J Comb Optim (2007) 14: 323–329
DOI 10.1007/s10878-007-9049-5

Packing 5-cycles into balanced complete *m*-partite graphs for odd *m*

Ming-Hway Huang · Chin-Mei Fu · Hung-Lin Fu

Published online: 31 March 2007

Abstract Let $K_{n_1,n_2,\ldots,n_m}$ be a complete m-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_m$. A complete m-partite graph is *balanced* if each partite set has n vertices. We denote this complete m-partite graph by $K_{m(n)}$. In this paper, we completely solve the problem of finding a maximum packing of the balanced complete m-partite graph $K_{m(n)}$, m odd, with edge-disjoint 5-cycles and we explicitly give the minimum leaves.

Keywords Complete m-partite graph · Balanced complete m-partite graph · 5-cycle · Packing · Leave · Decomposition

1 Introduction

A few definitions, although many of them are standard, are first given for clarity. Let K_m be a *complete graph* with m vertices. A graph G is *bipartite* if $V(G)$ is the union of two disjoint sets such that each edge consists of one vertex from each set. Let $K_{n_1,n_2,\ldots,n_m}$ be a complete m-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_m$. A complete m-partite graph is *balanced* if each partite set has n vertices. We denote

Dedicated to Professor Frank K. Hwang on the occasion of his 65th birthday.

Research of M.-H.W. was supported by NSC 93-2115-M-264-001.

M.-H. Huang
Department of Computer Science and Information Engineering, Yuanpei Institute of Science and Technology, Hsinchu, Taiwan

C.-M. Fu (✉)
Department of Mathematics, Tamkang University, Tamsui, Taipei Shien, Taiwan
e-mail: cmfu@math.tku.edu.tw

H.-L. Fu
Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan
e-mail: cmfu@mail.tku.edu.tw

this complete m-partite graph by $K_{m(n)}$. A subgraph of graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; an *induced subgraph* H of G is a subgraph of G such that $E(H)$ consists of all edges of G whose end points belong to $V(H)$. If S is a nonempty set of vertices of G, then the subgraph of G induced by S is the induced subgraph of G with vertex set S. This induced subgraph of G is denoted by $G[S]$.

A *Latin square* of order n based on an n-element set is an $n \times n$ array in which each cell contains a single element from the set such that each element occurs exactly once in each row and each column. A Latin square $A = [a_{i,j}]$ of order n based on the set $Z_n = \{0, \ldots, n-1\}$ is called *idempotent* if $a_{i,i} = i$ for each $i \in Z_n$.

A k-cycle is a cycle of length k. A *k-cycle packing* of a graph G is a set of edge-disjoint k-cycles in G. A k-cycle packing C of G is *maximum* if $|C| \geq |C'|$ for all other k-cycle packings C' of G. The *leave* L of a packing C is the subgraph induced by the set of edges of G that do not occur in any k-cycle of the packing C. The leave L of a maximum packing is referred to as a minimum leave, a leave with minimum number of edges. A packing with empty leave is known as a *k-cycle system* of G. A k-cycle system of a complete graph K_v with v vertices is referred to as a k-cycle system of order v.

Clearly, if K_v can be decomposed into a k-cycle system then v is odd and k divides $\binom{v}{2}$. To determine whether the above necessary condition is also sufficient is commonly referred to as the existence problem of k-cycle system.

The existence problem for *k-cycle system* of order v has been studied for more than 35 years. Recently, it has been completely solved by Alspach et al. see (Alspach and Gavlas 2001; Alspach and Marshall 1994; Wilson 1974). But, the packing of K_v with k-cycles is not that lucky, only partial results are obtained so far, see (Lindner and Rodger 1992). Mainly, $k \in \{3,4,5,6\}$ has been considered.

If we turn to the k-cycle packing of a complete multipartite graph, then the problem is getting more difficult. Even in the case $k = 3$, the existence problem is still unsolved; see (Lindner and Rodger 1992). Recently, Billington, Fu and Rodger completely solved the case $k = 4$, see (Billington et al. 2001, 2005). The cases other than $k = 4$ remain unsettled.

In this paper, we consider a 5-cycle packing of a balanced complete m-partite graph $K_{m(n)}$ for odd m and we obtain a minimum leave of a maximum packing of $K_{m(n)}$. The following two results obtained by Cavenagh and Billington, Rosa and Znám respectively are essential.

Theorem 1.1 (Cavenagh and Billington 2000b) *The complete tripartite graph K_{m_1,m_2,m_3} (with $m_1 \leq m_2 \leq m_3$) can be decomposed into* 5*-cycles only if m_1, m_2, m_3 are either all odd or all even,* 5 *divides $|E(K_{m_1,m_2,m_3})|$ and $m_3 \leq 4m_1m_2/(m_1+m_2)$. These necessary conditions are sufficient in the case when two partite sets have equal size or in the case when m_1 and m_2 are divisible by* 10.

Theorem 1.2 (Rosa and Znám 1994) *The minimum leaves of the maximum packings of K_v with* 5*-cycles are as follows in Table* 1. *v is considered to be the number modulo* 10. *F is a* 1*-factor, C_i is a cycle of length i, F_i is a graph with $v/2+i$ edges and each vertex has odd degree.*

Table 1 The minimum leaves of the maximum packings of K_v with 5-cycles

v	0	1	2	3	4	5	6	7	8	9
L	F	$\emptyset$	F	C_3	F_4	$\emptyset$	F_2	$2C_3$	F_4	$2C_3$

We note here that in the cases $v \equiv 7$ or 9 (mod 10) the leave $2C_3$ represents two C_3 with one vertex in common. It is also known as a *bowtie*.

2 The maximum 5-cycle packing of $K_{m(n)}$

First, we consider a maximum 5-cycle packing of $K_{n,n,n}$. Before that we need to solve some small cases:

Lemma 2.1 *There is a* 5*-cycle packing of* $K_{3,3,3}$ *with leave* $C_3 \cup C_4$.

Proof Let $Z_3 \times Z_3$ be the vertex set of $K_{3,3,3}$. Then $K_{3,3,3}$ can be packed with 5-cycles: $((0, j), (2, 1+j), (0, 2+j), (1, j), (1, 2+j))$, $j = 1, 2$, $((0, 0), (0, 1), (0, 2), (1, 0), (1, 2))$ and $((1, 0), (2, 1), (2, 0), (1, 1), (2, 2))$ with leave $C_3 \cup C_4$: $((2, 0), (2, 2), (2, 1), (1, 2)) \cup ((2, 1), (0, 0), (0, 2))$. □

Lemma 2.2 *There is a* 5*-cycle packing of* $K_{4,4,4}$ *with leave* C_3.

Proof Let $Z_4 \times Z_3$ be the vertex set of $K_{4,4,4}$. Then $K_{4,4,4}$ can be packed with 5-cycles: $((i, j), (2 + i, 1 + j), (i, 2 + j), (1 + i, j), (1 + i, 2 + j))$, $i = 0, 1$, $j \in Z_3$, $((0, 0), (3, 2), (2, 1), (3, 0), (3, 1))$, $((2, 0), (3, 2), (3, 0), (0, 2), (3, 1))$ and $((3, 0), (2, 2), (3, 1), (3, 2), (0, 1))$ with leave C_3: $((0, 0), (0, 1), (0, 2))$. □

Lemma 2.3 *There is a* 5*-cycle packing of* $K_{6,6,6}$ *with leave* C_3.

Proof Let $(\{\infty\} \cup Z_5) \times Z_3$ be the vertex set of $K_{6,6,6}$. Then $K_{6,6,6}$ can be packed with 5-cycles: $((i, j), (\infty, 1 + j), (i, 2 + j), (1 + i, j), (1 + i, 2 + j))$, $i \in Z_5$, $j \in Z_3$, and $((0, j), (3, 1+j), (1, j), (4, 1+j), (2, 2+j))$, $((4, j), (1, 1+j), (3, j), (0, 1 + j), (2, 2 + j))$, $j \in Z_3$ with leave C_3: $((\infty, 0), (\infty, 1), (\infty, 2))$. □

Lemma 2.4 *There is a* 5*-cycle packing of* $K_{7,7,7}$ *with leave* $C_3 \cup C_4$.

Proof Let $Z_7 \times Z_3$ be the vertex set of $K_{7,7,7}$. Since $K_{7,7,7}$ can be decomposed into $K_{5,5,5}$ and three copies of $K_{5,2,2}$. Let $Z_5 \times Z_3$ be the vertex set of $K_{5,5,5}$. Then $K_{5,5,5}$ can be decomposed into following 5-cycles: $((i, j), (2+i, 1 + j), (i, 2 + j), (1 + i, j), (1 + i, 2 + j))$, $i \in Z_5$, $j \in Z_3$. Since $K_{5,2,2}$ can not be decomposed into 5-cycles, we can pack $K_{5,2,2}$ with 5-cycles with leave a $K_{1,4}$ or $2K_{1,2}$. Therefore $K_{7,7,7}$ can be packed with 5-cycles with leave $2K_{1,4} \cup 2K_{1,2}$: $((4, 1), (5, 0))$, $((4, 1), (6, 0))$, $((4, 1), (5, 2))$, $((4, 1), (6, 2))$, $((0, 2), (5, 0))$, $((0, 2), (6, 0))$, $((0, 2), (5, 1))$, $((0, 2), (6, 1))$, $((3, 0), (6, 1))$, $((3, 0), (6, 2))$, $((4, 0), (5, 1))$, $((4, 0), (5, 2))$. In the above packing of $K_{5,5,5}$, there is a 5-cycle C_5: $((3, 1), (0, 2), (3, 0), (4, 1), (4, 0))$. Then $2K_{1,4} \cup 2K_{1,2} \cup C_5$ can be packed with two 5-cycles: $((3, 0), (6, 1), (0, 2), (5, 0), (4, 1))$ and $((6, 0), (4, 1), (6, 2), (3, 0), (0, 2))$ with leave $C_3 \cup C_4$: $((4, 0), (5, 2), (4, 1)) \cup ((4, 0), (5, 1), (0, 2), (3, 1))$. □

Lemma 2.5 *There is a 5-cycle packing of $K_{9,9,9}$ with leave C_3.*

Proof Let $(\{\infty\} \cup Z_8) \times Z_3$ be the vertex set of $K_{9,9,9}$. Then $K_{9,9,9}$ can be packed with 5-cycles: $((i, j), (\infty, 1 + j), (i, 2 + j), (1 + i, j), (1 + i, 2 + j))$, and $((i, j), (4 + i, 1 + j), (1 + i, 2 + j), (4 + i, j), (6 + i, 2 + j))$, $i \in Z_8$, $j \in Z_3$ with leave C_3: $((\infty, 0), (\infty, 1), (\infty, 2))$. □

Lemma 2.6 *There is a 5-cycle packing of $K_{11,11,11}$ with leave C_3.*

Proof Let the vertex set of $K_{11,11,11}$ be $(\{\infty\} \cup Z_{10}) \times Z_3$, and let $A_i = \{5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4\}$, for each $i \in Z_2$. Since $K_{11,11,11}$ can be decomposed into $K_{6,6,6}$ with vertex set $(\{\infty\} \cup A_0) \times Z_3$ and three copies of $K_{5,5,6}$, we can pack $K_{6,6,6}$ with leave C_3: $((\infty, 0), (\infty, 1), (\infty, 2))$ and $K_{5,5,6}$ with leave $6K_{1,5}$: $((\infty, i), (5, j))$, $((\infty, i), (6, j))$, $((\infty, i), (7, j))$, $((\infty, i), (8, j))$, $((\infty, i), (9, j))$, for $j = i + 1, i + 2, i = 0, 1, 2$. Using the same construction as in the proof of Lemma 2.4, we can get two C_5 from $K_{5,5,6}$: $((2, 1), (5, 0), (6, 2), (6, 0), (5, 2))$, $((4, 0), (7, 2), (8, 1), (8, 2), (7, 1))$. Then $C_3 \cup 6K_{1,5} \cup 2C_5$ can be packed with $8C_5$: $((\infty, 0), (\infty, 1), (9, 0), (\infty, 2), (7, 1))$, $((\infty, 1), (9, 2), (\infty, 0), (9, 1), (\infty, 2))$, $((\infty, 2), (\infty, 0), (5, 2), (\infty, 1), (8, 0))$, $((\infty, 0), (7, 2), (8, 1), (\infty, 2), (6, 1))$, $((4, 0), (7, 2), (\infty, 1), (8, 2), (7, 1))$, $((\infty, 0), (6, 2), (6, 0), (\infty, 2), (5, 1))$, $((\infty, 1), (6, 2), (5, 0), (\infty, 2), (7, 0))$, $((\infty, 1), (6, 0), (5, 2), (2, 1), (5, 0))$ with leave C_3: $((\infty, 0), (8, 1), (8, 2))$. □

Lemma 2.7 *There is a 5-cycle packing of $K_{n,n,n}$ with leave* (i) C_3 *when* $n \equiv 1$ *or* 4 (mod 5) *and* (ii) $C_3 \cup C_4$ *when* $n \equiv 2$ *or* 3 (mod 5).

Proof (i) $n = 5k + 1$, $k \geq 1$. Let $(\{\infty\} \cup Z_{5k}) \times Z_3$ be the vertex set of $K_{n,n,n}$, and $A_i = \{5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4\}$, for each $i \in Z_k$. When $k = 1$ and $k = 2$, it can be seen in Lemmas 2.3 and 2.6 respectively. If $k \geq 3$, let $M = [m_{i,j}]$ be an idempotent Latin square of order k based on Z_k. For each $i \in Z_k$, the induced subgraph $K_{n,n,n}[(\{\infty\} \cup A_i) \times Z_3]$ is isomorphic to $K_{6,6,6}$. By Lemma 2.3, $K_{n,n,n}[(\{\infty\} \cup A_i) \times Z_3]$ can be packed with 5-cycles with leave $C_3 : ((\infty, 0), (\infty, 1), (\infty, 2))$ for each i. By Theorem 1.1, the induced subgraph $K_{n,n,n}[(A_i \times \{0\}) \cup (A_j \times \{1\}) \cup (A_{m_{i,j}} \times \{2\})]$, which is isomorphic to $K_{5,5,5}$, can be decomposed into 5-cycles for each $i \neq j$. This implies that $K_{n,n,n}$ can be packed with 5-cycles with leave C_3.

(ii) $n = 5k + 2$, $k \geq 0$. If $n = 2$, it is easy to see that $K_{2,2,2}$ can be packed with one 5-cycle which has leave $C_3 \cup C_4$. If $k \geq 1$, let the vertex set of $K_{n,n,n}$ be $(\{\infty_1, \infty_2\} \cup Z_{5k}) \times Z_3$. Let $A_0 = \{\infty_1, \infty_2\} \cup \{0, 1, 2, 3, 4\}$ and $A_i = \{5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4\}$, for each $i \in Z_k$ and $i \geq 1$. Then the induced subgraph $K_{n,n,n}[(A_i \times \{0\}) \cup (A_j \times \{1\}) \cup (A_{i+j} \times \{2\})]$ is isomorphic to $K_{7,7,7}$ if $i = 0$ and $j = 0$, and isomorphic to $K_{5,5,5}$ or $K_{5,5,7}$ otherwise. Therefore by Theorem 1.1 and Lemma 2.4, we obtain that $K_{n,n,n}$ can be packed with 5-cycles with leave $C_3 \cup C_4$.

(iii) $n = 5k + 3$, $k \geq 0$. Let $(\{\infty_1, \infty_2, \infty_3\} \cup Z_{5k}) \times Z_3$ be the vertex set of $K_{n,n,n}$. Let $A_0 = \{\infty_1, \infty_2, \infty_3\}$ and $A_{i+1} = \{5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4\}$, for each $i \in Z_k$. Then the induced subgraph $K_{n,n,n}[(A_i \times \{0\}) \cup (A_j \times \{1\}) \cup (A_{i+j} \times \{2\})]$ is isomorphic to $K_{3,3,3}$ if $i = 0$ and $j = 0$, and isomorphic to $K_{5,5,5}$ or $K_{5,5,3}$ otherwise.

By Theorem 1.1 and Lemma 2.1, we obtain that $K_{n,n,n}$ can be packed with 5-cycles with leave $C_3 \cup C_4$.

(iv) $n = 5k + 4$, $k \geq 0$. Let the vertex set of $K_{n,n,n}$ be $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup Z_{5k}) \times Z_3$. Let $A_0 = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{0, 1, 2, 3, 4\}$ and $A_i = \{5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4\}$, for each $i \in Z_k$ and $i \geq 1$. Then the induced subgraph $K_{n,n,n}[(A_i \times \{0\}) \cup (A_j \times \{1\}) \cup (A_{i+j} \times \{2\})]$ is isomorphic to $K_{9,9,9}$ if $i = 0$ and $j = 0$, and isomorphic to $K_{5,5,5}$ or $K_{5,5,9}$ otherwise. By Theorem 1.1 and Lemma 2.5, we obtain that $K_{n,n,n}$ can be packed with 5-cycles with leave C_3. □

Since the number of edges in the leave of above 5-cycle packing of $K_{n,n,n}$ is the minimum, we have finished the maximum 5-cycle packing of the balance complete tripartite graphs. Now, we go on to consider the following special graphs.

Lemma 2.8 *Let $n \geq 2$, and $C_{5(n)}$ denote the graph with vertex set $Z_n \times Z_5$ and edge set $E(C_{5(n)})$, where $\{(i_1, j_1), (i_2, j_2)\} \in E(C_{5(n)})$ if and only if $j_2 \equiv j_1 + 1 \pmod 5$. Then $C_{5(n)}$ can be decomposed into 5-cycles.*

Proof $C_{5(n)}$ can be decomposed into n^2 5-cycles: $\{((i, 0), (j, 1), (i, 2), (j, 3), (i + j, 4)) \mid i, j, i + j \in Z_n\}$. □

Lemma 2.8 gives us a good idea to pack a balanced complete m-partite graph $K_{m(n)}$ with 5-cycles. If we view each partite set of $K_{m(n)}$ as a point, then it will turn to be a complete graph K'_m of order m. By Theorem 1.2, we can pack the complete graph K'_m with 5-cycles which has leave an L'_m. Thus the leave of the packing of $K_{m(n)}$ with 5-cycles depends on the leave of the packing of $L'_{m(n)}$ with 5-cycles. Since the leave of the packing contains the fewest number of edges, we will get the maximum packing. Now we are ready for the maximum packing of $K_{m(n)}$ with 5-cycles, where m is odd.

Theorem 2.9 *Let m be an odd integer. Then the minimum leaves of the maximum packings of $K_{m(n)}$ with 5-cycles are as follows in Table 2. m is considered to be the number modulo 10, n is considered to be the number modulo 5.*

Proof If we consider each partite set of $K_{m(n)}$ as a vertex, then $K_{m(n)}$ can be viewed as the complete graph K'_m.

(1) $m \equiv 1$ or $5 \pmod{10}$. By Theorem 1.2, K'_m can be decomposed into 5-cycles. By Lemma 2.8, $K_{m(n)}$ can be decomposed into 5-cycles.

Table 2 The minimum leaves of the maximum packings of $K_{m(n)}$ with 5-cycle

m	n				
	0	1	2	3	4
1	∅	∅	∅	∅	∅
3	∅	C_3	$C_3 \cup C_4$	$C_3 \cup C_4$	C_3
5	∅	∅	∅	∅	∅
7	∅	$2C_3$	C_4	C_4	$2C_3$
9	∅	$2C_3$	C_4	C_4	$2C_3$

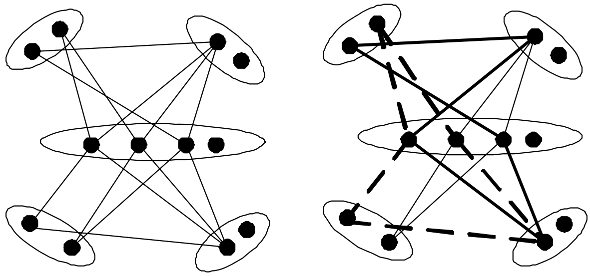

Fig. 1 Pack $2(C_3 \cup C_4)$ with two 5-cycles with leave a C_4

Since $K_{m(n)}$ is a simple graph and m is odd, the degree of each vertex in $K_{m(n)}$ is even. A nonempty leave of a 5-cycle packing of $K_{m(n)}$ contains at least 3 edges.

(2) $m \equiv 3 \pmod{10}$. Then $5|(|E(K'_m)| - 3)$. By Theorem 1.2, K'_m can be packed with 5-cycles with leave C_3. Therefore, $K_{m(n)}$ can be packed with 5-cycles with leave $C_{3(n)}$. $C_{3(n)}$ is isomorphic to $K_{n,n,n}$. By Theorem 1.1 and Lemma 2.7, if $n \equiv 0 \pmod 5$, then $K_{n,n,n}$ can be decomposed into 5-cycles. Thus $K_{m(n)}$ can be decomposed into 5-cycles. If $n \equiv 1$ or $4 \pmod 5$, then $K_{n,n,n}$ can be packed with 5-cycles with leave C_3. Thus $K_{m(n)}$ can be packed with 5-cycles with leave C_3. If $n \equiv 2$ or $3 \pmod 5$, then $K_{n,n,n}$ can be packed with 5-cycles with leave $C_3 \cup C_4$. Thus $K_{m(n)}$ can be packed with 5-cycles with leave $C_3 \cup C_4$.

(3) $m \equiv 7$ or $9 \pmod{10}$. Then $5|(|E(K'_m)| - 6)$. By Theorem 1.2, K'_m can be packed with 5-cycles with leave $2C_3$. As noted earlier $2C_3$ is the union of two C_3 with one vertex in common. By Theorem 1.1 and Lemma 2.7, if $n \equiv 0 \pmod 5$, then $K_{m(n)}$ can be decomposed into 5-cycles. If $n \equiv 1$ or $4 \pmod 5$, then $K_{m(n)}$ can be packed with 5-cycles with leave two 3-cycles. If $n \equiv 2$ or $3 \pmod 5$, then $K_{n,n,n}$ can be packed with 5-cycles with leave $C_3 \cup C_4$. From Fig. 1, the two $C_3 \cup C_4$ can be decomposed into two 5-cycles and one C_4. Thus $K_{m(n)}$ can be packed with 5-cycles with leave a C_4. □

3 Concluding remark

A k-cycle covering of G is a triple $(V(G), \mathcal{C}, P)$, where $P \subseteq E(G)$ is called the padding, and $\mathcal{C}$ is a collection of k-cycles that partition $E(G) + P$. If $|P|$ is the minimum, then $(V(G), \mathcal{C}, P)$ is called a minimum covering of G with k-cycles. Therefore, a k-cycle system of G is a k-cycle covering of G with padding $P = \emptyset$. A bit of reflection, if we have a k-cycle packing of G with leave L, then P is a padding provided that $L \cup P$ can be decomposed into k-cycles. Therefore, by the result obtained in this paper we can also find a minimum 5-cycle covering of $K_{m(n)}$ for odd m without too much difficulty.

As to the 5-cycle packing of $K_{m(n)}$ when m is even, due to the complexity of leaves, it is much more complicated. We wish the problem can be solved in the near future.

Acknowledgement The authors wish to extend their gratitude to the referees for their helpful comments in revising this paper.

References

Alspach B, Gavlas H (2001) Cycle decompositions of K_n and $K_n - I$. J Comb Theory Ser B 81:77–99
Alspach B, Marshall S (1994) Even cycle decompositions of complete graphs minus a 1-factor. J Comb Des 2:441–458
Billington EJ, Fu H-L, Rodger CA (2001) Packing complete multipartite graphs with 4-cycles. J Comb Des 9:107–127
Billington EJ, Fu H-L, Rodger CA (2005) Packing λ-fold complete multipartite graphs with 4-cycles. Graphs Comb 21:169–185
Cavenagh NJ, Billington EJ (2000b) On decomposing complete tripartite graphs into 5-cycles. Australas J Comb 22:41–62
Lindner CC, Rodger CA (1992) Decomposition into cycle II: cycle systems. In: Dinitz JH, Stinson DR (eds) Contemporary design theory: a collection of surveys. Wiley, New York, pp 325–369
Rosa A, Znám S (1994) Packing pentagons into complete graphs: how clumsy can you get. Discrete Math 128:305–316
Wilson RM (1974) Some partitions of all triples into Steiner triple systems. In: Lecture notes in mathematics, vol 411. Springer, Berlin, pp 267–277