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GENERALIZED HOMOMORPHISM GRAPH FUNCTIONS*

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A real-valued function f defined on the set of all graphs, $\mathscr{G}$, such that $f(G \times H) = f(G)f(H)$ for all G, $H \in \mathscr{G}$ is called multiplicative; and $f(G) \leq f(H)$ whenever G is a subgraph of H is called increasing. The classification of multiplicative increasing graph functions is still open. Up to now, there are a lot of known multiplicative increasing graph functions. In this paper, we introduce a new class of multiplicative increasing graph functions, namely, $\varphi_{G,S}$ for all $G \in \mathscr{G}$ and $\emptyset \neq S \subseteq V(G)$, defined to be the number of all possible homomorphic images of S for the homomorphism from G into H. Several properties of additive multiplicative increasing graph functions are also discussed in this paper.

1. Introduction and definition

$G = (V, E)$ is called a *graph* if V is a finite set and E is a subset of $\{(a, b) \mid a \neq b,\ (a, b)$ is an unordered pair of $V\}$. We say $V = V(G)$ is the *vertex set* of G, $E = E(G)$ is the *edge set* of G.

Let $G = (X, E)$, $H = (Y, F)$ be two graphs. The *sum* of G and H is the graph $G + H = (W, B)$ with $W = X_1 \cup Y_1$, $B = E_1 \cup F_1$, where $G_1 = (X_1, E_1) \cong G$, $H_1 = (Y_1, F_1) \cong H$ and $X_1 \cap Y_1 = \emptyset$; the *weak product* of G and H is the graph $G \times H = (Z, K)$, where $Z = X \times Y$, the Cartesian product of X and Y, and $K = \{((x_1, y), (x_2, y_2)) \mid (x_1, x_2) \in E \text{ and } (y_1, y_2) \in F\}$. We let kG denote $G + G + \cdots + G$ (k times) and let G^k denote $G \times G \times \cdots \times G$ (k times). A real-value function f defined on the set of all graphs, $\mathscr{G}$, is *increasing* if $f(G) \leq f(H)$ whenever G is a subgraph of H; f is *multiplicative* if $f(G \times H) = f(G) \times f(H)$ for any G, $H \in \mathscr{G}$; and f is *additive* if $f(G + H) = f(G) + f(H)$ for any G, $H \in \mathscr{G}$. We use MI to denote the set of all multiplicative increasing graph functions and use AMI to denote the set of all additive multiplicative increasing graph functions. The classification of all multiplicative increasing graph functions is still an interesting open problem [1, 2, 5, 7]. In this paper, we first review some previous work, then introduce a class of multiplicative increasing graph functions which has not been explored before.

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2. Previous work

2.1. *Homomorphism functions*

Let $G=(X,E)$, $H=(Y,F)$ be two graphs. A map $\psi: X\rightarrow Y$ is called a *homomorphism* if it satisfies $(x_1,x_2)\in E$ implies $(\psi(x_1),\psi(x_2))\in F$. For a fixed graph G, we can define h_G from $\mathscr{G}$ into R such that $h_G(H)$ equals the number of homomorphisms from G into H. It is easy to see that h_G is an element of MI for every G in $\mathscr{G}$. Since MI is closed under taking the positive power, finite product and pointwise convergence, the following functions are element in MI:

(1) $h_G^{\alpha}\quad \alpha\geqslant 0,\ G\in\mathscr{G}$.

(2) $\prod_{i=1}^{k} h_{G_i}^{\alpha_i}\quad \alpha_i\geqslant 0,\ G_i\in\mathscr{G}$.

(3) $\lim_{m\rightarrow\infty} f_m$ where f_m is of type (1) or (2).

Lovász [7] observed these facts and he asked whether nonzero multiplicative increasing graph functions are of these forms. However, the conjecture is not true [1, 2]. Let $S\subseteq$ MI. We use $\langle S\rangle$ to denote the set of all functions obtained by taking the positive power, finite product and pointwise convergence from elements of S. It is easy to see that $\langle\{h_G \mid G\in\mathscr{G}\}\rangle=\langle\{h_G \mid G \text{ is connected}\}\rangle$. Moreover, h_G is additive if G is connected, and $h_G=h_H$ if and only if G is isomorphic to H.

2.2. *Generalized homomorphism functions*

For any graph H and any integer $m\geqslant 1$, let H_m be the induced subgraph of H such that $x\in V(H_m)$ if and only if x is in an m-clique of H. For any graph function f, we can define another graph function f_m by $f_m(H)=f(H_m)$ for any graph H. In [5], we have the following theorem.

Theorem 2.1. *If f is additive (respectively, multiplicative, increasing), then f_m is also additive (respectively, multiplicative, increasing).*

Thus $(h_G)_m\in$ MI for all $G\in\mathscr{G}$ and $m\geqslant 1$, and $(h_G)_m\in$ AMI if G is connected. In [1], we have $\langle\{h_G \mid G\in\mathscr{G}\}\rangle\subset\langle\{(h_G)_m \mid G \text{ is connected and } m\geqslant 1\}\rangle$. (The notation "$A\subset B$" means A is a proper subset of B.) However, not all multiplicative increasing functions are generated by $\{(h_G)_m \mid G\in\mathscr{G},\ m\geqslant 1\}$. In [1], we have the following functions.

2.3. *The δ function*

A *bipartite graph* is a graph whose vertex set $V(G)$ can be partitioned into two subsets A and B such that every edge of G joins A with B and vice versa. If G is connected bipartite, such a partition is unique; we say G is of (r, s) *type* if $|A| = r$ and $|B| = s$. For an arbitrary bipartite graph G with connected components $C_1, C_2, \ldots, C_m$, where each C_i is bipartite, we say G is $\sum_{i=1}^{n}(r_i, s_i)$ *type* if C_i is of (r_i, s_i) type for every i. Let θ be a function defined on the set of bipartite graph which is defined as $\theta(G) = 2\ (\sum_{i=1}^{n}(r_i, s_i)^{\frac{1}{2}})$ where G is of $\sum_{i=1}^{n}(r_i, s_i)$ type. Then we can define $\delta : \mathscr{G} \to R$ by $\delta(G) = \frac{1}{2}\ \theta(G \times K_{1,1})$. In [1], we know $\delta \in \mathrm{AMI}$. Hence $\delta_m \in \delta = \delta_1 = \delta_2$ and $\delta_m = (h_{K_1})_m$ for every $m \geq 2$. Thus, only one AMI function added. However, in the following, we can find more AMI functions.

2.4. *The capacity functions*

For a fixed graph G, we can define the *capacity function* for G, P_G, from $\mathscr{G}$ into R as $P_G(H) = \lim_{n\to\infty} [\gamma_G(H^n)]^{1/n}$ where $\gamma_G(H)$ is the maximum number of disjoint G's in H. Not all capacity functions are AMI [3, 4, 5]. In [5], Hsu et al. proved that the capacity function for a primary uniform graph is AMI. The term of primary and uniform is defined below.

Definition. We say G is *primary* if for any homomorphic image G' of G we have $P_{G'} \leq P_G$. Let $D = \{(a_1, a_2, \ldots, a_v) \mid 0 \leq a_i \leq 1, \sum_{i=1}^{v} a_i = 1\}$. Let $\mathscr{H} : D \to R$ be a function defined by

$$\mathscr{H}(\vec{a}) = \prod_{i=1}^{v} a_i^{-a_i} \quad \text{where } \vec{a} = (a_1, a_2, \ldots, a_v) \in D.$$

Let G, H be graphs with $V(G) = \{x_1, x_2, \ldots, x_u\}$ and $V(H) = \{y_1, y_2, \ldots, y_v\}$. Let m be a positive integer and $\vec{z} = (z_1, z_2, \ldots, z_m)$ be a vertex in H^m. We call $\vec{a} = (a_1, a_2, \ldots, a_v) \in D$ with $a_i = |\{j \mid z_j = y_i, 1 \leq j \leq m\}|/m$ the *distribution* of $\vec{z}$. For any graph H, we can define a u-ary relation $R_G(H)$ on D such that $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u) \in R_G(H)$ with $\vec{a}_i \in D$ if and only if either (i) there exists a positive integer m such that in H^m we can find $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u \in V(H_m)$ with the distribution $\vec{y}_i$ to be $\vec{a}_i$ for every i and the induced subgraph of $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u\}$ in H^m containing a subgraph isomorphic to G with $\vec{y}_i$ corresponding to x_i for every i, or (ii) there exists a sequence $\{(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u})\}_{i=1}^{\infty}$ in $R_G(H)$ of type (i) such that $\lim_{i\to\infty} (\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u}) = (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$.

We say $\vec{a} \in I_G(G)$ is of *type* (i) if its corresponding vector in $R_G(H)$ is of type (i). A graph G with u vertices is called *uniform* if for any graph H, $\sum_{i=1}^{n} \vec{a}/u$ is of type (i) in $I_G(H)$ whenever $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$ is of type (i) in $R_G(H)$.

In [4, 5], it was proved that $P_G(H) = \max_{\vec{a} \in I_G(H)} \mathscr{H}(\vec{a})$ for primary uniform graphs. thus we can calculate such capacity functions using the Lagrange multiplier. We note that $P_{K_1} = h_{K_1}$ and the capacity function for primary uniform graphs can be viewed as a lower bound for AMI graph functions.

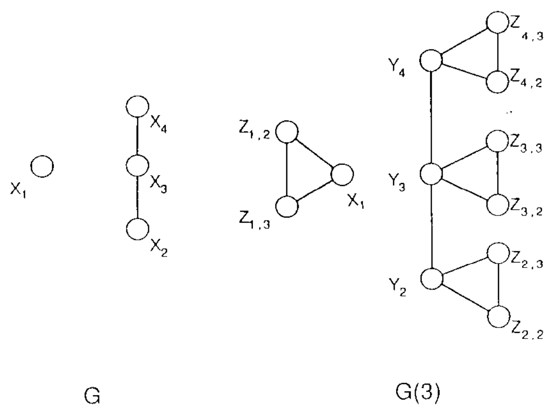

Fig. 1.

3. The $\varphi_{G,S}$ functions

In this section, we are going to present a class of MI functions which is in fact a generalization of the functions introduced in Sections 2.1 and 2.2.

Let G be a graph and $\emptyset \neq S = \{s_1, s_2, \ldots, s_k\} \subseteq V(G)$. We can define $\varphi_{G,S}: \mathscr{G} \to R$ by $\varphi_{G,S}(H) = |\{(f(s_1), f(s_2), \ldots, f(s_k)) \mid f$ is a homomorphism from G into $H\}|$. It is easy to get the following theorem.

Theorem 3.1. (1) *$\varphi_{G,S}$ is* MI.

(2) *$\varphi_{G,S}$ is* AMI *if G is connected.*

(3) *If $\emptyset \neq S_1 \subseteq V(G)$ and $\emptyset \neq S_2 \subseteq V(H)$, then*

$$\varphi_{G+H,S_1 \cup S_2} = \varphi_{G,S_1} \times \varphi_{H,S_2}.$$

(4) $\varphi_{G,V(G)} = h_G$.

Let G be a graph with $V(G) = \{x_1, x_2, \ldots, x_u\}$. Let m be a positive integer. Construct a graph $G(m)$ with $V(G(m)) = \{y_1, y_2, \ldots, y_u, z_{1,2}, z_{1,3}, \ldots, z_{1,m}, z_{2,2}, z_{2,3}, \ldots, z_{2,m}, \ldots, z_{u,2}, z_{u,3}, \ldots, z_{u,m}\}$ such that the induced subgraph of $S' = \{y_1, y_2, \ldots, y_u\}$ is isomorphic to G and the induced subgraph of $\{y_i, z_{i,2}, \ldots, z_{i,m}\}$ is isomorphic to K_m for every i. See *Fig. 1* for the case $m = 3$. Then $\varphi_{G(m),S'} = (\varphi_{G,S})_m$. Applying Theorem 3.1 (4), we hve $\{(h_G)_m \mid G \in \mathscr{G}, m \in N\} \subseteq \{\varphi_{G,S} \mid G \in \mathscr{G}, \emptyset \neq S \subseteq V(G)\}$. Therefore the $\varphi_{G,S}$ functions are a generalization of $(h_H)_m$. However, $\langle\{(h_H)_m \mid H \in \mathscr{G}, m \in N\}\rangle \neq \langle\{\varphi_{G,S} \mid G \in \mathscr{G}, \emptyset \neq S \subseteq V(G)\}\rangle$. We will prove this fact in the following section.

4. Properties of AMI functions

In order to prove $\langle\{(h_H)_m \mid H \in \mathscr{G}, m \in N\}\rangle \subset \langle\{\varphi_{G,S} \mid G \in \mathscr{G}, \emptyset \neq S \subseteq V(G)\}\rangle$, we need the following results.

Lemma 4.1. *If $f_1, f_2, \ldots, f_k$ are nonzero* AMI *and if $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_k^{\alpha_k}$ is* AMI, *then $\sum_{i=1}^{k} \alpha_i = 1$.*

Proof. Since

$$\begin{aligned} 2f(G) = f(2G) &= f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_k^{\alpha_k}(2G) \\ &= (2f(G))^{\sum_{i=1}^{k} \alpha_i}, \end{aligned}$$

we have $\sum_{i=1}^{k} \alpha_i = 1$. □

Lemma 4.2. *Let f_1 and f_2 be nonzero* AMI *and $f = f_1^{\alpha_1} f_2^{\alpha_2}$ with $\alpha_1 + \alpha_2 = 1$ and $\alpha_i \geq 0$. Then f cannot be* AMI *except $f = f_1$ or $f = f_2$.*

Proof. Let G and H be any two graphs such that $f_1(G) \neq 0$ and $f_2(H) \neq 0$. Let $f_1(G) = u$, $f_1(H) = v$, $f_2(G) = x$ and $f_2(H) = y$. We have

$$f_1^{\alpha_1} f_2^{\alpha_2}(G + H) = f_1^{\alpha_1} f_2^{\alpha_2}(G) + f_1^{\alpha_1} f_2^{\alpha_2}(H) = u^{\alpha_1} v^{\alpha_2} + x^{\alpha_1} y^{\alpha_2}.$$

But

$$\begin{aligned} f_1^{\alpha_1} f_2^{\alpha_2}(G + H) &= f_1^{\alpha_1}(G + H) f_2^{\alpha_2}(G + H) \\ &= (f_1(G) + f_1(H))^{\alpha_1} (f_2(G) + f_2(H))^{\alpha_2} = (u + v)^{\alpha_1} (x + y)^{\alpha_2}. \end{aligned}$$

By Hölder's inequality, we have

$$u^{\alpha_1} v^{\alpha_2} + z^{\alpha_1} y^{\alpha_2} \leq (u + v)^{\alpha_1} (x + y)^{\alpha_2},$$

with the equality only for $u = x$ and $v = y$.

Hence f is AMI only for $f = f_1$ or $f = f_2$. □

We can generalize Lemma 4.2 to the following theorem.

Theorem 4.3. *Let $f_1, f_2, \ldots, f_k$ be nonzero* AMI *and $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_k^{\alpha_k}$ with $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$. Then f cannot be* AMI *except $f = f_i$ for some i.*

Let f be an AMI function expressible as $f = \lim_{m \to \infty} g_m$ with $g_m = f_1^{\alpha_{m,1}} f_2^{\alpha_{m,2}} \cdots f_{m(f)}^{\alpha_{m,m(f)}}$. Then $2f(G) = f(2G) = \lim_{m \to \infty} g_m(2G) = \lim_{m \to \infty} f_1^{\alpha_{m,1}} f_2^{\alpha_{m,2}} \cdots f_{m(f)}^{\alpha_{m,m(f)}}(2G) = (2(f(G))^{\lim_{m \to \infty} \sum_k \alpha_{m,k}}$.

Hence $\lim_{m \to \infty} \sum_k \alpha_{m,k} = 1$. Thus we have the following theorem.

Theorem 4.4. *Let $\{f, f_1, f_2, \ldots\}$ be a set of nonzero* AMI *functions and f be expressed as $f = \lim_{m \to \infty} g_m$ with $g_m = f_1^{\alpha_{m,1}} f_2^{\alpha_{m,2}} \cdots f_{m(f)}^{\alpha_{m,m(f)}}$. Then f can be expressed as $f = \lim_{m \to \infty} k_m$ with $k_m = f_1^{\beta_{m,1}} f_2^{\beta_{m,2}} \cdots f_{m(f)}^{\beta_{m,m(f)}}$ such that $\sum_k \beta_{m,k} = 1$ for every m.*

Corollary 4.5. *Let $F = \{f_1, f_2, \ldots, f_k\}$ be a set of nonzero* AMI *functions. Then f cannot generate any other* AMI *functions except $f_1, f_2, \ldots, f_k$.*

Proof. Let $f \in \langle F \rangle$ be an AMI function other than $f_1, f_2, \ldots, f_k$. By Theorem 4.3, we know that f cannot be the form $f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_k^{\alpha_k}$. Thus f must be of the form $f = \lim_{m\to\infty} g_m$ with $g_m = f_1^{\alpha_{m,1}} f_2^{\alpha_{m,2}} \cdots f_k^{\alpha_{m,k}}$ and $\sum_j \alpha_{m,j} = 1$. Since $\alpha_{m,i} \in [0, 1]$ for every m, i, we have an accumulation point β_i for $\{\alpha_{m,i}\}_{m=1}^{\infty}$ for every i. Then $f = f_1^{\beta_1} f_2^{\beta_2} \cdots f_k^{\beta_k}$ which contradicts to Theorem 4.3. $\square$

Thus it is impossible to generate other AMI functions by a finite number of AMI functions. However, it is possible to generate another AMI function by an infinite number of AMI functions. We have the following example.

Example. The function $\mathcal{Q}: \mathcal{G} \to R$ is defined by $\mathcal{Q}(G) = |\{v \in V(G) \mid v$ is incident with a homomorphic image of an odd cycle of $G\}|$. Obviously, $\mathcal{Q}$ is an AMI. Let x_{2n+1} be any vertex in the odd cycle C_{2n+1}. Then it is easy to check that $\mathcal{Q} = \lim_{n\to\infty} \varphi_{C_{2n+1},\{x_{2n+1}\}}$.

Now, we are going to prove that $\langle \{(h_H)_m \mid H \in \mathcal{G}, m \in N\} \rangle \neq \langle \{\varphi_{G,S} \mid G \in \mathcal{G}, \emptyset \neq S \subseteq V(G)\}$. Let W_5 be the 5-wheel graph with its center vertex o. Since $\mathrm{Aut}(W_5) = D_5$, we have either $h_H(W_5) = 0$, $h_H(W_5) = 6$ (if $H \cong K_1$) or $h_H(W_5) \geq 10$ for other cases. Let f be any AMI function in $\langle \{(h_H)_m \mid H \in \mathcal{G}, m \in N\} \rangle$. By Theorem 4.3 and Theorem 4.4, we have either $f(W_5) = 0$ or $f(W_5) \geq 6$. But $\varphi_{W_5,\{o\}}$ is an AMI function with $\varphi_{W_5,\{o\}}(W_5) = 1$. Hence $\varphi_{W_5,\{o\}} \notin \langle \{(h_H)_m \mid H \in \mathcal{G}, m \in N\} \rangle$.

Note that $G \subseteq G^2$ for any graph G. Any multiplicative increasing function f such that $f(G) \neq 0$ must satisfy $f^2(G) = f(G^2) \geq f(G)$. Hence $f(G) \geq 1$. Thus $\varphi_{W_5,\{o\}}$ meets this bound.

It would be interesting to know whether $\varphi_{W_5,\{o\}}$ can be generated by all functions introduced in Section 2. However, it is a difficult job. We do not have the property of "change base" as in matroid theory. It is interesting to point out that $\delta(W_5) = 6$ and $P_{W_5}(W_5) = 1$. But P_{W_5} is not AMI [3].

Since $\varphi_{G,S}(K_{1,2})$ is either 0 or greater than 3, any AMI function f generated by $\{\varphi_{G,S} \mid G \in \mathcal{G}, \emptyset \neq S \subseteq V(G)\}$, either $f(K_{1,2}) \geq 3$ or $f(K_{1,2}) = 0$. Since $\delta(K_{1,2}) = P_{K_2}(K_{1,2}) = 2\sqrt{2}$, we still need δ and P_{K_2}.

5. The strong product

There is another product defined on the set of all graphs. Let $G = (X, E)$, $H = (Y, F)$ be two graphs. The *strong product* of G and H is the graph $G \cdot H = (Z, K)$ where $Z = X \times Y$, the Cartesian product of X and Y and $K = \{((x_1, y_1), (x_2, y_2)) \mid ((x_1, x_2) \in E$ and $(y_1, y_2) \in F)$ or $((x_1 = x_2$ and $(y_1, y_2) \in F)$ or $((x_1 = x_2) \in \mathrm{E}$ and $y_1 = y_2)\}$. With this strong product, the terminology of *strongly multiplicative increasing graph function* (SMI) and *strongly additive multiplicative increasing graph function* (SAMI) can be similarly defined.

Let $G=(X,E)$, $H=(Y,F)$ be two graphs. A map $\phi: X\to Y$ is called a *strongly homomorphism* from G into H if $(x_1,x_2)\in E$ implies $(\phi(x_1),\phi(x_2))\in F$ or $\phi(x_1)=\phi(x_2)$. For a fixed graph, we can define $\hbar_G$ as a function from $\mathscr{G}$ into R such that $\hbar_G(H)$ equals the number of strongly homomorphisms from G into H. It is easy to see that $\hbar_G$ is SMI for any $G\in\mathscr{G}$ and that $\hbar_G$ is SAMI if G is connected.

Similar to the weak product, Lovász asked whether nonzero strongly multiplicative increasing functions are generated by $\hbar_G$. Again, the conjecture is false. In [2], we showed that the function $\mathscr{Q}:\mathscr{G}\to R$ defined, for any $G\in\mathscr{G}$, as the size of the largest connected component in G is a counterexample.

Let G be a graph and $\emptyset\neq S\subseteq V(G)$. We can similarly define $\Phi_{G,S}:\mathscr{G}\to R$ by $\Phi_{G,S}(H)=|\{(f(s_1),f(s_2),\ldots,f(s_k))\mid f$ is a strongly homomorphism from G into $H\}|$. Let T_n be the graph obtained from the star graph S_n ($\cong K_{1,n}$) by replacing each edge with a path of length n. Let P_n denote the set of pendant vertices of T_n.

Theorem 5.1. $\mathscr{Q}=\lim_{n\to\infty}(\Phi_{T_n,P_n})^{1/n}$. *Thus* $\langle\{\hbar_G\mid G\in\mathscr{G}\}\rangle$ *is a proper subset of* $\langle\{\Phi_{G,S}\in G\in\mathscr{G},\emptyset\neq S\subseteq V(G)\}\rangle$.

Proof. Let G be a graph with its connected component $C_1,C_2,\ldots,C_k$ assume $|C_i|=r_i$ for all i and $r_1\geqslant r_2\geqslant\cdots\geqslant r_k$. Thus $\mathscr{Q}(G)=r_1$. For all $n\geqslant r_1$, we have $\Phi_{T_n,P_n}(G)=\sum_{i=1}^k r_i^n$. But $\lim_{n\to\infty}(\sum_{i=1}^k r_i^n)^{1/n}=r_1$. We have $\mathscr{Q}=\lim_{n\to\infty}(\Phi_{T_n,P_n})^{1/n}$. $\square$

We note that Lemma 4.1, Lemma 4.2, Theorem 4.3, Theorem 4.4 and Corollary 4.5 also hold for the strong product version.

6. Conjecture

We would like to pose the following conjectures:

(1) The set of multiplicative increasing graph functions (for both weak product and strong product version) is generated by additive multiplicative increasing graph functions.

(2) The set of strongly additive multiplicative increasing graph functions is $\{\Phi_{G,S}\mid G\in\mathscr{G},\emptyset\neq S\subseteq V(G)\ G$ is connected$\}$.

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