ in $(s_0,s_1)\subset(s_0,s_1(d_1,\lambda)]$, i.e., $d_2\in D$, a contradiction. This proves $s_1(d_1,\lambda)=\infty$. Hence, for all $d>d_1$ with $s_1(d,\lambda)<\infty$, by Proposition 2.9, we have $d\notin D$. This proves $D=(0,d_1)$. The proof is complete.

Now, by modifying an argument of Ni and Nussbaum [20], we can prove that if the annuli are narrow enough then there are exactly two positive solutions for (2.1), (2.2) for all $\lambda\in(0,\lambda^*)$.

THEOREM 2.11. *Assume conditions* (A-1)–(A-3) *are satisfied. If* $B/A\leqslant(n-1)^{1/(n-2)}$ *for* $n\geqslant 3$ *and* $B/A\leqslant e$ *for* $n=2$, *then* (2.1), (2.2) *has exactly two positive solutions for any* $\lambda\in(0,\lambda^*)$, *exactly one at* $\lambda=\lambda^*$, *and none for* $\lambda>\lambda^*$.

Proof. Set $g(s,u)=\lambda\rho(s)f(u)$, $X(s)=(s-s_0)\,u'(s,d,\lambda)$, and $Y(s)=u'(s,d,\lambda)$ as in [20]. Then

$$\frac{d}{ds}(Y\varphi'-Y'\varphi)=g_s\varphi, \tag{2.23}$$

$$\frac{d}{ds}(X'\varphi-\varphi'X)=-\varphi[(s-s_0)\,g_s+2g]. \tag{2.24}$$

We first prove that if $d\in(d(\lambda),\infty)$ with $s_1(d,\lambda)<\infty$ and $(s-s_0)\,g_s+2g\geqslant 0$ on $(s_0,s_1(d,\lambda))$ then

$$\frac{\partial s_1}{\partial d}(d,\lambda)<0. \tag{2.25}$$

By Lemma 2.10, $\varphi(\cdot,d,\lambda)$ has a zero in $(s_0,s_1(d,\lambda))$. Let $\xi(d,\lambda)$ be the zero of φ such that $\varphi>0$ in $(s_0,\xi(d,\lambda))$. Since $(s-s_0)\,g_s+2g\geqslant 0$ on $(s_0,s_1(d,\lambda))$, (2.24) implies that $\xi(d,\lambda)>\tau(d,\lambda)$ where $u'(\tau(d,\lambda),d,\lambda)=0$. Since $\rho'(s)<0$, $\partial g/\partial s<0$. Therefore (2.23) implies $\varphi(s_1(d,\lambda),d,\lambda)<0$. By (2.21), $(\partial s_1/\partial d)(d,\lambda)<0$.

It is clear that condition $(s-s_0)\,g_s+2g\geqslant 0$ is equivalent to

$$(s-s_0)\,\rho'(s)+2\rho(s)\geqslant 0, \tag{2.26}$$

which only depends on s. It is easy to verify that (2.26) holds if $B/A \leqslant (n-1)^{1/(n-2)}$ for $n \geqslant 3$ and $B/A \leqslant e$ if $n=2$.

Now, for $\lambda \in (0, \lambda^*)$, by Theorem 2.5 there are $d_1 \in D$ and $d_2 \notin D$ such that $s_1(d_1, \lambda) = s_1(d_2, \lambda) = s_1$. By (2.25) and (2.26), we have $s_1(d, \lambda) < s_1$ for all $d \in (d_2, \infty)$. This proves that there exists a unique non-minimum solution for (2.3), (2.4) on $[s_0, s_1]$. The proof is complete.

COROLLARY 2.12. *Let the conditions of Theorem* 2.11 *be satisfied. Then the family of non-minimum solutions* $u(\lambda)$ *is smooth in* $\lambda \in (0, \lambda^*)$ *with* $\mu_2(\lambda) > 0$ *for all* $\lambda \in (0, \lambda^*)$, *where* $\mu_2(\lambda)$ *is the second eigenvalue of* (2.17), (2.18) *with* $u = u(\lambda)$.

Proof. Set $H(d, \lambda) = s_1(d, \lambda) - s_1$, where $d \in (d(\lambda), \infty)$ with $s_1(d, \lambda) < \infty$, and $\lambda \in (0, \lambda^*)$. If $H(d, \lambda) = 0$, then $(\partial H/\partial d)(d, \lambda) = (\partial s_1/\partial d)(d, \lambda) < 0$ by (2.25). Hence, by the implicit function theorem $d = d(\lambda)$ is smooth in λ with $H(d(\lambda), \lambda) = 0$. Therefore, the family of non-minimum solutions $u(\lambda)$ is smooth in λ.

To prove $\mu_2(\lambda) > 0$ for all $\lambda \in (0, \lambda^*)$, it is easy to see $\mu_2(\lambda) \neq 0$ since $\varphi(\cdot, d, \lambda) < 0$ in $(\xi(d, \lambda), s_1]$ (as in the proof of the last theorem). If $\mu_2 < 0$, then by the Sturm comparison principle, φ must have at least two zeros in (s_0, s_1), a contradiction. The proof is complete.

Remark 2.13. If the annuli are wide enough, i.e., the aspect ratio B/A is large enough, then (2.1), (2.2) may have many solutions for certain $\lambda \in (0, \lambda^*)$, e.g., $f(u) = e^u$ and $3 \leqslant n \leqslant 9$, the details will appear elsewhere.

3. MINIMIZING SOLUTIONS

In this section we shall prove that there exists a variational solution for (2.1), (2.2) provided that f satisfies the following conditions:

(H-1) $f(u) > 0$ for $u > 0$, $f(0) = 0$, and $\lim_{u \to 0} (f(u)/u) = 1$,

(H-2) $uf'(u) > f(u)$ for $u > 0$,

(H-3) $uf(u) \geqslant 2(1+\varepsilon) \int_0^u f(t)\, dt$ for u large and $\varepsilon > 0$,

and $\lambda \in (0, \lambda_1)$, where λ_1 is the least eigenvalue of $-\Delta$ with the Dirichlet boundary condition.

By the result of Crandall and Rabinowitz [7], there is a family of positive radial solutions of (1.1), (1.2) which bifurcates from the trivial solution $u \equiv 0$ at $\lambda = \lambda_1$; however, we would like to have some more information about these solutions.

We first prove the following lemma.

LEMMA 3.1. *Assume* (H-1), (H-2). *If* (1.1), (1.2) *has a positive solution then* $\lambda \in (0, \lambda_1)$.

Proof. It is clear that (H-1), (H-2) implies that $f(u) > u$ for $u > 0$.

Let u be a positive solution of (1.1), (1.2) and $v_1 > 0$ be an associated eigenfunction of λ_1. Then $0 = \int_\Omega uv_1(\lambda(f(u)/u) - \lambda_1)$. Hence $\lambda < \lambda_1$. The proof is complete.

Set $a = B^{2-n}$ and $b = A^{2-n}$ if $n \geq 3$ and $a = -\log B$ and $b = -\log A$ if $n = 2$. Since we are only interested in the positive solution, we may assume that f is an odd function defined on $\mathbb{R}^1$.

Set

$$H(u) = \int_0^u f(t)\, dt, \tag{3.1}$$

and consider the functionals

$$J(u) \equiv J_\lambda(u) \equiv \int_a^b \left\{ \frac{1}{2} u'^2 - \lambda H(u)\, \rho \right\} ds, \tag{3.2}$$

and

$$I(u) \equiv I_\lambda(u) \equiv \int_a^b \{ u'^2 - \lambda \rho f(u)\, u \}\, ds. \tag{3.3}$$

Let

$D = \{u$ is an absolutely continuous function on $[a, b]$ with $u(a) = 0 = u(b)\}$

and

$$M \equiv M_\lambda \equiv \{ u \in D : u \not\equiv 0 \text{ and } I_\lambda(u) = 0 \}. \tag{3.4}$$

Note that if $u \in M_\lambda$, then

$$J_\lambda(u) = \lambda \int_a^b \left\{ \frac{1}{2} uf(u) - H(u) \right\} \rho\, ds. \tag{3.5}$$

The variational problem referred to above is the problem of minimizing J_λ over M_λ. It can be shown that a variational solution must satisfy (2.3), (2.4) (see, e.g., [18, 19]).

LEMMA 3.2. *Assume* (H-1)–(H-3). *If* $u \in D$ *with* $u \geq 0$ *and* $u \not\equiv 0$, *and* $\lambda \in (0, \lambda_1)$, *then there exists a* $t = t(u, \lambda) > 0$ *such that* $tu \in M_\lambda$.

Proof. For $\lambda \in (0, \lambda_1)$,

$$I(tu) = t^2 \int_a^b u'^2 - \lambda \int_a^b \rho f(tu)\, tu$$

$$\geqslant t^2 \lambda_1 \int_a^b \rho u^2 - \lambda \int_a^b \rho f(tu)\, tu$$

$$= t^2 \lambda_1 \int_a^b \rho u^2 \left\{ 1 - \frac{\lambda}{\lambda_1} \frac{f(tu)}{tu} \right\} > 0,$$

for $t > 0$ and is sufficiently small, here the Poincaré inequality has been used.

It is easy to see that (H-1)–(H-3) imply

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty. \tag{3.6}$$

Hence, for t sufficiently large, we have

$$I(tu) = t^2 \left\{ \int_a^b u'^2 - \lambda \int_a^b \rho \frac{f(tu)}{tu} u^2 \right\} < 0.$$

Therefore, there exists $t > 0$ such that $I(tu) = 0$, i.e., $tu \in M_\lambda$. The proof is complete.

Remark 3.3. At λ_1, we have $I_{\lambda_1}(tv_1) < 0$ for all $t > 0$. In fact,

$$I_{\lambda_1}(tv_1) = t^2 \lambda_1 \int_a^b \rho v_1^2 \left(1 - \frac{f(tv_1)}{tv_1} \right) < 0 \qquad \text{for all} \quad t > 0.$$

LEMMA 3.4. *Assume* (H-1)–(H-3). *Then, for any* $\lambda \in (0, \lambda_1)$,

$$m_\lambda \equiv \inf\{J_\lambda(u) : u \in M_\lambda\} > -\infty.$$

Furthermore, if u_k is a sequence in M_λ such that

$$J_\lambda(u_k) \to m_\lambda \qquad \text{as} \quad k \to \infty, \tag{3.7}$$

then there exists a constant $C_\lambda < \infty$ such that

$$\int_a^b u_k'^2 \leqslant C_\lambda. \tag{3.8}$$

Proof. By (H-3) and (3.5), it is easy to see $m_\lambda > -\infty$. Suppose that (3.8) were false, i.e.,

$$\int_a^b u_k'^2 \to \infty \qquad \text{as} \quad k \to \infty,$$

then (3.7) implies that $\lambda \int_a^b H(u_k)\,\rho \to \infty$ as $k \to \infty$. By (H-3) and (3.5) again, we have

$$J_\lambda(u_k) \geqslant \lambda\varepsilon \int_a^b \rho H(u_k) - C,$$

for some constant C, a contradiction to (3.7). The proof is complete.

After these preparations, we can now prove the following theorem.

THEOREM 3.5. *Assume conditions* (H-1)–(H-3) *are satisfied. Then for any* $\lambda \in (0, \lambda_1)$, *there exists a minimizer of* $J_\lambda(u)$ *over* M_λ *which is also a solution of* (2.1), (2.2).

Proof. Pick up a sequence $\{\tilde{u}_k\} \subset M_\lambda$ which satisfies (3.7). Without loss of generality, we may assume that $\tilde{u}_k \geqslant 0$. By (3.8), $\{\tilde{u}_k\}$ is equicontinuous on $[a, b]$. Hence, there is a subsequence $\{u_k\}$ of $\{\tilde{u}_k\}$ satisfying (3.7) and $u_k \to u_0$ uniformly on $[a, b]$.

Let $\tilde{w}_k \in C^2([a, b])$ solve the problem

$$\tilde{w}_k'' + \lambda\rho f(u_k) = 0 \qquad \text{in} \quad (a, b),$$
$$\tilde{w}_k(a) = 0 = \tilde{w}_k(b).$$

It is clear that $\tilde{w}_k > 0$ in (a, b). Since $\lambda \in (0, \lambda_1)$, by Lemma 3.2, there exists a $t_k > 0$ such that $w_k = t_k \tilde{w}_k \in M_\lambda$; i.e., w_k satisfies

$$w_k'' = -t_k \lambda\rho f(u_k) \qquad \text{in} \quad (a, b), \tag{3.9}$$

$$w_k(a) = 0 = w_k(b). \tag{3.10}$$

Then by the result of Nehari [18, (27) of p. 112], we have

$$J_\lambda(w_k) \leqslant J_\lambda(u_k). \tag{3.11}$$

In fact, if we let $f(u) = u(1 + h(u^2))$ and $F(u, s) = \lambda\rho(s)(1 + h(u))$, then $uF(u^2, s) = \lambda\rho(s) f(u)$, and Eq. (2.3), (2.4) is equivalent to

$$u'' + uF(u^2, s) = 0 \qquad \text{in} \quad (a, b),$$
$$u(a) = 0 = u(b),$$

which was studied by Nehari in [18]. Therefore, we have

$$J_\lambda(w_k) \to m_\lambda \qquad \text{as} \quad k \to \infty. \tag{3.12}$$

We shall prove t_k is bounded. Since

$$t_k^2 \left\{ \int_a^b \lambda \rho f(u_k)\, u_k \right\}^2 = \left\{ \int_a^b w_k'' u_k \right\}^2 = \left\{ \int_a^b w_k' u_k' \right\}^2$$
$$\leqslant \int_a^b w_k'^2 \int_a^b u_k'^2 = \int_a^b \lambda \rho f(w_k)\, w_k \int_a^b \lambda \rho f(u_k)\, u_k,$$

we have

$$t_k^2 \int_a^b \lambda \rho f(u_k)\, u_k \leqslant \int_a^b \lambda \rho f(w_k)\, w_k. \tag{3.13}$$

By (3.12), (3.13), and (3.8), we have

$$t_k^2 \int_a^b \lambda \rho f(u_k)\, u_k \leqslant C_\lambda, \tag{3.14}$$

for some constant C_λ. If t_k were unbounded, then

$$\liminf_{k \to \infty} \int_a^b \lambda \rho f(u_k)\, u_k = 0,$$

which implies $\liminf_{k \to \infty} \int_a^b u_k'^2 = 0$ and so $u_k \to 0$ uniformly on $[a, b]$. Hence, by (H-1), we have

$$\int_a^b \rho f(u_k)\, u_k \Big/ \int_a^b \rho u_k^2 \to 1 \qquad \text{as} \quad k \to \infty.$$

But

$$\lambda_1 \int_a^b \rho u_k^2 \leqslant \int_a^b u_k'^2 = \lambda \int_a^b \rho f(u_k)\, u_k,$$

we have $\lambda_1 \leqslant \lambda$, a contradiction. Therefore, t_k is bounded. Note that we have proved $\liminf_{k \to \infty} \int_a^b u_k'^2 > 0$, which implies $u_0 \not\equiv 0$.

Since t_k is bounded, we may assume it tends to a limit $t_0 \geqslant 0$. Since $u_k \to u_0$ uniformly on $[a, b]$, by (3.9), (3.10), there exists a $w_0 \in C^2([a, b])$ such that $w_k \to w_0$ in $C^2([a, b])$. It can be verified that $t_0 > 0$ and so $w_0 \not\equiv 0$. In fact, if $t_0 = 0$ then $w_k \to 0$ uniformly on $[a, b]$ and so

$$\int_a^b \rho f(w_k)\, w_k \Big/ \int_a^b \rho w_k^2 \to 1 \qquad \text{as} \quad k \to \infty.$$

Therefore, by the Poincaré inequality, for k sufficiently large we have

$$I_\lambda(w_k) \geqslant \frac{1}{2}\int_a^b \rho w_k^2(\lambda_1 - \lambda) > 0,$$

a contradiction. Hence, $w_0 \in M_\lambda$ and $J_\lambda(w_k) \to J_\lambda(w_0)$ implies $J_\lambda(w_0) = m_\lambda$. This proves w_0 is a minimizer of J_λ over M_λ. It can be shown that any minimizer is also a solution of (2.3), (2.4) (see, e.g., [18, 19].). The proof is complete.

We have the following stability results for minimizing solutions of J_λ over M_λ.

PROPOSITION 3.6. *Assume conditions* (H-1)–(H-3) *are satisfied. Let* u_λ *be a minimizer of* J_λ *over* M_λ, $\lambda \in (0, \lambda_1)$. *Then* $\mu_2(\lambda) \geqslant 0 > \mu_1(\lambda)$, *where* $\mu_l(\lambda)$ *is the lth eigenvalue of the linearized eigenvalue problem*

$$\varphi'' + \lambda\rho f(u_\lambda)\,\varphi = -\mu\rho\varphi \qquad \text{in} \quad (a, b),$$
$$\varphi(a) = 0 = \varphi(b).$$

Proof. It is easy to see that (H-2) implies $\mu_1(\lambda) < 0$ (see, e.g., [20]). Let $v_l(\lambda)$ be the lth eigenvalue of the linear eigenvalue problem

$$\psi'' + v\lambda\rho f'(u_\lambda)\,\psi = 0 \qquad \text{in} \quad (a, b),$$
$$\psi(a) = 0 = \psi(b).$$

By an argument similar to proving Lemma 6.3 of [2], we can prove $v_2(\lambda) \geqslant 1$ and omit the details here. Now the result follows from the following lemma.

LEMMA 3.7. *Assume* $g(x) \in C^0(\bar{\Omega})$ *and* $g(x) \geqslant c > 0$ *on* $\bar{\Omega}$, *where* Ω *is a bounded smooth domain in* $\mathbb{R}^n$. *Let* μ_l *be the lth eigenvalue of the linear eigenvalue problem*

$$\Delta\varphi + g\varphi = -\mu\varphi \qquad \text{in} \quad \Omega,$$
$$\varphi = 0 \qquad \text{on} \quad \partial\Omega,$$

and let v_l *be the lth eigenvalue of the linear eigenvalue problem*

$$\Delta\psi + vg\psi = 0 \qquad \text{in} \quad \Omega,$$
$$\psi = 0 \qquad \text{on} \quad \partial\Omega.$$

Then $\mu_l < 0$ *if and only if* $v_l < 1$.

Proof. The lemma can be proved by using the mini–max principle of eigenvalues (see, e.g., [6]); the detail is omitted.

Finally, we also have a unique positive radial solution of (1.1), (1.2) if the annuli are narrow.

THEOREM 3.8. *Assume conditions* (H-1)–(H-3) *are satisfied. If* $B/A \leqslant (n-1)^{1/(n-2)}$ *for* $n \geqslant 3$ *and* $B/A \leqslant e$ *for* $n=2$, *then there exists an unique positive radial solution* u_λ *of* (1.1), (1.2) *for* $\lambda \in (0, \lambda_1)$, *which is also the minimizer of* J_λ *over* M_λ. *Furthermore,*

$$\|u_\lambda\|_\infty \to \infty \qquad as \quad \lambda \to 0^+. \tag{3.15}$$

Proof. The existence of the minimizer of J_λ over M_λ was proved in Theorem 3.5 and the uniqueness of the positive radial solution was proved in Theorem 1.7 of [20].

Since

$$\lambda_1 \int_\Omega u_\lambda^2 \leqslant \int_\Omega |\nabla u_\lambda|^2 = \lambda \int_\Omega f(u_\lambda)\, u_\lambda,$$

we have

$$\lambda_1/\lambda \leqslant \sup\{f(u_\lambda(x))/u_\lambda(x) : x \in \Omega\},$$

so (3.15) follows. The proof is complete.

Remark 3.9. If the annuli are wide enough, then (2.1), (2.2) also may have many solutions for certain $\lambda \in (0, \lambda_1)$, e.g., $f(u) = u + u^p$, $p > (n+2)/(n-2)$, and $3 \leqslant n \leqslant 9$ (see, e.g., [5]).

4. NON-RADIAL BIFURCATION

In this section we shall study the problem of non-radial bifurcation (symmetry breaking) from a certain family of positive radial solutions of (1.1), (1.2).

We first study the linearized eigenvalue problem of (1.1), (1.2) at the positive radial solution u_λ:

$$\Delta w + \lambda f'(u_\lambda)\, w = -\mu w \qquad \text{in} \quad \Omega, \tag{4.1}$$

$$w = 0 \qquad \text{on} \quad \partial\Omega. \tag{4.2}$$

In spherical coordinates, (4.1), (4.2) can be reduced to

$$\varphi''(r) + \frac{n-1}{r}\,\varphi'(r) + \left\{\lambda f'(u_\lambda) - \frac{\alpha_k}{r^2}\right\} \varphi(r) = -\mu_{k,l}(\lambda)\, \varphi(r), \qquad r \in (A, B),$$

$$\varphi(A) = 0 = \varphi(B), \tag{4.3}$$

where $\alpha_k = k(k+n-2)$, $k = 0, 1, 2, ...$, $l = 1, 2, ...$ Note that α_k is the eigenvalue of Laplacian $-\Delta$ on S^{n-1}, the unit sphere, and the dimension of the eigenspace S_k of associated eigenfunctions is $l_k = \binom{k+n-2}{k}((n+2k-2)/(n+k-2))$. Let $\bar{x} = (x_1, ..., x_{n-1})$; a function v defined on S^{n-1} or Ω is called $O(n-1)$ invariant if $v(T\bar{x}, x_n) = v(\bar{x}, x_n)$ for all $T \in O(n-1)$. Then, for any positive integer k, the dimension of $V_k = \{v \in S_k \mid v \textit{ is } O(n-1)$ *invariant*$\}$ is one, for details see [24].

Denoted by $C_0^{1+\gamma}(\bar{\Omega})$ the set of continuously differentiable functions on $\bar{\Omega}$ which vanish on $\partial\Omega$ and whose first order derivatives are Hölder continuous on $\bar{\Omega}$ with exponent $\gamma \in (0, 1)$. $C_0^{1+\gamma}(\bar{\Omega})$ is a Banach space under the usual norm $\|\cdot\| \equiv \|\cdot\|_{1,\gamma}$.

Let u_λ be a family of positive radial solutions of (1.1), (1.2) which is smooth in $\lambda \in (\Lambda_1, \Lambda_2) \subset (0, \infty)$. Note that u_λ is called smooth in λ if u_λ is C^l in λ for some positive integer l. $\lambda_0 \in (\Lambda_1, \Lambda_2)$ is called a (non-radial) bifurcation point if every neighborhood of $(\lambda_0, u_{\lambda_0})$ in $\mathbb{R}^- \times C_0^{1+\gamma}(\bar{\Omega})$ contains a (non-radial) positive solution other than $u(\lambda)$. $[\lambda_0, \bar{\lambda}_0] \subset (\Lambda_1, \Lambda_2)$ is called a (non-radial) bifurcation interval if every neighborhood of $\{(\lambda, u_\lambda) : \lambda \in [\lambda_0, \bar{\lambda}_0]\}$ contains a (non-radial) positive solution other than $u(\lambda)$.

The following theorem is a variant of bifurcation theorems of Krasnosel'skii, and Rabinowitz [21].

THEOREM 4.1. *Let u_λ be a family of positive radial solutions of* (1.1), (1.2) *which is smooth in $\lambda \in (\Lambda_1, \Lambda_2)$. If $\lambda_0 \in (\Lambda_1, \Lambda_2)$ and $\varepsilon > 0$ such that*

(i) $\mu_{k,1}(\lambda_0) = 0$, $\mu_{k,1}(\lambda)\,\mu_{k,1}(\lambda') < 0$ *for* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ *and* $\lambda' \in (\lambda_0, \lambda_0 + \varepsilon)$, *for some positive integer k,*

(ii) $\mu_{j,2}(\lambda) > 0$ *for* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ *and all non-negative integers j,*

then λ_0 is a non-radial bifurcation point.

Similarly, if (i) *is replaced by*

(i) $\mu_{k,1}(\lambda) = 0$ *on* $[\lambda_0, \bar{\lambda}_0]$, *and* $\mu_{k,1}(\lambda)\,\mu_{k,1}(\lambda') < 0$ *for* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ *and* $\lambda' \in (\bar{\lambda}_0, \bar{\lambda}_0 + \varepsilon)$,

then $[\lambda_0, \bar{\lambda}_0]$ *is a non-radial bifurcation interval.*

Proof. Let $w = u - u_\lambda$. Then (1.1), (1.2) can be written as

$$\Delta w + \lambda\{f(u_\lambda + w) - f(u_\lambda)\} = 0 \quad \text{in} \quad \Omega, \tag{4.4}$$

$$w = 0 \quad \text{on} \quad \partial\Omega, \tag{4.5}$$

which can also be written as an operator equation

$$w - \Phi_\lambda(w) = 0 \tag{4.6}$$

on $C_0^{1+\gamma}(\bar{\Omega})$, where $\Phi_\lambda(w)=\lambda G\{f(u_\lambda+w)-f(u_\lambda)\}$ and $G=(-\Delta)^{-1}$. It is clear that $\Phi_\lambda(w)$ is a compact operator for each $\lambda>0$. We shall work (4.6) on the $O(n-1)$ invariant subspace $X=\{w\in C_0^{1+\gamma}(\bar{\Omega}): w$ is $O(n-1)$ invariant$\}$.

Assume conditions (i) and (ii) are satisfied. If λ_0 is not a bifurcation point, then there exists $\delta>0$ ($\delta<\varepsilon$) such that there is no solution of (4.6) in X for $|\lambda-\lambda_0|\leqslant\delta$ and $\|u-{}_{\lambda_0}\|\leqslant\delta$ except u_λ. Here δ can also be chosen small enough such that

$$\mu_{j,1}(\lambda)\neq 0 \qquad \text{for} \quad \lambda\in(\lambda_0-\delta,\lambda_0+\delta) \text{ and } j\neq k. \tag{4.7}$$

Let $B_\delta(0)$ be a δ-ball in X. Then

$$\deg(I-\Phi_\lambda, B_\delta(0), 0) \text{ is constant on } [\lambda_0-\delta,\lambda_0+\delta]. \tag{4.8}$$

By (ii) and (4.7),

$$\deg(I-\Phi_\lambda, B_\delta(0), 0)=\deg(I-\Phi'_\lambda(0), B_\delta(0), 0), \tag{4.9}$$

for all $\lambda\in[\lambda_0-\delta,\lambda_0+\delta]\backslash\{\lambda_0\}$.

By Lemma 3.7, it is easy to verify that

$$\deg(I-\Phi'_\lambda(0), B_\delta(0), 0)=(-1)^k \quad\text{and}\quad \deg(I-\Theta'_{\lambda'}(0), B_\delta(0), 0)=(-1)^{k+1}$$

when $\mu_{k,1}(\lambda)>0$ and $\mu_{k,1}(\lambda')<0$, and

$$\deg(I-\Phi'_\lambda(0), B_\delta(0), 0)=(-1)^{k+1} \quad\text{and}\quad \deg(I-\Phi'_{\lambda'}(0), B_\delta(0), 0)=(-1)^k$$

when $\mu_{k,1}(\lambda)<0$ and $\mu_{k,1}(\lambda')>0$. Which leads to a contradiction in view of (4.8) and (4.9). This proves λ_0 is a bifurcation point.

By a similar argument, if conditions (i)′ and (ii) are satisfied, then (1.1), (1.2) bifurcates on interval $[\lambda_0, \bar{\lambda}_0]$. The proof is complete.

Remark 4.2. The problem of bifurcation on an interval for certain non-linear differential equations has been studied by Berestycki [3]. Although (i)′ seems unlikely in our problem, we cannot rule out this possibility for the time being. In the case $n=2$ and $f(u)=e^u$, only (i) is possible, see Lin [14].

Let $\tilde{u}_1$ be the least eigenvalue of

$$\Delta w=-\tilde{u}r^{-2}w \qquad \text{in} \quad \Omega, \tag{4.10}$$

$$w=0 \qquad \text{on} \quad \partial\Omega. \tag{4.11}$$

Then, by the Poincaré inequality, we have

$$\tilde{\mu}_1 \int_\Omega r^{-2} w^2 \, dx \leqslant \int_\Omega |\nabla w|^2 \, dx \tag{4.12}$$

for all $w \in C_0^{1+\gamma}(\bar{\Omega})$.

The following lemma indicates that there is a great possibility to have a non-radial bifurcation of (1.1), (1.2) when the growth of $f(u)$ is rapid enough as $u \to \infty$.

LEMMA 4.3. *Let u_λ be a family of positive radial solutions of* (1.1), (1.2) *which is smooth in $\lambda \in (0, \Lambda)$. Assume $f(u)$ satisfies the following growth condition:*

(G) *$uf'(u) \geqslant pf(u)$ for u large,*

where $p > 1 + \alpha_k/\tilde{\mu}_1$ and k is a positive integer. If $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$, then $\mu_{k,1}(\lambda) < 0$ for λ sufficiently small.

Proof. It is easy to verify that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$ implies that

$$\int_\Omega |\nabla u_\lambda|^2 \to \infty \qquad \text{as} \quad \lambda \to 0^+. \tag{4.13}$$

In fact, for any $r \in (A, B)$,

$$\begin{aligned} u_\lambda(r) &= \int_A^r u'_\lambda(s) \, ds \\ &\leqslant (B-A)^{1/2} \left\{ \int_A^B |u'_\lambda(s)|^2 \, ds \right\}^{1/2} \\ &\leqslant (B-A)^{1/2} A^{-(n-1)/2} \omega_n^{-1/2} \left\{ \int_\Omega |\nabla u_\lambda|^2 \right\}^{1/2}, \end{aligned}$$

where ω_n is the area of unit sphere S^{n-1}. Hence,(4.13) follows.

The eigenvalue $\mu_{k,1}$ can be characterized as

$$\mu_{k,1} = \inf\{R(\psi) : \psi \in C_0^1([A, B])\},$$

where

$$R(\psi) = R_k(\psi) = Q(\psi)/I(\psi),$$

$$Q(\psi) = Q_{k,\lambda}(\psi) = \int_A^B r^{n-1} \{\psi'^2(r) - \lambda f'(u_\lambda(r)) \psi^2(r) + \alpha_k r^{-2} \psi^2(r)\} \, dr,$$

and

$$I(\psi) = \int_A^B r^{n-1} \psi^2(r)\, dr.$$

Since u_λ is a solution of (1.1), (1.2), we have

$$\int_\Omega |\nabla u_\lambda|^2 = \lambda \int_\Omega f(u_\lambda)\, u_\lambda. \tag{4.14}$$

Then, by (G), (4.12), (4.13), and (4.14), we have

$$\begin{aligned}
\omega_n Q(u_\lambda) &= \lambda \int_\Omega \{f(u_\lambda) - f'(u_\lambda)\, u_\lambda\}\, u_\lambda + \alpha_k \int_\Omega r^{-2} u_\lambda^2 \\
&\leqslant \lambda(1-p) \int_\Omega f(u_\lambda)\, u_\lambda + \alpha_k/\tilde{\mu}_1 \int_\Omega |\nabla u_\lambda|^2 + M \\
&= (1 + \alpha_k/\tilde{\mu}_1 - p) \int_\Omega |\nabla u_\lambda|^2 + M
\end{aligned}$$

for some constant $M > 0$ which is independent of λ. Hence, $Q(u_\lambda) < 0$ if λ is sufficiently small. The proof is complete.

DEFINITION 4.4. λ_0 is called a non-radial bifurcation point with mode k if $\mu_{k,1}(\lambda_0) = 0$ where k is a positive integer. A similar definition can also be given to non-radial bifurcation interval $[\lambda_0, \bar{\lambda}_0]$.

Now, we can apply the above results to the problems which have been studied in the previous sections.

THEOREM 4.5. *Assume* $B/A \leqslant (n-1)^{1/(n-2)}$ *for* $n \geqslant 3$ *and* $B/A \leqslant e$ *for* $n = 2$.

(i) *If conditions* (A-1)–(A-3) *and* (G) *are satisfied, then for each positive integer* j, $1 \leqslant j \leqslant k$, (1.1), (1.2) *has a non-radial bifurcation with mode* j *along the upper* (*non-minimum*) *branch of the positive radial solution.*

(ii) *If conditions* (H-1)–(H-3) *and* (G) *are satisfied, then for each positive integer* j, $1 \leqslant j \leqslant k$, (1.1), (1.2) *has a non-radial bifurcation with mode* j *along the positive radial solution.*

Proof. (i) Since $\mu_{0,1}(\lambda^*) = \mu_1(\lambda^*) = 0$, by Theorem 2.6 and Lemma 4.3, for any $1 \leqslant j \leqslant k$, there exists a $\lambda_j \in (0, \lambda^*)$ such that $\mu_{j,1}(\lambda_j) = 0$ and satisfies either (i) or (i)′ of Theorem 4.1. Condition (ii) of Theorem 4.1 follows from Corollary 2.12. Hence, there is a non-radial bifureation at λ_j or on $[\lambda_j, \bar{\lambda}_j]$. Similarly, (ii) can be proved by using Proposition 3.6, Theorem 3.8, Lemma 4.3, and Theorem 4.1. The proof is complete.

Remark 4.6. For $f(u)=e^u$, $n=2$, (G) holds for all positive integers k. It has been proved in [14] that there exists a sequence $\lambda_j \to 0$ as $j \to \infty$ such that (1.1), (1.2) has a non-radial bifurcation at each λ_j with mode j.

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