國立交通大學

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博 士 論 文

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摘要

物理系統自然形成非線性，因此，所有的控制系統都是具有某些程度的非線性。過去超過二十年時間，模糊技術已經廣泛地成功被利用在非線性系統模型建立與控制器設計。近十年來，T-S 模糊模型在處理複雜的非線性系統是一個廣為流傳且使用方便的工具。同樣地，對於非線性系統的模糊迴授控制設計問題已經廣泛地被研究藉由使用 T-S 模型，其中用簡單的局部線性模型被組合去描述非線性系統的全域行為。實際上，不可避免的不確定性也許會以一種非常複雜的方式進入到一個非線性系統模型。此不確定性也許包含模型誤差、參數變化、外部干擾和模糊近似誤差。在如此的一個情況下，模糊迴授控制設計方法也許不再運作良好。

在本論文中，我們首先提出強健適應滑差控制（包含滑差控制和適應控制）應用於具有範數界限外部干擾的 T-S 模糊模式，同時放寬每一個正規的局部系統模式擁有相同輸入通道的限制假設，這個限制假設是傳統可變結構模糊控制設計方法所需要的。然後，提出具有非相配參數變動和外部擾動的 T-S 模糊模式之強健適應滑差控制。此外，針對具有非相配參數變動和外部擾動的 T-S 模糊時間延遲模式，其強健適應滑差控制亦被提出。最後，利用一些例子來驗證本論文所提出方法的有效性和可行性。

關鍵字：T-S 模糊模式，範數界限變動，參數變動，外部擾動，滑差控制，適應控制。

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ABSTRACT

Physical systems are inherently nonlinear. Thus, all control systems are nonlinear to a certain extent. Over the past two decades, fuzzy techniques have been widely and successfully exploited in nonlinear system modeling and control. In last ten years, the Takagi-Sugeno (T-S) fuzzy model is a popular and convenient tool for handling complex nonlinear systems. Correspondingly, the fuzzy feedback control design problem for a nonlinear system has been studied extensively by using the T-S model where simple local linear models are combined to describe the global behavior of the nonlinear system. In practice, the inevitable uncertainties may enter a nonlinear system model in a very complicated way. The uncertainty may include modeling errors, parameter variations, external disturbances, and fuzzy approximation errors. In such a situation, the fuzzy feedback control design methods may not work well anymore.

In this dissertation, firstly, we propose two kinds of LMI-based robust adaptive sliding control, including a robust sliding control method and a robust adaptive control method, for uncertain Takagi-Sugeno fuzzy models with norm-bounded uncertainties, and meantime relax the restrictive assumption that each nominal local system model shares the same input channel, which is required in the traditional VSS-based fuzzy control design methods. Then, two kinds of LMI-based robust adaptive sliding control are developed for uncertain T-S fuzzy models which include mismatched parameter uncertainties and external disturbances. Moreover, two kinds of LMI-based robust adaptive sliding control are proposed for the uncertain T-S fuzzy time-delay model which includes mismatched parameter uncertainties in the state matrix and norm-bounded external disturbances. Finally, some examples are used to illustrate the effectiveness and usefulness of the proposed methods in this dissertation.

Keywords: T-S fuzzy models, norm-bounded uncertainties, parameter uncertainties, external disturbances, sliding control, adaptive control

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Symbol List

$\psi(t), x(t), u(t)$	initial condition, state, and control input
$A_i, A_{\tau i}, B_i$	state matrices, delayed state matrices, and input matrices
$\theta_j, \mu_{ij}, s, r, \beta_i(\theta)$	premise variables, fuzzy sets, number of premise variables, number of IF-THEN rules, and membership function
$h(t,x), \hat{h}(t,x), \rho_k, \hat{\rho}_k$	unknown function, estimate of $h(t,x)$, constants, and estimate of ρ_k
S, σ, E_g	linear sliding surface parameter matrix, linear sliding surface, and Lyapunov function
$\kappa, \Lambda, \alpha_{A_i}$	$\kappa = \lambda_{\min}(B^T B)$, any full rank matrix such that $B^T \Lambda = 0$ and $\Lambda^T \Lambda = I$, and known constant
$Y, c_0, c_1, c_2, \delta, \eta$	decision variables
$T_i, \Pi_i(t)$	constant matrix and time-varying matrix
$d_\tau(t), \tau, d_m$	time-delay unknown function and known constants.
$x_d(t), \Delta A_i, \Delta A_{\tau i}(t), h_i(t, x, x_d, u)$	delayed state, parameter uncertainties in A_i, parameter uncertainties in $A_{\tau i}$, and external disturbances
$\xi_i(t), \phi_i$	known function and known constant
ϕ_m, ε_i	known constants
K, X_i, Z_i	decision variables
$\rho_{dk}, \delta_{dk}, \hat{\rho}_{dk}, \hat{\delta}_{dk}, t_{sampling}$	constants, estimate of ρ_{dk}, δ_{dk}, and sampling time

Chapter 1

Introduction

Up to now, fuzzy systems have been applied with great success to numerous real world applications, such as Penicillin-G conversion [1], prediction of river water flow [2], and many other examples in ecological systems and biomedical field [1], [3]. In the meantime, numerous publications have been reported in providing theoretical support. Various methodologies have been proposed for analysis, modeling, design, control and monitor of fuzzy systems. Fuzzy ideas are useful for modeling complex nonlinear systems in which, due to the complexity or the uncertainty, classical tools are unsuccessful. The truth model is too complicated for use in the controller design. Thus, we need to develop a simplified model that can be used to design a controller. Such a simplified model is labeled by Friedland [4] as the design model. The design model should capture the essential features of the process. In practice, the inevitable uncertainties may enter a nonlinear system model in a very complicated way. The uncertainty may include modeling errors, parameter variations, external disturbances, and fuzzy approximation errors. In such a situation, some fuzzy feedback control design methods may not work well anymore. To deal with the problem, this dissertation provides two kinds of LMI-based robust adaptive sliding control, including a sliding control method and an adaptive control method.

The introduction of this dissertation is introduced in this chapter. The motivation of this dissertation is discussed in Section 1.1. In Section 1.2, related works are introduced. The approach of this dissertation is described in Section 1.3. In Section 1.4, the organization of the dissertation is introduced.

1.1 Motivation

The first step in the controller design procedure is the construction of a “truth model” of the dynamics of the process to be controlled. The truth model is a simulation model that includes all the relevant characteristics of the process. The truth model is too complicated for use in the controller design. Thus, we need to develop a simplified model that can be used to design a controller. Such a simplified model is labeled by Friedland [4] as the design model. The design model should capture the essential features of the process. In many situations, there may be human experts who can provide a linguistic description of the process in terms of IF-THEN rules. Combining the available mathematical description of the process with its linguistic description results in a fuzzy system model. Such an approach to modeling was proposed by Takagi and Sugeno [5] and further developed by Sugeno and Kang [6]. This type of model is called the Takagi-Sugeno (T-S) or Takagi-Sugeno-Kang (TSK) fuzzy model.

T-S fuzzy models are popular and well used tools in recent years. A general T-S fuzzy model employs an affine fuzzy model with a constant in the consequence [5]. It is known that smooth nonlinear dynamic systems can be approximated by affine T-S fuzzy models [7,8]. Most recent developments are based on T-S models with linear rule consequences (here and after, such models are generally called T-S fuzzy models). The main feature of T-S fuzzy models is to represent the nonlinear dynamics by simple (usually linear) models according to the so-called fuzzy rules and then to blend all the simple models into an overall single model through nonlinear fuzzy membership functions. Each simple model is called a local model or a sub-model. The output of the overall fuzzy model is calculated as a gradual activation of the local models by using proper defuzzification schemes [5], [9,10]. It has been proved that T-S fuzzy models can approximate any smooth nonlinear dynamic systems [11,12]. Based on the sector nonlinearity concept [12], the uncertain nonlinear system can be systematically constructed by T–S fuzzy models.

In practice, the inevitable uncertainties may enter a nonlinear system model in a very complicated way. The uncertainty may include modeling errors, parameter variations, external disturbances, and fuzzy approximation errors. In such a situation, some fuzzy feedback control design methods may not work well anymore.

On the other hand, time-delay is often encountered in various industrial systems, such as the turbojet engine, electrical networks, nuclear reactor, rolling mill, and chemical process, etc. Recently, the feedback stabilization problem for uncertain time-delay systems is also a problem of interest because the existence of a delay is frequently a source of poor system performance or instability. However, they are sensitive to the uncertainty, which directly affects the control systems.

1.2 Related Works

The history of the so-called parallel distributed compensation (PDC) began with a model-based design procedure proposed by [13]. However, the stability of the control systems was not addressed in the design procedure. The design procedure was improved and the stability of the control systems was analyzed in [14]. The design procedure is named “parallel distributed compensation” in [15]. The PDC [14-16] offers a procedure to design a fuzzy controller from a given T-S fuzzy model. It should be noted that many real systems, for example, mechanical systems and chaotic systems, can be and have been represented by T-S fuzzy models.

It is well-known that time-delay is a common and complex phenomenon in many industrial and engineering systems, such as communication systems, rolling mill systems and transportation systems. Since 2000, the T-S fuzzy model has been extended to undertake analysis and control problems for nonlinear systems with time-delay. More recently, great progress has been made in the analysis and synthesis of T-S fuzzy systems with time-delay, such as stability and stabilization based on the parallel distributed compensation (PDC) method [17-19].

On H_∞ control, the problem of static output feedback control was developed in [20,21]. Robust stability and guaranteed cost control were treated in [22]. Chen and Liu [21] proposed a robust H_∞ control by using the Lyapunov-Krasovskii function (LKF)

$$V(x(t)) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Q\,x(s)ds + \int_{-\bar{\tau}}^{0}\int_{t+\eta}^{t} \dot{x}^T(s)R\,\dot{x}(s)dsd\eta$$

and model transformation technique which introduced conservatism.

1.3 Approach

Over the past two decades, fuzzy techniques have been widely and successfully exploited in nonlinear system modeling and control. The Takagi-Sugeno (T-S) model [5] is a popular and convenient tool for handling complex nonlinear systems. Correspondingly, the fuzzy feedback control design problem for a nonlinear system has been studied extensively by using T-S model where simple local linear models are combined to describe the global behavior of the nonlinear system [23-29]. In practice, the inevitable uncertainties may enter a nonlinear system model in a very complicated way. The uncertainty may include modeling errors, parameter variations, external disturbances, and fuzzy approximation errors. In such a situation, the fuzzy feedback control design methods of [23-29] may not work well anymore. To deal with the problem, some authors [30,31] have exploited the variable structure system (VSS) theory which has provided an effective means to design robust controllers for uncertain nonlinear systems where the uncertainties are bounded by known scalar valued functions.

In the VSS, the control design of the plant is intentionally changed by using a viable high-speed switching feedback control to obtain a desired system response, from which the VSS arises in finite time. The VSS drives the trajectory of the system onto a specified and user-design surface, which is called the sliding surface or the switching surface, and maintains the trajectory on this sliding surface for all subsequent time. The closed-loop response obtained from using a VSS control law comprises two distinct modes. The first is the reaching mode, also called nonsliding mode, in which the trajectory starting from anywhere on the state space is being driven towards the switching surface. The second is the sliding mode in which the trajectory asymptotically tends to the origin. The central feature of the VSS is the sliding mode on the sliding surface on which the system remains insensitive to internal parameter variations and external disturbance. In sliding mode, the order of the system dynamics is reduced. This enables simplification and decoupling design procedure [32-35]. However, all the VSS-based fuzzy control system design methods are based on the assumption that each nominal local system model shares the same input channel. This assumption is very restrictive and inadequate to modeling uncertainty/nonlinearity in various mechanical systems such as an inverted pendulum on a cart.

Some authors [36-40] have relaxed the assumption and they have proposed adaptive laws to estimate the upper norm bounds. However, the previous VSC-based fuzzy control methods have considered the problem of adaptive control design and stability analysis for uncertain T-S fuzzy models where the input matrices of the local system models satisfy the assumption that each nominal local system shares the same input channel. It is practically difficult to satisfy this assumption.

On the other hand, time-delay is often encountered in various industrial systems, such as the turbojet engine, electrical networks, nuclear reactor, rolling mill, and chemical process, etc. Recently, the feedback stabilization problem for uncertain time-delay systems is also a problem of interest because the existence of a delay is frequently a source of poor system performance or instability [41-43]. However, they are sensitive to the uncertainty, which directly affects the control systems. These years, other authors [44-46] have exploited the SMC approach theory which has provided an effective means to design robust controllers for uncertain fuzzy time-delay systems where external disturbances are bounded by known upper norm bounds.

In this dissertation, we propose two kinds of LMI-based robust adaptive sliding control, including a robust sliding control method and a robust adaptive control method, for uncertain Takagi-Sugeno fuzzy models with norm-bounded uncertainties, and meantime relax the restrictive assumption that each nominal local system model shares the same input channel, which is required in the traditional VSS-based fuzzy control design methods. Then, two kinds of LMI-based robust adaptive sliding control are developed for uncertain T-S fuzzy models which include mismatched parameter uncertainties and external disturbances. Moreover, two kinds of LMI-based robust adaptive sliding control are proposed for the uncertain T-S fuzzy time-delay model which includes mismatched parameter uncertainties in the state matrix and norm-bounded external disturbances. Finally, some examples are used to illustrate the effectiveness and usefulness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection.

1.4 Organization of this Dissertation

This dissertation comprises five chapters. In Chapter 1, the introduction comprises motivation, related works, approach, and organization of this dissertation. In Chapter 2, foundations are described by providing concepts of Lyapunov stability and linear matrix inequality. In Chapter 3, LMI-based robust sliding control design methods are developed for different uncertain Takagi-Sugeno fuzzy models with matched/mismatched parameter uncertainties and external disturbances which are bounded by known scalar valued functions and meantime we relaxed the restrictive assumption that each nominal local system model shares the same input channel, which is required in the traditional VSS-based fuzzy control design methods. Besides, a robust sliding control design method is also presented for the uncertain T-S time-delay model with mismatched parameter uncertainties and external disturbances. Finally, some examples are used to illustrate the effectiveness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection. In Chapter 4, LMI-based robust adaptive control design methods are proposed for distinct uncertain T-S fuzzy models which include matched/mismatched parameter uncertainties and unknown norm-bounded external disturbances. Moreover, a robust adaptive control design method is also proposed for the uncertain T-S time-delay model with mismatched parameter uncertainties and external disturbances. Finally, some examples are used to illustrate the effectiveness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection. In Chapter 5, the contributions are discussed and suggestions for future work are proposed.

Chapter 2

Foundations

In this chapter, the basic concepts that relate to the proposed control methods are introduced. The Lyapunov stability is discussed in the first section. Section 2.2 introduces the concept of linear matrix inequality (LMI).

2.1 Lyapunov Stability

Consider a general nonlinear system [47]

$$\dot{x} = A(x) \tag{2.1}$$

where $x \in R^n$ are the state variables and $A: R^n \rightarrow R^n$ is a nonlinear function. We assume that A is such that system (2.1) has a unique solution $x(t)$ over $[0,\infty)$ for all initial conditions $x(0)$ and that the solution depends continuously on $x(0)$. A vector $x_0 \in R^n$ is an equilibrium point of the system (2.1) if $A(x_0) = 0$.

Without loss of generality, we can assume that $x_0 = 0$ is an equilibrium point of the system (2.1); that is, $A(0) = 0$. Otherwise, we can perform a simple state transformation $z = x - x_0$ to obtain a new state equation $\dot{z} = \tilde{A}(z) = A(z + x_0)$ where $z_0 = 0$ is an equilibrium point, that is, $\tilde{A}(0) = A(x_0) = 0$. Clearly, the solution of the differential equation (2.1) shows that if $x(0) = 0$, then $x(t) = 0$, for all $t > 0$. However, this solution may or may not be stable.

Definition 2.1.1:

Stability: The equilibrium point $x_0 = 0$ of the system (2.1) is stable if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\|x(0)\| < \delta(\varepsilon) \Rightarrow \|x(t)\| < \varepsilon$, $\forall t \geq 0$.

In other words, the equilibrium point $x_0 = 0$ is stable if arbitrarily small perturbations of the initial state $x(0) = 0$ from the equilibrium point result in arbitrarily small perturbation of the corresponding state trajectory $x(t)$.

Definition 2.1.2:

Asymptotic Stability: The equilibrium point $x_0 = 0$ of the system (2.1) is asymptotically stable if it is stable and there exists some $\gamma > 0$ such that if $\|x(0)\| < \gamma$, then $x(t) \to 0$ as $t \to \infty$.

In other words, the equilibrium point $x_0 = 0$ is asymptotically stable if there exists a neighborhood of $x_0 = 0$ such that if the system starts in the neighborhood, then its trajectory converges to the equilibrium point $x_0 = 0$ as $t \to \infty$.

The equilibrium point $x_0 = 0$ of the system (2.1) is globally asymptotically stable if $\gamma > 0$ can be arbitrarily large; that is, all trajectories converges to the equilibrium point $x_0 = 0$.

Determining stability of a system may not be an easy task if the system is nonlinear. One approach often used to determine stability is that of Lyapunov. Intuitively, the Lyapunov stability theorem can be explained as follows. Given a system with an equilibrium point $x_0 = 0$, let us define some suitable "energy" function of the system. The function must have the property that is zero at an equilibrium point $x_0 = 0$ and positive elsewhere. Assume further that the dynamic system is such that the energy of the system is monotonically decreasing with time and hence eventually reduces to zero. Then, the trajectories of the system have no other places to go but the origin. Therefore, the system is asymptotically stable. This generalized energy function is called a Lyapunov function. If there exists a Lyapunov function, then we can prove the asymptotic stability using the following Lyapunov stability theorem.

Theorem 2.1.1

The equilibrium point $x_0 = 0$ of the system (2.1) is asymptotically stable if there exists a Lyapunov function $V: R^n \to R$ such that $V(x) > 0$, $x \neq 0$, $V(x) = 0$, $x = 0$, $\dot{V}(x) < 0$, and $x \neq 0$, $\dot{V}(x) = 0$, $x = 0$ is true in a neighborhood of $x_0 = 0$, $N = \{x : \|x\| < \gamma\}$ for some $\gamma > 0$.

Proof:

We provide the following intuitive proof by contradiction. If the equilibrium point $x_0 = 0$ of the system (2.1) is not asymptotically stable; that is, $x(t) \to 0$ as $t \to \infty$ is not true even if $\|x(0)\| < \gamma$ for some $\gamma > 0$, then $\dot{V}(x) < -\alpha$ for some $\alpha > 0$. Since

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x)\, d\tau = V(x(0)) + \int_0^t -\alpha\, d\tau = V(x(0)) - \alpha t .$$

For a sufficiently large t, $V(x(t)) < 0$. This contradicts the assumption $V(x(t)) \geq 0$.

The key to proving stability of a system using the Lyapunov stability theorem is to construct a Lyapunov function. This construction must be done in a case-by-case basis. There is no general method for the construction. The following example illustrates the application of the Lyapunov stability theorem.

Example 2.1.1

Let us consider the following system:

$$\dot{x}_1 = x_2 - 3x_1, \quad \dot{x}_2 = -x_2^3 - 2x_1 .$$

To prove it is asymptotically stable, let us consider the following Lyapunov function:

$$V(x) = 2x_1^2 + x_2^2 .$$

Clearly, $V(x) > 0$, $x \neq 0$, $V(x) = 0$, $x = 0$.

On the other hand,

$$\dot{V}(x) = 4x_1\dot{x}_1 + 2x_2\dot{x}_2 = 4x_1(x_2 - 3x_1) + 2x_2(-x_2^3 - 2x_1)$$

$$= 4x_1x_2 - 12x_1^2 - 2x_2^4 - 4x_1x_2 = -12x_1^2 - 2x_2^4.$$

Therefore, $\dot{V}(x) < 0$, $x \neq 0$, $\dot{V}(x) = 0$, $x = 0$.

Finally, we can conclude that the system is asymptotically stable.

2.2 Linear Matrix Inequality

A linear matrix inequality (LMI) has the form [48]

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{2.2}$$

where $x \in R^m$ is the variable and the symmetric matrices $F_i = F_i^T \in R^{n\times n}$, $i = 0,\ldots,m$, are given. The inequality symbol in (2.2) means that $F(x)$ is positive-definite, i.e., $u^T F(x) u > 0$ for all nonzero $u \in R^n$. Thus, the LMI (2.2) is equivalent to a set of n polynomial inequalities in x, i.e., the leading principal minors of $F(x)$ must be positive. We will also encounter nonstrict LMIs, which have the form

$$F(x) \geq 0. \tag{2.3}$$

The strict LMI (2.2) and the nonstrict LMI (2.3) are closely related.

The LMI (2.2) is a convex constraint on x, i.e., the set $\{x \mid F(x) > 0\}$ is convex. Though the LMI (2.2) may seem to have a specialized form, it can represent a wide variety of convex constraints on x. In particular, linear inequalities, quadratic inequalities, matrix norm inequalities, and constraints that arise in control theory, such as Lyapunov and convex quadratic matrix inequalities, can all be cast in the form of an LMI.

Multiple LMIs $F^{(1)}(x) > 0$,⋯, $F^{(p)}(x) > 0$ can be expressed as the single LMI

$diag(F^{(1)}(x),\cdots,F^{(p)}(x))>0$. Therefore we will make no distinction between a set of LMIs and single LMI, i.e., "the LMI $F^{(1)}(x)>0$,$\cdots$, $F^{(p)}(x)>0$" will mean "the LMI $diag(F^{(1)}(x),\cdots,F^{(p)}(x))>0$".

When the matrices F_i are diagonal, the LMI $F(x)>0$ is just a set of linear inequalities. Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ s(x)^T & R(x) \end{bmatrix} > 0 \tag{2.4}$$

where $Q(x)=Q(x)^T$, $R(x)=R(x)^T$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x)>0,\ \ Q(x)-S(x)R(x)^{-1}S(x)^T>0. \tag{2.5}$$

In other words, the set of nonlinear inequalities (2.5) can be represented as the LMI (2.4).

As an example, the matrix norm constraint $\|Z(x)\|<1$, where $Z(x)\in R^{p\times q}$ and depends affinely on x, is represented as the LMI

$$\begin{bmatrix} I & Z(x) \\ Z(x)^T & I \end{bmatrix} > 0$$

Since $\|Z\|<1$ is equivalent to $I-ZZ^T>0$.

We will often encounter problems in which the variables are matrices, e.g., the Lyapunov inequality

$$A^TP+PA<0 \tag{2.6}$$

where $A\in R^{n\times n}$ is given and $P=P^T$ is the variable. In this case we will not write out the LMI explicitly in the form $F(x)>0$, but instead make clear which matrices are the variables. The phrase "the LMI $A^TP+PA<0$ in P" means that the matrix P is a variable. Of course, the Lyapunov inequality (2.6) is readily put in the form (2.2), as follows. Let $P_1, \cdots, P_m$ be a basis for symmetric $n \times n$ matrices. Then take $F_0 = 0$ and $F_i = -A^T P_i - P_i A$. Leaving LMIs in a condensed form such as (2.6), in addition to saving notation, may lead to more efficient computation.

As another related example, consider the quadratic matrix inequality

$$A^T P + PA + PBR^{-1}B^T P + Q < 0 \tag{2.7}$$

where A, B, $Q = Q^T$, $R = R^T > 0$ are given matrices of appropriate sizes, and $P = P^T$ is the variable. Note that this is a quadratic matrix inequality in the variable P. It can be expressed as the linear matrix inequality

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0.$$

This representation also clearly shows that the quadratic matrix inequality (2.7) is convex in P, which is not obvious.

Finally, given an LMI $F(x) > 0$, the corresponding LMI Problem (LMIP) is to find x^{feas} such that $F(x^{feas}) > 0$ or determine that the LMI is infeasible. Of course, this is a convex feasibility problem. We will say "solving the LMI $F(x) > 0$" to mean solving the corresponding LMIP.

As an example of an LMIP, consider the "simultaneous Lyapunov stability problem": We are given $A_i \in R^{n \times n}$, $i = 1, \cdots, L$, and need to find P satisfying the LMI

$$P > 0, \quad A_i^T P + PA_i < 0, \quad i = 1, \cdots, L$$

or determine that no such P exists.

Chapter 3

LMI-Based Robust Sliding Control

In this chapter, LMI-based robust sliding control methods are developed for different uncertain Takagi-Sugeno fuzzy models/time-delay models. The introduction of this chapter is introduced in Section 3.1. In Section 3.2, a robust sliding control method is proposed for T-S fuzzy systems. Section 3.3 presents two kinds of robust sliding control methods for mismatched T-S fuzzy systems. A robust sliding control method is presented for mismatched T-S fuzzy time-delay systems in Section 3.4.

3.1 Introduction

Over the past two decades, fuzzy techniques have been widely and successfully exploited in nonlinear system modeling and control. The Takagi-Sugeno (T-S) model [5] is a popular and convenient tool for handling complex nonlinear systems. Correspondingly, the fuzzy feedback control design problem for a nonlinear system has been studied extensively by using T-S model where simple local linear models are combined to describe the global behavior of the nonlinear system [23-29]. In practice, the inevitable uncertainties may enter a nonlinear system model in a very complicated way. The uncertainty may include modeling errors, parameter variations, external disturbances, and fuzzy approximation errors. In such a situation, the fuzzy feedback control design methods of [23-29] may not work well anymore. To deal with the problem, some authors [30,31] have exploited the variable structure system (VSS) theory which has provided an effective means to design robust controllers for uncertain nonlinear systems where the uncertainties are bounded by known scalar valued functions.

In the VSS, the control design of the plant is intentionally changed by using a viable high-speed switching feedback control to obtain a desired system response, from which the VSS arises in finite time. The VSS drives the trajectory of the system onto a specified and user-design surface, which is called the sliding surface or the switching surface, and maintains the trajectory on this sliding surface for all subsequent time. The closed-loop response obtained from using a VSS control law comprises two distinct modes. The first is the reaching mode, also called nonsliding mode, in which the trajectory starting from anywhere on the state space is being driven towards the switching surface. The second is the sliding mode in which the trajectory asymptotically tends to the origin. The central feature of the VSS is the sliding mode on the sliding surface on which the system remains insensitive to internal parameter variations and external disturbance. In sliding mode, the order of the system dynamics is reduced. This enables simplification and decoupling design procedure [32-35]. However, all the VSS-based fuzzy control system design methods are based on the assumption that each nominal local system model shares the same input channel. This assumption is very restrictive and inadequate to modeling uncertainty/nonlinearity in various mechanical systems such as an inverted pendulum on a cart.

On the other hand, time-delay is often encountered in various industrial systems, such as the turbojet engine, electrical networks, nuclear reactor, rolling mill, and chemical process, etc. Recently, the feedback stabilization problem for uncertain time-delay systems is also a problem of interest because the existence of a delay is frequently a source of poor system performance or instability [41-43]. However, they are sensitive to the uncertainty, which directly affects the control systems.

In this chapter, we propose robust sliding control design methods for different uncertain T-S fuzzy models with matched/mismatched parameter uncertainties and external disturbances which are bounded by known scalar valued functions. Each nominal local system model of the uncertain system under consideration may not share the same input channel. As the local controller, we use a sliding mode controller with a nonlinear switching feedback control term. We derive LMI conditions for existence of linear sliding surfaces guaranteeing asymptotic stability of the reduced order equivalent sliding mode dynamics, and we give an explicit formula of the switching surface parameter matrix in terms of the solution of the LMI existence conditions. The nonlinear switching feedback control term is also designed to drive the system trajectories so that a stable sliding motion is induced in finite time on the switching surface and the state converges to zero. Besides, a robust sliding control design method is also presented for the uncertain T-S time-delay model with mismatched parameter uncertainties and external disturbances. Finally, some examples are used to illustrate the effectiveness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection.

3.2 Robust Sliding Control for T-S Fuzzy Systems

In this section, system formulation for the uncertain T-S fuzzy model is described in Section 3.2.1. A robust sliding control method via LMI is proposed in Section 3.2.2. Some examples are used to illustrate the effectiveness of the proposed methods and to compare with the existing methods in Section 3.2.3.

3.2.1 System Formulation

Consider the following uncertain T-S fuzzy model [49]:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)\left[A_i x(t) + B_i u(t) + B_i h(t,x)\right] \tag{3.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, A_i, B_i are constant matrices of appropriate dimensions, $\theta = [\theta_1, \cdots, \theta_s], \theta_j (j = 1, \cdots, s)$ are the premise variables, s is the number of the premise variables, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta)$, $\omega_i : R^s \to [0,1], i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i, r is the number of the IF-THEN rules, β_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0$, $\sum_{i=1}^{r} \beta_i(\theta) = 1$, $h(t,x) \in R^m$ stands for the lumped nonlinearities or uncertainties. We will assume that the followings are satisfied:

A1: The $n \times m$ matrix B defined by $B = \frac{1}{r} \sum_{i=1}^{r} B_i$ satisfies the rank constraint $\text{rank}(B) = m$, i.e., the matrix B has full column rank m.

A2: The function $h(t,x)$ is unknown but bounded as $\left\| h(t,x) - \hat{h}(t,x) \right\| \leq \sum_{k=0}^{l} \rho_k \left\| x \right\|^k$ where $\rho_0, \cdots, \rho_l$ are known constants, $\hat{h}(t,x)$ is an estimate of $h(t,x)$, and l is a known positive integer.

The system (3.1) does not have to satisfy the restrictive assumption that all the input matrices of the local system models are in the same range space. It should be noted that the assumption A1 implies that $rank(B_i) \leq m$ and each nominal local system model may not share the same input channel. The assumption A2 with $l = 1$ and $\hat{h}(t,x) = 0$ has been used in the literature [50]. We can set $\hat{h}(t,x)$ as the nominal value of $h(t,x)$. Using the above assumptions, the uncertain T-S fuzzy model (3.1) can be written as follows.

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta) A_i x(t) + [B + HF(\beta)G][u + h(t,x)] \tag{3.2}$$

where $\beta = [\beta_1(\theta), \cdots, \beta_r(\theta)]$, and the matrices $H, G, F(\beta)$ are defined by

$$H = \frac{1}{2}\left[(B - B_1), \cdots, (B - B_r)\right], \; G = \left[I, \cdots, I\right]^T,$$

$$F(\beta) = diag\left[(1 - 2\beta_1(\theta))I, \cdots, (1 - 2\beta_r(\theta))I\right] \qquad (3.3)$$

It should be noted that the system (3.1) does not have to satisfy $B_1 = B_2 = \cdots = B_r$, which is used in almost all published results on VSS design methods including the VSS-based fuzzy control design methods of [33,34]. Hence the methods [30,31] cannot be applied to the above model (3.1). Since $\beta_i(\theta) \geq 0$ and $\sum_{i=1}^{r} \beta(\theta) = 1$, we can see that the following inequality always holds:

$$F^T(\beta)F(\beta) = F(\beta)F^T(\beta) \leq I. \qquad (3.4)$$

Many examples in the literature and various mechanical systems such as motors and robots do not satisfy the restrictive assumptions that each nominal local system model shares the same input channel and they fall into the special cases of the above model [49].

3.2.2 Sliding Control Design via LMI

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions: The first involves the design of $m(n-1)$ – dimensional switching surfaces for the sliding mode such that the reduced order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second is concerned with the selection of a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first characterize linear sliding surfaces using LMIs.

Let us define the linear sliding surface as $\sigma = Sx = 0$ where S is a $m \times n$ matrix. Referring to the previous results [33], [51], we can see that for the system (3.2) it is reasonable to find a sliding surface such that

P1 $[SB + SHF(\beta)G]$ is nonsingular for any β satisfying $\beta_i(\theta) \geq 0, i = 1,\cdots,r$, and $\sum_{i=1}^{r}\beta_i(\theta) = 1$.

P2 The reduced $(n-m)$ order sliding mode dynamics restricted to the sliding surface $Sx = 0$ is asymptotically stable for all admissible uncertainties.

It should be noted that P1 is necessary for the existence of the unique equivalent control [33] and the assumption A1 is necessary for the nonsingularity of SB.

Define a transformation matrix and the associated vector v as $M = [\Lambda(\Lambda^T Y\Lambda)^{-1}, Y^{-1}B(B^T Y^{-1}B)^{-1}]^T = [V^T, S^T]^T$, $v = [v_1^T, v_2^T]^T = Mx$ where $v_1 \in R^{n-m}$, $v_2 \in R^m$. By the above transformation, we can see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. Then, from system (3.2), we can obtain

$$\begin{bmatrix} \dot{v}_1 \\ \dot{\sigma} \end{bmatrix} = \sum_{i=1}^{r}\beta_i(\theta)\begin{bmatrix} VA_iY\Lambda & VA_iB \\ SA_iY\Lambda & SA_iB \end{bmatrix}\begin{bmatrix} v_1 \\ \sigma \end{bmatrix} + \begin{bmatrix} VHF(\beta)G \\ I + SHF(\beta)G \end{bmatrix}[u + h(t,x)]. \tag{3.5}$$

From the equivalent control method [33], we can see that the equivalent control is given by $u_{eq}(t) = -\sum_{i=1}^{r}\beta_i(\theta)[I + SHF(\beta)G]^{-1}SA_ix - h(t,x)$. By setting $\dot{\sigma} = \sigma = 0$ and substituting $u(t)$ with $u_{eq}(t)$, we can obtain that the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma = Sx = 0$ is given by

$$\dot{v}_1 = \sum_{i=1}^{r}\beta_i(\theta)(\Lambda^T Y\Lambda)^{-1}\Lambda^T D(\beta)A_iY\Lambda v_1 \tag{3.6}$$

where $D(\beta) = I - HF(\beta)G[I + SHF(\beta)G]^{-1}S$.

Theorem 3.1 Let us consider the sliding mode dynamics (3.6). If there exist matrices $Y \in R^{n\times n}$, $\Lambda \in R^{n\times(n-m)}$ satisfying $B^T\Lambda = 0, \Lambda^T\Lambda = I$, scalars $c_1 \in R, c_2 \in R, \eta \in R$, $\kappa = \lambda_{\min}(B^T B)$, and $*$ represents blocks that are readily inferred by symmetry such that the following LMIs holds:

$$\begin{bmatrix} \Lambda^T (A_i Y + *)\Lambda & * & * \\ \eta H^T \Lambda & -I & * \\ A_i Y \Lambda & \eta H & -I \end{bmatrix} < 0, \quad \forall i \tag{3.7}$$

$$\begin{bmatrix} Y & I & 0 \\ I & c_1 I & 0 \\ 0 & 0 & c_2 I - Y \end{bmatrix} > 0, \tag{3.8}$$

$$\begin{bmatrix} 2\eta\kappa & * & * \\ rc_1 & r\eta & 0 \\ rc_2 & 0 & r\eta \end{bmatrix} > 0 \tag{3.9}$$

then, there exists a linear sliding surface parameter matrix S satisfying P1-P2 and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \tag{3.10}$$

will guarantee that the sliding mode dynamics (3.6) is asymptotically stable.

Proof: By using Schur complement formula [48], we can easily show that in fact the following LMIs are incorporated in the LMIs (3.7)-(3.9)

$$c_1 > 0, \;\; c_2 > 0, \;\; \eta > 0, \;\; \eta^2 HH^T < I, \; 2\eta^2\kappa > r(c_1^2 + c_2^2). \tag{3.11}$$

It is clear that if the following inequality (3.12) holds, then $SB + SHF(\beta)G$ $= I + SHF(\beta)G$ is nonsingular and hence P1 holds

$$SHF(\beta)GG^T F^T(\beta)H^T S^T < I. \tag{3.12}$$

Using (3.3), (3.4), (3.11) and $GG^T \le \|G\|^2 I = rI$, we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T. \tag{3.13}$$

By using the Schur complement formula, we can see that (3.8) and (3.11) imply

$$0 < c_1^{-1} I < Y < c_2 I, \qquad 0 < c_2^{-1} I < Y^{-1} < c_1 I \tag{3.14}$$

and this leads to

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T \le \frac{rc_1c_2}{\eta^2}(B^T B)^{-1} \le \frac{rc_1c_2}{\kappa\eta^2} I. \tag{3.15}$$

Using the inequality $2ab \le a^2 + b^2$ where a and b are scalars, we can show that (3.15) implies

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{2\kappa\eta^2}(c_1^2 + c_2^2)I. \tag{3.16}$$

Finally, by using the above inequalities (3.11) and (3.16), we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T < I \tag{3.17}$$

which implies that $[SB + SHF(\beta)G]$ is nonsingular, i.e., P1 holds.

Now, we will show that S of (3.10) guarantees P2. Using the matrix inversion lemma:

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B$$

where A and B are compatible constant matrices such that $(I + AB)$ is nonsingular, we can show that the sliding mode dynamics (3.6) is equivalent to

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T C(\beta) A_i Y \Lambda v_1 \tag{3.18}$$

where $C(\beta) = I - H[I + F(\beta)GSH]^{-1} F(\beta)GS = [I + HF(\beta)GS]^{-1}$

$$= I - HF(\beta)G[I + SHF(\beta)G]^{-1} S = D(\beta).$$

The sliding mode dynamics (3.18) is asymptotically stable if there exists a positive definite matrix $P_0 \in R^{(n-m)\times(n-m)}$ such that the time derivative of the Lyapunov function $E_g(t) = v_1^T P_0 v_1$ satisfies for some positive scalar τ

$$\dot{E}_g(t) = 2\sum_{i=1}^{r} \beta_i(\theta) v_1^T P_0 Z_i(\beta) v_1 \le -\tau v_1^T v_1 \tag{3.19}$$

where $Z_i(\beta) = (A_{i0} + B_0[I - N(\beta)D_0]^{-1} N(\beta)C_{i0})$,

$A_{i0} = (\Lambda^T Y \Lambda)^{-1} \Lambda^T A_i Y \Lambda$, $B_0 = (\Lambda^T Y \Lambda)^{-1} \Lambda^T H$, $C_{i0} = A_i Y \Lambda$, $D_0 = H$, $N(\beta) = -F(\beta)GS$.

It should be noted that the inequalities (3.4), (3.11), (3.17) and $GG^T \le \|G\|^2 I = rI$ imply

$$N(\beta)N^T(\beta) = F(\beta)GSS^TG^TF^T(\beta) \le \eta^2 I,\ \eta^2 D_0^T D_0 = \eta^2 H^T H < I \qquad (3.20)$$

This and (3.19) imply that (3.18) is asymptotically stable if there exists a positive definite matrix P_0 such that

$$P_0 A_{i0} + P_0 B_0 [I - N(\beta)D_0]^{-1} \mathrm{N}(\beta) C_{i0} + * < 0\,, \quad \forall i \qquad (3.21)$$

where $*$ represents blocks that are readily inferred by symmetry.

Let z_i be $z_i = [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} y$ where $y \in R^{(n-m)}$.

Then z_i can be rewritten as $z_i = N(\beta)[C_{i0} y + D_0 z_i]$.

This equality and (3.20) imply $z_i^T z_i \le \eta^2 [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i]$ and this leads to

$$2y^T P_0 B_0 [I - N(\beta)D_0]^{-1} \mathrm{N}(\beta) C_{i0} y$$

$$= 2y^T P_0 B_0 z_i \le 2y^T P_0 B_0 z_i + [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i] - \eta^{-2} z_i^T z_i$$

$$= y^T C_{i0}^T C_{i0} y + 2y^T [P_0 B_0 + C_{i0}^T D_0] z_i - \eta^{-2} z_i^T \Omega z_i \quad \text{where } \Omega = I - \eta^2 D_0^T D_0\,. \qquad (3.22)$$

Since $\Omega > 0$, the following inequality holds for any (y, z_i):

$$2y^T [P_0 B_0 + C_{i0}^T D_0] z_i \le \eta^{-2} z_i^T \Omega z_i + \eta^2 y^T [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T y \quad (3.23)$$

Using (3.22) and (3.23), we can show that the Lyapunov inequality (3.21) is satisfied if the following inequality holds:

$$P_0 A_{i0} + A_{i0}^T P_0 + C_{i0}^T C_{i0} + \eta^2 [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T < 0\,.$$

Using the Schur complement formula, we can rewrite the above inequality as

$$\begin{bmatrix} A_{i0}^T P_o + * & * & * \\ \eta B_0^T P_0 & -I & * \\ C_{i0} & \eta D_0 & -I \end{bmatrix} < 0\,, \quad \forall i\,. \qquad (3.24)$$

Let the positive definite matrix P_0 be $P_0 = \Lambda^T Y \Lambda$ where Y is a solution to LMIs (3.7)-(3.9), which implies that the sliding mode dynamics (3.18) is asymptotically stable. Hence, the sliding mode dynamics (3.6) is asymptotically stable.

After the switching surface parameter matrix S is designed so that the reduced $(n-m)$ order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintains it there for all subsequent time. With σ of (3.10), we design a sliding fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear sliding switching feedback control law as the local controller.

Control rule i: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\hat{h}(t,x) - \chi_i \sigma - SA_i x - \frac{1}{1-\omega}\delta_i(t,x)\frac{\sigma}{\|\sigma\|}$$

where $\delta_i(t,x) = \alpha_i + \omega\|SA_i x\| + (1+\omega)\sum_{k=0}^{l}\rho_k\|x\|^k$ (3.25)

and $\sigma = Sx, \omega = \sqrt{r}\|SH\|, \alpha_i > 0, \chi_i > 0$. It should be noted that (3.17) implies $\omega = \sqrt{r}\|SH\| \le \sqrt{r}\|S\|\cdot\|H\| \le \eta\|H\|$. This and (3.11) guarantee $0 \le \omega < 1$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\hat{h}(t,x) - \sum_{i=1}^{r}\beta_i(\theta)\left(\chi_i\sigma + SA_i x + \frac{1}{1-\omega}\delta_i(t,x)\frac{\sigma}{\|\sigma\|}\right) \tag{3.26}$$

and we can establish the following theorem.

Theorem 3.2 Consider the closed-loop control system of the uncertain system (3.2) with control (3.26). Suppose that the LMIs (3.7)-(3.9) has a solution vector

(Y, c_1, c_2, η) and the linear sliding surface is given by (3.10). Then the state converges to zero.

Proof: Since Theorem 3.1 implies that the linear sliding surface (3.10) guarantees P1-P2, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T\sigma$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T\dot{\sigma}$. From (3.2), (3.10), (3.26), $\|SHF(\beta)G\| \le \sqrt{r}\|SH\| = \omega$,$0 \le \omega < 1$, and A2, we obtain

$$\sigma^T\dot{\sigma} = \sigma^T\sum_{i=1}^{r}\beta_i(\theta)SA_i x(t) + \sigma^T[I + SHF(\beta)G][u + h(t,x)]$$

$$\le \sum_{i=1}^{r}\beta_i(\theta)\sigma^T SA_i x(t) + \sigma^T u + \{\omega\|u\| + (1+\omega)\|h(t,x)\|\}\|\sigma\|.$$

This implies that $\dot{E}_g \le -(1-\omega)\sum_{i=1}^{r}\beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i(\theta)\alpha_i\|\sigma\| \le 0$ which indicates that $E_g \in L_2 \cap L_\infty, \dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 3.1 Theorem 3.1 and 3.2 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Obtain $B = \frac{1}{r}\sum_{i=1}^{r} B_i$ and $H = \frac{1}{2}[(B - B_1), \cdots, (B - B_r)]$ for given B_i .

Step 2: Check that (A_i, B) is stabilization. If not, exit.

Step 3: Find a solution vector (Y, c_1, c_2, η) to LMI (3.7)-(3.9).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (3.10).

Step 5: The controller is given by (3.26).

3.2.3 Numerical Examples

Example 3.1 Consider the following inverted pendulum on a cart [49]

$$\dot{x}_1 = x_2,\ \dot{x}_3 = x_4,\ \dot{x}_2 = \frac{1}{l\psi}(3g\sin x_1 - 3a\cos x_1[u + d(t) + \phi]),$$

$$\dot{x}_4 = -\frac{1}{\psi}(1.5mag\sin 2x_1 - 4a[u + d(t) + \phi]) \tag{3.27}$$

where x_1 is the angle (rad) of the pendulum from the vertical, $x_2 = \dot{x}_1$, x_3 is the displacement (m) of the cart, $x_4 = \dot{x}_3$, $\psi = 4 - 3ma\cos^2 x_1$, $\phi = mlx_2^2 \sin x_1$, u is the input, and $d(t)$ is related to external disturbances which may be caused by the frictional force. $a = 1/(m+M)$, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, $g = 9.8m/s^2$ is the gravity constant. We set $M = 9kg$,$m = 1kg$,$l = 1m$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1\|x\|$ where ρ_0 and ρ_1 are known constants. To design the fuzzy controller (3.26), we must have a fuzzy model. Here, we approximate the system (3.27) by the following two-rule fuzzy model.

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = A_1 x + B_1[u + h(t,x)]$$

Plant Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

$$\dot{x} = A_2 x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.9459 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.7946 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -0.0811 \\ 0 \\ 0.1081 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3097 & 0 & 0 & 0 \end{bmatrix}$,

$$B_2 = \begin{bmatrix} 0 \\ -0.0382 \\ 0 \\ 0.1019 \end{bmatrix},\ h(t,x) = d(t) + x_2^2 \sin x_1,\ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}},\ \beta_2 = 1 - \beta_1. \tag{3.28}$$

The inverted pendulum on a cart (3.27) can be cast as (3.2) with data (3.28). Because B_1 is not in the range space of B_2, all existing VSS-based fuzzy control system design methods cannot be applied to the above system (3.28). Via LMI optimization with (3.28), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x) = x_2^2 \sin x_1, \chi_i = 5, \alpha_i = 1, r = 2, l = 1, \rho_k = 1$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_1 x - \frac{1}{1-\omega}\delta_1 \operatorname{sgn}(\sigma).$$

Control Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

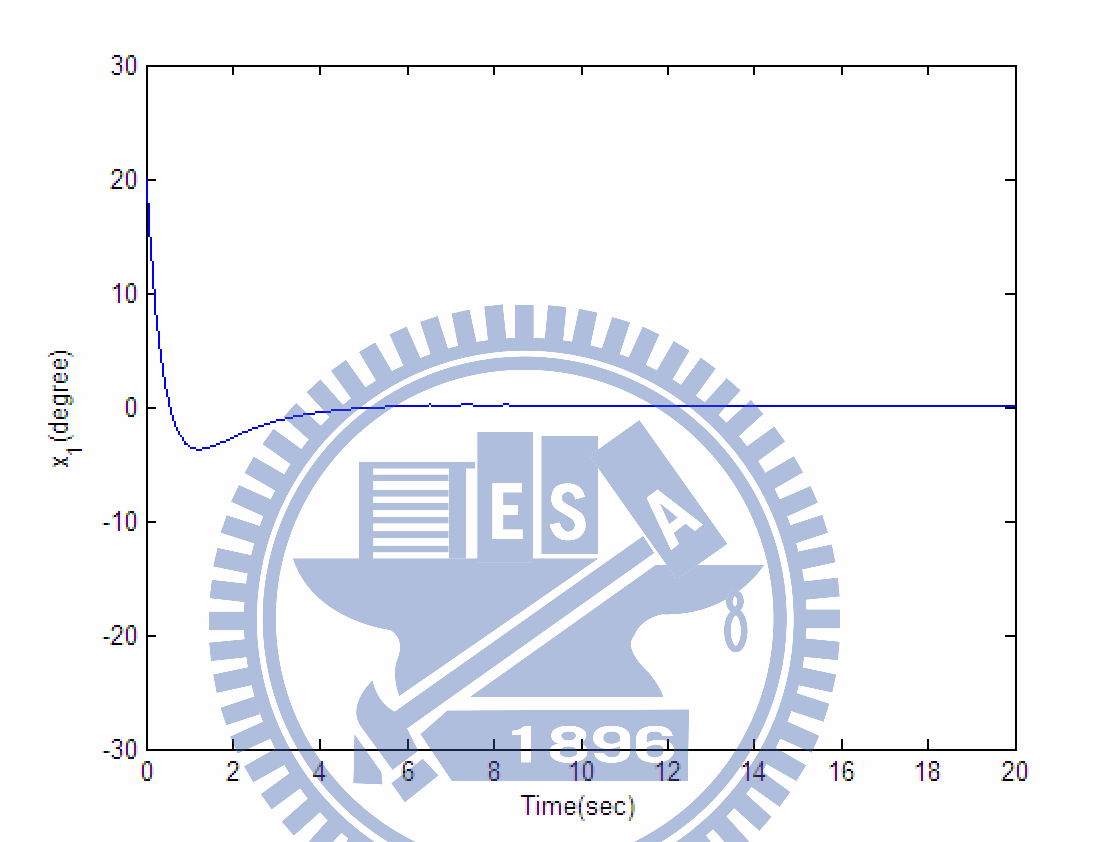
$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_2 x - \frac{1}{1-\omega}\delta_2 \operatorname{sgn}(\sigma).$$

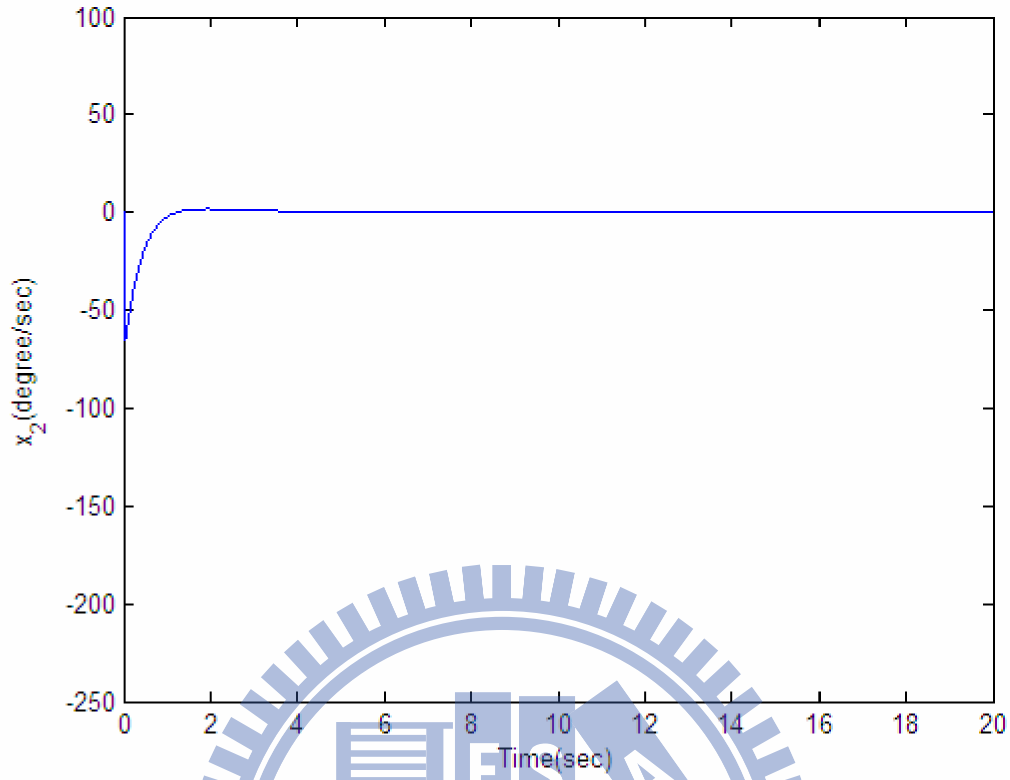
The final controller inferred as the weighted average of each local controller is given by

$$u(t) = -x_2^2 \sin x_1 - \sum_{i=1}^{r} \beta_i(\theta)\left[5\sigma + SA_i x + \frac{1}{1-\omega}\delta_i \operatorname{sgn}(\sigma)\right]. \qquad (3.29)$$

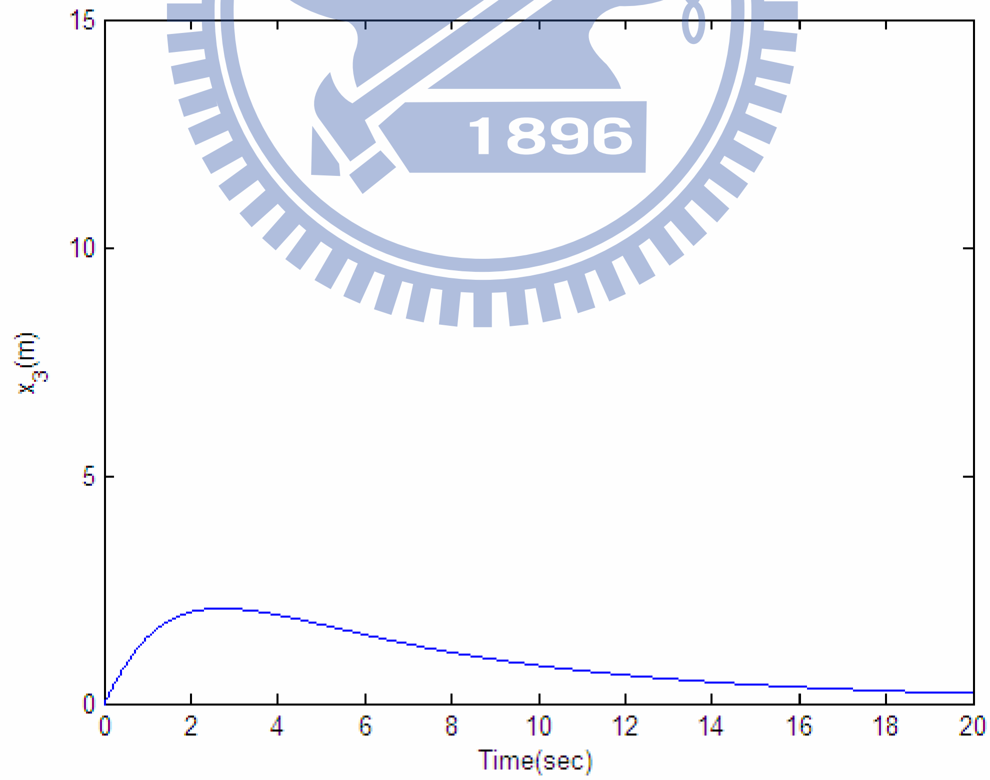
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (3.28) with nonzero $d(t)$. We assume that $d(t) = x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)$. Figure 3.1 shows the time histories of the state, the sliding variable σ, and the input (3.29) when $x_1(0) = 20^\circ (\pi/9\ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$. Figure 3.2 shows the time histories of the state, the sliding variable σ, and the input (3.29) when $x_1(0) = 40^\circ (2\pi/9\ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$. Figure 3.3 shows the time histories of the state, the sliding variable σ, and the input (3.29) when $x_1(0) = 60^\circ (\pi/3\ rad)$, $x_2(0) =$

$x_3(0) = x_4(0) = 0$. In Figure 3.1, Figure 3.2, and Figure 3.3, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. It is observed that in our simulations the proposed controller (3.29) stabilizes the following two-rule fuzzy model (3.28).

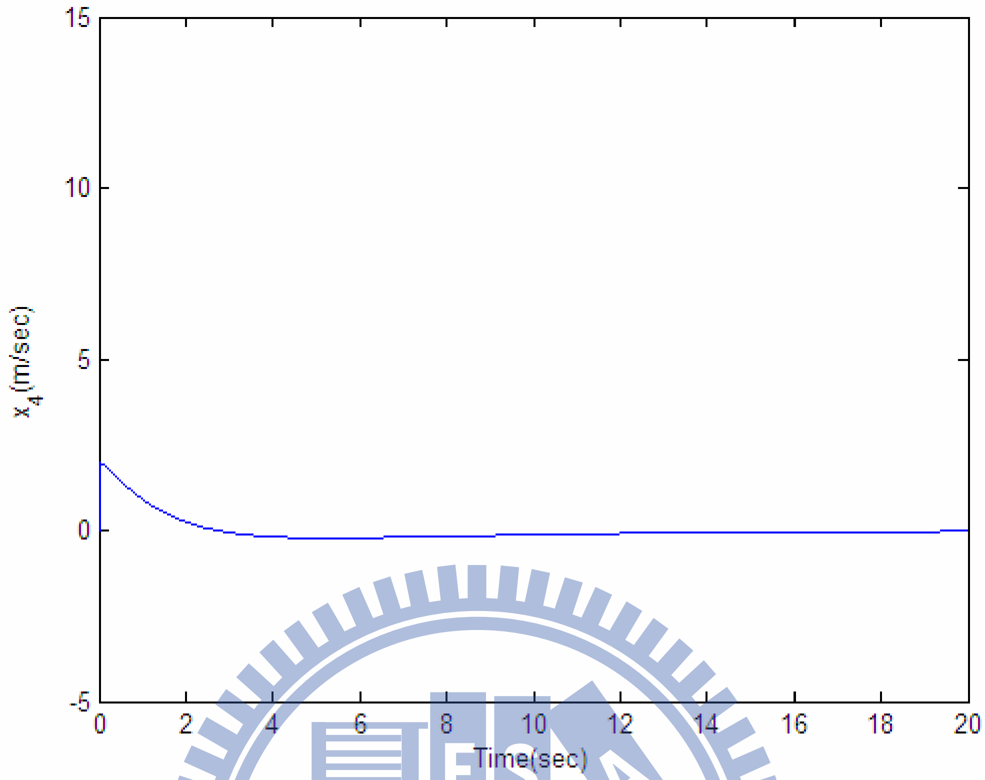


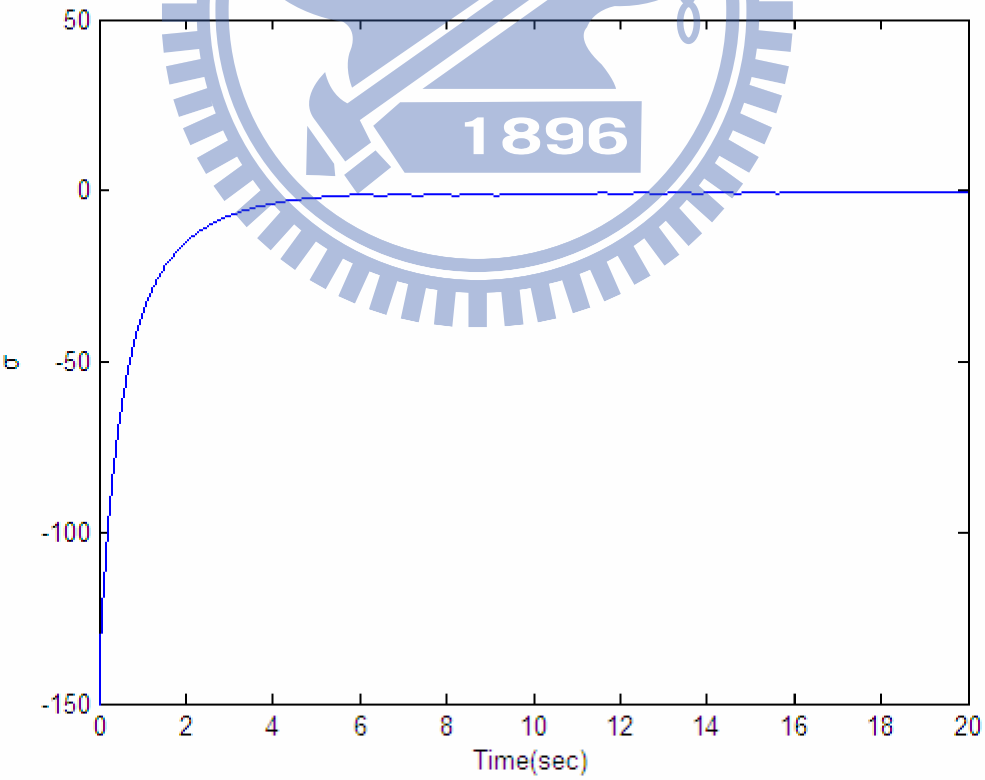


100
50
0
-50
-100
-150
-200
-250
0
2
4
6
8
10
12
14
16
18
20
Time(sec)
x_2(degree/sec)



15
10
5
0
0
2
4
6
8
10
12
14
16
18
20
Time(sec)
x_3(m)


15
10
5
0
-5
x_4(m/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


50
0
-50
-100
-150
σ
0
2
4
6
8
10
12
14
16
18
20
Time(sec)

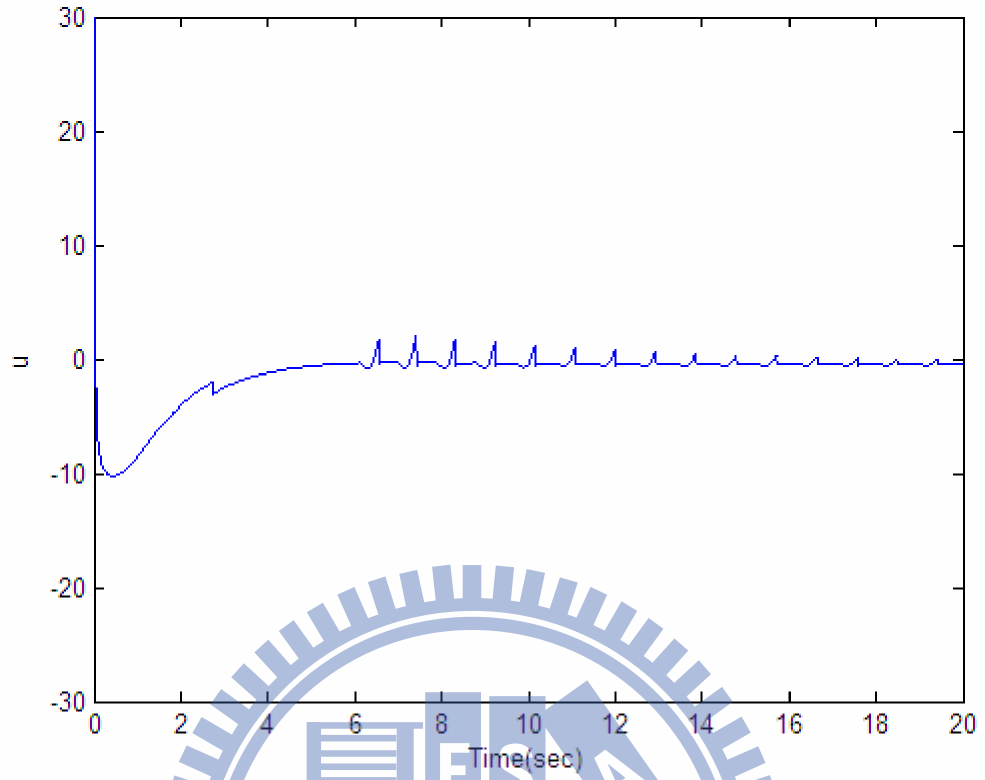
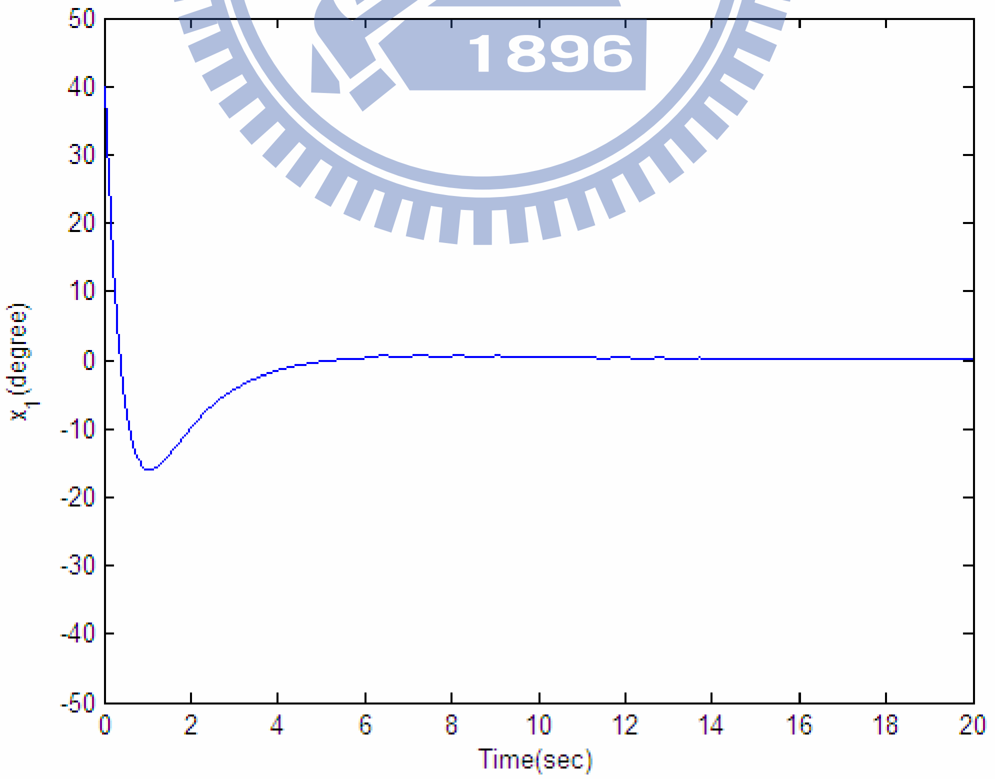
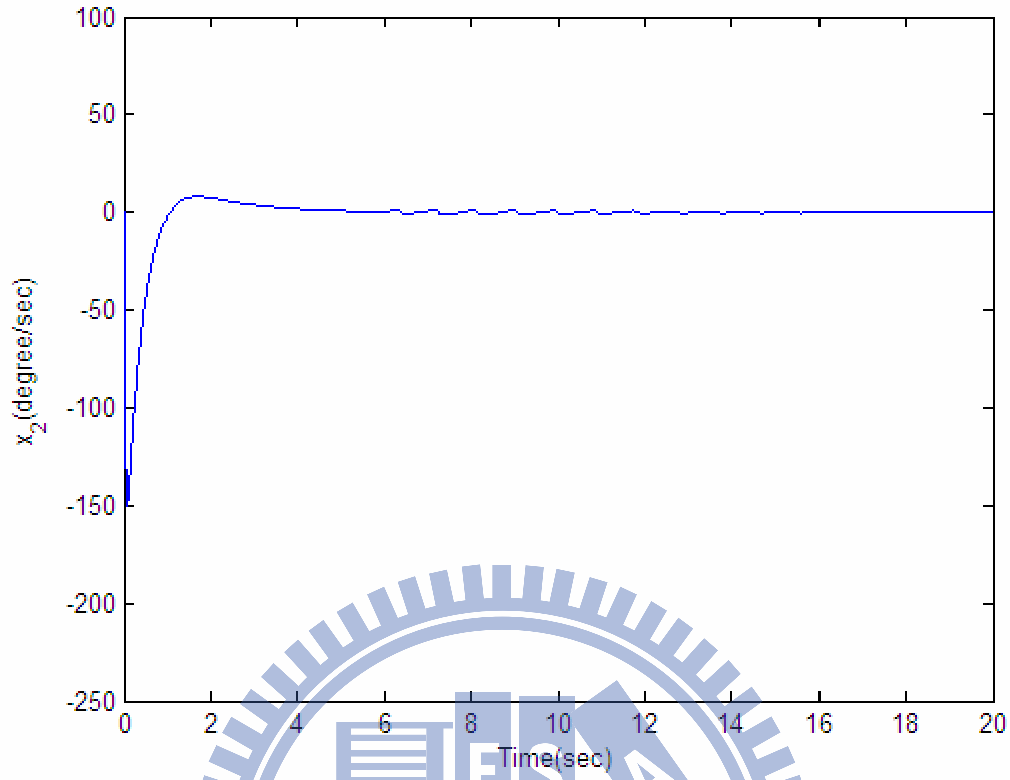
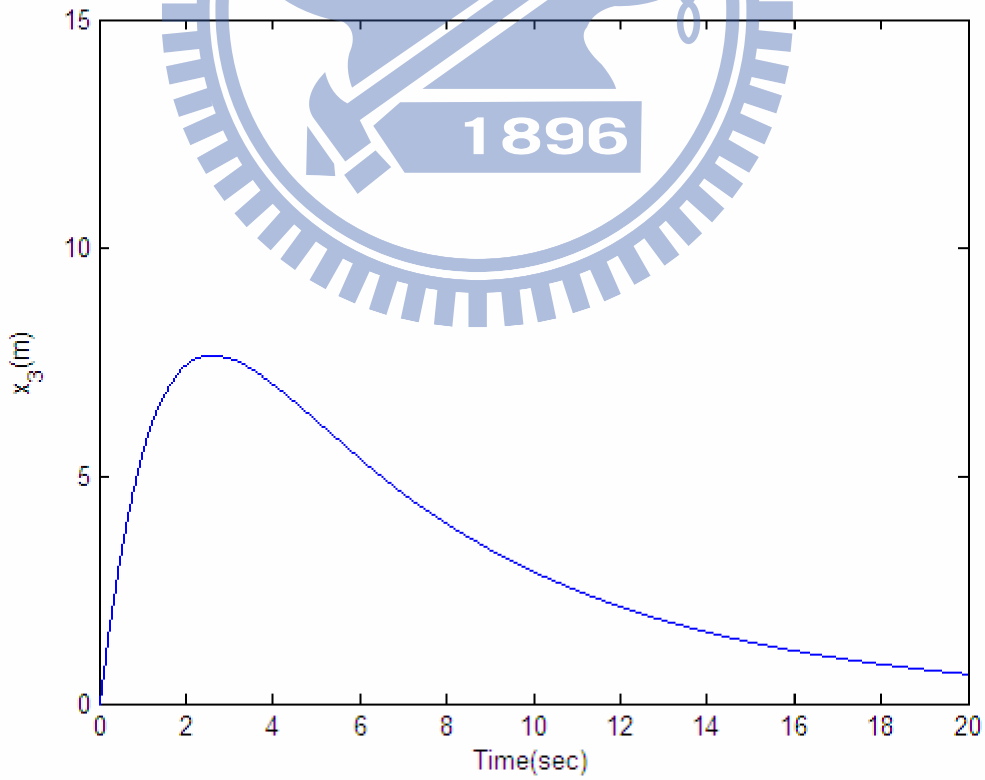


Figure 3.1 Simulation results with $x_1(0)=20^\circ(\pi/9\ \ rad),\ x_2(0)=x_3(0)=x_4(0)=0$.

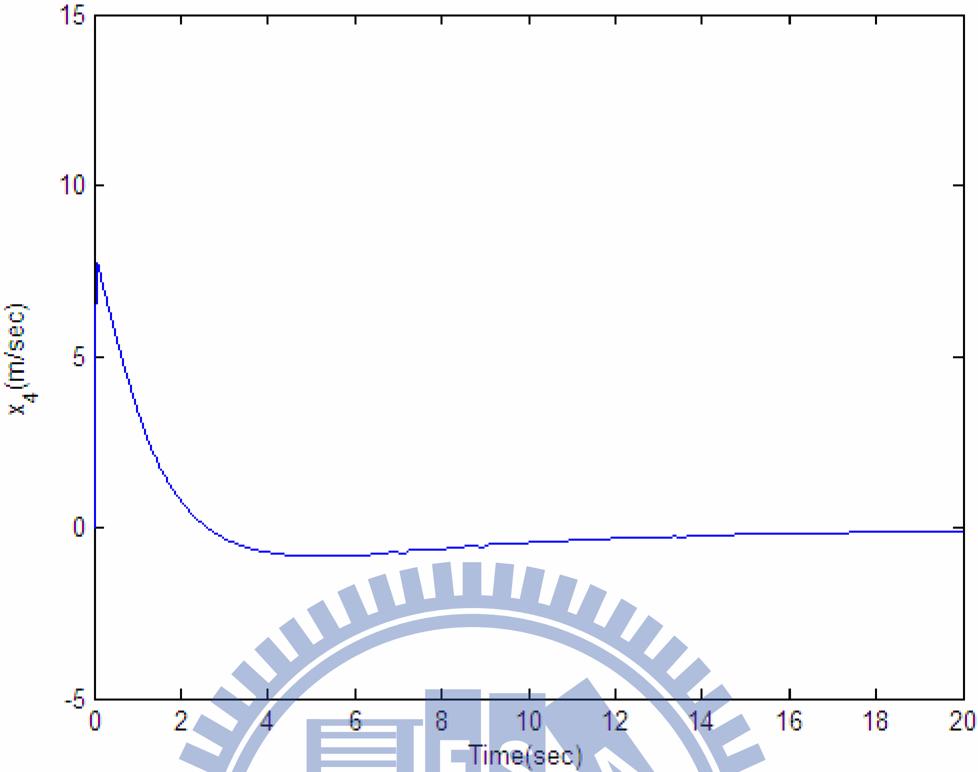




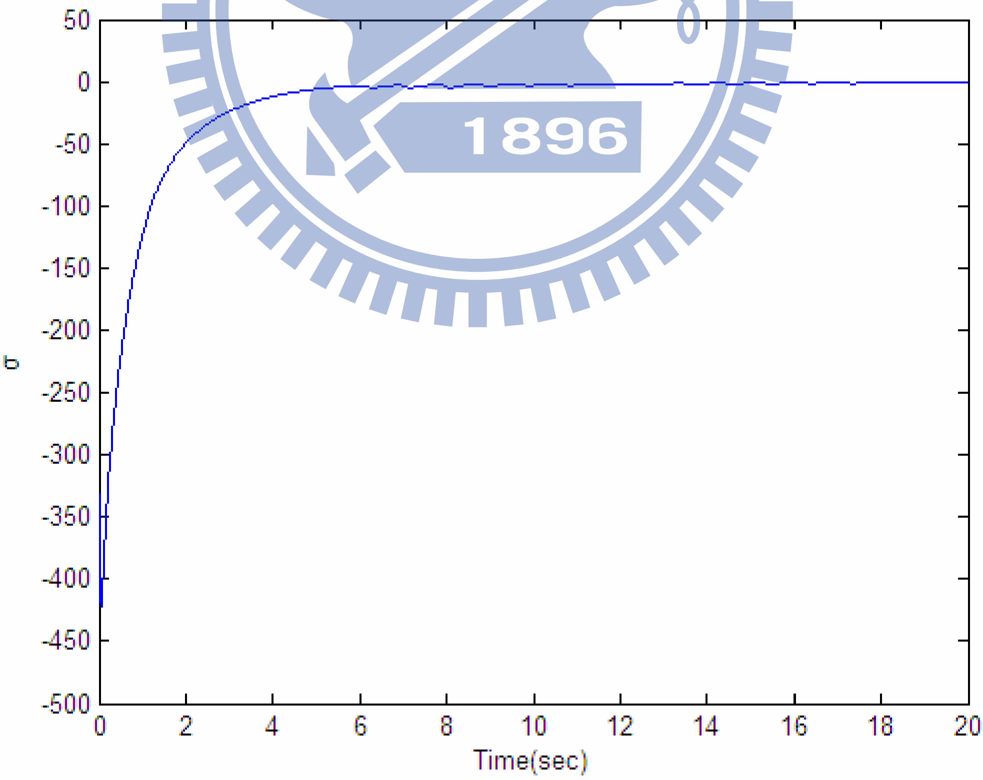
100
50
0
-50
-100
-150
-200
-250
0
2
4
6
8
10
12
14
16
18
20
x_2(degree/sec)
Time(sec)



15
10
5
0
0
2
4
6
8
10
12
14
16
18
20
x_3(m)
Time(sec)



15
10
5
0
-5
0
2
4
6
8
10
12
14
16
18
20
x4 (m/sec)
Time(sec)



50
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
0
2
4
6
8
10
12
14
16
18
20
σ
Time(sec)

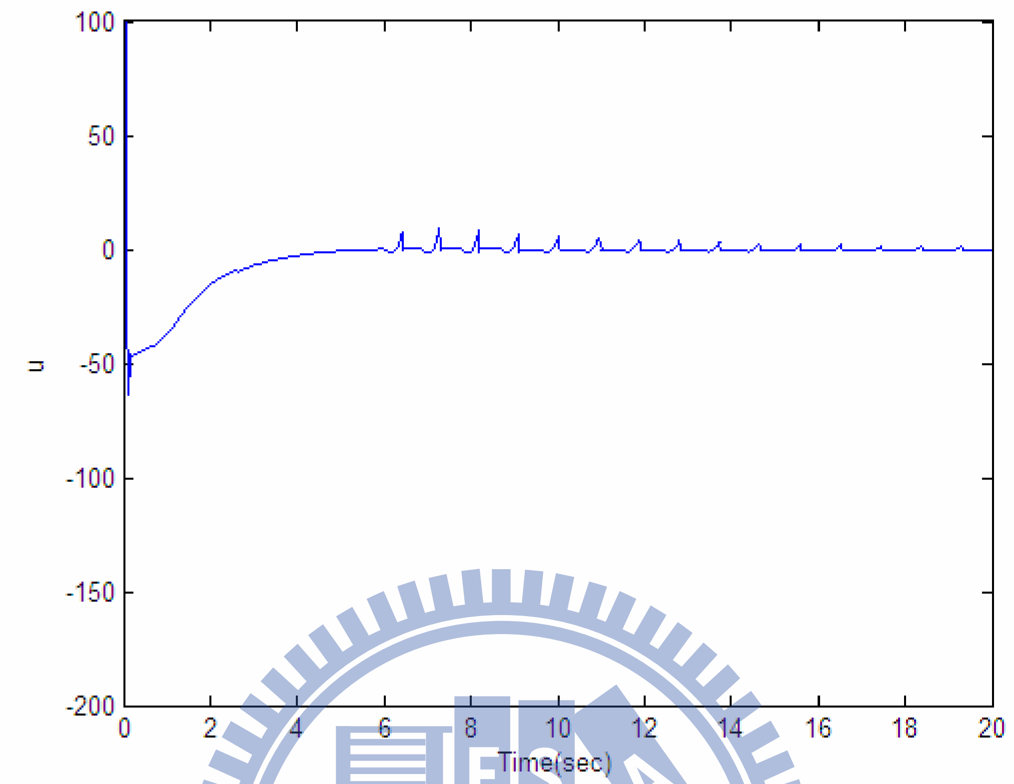
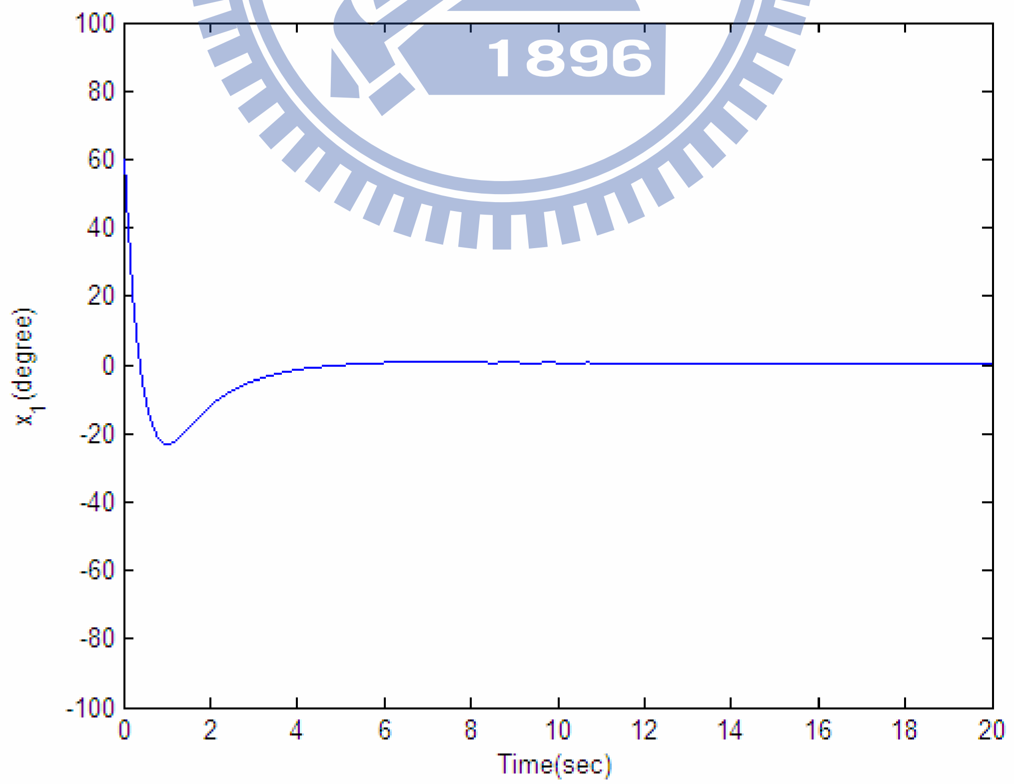
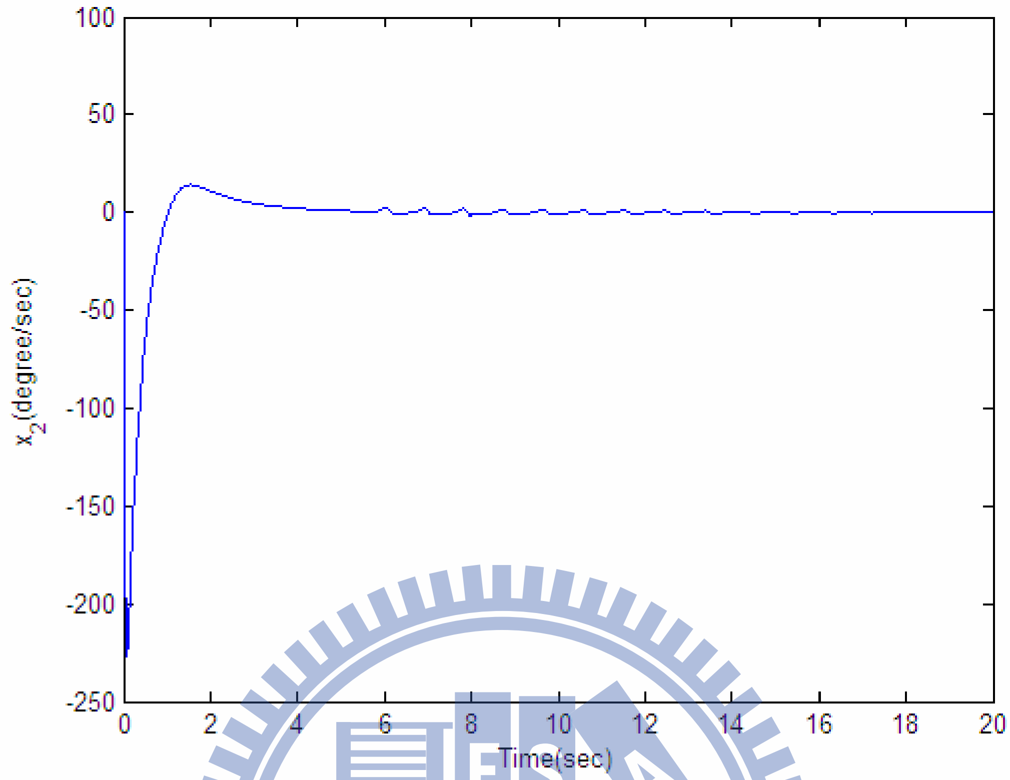
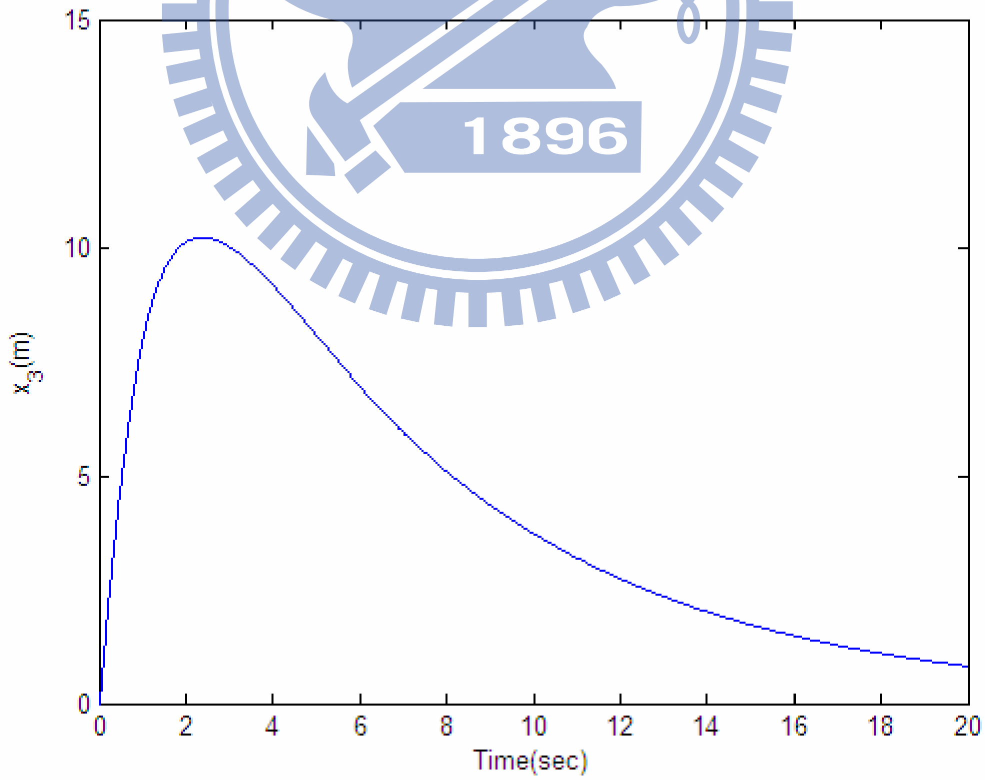


Figure 3.2 Simulation results with $x_1(0) = 40^{\circ}(2\pi/9\ \ rad),\ x_2(0) = x_3(0) = x_4(0) = 0.$

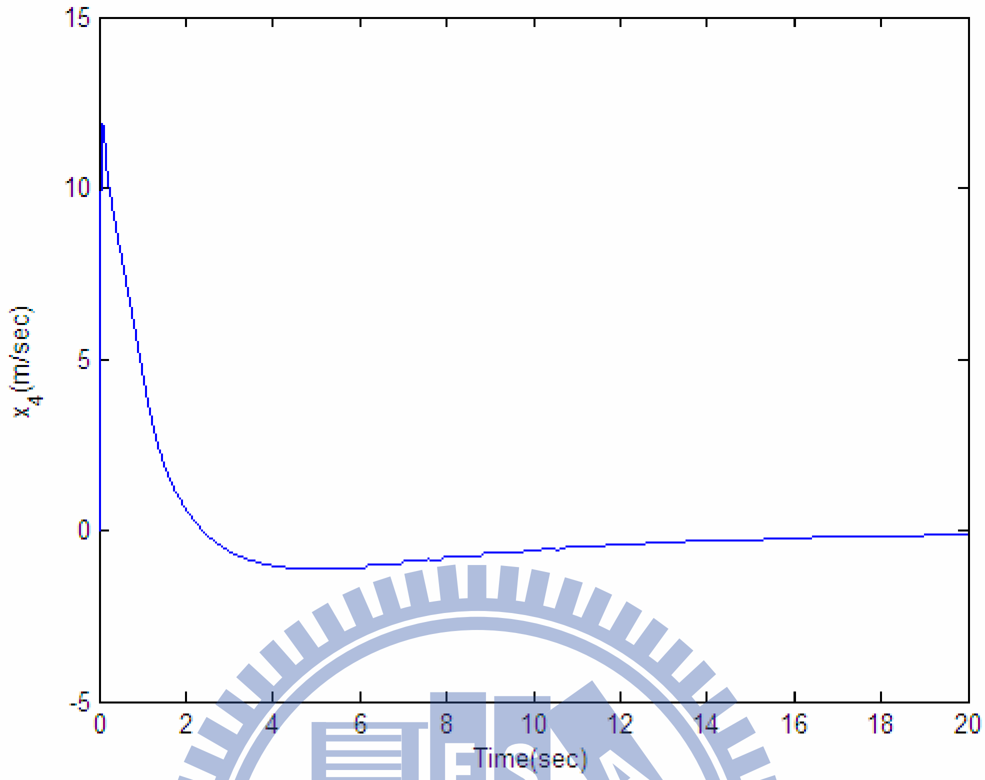




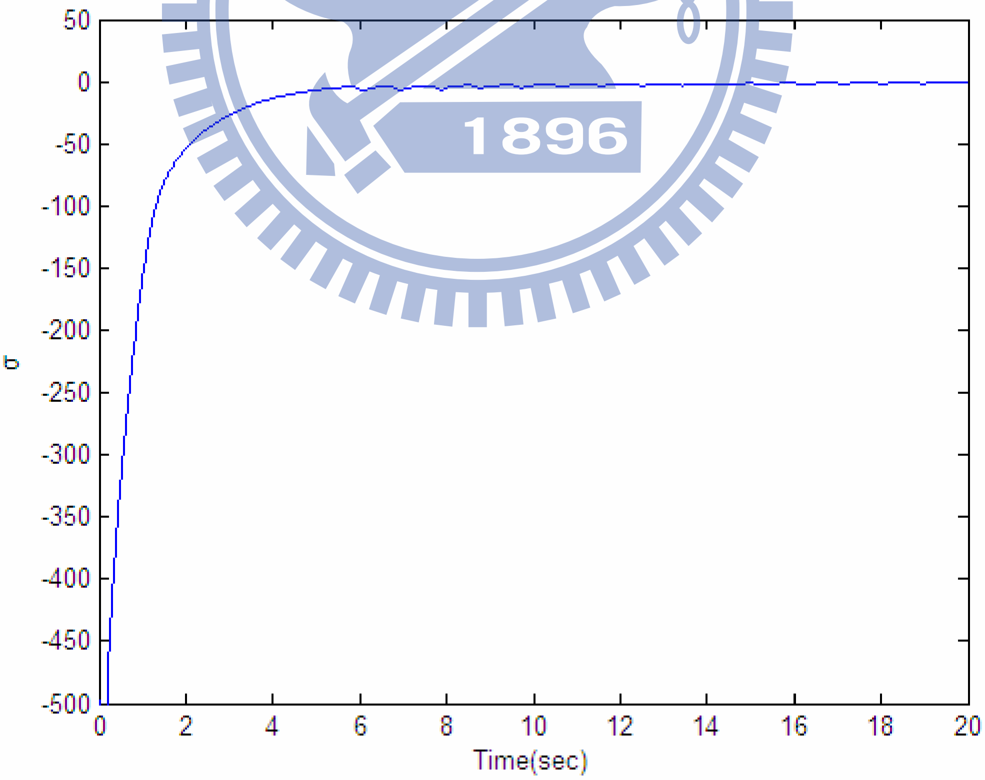
100
50
0
-50
-100
-150
-200
-250
x_2(degree/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



15
10
5
0
x_3(m)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



15
10
5
0
-5
x_4(m/sec)
0 2 4 6 8 10 12 14 16 18 20
Time(sec)



50
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
σ
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

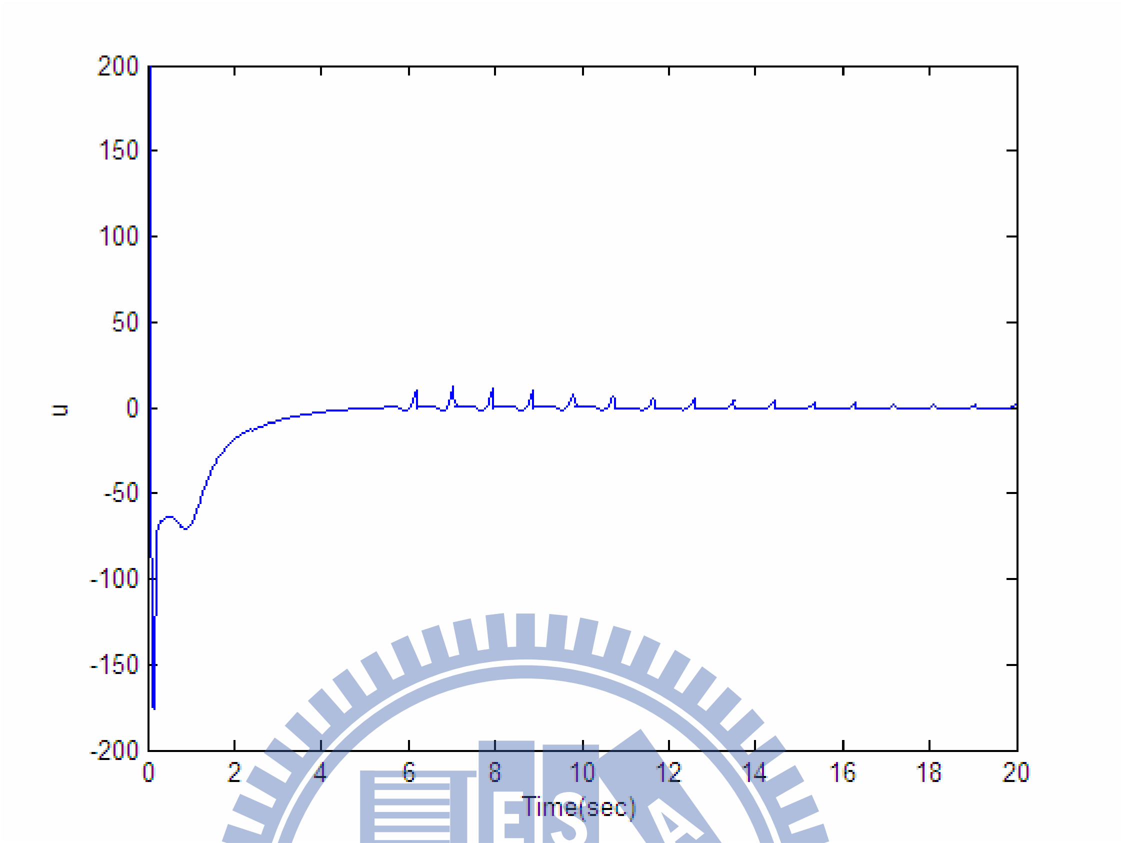


Figure 3.3 Simulation results with $x_1(0) = 60^\circ (\pi/3 \ \ rad),\ x_2(0) = x_3(0) = x_4(0) = 0.$

Example 3.2 Consider the following example of a ball and beam system [52], whose dynamic equations are described as follows:

$$\left(\frac{J_b}{R} + M\right)\ddot{r} + MG\sin\theta - Mr\dot{\theta}^2 = 0,\ \left(Mr^2 + J + J_b\right)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MGr\cos\theta = \tau \tag{3.30}$$

where r is the ball position, θ is the beam angle, J is the moment of inertia of the beam, M, J_b, and R are the mass, the moment of inertia, and the radius of the ball respectively, G is the acceleration of gravity, and τ is the torque applied to the beam.

Define $B = M/(J_b/R^2 + M)$ and change the coordinates in the input space by using the invertible transformation

$$\tau = 2Mr\dot{r}\dot{\theta} + MGr\cos\theta + Mr^2 + J + J_b)u \tag{3.31}$$

where u is the new input, then the ball and beam system can be written in the following state-space form:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = B(x_1 x_4^2 - G\sin x_3),\ \dot{x}_3 = x_4,\ \dot{x}_4 = u + d(t) \tag{3.32}$$

where $x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [r \quad \dot{r} \quad \theta \quad \dot{\theta}]^T$. The system parameters used for simulation are $M = 0.05kg$, $R = 0.01m$, $J = 0.02kgm^2$, $J_b = 2\times10^{-6}kgm^2$, $G = 9.81m/s^2$ and $B = 0.7143$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1\|x\|$ where ρ_0 and ρ_1 are known constants. Then, we approximate the system by the following two-rule fuzzy model:

Plant rule 1: IF x_1 is greater than 0, THEN

$$\dot{x} = A_1 x + B_1[u + h(t,x)].$$

Plant rule 2: IF x_1 is smaller than 0, THEN

$$\dot{x} = A_2 x + B_2[u + h(t,x)].$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & -2B\mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 2B\mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

$$\mu = 0.01,\ h(t,x) = d(t),\ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}},\quad \beta_2 = 1 - \beta_1. \tag{3.33}$$

By setting $\chi_i = 0.2, \alpha_i = 240, r = 2, k = 1, \rho_k = 1$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is greater than 0, THEN

$$u(t) = -0.2\sigma - SA_1 x - \frac{1}{1-\omega}\delta_1 \operatorname{sgn}(\sigma).$$

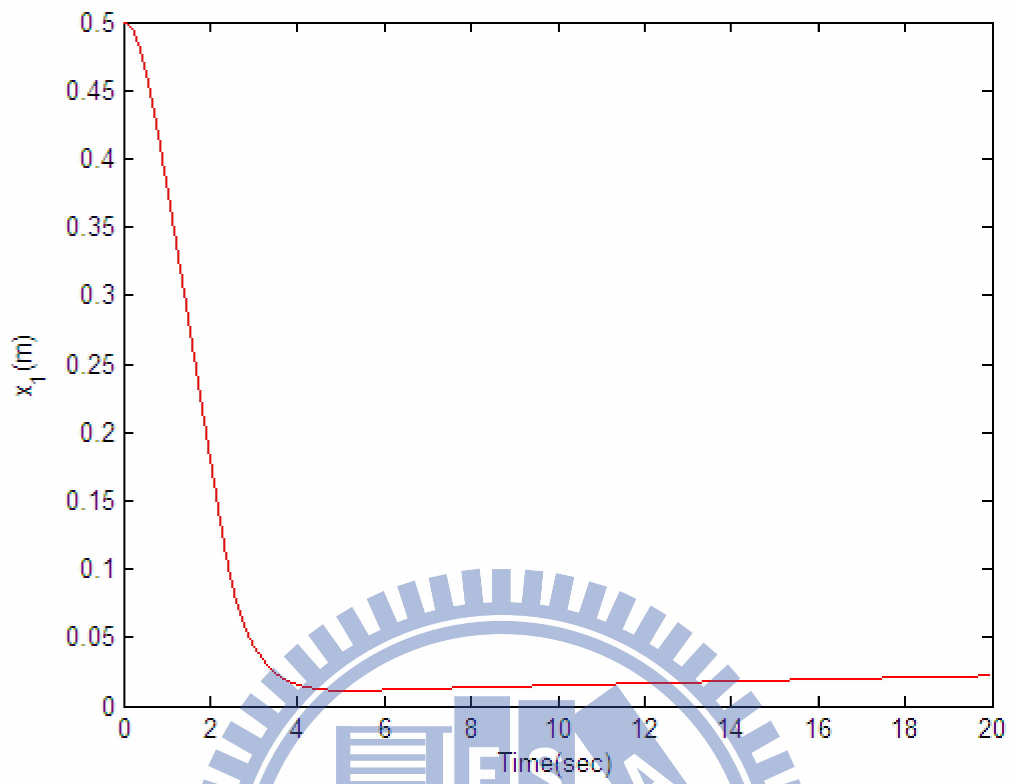
Control Rule 2: IF x_1 is smaller than 0, THEN

$$u(t) = -0.2\sigma - SA_2 x - \frac{1}{1-\omega}\delta_2 \operatorname{sgn}(\sigma).$$

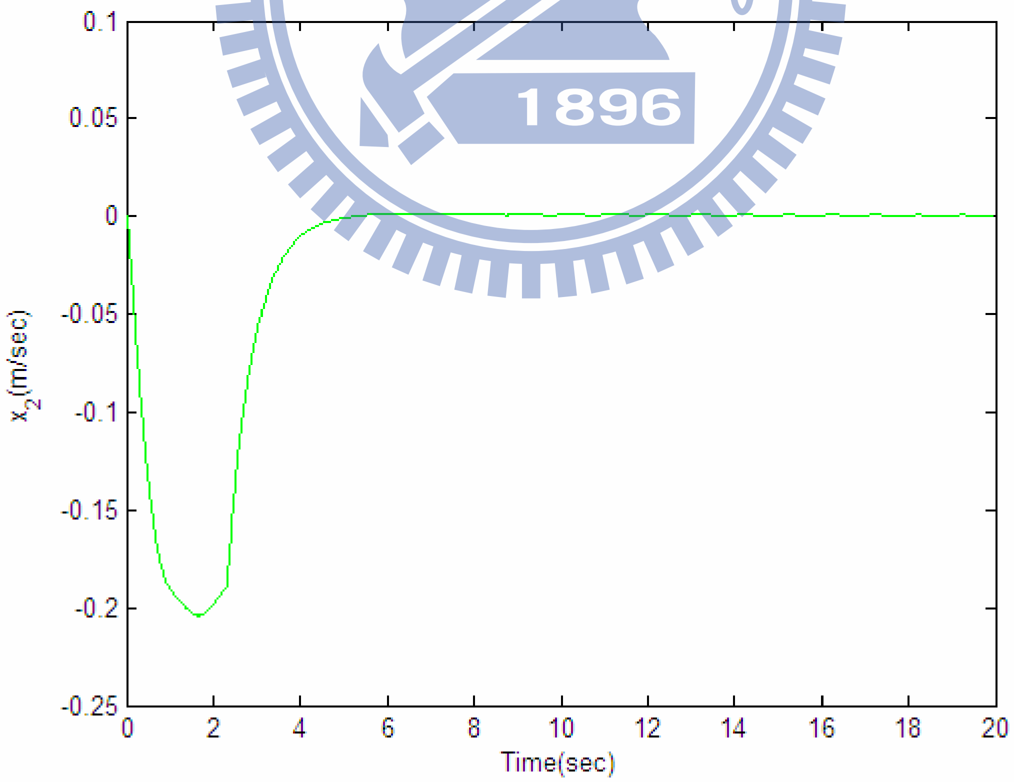
The final controller inferred as the weighted average of each local controller is given by

$$u(t) = -\sum_{i=1}^{r} \beta_i(\theta)\left[0.2\sigma + SA_i x + \frac{1}{1-\omega}\delta_i \operatorname{sgn}(\sigma)\right]. \tag{3.34}$$

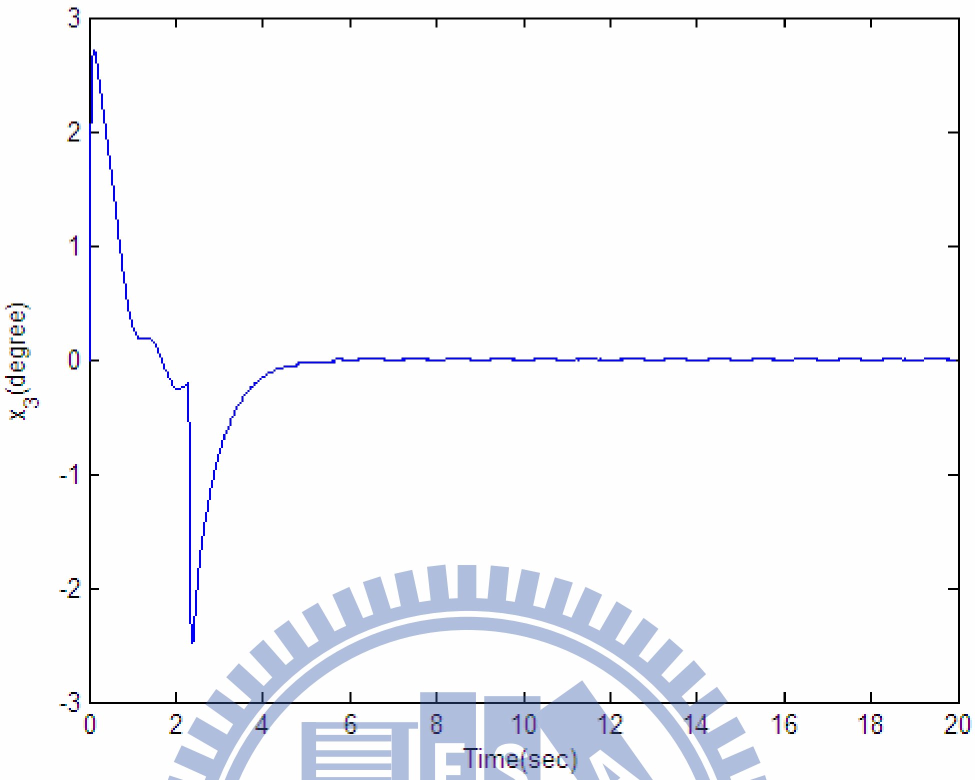
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (3.33) with nonzero $d(t)$. We assume that $d(t) = x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)$. Figure 3.4 shows the time histories of the state, the sliding variable σ, and the input (3.34) when $x_1(0) = 0.5$, $x_2(0) = x_3(0) = x_4(0) = 0$. Figure 3.5 shows the time histories of the state, the sliding variable σ, and the input (3.34) when $x_1(0) = 1$, $x_2(0) = x_3(0) = x_4(0) = 0$. In Figure 3.4 and Figure 3.5, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 3.4 and Figure 3.5, the proposed controller (3.34) also stabilizes the following two-rule fuzzy model (3.33).

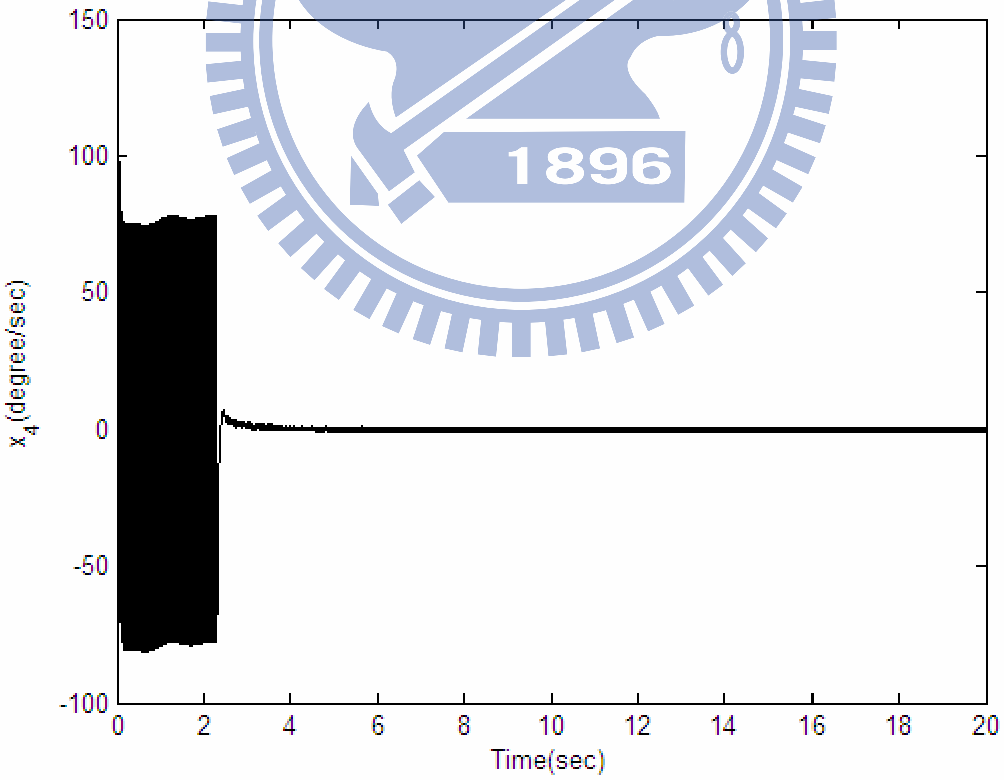


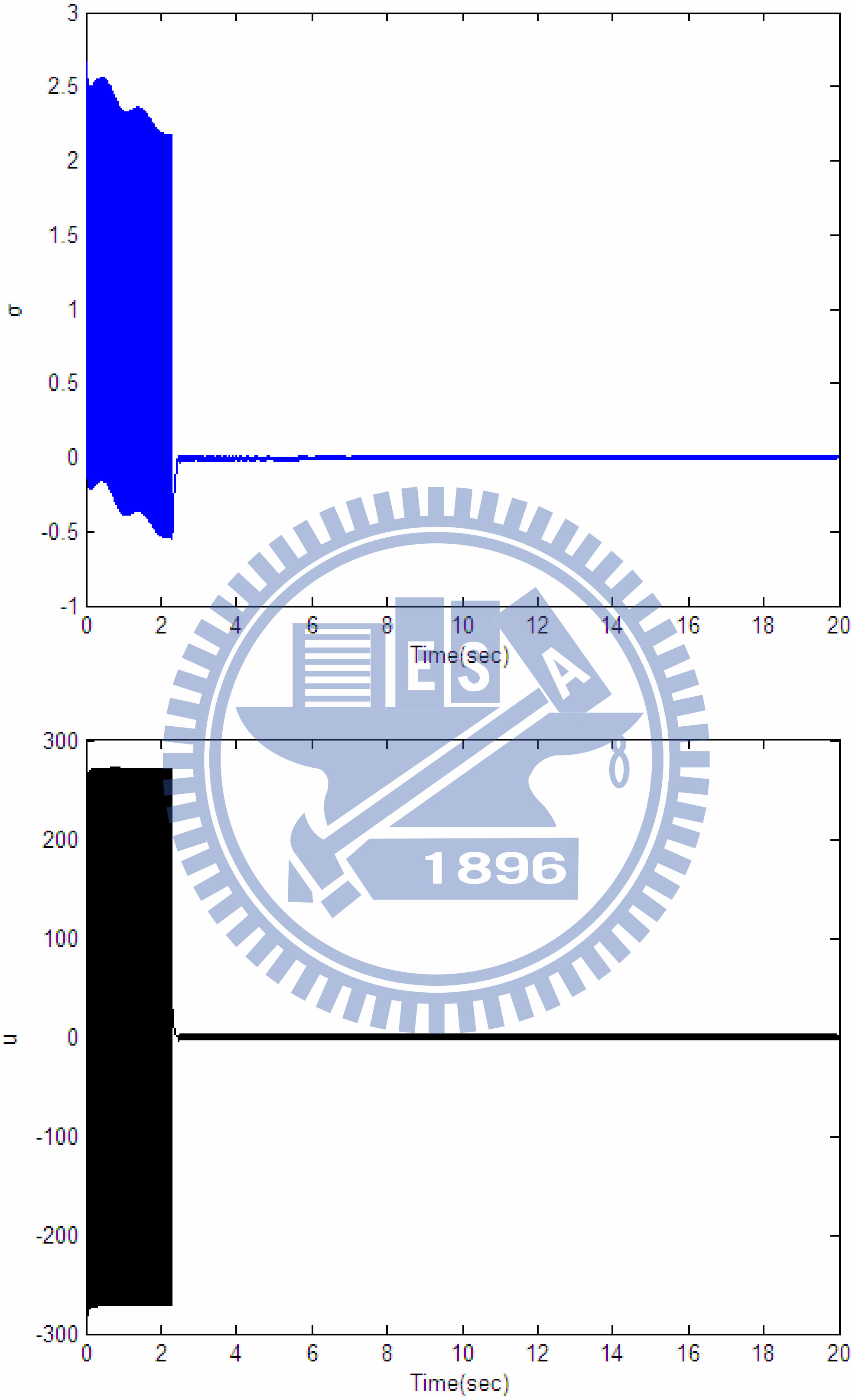
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
x_1(m)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
x_2(m/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


3
2
1
0
-1
-2
-3
x3(degree)
0 2 4 6 8 10 12 14 16 18 20
Time(sec)


150
100
50
0
-50
-100
x4(degree/sec)
0 2 4 6 8 10 12 14 16 18 20
Time(sec)



σ
Time(sec)
u
Time(sec)

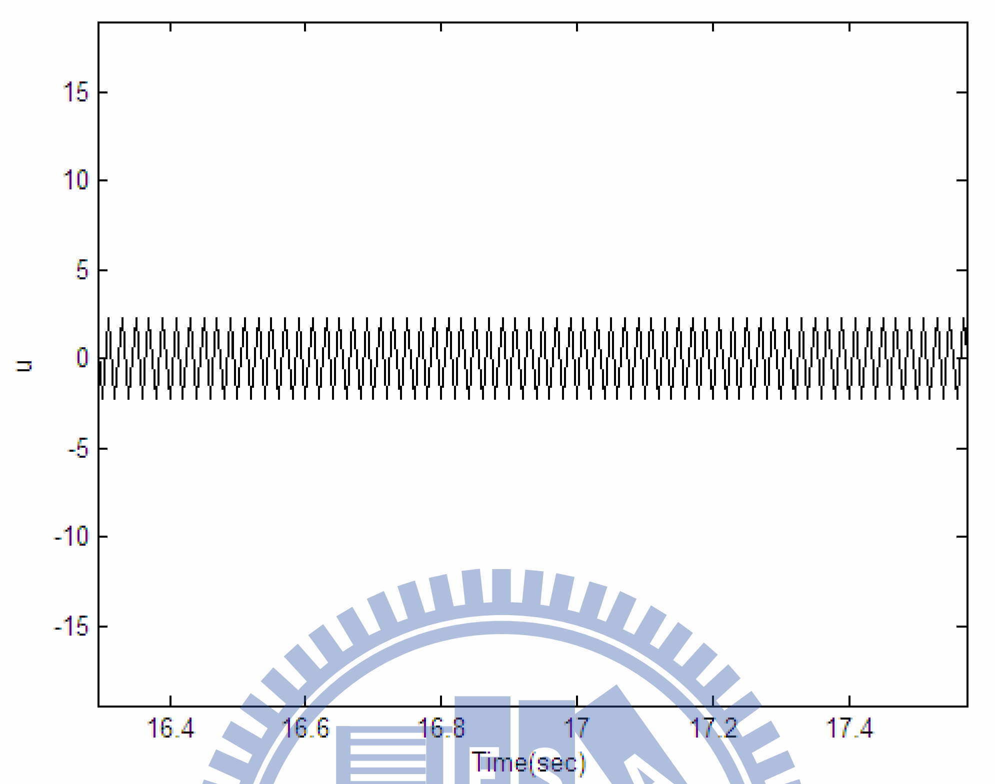
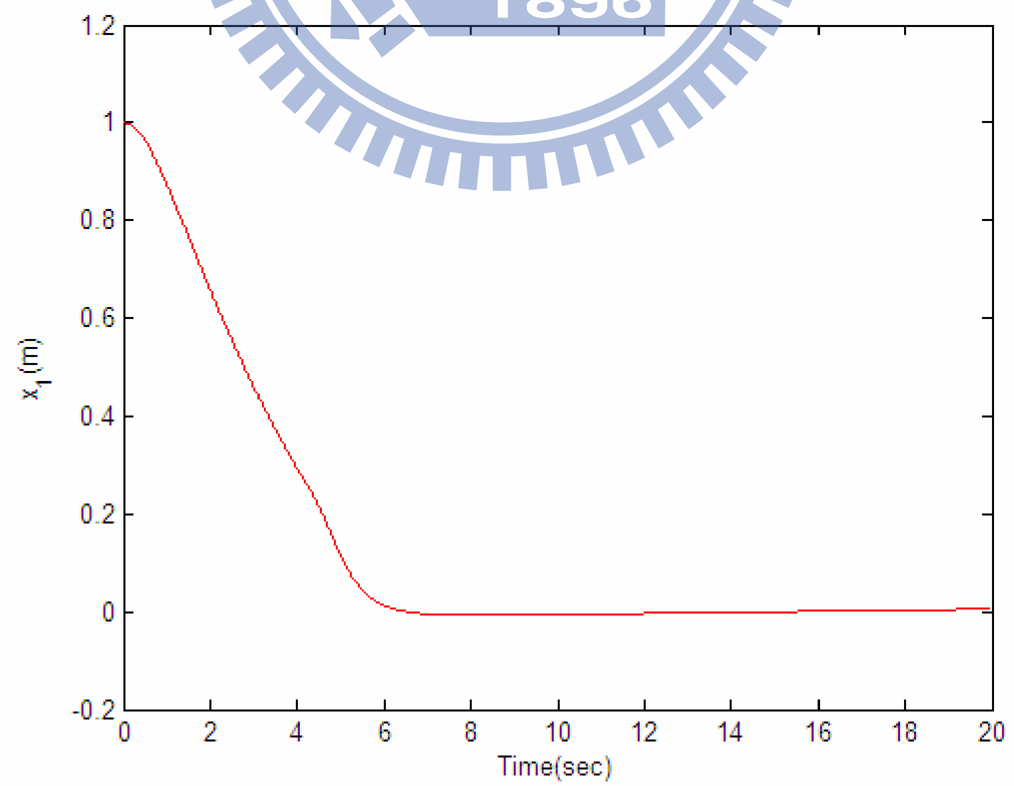
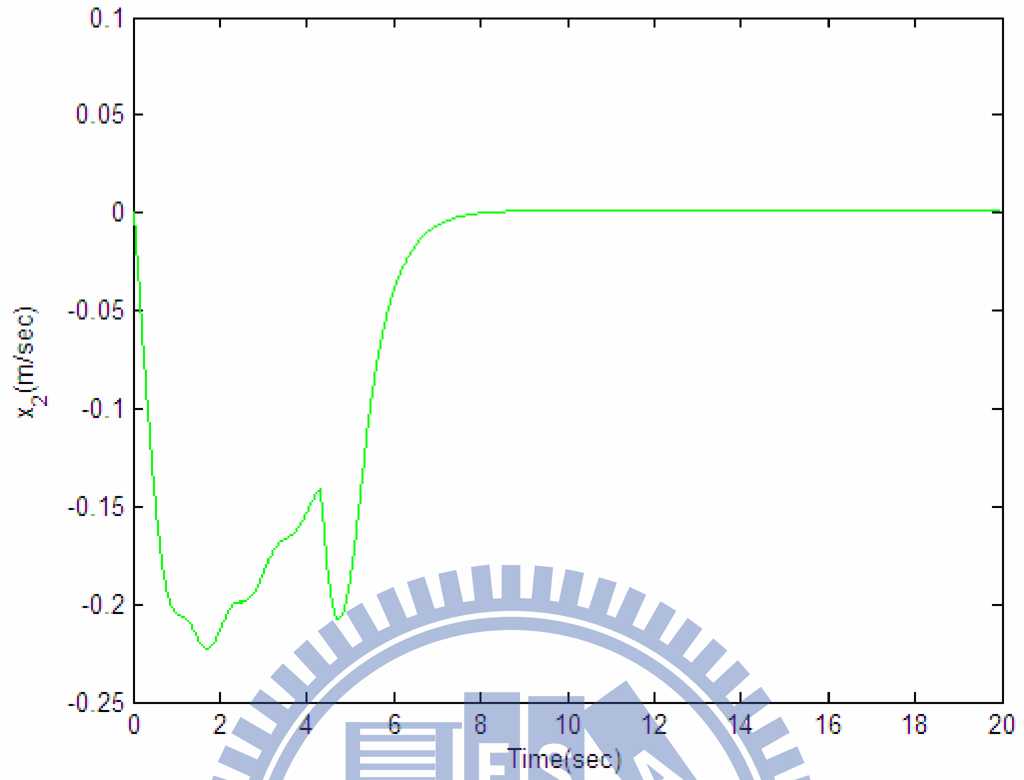
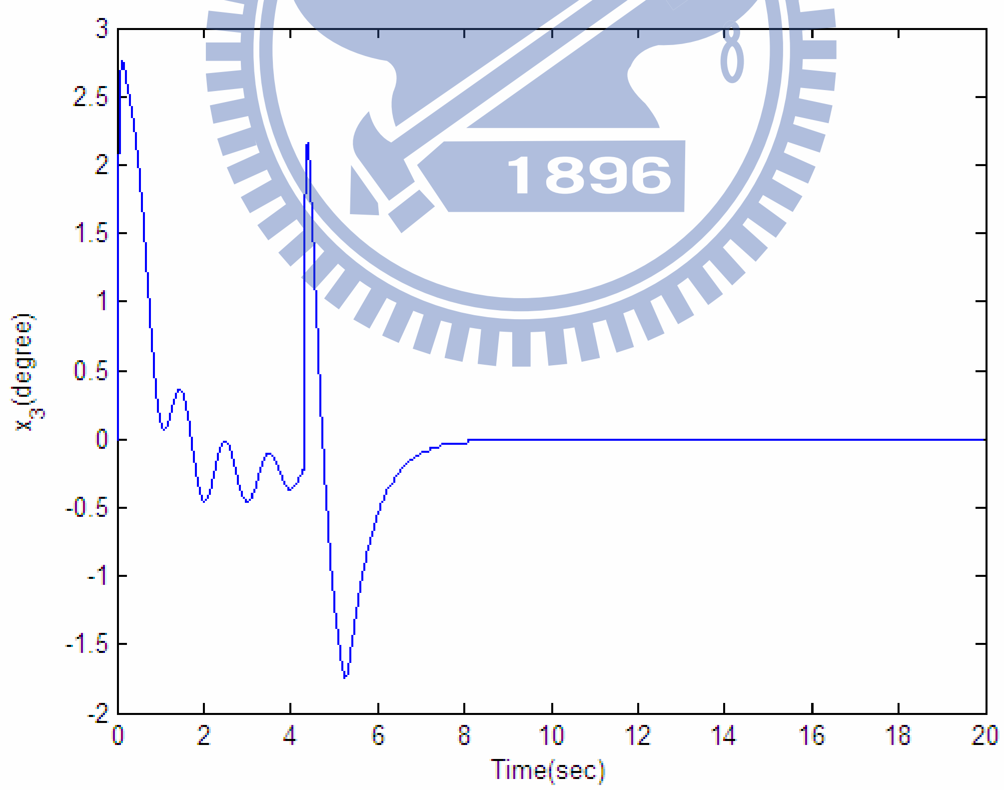


Figure 3.4 Simulation results with $x_1(0) = 0.5, x_2(0) = x_3(0) = x_4(0) = 0$, including amplifying the input u scale

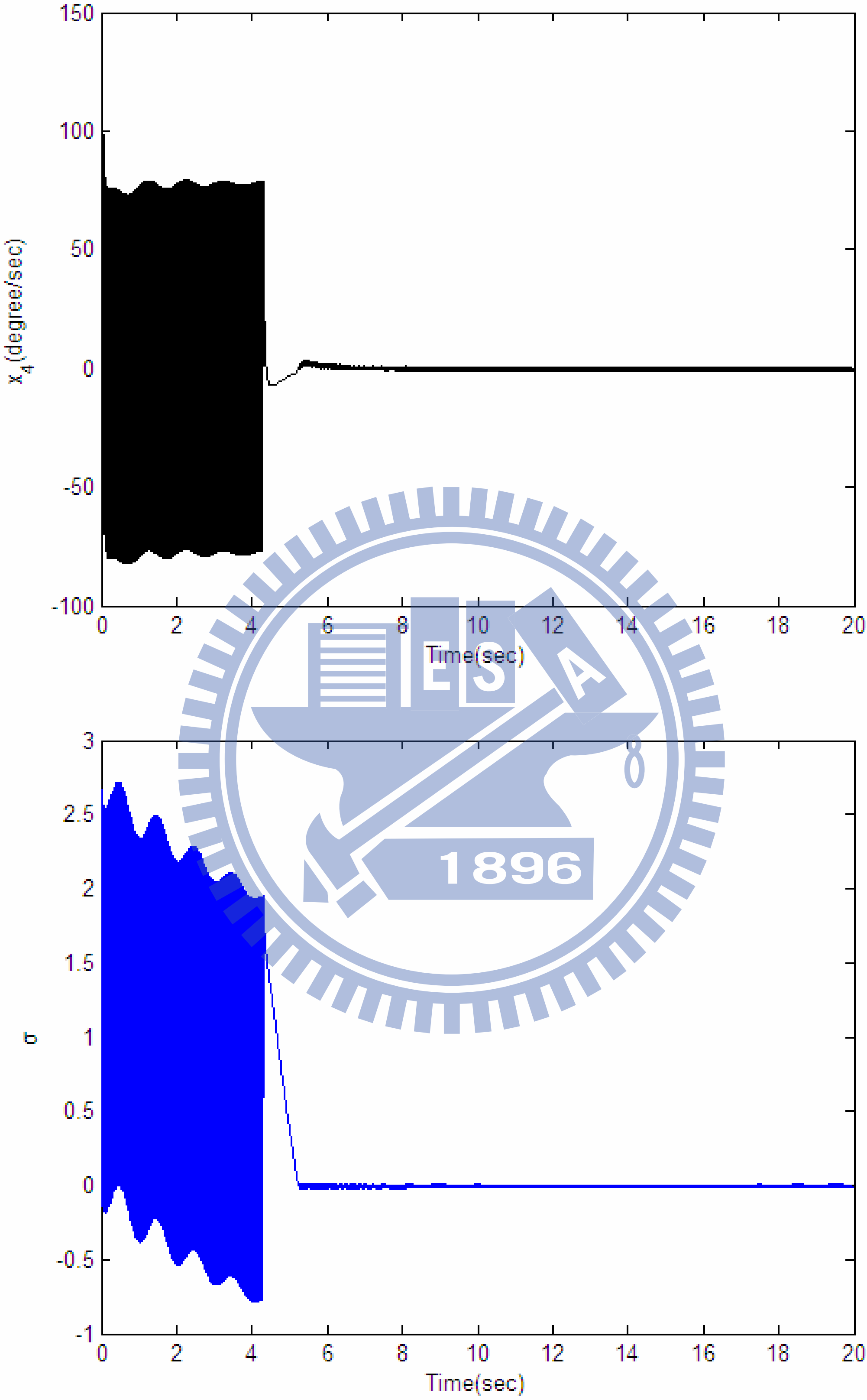




0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
2
4
6
8
10
12
14
16
18
20
Time(sec)
x_2(m/sec)



3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
2
4
6
8
10
12
14
16
18
20
Time(sec)
x_3(degree)


150
100
50
0
-50
-100
x_4(degree/sec)
0 2 4 6 8 10 12 14 16 18 20
Time(sec)
3
2.5
2
1.5
1
0.5
0
-0.5
-1
σ
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

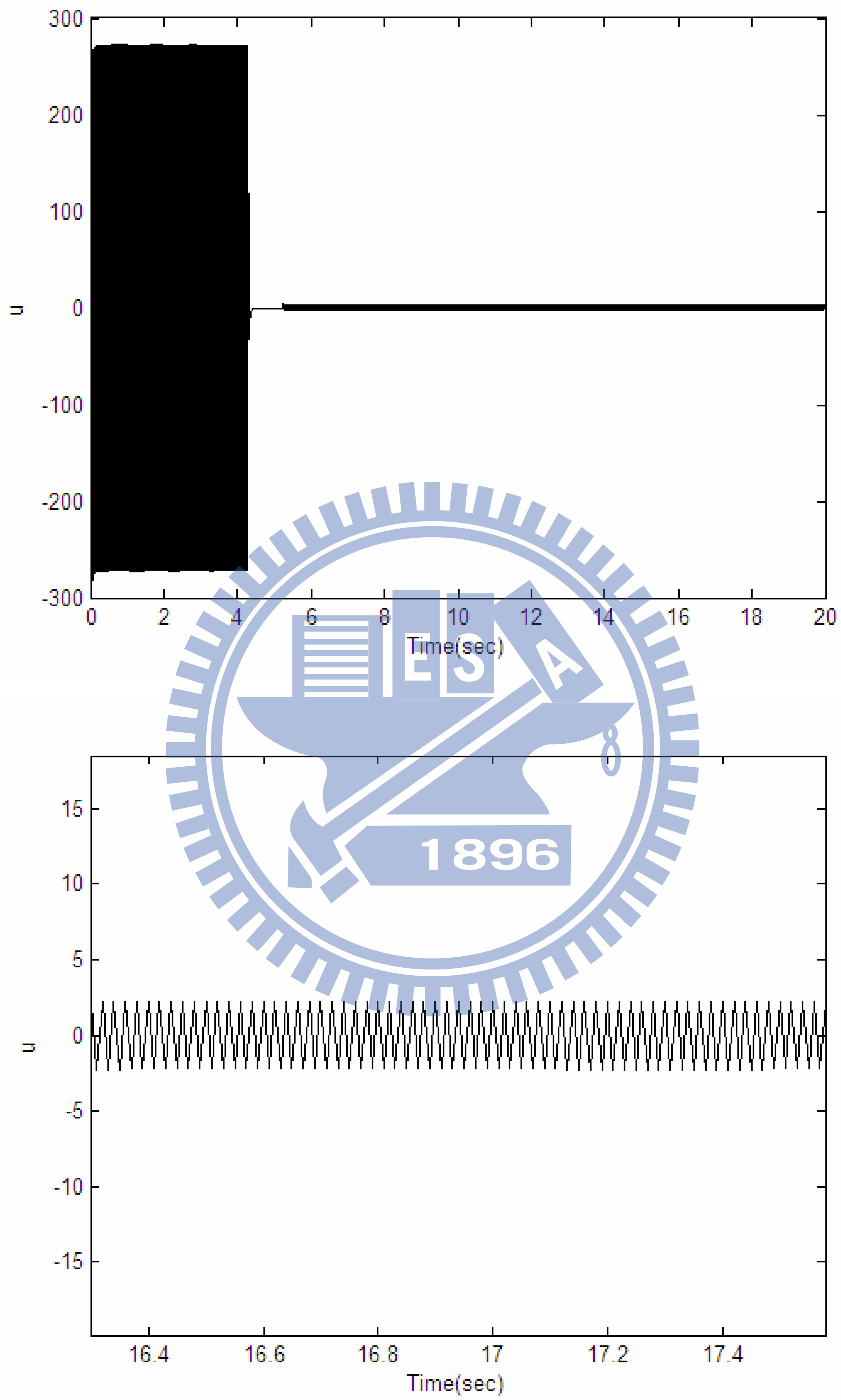


Figure 3.5 Simulation results with $x_1(0)=1, x_2(0)=x_3(0)=x_4(0)=0,$ including amplifying the input u scale

3.3 Robust Sliding Control for Mismatched T-S Fuzzy Systems

In this section, two kinds of system formulation for mismatched uncertain T-S fuzzy models are described in Section 3.3.1 and in Section 3.3.4, respectively. Two kinds of robust sliding control methods via LMI are proposed in Section 3.3.2 and in Section 3.3.5, respectively. Some examples are used to illustrate the effectiveness of the proposed methods and to compare with the existing methods in Section 3.3.3 and Section 3.3.6, respectively.

3.3.1 System Formulation I

Consider the following uncertain T-S fuzzy model [49], including parameter uncertainties and external disturbances:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)([A_i + \Delta A_i(t)]x(t) + B_i[u(t) + h(t,x)]) \tag{3.35}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, A_i, B_i are constant matrices of appropriate dimensions, $\Delta A_i(t)$ represents the parameter uncertainties in $A_i, h(t,x) \in R^m$ denotes external disturbances. $\theta = [\theta_1, \cdots, \theta_s], \theta_j (j = 1, \cdots, s)$ are the premise variables, s is the number of the premise variables, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta), \omega_i : R^s \to [0,1], i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i, r is the number of the IF-THEN rules, β_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0. \sum_{i=1}^{r} \beta_i(\theta) = 1$. We will assume that the followings are satisfied:

A1: The $n \times m$ matrix B defined by $B = \frac{1}{r} \sum_{i=1}^{r} B_i$ satisfies the rank constraint

$\text{rank}(B) = m$, i.e., the matrix B has full column rank m.

A2: The function $h(t,x)$ is unknown but bounded as $\left\| h(t,x) - \hat{h}(t,x) \right\| \le \sum_{k=0}^{l} \rho_k \left\| x \right\|^k$ where $\rho_0, \cdots, \rho_l$ are known constants, $\hat{h}(t,x)$ is an estimate of $h(t,x)$, and l is a known positive integer.

A3: $\Delta A_i(t)$ is of the form $T_i \Pi_i(t)$ where $\Pi_i(t)$ is a known time-varying matrix but bounded as $\left\| \Pi_i(t) \right\| \le 1$.

The system (3.35) does not have to satisfy the restrictive assumption that all the input matrices of the local system models are in the same range space. It should be noted that the assumption A1 implies that $rank(B_i) \le m$ and each nominal local system model may not share the same input channel. The assumption A2 with $l = 1$ and $\hat{h}(t,x) = 0$ has been used in the literature [50]. We can set $\hat{h}(t,x)$ as the nominal value of $h(t,x)$. Using the above assumptions, the uncertain T-S fuzzy model (3.35) can be written as follows.

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)(A_i + T_i \Pi_i(t))x(t) + [B + HF(\beta)G][u + h(t,x)] \tag{3.36}$$

where $\beta = [\beta_1(\theta), \cdots, \beta_r(\theta)]$, and the matrices $H, G, F(\beta)$ are defined by

$$H = \frac{1}{2}\left[(B - B_1), \cdots, (B - B_r)\right] G = \left[I, \cdots, I\right]^T,$$

$$F(\beta) = diag\left[(1 - 2\beta_1(\theta))I, \cdots, (1 - 2\beta_r(\theta))I\right] \tag{3.37}$$

It should be noted that the system (3.35) does not have to satisfy $B_1 = B_2 = \cdots = B_r$, which is used in almost all published results on VSS design methods including the VSS-based fuzzy control design methods of [33,34]. Hence the methods [30] and [31] cannot be applied to the above model (3.35). Since $\beta_i(\theta) \ge 0$ and

$\sum_{i=1}^{r}\beta(\theta)=1$, we can see that the following inequality always holds:

$$F^T(\beta)F(\beta)=F(\beta)F^T(\beta)\le I \tag{3.38}$$

Many examples in the literature and various mechanical systems such as motors and robots do not satisfy the restrictive assumptions that each nominal local system model shares the same input channel and they fall into the special cases of the above model [49].

3.3.2 LMI-based Sliding Control Design I

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions: The first involves the design of $m(n-1)-$dimensional switching surfaces for the sliding mode such that the reduced order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second is concerned with the selection of a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first characterize linear sliding surfaces using LMIs.

Let us define the linear sliding surface as $\sigma=Sx=0$ where S is a $m\times n$ matrix. Referring to the previous results [33], [51], we can see that for the system (3.36) it is reasonable to find a sliding surface such that

P1 $[SB+SHF(\beta)G]$ is nonsingular for any β satisfying $\beta_i(\theta)\ge 0, i=1,\cdots,r$, and $\sum_{i=1}^{r}\beta_i(\theta)=1$.

P2 The reduced $(n-m)$ order sliding mode dynamics restricted to the sliding surface $Sx=0$ is asymptotically stable for all admissible uncertainties.

It should be noted that P1 is necessary for the existence of the unique equivalent control [33] and the assumption A1 is necessary for the nonsingularity of SB

Define a transformation matrix and the associated vector v as $M=[\Lambda(\Lambda^T Y\Lambda)^{-1}, Y^{-1}B(B^T Y^{-1}B)^{-1}]^T = [V^T, S^T]^T$, $v=[v_1^T, v_2^T]^T = Mx$ where $v_1 \in R^{n-m}, v_2 \in R^m$. By the above transformation, we can see that $M^{-1}=[Y\Lambda, B]$ and $v_2=\sigma$. Then, from (3.36), we can obtain

$$\begin{bmatrix} \dot{v}_1 \\ \dot{\sigma} \end{bmatrix} = \sum_{i=1}^{r} \beta_i(\theta) \begin{bmatrix} V(A_i + T_i\Pi_i(t))Y\Lambda & V(A_i + T_i\Pi_i(t))B \\ S(A_i + T_i\Pi_i(t))Y\Lambda & S(A_i + T_i\Pi_i(t))B \end{bmatrix} \begin{bmatrix} v_1 \\ \sigma \end{bmatrix}$$

$$+ \begin{bmatrix} VHF(\beta)G \\ I + SHF(\beta)G \end{bmatrix} [u + h(t,x)]. \tag{3.39}$$

From the equivalent control method [33], we can see that the equivalent control is given by $u_{eq}(t) = -\sum_{i=1}^{r} \beta_i(\theta)[I + SHF(\beta)G]^{-1} S(A_i + T_i\Pi_i(t))x - h(t,x)$. By setting $\dot{\sigma} = \sigma = 0$ and substituting $u(t)$ with $u_{eq}(t)$, we can show that the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma = Sx = 0$ is given by

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y\Lambda)^{-1}\Lambda^T D(\beta)(A_i + T_i\Pi_i(t))Y\Lambda v_1 \tag{3.40}$$

where $D(\beta) = I - HF(\beta)G[I + SHF(\beta)G]^{-1} S$.

Theorem 3.3 Let us consider the sliding mode dynamics (3.40). If there exist matrices $Y \in R^{n\times n}$, $\Lambda \in R^{n\times(n-m)}$ satisfying $B^T\Lambda = 0, \Lambda^T\Lambda = I$, scalars $c_1 \in R, c_2 \in R, \eta \in R$, $\kappa = \lambda_{\min}(B^T B)$, and $*$ represents blocks that are readily inferred by symmetry such that the following LMIs holds:

$$\begin{bmatrix} \Lambda^T[(A_i + T_i\Pi_i(t))Y + *]\Lambda & * & * \\ \eta H^T\Lambda & -I & * \\ (A_i + T_i\Pi_i(t))Y\Lambda & \eta H & -I \end{bmatrix} < 0, \quad \forall i \tag{3.41}$$

$$\begin{bmatrix} Y & I & 0 \\ I & c_1 I & 0 \\ 0 & 0 & c_2 I - Y \end{bmatrix} > 0, \tag{3.42}$$

$$\begin{bmatrix} 2\eta\kappa & * & * \\ rc_1 & r\eta & 0 \\ rc_2 & 0 & r\eta \end{bmatrix} > 0 \tag{3.43}$$

then, there exists a linear sliding surface parameter matrix S satisfying P1-P2 and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \tag{3.44}$$

will guarantee that the sliding mode dynamics (3.40) is asymptotically stable.

Proof: By using Schur complement formula [48], we can easily show that in fact the following LMIs are incorporated in the LMIs (3.41)-(3.43)

$$c_1 > 0,\ \ c_2 > 0,\ \ \eta > 0,\ \ \eta^2 HH^T < I,\ 2\eta^2\kappa > r(c_1^2 + c_2^2). \tag{3.45}$$

It is clear that if the following inequality (3.46) holds, then $SB + SHF(\beta)G = I + SHF(\beta)G$ is nonsingular and hence P1 holds

$$SHF(\beta)GG^T F^T(\beta)H^T S^T < I. \tag{3.46}$$

Using (3.37), (3.38), (3.45) and $GG^T \le \|G\|^2 I = rI$, we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T. \tag{3.47}$$

By using the Schur complement formula, we can see that (3.42) and (3.45) imply

$$0 < c_1^{-1} I < Y < c_2 I, \qquad 0 < c_2^{-1} I < Y^{-1} < c_1 I \tag{3.48}$$

and this leads to

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T \le \frac{rc_1c_2}{\eta^2}(B^T B)^{-1} \le \frac{rc_1c_2}{\kappa\eta^2} I. \tag{3.49}$$

Using the inequality $2ab \le a^2 + b^2$ where a and b are scalars, we can show that (3.49) implies

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{2\kappa\eta^2}(c_1^2 + c_2^2)I. \tag{3.50}$$

Finally, by using the above inequalities (3.45) and (3.50), we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T < I \tag{3.51}$$

which implies that $[SB + SHF(\beta)G]$ is nonsingular, i.e., P1 holds.

Now, we will show that S of (3.44) guarantees P2. Using the matrix inversion lemma:

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B$$

where A and B are compatible constant matrices such that $(I + AB)$ is nonsingular, we can show that the sliding mode dynamics (3.40) is equivalent to

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y\Lambda)^{-1}\Lambda^T C(\beta)(A_i + T_i\Pi_i(t))Y\Lambda v_1 \tag{3.52}$$

where $v_1 = (\Lambda^T Y\Lambda)^{-1}\Lambda^T x$ and $C(\beta) = I - H[I + F(\beta)GSH]^{-1}F(\beta)GS$.

The previous results [53,54] imply that sliding mode dynamics (3.52) is asymptotically stable. Hence, the sliding mode dynamics (3.40) is asymptotically stable.

After the switching surface parameter matrix S is designed so that the reduced $(n-m)$ order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the the reaching mode such that the reachability condition is met. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintains it there for all subsequent time. With σ of (3.52), we design a sliding fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear sliding switching feedback control law as the local controller.

Control rule *i*: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\hat{h}(t,x) - \chi_i\sigma - S(A_i + T_i\Pi_i(t))x - \frac{1}{1-\omega}\delta_i(t,x)\frac{\sigma}{\|\sigma\|}$$

where $\delta_i(t,x) = \alpha_i + \omega \| S(A_i + T_i \Pi_i(t))x \| + (1+\omega) \sum_{k=0}^{l} \rho_k \|x\|^k$ (3.53)

and $\sigma = Sx, \omega = \sqrt{r} \| SH \|, \alpha_i > 0, \chi_i > 0$. It should be noted that (3.51) implies $\omega = \le \sqrt{r} \|S\| \cdot \|H\| \le \eta \|H\|$. This and (3.45) guarantee $0 \le \omega < 1$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\hat{h}(t,x) - \sum_{i=1}^{r} \beta_i(\theta) \left(\chi_i \sigma + S(A_i + T_i \Pi_i(t))x + \frac{1}{1-\omega} \delta_i(t,x) \frac{\sigma}{\|\sigma\|} \right) \tag{3.54}$$

and we can establish the following theorem.

Theorem 3.4 Consider the closed-loop control system of the uncertain system (3.36) with control (3.54). Suppose that the LMIs (3.41)-(3.43) has a solution vector (Y, c_1, c_2, η) and the linear sliding surface is given by (3.44). Then the state converges to zero.

Proof: Since Theorem 3.3 implies that the linear sliding surface (3.44) guarantees P1-P2, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T \sigma$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T \dot{\sigma}$. From (3.36), (3.44), (3.54), $\|SHF(\beta)G\| \le \sqrt{r} \| SH \| = \omega, 0 \le \omega < 1$, and A2, we obtain

$$\sigma^T \dot{\sigma} = \sigma^T \sum_{i=1}^{r} \beta_i(\theta) S(A_i + T_i \Pi_i(t))x(t) + \sigma^T [I + SHF(\beta)G][u + h(t,x)]$$

$$\le \sum_{i=1}^{r} \beta_i(\theta) \sigma^T S(A_i + T_i \Pi_i(t))x(t) + \sigma^T u + \{\omega \|u\| + (1+\omega) \|h(t,x)\|\} \|\sigma\|.$$

This implies that $\dot{E}_g \le -(1-\omega) \sum_{i=1}^{r} \beta_i(\theta) \chi_i \|\sigma\|^2 - \sum_{i=1}^{r} \beta_i(\theta) \alpha_i \|\sigma\| \le 0$ which indicates that $E_g \in L_2 \cap L_\infty, \dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 3.2 Theorem 3.3 and 3.4 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Obtain $B = \frac{1}{r}\sum_{i=1}^{r} B_i$ and $H = \frac{1}{2}\left[(B - B_1), \cdots, (B - B_r)\right]$ for given B_i.

Step 2: Check that (A_i, B) is stabilization. If not, exit.

Step 3: Find a solution vector (Y, c_1, c_2, η) to LMI (3.41)-(3.43).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (3.44).

Step 5: The controller is given by (3.54).

3.3.3 Numerical Examples I

Example 3.3 Consider the following two-rule fuzzy model from a VTOL helicopter model [55]

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = (A_1 + T_1\Pi_1(t))x + B_1[u + h(t, x)]$$

Plant Rule2: IF x_1 is about ± 2, THEN

$$\dot{x} = (A_2 + T_2\Pi_2(t))x + B_2[u + h(t, x)]$$

where $A_1 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3181 & -0.7070 & 1.4100 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $T_1 = T_2 = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} -0.0366 & 0.0271 & .0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.4181 & -0.7070 & 1.4300 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.6446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \; \Pi_1(t) = \Pi_2(t) = \begin{bmatrix} 0 & 0 & \sin t & 0 \end{bmatrix},$$

$$h(t,x)=d(t)+\begin{bmatrix}0.9\sin 3t & 0.9\sin 3t\end{bmatrix}^T,\ \beta_1=\frac{1-1/(1+e^{-14(x_1-1)})}{1+e^{-14(x_1+1)}},\ \beta_2=1-\beta_1. \tag{3.55}$$

It should be noted that T_1 and T_2 are not matched and thus the previous VSS-based fuzzy control design methods cannot be applied to the above system (3.55). Via LMI optimization with (3.55), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x)=\begin{bmatrix}0.9\sin 3t & 0.9\sin 3t\end{bmatrix}^T$, $\chi_i=5, \alpha_i=0.1, r=2, l=1, \rho_k=1$, and $t_{sampling}=0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t)=\begin{bmatrix}-0.9\sin 3t & -0.9\sin 3t\end{bmatrix}^T-5\sigma-S(A_1+T_1\Pi_1(t))x-\frac{1}{1-\omega}\delta_1\,\mathrm{sgn}(\sigma).$$

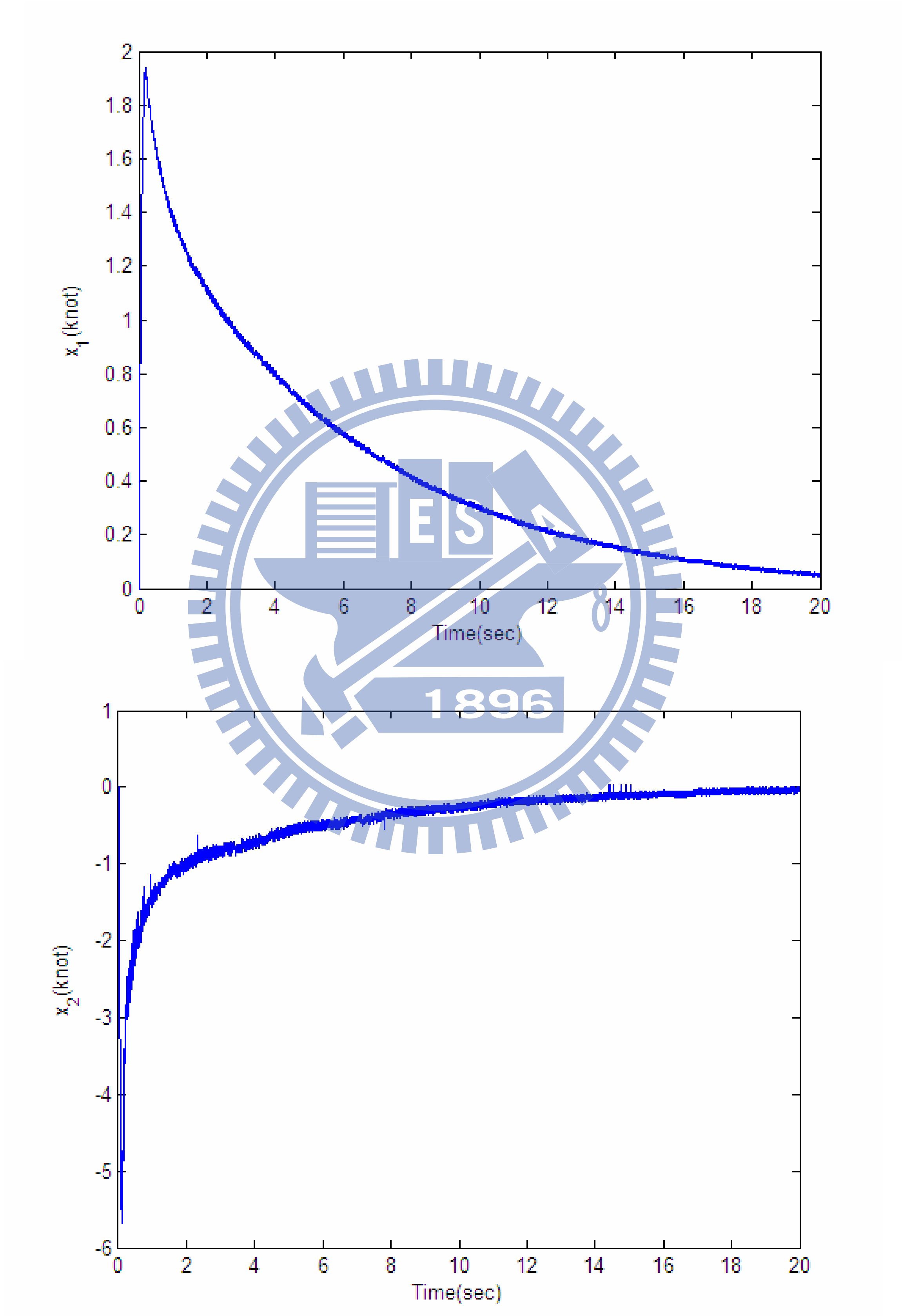
Control Rule 2: IF x_1 is about ± 2, THEN

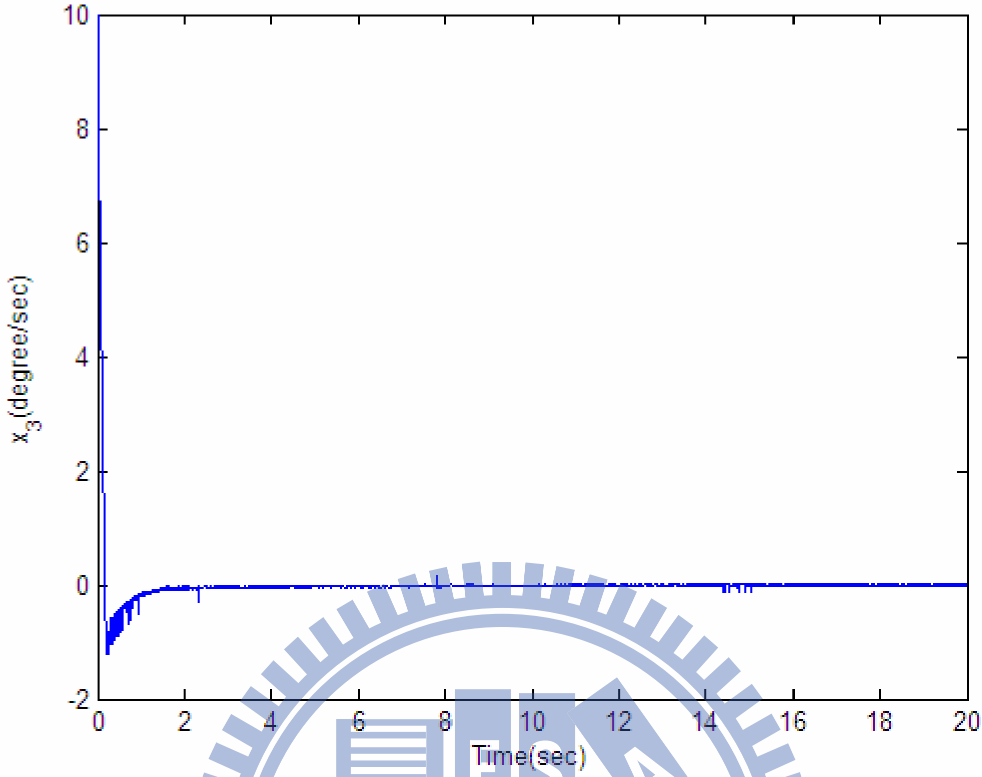
$$u(t)=\begin{bmatrix}-0.9\sin 3t & -0.9\sin 3t\end{bmatrix}^T-5\sigma-S(A_2+T_2\Pi_2(t))x-\frac{1}{1-\omega}\delta_2\,\mathrm{sgn}(\sigma).$$

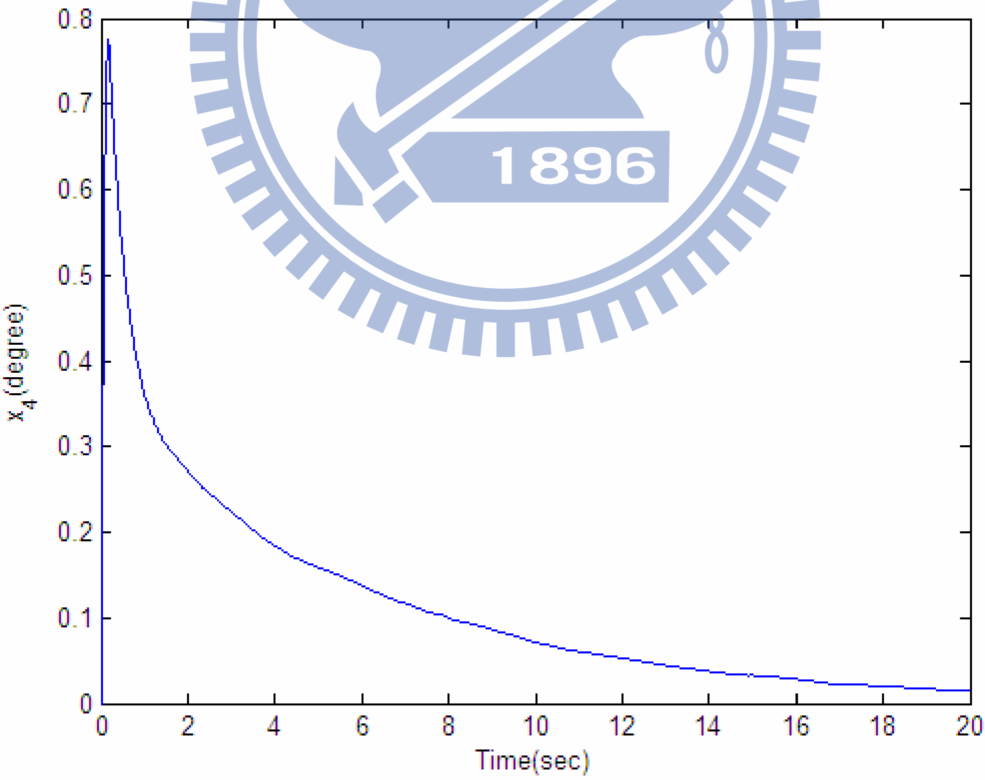
The final controller inferred as the weighted average of each local controller is given by

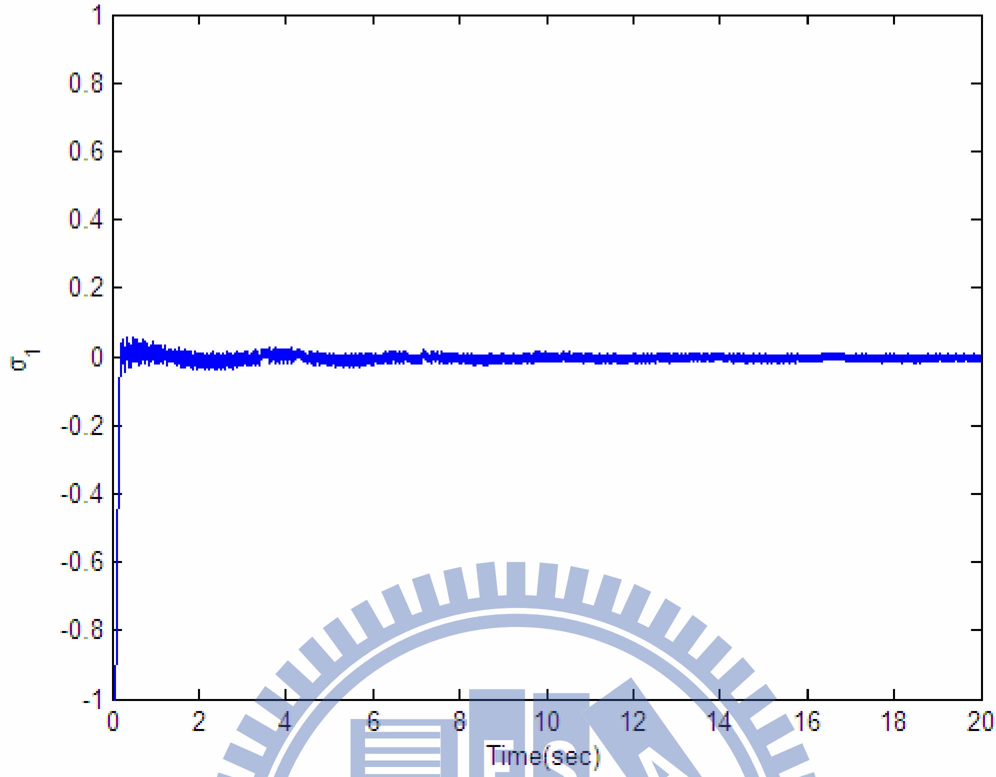
$$u(t)=\begin{bmatrix}-0.9\sin 3t & -0.9\sin 3t\end{bmatrix}^T-\sum_{i=1}^{r}\beta_i(\theta)\left[5\sigma+S(A_i+T_i\Pi_i(t))x+\frac{1}{1-\omega}\delta_i\,\mathrm{sgn}(\sigma)\right]. \tag{3.56}$$

To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (3.55) with nonzero $d(t)$. We assume that $d(t)=\begin{bmatrix}x_1\sin 2t-0.5\,\mathrm{sgn}(x_4) & x_1\sin 2t-0.5\,\mathrm{sgn}(x_4)\end{bmatrix}^T$. The time histories of the state, the sliding variable σ, and the input (3.56) are shown in Figure 3.6 when $x_1(0)=x_2(0)=x_4(0)=0,\ x_3(0)=10$. In Figure 3.6, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 3.6, the proposed controller is applicable to T-S fuzzy systems with mismatched parameter uncertainties in the state matrix and external disturbances. The control performances are satisfactory. It should be noted that all existing VSS-based fuzzy control system design methods cannot be applied to the two-rule fuzzy model (3.55) because B_1 is not in the range space of B_2.

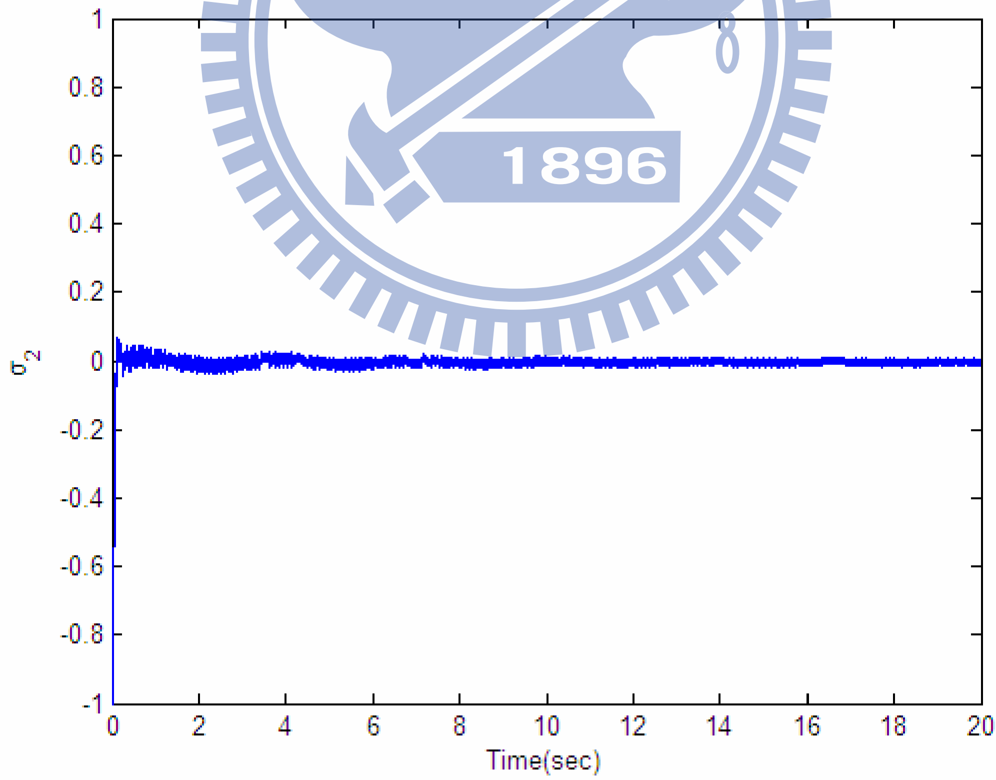



x_3(degree/sec)
Time(sec)


x_4(degree)
Time(sec)



σ1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0 2 4 6 8 10 12 14 16 18 20
Time(sec)



σ2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

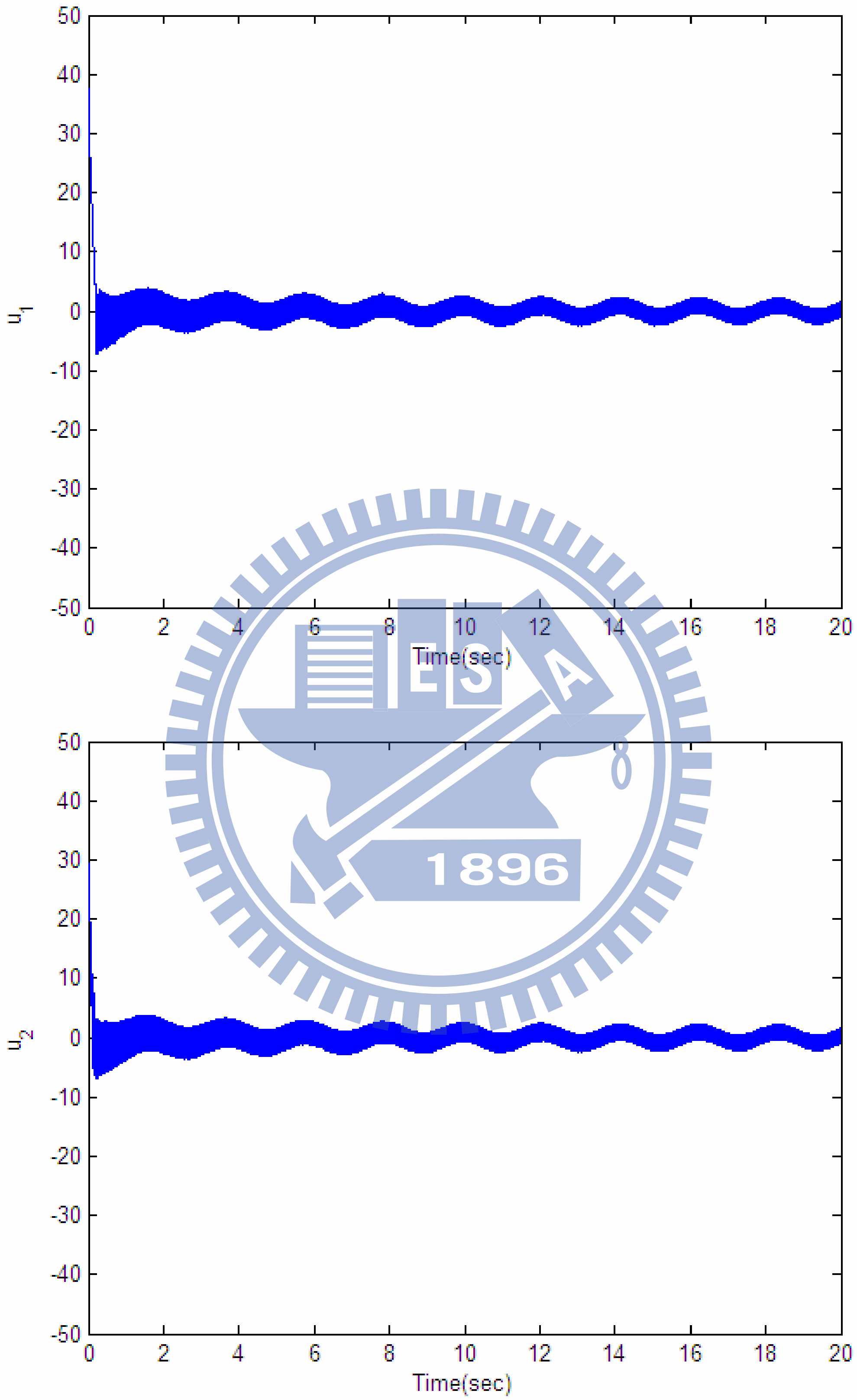


Figure 3.6 Simulation results with $x_1(0) = x_2(0) = x_4(0) = 0,\ x_3(0) = 10$.

Example 3.4 For the special case of $\Pi_i(t) \equiv 0$, the robust sliding controller design is proposed in [54]. Consider the following inverted pendulum on a cart

$$\dot{x}_1 = x_2, \quad \dot{x}_3 = x_4, \quad \dot{x}_2 = \frac{1}{l\psi}(3g\sin x_1 - 3a\cos x_1[u + d(t) + \phi]),$$

$$\dot{x}_4 = -\frac{1}{\psi}(1.5mag\sin 2x_1 - 4a[u + d(t) + \phi]) \tag{3.57}$$

where x_1 is the angle (rad) of the pendulum from the vertical, $x_2 = \dot{x}_1$, x_3 is the displacement (m) of the cart, $x_4 = \dot{x}_3$, $\psi = 4 - 3ma\cos^2 x_1$, $\phi = mlx_2^2 \sin x_1$, u is the input, and $d(t)$ is related to external disturbances which may be caused by the frictional force. $a = 1/(m + M)$, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, $g = 9.8m/s^2$ is the gravity constant. We set $M = 9kg$, $m = 1kg$, $l = 1m$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1 \|x\|$ where ρ_0 and ρ_1 are known constants. Here, we approximate the system (23) by the following two-rule fuzzy model.

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = A_1 x + B_1[u + h(t,x)].$$

Plant Rule2: IF x_2 is about $\pm 60^\circ (\pm\pi/3 \ rad)$, THEN

$$\dot{x} = A_2 x + B_2[u + h(t,x)].$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.9459 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.7946 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -0.0811 \\ 0 \\ 0.1081 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3097 & 0 & 0 & 0 \end{bmatrix}$,

$$B_2 = \begin{bmatrix} 0 \\ -0.0382 \\ 0 \\ 0.1019 \end{bmatrix}, h(t,x) = d(t) + x_2^2 \sin x_1, \ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}}, \ \beta_2 = 1 - \beta_1. \quad (3.58)$$

Because B_1 is not in the range space of B_2, all existing VSS-based fuzzy control system design methods cannot be applied to the above system (3.58). Via LMI optimization with (3.58), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x) = x_2^2 \sin x_1, \chi_i = 5, \alpha_i = 1, r = 2, l = 1, \rho_k = 1$, and $t_{sampling} = 0.01\text{sec}$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_1 x - \frac{1}{1-\omega}\delta_1 \,\text{sgn}(\sigma).$$

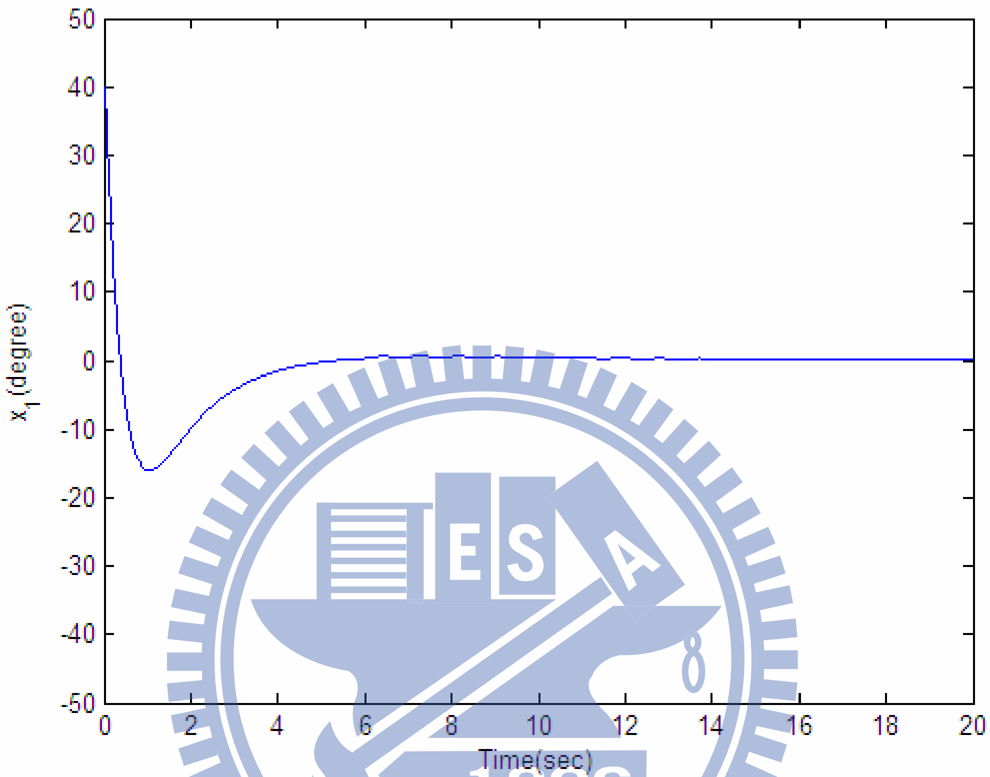
Control Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3 \ rad)$, THEN

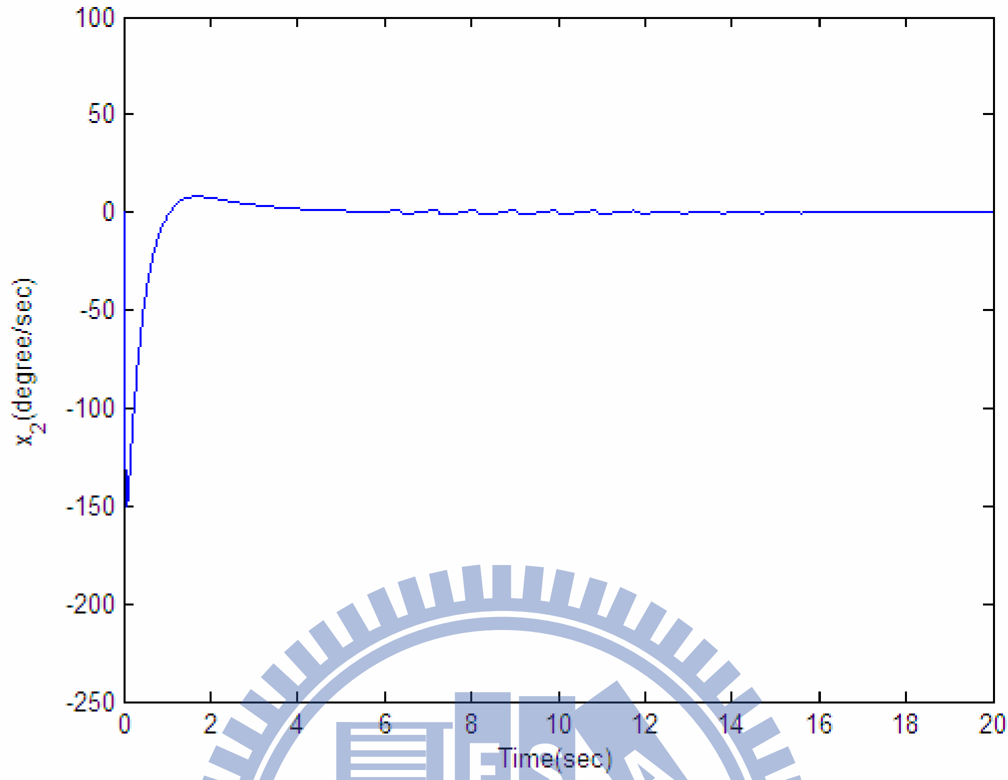
$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_2 x - \frac{1}{1-\omega}\delta_2 \,\text{sgn}(\sigma).$$

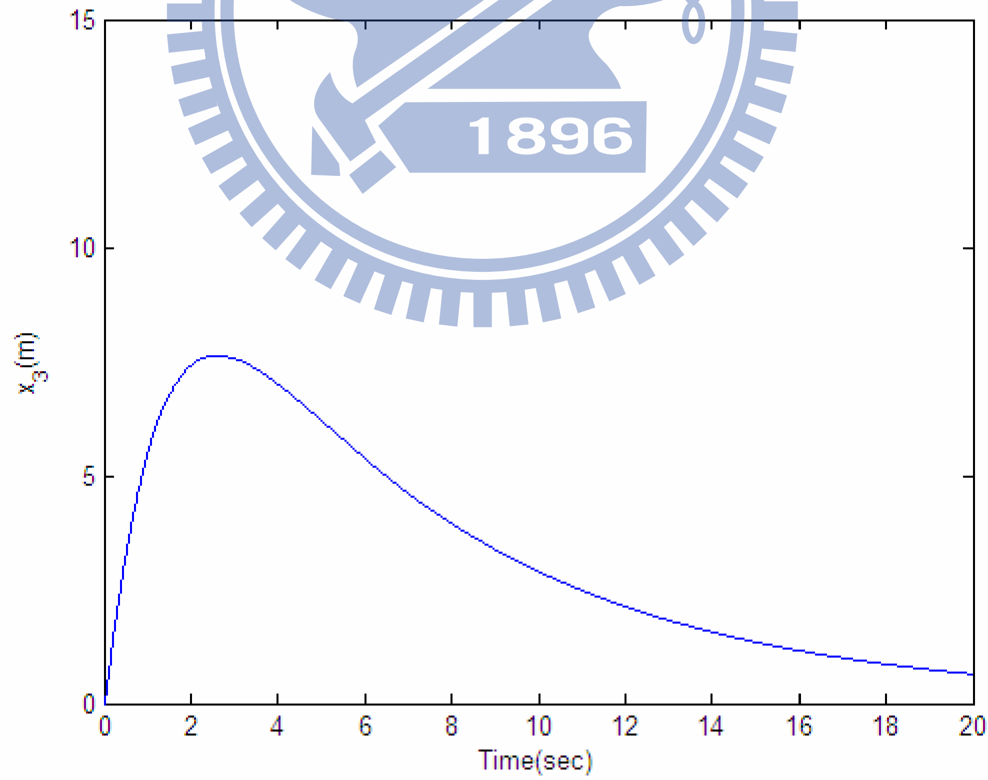
The final controller inferred as the weighted average of each local controller is given by

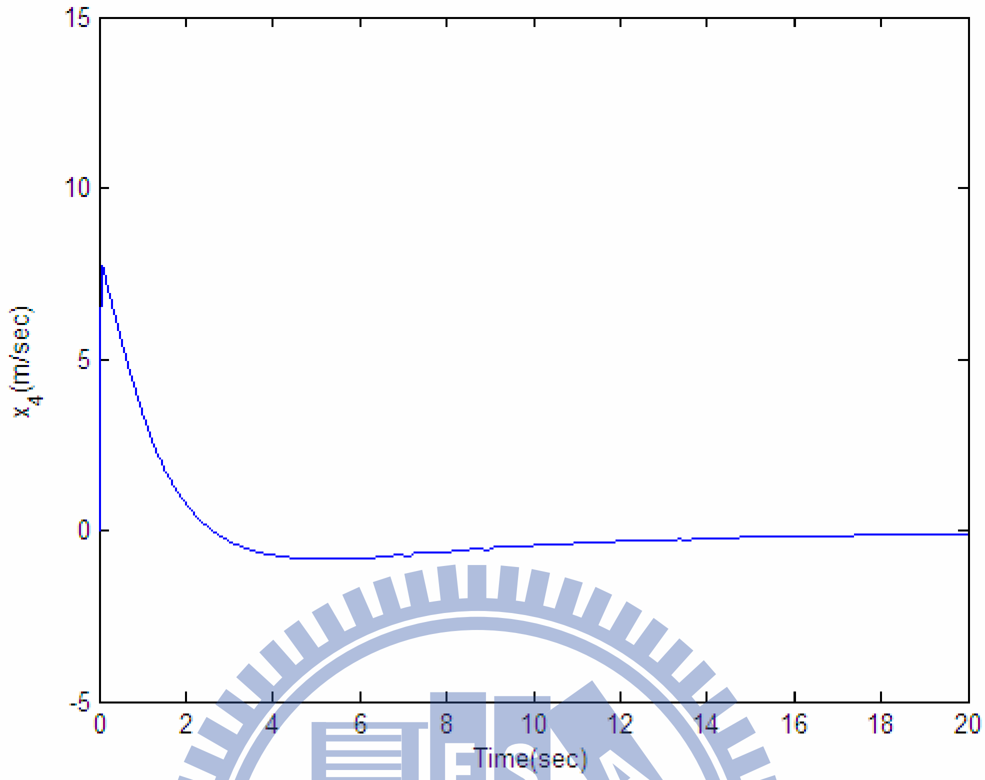
$$u(t) = -x_2^2 \sin x_1 - \sum_{i=1}^{r} \beta_i(\theta) \left[5\sigma + SA_i x + \frac{1}{1-\omega}\delta_i \,\text{sgn}(\sigma) \right]. \quad (3.59)$$

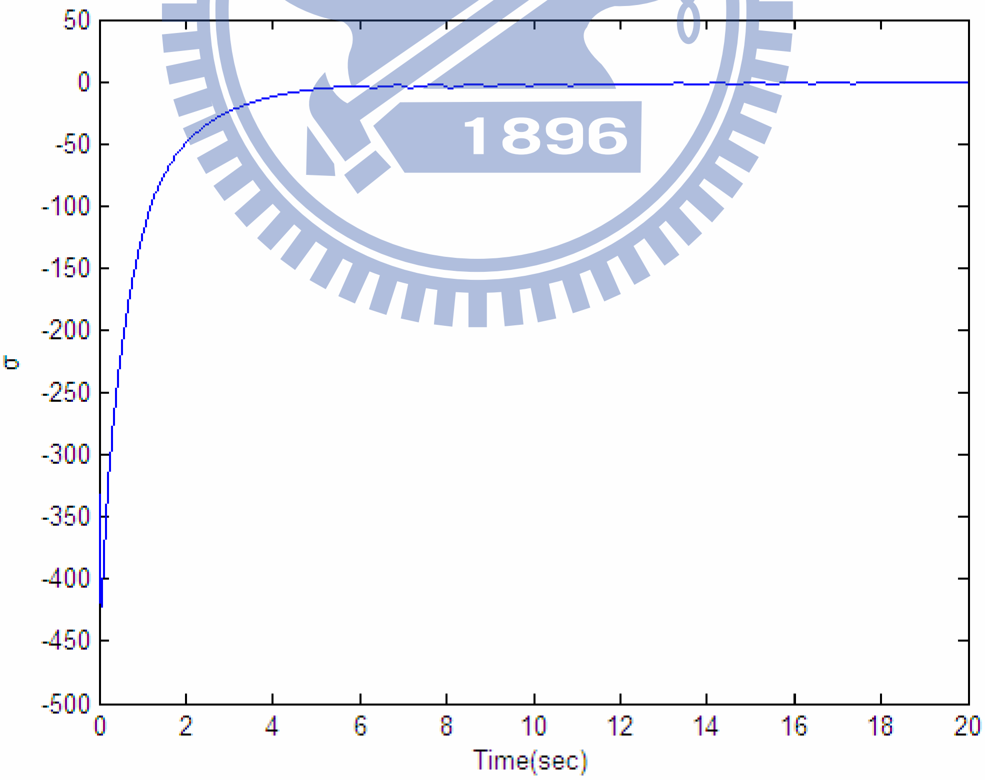
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (3.58) with nonzero $d(t)$. We assume that $d(t) = x_1 \sin 2\pi t - 0.5\,\text{sgn}(x_4)$. The time histories of the state, the sliding variable σ, and the input (3.59) are shown in Figure 3.7. In Figure 3.7, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 3.7, the control performances of the proposed controller are also satisfactory for the two-rule fuzzy model (3.58).




100
50
0
-50
-100
-150
-200
-250
x2(degree/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


15
10
5
0
x3(m)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


x4(m/sec)
Time(sec)


σ
Time(sec)

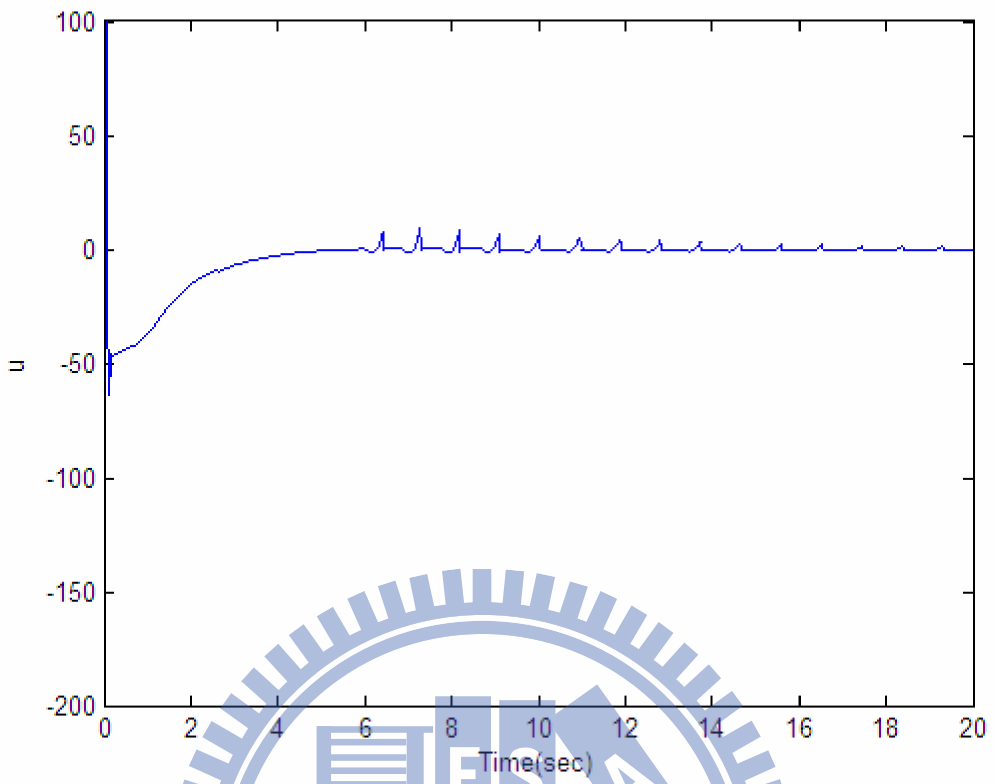


Figure 3.7 Simulation results with $x_1(0) = 40^\circ(2\pi/9\ rad), x_2(0) = x_3(0) = x_4(0) = 0.$

3.3.4 System Formulation II

Consider the following uncertain T-S fuzzy model [49], including parameter uncertainties and external disturbances:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)([A_i + \Delta A_i(t)]x(t) + B_i[u(t) + h(t,x)]) \tag{3.60}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, A_i, B_i are constant matrices of appropriate dimensions, $\Delta A_i(t)$ represents the parameter uncertainties in $A_i, h(t,x) \in R^m$ denotes external disturbances. $\theta = [\theta_1, \cdots, \theta_s], \theta_j (j = 1, \cdots, s)$ are the premise variables, s is the number of the premise variables, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta), \omega_i : R^s \to [0,1], i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i, r is the number of the IF-THEN rules, β_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0$. $\sum_{i=1}^{r} \beta_i(\theta) = 1$. We will assume that the followings are satisfied:

A1: The $n \times m$ matrix B defined by $B = \frac{1}{r}\sum_{i=1}^{r} B_i$ satisfies the rank constraint rank$(B) = m$, i.e., the matrix B has full column rank m.

A2: The function $h(t,x)$ is unknown but bounded as $\|h(t,x) - \hat{h}(t,x)\| \leq \sum_{k=0}^{l} \rho_k \|x\|^k$ where $\rho_0, \cdots, \rho_l$ are known constants, $\hat{h}(t,x)$ is an estimate of $h(t,x)$, and l is a known positive integer.

A3: $\Delta A_i(t)$ is of the form $T_i \Pi_i(t)$ where $\Pi_i(t)$ is unknown, $\|\Delta A_i(t)\| \leq \alpha_{A_i}$, and $T_i T_i^T \geq T_i \Pi_i(t)$.

The system (3.60) does not have to satisfy the restrictive assumption that all the input matrices of the local system models are in the same range space. It should be noted that the assumption A1 implies that $rank(B_i) \leq m$ and each nominal local system model may not share the same input channel. The assumption A2 with $l = 1$ and $\hat{h}(t,x) = 0$ has been used in the literature [50]. We can set $\hat{h}(t,x)$ as the nominal value of $h(t,x)$. Using the above assumptions, the uncertain T-S fuzzy model (3.60) can be written as follows.

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)(A_i + T_i \Pi_i(t))x(t) + [B + HF(\beta)G][u + h(t,x)] \tag{3.61}$$

where $\beta = [\beta_1(\theta), \cdots, \beta_r(\theta)]$, and the matrices $H, G, F(\beta)$ are defined by

$$H = \frac{1}{2}[(B - B_1), \cdots, (B - B_r)], \; G = [I, \cdots, I]^T,$$

$$F(\beta) = diag[(1 - 2\beta_1(\theta))I, \cdots, (1 - 2\beta_r(\theta))I]. \tag{3.62}$$

It should be noted that the system (3.61) does not have to satisfy $B_1 = B_2 = \cdots = B_r$, which is used in almost all published results on VSS design methods including the VSS-based fuzzy control design methods of [33,34]. Hence the methods [30,31] cannot be applied to the above model (3.61). Since $\beta_i(\theta) \geq 0$ and $\sum_{i=1}^{r} \beta(\theta) = 1$, we can see that the following inequality always holds:

$$F^T(\beta)F(\beta) = F(\beta)F^T(\beta) \leq I. \tag{3.63}$$

The following lemma will be used to establish our main results.

Lemma 3.1 For any vectors a and b with appropriate dimensions, the following inequalities hold for any $W > 0$:

$$2a^T b \leq a^T W a + b^T W^{-1} b.$$

Proof: The above inequality is derived from $(Wa-b)^T W^{-1}(Wa-b) = a^T W a + b^T W^{-1} b - 2a^T b \geq 0.$

3.3.5 LMI-based Sliding Control Design II

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions: The first involves the design of $m(n-1)-$dimensional switching surfaces for the sliding mode such that the reduced order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second is concerned with the selection of a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first characterize linear sliding surfaces using LMIs.

Let us define the linear sliding surface as $\sigma = Sx = 0$ where S is a $m \times n$ matrix. Referring to the previous results [33], [51], we can see that for the system (3.61) it is reasonable to find a sliding surface such that

P1 $[SB + SHF(\beta)G]$ is nonsingular for any β satisfying $\beta_i(\theta) \geq 0, i = 1, \cdots, r$, and $\sum_{i=1}^{r} \beta_i(\theta) = 1$.

P2 The reduced $(n-m)$ order sliding mode dynamics restricted to the sliding surface $Sx = 0$ is asymptotically stable for all admissible uncertainties.

It should be noted that P1 is necessary for the existence of the unique equivalent control [33] and the assumption A1 is necessary for the nonsingularity of SB.

Define a transformation matrix and the associated vector v as $M = [\Lambda(\Lambda^T Y \Lambda)^{-1}, Y^{-1}B(B^T Y^{-1} B)^{-1}]^T = [V^T, S^T]^T$, $v = [v_1^T, v_2^T]^T = Mx$ where $v_1 \in R^{n-m}$, $v_2 \in R^m$. By the above transformation, we can see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. Then from system (3.61), we can obtain

$$\begin{bmatrix} \dot{v}_1 \\ \dot{\sigma} \end{bmatrix} = \sum_{i=1}^{r} \beta_i(\theta) \begin{bmatrix} V(A_i + T_i \Pi_i(t))Y\Lambda & V(A_i + T_i \Pi_i(t))B \\ S(A_i + T_i \Pi_i(t))Y\Lambda & S(A_i + T_i \Pi_i(t))B \end{bmatrix} \begin{bmatrix} v_1 \\ \sigma \end{bmatrix} + \begin{bmatrix} VHF(\beta)G \\ I + SHF(\beta)G \end{bmatrix} [u + h(t,x)]. \tag{3.64}$$

Then from the equivalent control method [33], we can see that the equivalent control is given by $u_{eq}(t) = -\sum_{i=1}^{r} \beta_i(\theta)[I + SHF(\beta)G]^{-1} S(A_i + T_i \Pi_i(t))x - h(t,x)$. By setting $\dot{\sigma} = \sigma = 0$ and substituting $u(t)$ with $u_{eq}(t)$, we can show that the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma = Sx = 0$ is given by

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T D(\beta)(A_i + T_i \Pi_i(t)) Y \Lambda v_1 \tag{3.65}$$

where $D(\beta) = I - HF(\beta)G[I + SHF(\beta)G]^{-1} S$.

Theorem 3.5 Let us consider the sliding mode dynamics (3.65). If there exist matrices $Y \in R^{n \times n}$, $\Lambda \in R^{n \times (n-m)}$ satisfying $B^T \Lambda = 0, \Lambda^T \Lambda = I$, scalars $c_0 \in R, c_1 \in R$,

$c_2 \in R,\ \delta \in R, \eta \in R,\ \ \kappa = \lambda_{\min}(B^T B),\ \ \|\Delta A_i(t)\| \le \alpha_{A_i}$, and $*$ represents blocks that are readily inferred by symmetry such that the following LMIs holds:

$$\begin{bmatrix} \Lambda^T(A_i Y + Y A_i^T + c_o I)\Lambda & \eta \Lambda^T H & \Lambda^T Y A_i^T & \alpha_{A_i}\Lambda^T Y & \alpha_{A_i}\Lambda^T Y \\ \eta H^T \Lambda & -I & \eta H^T & 0 & 0 \\ A_i Y \Lambda & \eta H & -(1-\delta) I & 0 & 0 \\ \alpha_{A_i} Y \Lambda & 0 & 0 & -c_0 I & 0 \\ \alpha_{A_i} Y \Lambda & 0 & 0 & 0 & -\delta I \end{bmatrix} < 0, \quad \forall i \qquad (3.66)$$

$$\begin{bmatrix} Y & I & 0 \\ I & c_1 I & 0 \\ 0 & 0 & c_2 I - Y \end{bmatrix} > 0, \qquad (3.67)$$

$$\begin{bmatrix} 2\eta\kappa & * & * \\ r c_1 & r\eta & 0 \\ r c_2 & 0 & r\eta \end{bmatrix} > 0. \qquad (3.68)$$

Suppose that the LMIs (3.66)-(3.68) have a solution vector $(Y, c_0, c_1, c_2, \delta, \eta)$, then there exists a linear sliding surface parameter matrix S satisfying P1-P2 and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \qquad (3.69)$$

will guarantee that the sliding mode dynamics (3.65) is asymptotically stable.

Proof: By using Schur complement formula [48], we can easily show that in fact the following LMIs are incorporated in the LMIs (3.66)-(3.68)

$$c_1 > 0,\ \ c_2 > 0,\ \ \eta > 0,\ \ \eta^2 H H^T < I,\ \ 2\eta^2 \kappa > r(c_1^2 + c_2^2)\,. \qquad (3.70)$$

It is clear that if the following inequality (3.71) holds, then $SB + SHF(\beta)G$ $= I + SHF(\beta)G$ is nonsingular and hence P1 holds

$$SHF(\beta) G G^T F^T(\beta) H^T S^T < I. \qquad (3.71)$$

Using (3.62), (3.63), (3.70) and $GG^T \le \|G\|^2 I = rI$, we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T. \quad (3.72)$$

By using the Schur complement formula, we can see that (3.67) and (3.70) imply

$$0 < c_1^{-1} I < Y < c_2 I,\ 0 < c_2^{-1} I < Y^{-1} < c_1 I \quad (3.73)$$

and this leads to

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T \le \frac{rc_1c_2}{\eta^2}(B^T B)^{-1} \le \frac{rc_1c_2}{\kappa\eta^2} I. \quad (3.74)$$

Using the inequality $2ab \le a^2 + b^2$ where a and b are scalars, we can show that (3.74) implies

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{2\kappa\eta^2}(c_1^2 + c_2^2) I. \quad (3.75)$$

Finally, by using the above inequalities (3.70) and (3.75), we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T < I \quad (3.76)$$

which implies that $[SB + SHF(\beta)G]$ is nonsingular, i.e., P1 holds.

Now, we will show that S of (3.69) guarantees P2. Using the matrix inversion lemma:

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B$$

where A and B are compatible constant matrices such that $(I + AB)$ is nonsingular, we can show that the sliding mode dynamics (3.65) is equivalent to

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y\Lambda)^{-1}\Lambda^T C(\beta)(A_i + T_i\Pi_i(t))Y\Lambda v_1 \quad (3.77)$$

where $C(\beta) = I - H[I + F(\beta)GSH]^{-1} F(\beta)GS = [I + HF(\beta)GS]^{-1}$

$$= I - HF(\beta)G\,[I + SHF(\beta)G]^{-1} S = D(\beta).$$

The sliding mode dynamics (3.77) is asymptotically stable if there exists a positive definite matrix $P_0 \in R^{(n-m)\times(n-m)}$ such that the time derivative of the Lyapunov function $E_g(t) = v_1^T P_0 v_1$ satisfies for some positive scalar τ

$$\dot{E}_g(t) = 2\sum_{i=1}^{r} \beta_i(\theta) v_1^T P_0 Z_i(\beta) v_1 \le -\tau v_1^T v_1 \tag{3.78}$$

where $Z_i(\beta) = (A_{i0} + B_0[I - N(\beta)D_0]^{-1} N(\beta)C_{i0})$, $A_{i0} = (\Lambda^T Y\Lambda)^{-1}\Lambda^T(A_i + T_i\Pi_i(t))Y\Lambda$, $B_0 = (\Lambda^T Y\Lambda)^{-1}\Lambda^T H$, $C_{i0} = (A_i + T_i\Pi_i(t))Y\Lambda$, $D_0 = H$, $N(\beta) = -F(\beta)GS$.

It should be noted that the inequalities (3.63), (3.70), (3.76) and $GG^T \le \|G\|^2 I = rI$ imply

$$N(\beta)N^T(\beta) = F(\beta)GSS^T G^T F^T(\beta) \le \eta^2 I,\ \eta^2 D_0^T D_0 = \eta^2 H^T H < I. \tag{3.79}$$

This and (3.78) imply that (3.77) is asymptotically stable if there exists a positive definite matrix P_0 such that

$$P_0 A_{i0} + P_0 B_0[I - N(\beta)D_0]^{-1} N(\beta)C_{i0} + * < 0, \quad \forall i \tag{3.80}$$

where $*$ represents blocks that are readily inferred by symmetry. Let z_i be $z_i = [I - N(\beta)D_0]^{-1} N(\beta)C_{i0} y$ where $y \in R^{(n-m)}$. Then z_i can be rewritten as $z_i = N(\beta)[C_{i0}y + D_0 z_i]$. This equality and (3.79) imply $z_i^T z_i \le \eta^2 [C_{i0}y + D_0 z_i]^T [C_{i0}y + D_0 z_i]$ and this leads to

$$2y^T P_0 B_0[I - N(\beta)D_0]^{-1} N(\beta)C_{i0} y$$

$$= 2y^T P_0 B_0 z_i \le 2y^T P_0 B_0 z_i + [C_{i0}y + D_0 z_i]^T [C_{i0}y + D_0 z_i] - \eta^{-2} z_i^T z_i$$

$$= y^T C_{i0}^T C_{i0} y + 2y^T [P_0 B_0 + C_{i0}^T D_0] z_i - \eta^{-2} z_i^T \Omega z_i \quad \text{where} \quad \Omega = I - \eta^2 D_0^T D_0. \tag{3.81}$$

Since $\Omega > 0$, the following inequality holds for any (y, z_i):

$$2y^T [P_0 B_0 + C_{i0}^T D_0] z_i \le \eta^{-2} z_i^T \Omega z_i + \eta^2 y^T [P_0 B_0 + C_{i0}^T D_0]\Omega^{-1}[P_0 B_0 + C_{i0}^T D_0]^T y \tag{3.82}$$

Using (3.81) and (3.82), we can show that the Lyapunov inequality (3.80) is satisfied if the following inequality holds:

$$P_0 A_{i0} + A_{i0}^T P_0 + C_{i0}^T C_{i0} + \eta^2 [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T < 0 .$$

Using the Schur complement formula, we can rewrite the above inequality as

$$\begin{bmatrix} A_{i0}^T P_o + * & * & * \\ \eta B_0^T P_0 & -I & * \\ C_{i0} & \eta D_0 & -I \end{bmatrix} < 0, \quad \forall i . \tag{3.83}$$

Let the positive definite matrix P_0 be $P_0 = \Lambda^T Y \Lambda$ where Y is a solution to LMIs (3.66)-(3.68), then the above matrix inequality (3.83) can be rewritten as

$$\begin{bmatrix} \Lambda^T [(A_i + \Delta A_i(t))Y + *]\Lambda & \eta \Lambda^T H & \Lambda^T Y (A_i + \Delta A_i(t))^T \\ \eta H^T \Lambda & -I & \eta H^T \\ (A_i + \Delta A_i(t))Y\Lambda & \eta H & -I \end{bmatrix} < 0, \quad \forall i \tag{3.84}$$

where $\Delta A_i(t) = T_i \Pi_i(t)$. The matrix inequality (3.84) is satisfied if the following inequality holds for any nonzero vectors: $z^T = \begin{bmatrix} z_1^T & z_2^T & z_3^T \end{bmatrix}$

$$2z_1^T \Lambda^T (A_i + \Delta A_i(t))Y\Lambda z_1 + 2z_3^T (A_i + \Delta A_i(t))Y\Lambda z_1 + 2\eta z_2^T H^T \Lambda z_1 + 2\eta z_3^T H z_2 - z_2^T z_2 - z_3^T z_3 < 0. \tag{3.85}$$

Lemma 3.1 implies that if $\|\Delta A_i(t)\| \le \alpha_{A_i}$, the following inequalities hold:

$$2z_1^T \Lambda^T \Delta A_i(t) Y \Lambda z_1 \le c_0 z_1^T \Lambda^T \Lambda z_1 + \alpha^2 c_0^{-1} z_1^T \Lambda^T Y^2 \Lambda z_1 \tag{3.86}$$

$$2_3^T \Delta A_i(t) Y \Lambda z_1 \le \delta z_3^T z_3 + \alpha^2 \delta^{-1} z_1^T \Lambda^T Y^2 Y z_1 . \tag{3.87}$$

The previous inequalities (3.86) and (3.87) imply that for all admissible $\|\Delta A_i(t)\| \le \alpha_{A_i}$, the inequality condition (3.85) holds if

$$\begin{aligned} &2z_1^T \Lambda^T A_i Y \Lambda z_1 + c_0 z_1^T \Lambda^T \Lambda z_1 + \alpha^2 \delta^{-1} z_1^T \Lambda^T Y^2 \Lambda z_1 \\ &+ \alpha^2 c_0^{-1} z_1^T \Lambda^T Y^2 \Lambda z_1 + 2z_3^T A_i Y \Lambda z_1 + 2\eta z_2^T H^T \Lambda z_1 \\ &+ 2\eta z_3^T H z_2 + \delta z_3^T z_3 - z_2^T z_2 - z_3^T z_3 < 0. \end{aligned} \tag{3.88}$$

This implies that (3.84) holds if the following LMI (3.89) holds

$$\begin{bmatrix} \Lambda^T(A_iY+YA_i^T+c_0I+\frac{\alpha_{A_i}^2}{c_0}Y^2+\frac{\alpha_{A_i}^2}{\delta}Y^2)\Lambda & \eta\Lambda^T H & \Lambda^T YA_i^T \\ \eta H^T\Lambda & -I & \eta H^T \\ A_iY\Lambda & \eta H & -(1-\delta)I \end{bmatrix} < 0 . \qquad (3.89)$$

By using Schur complement formula, the above inequality (3.89) can be rewritten as the LMI (3.66), which implies that the sliding mode dynamics (3.77) is asymptotically stable. Hence, the sliding mode dynamics (3.65) is asymptotically stable.

After the switching surface parameter matrix S is designed so that the reduced $(n-m)$ order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintain it there for all subsequent time. With σ of (3.69), we design a sliding fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear sliding switching feedback control law as the local controller.

Control rule i: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\hat{h}(t,x) - \chi_i\sigma - S(A_i + T_iT_i^T)x - \frac{1}{1-\omega}\delta_i(t,x)\frac{\sigma}{\|\sigma\|}$$

where $\delta_i(t,x) = \alpha_i + \omega\|S(A_i + T_iT_i^T)x\| + (1+\omega)\sum_{k=0}^{l}\rho_k\|x\|^k$ (3.90)

and $\sigma = Sx, \omega = \sqrt{r}\|SH\|, \alpha_i > 0, \chi_i > 0$. It should be noted that (3.76) implies $\omega = \sqrt{r}\|SH\| \le \sqrt{r}\|S\|\cdot\|H\| \le \eta\|H\|$. This and (3.70) guarantee $0 \le \omega < 1$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\hat{h}(t,x) - \sum_{i=1}^{r}\beta_i(\theta)\left(\chi_i\sigma + S(A_i + T_iT_i^T)x + \frac{1}{1-\omega}\delta_i(t,x)\frac{\sigma}{\|\sigma\|}\right) \qquad (3.91)$$

and we can establish the following theorem.

Theorem 3.6 Consider the closed-loop control system of the uncertain system (3.61) with control (3.91). Suppose that the LMIs (3.66)-(3.68) has a solution vector $(Y, c_0, c_1, c_2, \delta, \eta)$ and the linear sliding surface is given by (3.69). Then the state converges to zero.

Proof: Since Theorem 3.5 implies that the linear sliding surface (3.69) guarantees P1-P2, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T\sigma$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T\dot{\sigma}$. From (3.61), (3.69), (3.91), $\|SHF(\beta)G\| \leq \sqrt{r}\|SH\| = \omega$,$0 \leq \omega < 1$, and A2, we obtain

$$\sigma^T\dot{\sigma} = \sigma^T \sum_{i=1}^r \beta_i(\theta) S(A_i + T_i\Pi_i(t))x(t) + \sigma^T[I + SHF(\beta)G][u + h(t,x)]$$

$$\leq \sum_{i=1}^r \beta_i(\theta)\sigma^T S(A_i + T_i\Pi_i(t))x(t) + \sigma^T u + \{\omega\|u\| + (1+\omega)\|h(t,x)\|\}\|\sigma\|$$

$$\leq -\sum_{i=1}^r \beta_1(\theta)\sigma^T S(T_iT_i^T - T_i\Pi_i^T(t))x(t) - (1-\omega)\sum_{i=1}^r \beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^r \beta_i(\theta)\alpha_i\|\sigma\|$$

$$\leq -\sum_{i=1}^r \beta_1(\theta)\|x\|^{-2} x^T(T_iT_i^T - T_i\Pi_i^T(t))x\|\sigma\|^2 - (1-\omega)\sum_{i=1}^r \beta_i(\theta)\chi_i\|\sigma\|^2$$

$$- \sum_{i=1}^r \beta_i(\theta)\alpha_i\|\sigma\|.$$

From A3 and $\varepsilon_g = \|x\|^{-2} x^T(T_iT_i^T - T_i\Pi_i^T(t))x \geq 0$, this implies that $\dot{E}_g \leq -\sum_{i=1}^r \beta_1(\theta)\varepsilon_g\|\sigma\|^2 - (1-\omega)\sum_{i=1}^r \beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^r \beta_i(\theta)\alpha_i\|\sigma\| \leq 0$ which indicates that $E_g \in L_2 \cap L_\infty$, $\dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 3.3 Theorem 3.5 and 3.6 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Obtain $B = \frac{1}{r}\sum_{i=1}^r B_i$ and $H = \frac{1}{2}[(B - B_1), \cdots, (B - B_r)]$ for given B_i.

Step 2: Check that (A_i, B) is stabilization. If not, exit.

Step 3: Find a solution vector $(Y, c_0, c_1, c_2, \delta, \eta)$ to LMI (3.66)-(3.68).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (3.69).

Step 5: The controller is given by (3.91).

3.3.6 Numerical Examples II

Example 3.5 To illustrate the performance of the proposed sliding fuzzy control design method, consider the following two-rule fuzzy model from a vertical takeoff and landing (VTOL) helicopter model [55]

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = (A_1 + T_1 \Pi_1(t))x + B_1[u + h(t,x)]$$

Plant Rule2: IF x_1 is about ± 2, THEN

$$\dot{x} = (A_2 + T_2 \Pi_2(t))x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3181 & -0.7070 & 1.4100 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}$,

$A_2 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.4181 & -0.7070 & 1.4300 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.6446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}$,

$T_1 = T_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}$, $\Pi_1(t) = \Pi_2(t) = \begin{bmatrix} 0 & \sin t & 0 & \sin t \end{bmatrix}$,

$$h(t,x) = d(t) + \begin{bmatrix} 0.9\sin 3t & 0.9\sin 3t \end{bmatrix}^T, \ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - 1)})}{1 + e^{-14(x_1 + 1)}}, \ \beta_2 = 1 - \beta_1. \tag{3.92}$$

It should be noted that B_1 and B_2 are not matched and thus the previous VSS-based fuzzy control design methods cannot be applied to the above system (3.92). Via LMI optimization with (3.92), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x) = [0.9\sin 3t \quad 0.9\sin 3t]^T$ and $\chi_i = 1, \alpha_i = 0.0001, r = 2, \quad l = 1,$ $\rho_k = 1,$ and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = [-0.9\sin 3t \quad -0.9\sin 3t]^T - \sigma - S(A_1 + T_1 T_1^T)x - \frac{1}{1-\omega}\delta_1 \operatorname{sgn}(\sigma).$$

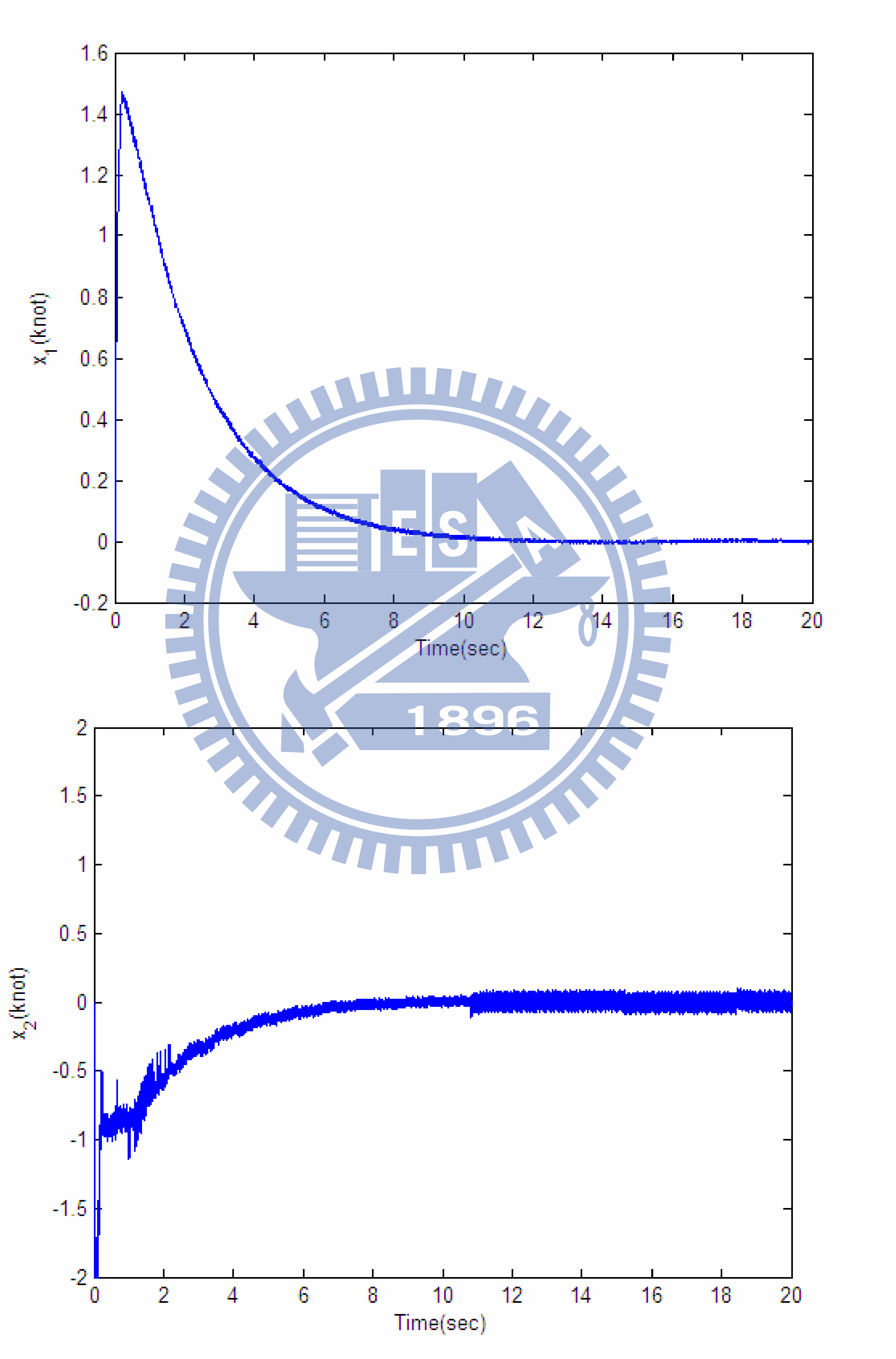
Control Rule 2: IF x_1 is about ± 2, THEN

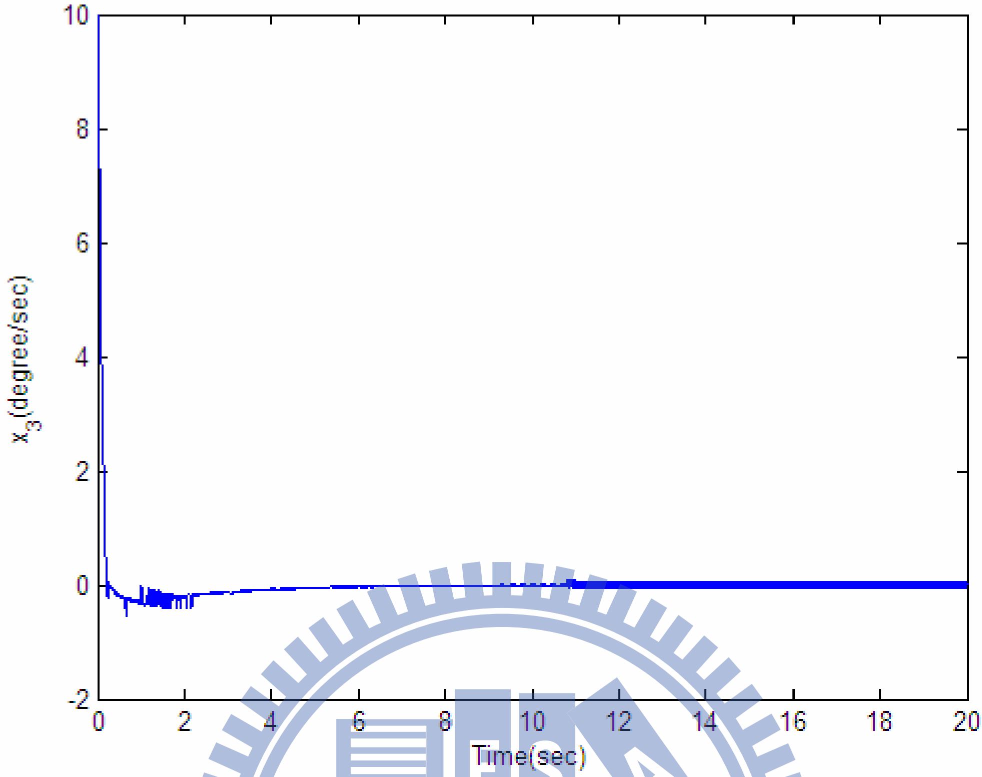
$$u(t) = [-0.9\sin 3t \quad -0.9\sin 3t]^T - \sigma - S(A_2 + T_2 T_2^T)x - \frac{1}{1-\omega}\delta_2 \operatorname{sgn}(\sigma).$$

The final controller inferred as the weighted average of each local controller is given by

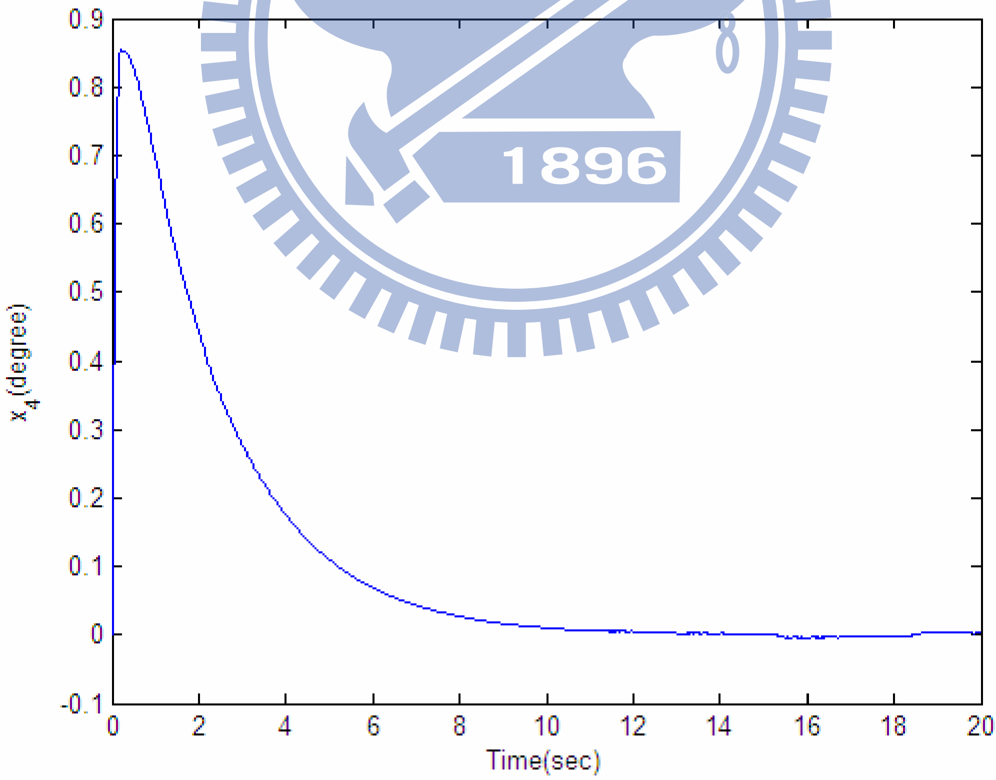
$$u(t) = [-0.9\sin 3t \quad -0.9\sin 3t]^T - \sum_{i=1}^{r}\beta_i(\theta)\left[\sigma + S(A_i + T_i T_i^T)x + \frac{1}{1-\omega}\delta_i \operatorname{sgn}(\sigma)\right]. \quad (3.93)$$

To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (3.92) with nonzero $d(t)$. We assume that $d(t)$ $=[x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4) \quad x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)]^T$. The time histories of the state, the sliding variable σ, and the input (3.93) are shown in Figure 3.8 when $x_1(0) = x_2(0) = x_4(0) = 0, \quad x_3(0) = 10.$ In Figure 3.8, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 3.8, the proposed controller is applicable to T-S fuzzy systems with mismatched parameter uncertainties in the state matrix and external disturbances. The control performances of the proposed controller are satisfactory for the two-rule fuzzy model (3.92). It should be noted that all existing VSS-based fuzzy control system design methods cannot be applied to the two-rule fuzzy model (3.92) because B_1 is not in the range space of B_2.

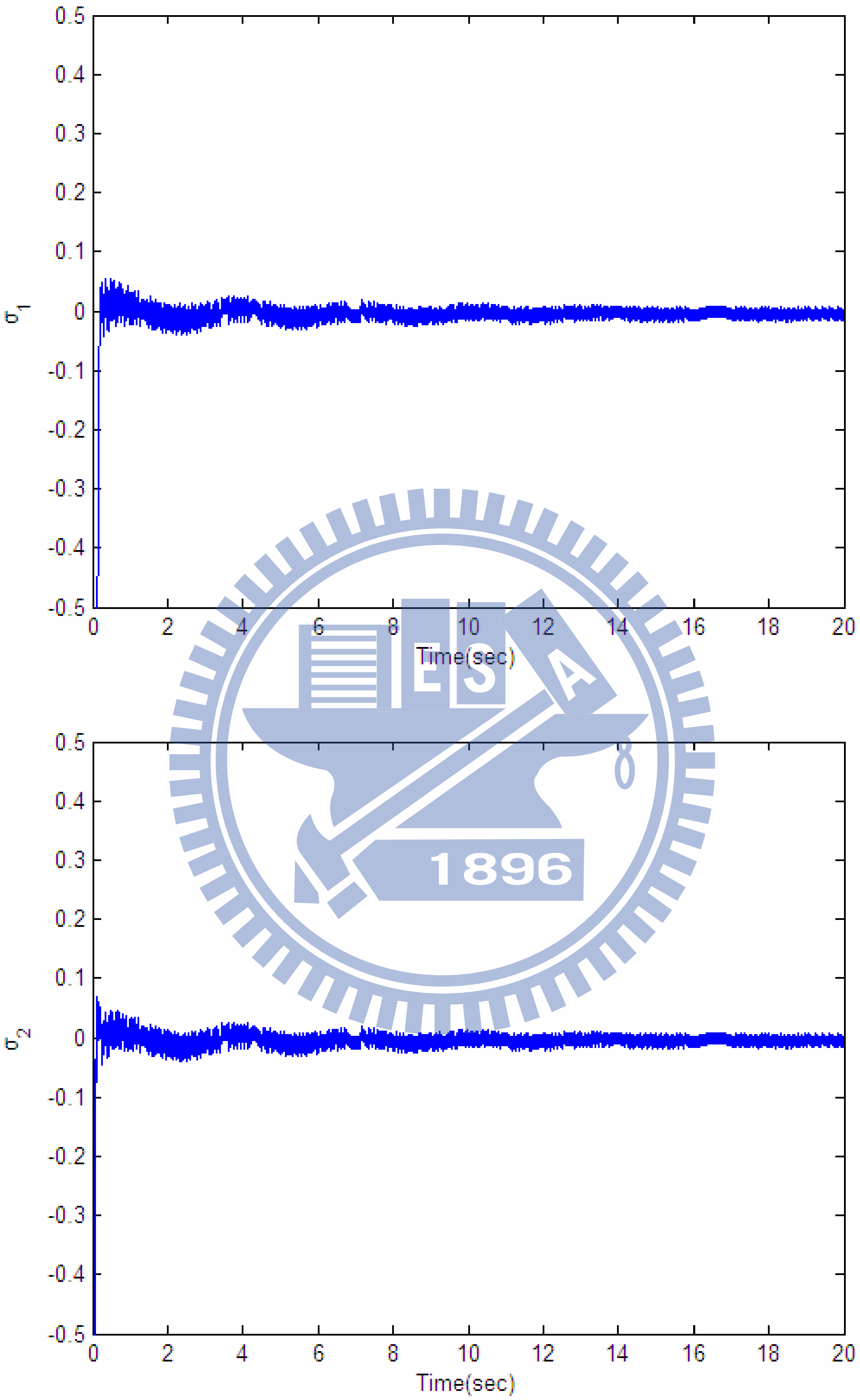




x3(degree/sec)
Time(sec)



x4(degree)
Time(sec)



0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
σ_1
0
2
4
6
8
10
12
14
16
18
20
Time(sec)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
σ_2
0
2
4
6
8
10
12
14
16
18
20
Time(sec)

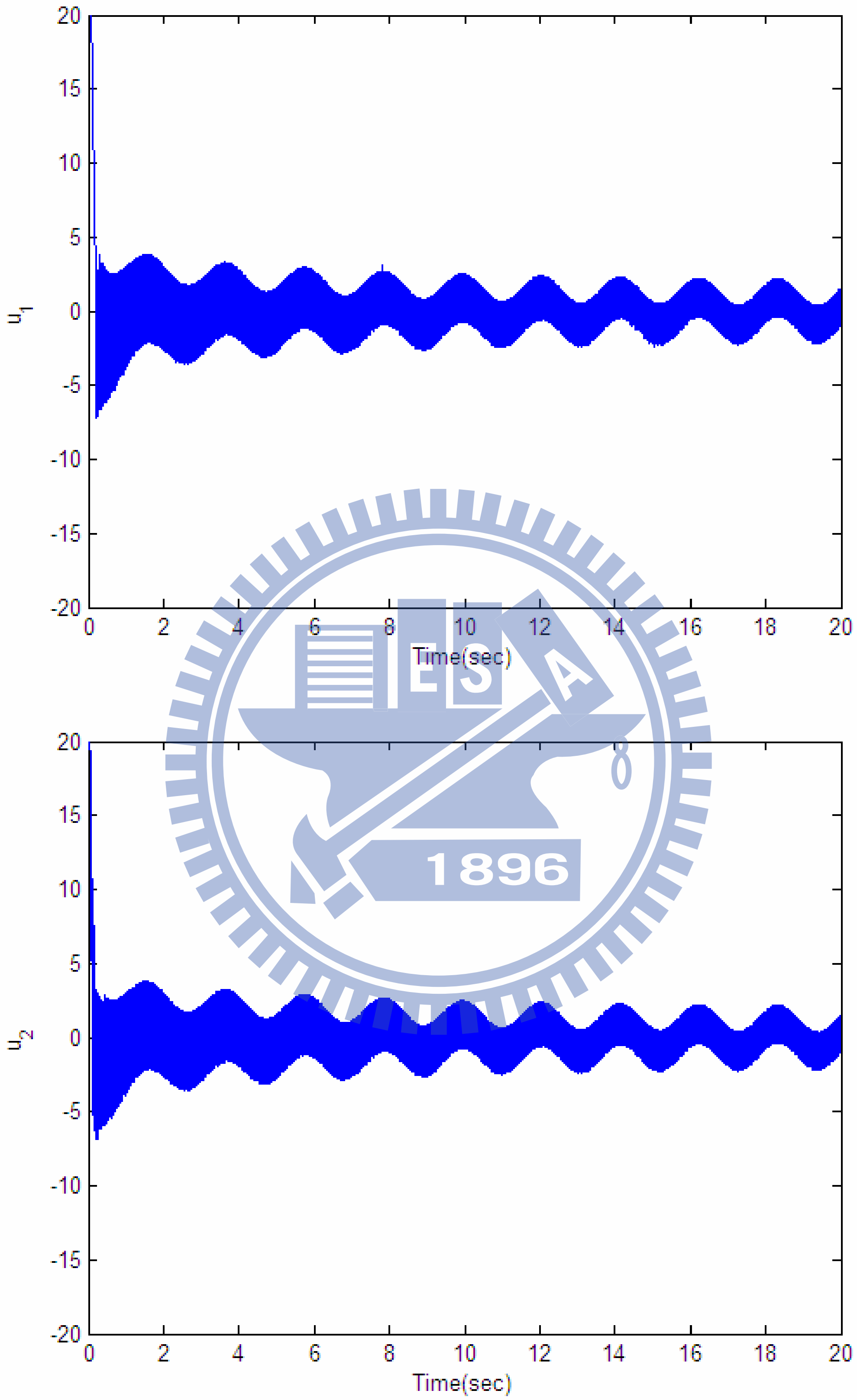


Figure 3.8 Simulation results with $x_1(0) = x_2(0) = x_4(0) = 0,\ x_3(0) = 10.$

Example 3.6 For the special case of $\Delta A_i(t) \equiv 0$, the robust sliding controller design is proposed in [54]. Consider the following inverted pendulum on a cart

$$\dot{x}_1 = x_2, \quad \dot{x}_3 = x_4, \quad \dot{x}_2 = \frac{1}{l\psi}(3g\sin x_1 - 3a\cos x_1[u + d(t) + \phi]),$$

$$\dot{x}_4 = -\frac{1}{\psi}(1.5mag\sin 2x_1 - 4a[u + d(t) + \phi]) \qquad (3.94)$$

where x_1 is the angle (rad) of the pendulum from the vertical, $x_2 = \dot{x}_1$, x_3 is the displacement (m) of the cart, $x_4 = \dot{x}_3$, $\psi = 4 - 3ma\cos^2 x_1$, $\phi = mlx_2^2 \sin x_1$, u is the input, and $d(t)$ is related to external disturbances which may be caused by the frictional force. $a = 1/(m+M)$, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, $g = 9.8m/s^2$ is the gravity constant. We set $M = 9kg$, $m = 1kg$, $l = 1m$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1\|x\|$ where ρ_0 and ρ_1 are known constants. Here, we approximate the system (3.94) by the following two-rule fuzzy model.

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = A_1 x + B_1[u + h(t,x)]$$

Plant Rule2: IF x_1 is about $\pm 60^\circ (\pm\pi/3 \ rad)$, THEN

$$\dot{x} = A_2 x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.9459 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.7946 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -0.0811 \\ 0 \\ 0.1081 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3097 & 0 & 0 & 0 \end{bmatrix}$,

$$B_2 = \begin{bmatrix} 0 \\ -0.0382 \\ 0 \\ 0.1019 \end{bmatrix}, \ h(t,x) = d(t) + x_2^2 \sin x_1, \ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}}, \ \beta_2 = 1 - \beta_1. \qquad (3.95)$$

Because B_1 is not in the range space of B_2, all existing VSS-based fuzzy control system design methods cannot be applied to the above system (3.95). Via LMI optimization with (3.95), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x) = x_2^2 \sin x_1, \chi_i = 5, \alpha_i = 1, r = 2, l = 1, \rho_k = 1$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_1 x - \frac{1}{1-\omega}\delta_1 \operatorname{sgn}(\sigma).$$

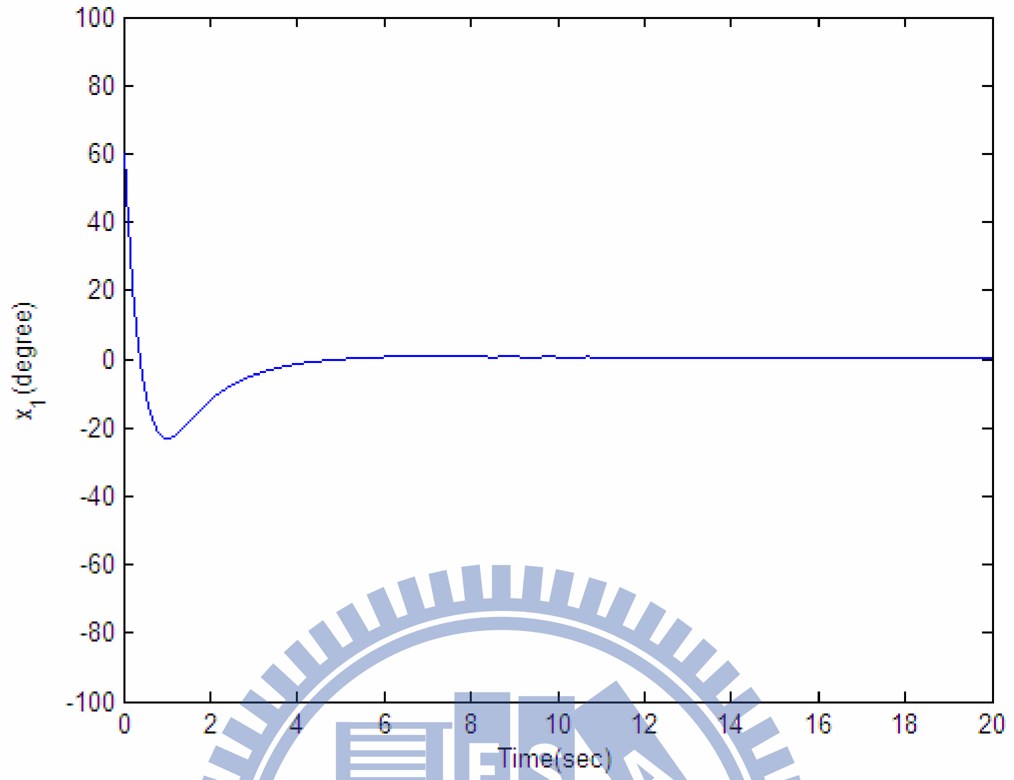
Control Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

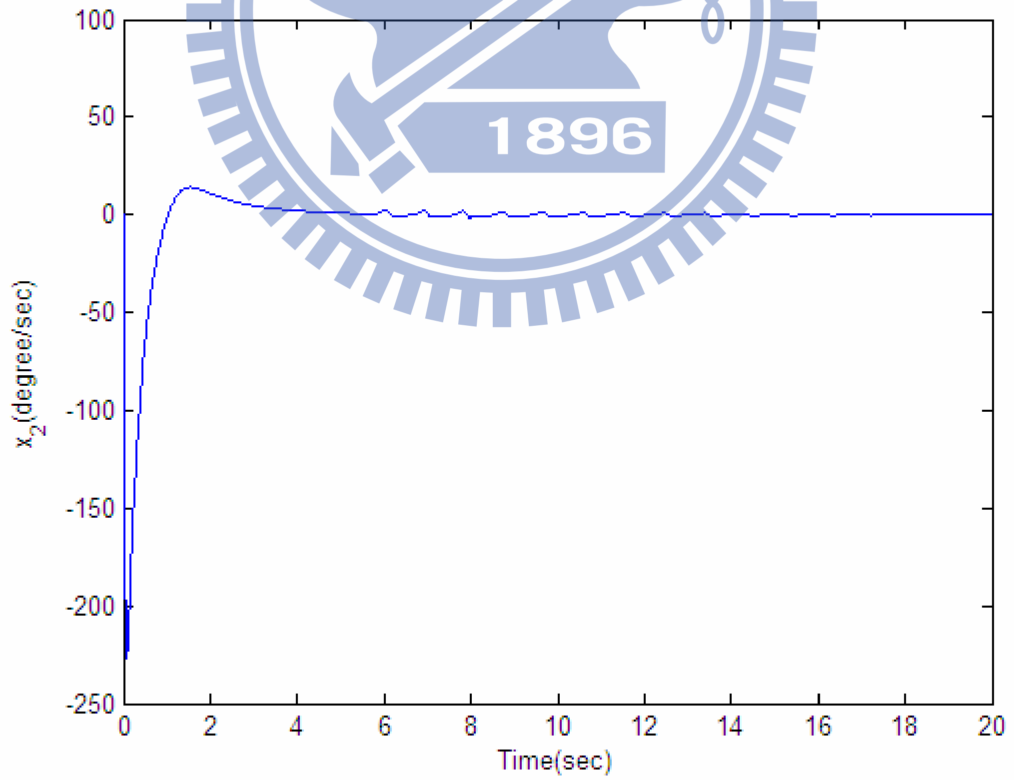
$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_2 x - \frac{1}{1-\omega}\delta_2 \operatorname{sgn}(\sigma).$$

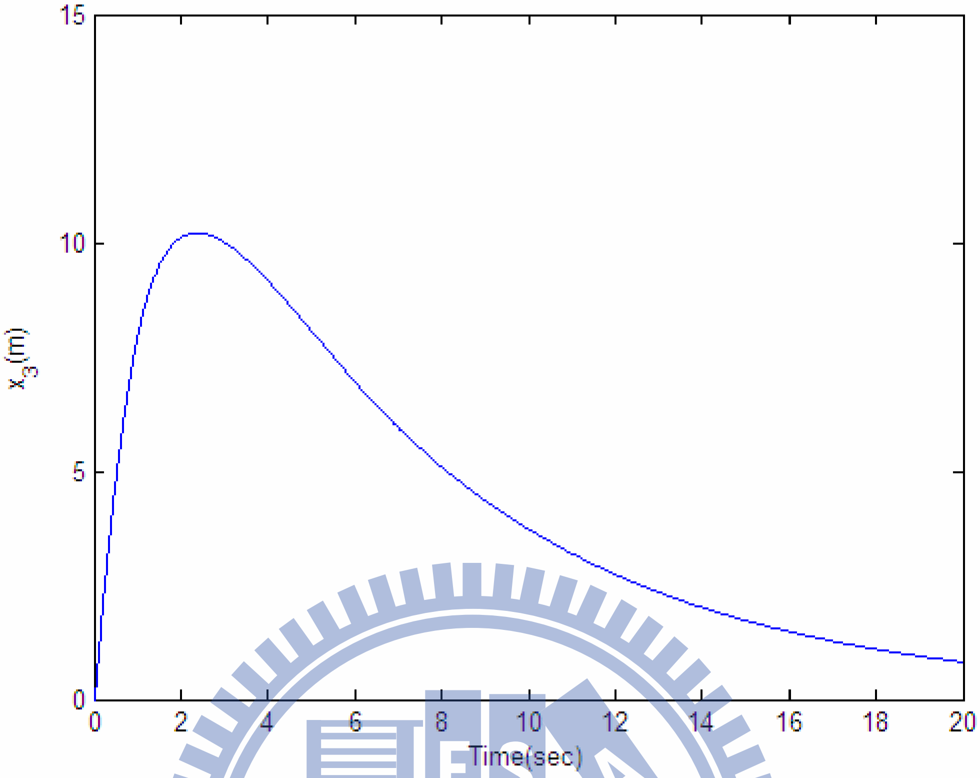
The final controller inferred as the weighted average of each local controller is given by

$$u(t) = -x_2^2 \sin x_1 - \sum_{i=1}^{r} \beta_i(\theta)\left[5\sigma + SA_i x + \frac{1}{1-\omega}\delta_i \operatorname{sgn}(\sigma)\right]. \quad (3.96)$$

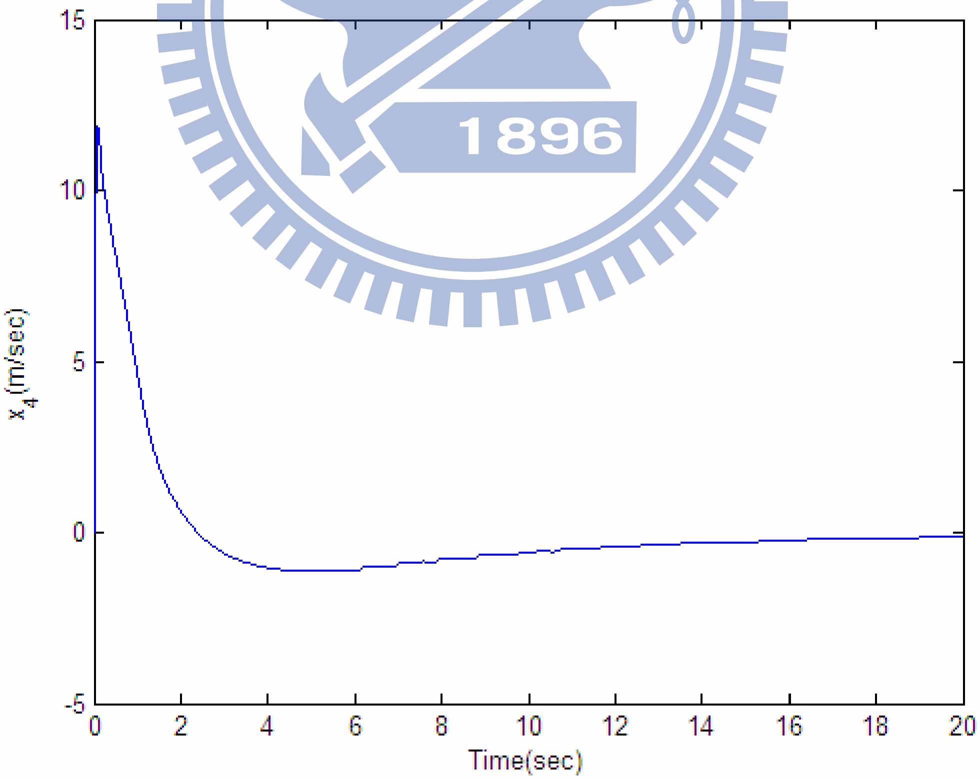
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (3.95) with nonzero $d(t)$. We assume that $d(t) = x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)$. The time histories of the state, the sliding variable σ, and the input (3.96) are shown in Figure 3.9. when $x_1(0) = 60^\circ (2\pi/9\ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$. In Figure 3.9, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 3.9, the control performances of the proposed controller are also satisfactory for the two-rule fuzzy model (3.95).


100
80
60
40
20
0
-20
-40
-60
-80
-100
0 2 4 6 8 10 12 14 16 18 20
Time(sec)
x_1(degree)


100
50
0
-50
-100
-150
-200
-250
0 2 4 6 8 10 12 14 16 18 20
Time(sec)
x_2(degree/sec)



x3(m)
Time(sec)



x4(m/sec)
Time(sec)

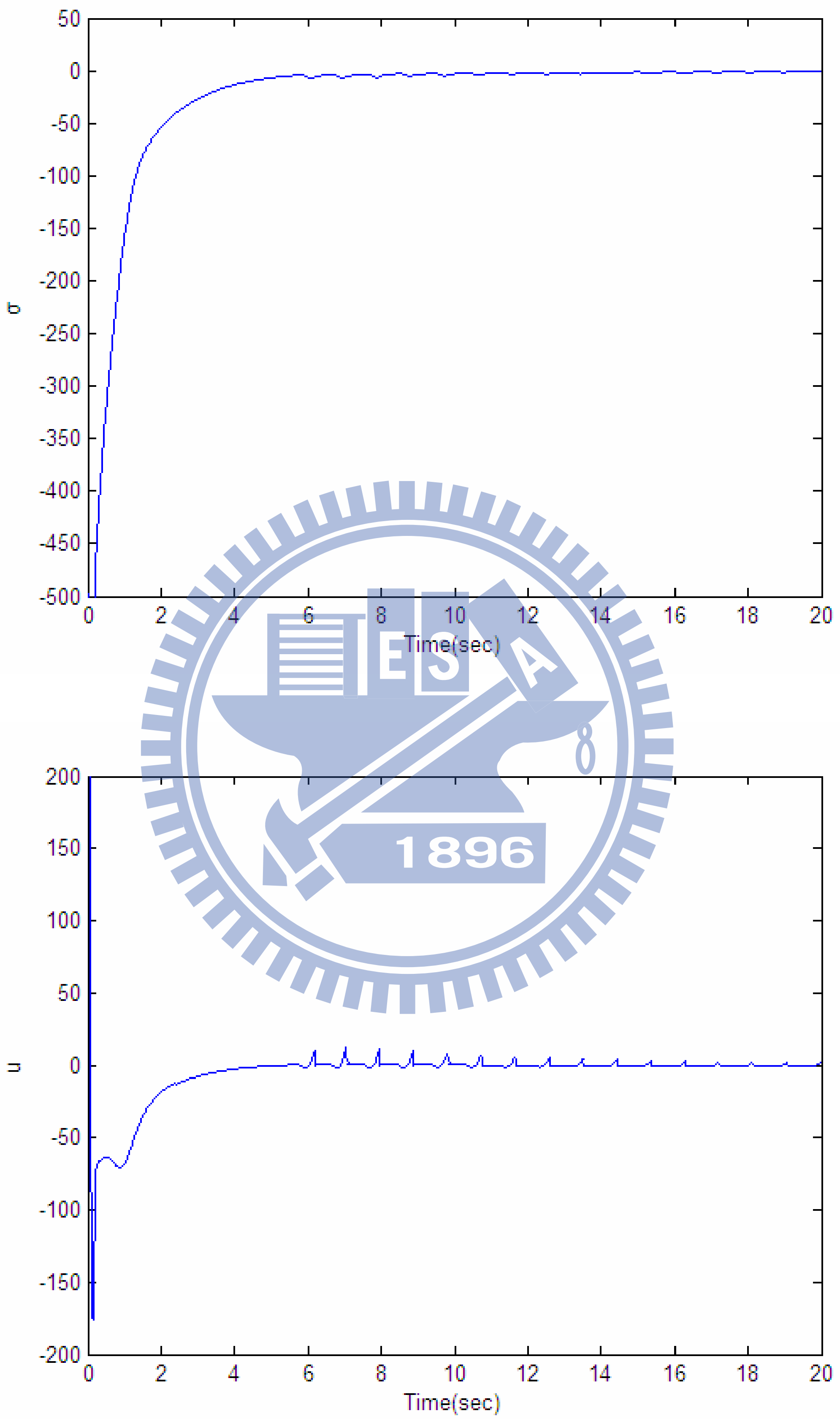


Figure 3.9 Simulation results with $x_1(0) = 60^\circ (\pi/3 \ \ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$.

3.4 Robust Sliding Control for Mismatched T-S Fuzzy Time-Delay Systems

In this section, system formulation for the uncertain T-S fuzzy time-delay model is described in Section 3.4.1. A robust sliding control method via LMI is proposed in Section 3.4.2. Some examples are used to illustrate the effectiveness of the proposed methods and to compare with the existing methods in Section 3.4.3.

3.4.1 System Formulation

The T-S fuzzy model is described by fuzzy IF-THEN rules, which represent local linear input-output relations of nonlinear systems. The ith rule of the T-S fuzzy time-delay model is of the following form:

Plant Rule i: IF θ_1 is μ_{i1} and … and θ_s is μ_{is} , THEN

$$\dot{x}(t) = A_i x(t) + A_{\tau i} x(t - d(t)) + B_i u(t), \quad x(t) = \psi(t), \quad t \in [-\tau, 0]$$

where $\psi(t)$ is the initial condition, $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $A_i \in R^{n\times n}$ are the state matrices, $A_{\tau i} \in R^{n\times n}$ are the delayed state matrices, $B_i \in R^{n\times m}$ are the input matrices, $\theta_j (j = 1,\cdots,s)$ are the premise variables, s is the number of the premise variables, μ_{ij} $(i = 1,\cdots,r; j = 1,\cdots,s)$ are the fuzzy sets that are characterized by membership function, r is the number of the IF-THEN rules. The time-varying delay $d(t)$ is bounded as $d(t) \le \tau$. The overall fuzzy model achieved by fuzzy synthesizing of each individual plant rule is given by

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)[A_i x(t) + A_{\tau i} x(t - d(t)) + B_i u(t)], \; x(t) = \psi(t), \quad t \in [-\tau, 0]$$

where $\theta = [\theta_1,\cdots,\theta_s]$, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta), \omega_i : R^s \to [0,1], i = 1,\cdots,r$ is the membership function of the system with respect to plant rule *i*. The function $\beta_i(\theta)$ can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0, \sum_{i=1}^{r} \beta_i(\theta) = 1$. To take into account parameter uncertainties and external disturbances, we consider the following uncertain T-S fuzzy time-delay model:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)[(A_i + \Delta A_i(t))x(t) + (A_{\tau i} + \Delta A_{\tau i}(t))x_d(t) + B_i(u(t) + h_i(t,x,x_d,u))]$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0] \tag{3.97}$$

where $x_d(t) = x(t - d(t))$, $\Delta A_i(t)$ represents the parameter uncertainties in A_i, $\Delta A_{\tau i}(t)$ represents the parameter uncertainties in $A_{\tau i}$, $h_i(t,x,x_d,u) \in R^m$ denotes external disturbances. We will assume that the following assumptions are satisfied:

A1: $B_1 = B_2 = ... = B_r := B$ and $\operatorname{rank}(B) = m$.

A2: The function $h_i(t,x,x_d,u)$ is unknown but bounded as $\|h_i(t,x,x_d,u)\| \leq \phi_i \|u\| + \xi_i(t)$ where $\xi_i(t)$ is a known function and ϕ_i satisfies $\phi_i \leq \phi_m < 1$ for a known constant ϕ_m.

A3: The time delay $d(t)$ is unknown but bounded as $d(t) \leq \tau$ and $\dot{d}(t) \leq d_m < 1$ where τ and d_m are known constants.

A4: $\Delta A_i(t)$ and $\Delta A_{\tau i}(t)$ are of the form $T_i \Pi_i(t)$ where $\Pi_i(t)$ is a known time-varying matrix but bounded as $\|\Pi_i(t)\| \leq 1$.

Using the above assumptions, the uncertain T-S fuzzy model (3.97) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)[(A_i + \mathrm{T}_i \Pi_i(t))x(t) + (A_{\tau i} + \mathrm{T}_i \Pi_i(t))x_d(t) + Bh_i(t,x,x_d,u)] + Bu(t)$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0] \tag{3.98}$$

A large number of examples in the literature and various mechanical systems, such as motors and robots, fall into the special cases of the above model (3.98), as reported in

[44], [56-60]. The above model (3.98) also involves the uncertain time-delay system models considered in the previous SMC design methods [44], [56-60]. The symbol $*$ will be used in some matrix expressions to induce a symmetric structure. For given symmetric matrices K and L of appropriate dimensions, the following holds:

$$\begin{bmatrix} K+X+* & * \\ Z & L \end{bmatrix} = \begin{bmatrix} K+X+X^T & Z^T \\ Z & L \end{bmatrix}$$

When no confusion arises, the arguments t, x, x_d, θ, etc… can be omitted for brevity.

3.4.2 Sliding Control Design via LMI

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions. The first is concerned with the design of a sliding surface for the sliding mode such that the reduced-order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second involves choosing a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first design a sliding surface that guarantees asymptotic stability of the reduced-order sliding mode dynamics using LMIs.

Defining a nonsingular transformation matrix M and the associated vector $v = Mx$ such that

$$M = \begin{bmatrix} (\Lambda^T Y \Lambda)^{-1}\Lambda^T \\ (B^T Y^{-1} B)^{-1} B^T Y^{-1} \end{bmatrix} = \begin{bmatrix} V \\ S \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Vx \\ Sx \end{bmatrix} = M\,x \tag{3.99}$$

where $v_1 \in R^{n-m}, v_2 \in R^m$. Then we can easily see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. By the above transformation we can obtain, we can transform (3.98) into the following regular form:

$$\dot{v} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} v + \begin{bmatrix} \bar{A}_{\tau 11} & \bar{A}_{\tau 12} \\ \bar{A}_{\tau 21} & \bar{A}_{\tau 22} \end{bmatrix} v_d + \begin{bmatrix} 0 \\ I \end{bmatrix} \left(u + \sum_i \beta_i h_i \right) \tag{3.100}$$

where $v_d = v(t-d(t))$ and

$$\overline{A}_{11} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,,\ \overline{A}_{12} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) B \,,$$

$$\overline{A}_{21} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,,\ \overline{A}_{22} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_i + \mathrm{T}_i \Pi_i(t)) B \,,$$

$$\overline{A}_{\tau 11} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,,\ \overline{A}_{\tau 12} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) B \,,$$

$$\overline{A}_{\tau 21} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,,\ \overline{A}_{\tau 22} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) B \,.$$

Thus, from the above regular form, by setting $\dot{\sigma} = \sigma = 0$, we can obtain the following sliding mode dynamics :

$$\dot{\alpha} = A_o \alpha + A_d \alpha_d \tag{3.101}$$

where $\alpha = v_1, \alpha_d = v_1(t-d(t)),\ A_0 = \overline{A}_{11}$, and $A_d = \overline{A}_{\tau 11}$.

Theorem 3.7 Let us consider the sliding mode dynamics (3.101). If the matrix $\Lambda \in R^{n\times(n-m)}$ is any full rank matrix such that $B^T\Lambda = 0, \Lambda^T\Lambda = I$, the matrices $Y \in R^{n\times n}, K \in R^{(n-m)\times(n-m)}, X_i \in R^{(n-m)\times(n-m)}$, and $Z_i \in R^{(n-m)\times(n-m)}$ are decision variables, and $*$ represents blocks that are readily inferred by symmetry such that the following LMI holds:

$$Y > 0, \qquad K \geq 0$$

$$\begin{bmatrix} \mathrm{N}_{11} & * & * & * \\ \mathrm{N}_{21} & \mathrm{N}_{22} & * & * \\ \tau X_i & \tau Z_i & -\tau\Lambda^T Y \Lambda & 0 \\ \mathrm{N}_{41} & \mathrm{N}_{42} & 0 & -\tau\Lambda^T Y\Lambda \end{bmatrix} < 0, \quad \forall i \tag{3.102}$$

where $\mathrm{N}_{11} = K + \Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda + X_i + *$,

$\mathrm{N}_{21} = \Lambda^T Y (A_{\tau i} + \mathrm{T}_i \Pi_i(t))^T \Lambda - X_i + Z_i^T$, $\mathrm{N}_{22} = -(1-d_m)K - Z_i - Z_i^T$,

$$\mathrm{N}_{41} = \tau\Lambda^T (A_i + \mathrm{T}_i\Pi_i(t))Y\Lambda, \quad \mathrm{N}_{42} = \tau\Lambda^T (A_{\tau i} + \mathrm{T}_i\Pi_i(t))Y\Lambda.$$

Suppose that the LMI (3.102) have a solution (Y, K, X_i, Z_i) for given $A_i, A_{\tau i}, B, d_m, \tau$, then there exists a linear sliding surface parameter matrix S and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \tag{3.103}$$

will guarantee that the sliding mode dynamics (3.101) is asymptotically stable.

Proof: Let us define a Lyapunov-Krasovskii function (LKF) as

$$V_g(t) = \alpha^T(t)P_0\alpha(t) + \int_{t-d}^{t} \alpha^T(s)F\alpha(s)ds + \int_{-\tau}^{0}\int_{t+\eta}^{t} \dot{\alpha}^T(s)P_0\dot{\alpha}(s)dsd\eta$$

where $P_0 = \Lambda^T Y\Lambda \in R^{n\times n}$ and $F \in R^{n\times n}$ are solution matrices for the LMI (3.102). It should be noted that a large number of previous methods such as the methods given in [42,43], have used similar Lyapunov-Krasovskii functions to obtain less-conservative stability conditions by exploiting information on the upper bounds of delay and its time derivative. None of the previous SMC design methods [44], [56-60] have used the term $\int_{-\tau}^{0}\int_{t+\eta}^{t} \dot{\alpha}^T(s)P_0\dot{\alpha}(s)dsd\eta$ in stability analysis. The time derivative of the Lyapunov-Krasovskii function is given by

$$\dot{V}_g = 2\alpha^T P_0(A_0\alpha + A_d\alpha_d) + \alpha^T F\alpha - (1-\dot{d})\alpha_d^T F\alpha_d + \tau\dot{\alpha}^T P_0\dot{\alpha} - \int_{t-\tau}^{t} \dot{\alpha}^T(s)P_0\dot{\alpha}(s)ds.$$

By using (3.101) and the Newton-Leibniz formula $\alpha - \alpha_d - \int_{t-d}^{t} \dot{\alpha}(s)ds = 0$, we have

$$\dot{V}_g = 2\alpha^T P_0(A_0\alpha + A_d\alpha_d) + \alpha^T F\alpha - (1-\dot{d})\alpha_d^T F\alpha_d + \tau(A_0\alpha + A_d\alpha_d)^T P_0(A_0\alpha + A_d\alpha_d)$$

$$-\int_{t-\tau}^{t} \dot{\alpha}^T(s)P_0\dot{\alpha}(s)ds + 2(\alpha^T X^T + \alpha_d^T Z^T)(\alpha - \alpha_d - \int_{t-\tau}^{t} \dot{\alpha}(s)ds)$$

where $X = \sum \beta_i X_i$ and $Z = \sum \beta_i Z_i$. By using the inequality $2x^T y \le x^T Hx + y^T H^{-1} Y$, where x and y are any vectors with appropriate dimensions and $H > 0$, we can obtain

$$2[\alpha^T(t)X^T + \alpha_d^T(t)Z^T]\int_{t-\tau}^{t}\dot{\alpha}(s)ds \le \tau[\alpha^T(t)X^T + \alpha_d^T(t)Z^T]P_0^{-1}[X\alpha(t) + Z\alpha_d(t)]$$

$+\int_{t-\tau}^{t}\dot{\alpha}^T(s)P_0\dot{\alpha}(s)ds$ which leads to

$$\dot{V}_g \le 2\alpha^T(P_0A_0\alpha + P_0A_d\alpha_d) + \alpha^T F\alpha - (1-d_m)\alpha_d^T F\alpha_d + \tau[\alpha^T X^T + \alpha_d^T Z^T]P_0^{-1}[X\alpha + Z\alpha_d]$$

$$+2(\alpha^T X^T + \alpha_d^T Z^T)(\alpha - \alpha_d) + \tau(P_0A_0\alpha + P_0A_d\alpha_d)^T P_0^{-1}(P_0A_0\alpha + P_0A_d\alpha_d).$$

By applying the Schur complement formula [48] to (3.102), we can obtain

$$\begin{bmatrix} \mathrm{N}_{11} & * \\ \mathrm{N}_{21} & \mathrm{N}_{22} \end{bmatrix} + \tau\begin{bmatrix} X_i^T \\ Z_i^T \end{bmatrix} P_0^{-1} \begin{bmatrix} X_i^T \\ Z_i^T \end{bmatrix}^T + \tau\begin{bmatrix} \Lambda^T Y(A_i + \mathrm{T}_i\Pi_i(t))^T\Lambda \\ \Lambda^T Y(A_{\tau i} + \mathrm{T}_i\Pi_i(t))^T\Lambda \end{bmatrix} P_o^{-1} \begin{bmatrix} \Lambda^T Y(A_i + \mathrm{T}_i\Pi_i(t))^T\Lambda \\ \Lambda^T Y(A_{\tau i} + \mathrm{T}_i\Pi_i(t))^T\Lambda \end{bmatrix}^T < 0. \tag{3.104}$$

This implies that $\dot{V}_g \le -\mu(\|\alpha\|^2 + \|\alpha_d\|^2)$ for some $\mu > 0$. After all, we can conclude that the sliding mode dynamics (3.101) is stable.

After the switching surface parameter matrix S is designed so that the reduced-order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met [33], [57], [61]. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintains it there for all subsequent time. We design a sliding fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear sliding switching feedback control law as the local controller:

Control Rule i: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -S(A_i + \mathrm{T}_i\Pi_i(t))x - S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d - \kappa_i(t)\frac{\sigma}{\|\sigma\|} \tag{3.105}$$

where $\kappa_i(t) = \dfrac{1}{1-\phi_m}(\xi_i(t) + \phi_m\|S(A_i + \mathrm{T}_i\Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d\| + \varepsilon_i)$ (3.106) and $\varepsilon_i > 0$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\sum_{i=1}^{r} \beta_i(\theta)\left(S(A_i + \mathrm{T}_i\Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d + \kappa_i(t)\frac{\sigma}{\|\sigma\|} \right) \qquad (3.107)$$

and we can establish the following theorem.

Theorem 3.8 Consider the closed-loop control system of the uncertain system (3.98) with control (3.107). Suppose that the LMI (3.102) is feasible and the sliding surface is given by (3.103). Then, the switching feedback control law (3.107) induces an ideal sliding motion on the sliding surface $\sigma = 0$ in finite time and the state converges to zero.

Proof: Since Theorem 3.7 implies that the sliding mode dynamics restricted to $\sigma = Sx = 0$ is stable, we only have to show that reachability condition $\sigma^T\dot{\sigma} < -\varepsilon\|\sigma\|$ is satisfied for some $\varepsilon > 0$. Using $SB = I$ and the assumption A2, we can obtain

$$\sigma^T\dot{\sigma} = \sigma^T \sum_{i=1}^{r} \beta_i (S(A_i + \mathrm{T}_i\Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d + h_i) + \sigma^T u$$

$$\le \sum_{i=1}^{r} \beta_i(\kappa_i - \phi_i\|u\| - \varsigma_i)\|\sigma\| \le -\sum_{i=1}^{r} \varepsilon_i\|\sigma\|.$$

After all, we can conclude that σ converges to zero.

Remark 3.4 Theorem 3.7 and 3.8 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Check that $(A_i + A_{\tau i}, B)$ is stabilization. If not, exit.

Step 2: Find a full-rank matrix $\Lambda \in R^{n\times(n-m)}$ such that $B^T\Lambda = 0, \Lambda^T\Lambda = I$.

Step 3: Find a solution vector (Y, c_1, c_2, η) to LMI (3.102).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (3.103).

Step 5: The controller is given by (3.107).

3.4.3 Numerical Examples

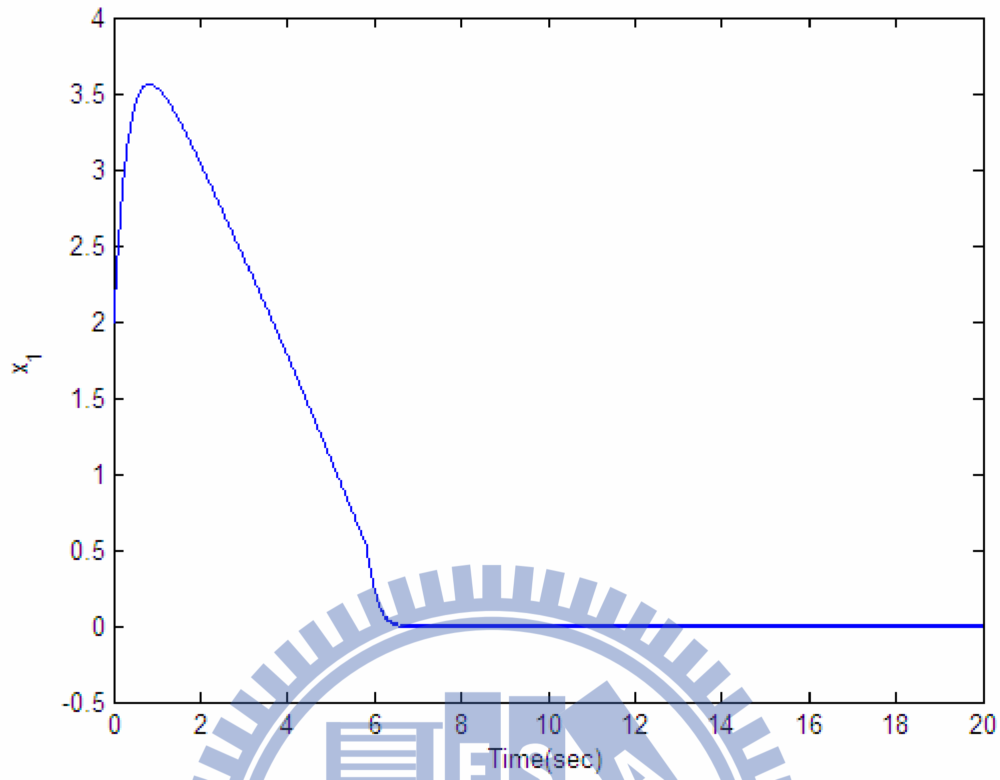
Example 3.7 To illustrate the performance of the proposed sliding fuzzy control design method, consider the following T-S fuzzy time-delay model [62] without mismatched parameter uncertainties and external disturbances.

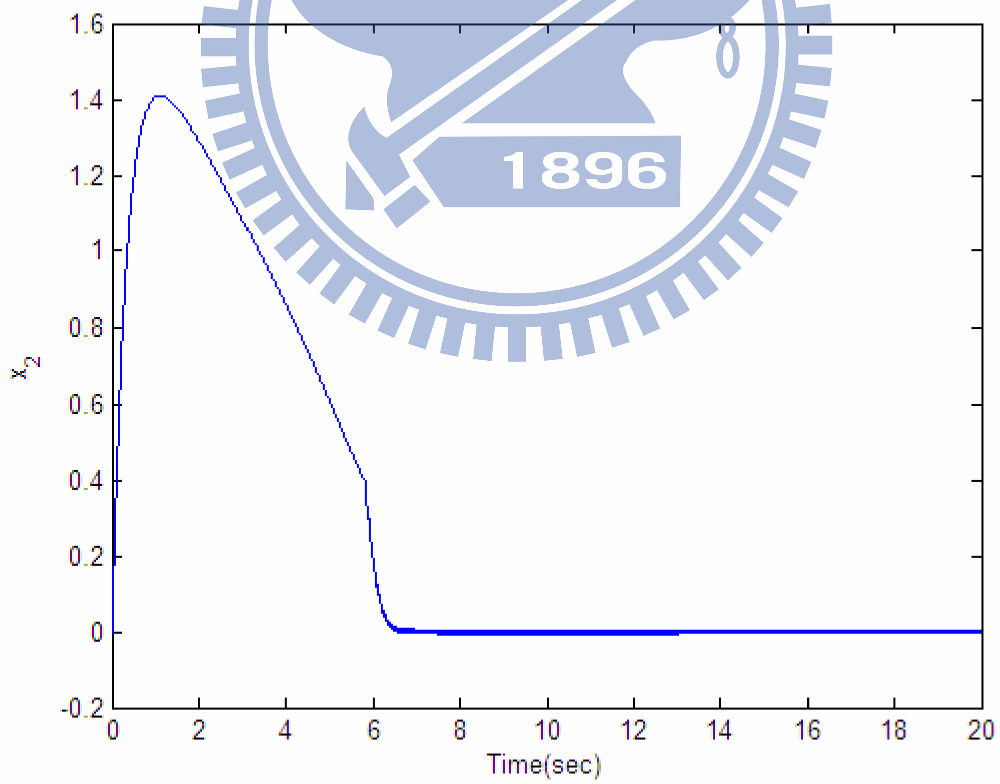
$$\dot{x}(t) = \sum_{i=1}^{2} \beta_i(\theta)[A_i x(t) + A_{\tau i} x_d(t)] + Bu(t) \tag{3.108}$$

where $x(t) = [x_1(t) \;\; x_2(t)]^T$ and $A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1 \end{bmatrix}, A_{\tau 1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$A_{\tau 2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \beta_1 = \dfrac{1}{1+e^{-2x_1(t)}}, \beta_2 = 1 - \beta_1.$

We assume that $d(t) = \tau = 0.4, \phi_i = 0, \xi_i = 1, \phi_m = 0, h_i = 0$ and $\varepsilon_i = 0.5$. Figure 3.10 shows the control results for system (3.108) via the proposed controller (3.107) under the initial condition $\varphi(t) = [2\ 0]^T$. In Figure 3.10, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system.


x1
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0 2 4 6 8 10 12 14 16 18 20
Time(sec)


x2
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

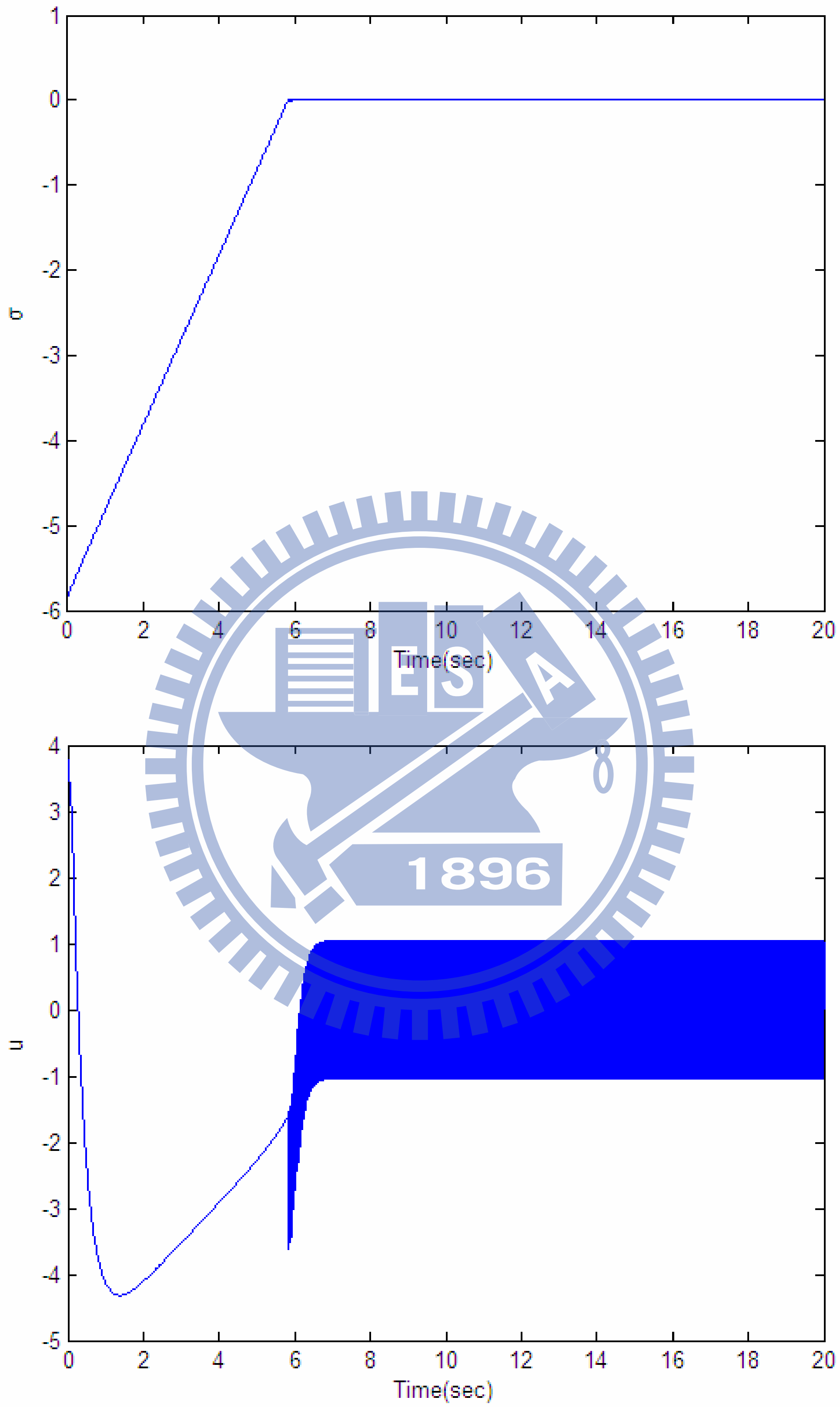


Figure 3.10 Control results for the system (3.108)

Example 3.8 Consider a well-studied example of a continuous-time truck-trailer with time-delay proposed in [63]. The time-delay model is given by

$$\dot{x}_1(t) = -a\frac{vT}{Lt_0}x_1(t) - (1-a)\frac{vT}{Lt_0}x_1(t-d) + \frac{vT}{lt_0}[u(t)+h(t)],$$

$$\dot{x}_2(t) = a\frac{vT}{Lt_0}x_1(t) + (1-a)\frac{vT}{Lt_0}x_1(t-d),$$

$$\dot{x}_3(t) = \frac{vT}{t_0}\sin\left[x_2(t) + a\frac{vT}{2L}x_1(t) + (1-a)\frac{vT}{2L}x_1(t-d)\right] \tag{3.109}$$

where $x_1(t)$ is the angle difference between truck and trailer (in radians), $x_2(t)$ is the angle of trailer (in radians), $x_3(t)$ is the vertical position of rear of trailer (in meters), $u(t)$ is the steering angle (in radians), $T = 2.0, l = 2.8$, $L = 5.5$, $v = -1.0$ and $t_0 = 0.5$. The constant parameter a is the retarded coefficient satisfying $a \in [0,1]$. The limits 1 and 0 correspond to a no-delay term and to a completed-delay term. We assume that the disturbance input $h(t)$ is unknown but bounded as $|h(t)| \le 1$. By using the fact that $\sin(x) \approx x$ if $x \approx 0$, we can represent the above model as the following two-rule T-S fuzzy model, including parameter uncertainties and external disturbances:

Plant Rule 1: IF $\theta(t)$ is about 0, THEN

$$\dot{x} = (A_1 + \mathrm{T}_1\Pi_1(t))x + (A_{\tau 1} + \mathrm{T}_1\Pi_1(t))x_d + Bu + Bh_1$$

Plant Rule 2: IF $\theta(t)$ is about $\pm\pi$, THEN

$$\dot{x} = (A_2 + \mathrm{T}_2\Pi_2(t))x + (A_{\tau 2} + \mathrm{T}_2\Pi_2(t))x_d + Bu + Bh_2$$

where $\theta(t) = x_2(t) + avT\, x_1(t)/2L + (1-a)vT\, x_1(t-d)/2L$

$$A_1 = \begin{bmatrix} -a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{v^2T^2}{2Lt_0} & \frac{vT}{t_0} & 0 \end{bmatrix}, \quad A_{\tau 1} = \begin{bmatrix} -(1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{v^2T^2}{2Lt_0} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{vT}{Lt_0} & 0 & 0 \\ a\frac{10v^2T^2}{2L\pi} & \frac{10vT}{\pi} & 0 \end{bmatrix}, \quad A_{\tau 2} = \begin{bmatrix} -(1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{vT}{Lt_0} & 0 & 0 \\ (1-a)\frac{10v^2T^2}{2L\pi} & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{vT}{lt_0} \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{T}_1 = \mathrm{T}_2 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad \Pi_1(t) = \Pi_2(t) = \begin{bmatrix} \sin t & 0 & 0 \end{bmatrix},$$

$$\beta_1 = \frac{1 - 1/(1 + e^{-2(\theta - 0.5\pi)})}{1 + e^{-2(\theta + 0.5\pi)}}, \; \beta_2 = 1 - \beta_1, \qquad h_1 = h_2 = h(t). \tag{3.110}$$

We assume that $d(t) = \tau = 0.1$. Considering LMI optimization with the data (3.110), $a = 0, \tau = 0.1$ and $d_m = 0$, we can obtain the sliding surface parameter vector $\sigma = Sx$. Since $|h_i(t)| \le 1$, we can set $\phi_i = 0, \xi_i = 1, \phi_m = 0$, $\varepsilon_i = 0.2$, and $t_{sampling} = 0.01\,\mathrm{sec}$. We can obtain the following fuzzy controller:

Control Rule 1: IF $\theta(t)$ is about 0, THEN

$$u(t) = -S(A_1 + \mathrm{T}_1\Pi_1(t))x - S(A_{\tau 1} + \mathrm{T}_1\Pi_1(t))x_d - 1.2\,\mathrm{sgn}(\sigma).$$

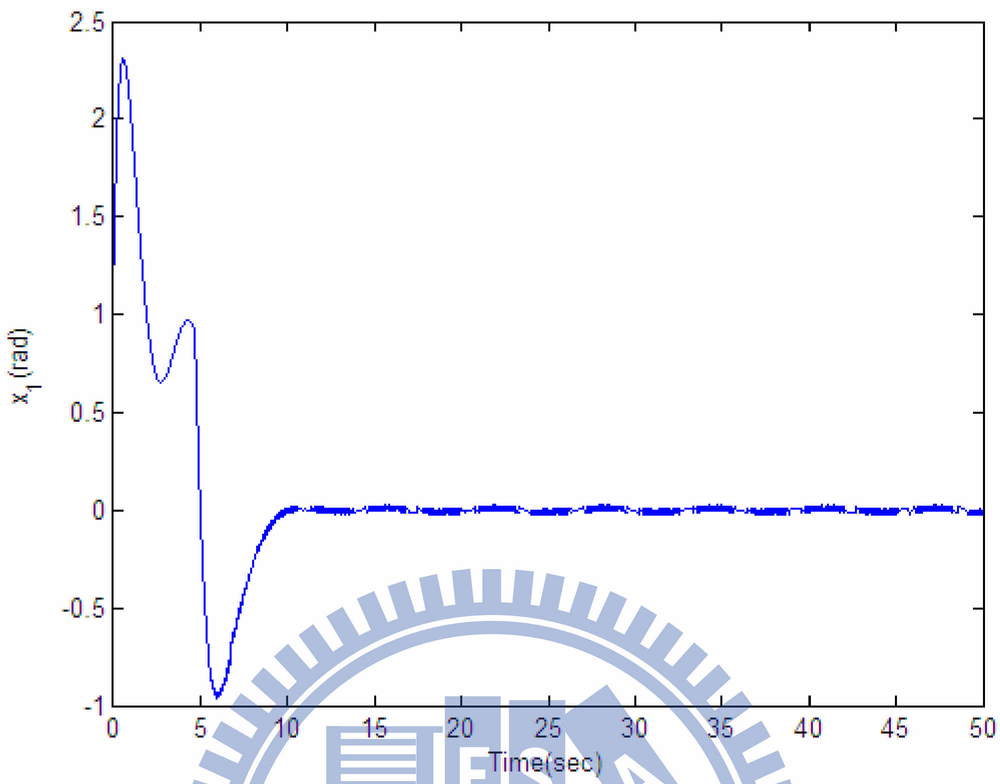
Control Rule 2: IF $\theta(t)$ is about $\pm\pi$, THEN

$$u(t) = -S(A_2 + \mathrm{T}_2\Pi_2(t))x - S(A_{\tau 2} + \mathrm{T}_2\Pi_2(t))x_d - 1.2\,\mathrm{sgn}(\sigma).$$

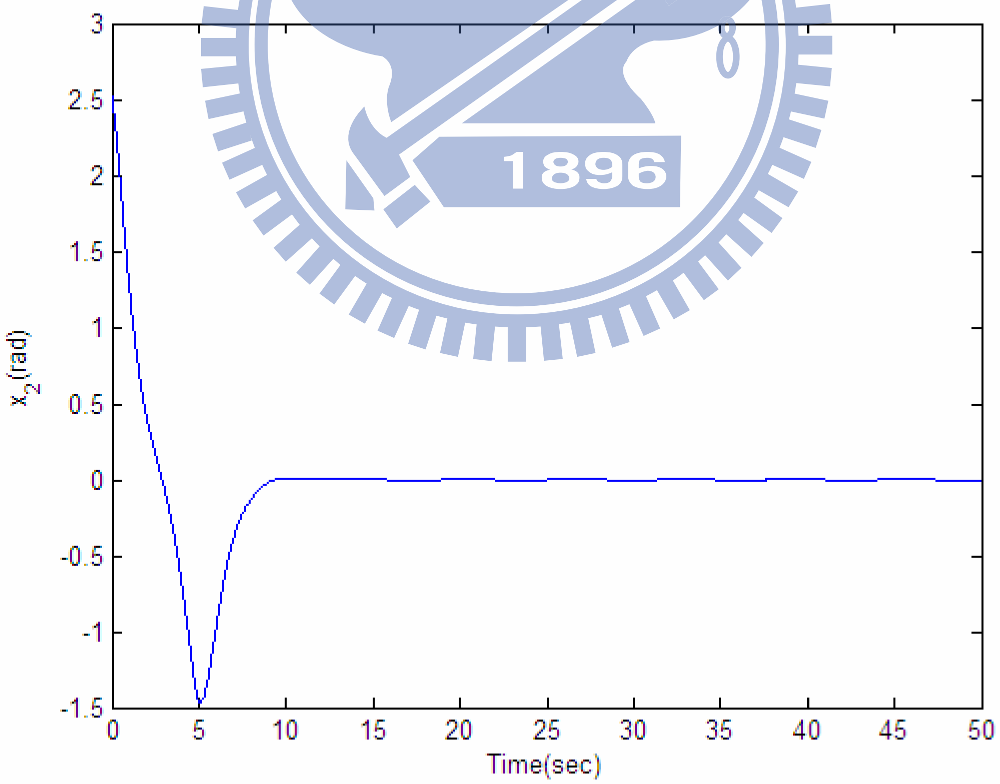
The final controller inferred as the weighted average of each local controller is given by

$$u(t) = -\sum_{i=1}^{2} \beta_i [S(A_i + \mathrm{T}_i \Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i \Pi_i(t))x_d + 1.2\,\mathrm{sgn}(\sigma)]. \tag{3.111}$$

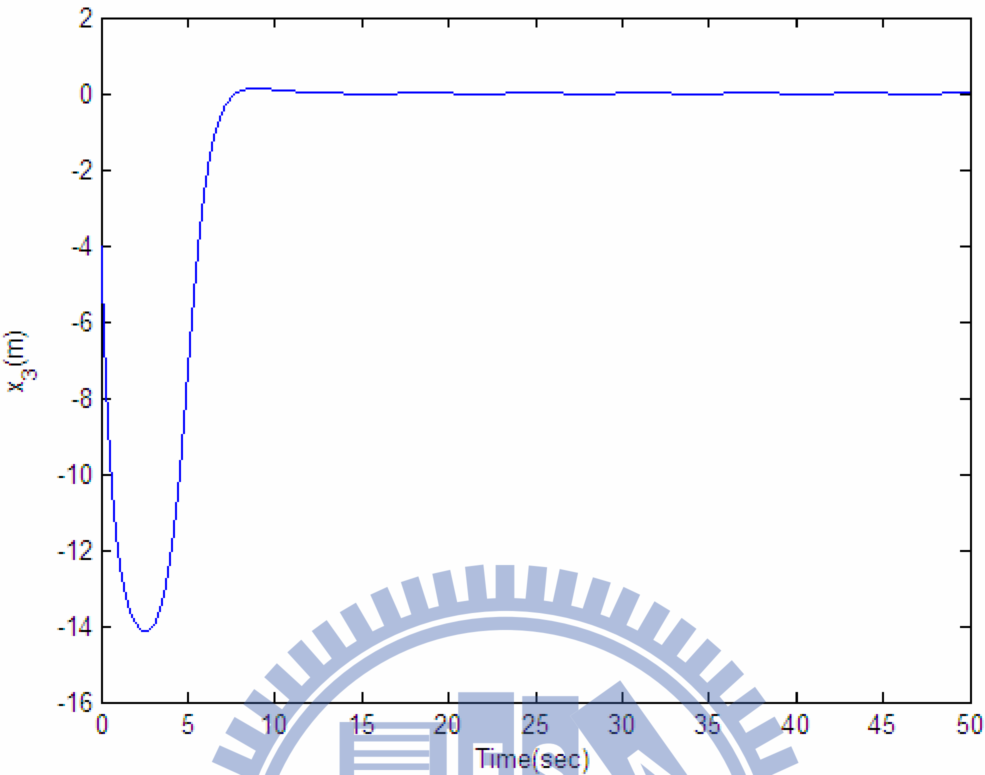
To demonstrate the controller ability, we apple the above fuzzy controller (3.111) to the system model (3.110) with $h(t) = \sin t$ and $d(t) = \tau = 0.1$. Figure 3.11 shows the closed-loop system responses of (3.110) and the proposed controller (3.111) with the initial condition $\psi(t) = [0.4\pi, 0.8\pi, -4]^T$. Moreover, the closed-loop system responses of the truth model (3.109) and the proposed controller (3.111) with the initial condition $\psi(t) = [0.4\pi, 0.8\pi, -4]^T$ are also shown in Figure 3.12. In Figure 3.11 and Figure 3.12, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 3.11 and Figure 3.12, the proposed controller is applicable to T-S fuzzy time-delay systems with mismatched parameter uncertainties in the state matrix and external disturbances and the nonlinear truth model. The control performances of the two-rule T-S fuzzy model (3.110) and the nonlinear truth model (3.109) are satisfactory.

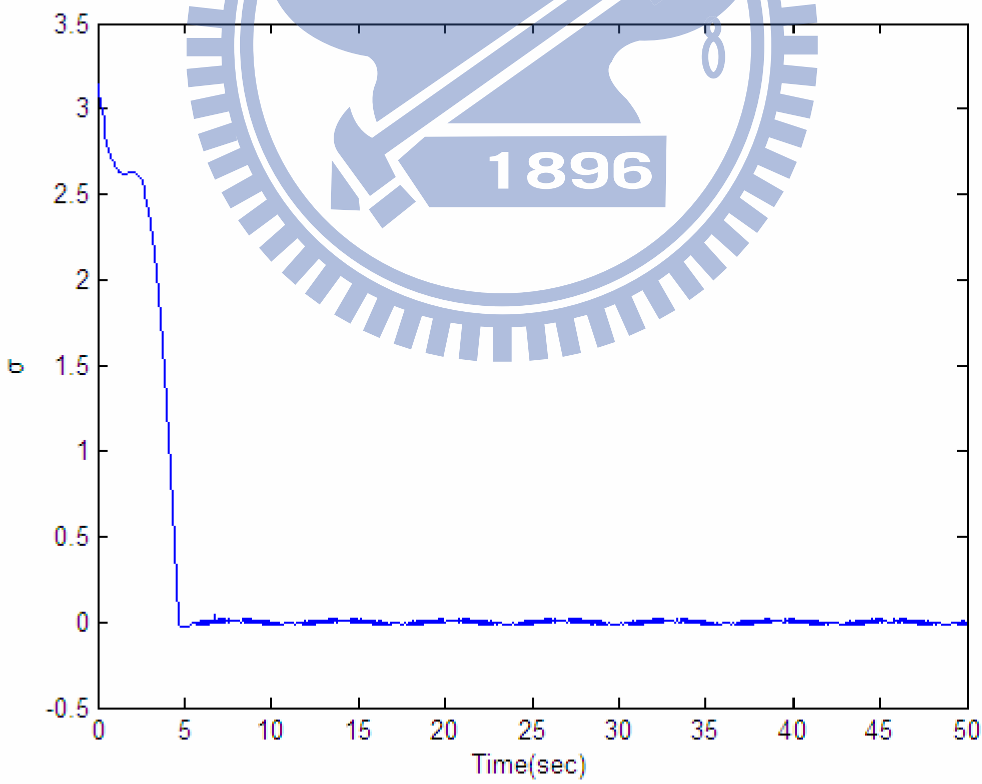


2.5
2
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30
35
40
45
50
x_1(rad)
Time(sec)



3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
40
45
50
x_2(rad)
Time(sec)


x3(m)
Time(sec)


σ
Time(sec)

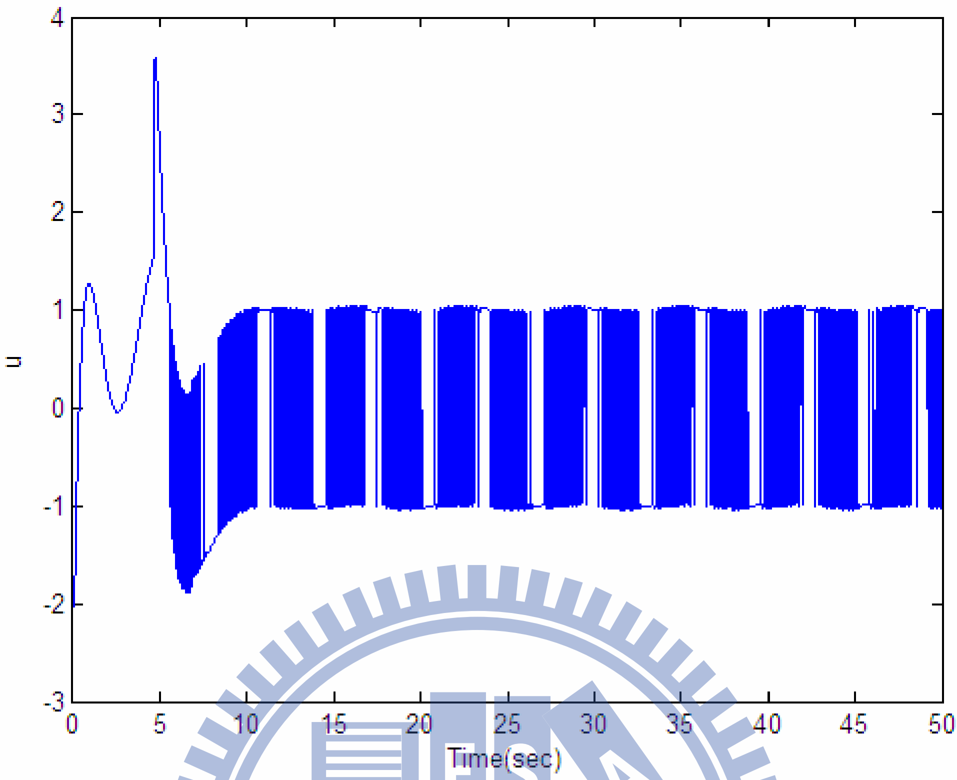
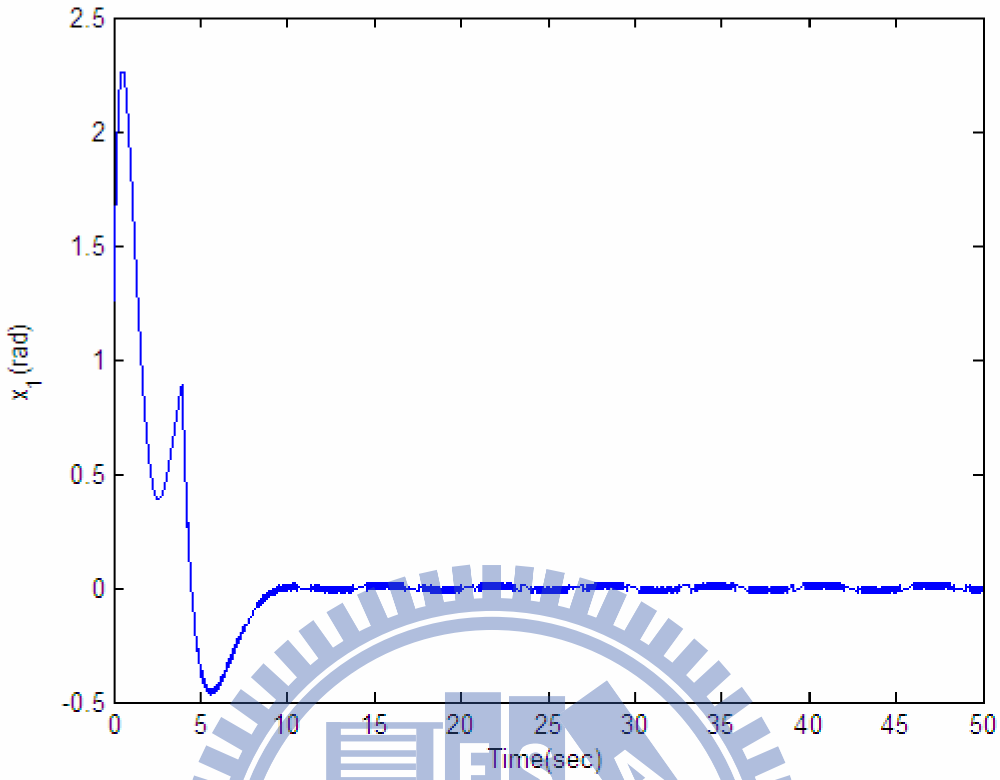
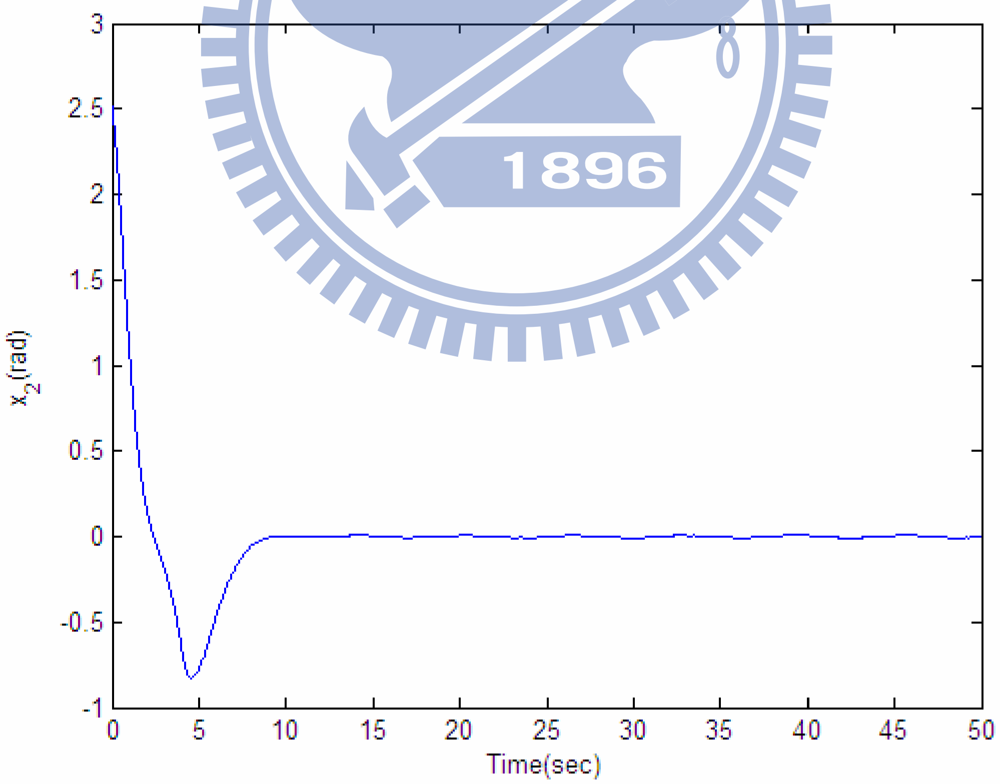


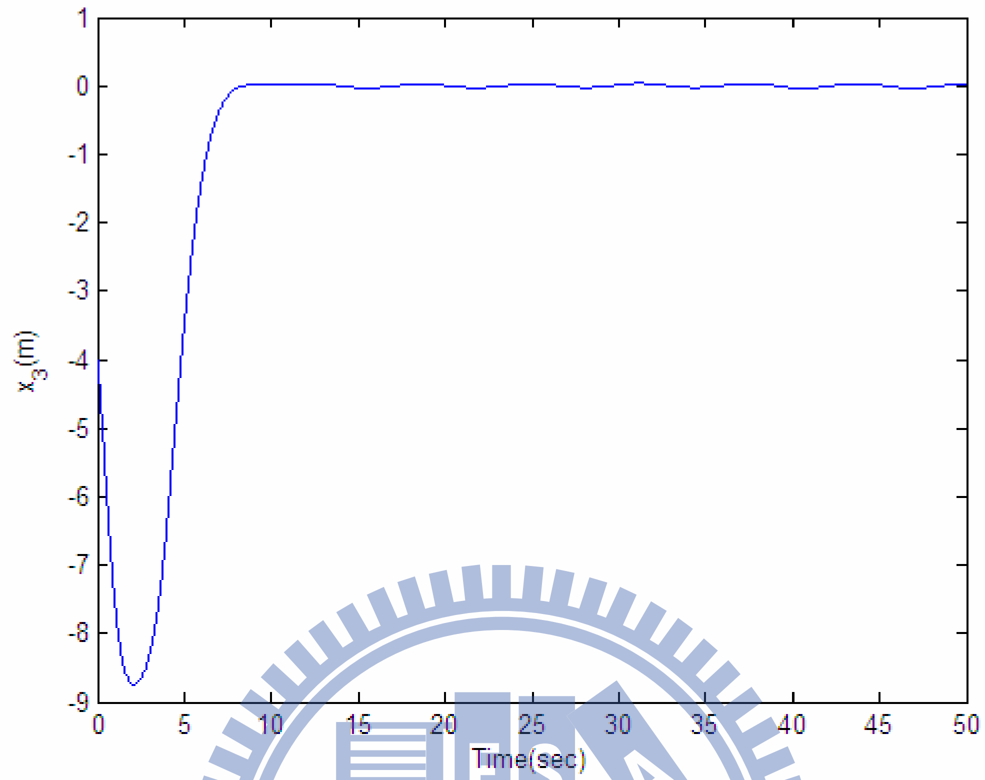
Figure 3.11 Simulation results with the proposed method on the two-rule T-S fuzzy model (3.110).



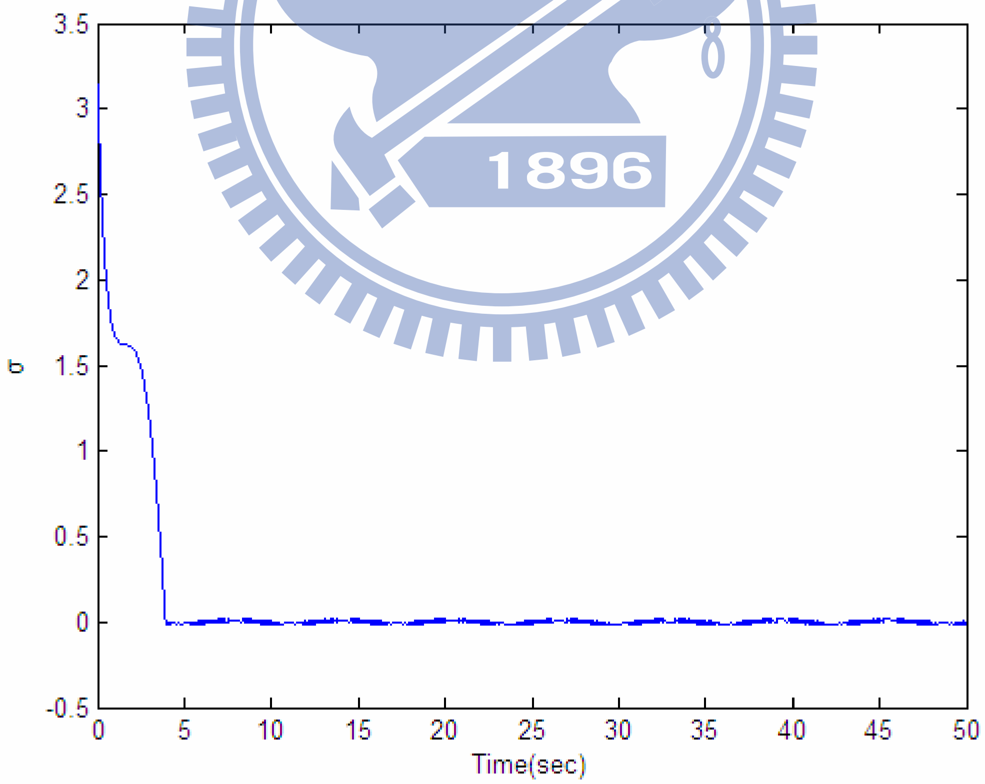
x_1(rad)
Time(sec)



x_2(rad)
Time(sec)



x_3(m)
Time(sec)



σ
Time(sec)

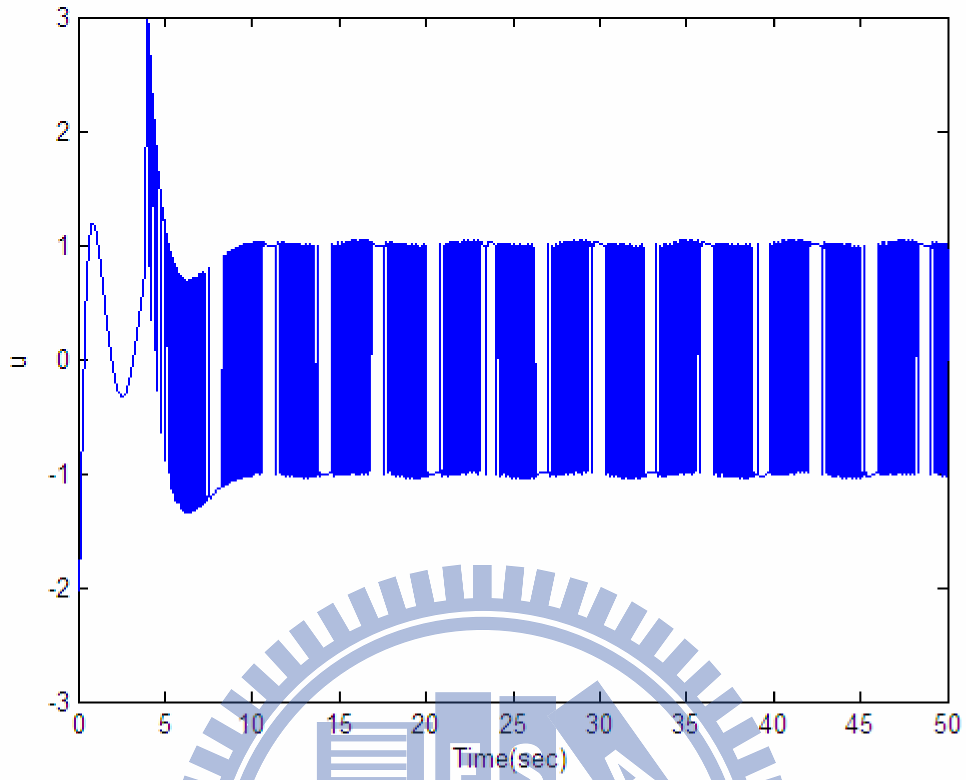


Figure 3.12 Simulation results with the proposed method on the nonlinear truth model (3.109).

Chapter 4

LMI-Based Robust Adaptive Control

In this chapter, LMI-based robust adaptive control methods are developed for distinct uncertain Takagi-Sugeno fuzzy models/time-delay models. The introduction of this chapter is introduced in Section 4.1. In Section 4.2, a robust adaptive control method is proposed for T-S fuzzy systems. Section 4.3 presents two kinds of robust adaptive control methods for mismatched T-S fuzzy systems. A robust adaptive control method is presented for mismatched T-S fuzzy time-delay systems in Section 4.4.

4.1 Introduction

Fuzzy techniques have been widely and successfully applied to nonlinear system modeling and control for over two decades. The feedback stabilization problem of a nonlinear system in the Takagi-Sugeno (T-S) model [5] has been studied extensively. In the T-S model, local models are combined to describe the global behavior of the nonlinear system. Some authors [23-29] have studied to solve the feedback stabilization problem based on the assumption that the local model can be described by a simple linear system. In practice, the inevitable uncertainties may enter a nonlinear system model in a very complicated way. The uncertainty may include modeling errors, parameter variations, external disturbances, and fuzzy approximation errors. In such a situation, the fuzzy feedback control design methods of [23-29] may not work well anymore. To deal with the problem, some authors [30,31] have exploited the variable structure system (VSS) theory which has proposed an effective method to design robust controllers for uncertain nonlinear systems where external disturbances are bounded by known upper norm bounds.

Some authors [36-40] have relaxed the assumption and they have proposed adaptive laws to estimate the upper norm bounds. However, the previous VSC-based fuzzy control methods have considered the problem of adaptive control design and stability analysis for uncertain T-S fuzzy models where the input matrices of the local system models satisfy the assumption that each nominal local system shares the same input channel. It is practically difficult to satisfy this assumption. Moreover, these years, other authors [44-46] have exploited the SMC approach theory which has provided an effective means to design robust controllers for uncertain fuzzy time-delay systems where external disturbances are bounded by known upper norm bounds.

In this chapter, we propose robust adaptive control design methods for different uncertain T-S fuzzy models with matched/mismatched parameter uncertainties and external disturbances which are bounded by unknown upper norm bounds. As the local controller, we use an adaptive controller with a nonlinear switching feedback control term and an adaptation law to specify unknown upper norm bounds. We derive LMI conditions for existence of linear sliding surfaces guaranteeing asymptotic stability of the reduced order equivalent sliding mode dynamics, and we give an explicit formula of the switching surface parameter matrix in terms of the solution of the LMI existence conditions. We also design the nonlinear switching feedback control term and an adaptation law to drive the system trajectories so that a stable sliding motion is induced in finite time on the switching surface and the state converges to zero. Moreover, a robust adaptive control design method is also presented for the uncertain T-S time-delay model with mismatched parameter uncertainties and external disturbances. Finally, some examples are used to illustrate the effectiveness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection.

4.2 Robust Adaptive Control for T-S Fuzzy Systems

In this section, system formulation for the uncertain T-S fuzzy model is described in Section 4.2.1. A robust adaptive control method via LMI is proposed in Section 4.2.2. Some examples are used to illustrate the effectiveness of the proposed methods and to compare with the existing methods in Section 4.2.3.

4.2.1 System Formulation

Consider the following uncertain T-S fuzzy model [49]:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)\left[A_i x(t) + B_i u(t) + B_i h(t,x)\right] \tag{4.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, A_i, B_i are constant matrices of appropriate dimensions, $\theta = [\theta_1, \cdots, \theta_s], \theta_j\ (j = 1, \cdots, s)$ are the premise variables, s is the number of the premise variables, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta)$, $\omega_i : R^s \to [0,1]$, $i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i, r is the number of the IF-THEN rules, β_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0$, $\sum_{i=1}^{r} \beta_i(\theta) = 1$, $h(t,x) \in R^m$ represents the lumped nonlinearities or uncertainties. We will assume that the followings are satisfied:

A1: The $n \times m$ matrix B defined by $B = 1/r \sum_{i=1}^{r} B_i$ satisfies the rank constraint rank$(B) = m$, i.e., the matrix B has full column rank m.

A2: The function $h(t,x)$ is unknown but bounded as $\left\| h(t,x) - \hat{h}(t,x) \right\| \leq \sum_{k=0}^{l} \rho_k \left\| x \right\|^k$ where $\rho_0, \cdots, \rho_l$ are unknown constants, $\hat{h}(t,x)$ is an estimate of $h(t,x)$, and l is a known positive integer.

The system (4.1) does not have to satisfy the restrictive assumption that all the input matrices of the local system models are in the same range space. It should be noted that the assumption A1 implies that $rank(B_i) \le m$ and each nominal local system model may not share the same input channel. The assumption A2 with $l = 1$ and $\hat{h}(t,x) = 0$ has been used in the literature [50]. We can set $\hat{h}(t,x)$ as the nominal value of $h(t,x)$. Using the above assumptions, the uncertain T-S fuzzy model (4.1) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta) A_i x(t) + [B + HF(\beta)G][u + h(t,x)] \tag{4.2}$$

where $\beta = [\beta_1(\theta), \cdots, \beta_r(\theta)]$, and the matrices $H, G, F(\beta)$ are defined by

$$H = \frac{1}{2}[(B - B_1), \cdots, (B - B_r)], G = [I, \cdots, I]^T,$$

$$F(\beta) = diag\left[(1 - 2\beta_1(\theta))I, \cdots, (1 - 2\beta_r(\theta))I\right]. \tag{4.3}$$

It should be noted that the system (4.1) does not have to satisfy $B_1 = B_2 = \cdots = B_r$, which is used in almost all published results on VSS design methods including the VSS-based fuzzy control design methods of [33,34]. Hence the methods [30,31] cannot be applied to the above model (4.1). Since $\beta_i(\theta) \ge 0$ and $\sum_{i=1}^{r} \beta(\theta) = 1$, we can see that the following inequality always holds:

$$F^T(\beta)F(\beta) = F(\beta)F^T(\beta) \le I. \tag{4.4}$$

Many examples in the literature and various mechanical systems such as motors and robots do not satisfy the restrictive assumptions that each nominal local system model shares the same input channel and they fall into the special cases of the above model [49].

4.2.2 Adaptive Control Design via LMI

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions: The first involves the design of $m(n-1)-$dimensional switching surfaces for the sliding mode such that the reduced order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second is concerned with the selection of a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first characterize linear sliding surfaces using LMIs.

Let us define the linear sliding surface as $\sigma = Sx = 0$ where S is a $m \times n$ matrix. Referring to the previous results [33], [51], we can see that for the system (4.2) it is reasonable to find a sliding surface such that

P1 $[SB + SHF(\beta)G]$ is nonsingular for any β satisfying $\beta_i(\theta) \geq 0, i = 1, \cdots, r,$ and $\sum_{i=1}^{r} \beta_i(\theta) = 1.$

P2 The reduced $(n-m)$ order sliding mode dynamics restricted to the sliding surface $Sx = 0$ is asymptotically stable for all admissible uncertainties.

It should be noted that P1 is necessary for the existence of the unique equivalent control [33] and the assumption A1 is necessary for the nonsingularity of SB.

Define a transformation matrix and the associated vector v as $M = [\Lambda(\Lambda^T Y \Lambda)^{-1}, Y^{-1}B(B^T Y^{-1} B)^{-1}]^T = [V^T, S^T]^T$, $v = [v_1^T, v_2^T]^T = Mx$ where $v_1 \in R^{n-m}$, $v_2 \in R^m$. By the above transformation, we can see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. Then, from system (4.2), we can obtain

$$\begin{bmatrix} \dot{v}_1 \\ \dot{\sigma} \end{bmatrix} = \sum_{i=1}^{r} \beta_i(\theta) \begin{bmatrix} VA_i Y\Lambda & VA_i B \\ SA_i Y\Lambda & SA_i B \end{bmatrix} \begin{bmatrix} v_1 \\ \sigma \end{bmatrix} + \begin{bmatrix} VHF(\beta)G \\ I + SHF(\beta)G \end{bmatrix} [u + h(t,x)]. \tag{4.5}$$

From the equivalent control method [33], we can see that the equivalent control is given by $u_{eq}(t) = -\sum_{i=1}^{r} \beta_i(\theta)[I + SHF(\beta)G]^{-1} SA_i x - h(t,x)$. By setting $\dot{\sigma} = \sigma = 0$ and substituting $u(t)$ with $u_{eq}(t)$, we can show that the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma = Sx = 0$ is given by

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T D(\beta) A_i Y \Lambda v_1 \tag{4.6}$$

where $D(\beta) = I - HF(\beta)G[I + SHF(\beta)G]^{-1}S$.

Theorem 4.1 Let us consider the sliding mode dynamics (4.6). If $Y \in R^{n\times n}$, $c_1 \in R, c_2 \in R, \eta \in R$ are decision variables, $\kappa = \lambda_{\min}(B^T B)$, $\Lambda \in R^{n\times(n-m)}$ is any full rank matrix satisfying $B^T \Lambda = 0, \Lambda^T \Lambda = I$, and $*$ represents blocks that are readily inferred by symmetry such that the following LMIs holds:

$$\begin{bmatrix} \Lambda^T (A_i Y + *)\Lambda & * & * \\ \eta H^T \Lambda & -I & * \\ A_i Y \Lambda & \eta H & -I \end{bmatrix} < 0, \quad \forall i \tag{4.7}$$

$$\begin{bmatrix} Y & I & 0 \\ I & c_1 I & 0 \\ 0 & 0 & c_2 I - Y \end{bmatrix} > 0, \tag{4.8}$$

$$\begin{bmatrix} 2\eta\kappa & * & * \\ rc_1 & r\eta & 0 \\ rc_2 & 0 & r\eta \end{bmatrix} > 0. \tag{4.9}$$

Suppose that the LMIs (4.7)-(4.9) have a solution vector (Y, c_1, c_2, η), then there exists a linear sliding surface parameter matrix S satisfying P1-P2 and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \tag{4.10}$$

will guarantee that the sliding mode dynamics (4.6) is asymptotically stable.

Proof: By using Schur complement formula [48], we can easily show that in fact the following LMIs are incorporated in the LMIs (4.7)-(4.9)

$$c_1 > 0, \quad c_2 > 0, \quad \eta > 0, \quad \eta^2 HH^T < I, \quad 2\eta^2\kappa > r(c_1^2 + c_2^2). \tag{4.11}$$

It is clear that if the following inequality (4.12) holds, then $SB + SHF(\beta)G = I + SHF(\beta)G$ is nonsingular and hence P1 holds

$$SHF(\beta)GG^T F^T(\beta)H^T S^T < I. \tag{4.12}$$

Using (4.3), (4.4), (4.11) and $GG^T \le \|G\|^2 I = rI$, we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T. \tag{4.13}$$

By using the Schur complement formula, we can see that (4.8) and (4.11) imply

$$0 < c_1^{-1} I < Y < c_2 I, \qquad 0 < c_2^{-1} I < Y^{-1} < c_1 I \tag{4.14}$$

and this leads to

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T \le \frac{rc_1c_2}{\eta^2}(B^T B)^{-1} \le \frac{rc_1c_2}{\kappa\eta^2} I. \tag{4.15}$$

Using the inequality $2ab \le a^2 + b^2$ where a and b are scalars, we can show that (4.15) implies

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{2\kappa\eta^2}(c_1^2 + c_2^2) I. \tag{4.16}$$

Finally, by using the above inequalities (4.11) and (4.16), we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T < I \tag{4.17}$$

which implies that $[SB + SHF(\beta)G]$ is nonsingular, i.e., P1 holds.

Now, we will show that S of (4.10) guarantees P2. Using the matrix inversion lemma:

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B$$

where A and B are compatible constant matrices such that $(I + AB)$ is nonsingular, we can show that the sliding mode dynamics (4.6) is equivalent to

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T C(\beta) A_i Y \Lambda v \tag{4.18}$$

Where $C(\beta) = I - H[I + F(\beta)GSH]^{-1} F(\beta)GS = [I + HF(\beta)GS]^{-1}$

$$= I - HF(\beta)G[I + SHF(\beta)G]^{-1} S = D(\beta).$$

The sliding mode dynamics (4.18) is asymptotically stable if there exists a positive definite matrix $P_0 \in R^{(n-m)\times(n-m)}$ such that the time derivative of the Lyapunov function

$E_g(t) = v_1^T P_0 v_1$ satisfies for some positive scalar τ

$$\dot{E}_g(t) = 2\sum_{i=1}^{r} \beta_i(\theta) v_1^T P_0 Z_i(\beta) v_1 \le -\tau v_1^T v_1 \quad (4.19)$$

where $Z_i(\beta) = (A_{i0} + B_0[I - N(\beta)D_0]^{-1} N(\beta)C_{i0})$, $A_{i0} = (\Lambda^T Y \Lambda)^{-1} \Lambda^T A_i Y \Lambda$,

$$B_0 = (\Lambda^T Y \Lambda)^{-1} \Lambda^T H, \ C_{i0} = A_i Y \Lambda, \ D_0 = H, N(\beta) = -F(\beta)GS.$$

It should be noted that the inequalities (4.4), (4.11), (4.17) and $GG^T \le \|G\|^2 I = rI$ imply

$$N(\beta)N^T(\beta) = F(\beta)GSS^T G^T F^T(\beta) \le \eta^2 I, \eta^2 D_0^T D_0 = \eta^2 H^T H < I. \quad (4.20)$$

This and (4.19) imply that (4.18) is asymptotically stable if there exists a positive definite matrix P_0 such that

$$P_0 A_{i0} + P_0 B_0 [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} + * < 0 \ \forall i \quad (4.21)$$

where $*$ represents blocks that are readily inferred by symmetry.

Let z_i be $z_i = [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} y$ where $y \in R^{(n-m)}$. Then z_i can be rewritten as $z_i = N(\beta)[C_{i0} y + D_0 z_i]$. This equality and (4.20) imply $z_i^T z_i \le \eta^2 [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i]$ and this leads to

$$2y^T P_0 B_0 [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} y$$

$$= 2y^T P_0 B_0 z_i \le 2y^T P_0 B_0 z_i + [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i] - \eta^{-2} z_i^T z_i$$

$$= y^T C_{i0}^T C_{i0} y + 2y^T [P_0 B_0 + C_{i0}^T D_0] z_i - \eta^{-2} z_i^T \Omega z_i \text{ where } \Omega = I - \eta^2 D_0^T D_0. \quad (4.22)$$

Since $\Omega > 0$, the following inequality holds for any (y, z_i):

$$2y^T [P_0 B_0 + C_{i0}^T D_0] z_i \le \eta^{-2} z_i^T \Omega z_i + \eta^2 y^T [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T y. \quad (4.23)$$

Using (4.22) and (4.23), we can show that the Lyapunov inequality (4.21) is satisfied if the following inequality holds:

$$P_0 A_{i0} + A_{i0}^T P_0 + C_{i0}^T C_{i0} + \eta^2 [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T < 0.$$

Using the Schur complement formula, we can rewrite the above inequality as

$$\begin{bmatrix} A_{i0}^T P_o + * & * & * \\ \eta B_0^T P_0 & -I & * \\ C_{i0} & \eta D_0 & -I \end{bmatrix} < 0, \forall i. \tag{4.24}$$

Let the positive definite matrix P_0 be $P_0 = \Lambda^T Y \Lambda$ where Y is a solution to LMIs (4.7)-(4.9), which implies that the sliding mode dynamics (4.18) is asymptotically stable. Hence, the sliding mode dynamics (4.6) is asymptotically stable.

After the switching surface parameter matrix S is designed so that the reduced $(n-m)$ order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintains it there for all subsequent time. With σ of (4.10), we design an adaptive fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear adaptive switching feedback control law as the local controller.

Control rule *i*: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\hat{h}(t,x) - \chi_i \sigma - SA_i x - \frac{1}{1-\omega} \hat{\delta}_i(t,x) \frac{\sigma}{\|\sigma\|}$$

where $$\hat{\delta}_i(t,x) = \alpha_i + \omega \|SA_i x\| + (1+\omega) \sum_{k=0}^{l} \rho_k \|x\|^k \tag{4.25}$$

$$\dot{\hat{\rho}}_k = \varepsilon_k \|\sigma\| \cdot \|x\|^k \tag{4.26}$$

and $\sigma = Sx, \omega = \sqrt{r}\|SH\|, \alpha_i > 0, \chi_i > 0, \varepsilon_k > 0$. It should be noted that (4.17) implies $\omega = \sqrt{r}\|SH\| \le \sqrt{r}\|S\| \cdot \|H\| \le \eta \|H\|$. This and (4.11) guarantee $0 \le \omega < 1$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\hat{h}(t,x) - \sum_{i=1}^{r} \beta_i(\theta) \left(\chi_i \sigma + SA_i x + \frac{1}{1-\omega} \hat{\delta}_i(t,x) \frac{\sigma}{\|\sigma\|} \right) \tag{4.27}$$

and we can establish the following theorem.

Theorem 4.2 Consider the closed-loop control system of the uncertain system (4.2) with control (4.27). Suppose that the LMIs (4.7)-(4.9) has a solution vector (Y, c_1, c_2, η) and the linear sliding surface is given by (4.10). Then the state converges to zero.

Proof: Since Theorem 4.1 implies that the linear sliding surface (4.10) guarantees P1-P2, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T\sigma + 0.5\xi\sum_{k=0}^{l}\tilde{\rho}_k^2$ where $\xi = 1+\omega$ and $\tilde{\rho}_k = \hat{\rho}_k - \rho_k$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T\dot{\sigma} + \xi\|\sigma\|\sum_{k=0}^{l}\tilde{\rho}_k\|x\|^k$. From (4.2), (4.10), (4.27), $\|SHF(\beta)G\| \le \sqrt{r}\|SH\| = \omega$,$0 \le \omega < 1$, and A2, we obtain

$$\sigma^T\dot{\sigma} = \sigma^T\sum_{i=1}^{r}\beta_i(\theta)SA_i x(t) + \sigma^T[I + SHF(\beta)G][u + h(t,x)]$$

$$\le \sum_{i=1}^{r}\beta_i(\theta)\sigma^T SA_i x(t) + \sigma^T u + \{\omega\|u\| + (1+\omega)\|h(t,x)\|\}\|\sigma\|$$

$$\le -(1-\omega)\sum_{i=1}^{r}\beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i(\theta)\alpha_i\|\sigma\| - \xi\|\sigma\|\sum_{k=0}^{l}\tilde{\rho}_k\|x\|^k .$$

This implies that $\dot{E}_g \le -(1-\omega)\sum_{i=1}^{r}\beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i(\theta)\alpha_i\|\sigma\| \le 0$ which indicates that $E_g \in L_2 \cap L_\infty, \dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 4.1 Theorem 4.1 and 4.2 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Obtain $B = \frac{1}{r}\sum_{i=1}^{r}B_i$ and $H = \frac{1}{2}[(B - B_1),\cdots,(B - B_r)]$ for given B_i .

Step 2: Check that (A_i, B) is stabilization. If not, exit.

Step 3: Find a solution vector (Y, c_1, c_2, η) to LMI (4.7)-(4.9).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (4.10).

Step 5: The controller is given by (4.27).

4.2.3 Numerical Examples

Example 4.1 Consider the following inverted pendulum on a cart [49]

$$\dot{x}_1 = x_2,\ \dot{x}_3 = x_4,\ \dot{x}_2 = \frac{1}{l\psi}(3g\sin x_1 - 3a\cos x_1[u + d(t) + \phi]),$$

$$\dot{x}_4 = -\frac{1}{\psi}(1.5mag\sin 2x_1 - 4a[u + d(t) + \phi]) \tag{4.28}$$

where x_1 is the angle (rad) of the pendulum from the vertical, $x_2 = \dot{x}_1$, x_3 is the displacement (m) of the cart, $x_4 = \dot{x}_3$, $\psi = 4 - 3ma\cos^2 x_1$, $\phi = mlx_2^2 \sin x_1$, u is the input, and $d(t)$ is related to external disturbances which may be caused by the frictional force. $a = 1/(m+M)$, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, $g = 9.8m/s^2$ is the gravity constant. We set $M = 9kg$, $m = 1kg$, $l = 1m$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1\|x\|$ where ρ_0 and ρ_1 are unknown constants. Here, we approximate the system (4.28) by the following two-rule fuzzy model.

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = A_1 x + B_1[u + h(t,x)]$$

Plant Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

$$\dot{x} = A_2 x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.9459 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.7946 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -0.0811 \\ 0 \\ 0.1081 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3097 & 0 & 0 & 0 \end{bmatrix}$,

$$B_2=\begin{bmatrix}0\\-0.0382\\0\\0.1019\end{bmatrix},\ h(t,x)=d(t)+x_2^2\sin x_1,\ \beta_1=\frac{1-1/(1+e^{-14(x_1-\pi/8)})}{1+e^{-14(x_1+\pi/8)}},\ \beta_2=1-\beta_1. \quad (4.29)$$

The inverted pendulum on a cart (4.28) can be cast as (4.2) with data (4.29). Because B_1 is not in the range space of B_2 and the previous adaptive fuzzy control system design methods cannot be applied to the above system (4.29). Via LMI optimization with (4.29), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x)=x_2^2\sin x_1$, $\chi_i=5$, $\alpha_i=1$, $r=2$, $l=1$, $\varepsilon_1=0.004, \varepsilon_2=0.001$, and $t_{sampling}=0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t)=-x_2^2\sin x_1-5\sigma-SA_1x-\frac{1}{1-\omega}\hat{\delta}_1\operatorname{sgn}(\sigma).$$

Control Rule 2: IF x_1 is about $\pm 60^\circ(\pm\pi/3\ rad)$, THEN

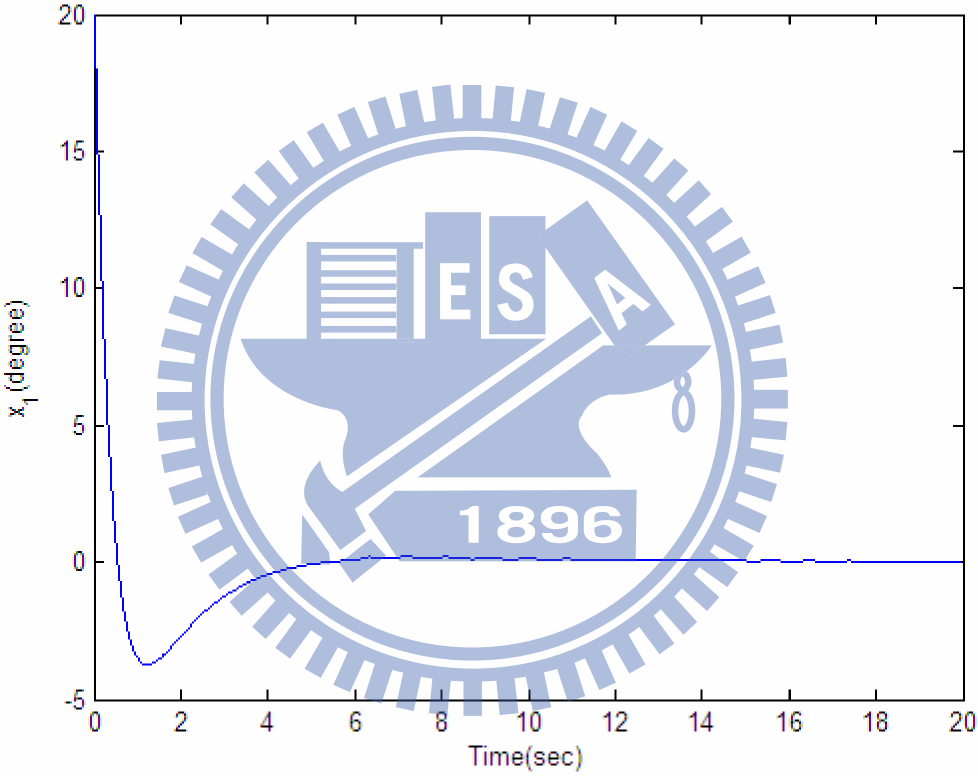
$$u(t)=-x_2^2\sin x_1-5\sigma-SA_2x-\frac{1}{1-\omega}\hat{\delta}_2\operatorname{sgn}(\sigma).$$

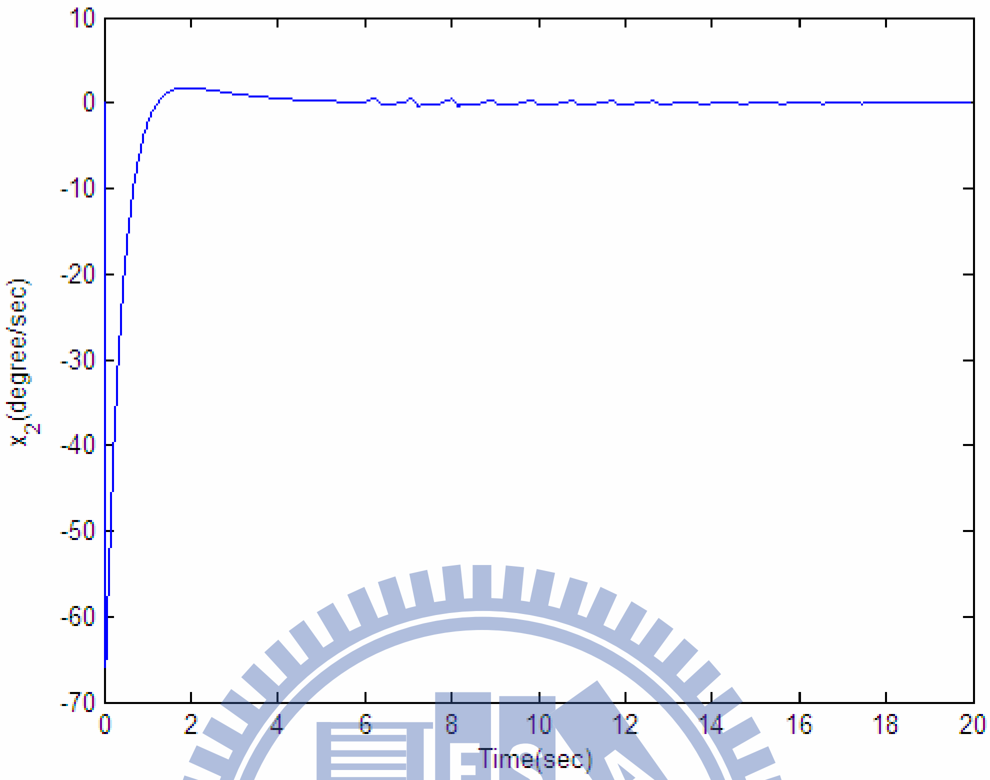
The final controller inferred as the weighted average of each local controller is given by

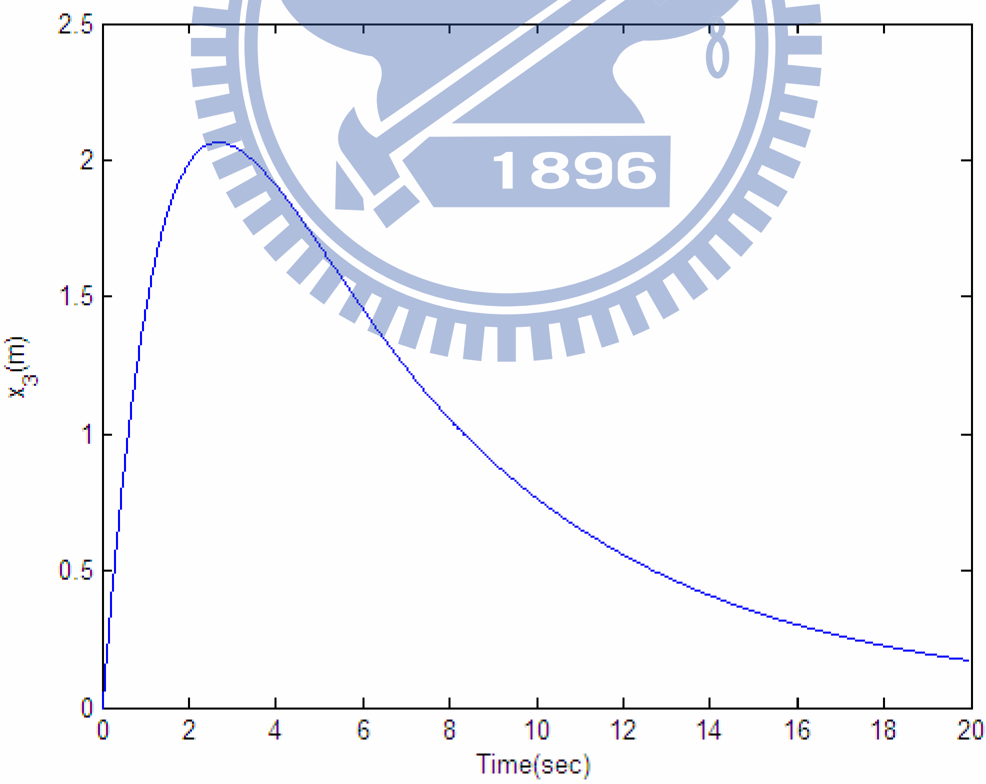
$$u(t)=-x_2^2\sin x_1-\sum_{i=1}^{r}\beta_i(\theta)\left[5\sigma+SA_ix+\frac{1}{1-\omega}\hat{\delta}_i\operatorname{sgn}(\sigma)\right]. \quad (4.30)$$

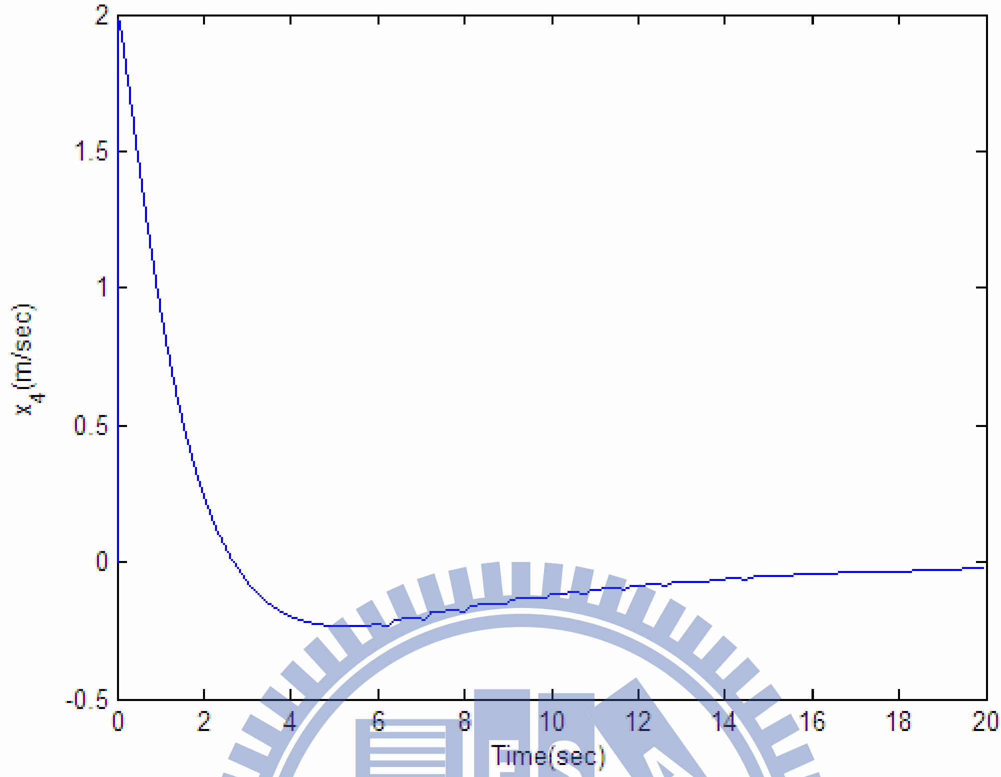
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (4.29) with nonzero $d(t)$. We assume that $d(t)=x_1\sin 2\pi t-0.5\operatorname{sgn}(x_4)$. Figure 4.1 shows the time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.30) when $x_1(0)=20^\circ(\pi/9\ rad)$, $x_2(0)=x_3(0)=x_4(0)=0$. Figure 4.2 shows the time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.30) when $x_1(0)=40^\circ(2\pi/9\ rad)$, $x_2(0)=x_3(0)=$

$x_4(0)=0$. Figure 4.3 shows the time histories of the state, $\hat{\rho}_k$, the sliding variable σ , and the input (4.30) when $x_1(0)=60^\circ(\pi/3\ rad)$, $x_2(0)=$ $x_3(0)=$ $x_4(0)=0$. In Figure 4.1, Figure 4.2, and Figure 4.3, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. It is observed that in our simulations the proposed controller (4.30) stabilizes the following two-rule fuzzy model (4.29).

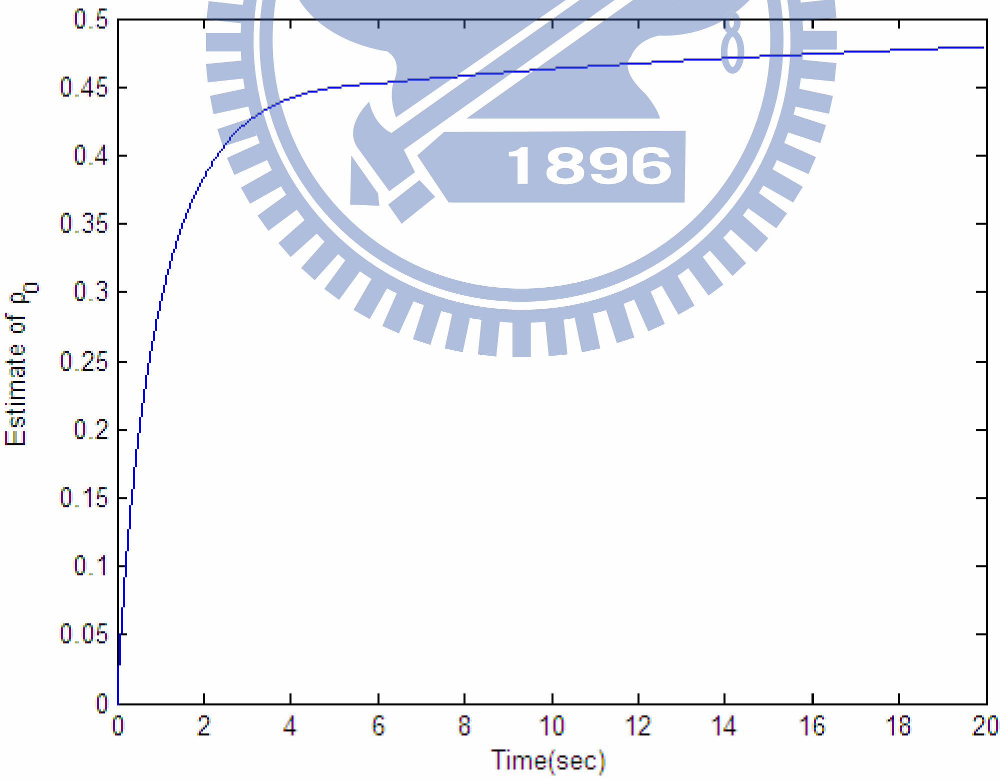



10
0
-10
-20
-30
-40
-50
-60
-70
0
2
4
6
8
10
12
14
16
18
20
Time(sec)

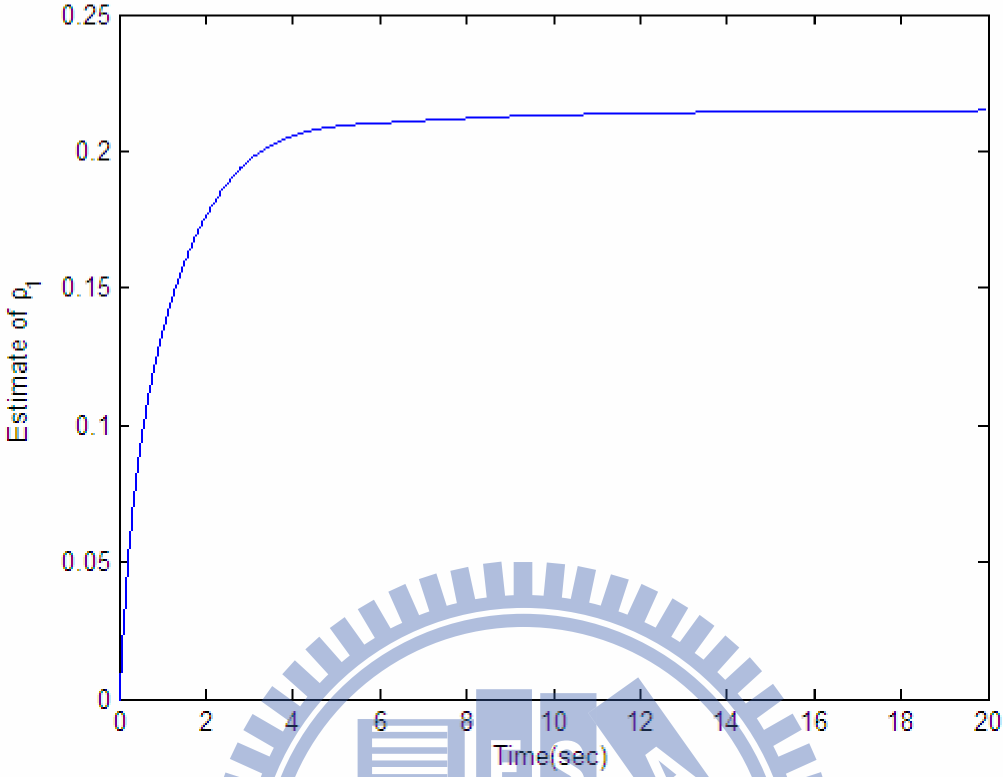

2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
12
14
16
18
20
Time(sec)

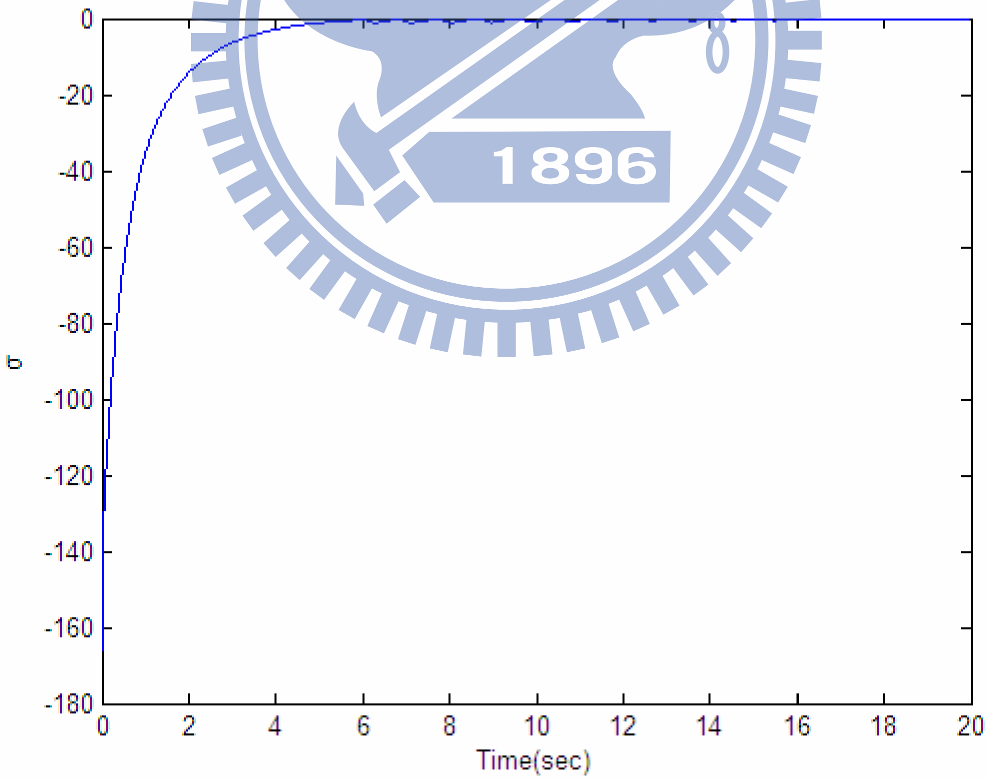


x_4(m/sec)
Time(sec)



Estimate of ρ_0
Time(sec)


0.25
0.2
0.15
0.1
0.05
0
Estimate of ρ_1
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


0
-20
-40
-60
-80
-100
-120
-140
-160
-180
σ
0
2
4
6
8
10
12
14
16
18
20
Time(sec)

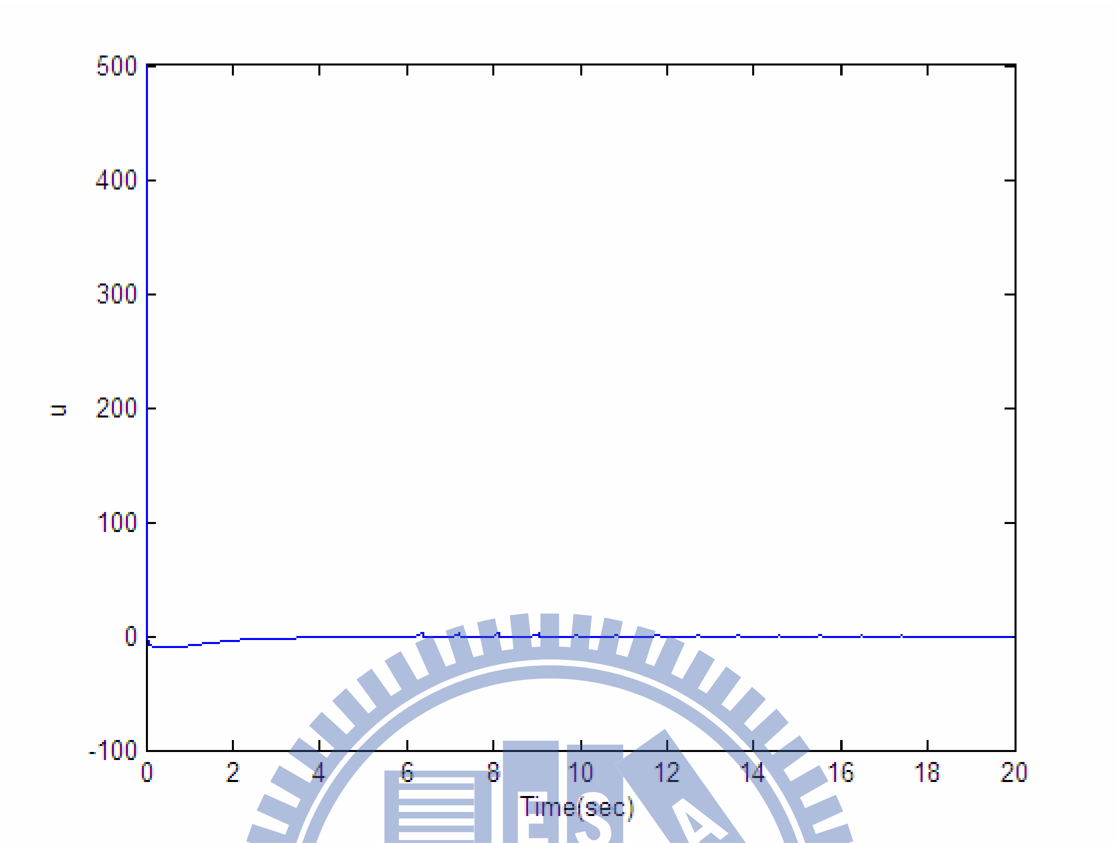
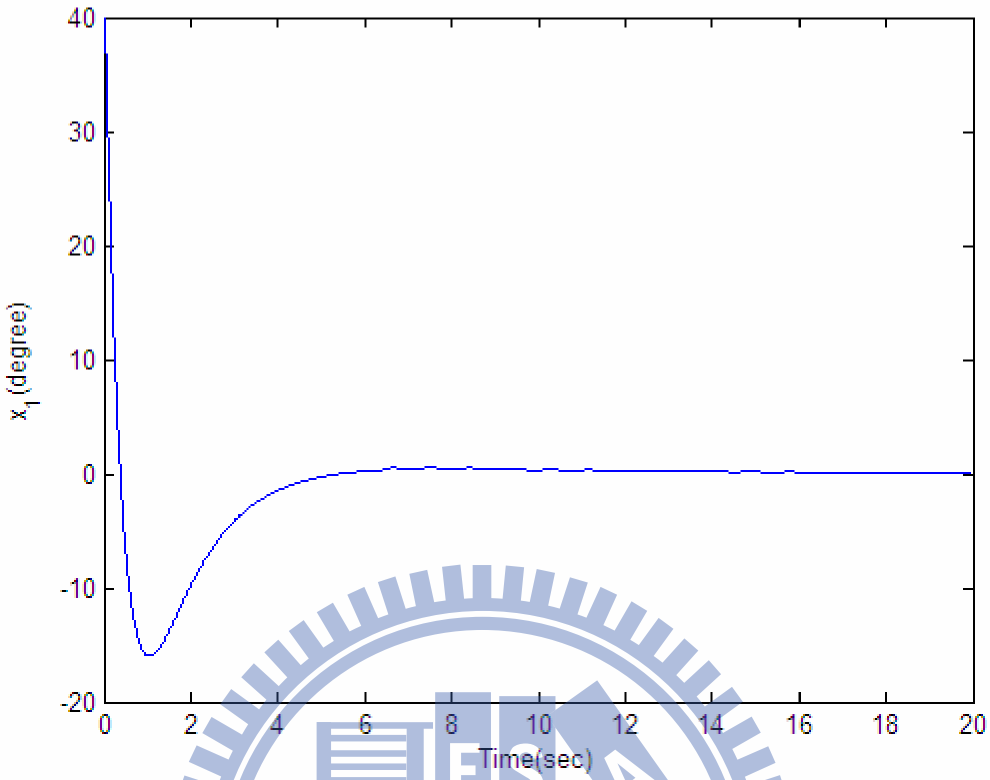
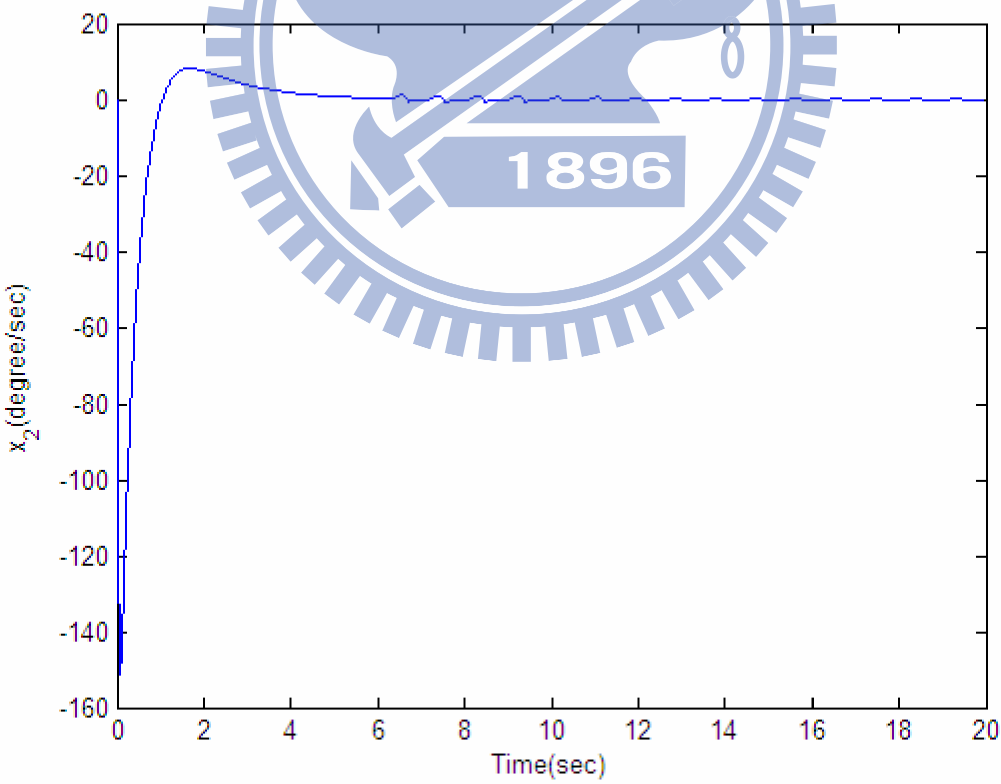


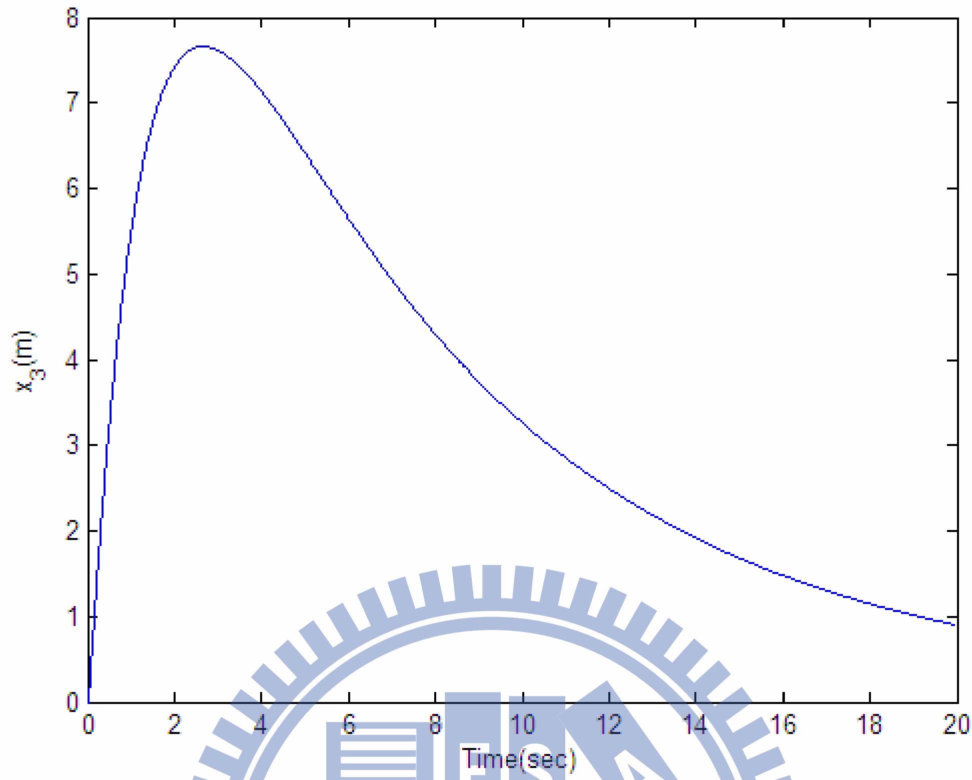
Figure 4.1 Simulation results with $x_1(0) = 20^\circ(\pi/9 \;\; rad),\ x_2(0) = x_3(0) = x_4(0) = 0.$



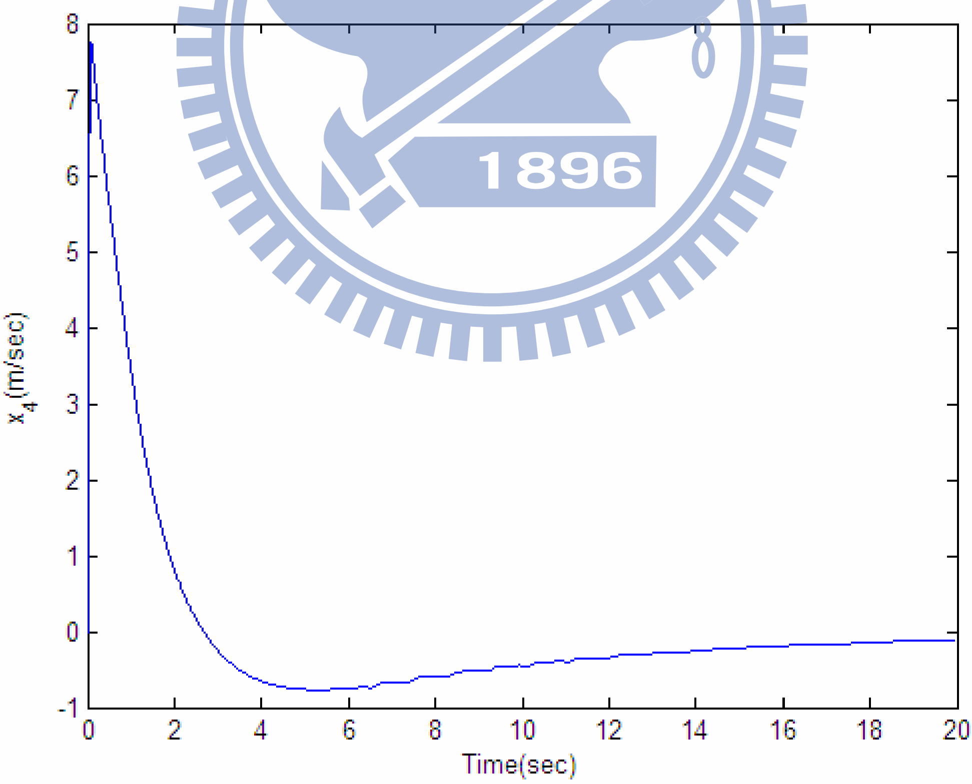
x1(degree)
40
30
20
10
0
-10
-20
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



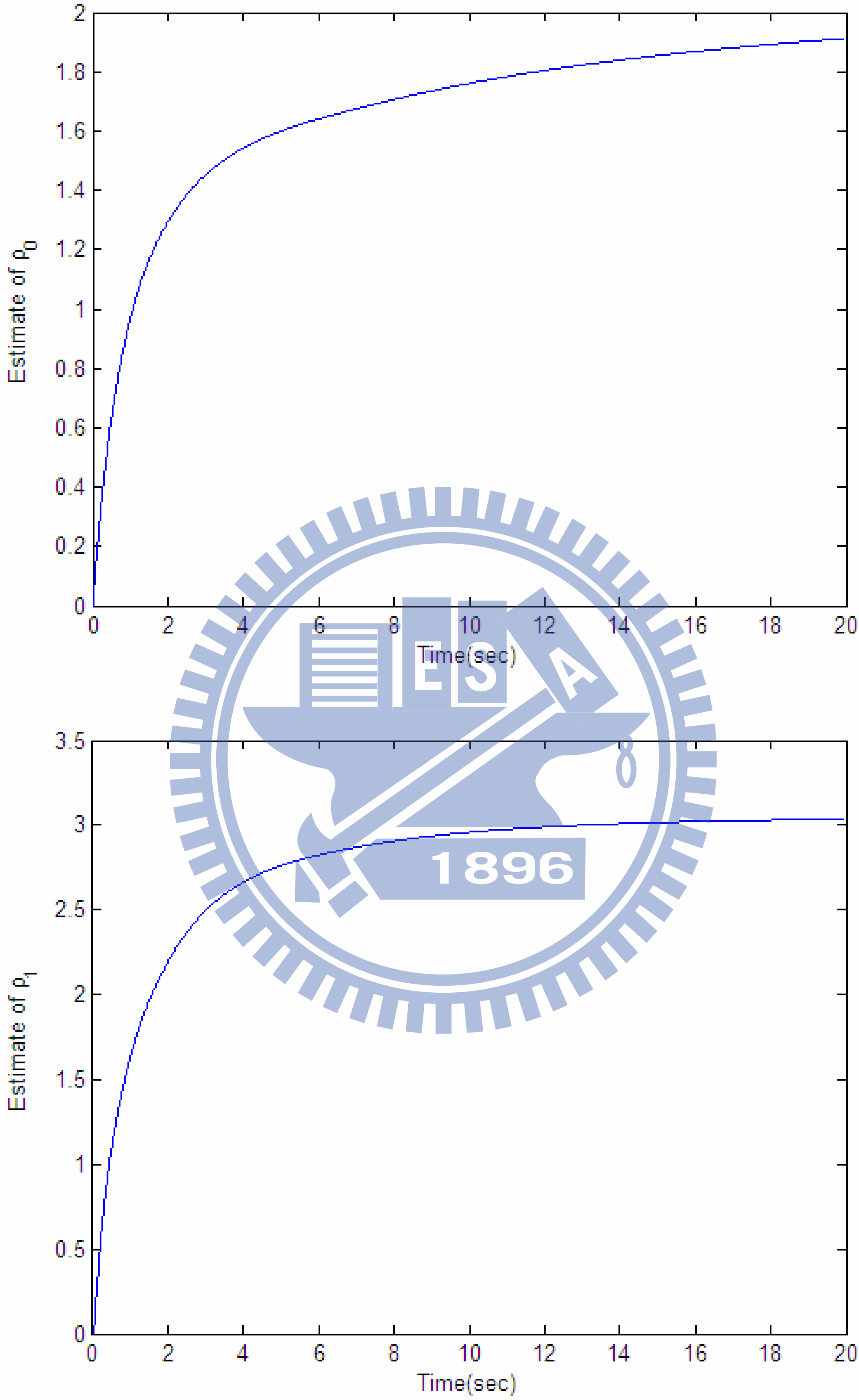
x2(degree/sec)
20
0
-20
-40
-60
-80
-100
-120
-140
-160
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



x3(m)
Time(sec)



x4(m/sec)
Time(sec)



Estimate of ρ_0
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0 2 4 6 8 10 12 14 16 18 20
Time(sec)
Estimate of ρ_1
3.5
3
2.5
2
1.5
1
0.5
0
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

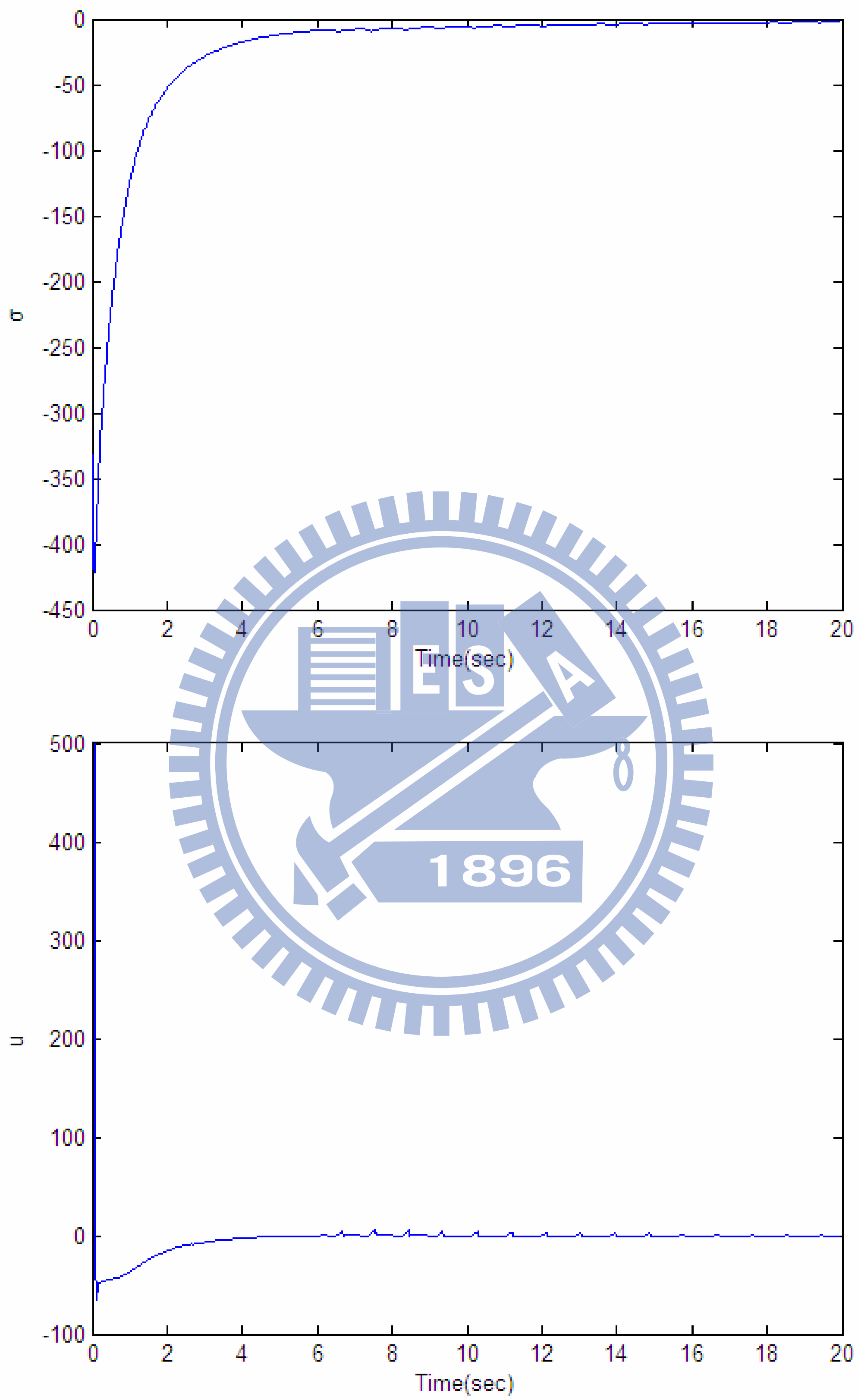
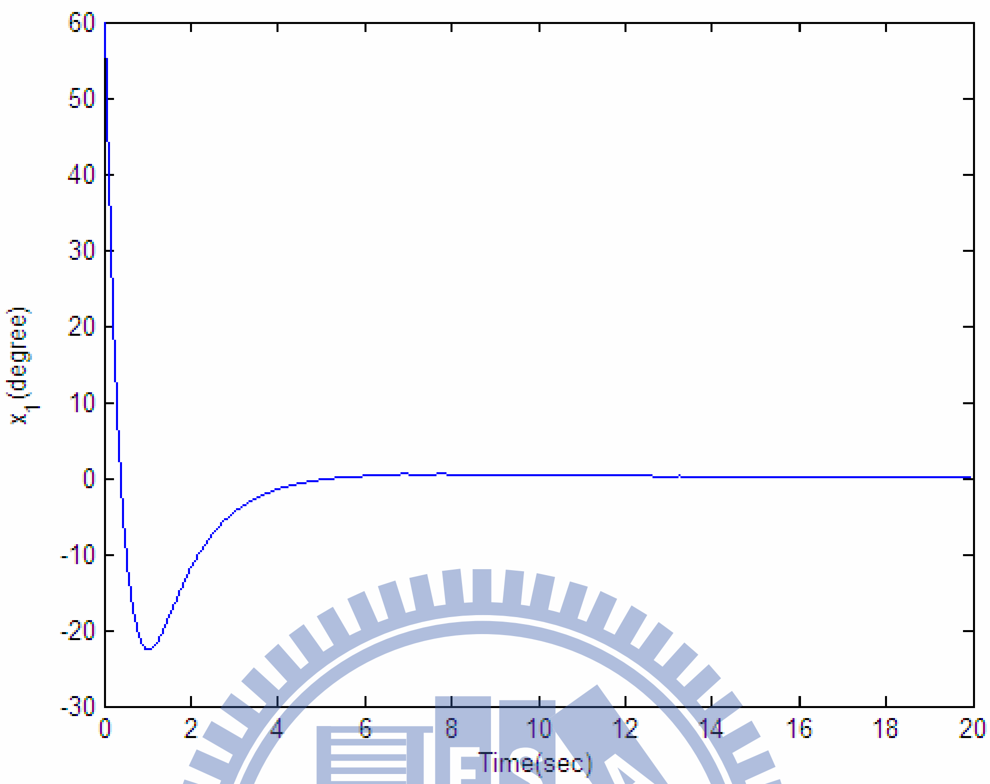
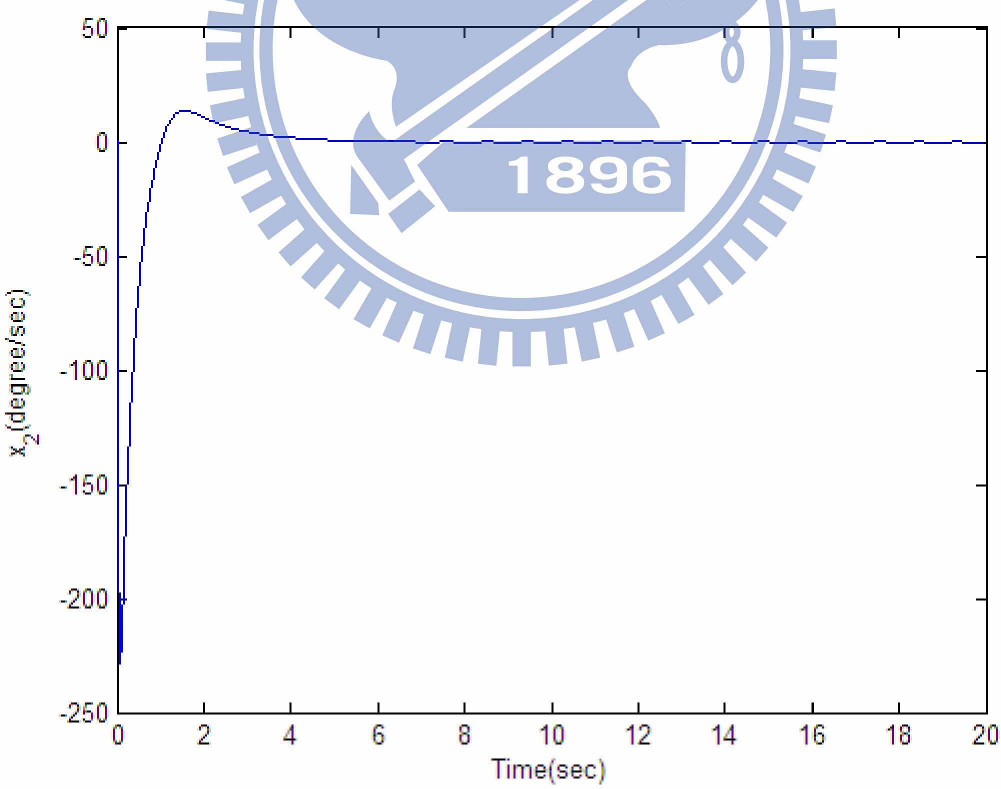
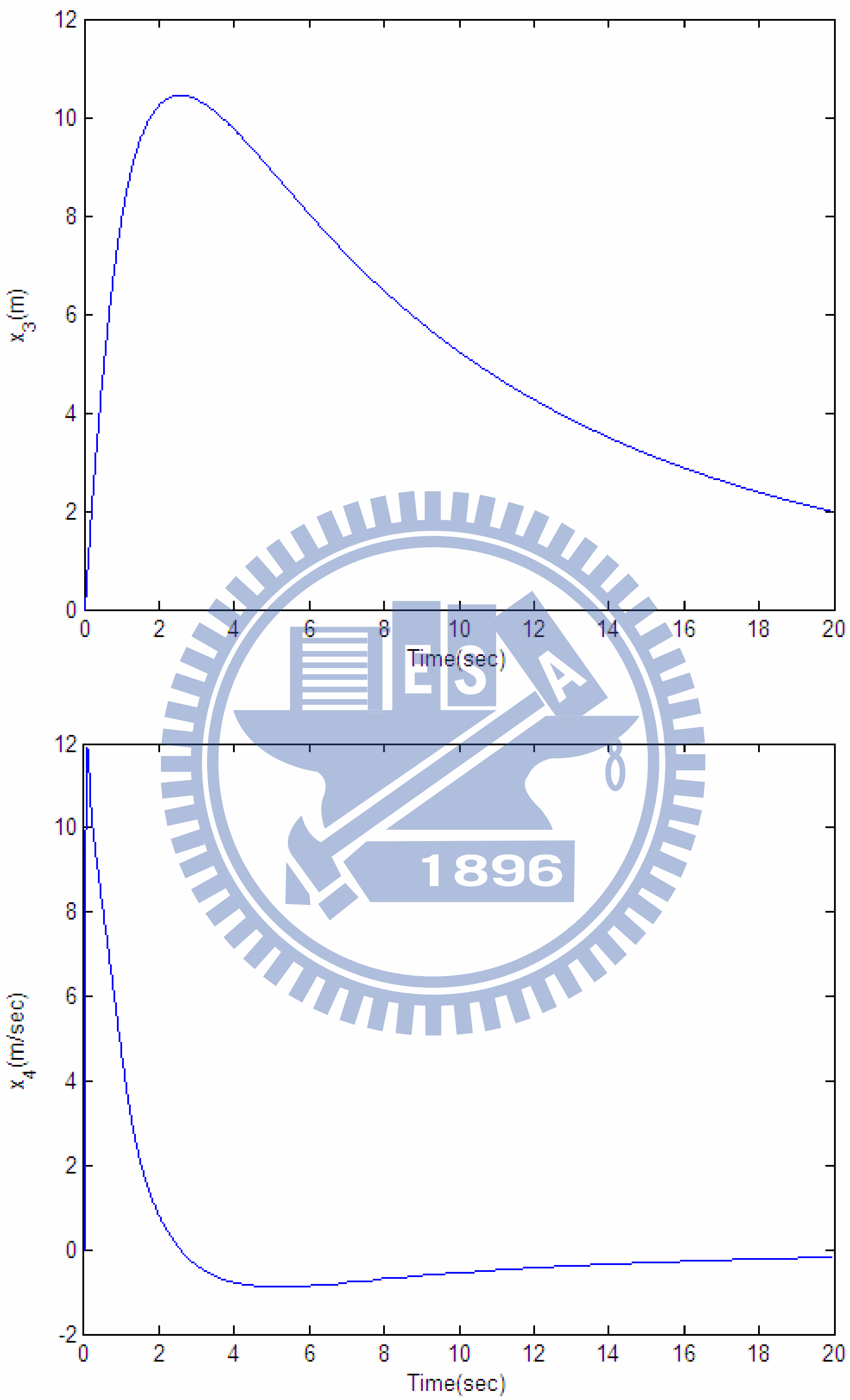
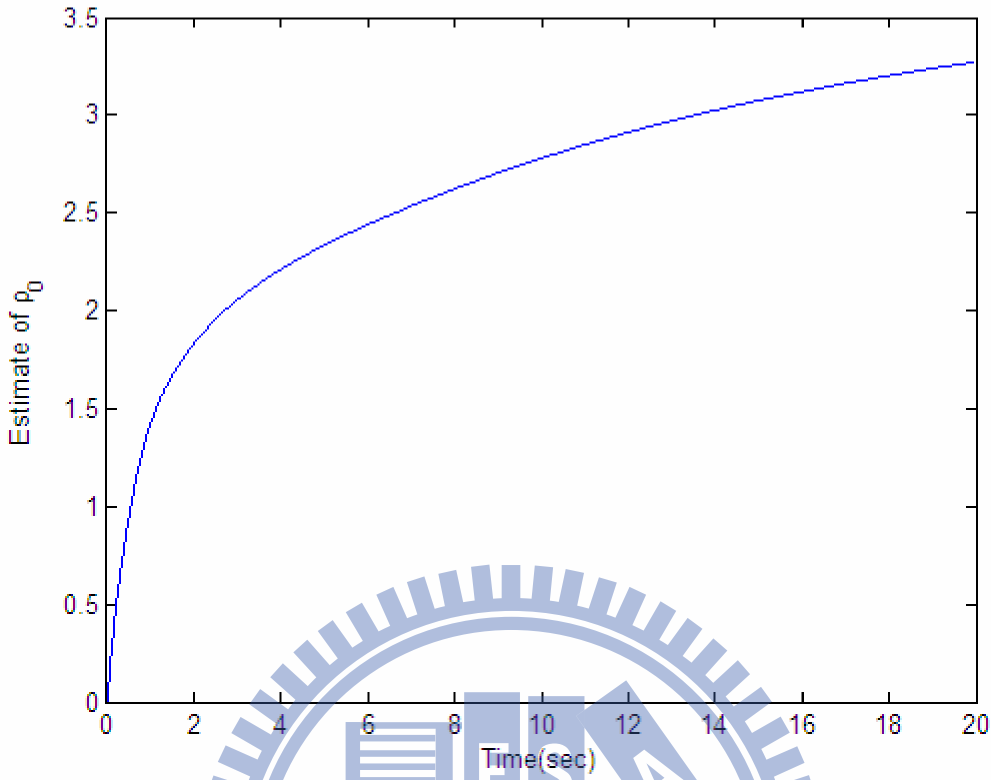


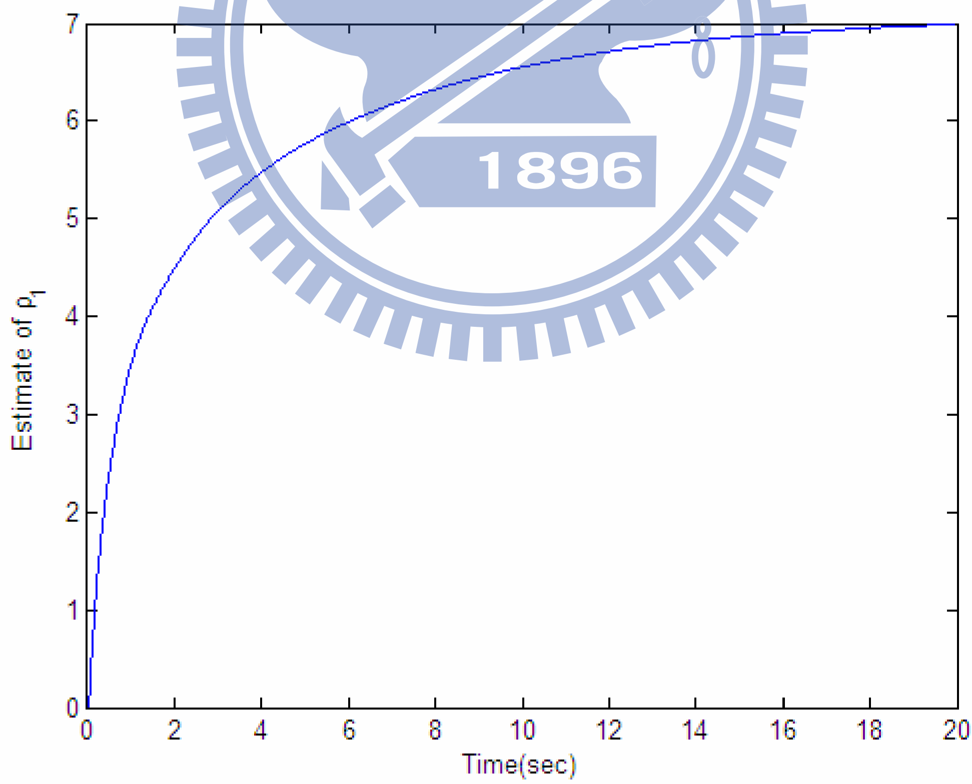
Figure 4.2 Simulation results with $x_1(0) = 40^\circ(2\pi/9\ \ rad),\ x_2(0) = x_3(0) = x_4(0) = 0.$


x1(degree)
Time(sec)


x2(degree/sec)
Time(sec)


x3(m)
Time(sec)
x4(m/sec)
Time(sec)


Estimate of ρ0
Time(sec)


Estimate of ρ1
Time(sec)

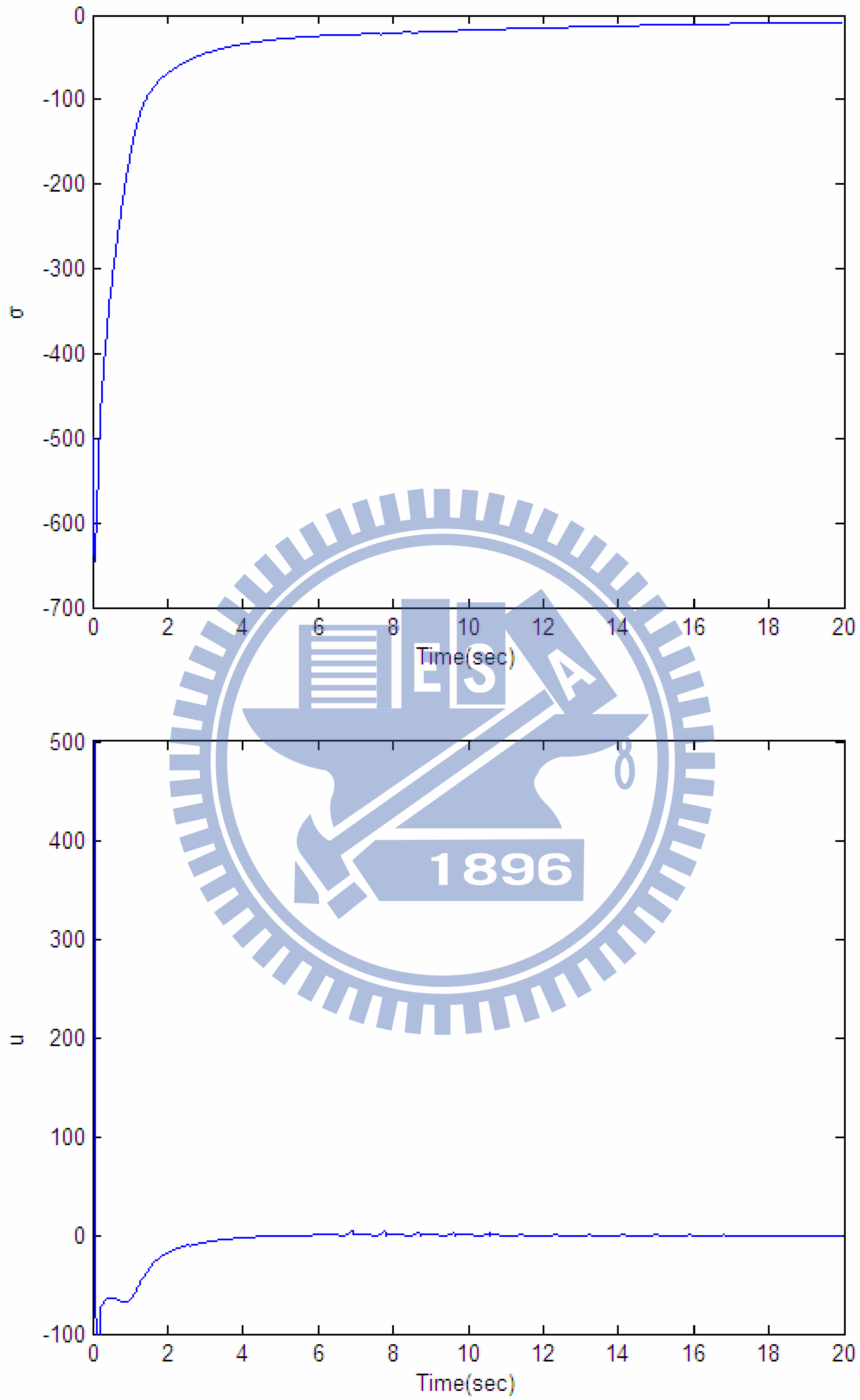


Figure 4.3 Simulation results with $x_1(0) = 60^\circ(\pi/3\ rad),\ x_2(0) = x_3(0) = x_4(0) = 0.$

Example 4.2 Consider the following example of a ball and beam system [52], whose dynamic equations are described as follows:

$$\left(\frac{J_b}{R}+M\right)\ddot{r}+MG\sin\theta-Mr\dot{\theta}^2=0,\ \left(Mr^2+J+J_b\right)\ddot{\theta}+2Mr\dot{r}\dot{\theta}+MGr\cos\theta=\tau \tag{4.31}$$

where r is the ball position, θ is the beam angle, J is the moment of inertia of the beam, M, J_b, and R are the mass, the moment of inertia, and the radius of the ball respectively, G is the acceleration of gravity, and τ is the torque applied to the beam.

Define $B=M/(J_b/R^2+M)$ and change the coordinates in the input space by using the invertible transformation

$$\tau=2Mr\dot{r}\dot{\theta}+MGr\cos\theta+(Mr^2+J+J_b)u \tag{4.32}$$

where u is the new input, then the ball and beam system can be written in the following state-space form:

$$\dot{x}_1=x_2,\ \dot{x}_2=B(x_1x_4^2-G\sin x_3),\ \dot{x}_3=x_4,\ \dot{x}_4=u+d(t) \tag{4.33}$$

where $x=\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T=\begin{bmatrix}r & \dot{r} & \theta & \dot{\theta}\end{bmatrix}^T$. The system parameters used for simulation are $M=0.05kg$, $R=0.01m$, $J=0.02kgm^2$, $J_b=2\times10^{-6}kgm^2$, $G=9.81m/s^2$ and $B=0.7143$. We assume that $d(t)$ is bounded as $|d(t)|\le\rho_0+\rho_1\|x\|$ where ρ_0 and ρ_1 are unknown constants. Then, we approximate the system by the following two-rule fuzzy model:

Plant rule 1: IF x_1 is greater than 0, THEN

$$\dot{x}=A_1x+B_1[u+h(t,x)].$$

Plant rule 2: IF x_1 is smaller than 0, THEN

$$\dot{x} = A_2 x + B_2[u + h(t, x)] .$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & -2B\mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 2B\mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$

$$\mu = 0.01,\ h(t,x) = d(t),\ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}},\quad \beta_2 = 1 - \beta_1. \tag{4.34}$$

By setting $\chi_i = 0.2, \alpha_i = 175, r = 2, l = 1, \varepsilon_k = 0.5$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is greater than 0, THEN

$$u(t) = -0.2\sigma - SA_1 x - \frac{1}{1-\omega}\hat{\delta}_1 \operatorname{sgn}(\sigma).$$

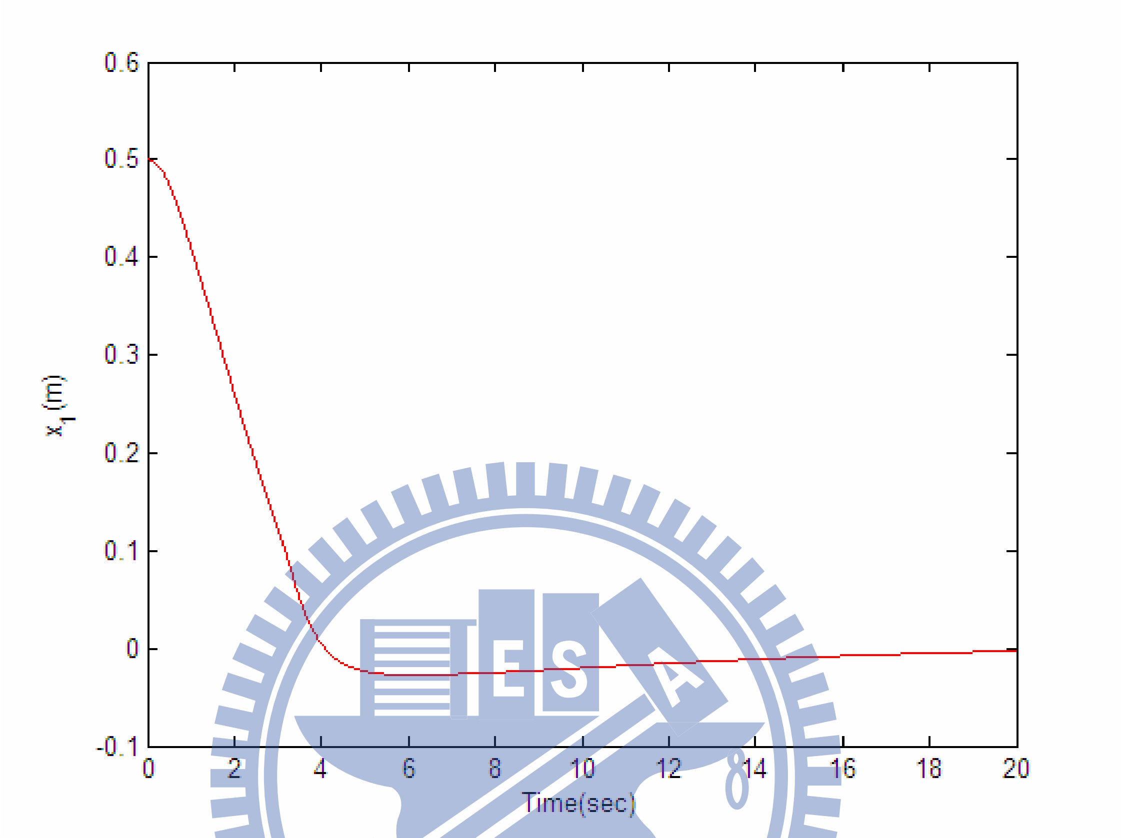
Control Rule 2: IF x_1 is smaller than 0, THEN

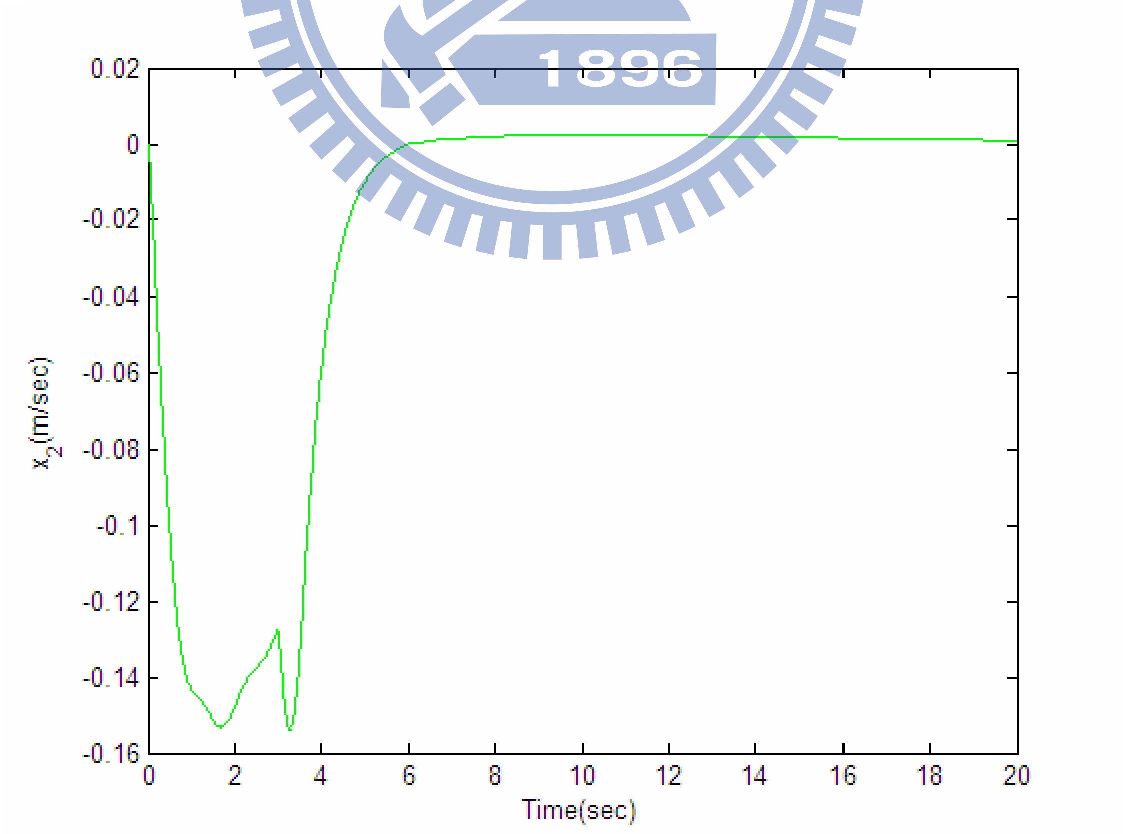
$$u(t) = -0.2\sigma - SA_2 x - \frac{1}{1-\omega}\hat{\delta}_2 \operatorname{sgn}(\sigma).$$

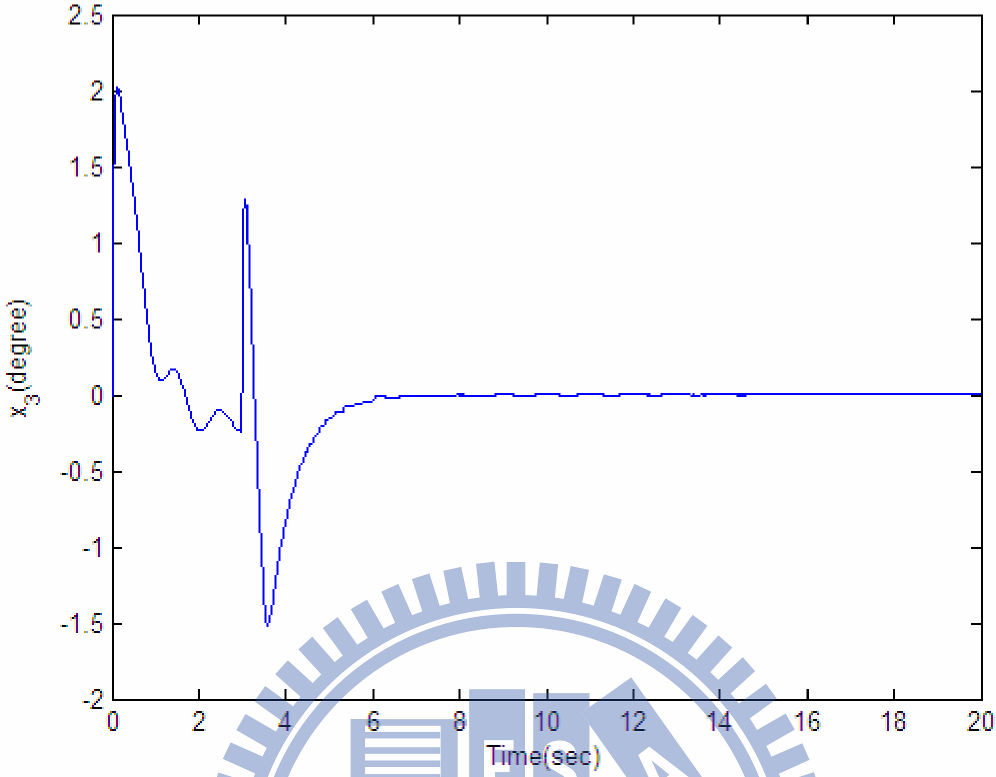
The final controller inferred as the weighted average of each local controller is given by

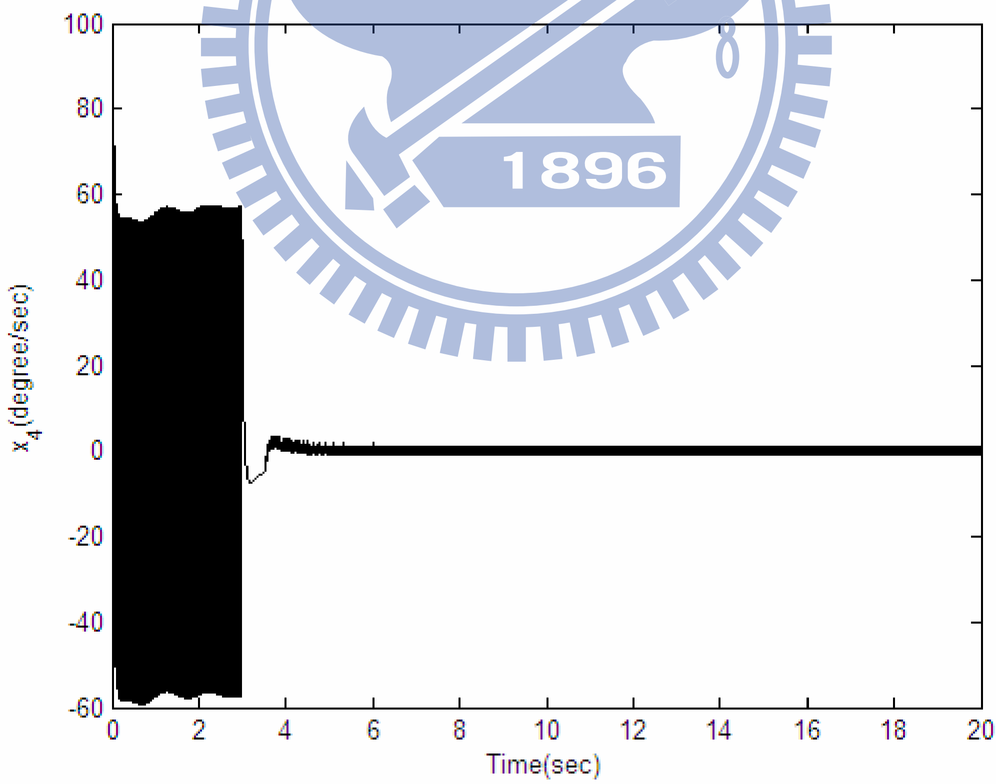
$$u(t) = -\sum_{i=1}^{r} \beta_i(\theta)\left[0.2\sigma + SA_i x + \frac{1}{1-\omega}\hat{\delta}_i \operatorname{sgn}(\sigma)\right]. \tag{4.35}$$

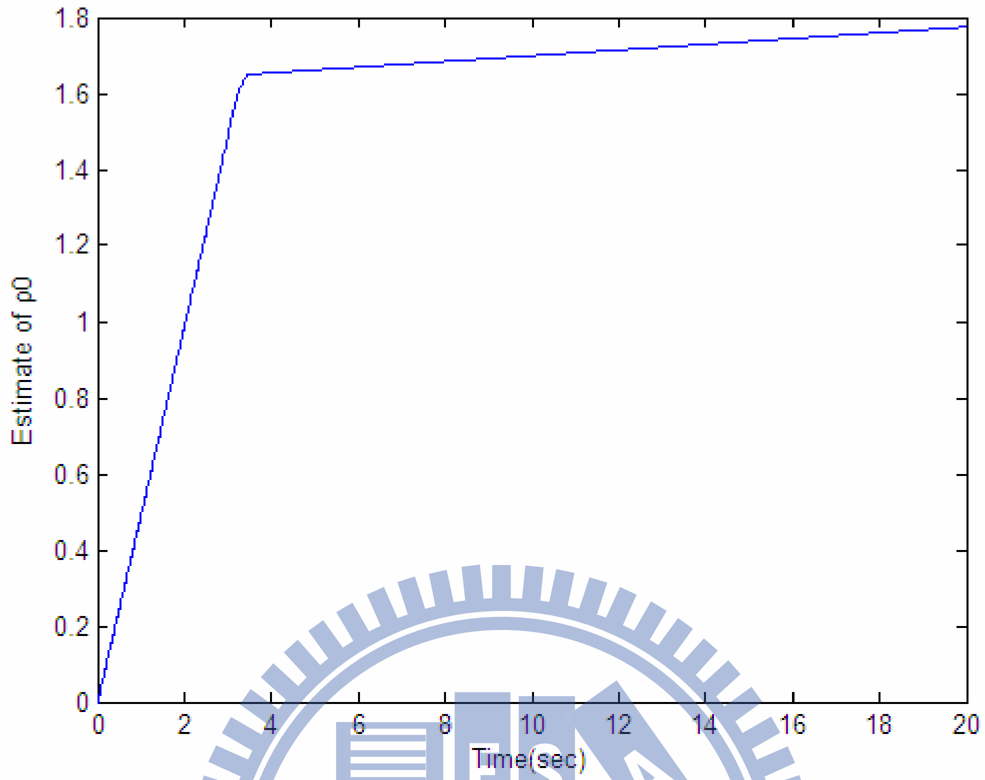
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (4.34) with nonzero *d*(*t*). We assume that $d(t) = x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)$. Figure 4.4 shows the time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.35) when $x_1(0) = 0.5,\ x_2(0) = x_3(0) = x_4(0) = 0$. Figure 4.5 shows the time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.35) when $x_1(0) = 1,\ x_2(0) = x_3(0) = x_4(0) = 0$. In Figure 4.4 and Figure 4.5, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 4.4 and Figure 4.5, the proposed controller (4.35) also stabilizes the following two-rule fuzzy model (4.34).

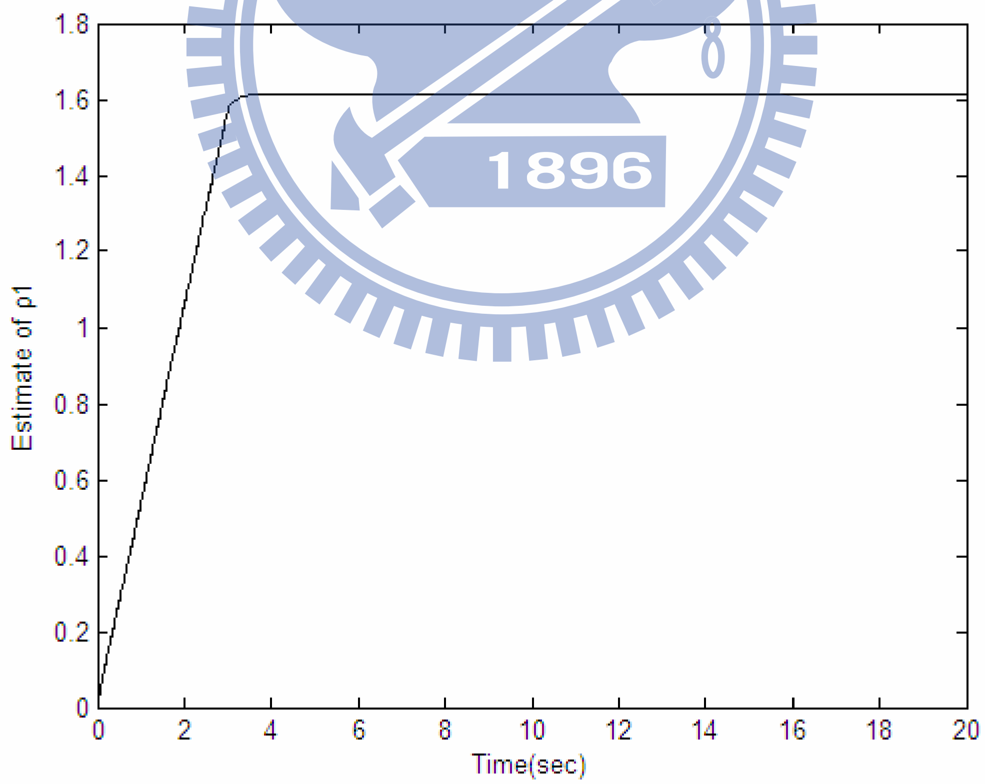


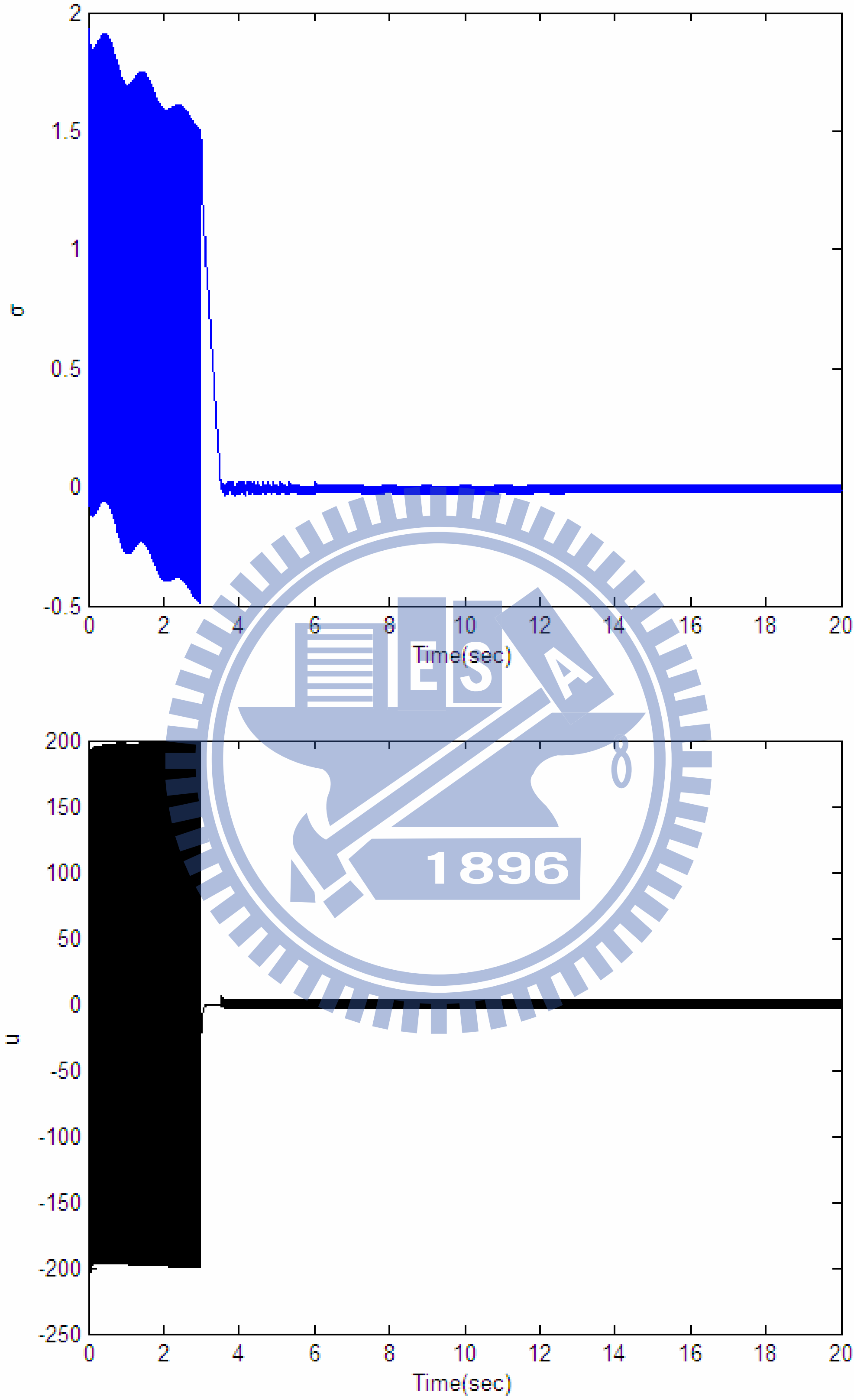



2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x_3(degree)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


100
80
60
40
20
0
-20
-40
-60
x_4(degree/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


Estimate of p0
Time(sec)


Estimate of p1
Time(sec)


σ
Time(sec)
u
Time(sec)

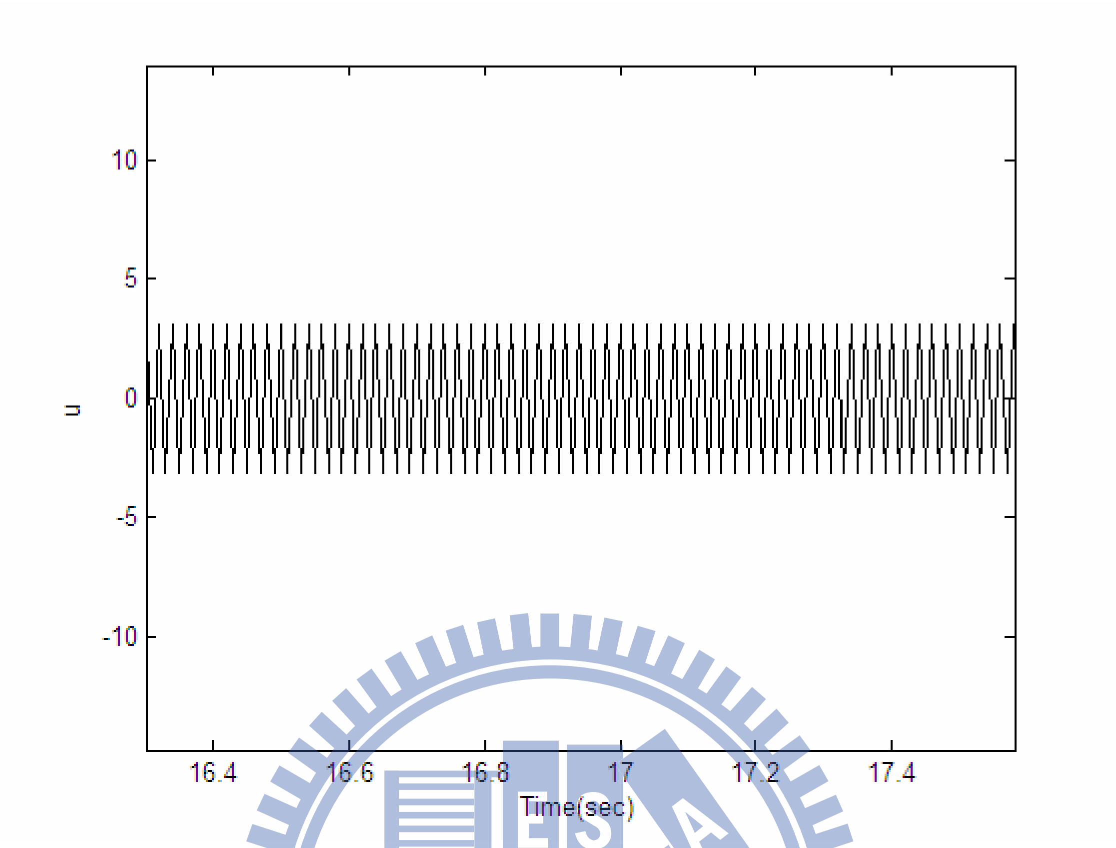
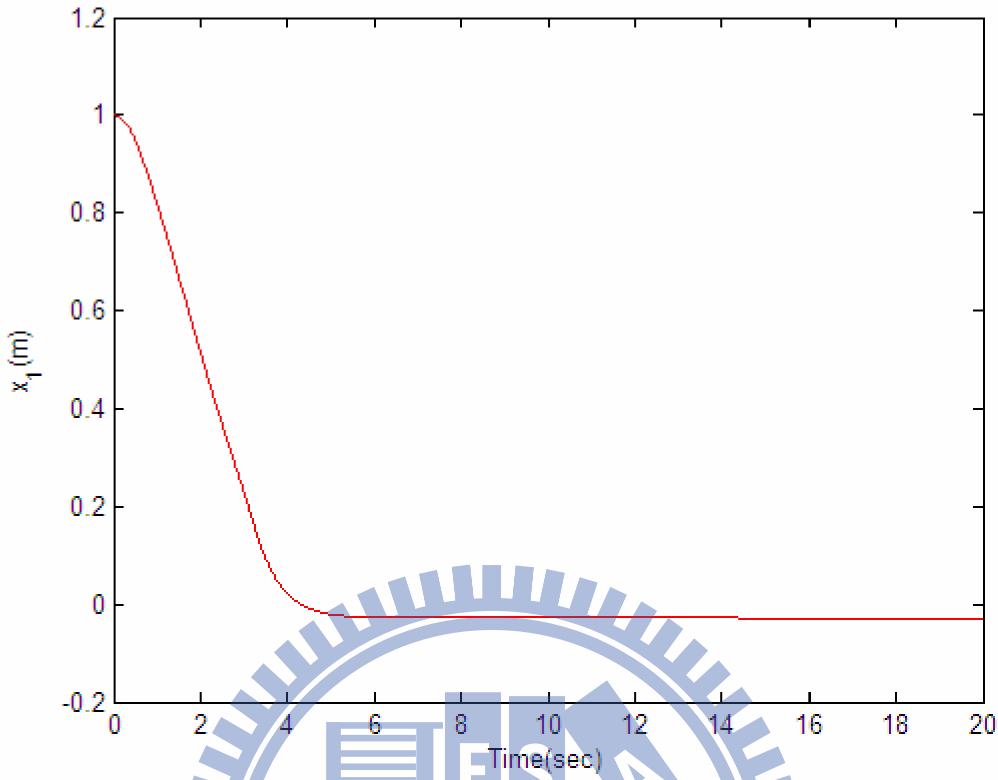
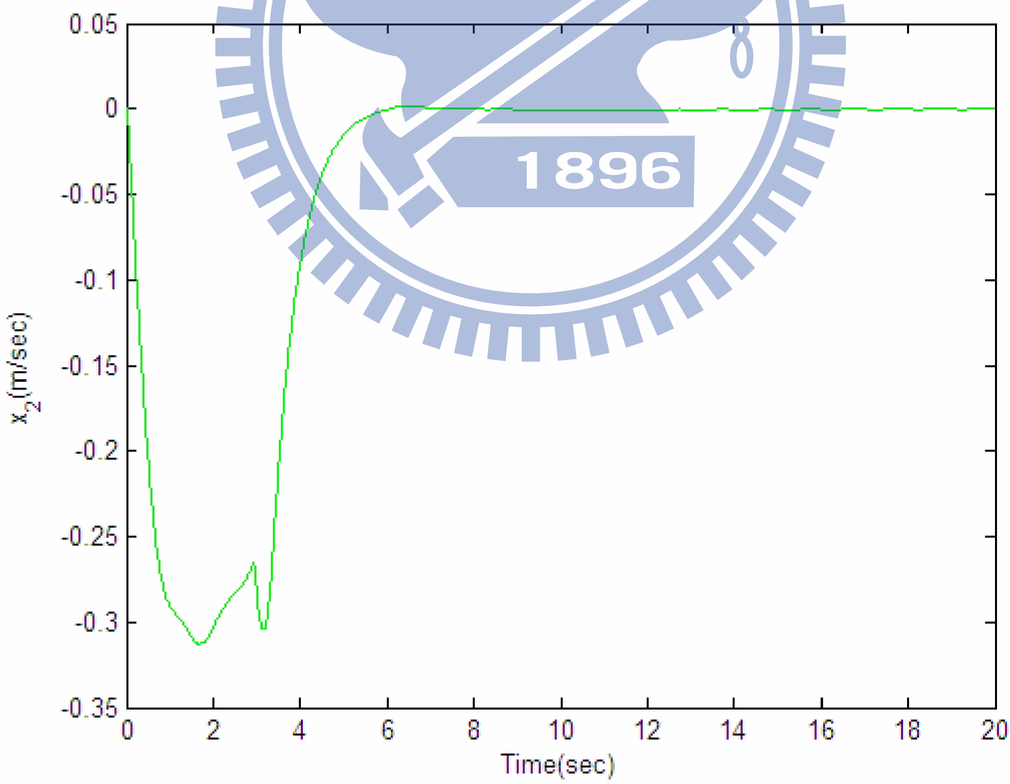
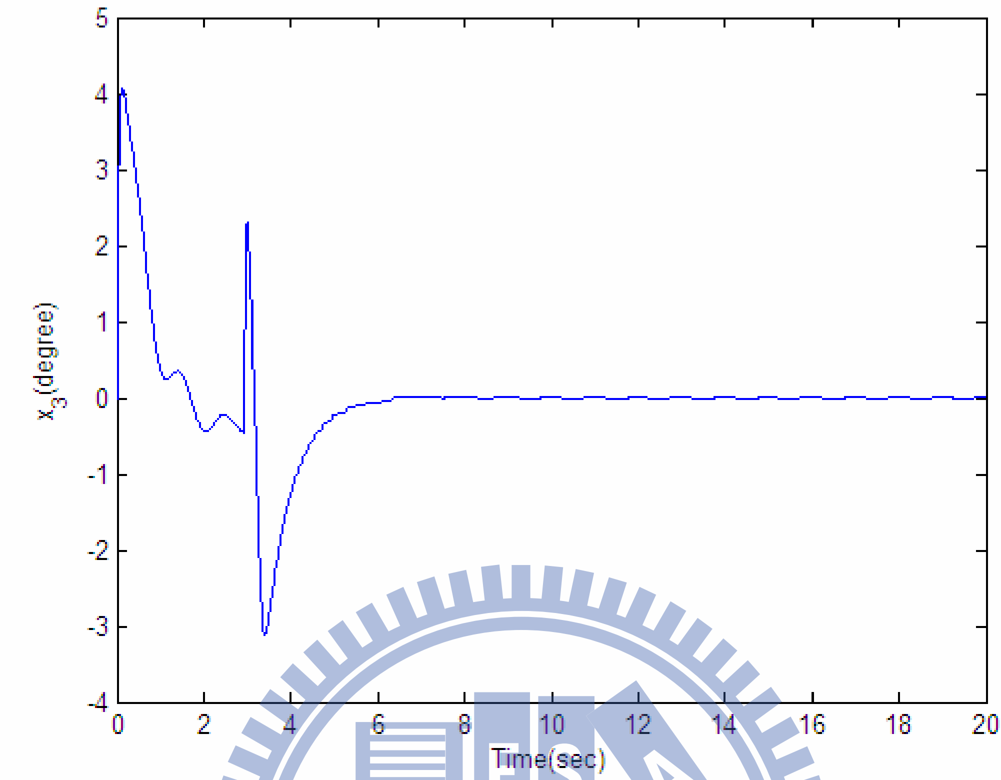


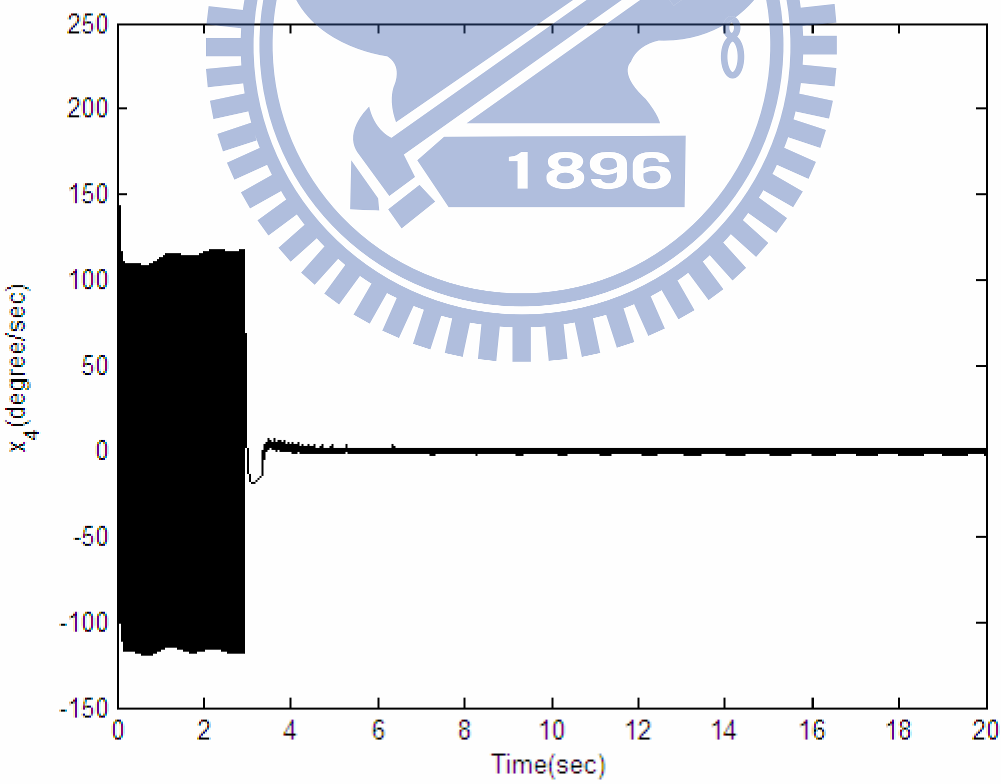
Figure 4.4 Simulation results with $x_1(0)=0.5, x_2(0)=x_3(0)=x_4(0)=0,$ including amplifying the input u scale


x1(m)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

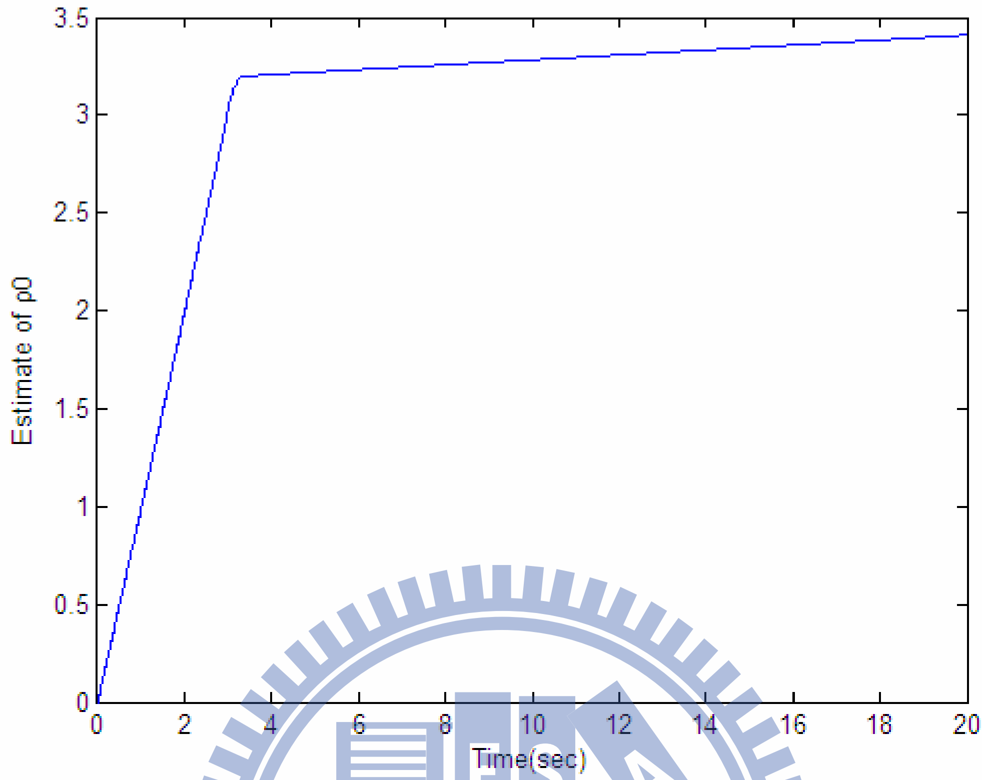

x2(m/sec)
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
0 2 4 6 8 10 12 14 16 18 20
Time(sec)



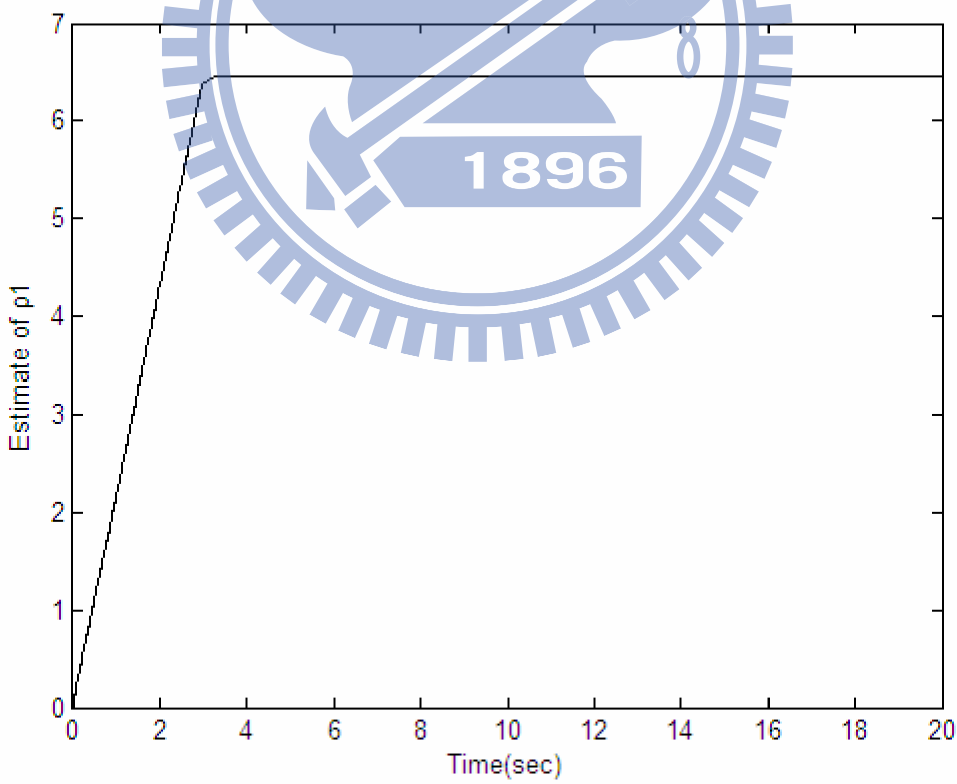
5
4
3
2
1
0
-1
-2
-3
-4
x_3(degree)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



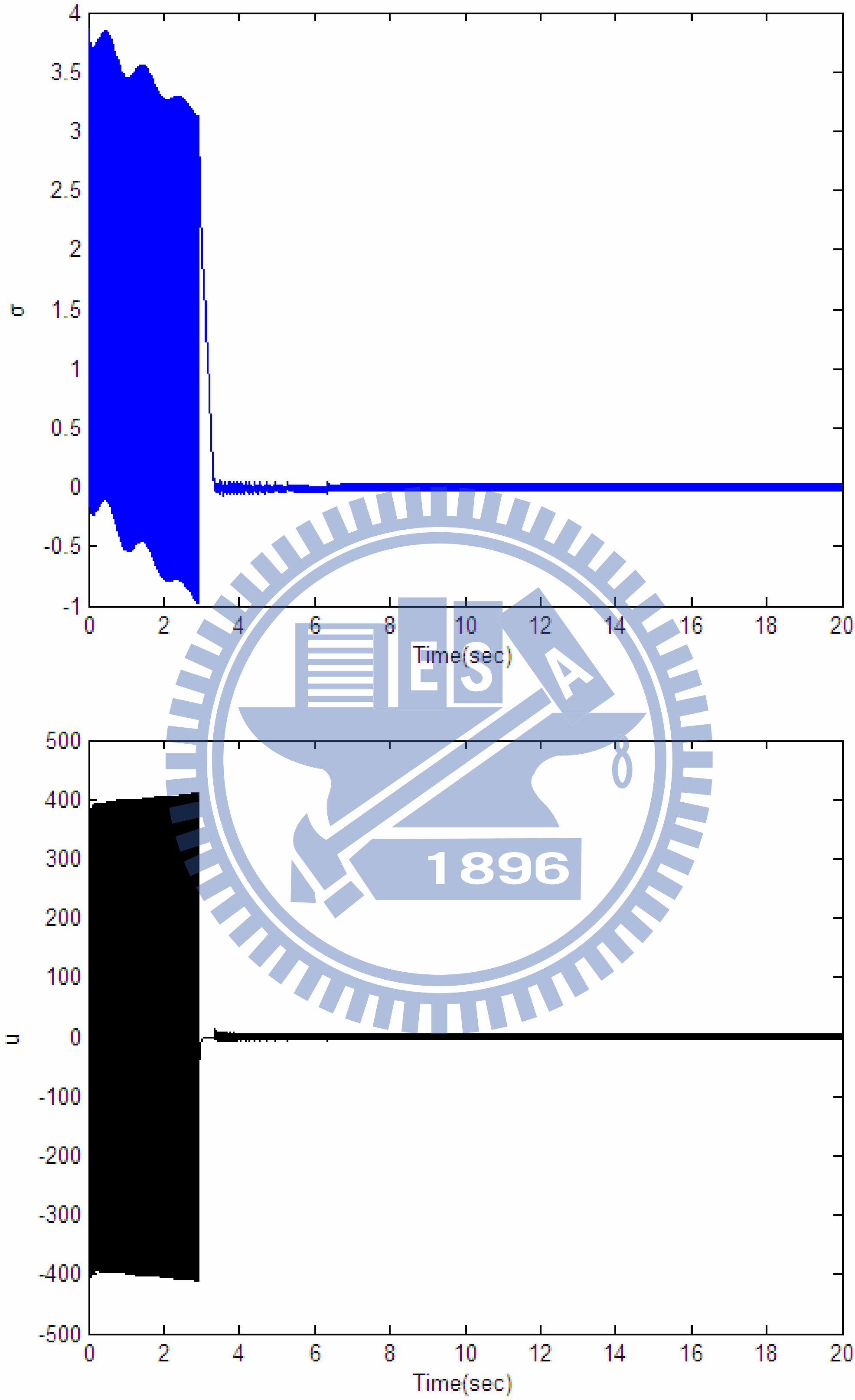
250
200
150
100
50
0
-50
-100
-150
x_4(degree/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



Estimate of ρ0
Time(sec)



Estimate of ρ1
Time(sec)


σ
Time(sec)
u
Time(sec)

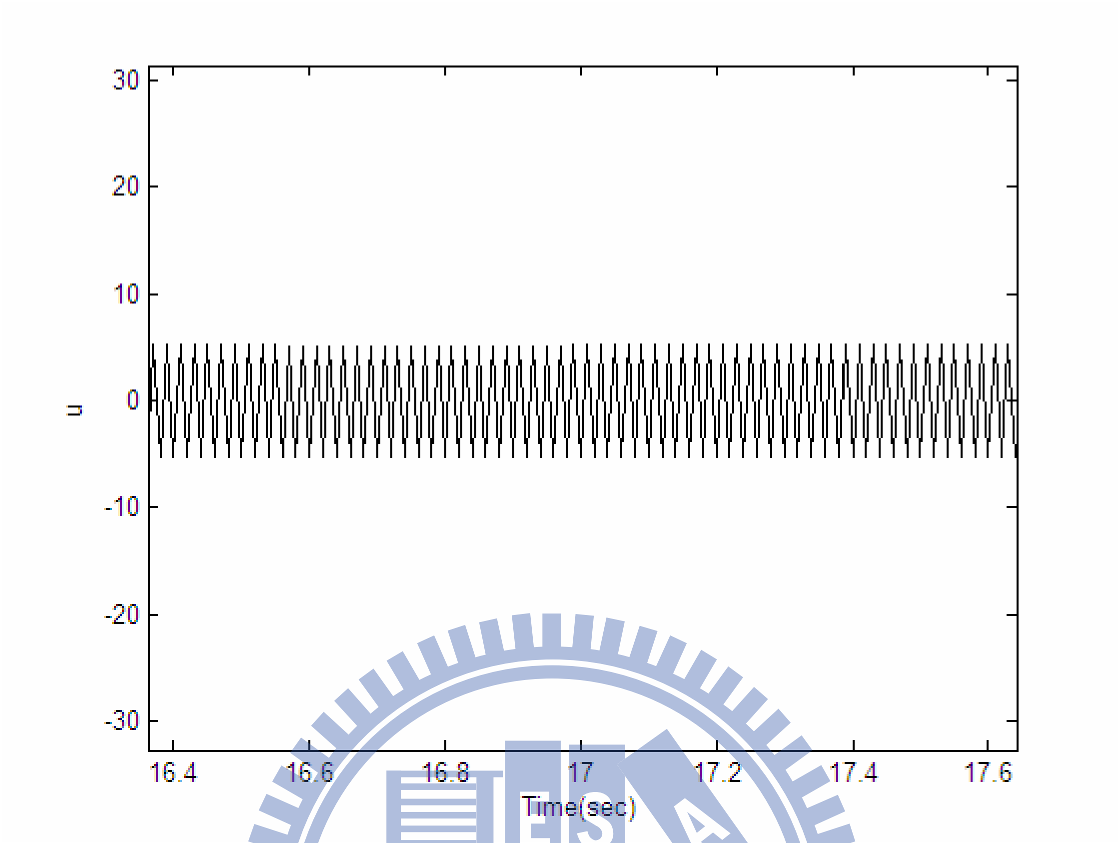


Figure 4.5 Simulation results with $x_1(0)=1, x_2(0)=x_3(0)=x_4(0)=0$, including amplifying the input u scale

4.3 Robust Adaptive Control for Mismatched T-S Fuzzy Systems

In this section, two kinds of system formulation for mismatched uncertain T-S fuzzy models are described in Section 4.3.1 and in Section 4.3.4, respectively. Two kinds of robust adaptive control methods via LMI are proposed in Section 4.3.2 and in Section 4.3.5, respectively. Some examples are used to illustrate the effectiveness of the proposed methods and to compare with the existing methods in Section 4.3.3 and Section 4.3.6, respectively.

4.3.1 System Formulation I

Consider the following uncertain T-S fuzzy model [49], including parameter uncertainties and unknown norm-bounded external disturbances:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)([A_i + \Delta A_i(t)]x(t) + B_i[u(t) + h(t,x)]) \tag{4.36}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, A_i, B_i are constant matrices of appropriate dimensions, $\Delta A_i(t)$ represents the parameter uncertainties in A_i, $h(t,x) \in R^m$ denotes external disturbances. $\theta = [\theta_1, \cdots, \theta_s], \theta_j (j = 1, \cdots, s)$ are the premise variables, s is the number of the premise variables, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta), \omega_i : R^s \to [0,1], i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i, r is the number of the IF-THEN rules, β_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0. \sum_{i=1}^{r} \beta_i(\theta) = 1$.

We will assume that the followings are satisfied:

A1: The $n \times m$ matrix B defined by $B = \frac{1}{r}\sum_{i=1}^{r} B_i$ satisfies the rank constraint $\text{rank}(B) = m$, i.e., the matrix B has full column rank m.

A2: The function $h(t,x)$ is unknown but bounded as $\left\| h(t,x) - \hat{h}(t,x) \right\| \leq \sum_{k=0}^{l} \rho_k \|x\|^k$ where $\rho_0, \cdots, \rho_l$ are unknown constants, $\hat{h}(t,x)$ is an estimate of $h(t,x)$, and l is a known positive integer.

A3: $\Delta A_i(t)$ is of the form $T_i \Pi_i(t)$ where $\Pi_i(t)$ is a known time-varying matrix but bounded as $\|\Pi_i(t)\| \leq 1$.

The system (4.36) does not have to satisfy the restrictive assumption that all the input matrices of the local system models are in the same range space. It should be noted that the assumption A1 implies that $rank(B_i) \le m$ and each nominal local system model may not share the same input channel. The assumption A2 with $l = 1$ and $\hat{h}(t,x) = 0$ has been used in the literature [50]. We can set $\hat{h}(t,x)$ as the nominal value of $h(t,x)$. Using the above assumptions, the uncertain T-S fuzzy model (4.36) can be written as follows.

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)(A_i + T_i\Pi_i(t))x(t) + [B + HF(\beta)G][u + h(t,x)] \tag{4.37}$$

where $\beta = [\beta_1(\theta), \cdots, \beta_r(\theta)]$, and the matrices $H, G, F(\beta)$ are defined by

$$H = \frac{1}{2}\left[(B - B_1), \cdots, (B - B_r)\right],\ G = \left[I, \cdots, I\right]^T,$$

$$F(\beta) = diag\left[(1 - 2\beta_1(\theta))I, \cdots, (1 - 2\beta_r(\theta))I\right]. \tag{4.38}$$

It should be noted that the system (4.36) does not have to satisfy $B_1 = B_2 = \cdots = B_r$, which is used in almost all published results on VSS design methods including the VSS-based fuzzy control design methods of [33,34]. Hence the methods [30,31] cannot be applied to the above model (4.36). Since $\beta_i(\theta) \ge 0$ and $\sum_{i=1}^{r} \beta(\theta) = 1$, we can see that the following inequality always holds:

$$F^T(\beta)F(\beta) = F(\beta)F^T(\beta) \le I. \tag{4.39}$$

Many examples in the literature and various mechanical systems such as motors and robots do not satisfy the restrictive assumptions that each nominal local system model shares the same input channel and they fall into the special cases of the above model [49].

4.3.2 LMI-based Adaptive Control Design I

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions: The first involves the design of $m(n-1)-$dimensional switching surfaces for the sliding mode such that the reduced order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second is concerned with the selection of a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first characterize linear sliding surfaces using LMIs.

Let us define the linear sliding surface as $\sigma = Sx = 0$ where S is a $m \times n$ matrix. Referring to the previous results [33], [51], we can see that for the system (4.37) it is reasonable to find a sliding surface such that

P1 $[SB + SHF(\beta)G]$ is nonsingular for any β satisfying $\beta_i(\theta) \geq 0, i = 1, \cdots, r$, and $\sum_{i=1}^{r} \beta_i(\theta) = 1$.

P2 The reduced $(n-m)$ order sliding mode dynamics restricted to the sliding surface $Sx = 0$ is asymptotically stable for all admissible uncertainties.

It should be noted that P1 is necessary for the existence of the unique equivalent control [33] and the assumption A1 is necessary for the nonsingularity of SB.

Define a transformation matrix and the associated vector v as $M = [\Lambda(\Lambda^T Y \Lambda)^{-1}, Y^{-1}B(B^T Y^{-1} B)^{-1}]^T = [V^T, S^T]^T$, $v = [v_1^T, v_2^T]^T = Mx$ where $v_1 \in R^{n-m}$, $v_2 \in R^m$. By the above transformation, we can see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. Then, from system (4.37), we can obtain

$$\begin{bmatrix} \dot{v}_1 \\ \dot{\sigma} \end{bmatrix} = \sum_{i=1}^{r} \beta_i(\theta) \begin{bmatrix} V(A_i + T_i \Pi_i(t)) Y \Lambda & V(A_i + T_i \Pi_i(t)) B \\ S(A_i + T_i \Pi_i(t)) Y \Lambda & S(A_i + T_i \Pi_i(t)) B \end{bmatrix} \begin{bmatrix} v_1 \\ \sigma \end{bmatrix}$$

$$+ \begin{bmatrix} VHF(\beta)G \\ I + SHF(\beta)G \end{bmatrix} [u + h(t,x)]. \qquad (4.40)$$

Then from the equivalent control method [33], we can see that the equivalent control is given by $u_{eq}(t) = -\sum_{i=1}^{r} \beta_i(\theta)[I + SHF(\beta)G]^{-1} S(A_i + T_i \Pi_i(t))x - h(t,x)$. By setting

$\dot{\sigma} = \sigma = 0$ and substituting $u(t)$ with $u_{eq}(t)$, we can show that the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma = Sx = 0$ is given by

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T D(\beta)(A_i + T_i \Pi_i(t)) Y \Lambda v_1 \qquad (4.41)$$

where $D(\beta) = I - HF(\beta)G[I + SHF(\beta)G]^{-1} S$.

Theorem 4.3 Let us consider the sliding mode dynamics (4.41). If $Y \in R^{n\times n}$, $c_1 \in R, c_2 \in R, \eta \in R$ are decision variables, $\kappa = \lambda_{\min}(B^T B)$, $\Lambda \in R^{n\times(n-m)}$ is any full rank matrix satisfying $B^T \Lambda = 0, \Lambda^T \Lambda = I$, and $*$ represents blocks that are readily inferred by symmetry such that the following LMIs holds:

$$\begin{bmatrix} \Lambda^T[(A_i + T_i \Pi_i(t))Y + *]\Lambda & * & * \\ \eta H^T \Lambda & -I & * \\ (A_i + T_i \Pi_i(t)) Y \Lambda & \eta H & -I \end{bmatrix} < 0, \quad \forall i \qquad (4.42)$$

$$\begin{bmatrix} Y & I & 0 \\ I & c_1 I & 0 \\ 0 & 0 & c_2 I - Y \end{bmatrix} > 0, \qquad (4.43)$$

$$\begin{bmatrix} 2\eta\kappa & * & * \\ rc_1 & r\eta & 0 \\ rc_2 & 0 & r\eta \end{bmatrix} > 0. \qquad (4.44)$$

Suppose that the LMIs (4.42)-(4.44) have a solution vector (Y, c_1, c_2, η), then there exists a linear sliding surface parameter matrix S satisfying P1-P2 and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \qquad (4.45)$$

will guarantee that the sliding mode dynamics (4.41) is asymptotically stable.

Proof: By using Schur complement formula [48], we can easily show that in fact the following LMIs are incorporated in the LMIs (4.42)-(4.44)

$$c_1 > 0,\ \ c_2 > 0,\ \ \eta > 0,\ \ \eta^2 HH^T < I,\ \ 2\eta^2\kappa > r(c_1^2 + c_2^2). \tag{4.46}$$

It is clear that if the following inequality (4.47) holds, then $SB + SHF(\beta)G = I + SHF(\beta)G$ is nonsingular and hence P1 holds

$$SHF(\beta)GG^T F^T(\beta)H^T S^T < I. \tag{4.47}$$

Using (4.38), (4.39), (4.46) and $GG^T \le \|G\|^2 I = rI$, we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T. \tag{4.48}$$

By using the Schur complement formula, we can see that (4.43) and (4.46) imply

$$0 < c_1^{-1} I < Y < c_2 I, \qquad 0 < c_2^{-1} I < Y^{-1} < c_1 I \tag{4.49}$$

and this leads to

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T \le \frac{rc_1c_2}{\eta^2}(B^T B)^{-1} \le \frac{rc_1c_2}{\kappa\eta^2} I. \tag{4.50}$$

Using the inequality $2ab \le a^2 + b^2$ where a and b are scalars, we can show that (4.50) implies

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{2\kappa\eta^2}(c_1^2 + c_2^2) I. \tag{4.51}$$

Finally, by using the above inequalities (4.46) and (4.51), we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T < I \tag{4.52}$$

which implies that $[SB + SHF(\beta)G]$ is nonsingular, i.e., P1 holds.

Now, we will show that S of (4.45) guarantees P2. Using the matrix inversion lemma:

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B$$

where A and B are compatible constant matrices such that $(I + AB)$ is nonsingular, we can show that the sliding mode dynamics (4.41) is equivalent to

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T C(\beta)(A_i + T_i \Pi_i(t)) Y \Lambda v_1 \tag{4.53}$$

where $C(\beta) = I - H[I + F(\beta)GSH]^{-1} F(\beta)GS = [I + HF(\beta)GS]^{-1}$

$$= I - HF(\beta)G[I + SHF(\beta)G]^{-1} S = D(\beta).$$

The sliding mode dynamics (4.53) is asymptotically stable if there exists a positive definite matrix $P_0 \in R^{(n-m)\times(n-m)}$ such that the time derivative of the Lyapunov function $E_g(t) = v_1^T P_0 v_1$ satisfies for some positive scalar τ

$$\dot{E}_g(t) = 2\sum_{i=1}^{r} \beta_i(\theta) v_1^T P_0 Z_i(\beta) v_1 \le -\tau v_1^T v_1 \tag{4.54}$$

where $Z_i(\beta) = (A_{i0} + B_0[I - N(\beta)D_0]^{-1} N(\beta) C_{i0})$, $A_{i0} = (\Lambda^T Y \Lambda)^{-1} \Lambda^T (A_i + T_i \Pi_i(t)) Y \Lambda$, $B_0 = (\Lambda^T Y \Lambda)^{-1} \Lambda^T H$, $C_{i0} = (A_i + T_i \Pi_i(t)) Y \Lambda$, $D_0 = H$, $N(\beta) = -F(\beta)GS$.

It should be noted that the inequalities (4.39), (4.46), (4.52) and $GG^T \le \|G\|^2 I = rI$ imply

$$N(\beta)N^T(\beta) = F(\beta)GSS^T G^T F^T(\beta) \le \eta^2 I,\ \eta^2 D_0^T D_0 = \eta^2 H^T H < I. \tag{4.55}$$

This and (4.54) imply that (4.53) is asymptotically stable if there exists a positive definite matrix P_0 such that

$$P_0 A_{i0} + P_0 B_0 [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} + * < 0, \quad \forall i \tag{4.56}$$

where $*$ represents blocks that are readily inferred by symmetry. Let z_i be $z_i = [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} y$ where $y \in R^{(n-m)}$. Then z_i can be rewritten as $z_i = N(\beta)[C_{i0} y + D_0 z_i]$ .This equality and (4.55) imply $z_i^T z_i \le \eta^2 [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i]$ and this leads to

$$2y^T P_0 B_0 [I - N(\beta)D_0]^{-1} N(\beta) C_{i0} y$$

$$= 2y^T P_0 B_0 z_i \le 2y^T P_0 B_0 z_i + [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i] - \eta^{-2} z_i^T z_i$$

$$= y^T C_{i0}^T C_{i0} y + 2y^T [P_0 B_0 + C_{i0}^T D_0] z_i - \eta^{-2} z_i^T \Omega z_i \quad \text{where} \quad \Omega = I - \eta^2 D_0^T D_0 . \quad (4.57)$$

Since $\Omega > 0$, the following inequality holds for any (y, z_i):

$$2y^T [P_0 B_0 + C_{i0}^T D_0] z_i \le \eta^{-2} z_i^T \Omega z_i + \eta^2 y^T [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T y \quad (4.58)$$

Using (4.57) and (4.58), we can show that the Lyapunov inequality (4.56) is satisfied if the following inequality holds:

$$P_0 A_{i0} + A_{i0}^T P_0 + C_{i0}^T C_{i0} + \eta^2 [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T < 0 .$$

Using the Schur complement formula, we can rewrite the above inequality as

$$\begin{bmatrix} A_{i0}^T P_o + * & * & * \\ \eta B_0^T P_0 & -I & * \\ C_{i0} & \eta D_0 & -I \end{bmatrix} < 0, \quad \forall i . \quad (4.59)$$

Let the positive definite matrix P_0 be $P_0 = \Lambda^T Y \Lambda$ where Y is a solution to LMIs (4.42)-(4.44), which implies that the sliding mode dynamics (4.53) is asymptotically stable. Hence, the sliding mode dynamics (4.41) is asymptotically stable.

After the switching surface parameter matrix S is designed so that the reduced $(n-m)$ order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintains it there for all subsequent time. With σ of (4.45), we design an adaptive fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear adaptive switching feedback control law as the local controller.

Control rule *i*: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\hat{h}(t,x) - \chi_i \sigma - S(A_i + T_i \Pi_i(t))x - \frac{1}{1-\omega} \hat{\delta}_i(t,x) \frac{\sigma}{\|\sigma\|}$$

where $$\hat{\delta}_i(t,x) = \alpha_i + \omega\|S(A_i + T_i\Pi_i(t))x\| + (1+\omega)\sum_{k=0}^{l}\hat{\rho}_k\|x\|^k \tag{4.60}$$

$$\dot{\hat{\rho}}_k = \varepsilon_k\|\sigma\|\cdot\|x\|^k \tag{4.61}$$

and $\sigma = Sx, \omega = \sqrt{r}\|SH\|, \alpha_i > 0, \chi_i > 0, \varepsilon_k > 0$. It should be noted that (4.52) implies $\omega = \sqrt{r}\|SH\| \le \sqrt{r}\|S\|\cdot\|H\| \le \eta\|H\|$. This and (4.46) guarantee $0 \le \omega < 1$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\hat{h}(t,x) - \sum_{i=1}^{r}\beta_i(\theta)\left(\chi_i\sigma + S(A_i + T_i\Pi_i(t))x + \frac{1}{1-\omega}\hat{\delta}_i(t,x)\frac{\sigma}{\|\sigma\|}\right) \tag{4.62}$$

and we can establish the following theorem.

Theorem 4.4 Consider the closed-loop control system of the uncertain system (4.37) with control (4.62).Suppose that the LMIs (4.42)-(4.44) has a solution vector (Y, c_1, c_2, η) and the linear sliding surface is given by (4.45). Then the state converges to zero.

Proof: Since Theorem 4.3 implies that the linear sliding surface (4.45) guarantees P1-P2, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T\sigma + 0.5\xi\sum_{k=0}^{l}\tilde{\rho}_k^2$ where $\xi = 1+\omega$ and $\tilde{\rho}_k = \hat{\rho}_k - \rho_k$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T\dot{\sigma} + \xi\|\sigma\|\sum_{k=0}^{l}\tilde{\rho}_k\|x\|^k$. From (4.37), (4.45), (4.62), $\|SHF(\beta)G\| \le \sqrt{r}\|SH\| = \omega$,$0 \le \omega < 1$, and A2, we obtain

$$\sigma^T\dot{\sigma} = \sigma^T\sum_{i=1}^{r}\beta_i(\theta)S(A_i + T_i\Pi_i(t))x(t) \quad + \sigma^T[I + SHF(\beta)G][u + h(t,x)]$$

$$\le \sum_{i=1}^{r}\beta_i(\theta)\sigma^T S(A_i + T_i\Pi_i(t))x(t) + \sigma^T u + \{\omega\|u\| + (1+\omega)\|h(t,x)\|\}\|\sigma\|$$

$$\le -(1-\omega)\sum_{i=1}^{r}\beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i(\theta)\alpha_i\|\sigma\| - \xi\|\sigma\|\sum_{k=0}^{l}\tilde{\rho}_k\|x\|^k.$$

This implies that $\dot{E}_g \le -(1-\omega)\sum_{i=1}^{r}\beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i(\theta)\alpha_i\|\sigma\| \le 0$ which indicates that $E_g \in L_2 \cap L_\infty$, $\dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 4.2 Theorem 4.3 and 4.4 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Obtain $B = \frac{1}{r}\sum_{i=1}^{r} B_i$ and $H = \frac{1}{2}\left[(B-B_1),\cdots,(B-B_r)\right]$ for given B_i.

Step 2: Check that (A_i, B) is stabilization. If not, exit.

Step 3: Find a solution vector (Y, c_1, c_2, η) to LMI (4.42)-(4.44).

Step 4: Compute the sliding surface parameter matrix *S* by using the formula of (4.45).

Step 5: The controller is given by (4.62).

4.3.3 Numerical Examples I

Example 4.3 To illustrate the performance of the proposed adaptive fuzzy control design method, consider the following two-rule fuzzy model from a vertical take-off and landing (VTOL) helicopter model [55]

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = (A_1 + T_1\Pi_1(t))x + B_1[u + h(t,x)]$$

Plant Rule2: IF x_1 is about ± 2, THEN

$$\dot{x} = (A_2 + T_2\Pi_2(t))x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3181 & -0.7070 & 1.4100 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $T_1 = T_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.4181 & -0.7070 & 1.4300 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.6446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \; \Pi_1(t) = \Pi_2(t) = [0 \quad \sin t \quad 0 \quad \sin t],$$

$$h(t,x) = d(t) + [0.9\sin 3t \quad 0.9\sin 3t]^T, \; \beta_1 = \frac{1 - 1/(1+e^{-14(x_1-1)})}{1+e^{-14(x_1+1)}}, \; \beta_2 = 1 - \beta_1. \qquad (4.63)$$

It should be noted that T_1 and T_2 are not matched and thus the previous VSS-based fuzzy control design methods cannot be applied to the above system (4.63). Via LMI optimization with (4.63), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x) = [0.9\sin 3t \quad 0.9\sin 3t]^T$ and $\chi_i = 1, \alpha_i = 0.01, r = 2, l = 1, \varepsilon_k = 1$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = [-0.9\sin 3t \quad -0.9\sin 3t]^T - \sigma - S(A_1 + T_1\Pi_1(t))x - \frac{1}{1-\omega}\hat{\delta}_1 \operatorname{sgn}(\sigma).$$

Control Rule 2: IF x_1 is about ± 2, THEN

$$u(t) = [-0.9\sin 3t \quad -0.9\sin 3t]^T - \sigma - S(A_2 + T_2\Pi_2(t))x - \frac{1}{1-\omega}\hat{\delta}_2 \operatorname{sgn}(\sigma).$$

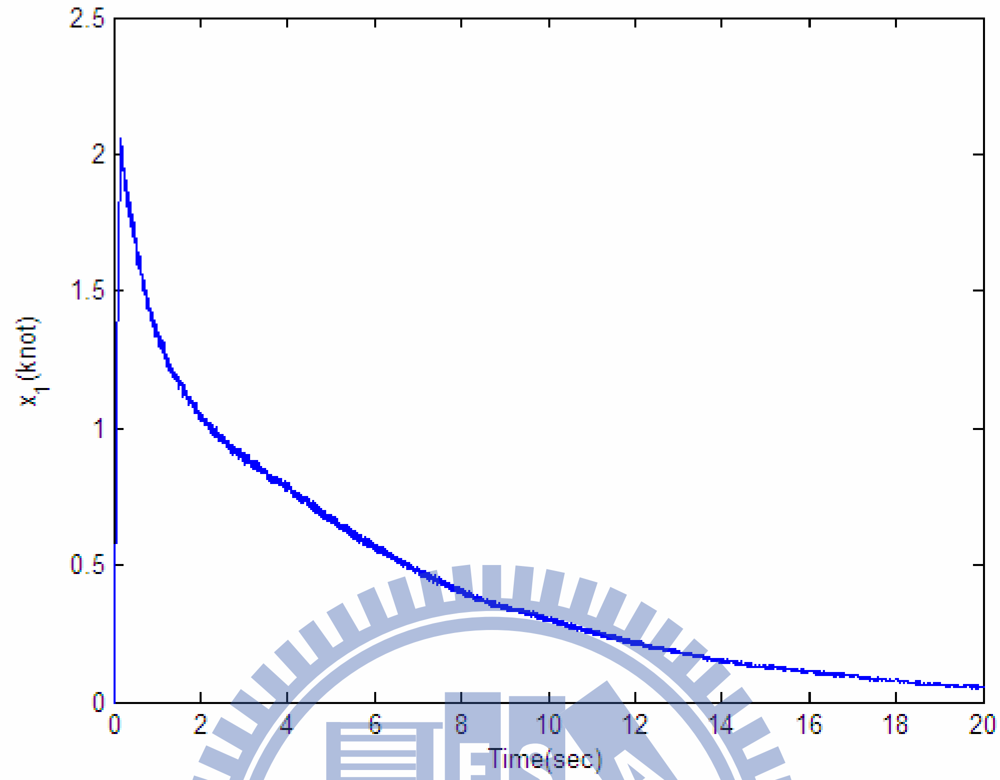
The final controller inferred as the weighted average of each local controller is given by

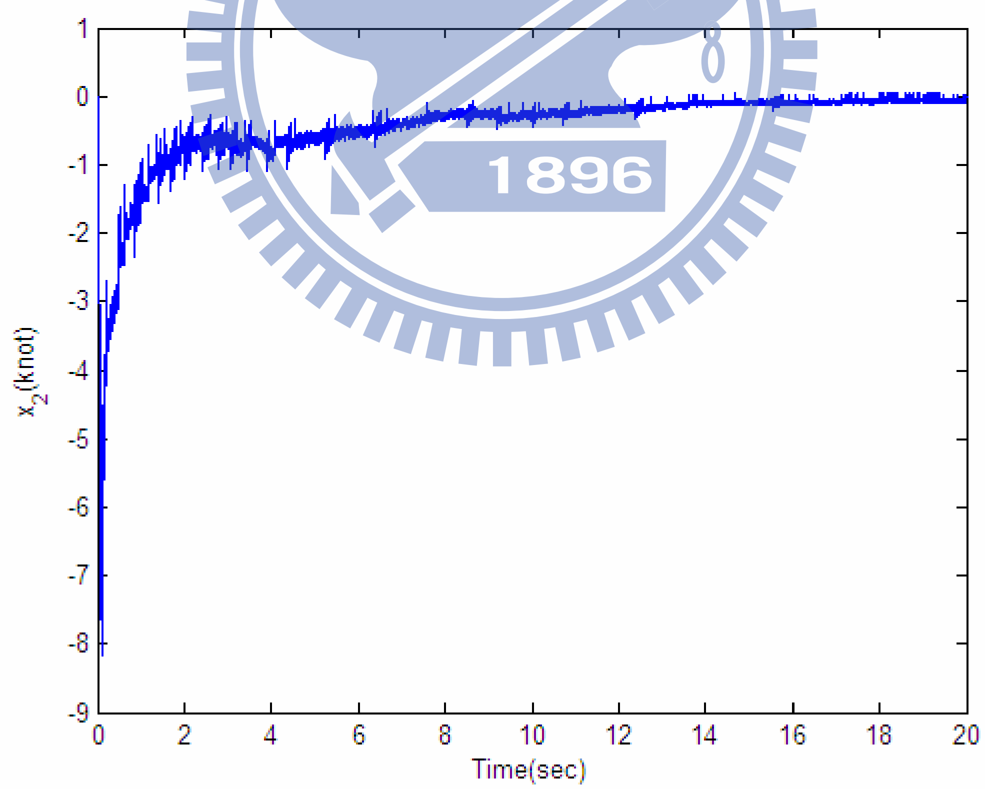
$$u(t) = [-0.9\sin 3t \quad -0.9\sin 3t]^T - \sum_{i=1}^{r} \beta_i(\theta)\left[\sigma + S(A_i + T_i\Pi_i(t))x + \frac{1}{1-\omega}\hat{\delta}_i \operatorname{sgn}(\sigma)\right].$$

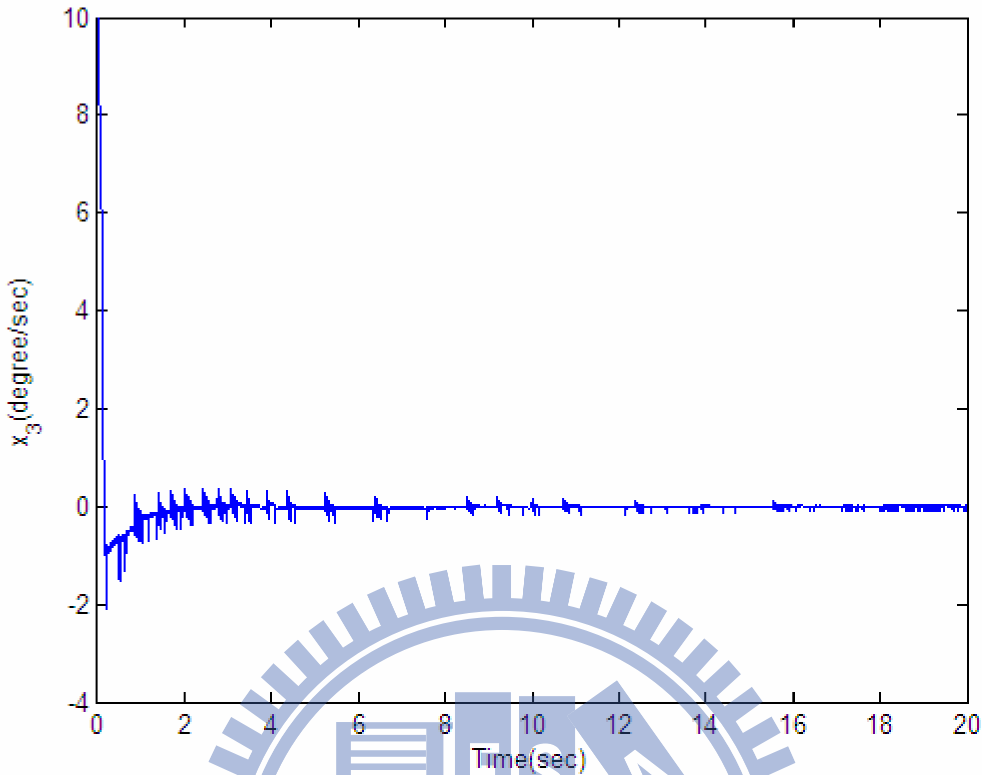
(4.64)

To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (4.63) with nonzero $d(t)$. We assume that $d(t) = [x_1 \sin 2t - 0.5\mathrm{sgn}(x_4) \quad x_1 \sin 2t - 0.5\mathrm{sgn}(x_4)]^T$. The time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.64) are shown in Figure 4.6 when $x_1(0) = x_2(0) = x_4(0) = 0, \quad x_3(0) = 10.$

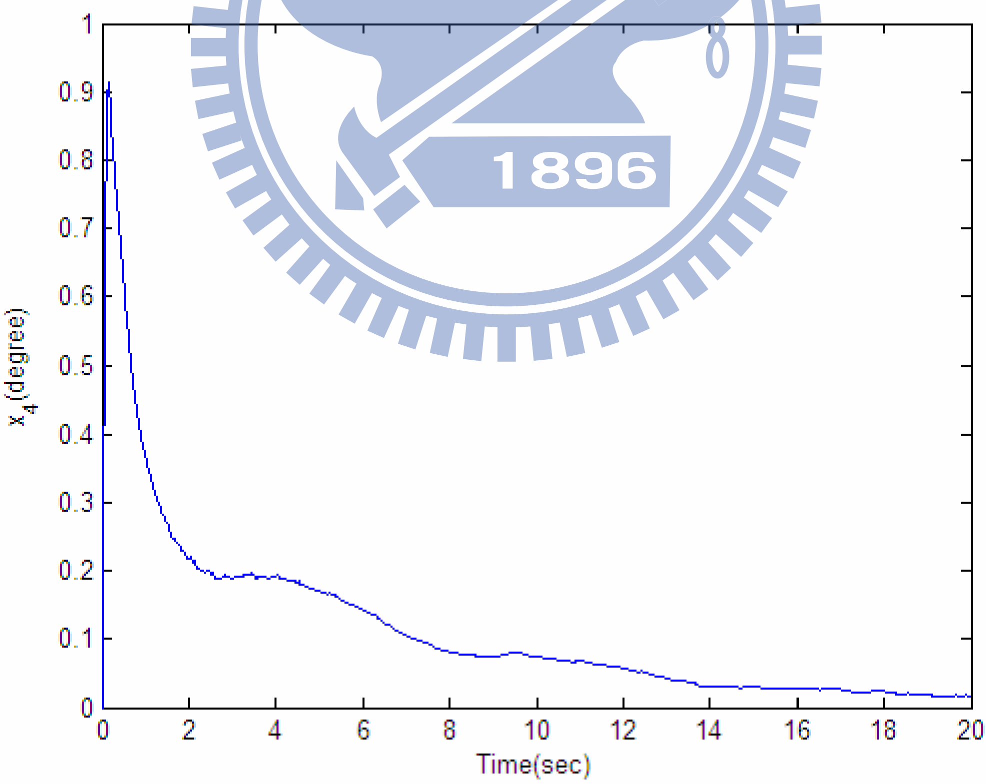
From Figure 4.6, the proposed controller is applicable to T-S fuzzy systems with mismatched parameter uncertainties in the state matrix and unknown norm-bounded external disturbances. The control performances are satisfactory. Besides, in Figure 4.6, since it is impossible to switch the input u instantaneously, oscillations on control input u always occur in the sliding mode of an SMC system. It should be noted that all existing VSS-based fuzzy control system design methods cannot be applied to the two-rule fuzzy model (4.63) because B_1 is not in the range space of B_2.


2.5
2
1.5
1
0.5
0
x_1(knot)
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

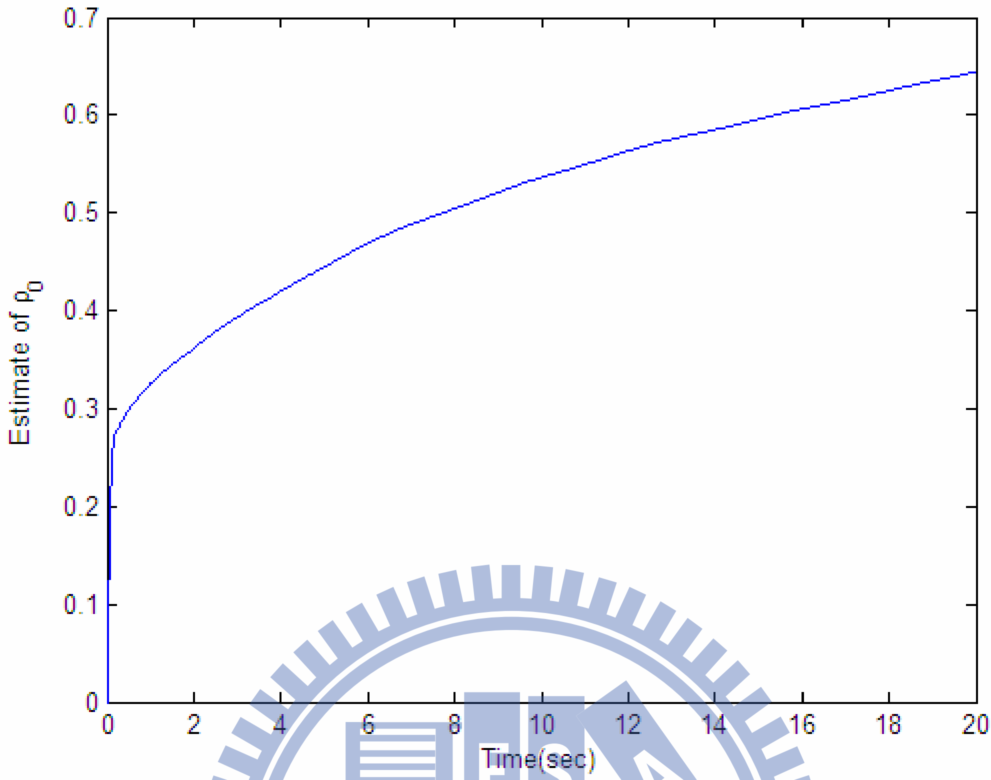

1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
x_2(knot)
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

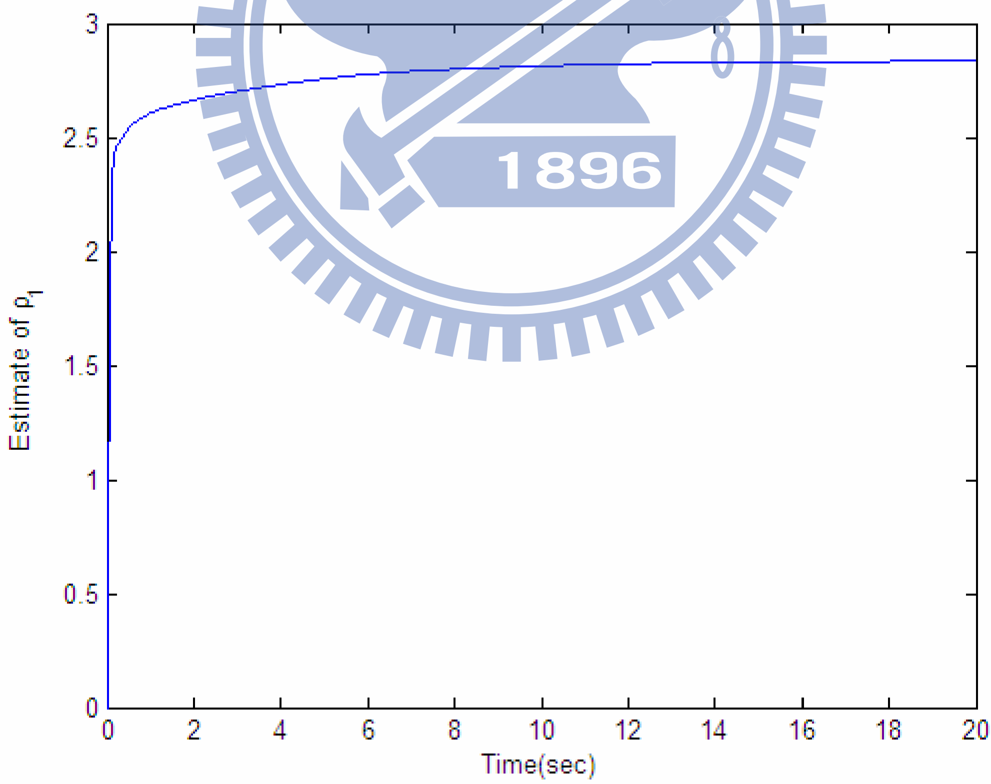


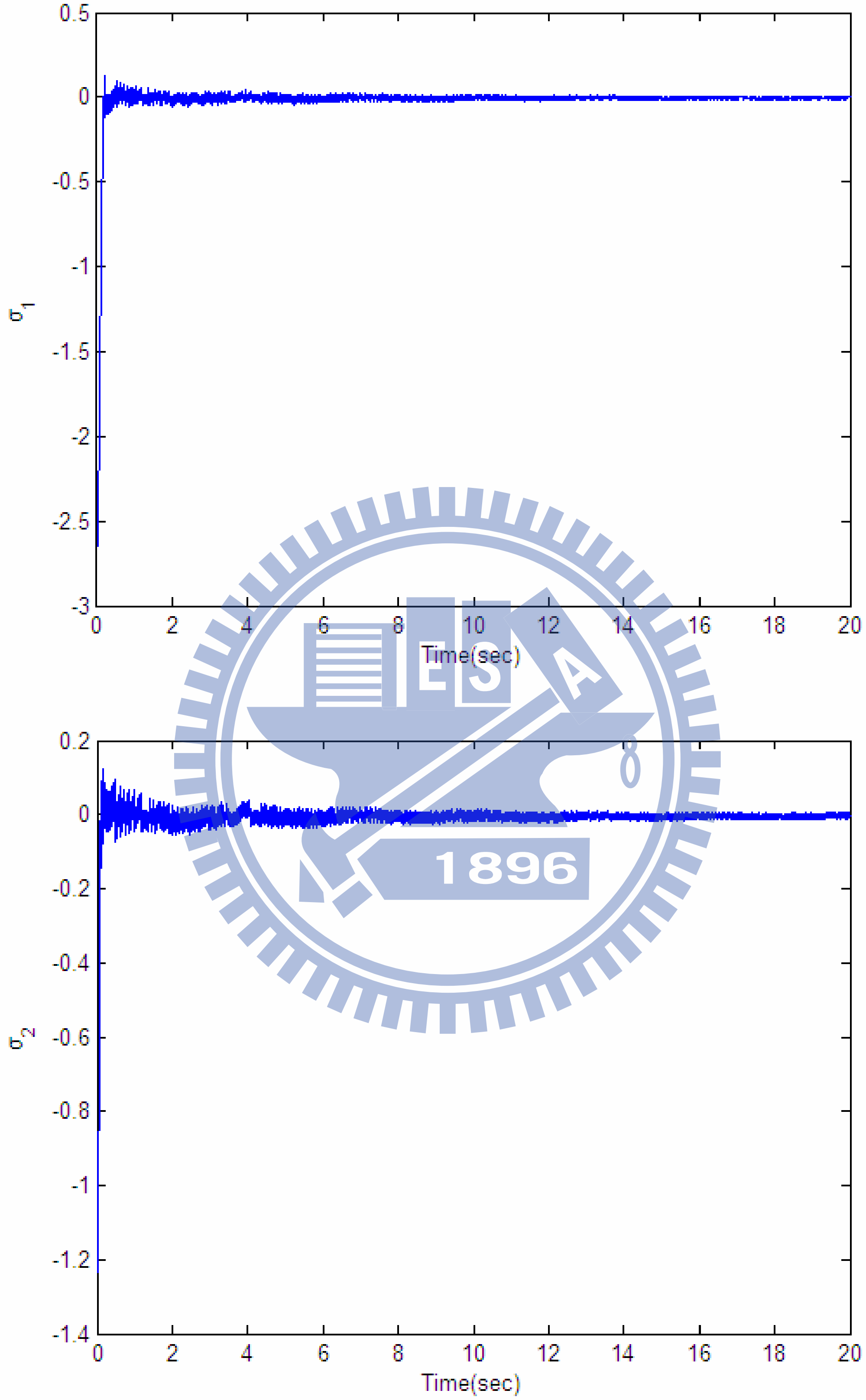
10
8
6
4
2
0
-2
-4
x_3(degree/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x_4(degree)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


Estimate of ρ_0
Time(sec)


Estimate of ρ_1
Time(sec)



0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
σ1
0 2 4 6 8 10 12 14 16 18 20
Time(sec)
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
σ2
0 2 4 6 8 10 12 14 16 18 20
Time(sec)

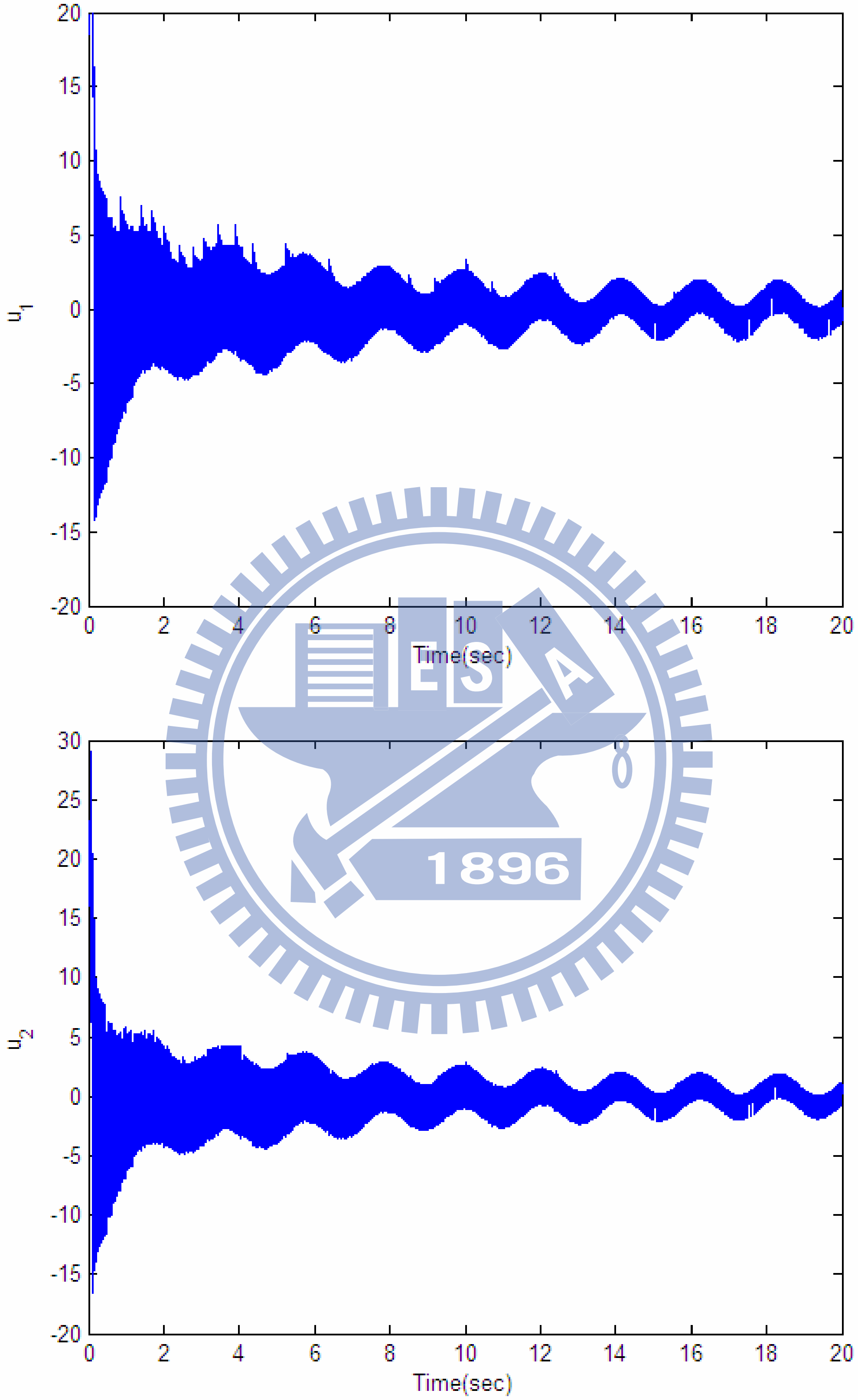


Figure 4.6 Simulation results with $x_1(0) = x_2(0) = x_4(0) = 0,\ x_3(0) = 10.$

Example 4.4 For the special case of $\Pi_i(t) \equiv 0$, the robust adaptive controller design is proposed in [64]. Consider the following inverted pendulum on a cart

$$\dot{x}_1 = x_2,\ \dot{x}_2 = \frac{1}{l\psi}(3g\sin x_1 - 3a\cos x_1[u + d(t) + \phi]),\ \dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{1}{\psi}(1.5mag\sin 2x_1 - 4a[u + d(t) + \phi]) \tag{4.65}$$

where x_1 is the angle (rad) of the pendulum from the vertical, $x_2 = \dot{x}_1$, x_3 is the displacement (m) of the cart, $x_4 = \dot{x}_3$, $\psi = 4 - 3ma\cos^2 x_1$, $\phi = mlx_2^2\sin x_1$, u is the input, and $d(t)$ is related to external disturbances which may be caused by the frictional force. $a = 1/(m+M)$, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, $g = 9.8m/s^2$ is the gravity constant. We set $M = 9kg$, $m = 1kg$, $l = 1m$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1\|x\|$ where ρ_0 and ρ_1 are unknown constants. Here, we approximate the system (4.65) by the following two-rule fuzzy model.

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = A_1x + B_1[u + h(t,x)]$$

Plant Rule2: IF x_2 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

$$\dot{x} = A_2x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.9459 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.7946 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -0.0811 \\ 0 \\ 0.1081 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3097 & 0 & 0 & 0 \end{bmatrix}$,

$$B_2 = \begin{bmatrix} 0 \\ -0.0382 \\ 0 \\ 0.1019 \end{bmatrix},\ h(t,x) = d(t) + x_2^2\sin x_1,\ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}},\ \beta_2 = 1 - \beta_1. \tag{4.66}$$

Because B_1 is not in the range space of B_2, all existing VSS-based fuzzy control system design methods cannot be applied to the above system (4.66). Via LMI optimization with (4.66), we can obtain the sliding surface $\sigma = Sx$. By setting $\hat{h}(t,x) = x_2^2 \sin x_1, \chi_i = 5, \alpha_i = 1, r = 2, l = 1, \varepsilon_k = 0.001$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_1 x - \frac{1}{1-\omega}\hat{\delta}_1 \operatorname{sgn}(\sigma).$$

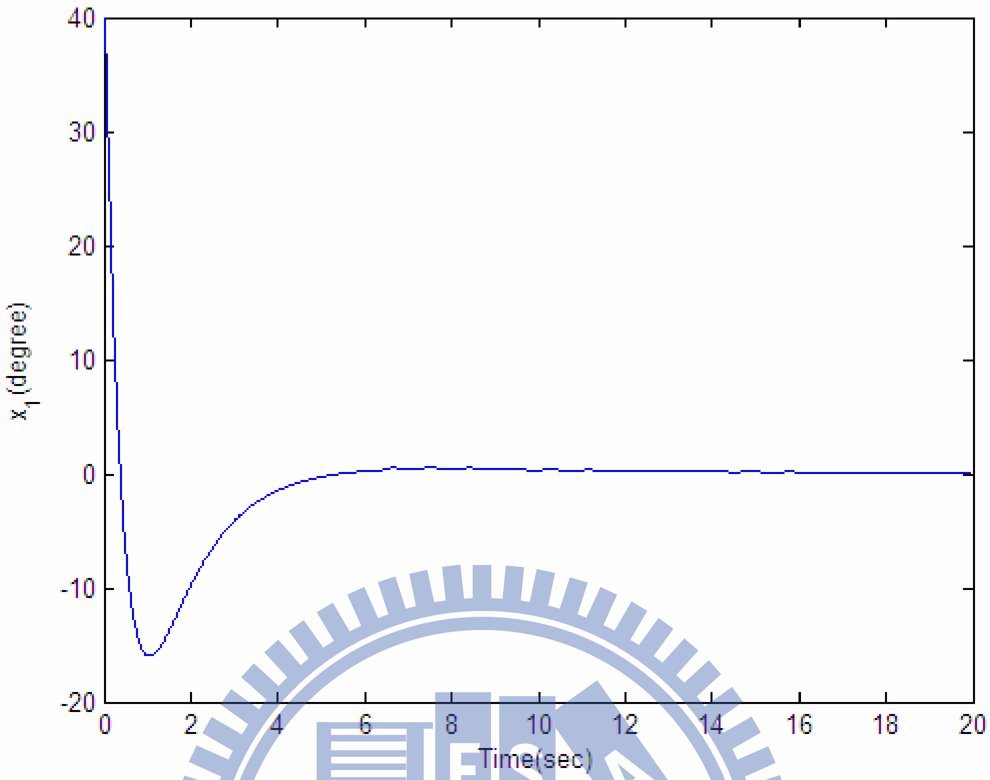
Control Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

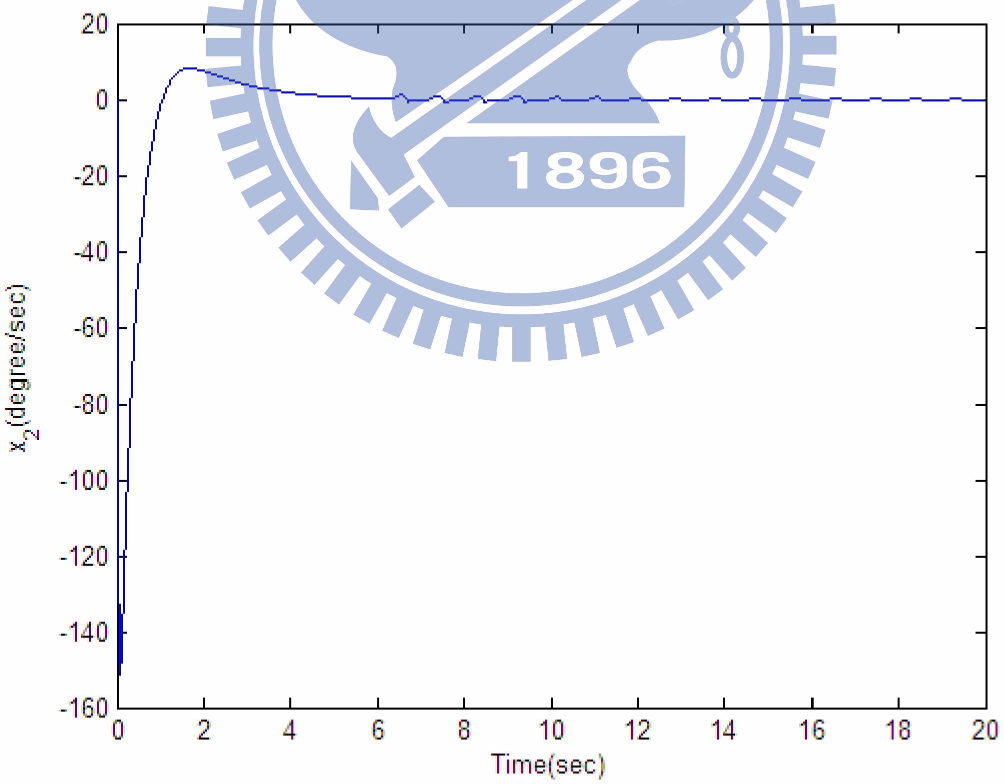
$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_2 x - \frac{1}{1-\omega}\hat{\delta}_2 \operatorname{sgn}(\sigma).$$

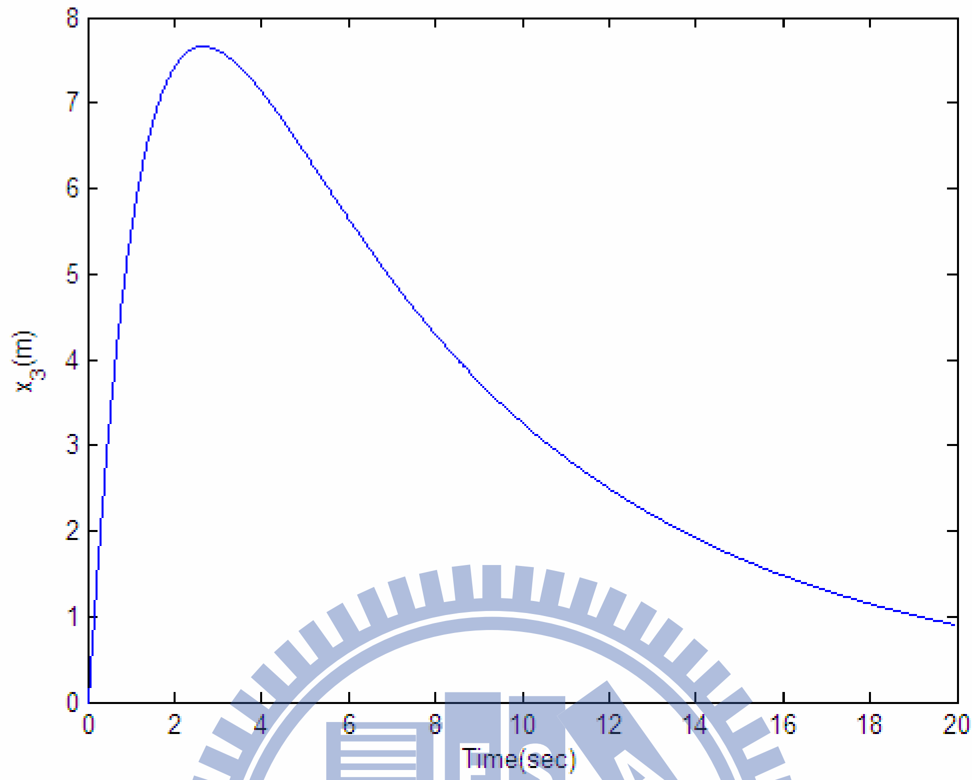
The final controller inferred as the weighted average of each local controller is given by

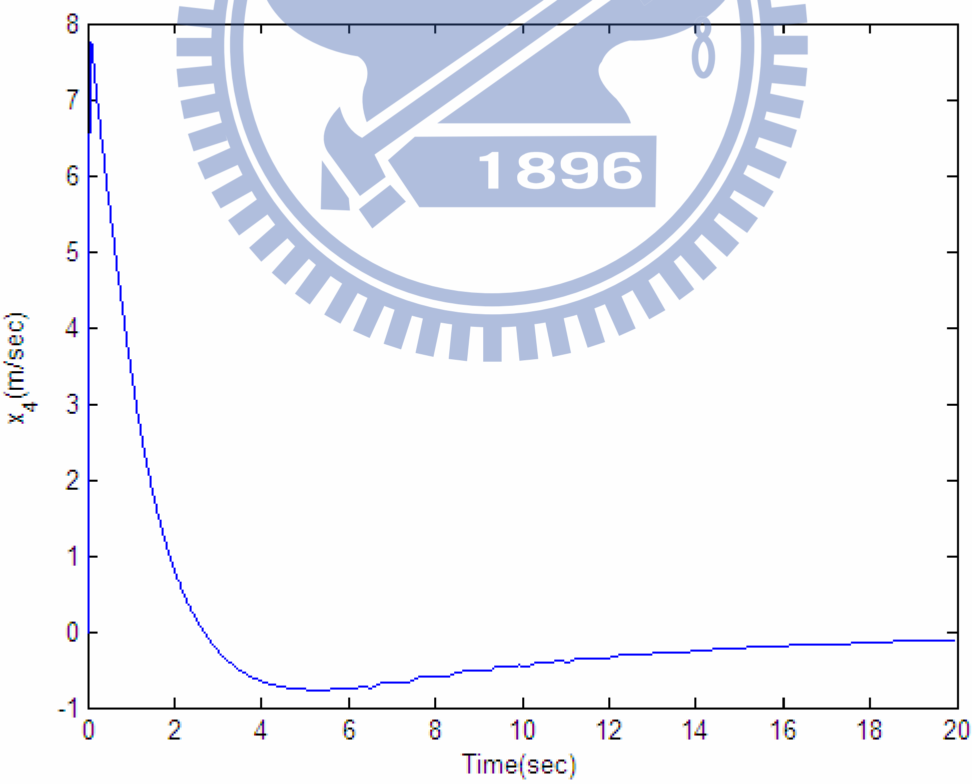
$$u(t) = -x_2^2 \sin x_1 - \sum_{i=1}^{r} \beta_i(\theta)\left[5\sigma + SA_i x + \frac{1}{1-\omega}\hat{\delta}_i \operatorname{sgn}(\sigma)\right]. \qquad (4.67)$$

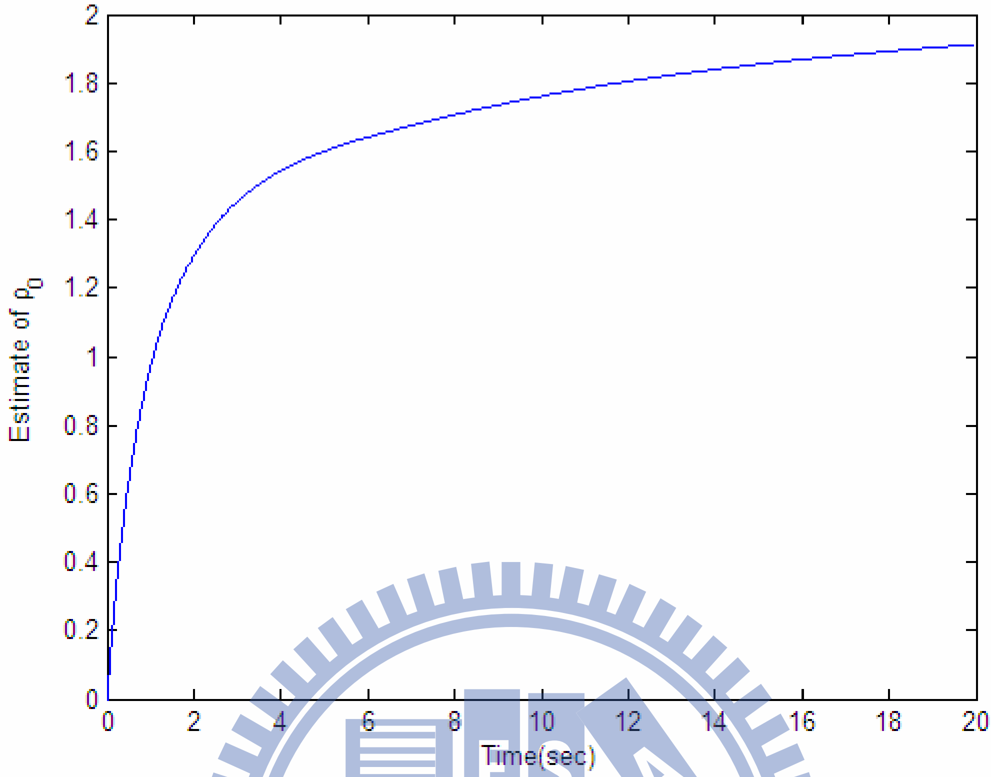
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (4.66) with nonzero $d(t)$. We assume that $d(t) = x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)$. The time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.67) are shown in Figure 4.7 when $x_1(0) = 40^\circ (2\pi/9\ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$. In Figure 4.7, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 4.7, the control performances of the proposed controller are also satisfactory for the two-rule fuzzy model (4.66).

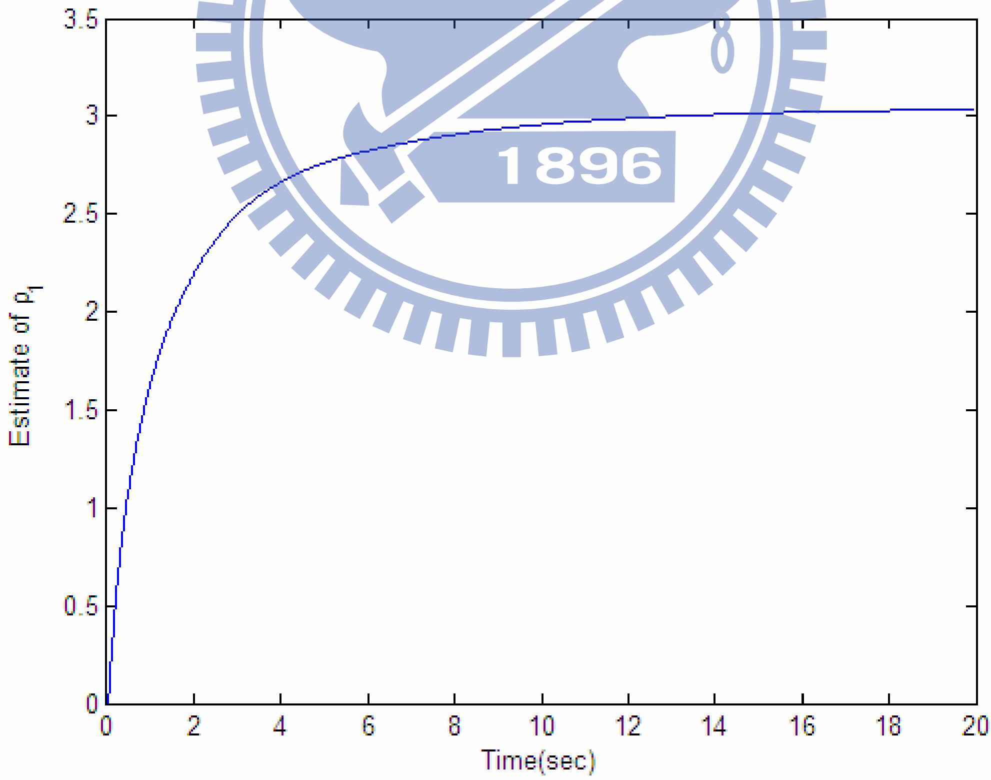

40
30
20
10
0
-10
-20
x_1(degree)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


20
0
-20
-40
-60
-80
-100
-120
-140
-160
x_2(degree/sec)
0
2
4
6
8
10
12
14
16
18
20
Time(sec)


x3(m)
Time(sec)


x4(m/sec)
Time(sec)


Estimate of ρ0
Time(sec)
0
2
4
6
8
10
12
14
16
18
20
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Estimate of ρ1
Time(sec)
0
2
4
6
8
10
12
14
16
18
20
0.5
1
1.5
2
2.5
3
3.5

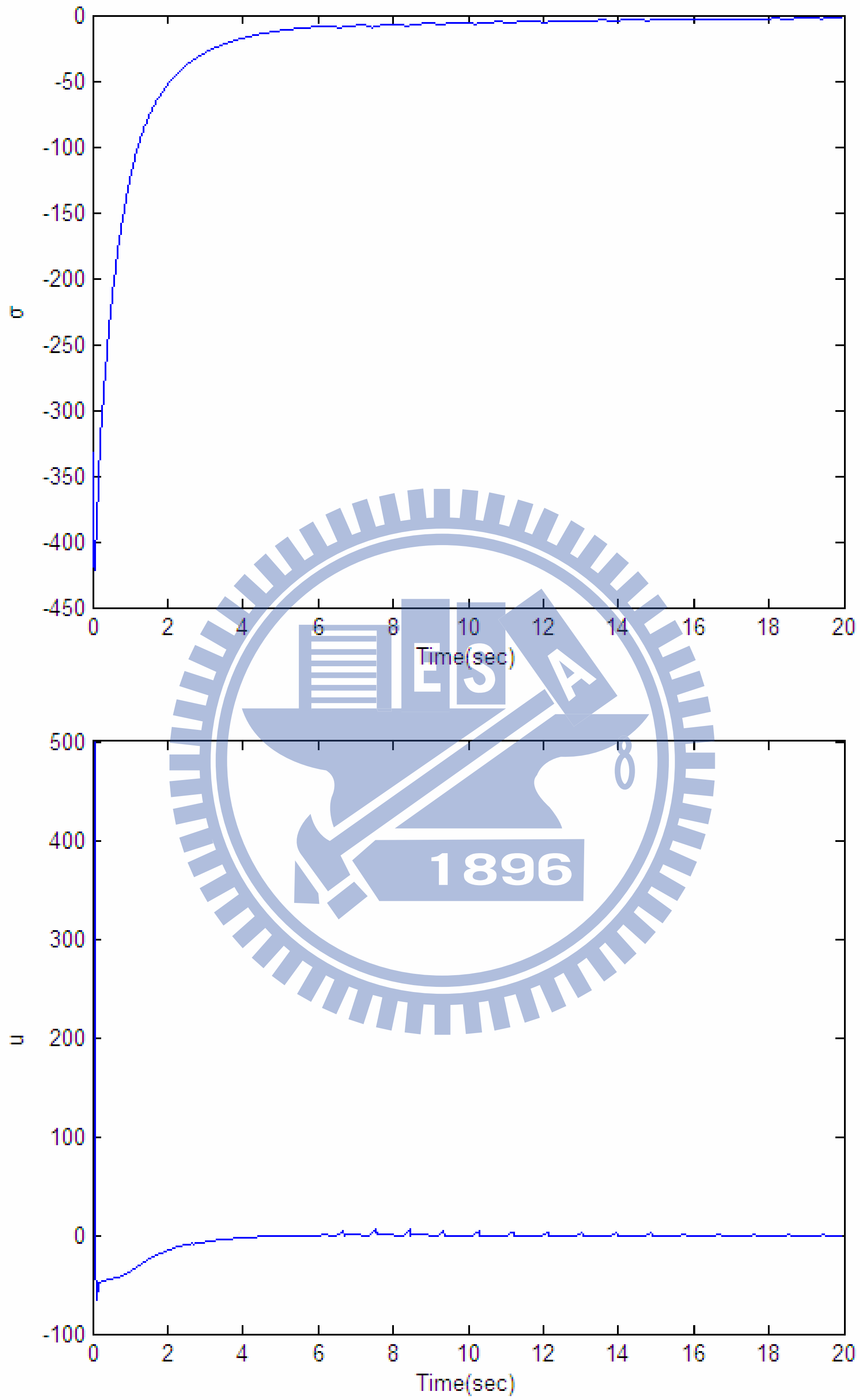


Figure 4.7 Simulation results with $x_1(0)=40^\circ(2\pi/9\ rad), x_2(0)=x_3(0)=x_4(0)=0.$

4.3.4 System Formulation II

Consider the following uncertain T-S fuzzy model [49], including parameter uncertainties and unknown norm-bounded external disturbances:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)([A_i + \Delta A_i(t)]x(t) + B_i[u(t) + h(t,x)]) \tag{4.68}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, A_i, B_i are constant matrices of appropriate dimensions, $\Delta A_i(t)$ represents the parameter uncertainties in $A_i, h(t,x) \in R^m$ denotes external disturbances. $\theta = [\theta_1, \cdots, \theta_s], \theta_j (j = 1, \cdots s)$ are the premise variables, s is the number of the premise variables, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta), \omega_i : R^s \to [0,1], i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i, r is the number of the IF-THEN rules, β_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0. \sum_{i=1}^{r} \beta_i(\theta) = 1$. We will assume that the followings are satisfied:

A1: The $n \times m$ matrix B defined by $B = \frac{1}{r} \sum_{i=1}^{r} B_i$ satisfies the rank constraint rank$(B) = m$, i.e., the matrix B has full column rank m.

A2: The function $h(t,x)$ is unknown but bounded as $\left\| h(t,x) - \hat{h}(t,x) \right\| \leq \sum_{k=0}^{l} \rho_k \left\| x \right\|^k$ where $\rho_0, \cdots, \rho_l$ are unknown constants, $\hat{h}(t,x)$ is an estimate of $h(t,x)$, and l is a known positive integer.

A3: $\Delta A_i(t)$ is of the form $T_i \Pi_i(t)$ where $\Pi_i(t)$ is unknown, $\left\| \Delta A_i(t) \right\| \leq \alpha_{A_i}$ and $T_i T_i^T \geq T_i \Pi_i(t)$.

The system (4.68) does not have to satisfy the restrictive assumption that all the input matrices of the local system models are in the same range space. It should be noted that the assumption A1 implies that $rank(B_i) \le m$ and each nominal local system model may not share the same input channel. The assumption A2 with $l = 1$ and $\hat{h}(t,x) = 0$ has been used in the literature [50]. We can set $\hat{h}(t,x)$ as the nominal value of $h(t,x)$. Using the above assumptions, the uncertain T-S fuzzy model (4.68) can be written as follows.

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)(A_i + T_i \Pi_i(t))x(t) + [B + HF(\beta)G][u + h(t,x)] \tag{4.69}$$

where $\beta = [\beta_1(\theta), \cdots, \beta_r(\theta)]$, and the matrices $H, G, F(\beta)$ are defined by

$$H = \frac{1}{2}[(B - B_1), \cdots, (B - B_r)],\ G = [I, \cdots, I]^T,$$

$$F(\beta) = diag[(1 - 2\beta_1(\theta))I, \cdots, (1 - 2\beta_r(\theta))I] \tag{4.70}$$

It should be noted that the system (4.68) does not have to satisfy $B_1 = B_2 = \cdots = B_r$, which is used in almost all published results on VSS design methods including the VSS-based fuzzy control design methods of [33,34]. Hence the methods [30,31] cannot be applied to the above model (4.68). Since $\beta_i(\theta) \ge 0$ and $\sum_{i=1}^{r} \beta(\theta) = 1$, we can see that the following inequality always holds:

$$F^T(\beta)F(\beta) = F(\beta)F^T(\beta) \le I. \tag{4.71}$$

The following lemma will be used to establish our main results.

Lemma 4.1 For any vectors a and b with appropriate dimensions, the following inequalities hold for any $W > 0$:

$$2a^T b \le a^T W a + b^T W^{-1} b.$$

Proof: The above inequality is derived from $(Wa - b)^T W^{-1}(Wa - b) = a^T W a + b^T W^{-1} b - 2a^T b \ge 0$.

Many examples in the literature and various mechanical systems such as motors and robots do not satisfy the restrictive assumptions that each nominal local system model shares the same input channel and they fall into the special cases of the above model [49]

4.3.5 LMI-based Adaptive Control Design II

The Sliding Mode Control (SMC) design is decoupled into two independent tasks of lower dimensions: The first involves the design of $m(n-1)-$dimensional switching surfaces for the sliding mode such that the reduced order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second is concerned with the selection of a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first characterize linear sliding surfaces using LMIs.

Let us define the linear sliding surface as $\sigma = Sx = 0$ where S is a $m \times n$ matrix. Referring to the previous results [33], [51], we can see that for the system (4.69) it is reasonable to find a sliding surface such that

P1 $\left[SB + SHF(\beta)G\right]$ is nonsingular for any β satisfying $\beta_i(\theta) \geq 0, i = 1,\cdots,r,$ and $\sum_{i=1}^{r} \beta_i(\theta) = 1$.

P2 The reduced $(n-m)$ order sliding mode dynamics restricted to the sliding surface $Sx = 0$ is asymptotically stable for all admissible uncertainties.

It should be noted that P1 is necessary for the existence of the unique equivalent control [33] and the assumption A1 is necessary for the nonsingularity of SB.

Define a transformation matrix and the associated vector v as $M = [\Lambda(\Lambda^T Y \Lambda)^{-1}, Y^{-1}B(B^T Y^{-1} B)^{-1}]^T = [V^T, S^T]^T$, $v = [v_1^T, v_2^T]^T = Mx$ where $v_1 \in R^{n-m}$, $v_2 \in R^m$. By the above transformation, we can see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. Then, from system (4.69), we can obtain

$$\begin{bmatrix} \dot{v}_1 \\ \dot{\sigma} \end{bmatrix} = \sum_{i=1}^{r} \beta_i(\theta) \begin{bmatrix} V(A_i + T_i\Pi_i(t))Y\Lambda & V(A_i + T_i\Pi_i(t))B \\ S(A_i + T_i\Pi_i(t))Y\Lambda & S(A_i + T_i\Pi_i(t))B \end{bmatrix} \begin{bmatrix} v_1 \\ \sigma \end{bmatrix}$$

$$+ \begin{bmatrix} VHF(\beta)G \\ I + SHF(\beta)G \end{bmatrix} [u + h(t,x)]. \tag{4.72}$$

Then from the equivalent control method [33], we can see that the equivalent control is given by $u_{eq}(t) = -\sum_{i=1}^{r} \beta_i(\theta)[I + SHF(\beta)G]^{-1} S(A_i + T_i\Pi_i(t))x - h(t,x)$. By setting $\dot{\sigma} = \sigma = 0$ and substituting $u(t)$ with $u_{eq}(t)$, we can show that the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma = Sx = 0$ is given by

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y \Lambda)^{-1} \Lambda^T D(\beta)(A_i + T_i\Pi_i(t))Y\Lambda v_1 \tag{4.73}$$

where $D(\beta) = I - HF(\beta)G[I + SHF(\beta)G]^{-1} S$.

Theorem 4.5 Let us consider the sliding mode dynamics (4.73). If $Y \in R^{n\times n}$ $, c_0 \in R, c_1 \in R, c_2 \in R, \delta \in R, \eta \in R$ are decision variables, $\kappa = \lambda_{\min}(B^T B)$, $\Lambda \in R^{n\times(n-m)}$ is any full rank matrix satisfying $B^T\Lambda = 0, \Lambda^T\Lambda = I$, $\|\Delta A_i(t)\| \le \alpha_{A_i}$, and $*$ represents blocks that are readily inferred by symmetry such that the following LMIs holds:

$$\begin{bmatrix} \Lambda^T(A_iY + YA_i^T + d_oI)\Lambda & \eta\Lambda^T H & \Lambda^T YA_i^T & \alpha_{A_i}\Lambda^T Y & \alpha_{A_i}\Lambda^T Y \\ \eta H^T\Lambda & -I & \eta H^T & 0 & 0 \\ A_iY\Lambda & \eta H & -(1-\delta)I & 0 & 0 \\ \alpha_{A_i}Y\Lambda & 0 & 0 & -c_0 I & 0 \\ \alpha_{A_i}Y\Lambda & 0 & 0 & 0 & -\delta I \end{bmatrix} < 0, \quad \forall i \tag{4.74}$$

$$\begin{bmatrix} Y & I & 0 \\ I & c_1 I & 0 \\ 0 & 0 & c_2 I - Y \end{bmatrix} > 0, \tag{4.75}$$

$$\begin{bmatrix} 2\eta\kappa & * & * \\ rc_1 & r\eta & 0 \\ rc_2 & 0 & r\eta \end{bmatrix} > 0. \tag{4.76}$$

Suppose that the LMIs (4.74)-(4.76) have a solution vector $(Y, c_0, c_1, c_2, \delta, \eta)$, then there exists a linear sliding surface parameter matrix S satisfying P1-P2 and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \tag{4.77}$$

will guarantee that the sliding mode dynamics (4.73) is asymptotically stable.

Proof: By using Schur complement formula [48], we can easily show that in fact the following LMIs are incorporated in the LMIs (4.74)-(4.76)

$$c_1 > 0, \;\; c_2 > 0, \;\; \eta > 0, \;\; \eta^2 HH^T < I, \;\; 2\eta^2 \kappa > r(c_1^2 + c_2^2)\,. \tag{4.78}$$

It is clear that if the following inequality (4.79) holds, then $SB + SHF(\beta)G = I + SHF(\beta)G$ is nonsingular and hence P1 holds

$$SHF(\beta)GG^T F^T(\beta)H^T S < I. \tag{4.79}$$

Using (4.70), (4.71), (4.78) and $GG^T \le \|G\|^2 I = rI$, we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T. \tag{4.80}$$

By using the Schur complement formula, we can see that (4.75) and (4.78) imply

$$0 < c_1^{-1} I < Y < c_2 I, \qquad 0 < c_2^{-1} I < Y^{-1} < c_1 I \tag{4.81}$$

and this leads to

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T \le \frac{rc_1c_2}{\eta^2}(B^T B)^{-1} \le \frac{rc_1c_2}{\kappa\eta^2} I. \tag{4.82}$$

Using the inequality $2ab \le a^2 + b^2$ where a and b are scalars, we can show that (4.82) implies

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{2\kappa\eta^2}(c_1^2 + c_2^2) I. \tag{4.83}$$

Finally, by using the above inequalities (4.78) and (4.83), we can obtain

$$SHF(\beta)GG^T F^T(\beta)H^T S^T \le \frac{r}{\eta^2} SS^T < I \tag{4.84}$$

which implies that $[SB + SHF(\beta)G]$ is nonsingular, i.e., P1 holds.

Now, we will show that S of (4.77) guarantees P2. Using the matrix inversion lemma:

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B$$

where A and B are compatible constant matrices such that $(I + AB)$ is nonsingular, we can show that the sliding mode dynamics (4.73) is equivalent to

$$\dot{v}_1 = \sum_{i=1}^{r} \beta_i(\theta)(\Lambda^T Y\Lambda)^{-1}\Lambda^T C(\beta)(A_i + T_i\Pi_i(t))Y\Lambda v_1 \tag{4.85}$$

where $C(\beta) = I - H[I + F(\beta)GSH]^{-1} F(\beta)GS = [I + HF(\beta)GS]^{-1}$

$$= I - HF(\beta)G[I + SHF(\beta)G]^{-1} S = D(\beta) \text{ and } v_1 = (\Lambda^T Y\Lambda)^{-1}\Lambda^T x.$$

The sliding mode dynamics (4.85) is asymptotically stable if there exists a positive definite matrix $P_0 \in R^{(n-m)\times(n-m)}$ such that the time derivative of the Lyapunov function $E_g(t) = v_1^T P_0 v_1$ satisfies for some positive scalar τ

$$\dot{E}_g(t) = 2\sum_{i=1}^{r} \beta_i(\theta) v_1^T P_0 Z_i(\beta) v_1 \le -\tau v_1^T v_1 \tag{4.86}$$

where $Z_i(\beta) = (A_{i0} + B_0[I - N(\beta)D_0]^{-1} N(\beta)C_{i0})$, $A_{i0} = (\Lambda^T Y\Lambda)^{-1}\Lambda^T(A_i + T_i\Pi_i(t))Y\Lambda$, $B_0 = (\Lambda^T Y\Lambda)^{-1}\Lambda^T H$, $C_{i0} = (A_i + T_i\Pi_i(t))Y\Lambda$, $D_0 = H$, $N(\beta) = -F(\beta)GS$.

It should be noted that the inequalities (4.71), (4.78), (4.84) and $GG^T \le \|G\|^2 I = rI$ imply

$$N(\beta)N^T(\beta) = F(\beta)GSS^T G^T F^T(\beta) \le \eta^2 I,\ \eta^2 D_0^T D_0 = \eta^2 H^T H < I. \tag{4.87}$$

This and (4.86) imply that (4.85) is asymptotically stable if there exists a positive definite matrix P_0 such that

$$P_0 A_{i0} + P_0 B_0 [I - N(\beta) D_0]^{-1} N(\beta) C_{i0} + * < 0\,, \quad \forall i \tag{4.88}$$

where $*$ represents blocks that are readily inferred by symmetry. Let z_i be $z_i = [I - N(\beta) D_0]^{-1} N(\beta) C_{i0} y$ where $y \in R^{(n-m)}$ .Then z_i can be rewritten as $z_i =\ N(\beta)[C_{i0} y + D_0 z_i]$ .This equality and (4.87) imply $z_i^T z_i \le \eta^2 [C_{i0} y + D_0 z_i]^T$ $[C_{i0} y + D_0 z_i]$and this leads to

$$2 y^T P_0 B_0 [I - N(\beta) D_0]^{-1} N(\beta) C_{i0} y$$

$$= 2 y^T P_0 B_0 z_i \le 2 y^T P_0 B_0 z_i + [C_{i0} y + D_0 z_i]^T [C_{i0} y + D_0 z_i] - \eta^{-2} z_i^T z_i$$

$$= y^T C_{i0}^T C_{i0} y + 2 y^T [P_0 B_0 + C_{i0}^T D_0] z_i - \eta^{-2} z_i^T \Omega z_i \quad \text{where} \quad \Omega = I - \eta^2 D_0^T D_0\,. \tag{4.89}$$

Since $\Omega > 0$, the following inequality holds for any (y, z_i):

$$2 y^T [P_0 B_0 + C_{i0}^T D_0] z_i \le \eta^{-2} z_i^T \Omega z_i + \eta^2 y^T [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T y \tag{4.90}$$

Using (4.89) and (4.90), we can show that the Lyapunov inequality (4.88) is satisfied if the following inequality holds:

$$P_0 A_{i0} + A_{i0}^T P_0 + C_{i0}^T C_{i0} + \eta^2 [P_0 B_0 + C_{i0}^T D_0] \Omega^{-1} [P_0 B_0 + C_{i0}^T D_0]^T < 0\,.$$

Using the Schur complement formula, we can rewrite the above inequality as

$$\begin{bmatrix} A_{i0}^T P_o + * & * & * \\ \eta B_0^T P_0 & -I & * \\ C_{i0} & \eta D_0 & -I \end{bmatrix} < 0\,, \quad \forall i\,. \tag{4.91}$$

Let the positive definite matrix P_0 be $P_0 = \Lambda^T Y \Lambda$ where Y is a solution to LMIs (3.74)-(3.76), then the above matrix inequality (4.91) can be rewrite as

$$\begin{bmatrix} \Lambda^T [(A_i + \Delta A_i(t)) Y + *] \Lambda & \eta \Lambda^T H & \Lambda^T Y (A_i + \Delta A_i(t))^T \\ \eta H^T \Lambda & -I & \eta H^T \\ (A_i + \Delta A_i(t)) Y \Lambda & \eta H & -I \end{bmatrix} < 0, \quad \forall i \tag{4.92}$$

where $\Delta A_i(t) = T_i \Pi_i(t)$. The matrix inequality (4.92) is satisfied if the following inequality holds for any nonzero vectors: $z^T = \begin{bmatrix} z_1^T & z_2^T & z_3^T \end{bmatrix}$

$$2z_1^T\Lambda^T(A_i+\Delta A_i(t))Y\Lambda z_1+2z_3^T(A_i+\Delta A_i(t))Y\Lambda z_1$$
$$+2\eta\, z_2^T H^T\Lambda\, z_1+2\eta\, z_3^T Hz_2-z_2^T z_2-z_3^T z_3<0. \qquad (4.93)$$

Lemma 4.1 implies that if $\|\Delta A_i(t)\|\le\alpha_{A_i}$, the following inequalities hold:

$$2z_1^T\Lambda^T\Delta A_i(t)\,Y\Lambda\, z_1\le c_0 z_1^T\Lambda^T\Lambda z_1+\alpha^2 c_0^{-1} z_1^T\Lambda^T Y^2\Lambda z_1 \qquad (4.94)$$

$$2_3^T\,\Delta A_i(t)Y\Lambda z_1\le\delta\, z_3^T z_3+\alpha^2\delta^{-1} z_1^T\Lambda^T Y^2 Yz_1. \qquad (4.95)$$

The previous inequalities (4.94) and (4.95) imply that for all admissible $\|\Delta A_i(t)\|\le\alpha_{A_i}$, the inequality condition (4.93) holds if

$$2z_1^T\Lambda^T A_i Y\Lambda z_1+c_0 z_1^T\Lambda^T\Lambda z_1+\alpha^2\delta^{-1}z_1^T\Lambda^T Y^2\Lambda z_1$$
$$+\alpha^2 c_0^{-1} z_1^T\Lambda^T Y^2\Lambda z_1+2z_3^T A_i Y\Lambda z_1+2\eta z_2^T H^T\Lambda z_1$$
$$+2\eta z_3^T Hz_2+\delta z_3^T z_3-z_2^T z_2-z_3^T z_3<0. \qquad (4.96)$$

This implies that (4.92) holds if the following LMI (4.97) holds

$$\begin{bmatrix}\Lambda^T(A_iY+YA_i^T+c_0I+\dfrac{\alpha_{A_i}^2}{c_0}Y^2+\dfrac{\alpha_{A_i}^2}{\delta}Y^2)\Lambda & \eta\Lambda^T H & \Lambda^T YA_i^T\\ \eta H^T\Lambda & -I & \eta H^T\\ A_iY\Lambda & \eta H & -(1-\delta)I\end{bmatrix}<0. \qquad (4.97)$$

By using Schur complement formula, the above inequality (4.97) can be rewritten as the LMI (4.74), which implies that the sliding mode dynamics (4.85) is asymptotically stable. Hence, the sliding mode dynamics (4.73) is asymptotically stable.

After the switching surface parameter matrix S is designed so that the reduced $(n-m)$ order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the the reaching mode such that the reachability condition is met. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma=Sx=0$ and maintains it there for all subsequent time. With σ of (4.77), we design an adaptive fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear adaptive switching feedback control law as the local controller.

Control rule i: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\hat{h}(t,x) - \chi_i\sigma - S(A_i + T_iT_i^T)x - \frac{1}{1-\omega}\hat{\delta}_i(t,x)\frac{\sigma}{\|\sigma\|}$$

where $$\hat{\delta}_i(t,x) = \alpha_i + \omega\|S(A_i + T_iT_i^T)x\| + (1+\omega)\sum_{k=0}^{l}\hat{\rho}_k\|x\|^k \tag{4.98}$$

$$\dot{\hat{\rho}}_k = \varepsilon_k\|\sigma\|\cdot\|x\|^k \tag{4.99}$$

and $\sigma = Sx, \omega = \sqrt{r}\|SH\|, \alpha_i > 0, \chi_i > 0$. It should be noted that (4.84) implies $\omega = \sqrt{r}\|SH\|$ $\omega = \sqrt{r}\|SH\| \le \sqrt{r}\|S\|\cdot\|H\| \le \eta\|H\|$. This and (4.78) guarantee $0 \le \omega < 1$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\hat{h}(t,x) - \sum_{i=1}^{r}\beta_i(\theta)\left(\chi_i\sigma + S(A_i + T_iT_i^T)x + \frac{1}{1-\omega}\hat{\delta}_i(t,x)\frac{\sigma}{\|\sigma\|}\right) \tag{4.100}$$

and we can establish the following theorem.

Theorem 4.6 Consider the closed-loop control system of the uncertain system (4.69) with control (4.100). Suppose that the LMIs (4.74)-(4.76) has a solution vector $(Y, c_0, c_1, c_2, \delta, \eta)$ and the linear sliding surface is given by (4.77). Then the state converges to zero.

Proof: Since Theorem 4.5 implies that the linear sliding surface (4.77) guarantees P1-P2, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T\sigma + 0.5\xi\sum_{k=0}^{l}\tilde{\rho}_k^2$ where $\xi = 1+\omega$ and $\tilde{\rho}_k = \hat{\rho}_k - \rho_k$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T\dot{\sigma} + \xi\|\sigma\|\sum_{k=0}^{l}\tilde{\rho}_k\|x\|^k$. From (4.69), (4.77), (4.100), $\|SHF(\beta)G\|$ $\le \sqrt{r}\|SH\| = \omega$,$0 \le \omega < 1$, and A2, we obtain

$$\sigma^T\dot{\sigma} = \sigma^T\sum_{i=1}^{r}\beta_i(\theta)S(A_i + T_i\Pi_i(t))x(t) + \sigma^T[I + SHF(\beta)G][u + h(t,x)]$$

$$\le \sum_{i=1}^{r} \beta_i(\theta)\sigma^T S(A_i + T_i\Pi_i(t))x(t) + \sigma^T u + \{\omega\|u\| + (1+\omega)\|h(t,x)\|\}\|\sigma\|$$

$$\le -\sum_{i=1}^{r} \beta_1(\theta)\sigma^T S(T_iT_i^T - T_i\Pi_i^T(t))x(t) - (1-\omega)\sum_{i=1}^{r} \beta_i(\theta)\chi_i\|\sigma\|^2$$

$$-\sum_{i=1}^{r} \beta_i(\theta)\alpha_i\|\sigma\| - \xi\|\sigma\|\sum_{k=0}^{l} \tilde{\rho}_k\|x\|^k$$

$$\le -\sum_{i=1}^{r} \beta_1(\theta)\|x\|^{-2} x^T(T_iT_i^T - T_i\Pi_i^T(t))x\|\sigma\|^2 - (1-\omega)\sum_{i=1}^{r} \beta_i(\theta)\chi_i\|\sigma\|^2$$

$$-\sum_{i=1}^{r} \beta_i(\theta)\alpha_i\|\sigma\| - \xi\|\sigma\|\sum_{k=0}^{l} \tilde{\rho}_k\|x\|^k .$$

From A3 and $\varepsilon_g = \|x\|^{-2} x^T(T_iT_i^T - T_i\Pi_i^T(t))x \ge 0$, this implies that $\dot{E}_g \le -\sum_{i=1}^{r} \beta_1(\theta)\varepsilon_{i5}\|\sigma\|^2 - (1-\omega)\sum_{i=1}^{r} \beta_i(\theta)\chi_i\|\sigma\|^2 - \sum_{i=1}^{r} \beta_i(\theta)\alpha_i\|\sigma\| \le 0$ which indicates that $E_g \in L_2 \cap L_\infty$, $\dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 4.3 Theorem 4.5 and 4.6 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Obtain $B = \frac{1}{r}\sum_{i=1}^{r} B_i$ and $H = \frac{1}{2}\left[(B - B_1), \cdots, (B - B_r)\right]$ for given B_i.

Step 2: Check that (A_i, B) is stabilization. If not, exit.

Step 3: Find a solution vector $(Y, c_0, c_1, c_2, \delta, \eta)$ to LMI (4.74)-(4.76).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (4.77).

Step 5: The controller is given by (4.100).

4.3.6 Numerical Examples II

Example 4.5 To demonstrate the performance of the proposed adaptive control design method, consider the following two-rule fuzzy model from a vertical take-off and landing (VTOL) helicopter model [55]

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = (A_1 + T_1\Pi_1(t))x + B_1[u + h(t,x)]$$

Plant Rule2: IF x_1 is about ± 2, THEN

$$\dot{x} = (A_2 + T_2\Pi_2(t))x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3181 & -0.7070 & 1.4100 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} -0.0366 & 0.0271 & .0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.4181 & -0.7070 & 1.4300 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.6446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},$$

$$T_1 = T_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}, \Pi_1(t) = \Pi_2(t) = \begin{bmatrix} 0 & \sin t & 0 & \sin t \end{bmatrix},$$

$$h(t,x) = d(t) + \begin{bmatrix} 0.9\sin 3t & 0.9\sin 3t \end{bmatrix}^T, \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1-1)})}{1 + e^{-14(x_1+1)}}, \beta_2 = 1 - \beta_1. \quad (4.101)$$

Note that B_1 and B_2 are not matched and almost existing VSS-based fuzzy control design methods cannot be applied to the above system (4.101). Via LMI optimization with (4.101), we can obtain the sliding surface $\sigma = Sx$.

By setting $\hat{h}(t,x) = \begin{bmatrix} 0.9\sin 3t & 0.9\sin 3t \end{bmatrix}^T$ and $\chi_i = 1, \alpha_i = 0.0001, r = 2, l = 1, \varepsilon_k = 2$, and $t_{sampling} = 0.01\sec$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = \begin{bmatrix} -0.9\sin 3t & -0.9\sin 3t \end{bmatrix}^T - \sigma - S(A_1 + T_1T_1^T)x - \frac{1}{1-\omega}\hat{\delta}_1 \operatorname{sgn}(\sigma).$$

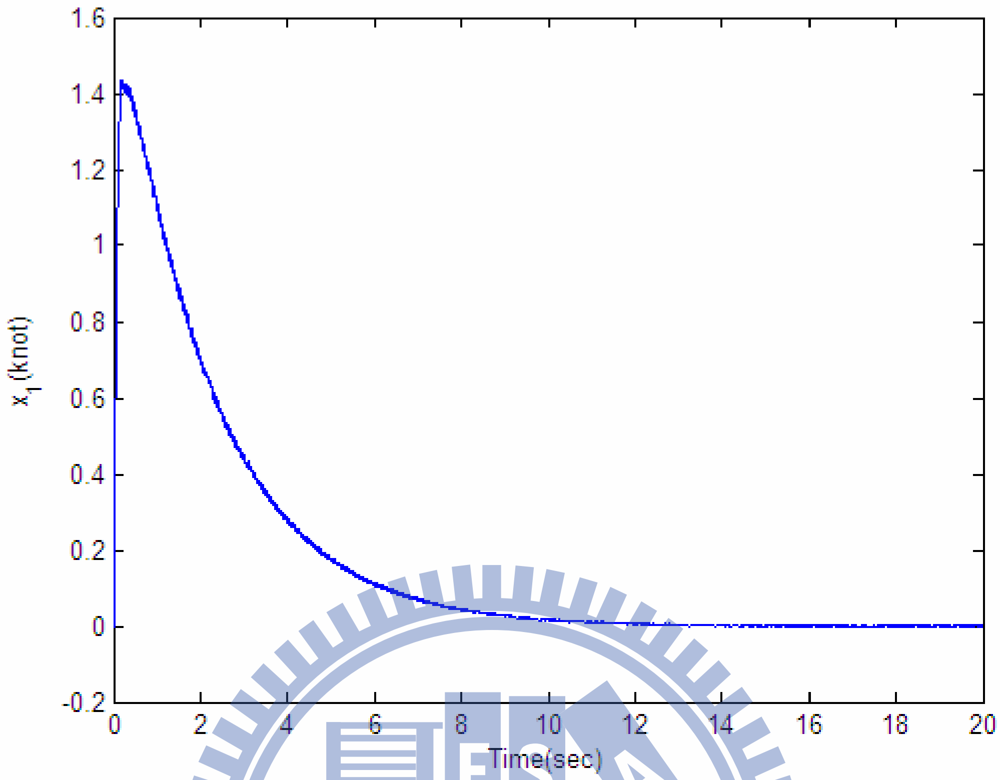
Control Rule 2: IF x_1 is about ± 2, THEN

$$u(t) = \begin{bmatrix} -0.9\sin 3t & -0.9\sin 3t \end{bmatrix}^T - \sigma - S(A_2 + T_2 T_2^T)x - \frac{1}{1-\omega}\hat{\delta}_2 \operatorname{sgn}(\sigma).$$

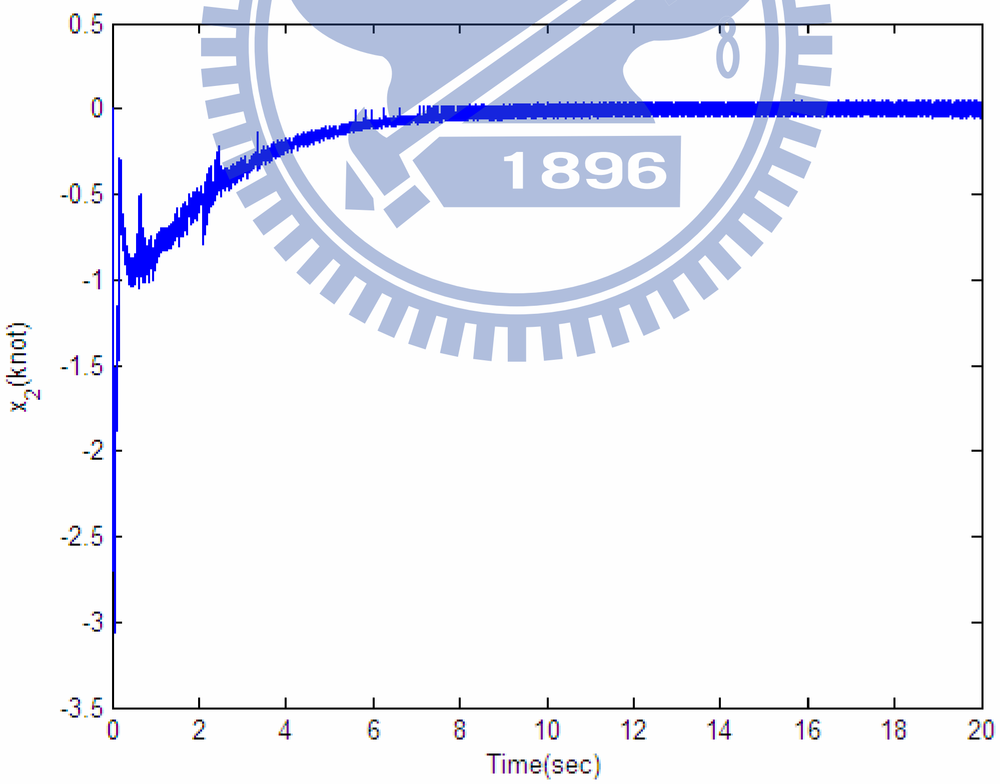
The final controller inferred as the weighted average of each local controller is given by

$$u(t) = \begin{bmatrix} -0.9\sin 3t & -0.9\sin 3t \end{bmatrix}^T - \sum_{i=1}^{r} \beta_i(\theta)\left[\sigma + S(A_i + T_i T_i^T)x + \frac{1}{1-\omega}\hat{\delta}_i \operatorname{sgn}(\sigma)\right]. \quad (4.102)$$

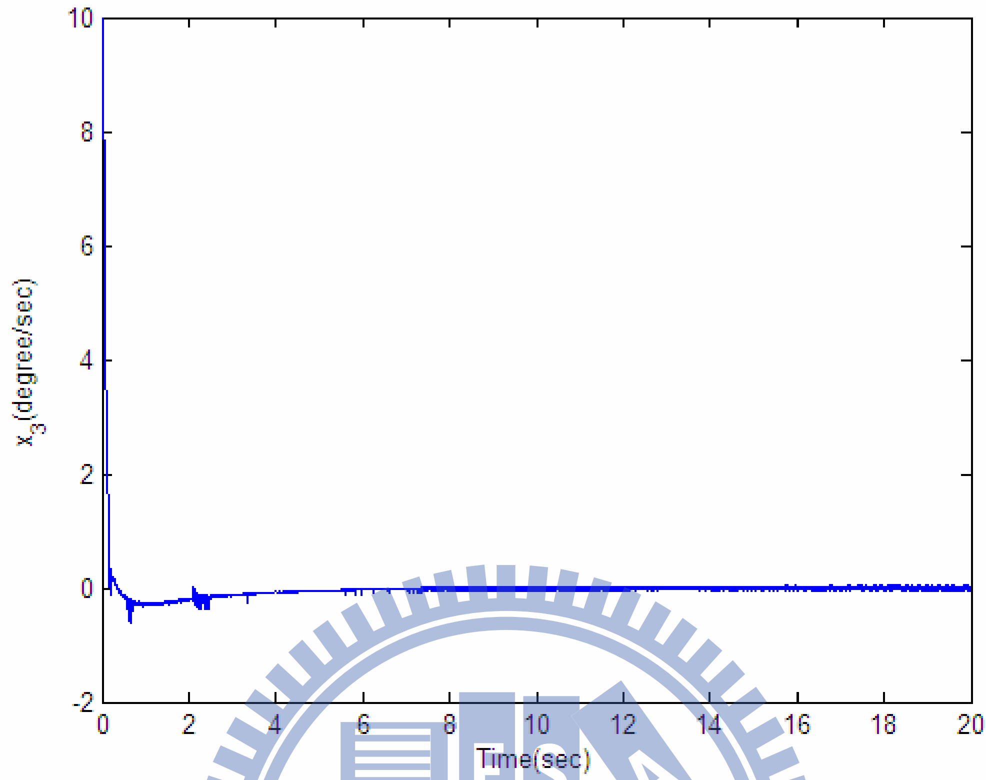
To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (4.101) with nonzero $d(t)$. We assume that $d(t) = \begin{bmatrix} 0.25x_1 \sin 2\pi t - 0.1\operatorname{sgn}(x_4) & 0.25x_1 \sin 2\pi t - 0.1\operatorname{sgn}(x_4) \end{bmatrix}^T$. The time histories of the state, $\hat{\rho}_k$, the sliding variable σ, and the input (4.102) are shown in Figure 4.8 when $x_1(0) = x_2(0) = x_4(0) = 0$, $x_3(0) = 10$. In Figure 4.8, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 4.8, the proposed controller is applicable to uncertain fuzzy systems with mismatched parameter uncertainties in the state matrix and unknown norm-bounded external disturbances. The control performances of the proposed controller are satisfactory for the two-rule fuzzy model (4.101). Note that almost existing VSS-based fuzzy control system design methods cannot be applied to the two-rule fuzzy model (4.101) because B_1 is distinct from B_2.

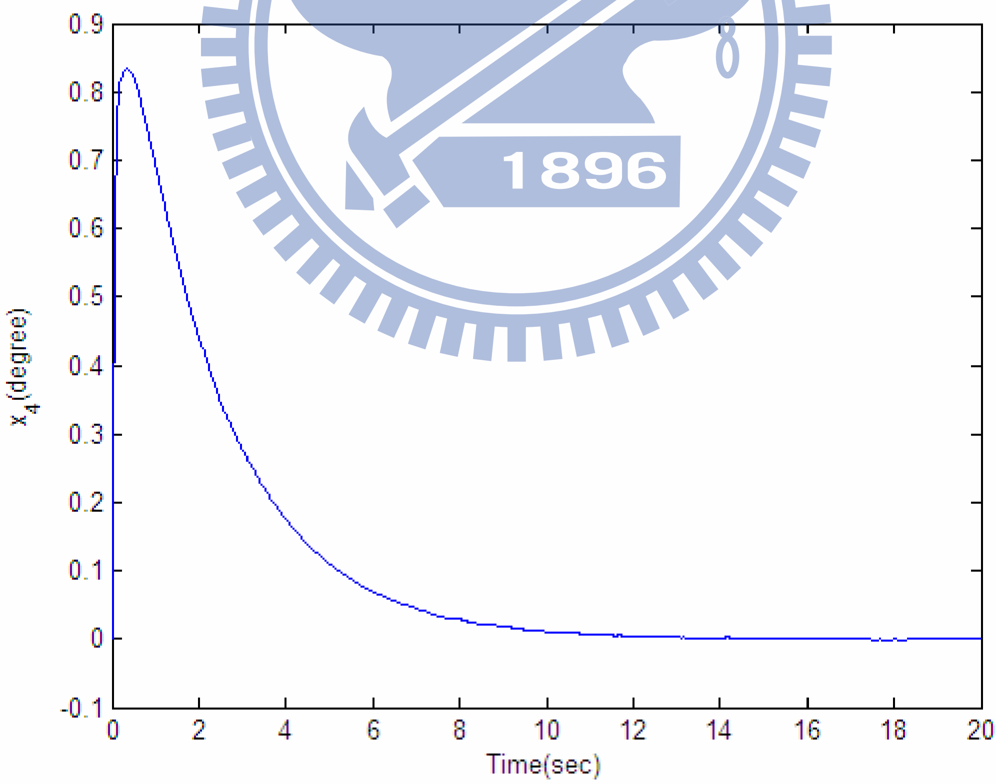


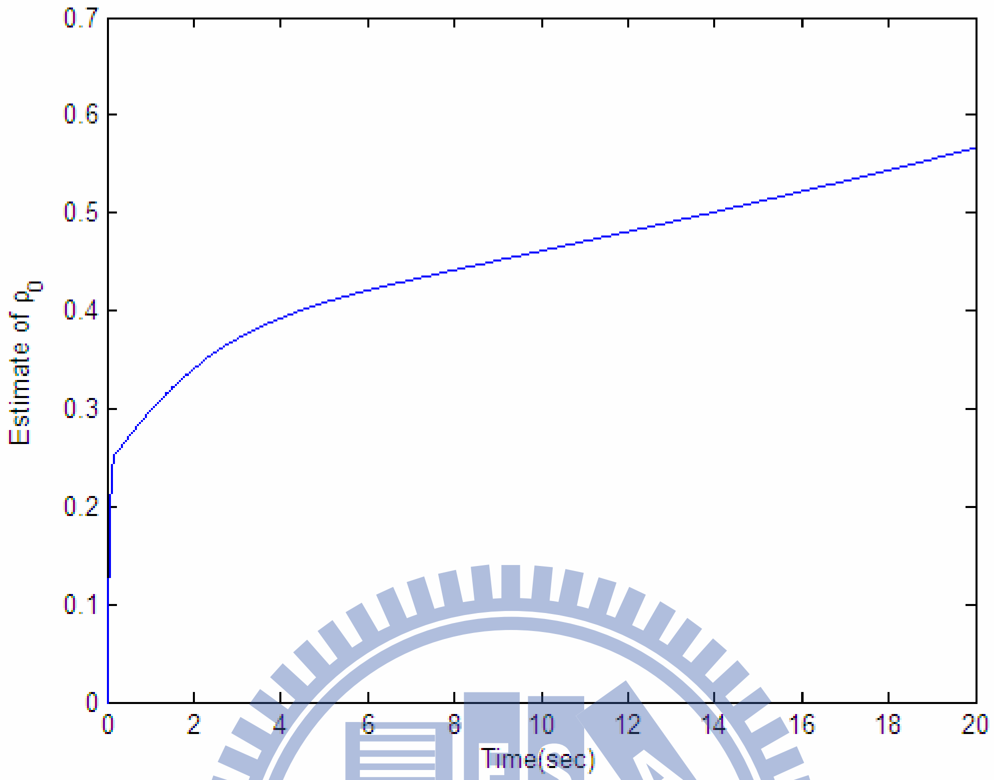
x1(knot)
Time(sec)



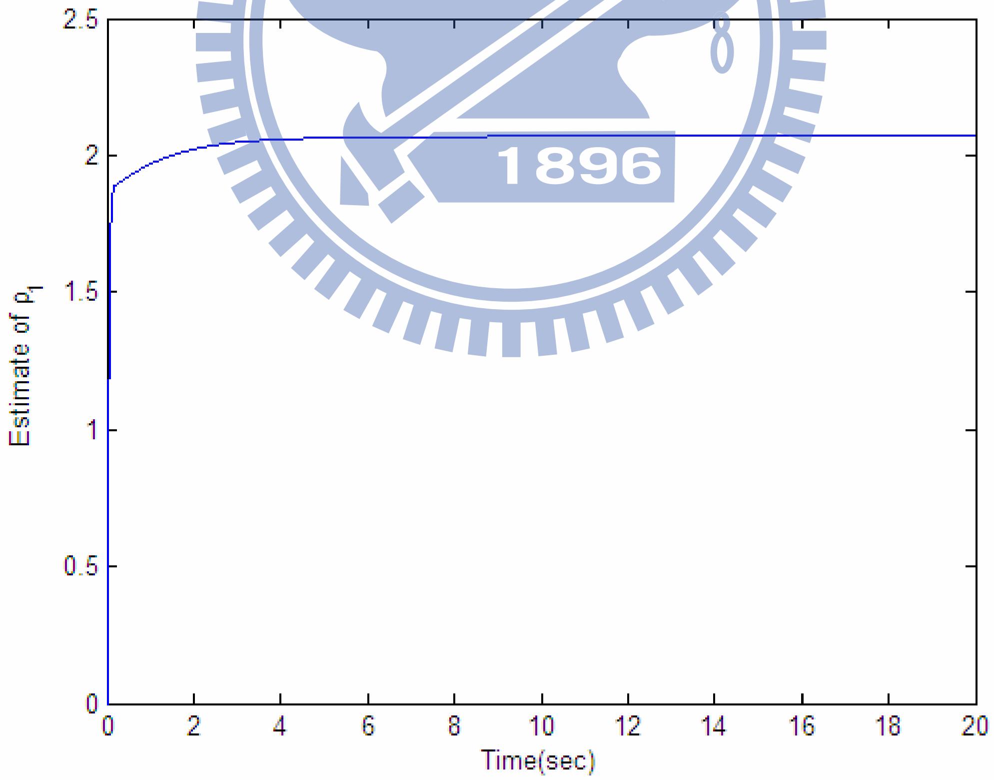
x2(knot)
Time(sec)


x3(degree/sec)
Time(sec)

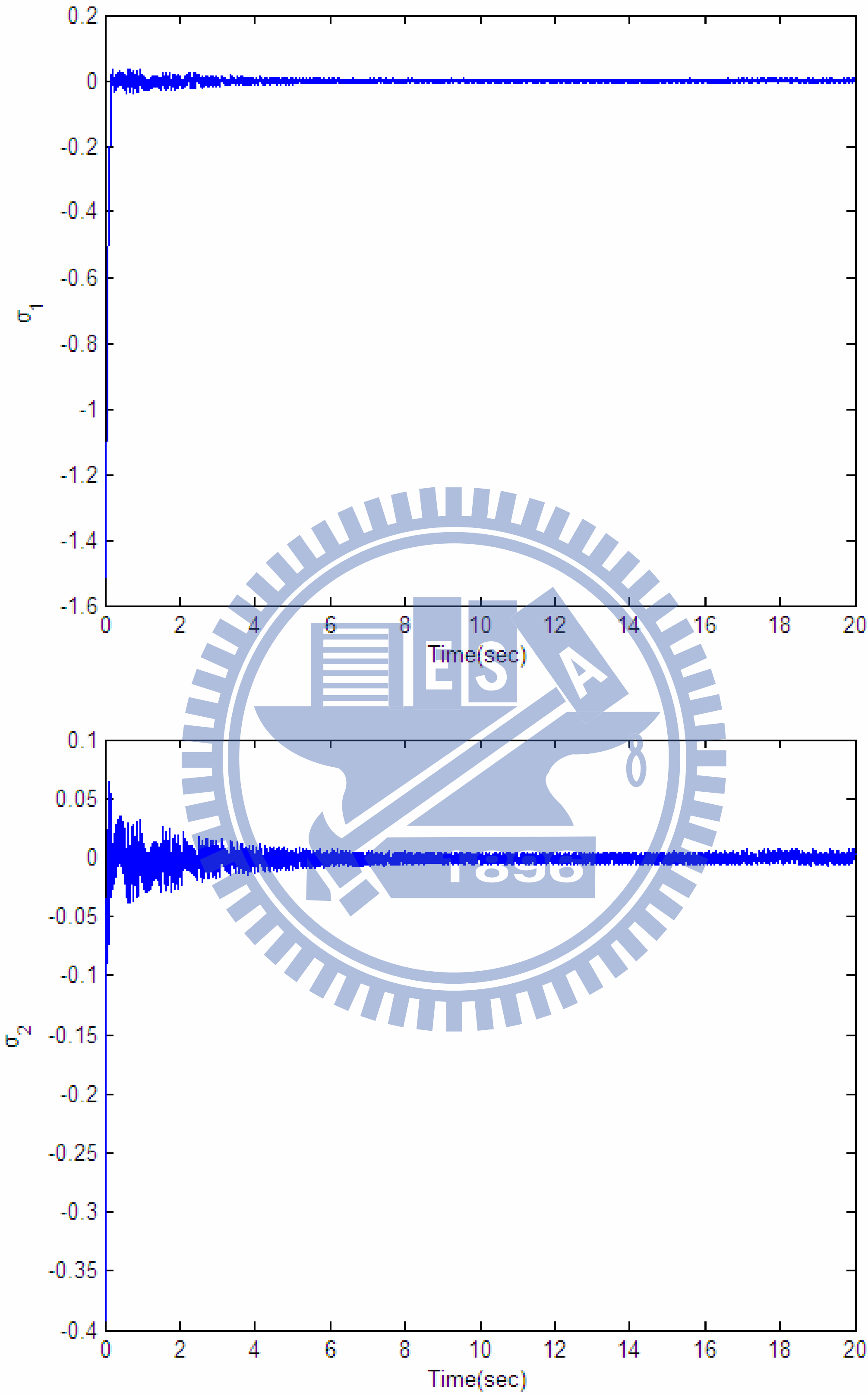

x4(degree)
Time(sec)



Estimate of ρ_0
Time(sec)



Estimate of ρ_1
Time(sec)


0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
σ1
0
2
4
6
8
10
12
14
16
18
20
Time(sec)
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
σ2
0
2
4
6
8
10
12
14
16
18
20
Time(sec)

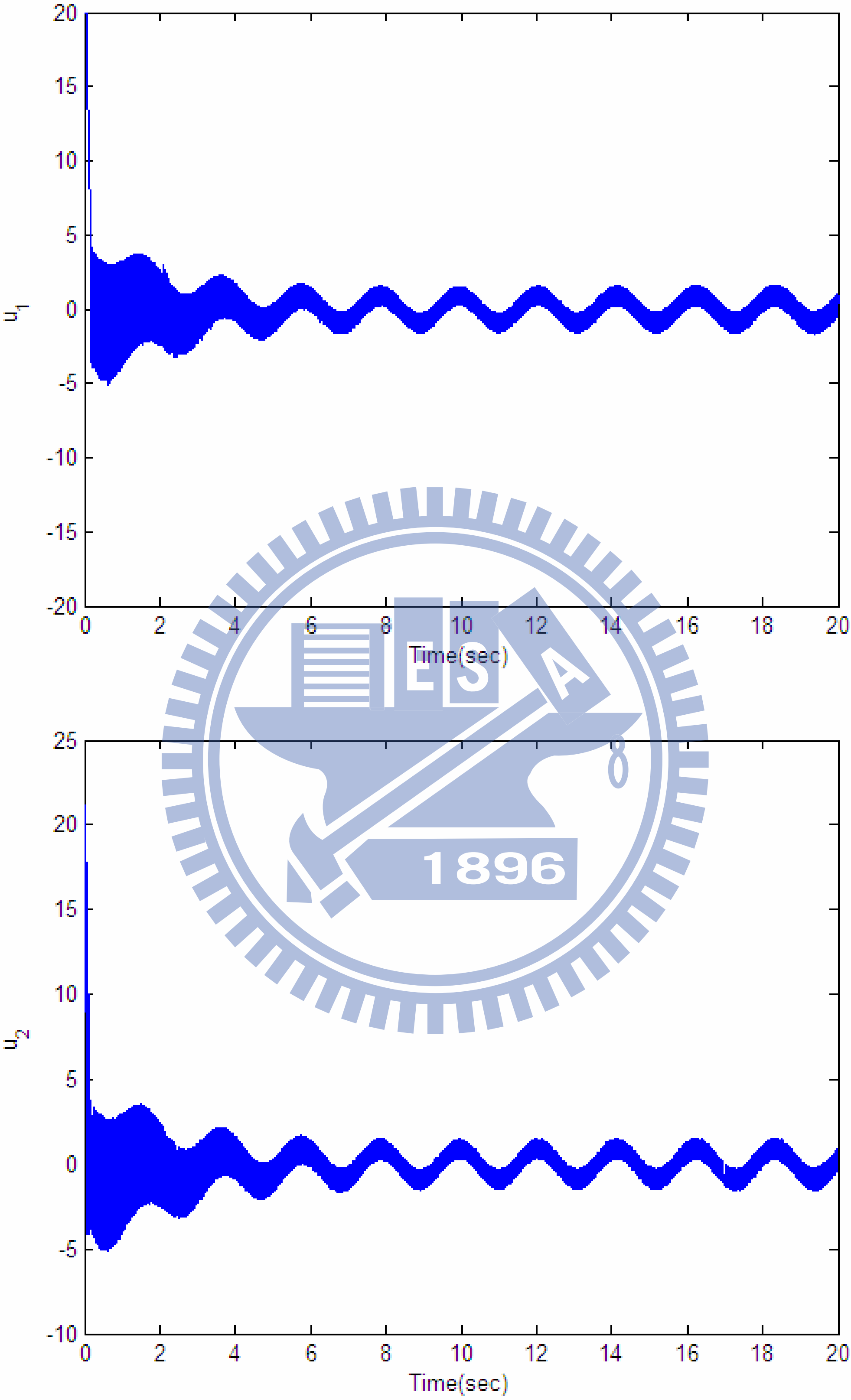


Figure 4.8 Simulation results with $x_1(0) = x_2(0) = x_4(0) = 0,\ x_3(0) = 10.$

Example 4.6 For the special case of $\Delta A_i(t) \equiv 0$, the robust adaptive controller design is proposed in [64]. Consider the following inverted pendulum on a cart

$$\dot{x}_1 = x_2,\ \dot{x}_2 = \frac{1}{l\psi}(3g\sin x_1 - 3a\cos x_1[u + d(t) + \phi]),\ \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -\frac{1}{\psi}(1.5mag\sin 2x_1 - 4a[u + d(t) + \phi]) \tag{4.103}$$

where x_1 is the angle (rad) of the pendulum from the vertical, $x_2 = \dot{x}_1$, x_3 is the displacement (m) of the cart, $x_4 = \dot{x}_3$, $\psi = 4 - 3ma\cos^2 x_1$, $\phi = mlx_2^2 \sin x_1$, u is the input, and $d(t)$ is related to external disturbances which may be caused by the frictional force. $a = 1/(m+M)$, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, $g = 9.8m/s^2$ is the gravity constant. We set $M = 9kg$, $m = 1kg$, $l = 1m$. We assume that $d(t)$ is bounded as $|d(t)| \le \rho_0 + \rho_1\|x\|$ where ρ_0 and ρ_1 are unknown constants. To design the fuzzy controller (40), we must have a fuzzy model. Here, we approximate the system (4.103) by the following two-rule fuzzy model.

Plant Rule 1: IF x_1 is about 0, THEN

$$\dot{x} = A_1 x + B_1[u + h(t,x)]$$

Plant Rule2: IF x_2 is about $\pm 60^\circ (\pm\pi/3\ rad)$, THEN

$$\dot{x} = A_2 x + B_2[u + h(t,x)]$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.9459 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.7946 & 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -0.0811 \\ 0 \\ 0.1081 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3097 & 0 & 0 & 0 \end{bmatrix}$,

$$B_2 = \begin{bmatrix} 0 \\ -0.0382 \\ 0 \\ 0.1019 \end{bmatrix}, \ h(t,x) = d(t) + x_2^2 \sin x_1, \ \beta_1 = \frac{1 - 1/(1 + e^{-14(x_1 - \pi/8)})}{1 + e^{-14(x_1 + \pi/8)}}, \ \beta_2 = 1 - \beta_1. \quad (4.104)$$

Via LMI optimization with (4.104), we can obtain the sliding surface $\sigma = Sx$. By setting $\hat{h}(t,x) = x_2^2 \sin x_1$, $\chi_i = 5, \alpha_i = 1$, $r = 2$, $l = 1$, $\varepsilon_k = 0.001$, and $t_{sampling} = 0.01\text{sec}$, we can obtain the following nonlinear controller:

Control Rule 1: IF x_1 is about 0, THEN

$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_1 x - \frac{1}{1-\omega} \hat{\delta}_1 \operatorname{sgn}(\sigma).$$

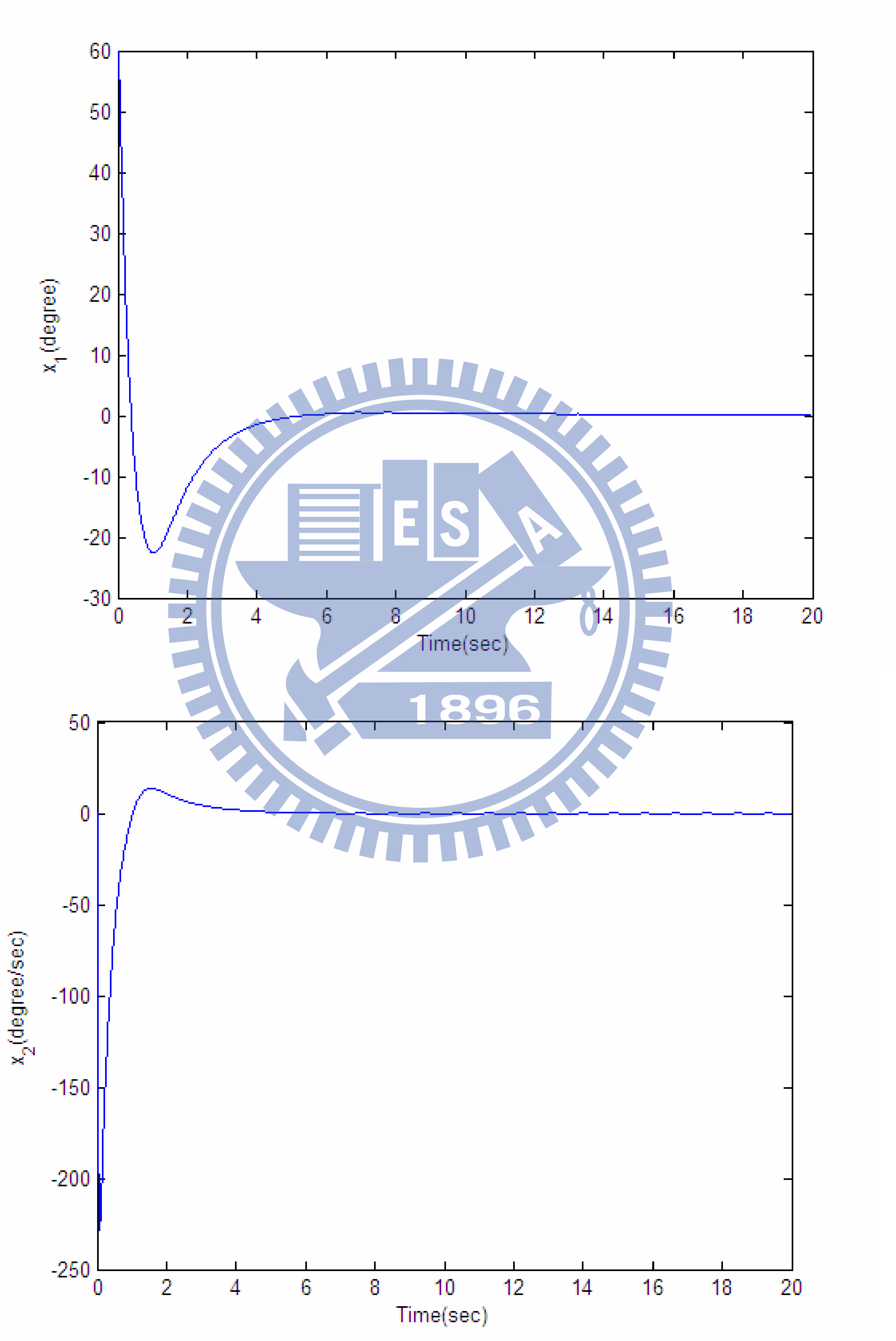
Control Rule 2: IF x_1 is about $\pm 60^\circ (\pm\pi/3 \ rad)$, THEN

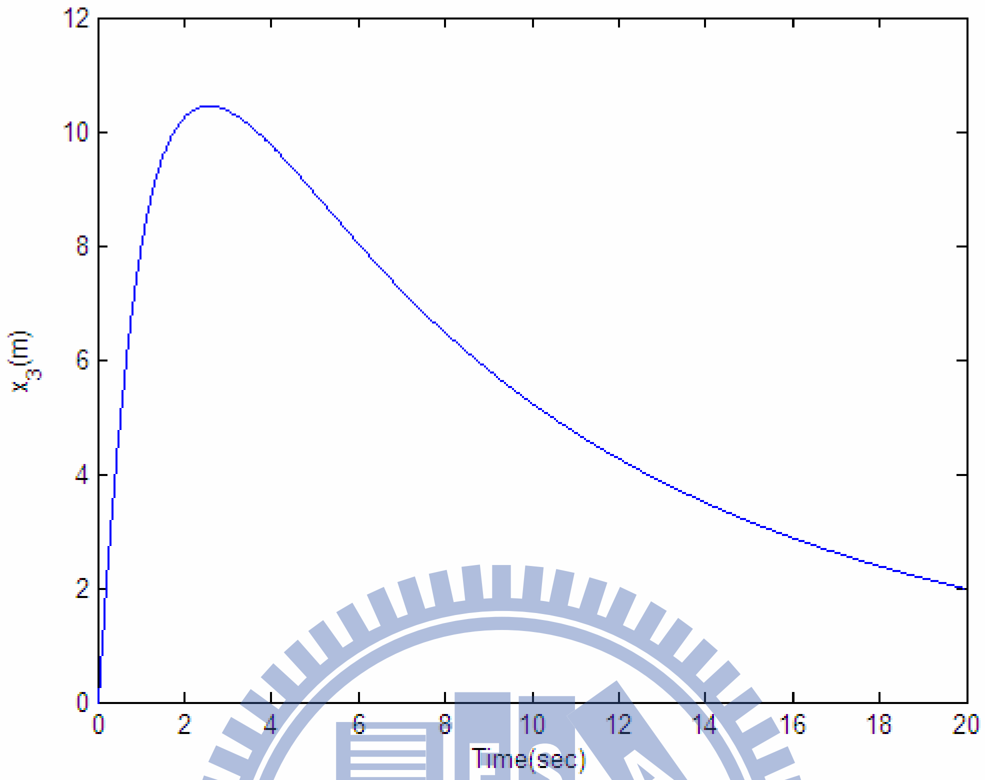
$$u(t) = -x_2^2 \sin x_1 - 5\sigma - SA_2 x - \frac{1}{1-\omega} \hat{\delta}_2 \operatorname{sgn}(\sigma).$$

The final controller inferred as the weighted average of each local controller is given by

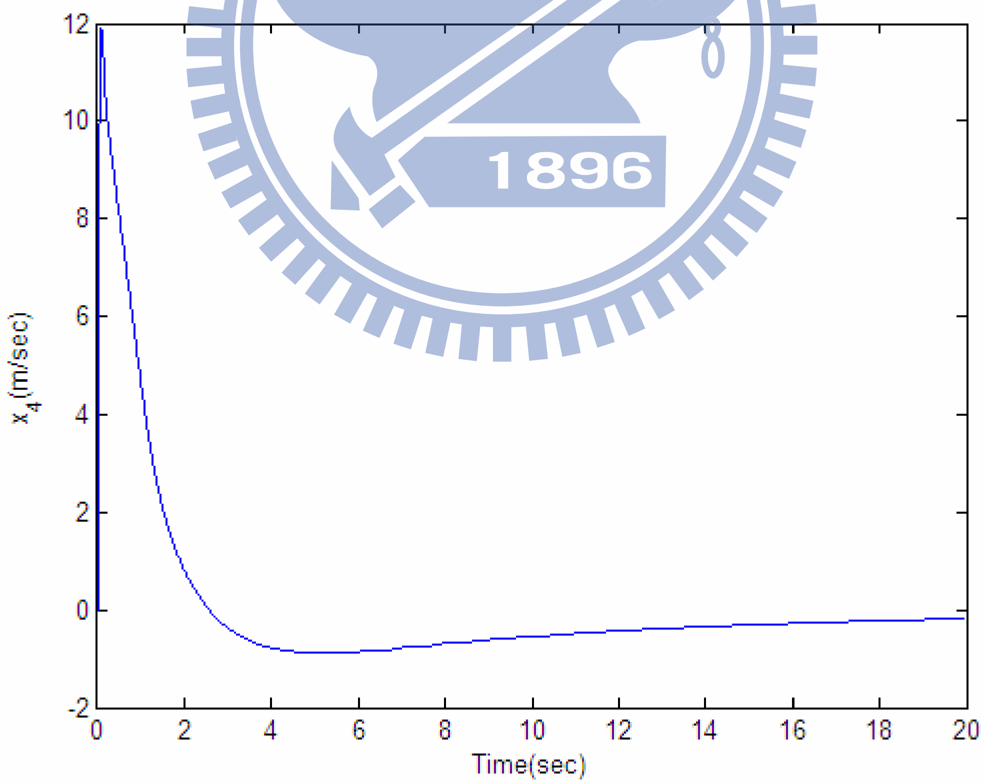
$$u(t) = -x_2^2 \sin x_1 - \sum_{i=1}^{r} \beta_i(\theta) \left[5\sigma + SA_i x + \frac{1}{1-\omega} \hat{\delta}_i \operatorname{sgn}(\sigma) \right]. \quad (4.105)$$

To assure the effectiveness of our fuzzy controller, we apply the controller to the two-rule fuzzy model (4.104) with nonzero $d(t)$. We assume that $d(t) = x_1 \sin 2\pi t - 0.5\operatorname{sgn}(x_4)$. The time histories of the state, $\hat{\rho}_k$, the sliding variable σ , and the input (4.105) are shown in Figure 4.9 when $x_1(0) = 60^\circ (2\pi/9 \ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$. In Figure 4.9, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 4.9, the control performances of the proposed controller are also satisfactory for the two-rule fuzzy model (4.104).

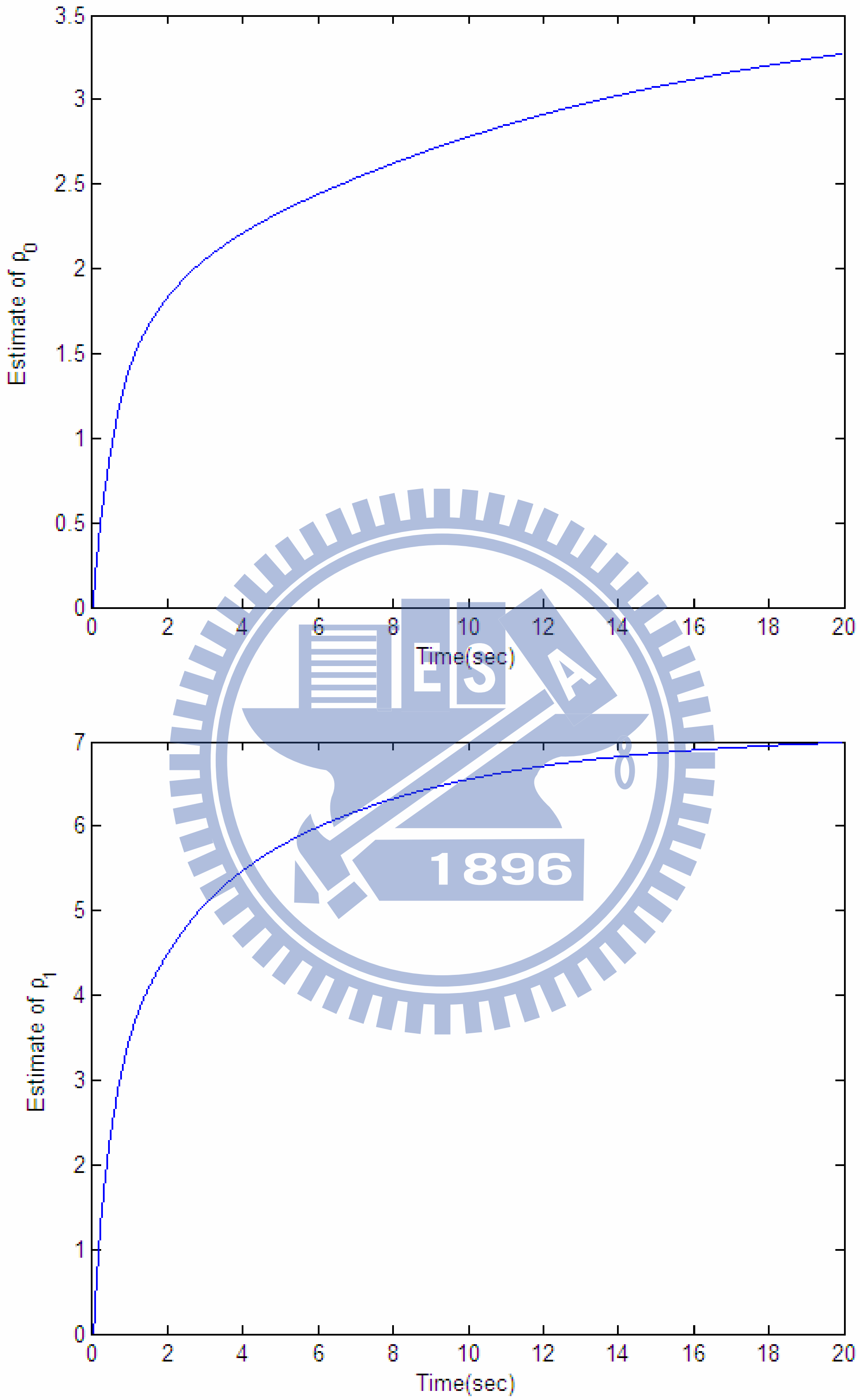




x3(m)
Time(sec)



x4(m/sec)
Time(sec)



Estimate of ρ0
Time(sec)
Estimate of ρ1
Time(sec)

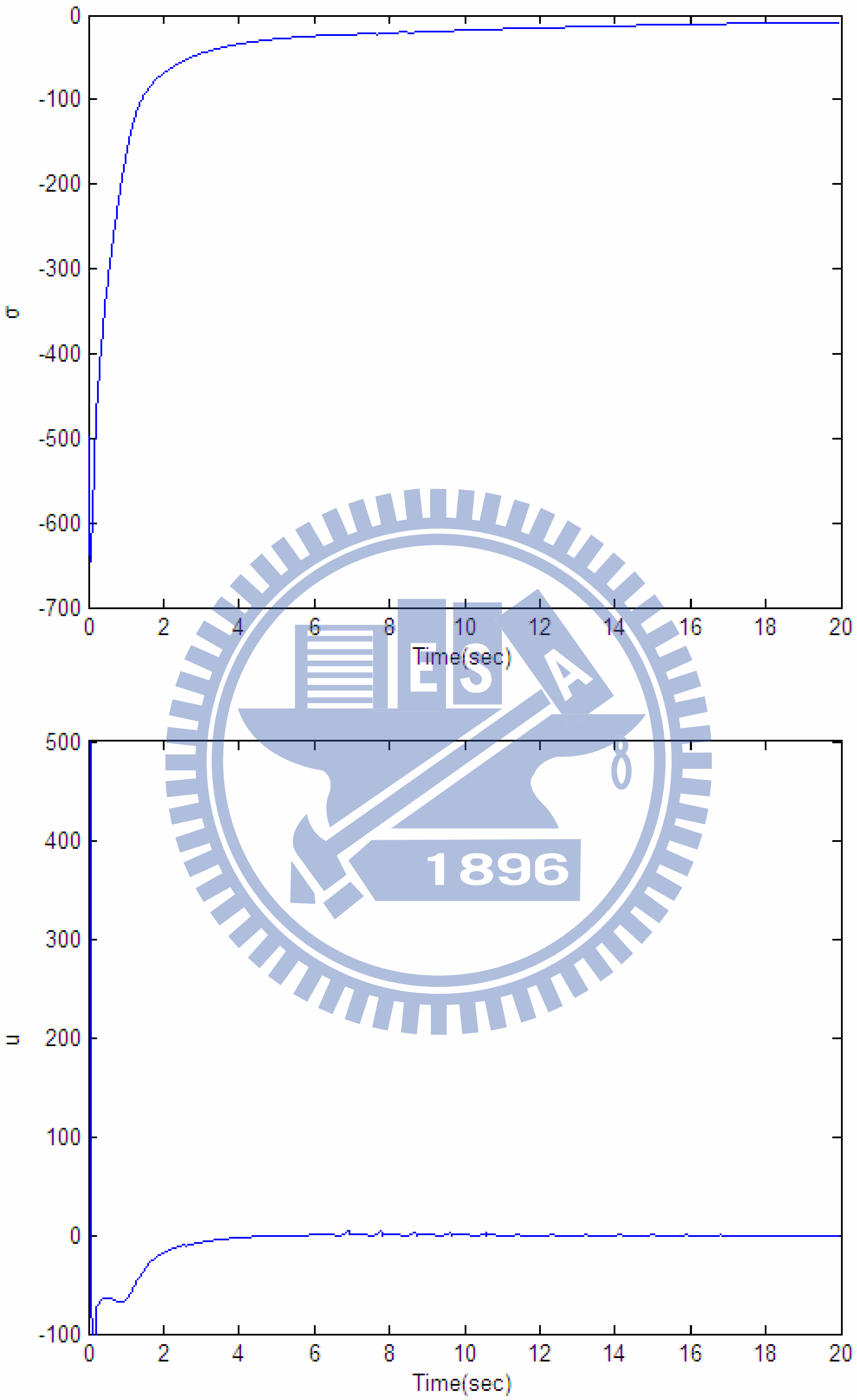


Figure 4.9 Simulation results with $x_1(0) = 60^\circ (2\pi/9\ rad)$, $x_2(0) = x_3(0) = x_4(0) = 0$.

4.4 Robust Adaptive Control for Mismatched T-S Fuzzy Time-Delay Systems

In this section, system formulation for the uncertain T-S fuzzy time-delay model is described in Section 4.4.1. A robust adaptive control method via LMI is proposed in Section 4.4.2. Some examples are used to illustrate the effectiveness of the proposed methods and to compare with the existing methods in Section 4.4.3.

4.4.1 System Formulation

The T-S fuzzy model is described by fuzzy IF-THEN rules, which represent local linear input-output relations of nonlinear systems. The ith rule of the T-S fuzzy time-delay model is of the following form:

Plant Rule i: IF θ_1 is μ_{i1} and … and θ_s is μ_{is}, THEN

$$\dot{x}(t) = A_i x(t) + A_{\tau i} x(t - d(t)) + B_i u(t), \quad x(t) = \psi(t), \quad t \in [-\tau, 0]$$

where $\psi(t)$ is the initial condition, $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $A_i \in R^{n \times n}$ are the state matrices, $A_{\tau i} \in R^{n \times n}$ are the delayed state matrices, $B_i \in R^{n \times m}$ are the input matrices, $\theta_j (j = 1, \cdots, s)$ are the premise variables, s is the number of the premise variables, $\mu_{ij} (i = 1, \cdots, r; j = 1, \cdots, s)$ are the fuzzy sets that are characterized by membership function, r is the number of the IF-THEN rules. The time-varying delay $d(t)$ is bounded as $d(t) \le \tau$. The overall fuzzy model achieved by fuzzy synthesizing of each individual plant rule is given by

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)[A_i x(t) + A_{\tau i} x(t - d(t)) + B_i u(t)], \quad x(t) = \psi(t), \quad t \in [-\tau, 0]$$

where $\theta = [\theta_1, \cdots, \theta_s]$, $\beta_i(\theta) = \omega_i(\theta) / \sum_{j=1}^{r} \omega_j(\theta), \omega_i : R^s \to [0,1], i = 1, \cdots, r$ is the membership function of the system with respect to plant rule i. The function $\beta_i(\theta)$ can be regarded as the normalized weight of each IF-THEN rule and it satisfies that $\beta_i(\theta) \geq 0$, $\sum_{i=1}^{r} \beta_i(\theta) = 1$. To take into account parameter uncertainties and external disturbances, we consider the following uncertain T-S fuzzy time-delay model:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)[(A_i + \Delta A_i(t))x(t) + (A_{\tau i} + \Delta A_{\tau i}(t))x_d(t) + B_i(u(t) + h(t,x,x_d))],$$

$$x(t) = \psi(t),\ t \in [-\tau, 0] \tag{4.106}$$

where $x_d(t) = x(t - d(t))$, $\Delta A_i(t)$ represents the parameter uncertainties in A_i, $\Delta A_{\tau i}(t)$ represents the parameter uncertainties in $A_{\tau i}$, $h(t,x,x_d) \in R^m$ denotes external disturbances. We will assume that the following assumptions are satisfied:

A1: $B_1 = B_2 = \ldots = B_r := B$ and $\text{rank}(B) = m$.

A2:The function $h(t,x,x_d)$ is unknown but bounded as $\|h(t,x,x_d)\| \leq \sum_{k=0}^{p} \rho_{dk}\|x\|^k + \sum_{k=0}^{q} \delta_{dk}\|x_d\|^k$ where $\rho_{d0},\cdots,\rho_{dp}$ and $\delta_{d0},\cdots,\delta_{dq}$ are unknown constants, and p, q are known positive integers.

A3: The time delay $d(t)$ is unknown but bounded as $d(t) \leq \tau$ and $\dot{d}(t) \leq d_m < 1$ where τ and d_m are known constants.

A4: $\Delta A_i(t)$ and $\Delta A_{\tau i}(t)$ are of the form $T_i\Pi_i(t)$ where $\Pi_i(t)$ is a known time-varying matrix but bounded as $\|\Pi_i(t)\| \leq 1$.

Using the above assumptions, the uncertain T-S fuzzy model (4.106) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \beta_i(\theta)[(A_i + T_i\Pi_i(t))x(t) + (A_{\tau i} + T_i\Pi_i(t))x_d(t) + Bh(t,x,x_d)] + Bu(t),$$

$$x(t) = \psi(t),\ t \in [-\tau, 0] \tag{4.107}$$

A large number of examples in the literature and various mechanical systems, such as motors and robots, fall into the special cases of the above model (4.96), as reported in [44], [56-60]. The above model (4.107) also involves the uncertain time-delay system models considered in the previous SMC design methods[44], [56-60]. The symbol $*$ will be used in some matrix expressions to induce a symmetric structure. For given symmetric matrices K and L of appropriate dimensions, the following holds:

$$\begin{bmatrix} K+X+* & * \\ Z & L \end{bmatrix} = \begin{bmatrix} K+X+X^T & Z^T \\ Z & L \end{bmatrix}$$

When no confusion arises, the arguments t, x, x_d, θ, etc… can be omitted for brevity.

4.4.2 Adaptive Control Design via LMI

The SMC design is decoupled into two independent tasks of lower dimensions. The first is concerned with the design of a sliding surface for the sliding mode such that the reduced-order sliding mode dynamics satisfies the design specifications such as stabilization, tracking, regulation, etc. The second involves choosing a switching feedback control for the reaching mode so that it can drive the system's dynamics into the switching surface [33]. We first design a sliding surface that guarantees asymptotic stability of the reduced-order sliding mode dynamics using LMIs.

Defining a nonsingular transformation matrix M and the associated vector $v = Mx$ such that

$$M = \begin{bmatrix} (\Lambda^T Y \Lambda)^{-1}\Lambda^T \\ (B^T Y^{-1} B)^{-1} B^T Y^{-1} \end{bmatrix} = \begin{bmatrix} V \\ S \end{bmatrix}, \; v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Vx \\ Sx \end{bmatrix} = Mx \tag{4.108}$$

where $v_1 \in R^{n-m}, v_2 \in R^m$. Then we can easily see that $M^{-1} = [Y\Lambda, B]$ and $v_2 = \sigma$. By the above transformation we can obtain, we can transform (4.107) into the following regular form:

$$\dot{v} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} v + \begin{bmatrix} \bar{A}_{\tau 11} & \bar{A}_{\tau 12} \\ \bar{A}_{\tau 21} & \bar{A}_{\tau 22} \end{bmatrix} v_d + \begin{bmatrix} 0 \\ I \end{bmatrix} \left(u + \sum_i \beta_i h_i \right) \tag{4.109}$$

where $v_d = v(t-d(t))$ and

$$\overline{A}_{11} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,, \overline{A}_{12} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) B \,,$$

$$\overline{A}_{21} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,, \overline{A}_{22} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_i + \mathrm{T}_i \Pi_i(t)) B \,,$$

$$\overline{A}_{\tau 11} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,, \overline{A}_{\tau 12} = \sum_{i=1}^{r} \beta_i P_0^{-1} \Lambda^T (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) B \,,$$

$$\overline{A}_{\tau 21} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) Y\Lambda \,, \overline{A}_{\tau 22} = \sum_{i=1}^{r} \beta_i B^T Y^{-1} (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) B \,.$$

Thus, from the above regular form, by setting $\dot{\sigma} = \sigma = 0$, we can obtain the following sliding mode dynamics :

$$\dot{\alpha} = A_o \alpha + A_d \alpha_d \tag{4.110}$$

where $\alpha = v_1, \alpha_d = v_1(t-d(t)),\ A_0 = \overline{A}_{11}$, and $A_d = \overline{A}_{\tau 11}$.

Theorem 4.7 Let us consider the sliding mode dynamics (4.110). If the matrix $\Lambda \in R^{n\times(n-m)}$ is any full rank matrix such that $B^T \Lambda = 0, \Lambda^T \Lambda = I$, the matrices $Y \in R^{n\times n}, K \in R^{(n-m)\times(n-m)}, X_i \in R^{(n-m)\times(n-m)}$ and $Z_i \in R^{(n-m)\times(n-m)}$ are decision variables, and $*$ represents blocks that are readily inferred by symmetry such that the following LMI holds:

$$Y > 0, \qquad K \geq 0$$

$$\begin{bmatrix} \mathrm{N}_{11} & * & * & * \\ \mathrm{N}_{21} & \mathrm{N}_{22} & * & * \\ \tau X_i & \tau Z_i & -\tau\Lambda^T Y\Lambda & 0 \\ \mathrm{N}_{41} & \mathrm{N}_{42} & 0 & -\tau\Lambda^T Y\Lambda \end{bmatrix} < 0, \quad \forall i \tag{4.111}$$

where $\mathrm{N}_{11} = K + \Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda + X_i + *,$

$\mathrm{N}_{21} = \Lambda^T Y (A_{\tau i} + \mathrm{T}_i \Pi_i(t))^T \Lambda - X_i + Z_i^T,\ \mathrm{N}_{22} = -(1-d_m)K - Z_i - Z_i^T,$

$\mathrm{N}_{41} = \tau\Lambda^T (A_i + \mathrm{T}_i \Pi_i(t)) Y\Lambda,\ \mathrm{N}_{42} = \tau\Lambda^T (A_{\tau i} + \mathrm{T}_i \Pi_i(t)) Y\Lambda.$

Suppose that the LMI (4.111) have a solution (Y, K, X_i, Z_i) for given $A_i, A_{\tau i}, B, d_m, \tau$, then, there exists a linear sliding surface parameter matrix S and the sliding surface

$$\sigma(x) = Sx = (B^T Y^{-1} B)^{-1} B^T Y^{-1} x = 0 \tag{4.112}$$

will guarantee that the sliding mode dynamics (4.110) is asymptotically stable.

Proof: Let us define a Lyapunov-Krasovskii function (LKF) as

$$V(t) = \alpha^T(t) P_0 \alpha(t) + \int_{t-d}^{t} \alpha^T(s) F \alpha(s) ds + \int_{-\tau}^{0} \int_{t+\eta}^{t} \dot{\alpha}^T(s) P_0 \dot{\alpha}(s) ds d\eta$$

where $P_0 = \Lambda^T Y \Lambda \in R^{n \times n}$ and $F \in R^{n \times n}$ are solution matrices for the LMIs (4.111). It should be noted that a large number of previous methods such as the methods given in [42,43], have used similar Lyapunov-Krasovskii functions to obtain less-conservative stability conditions by exploiting information on the upper bounds of delay and its time derivative. None of the previous SMC design methods [44], [56-60] have used the term $\int_{-\tau}^{0} \int_{t+\eta}^{t} \dot{\alpha}^T(s) P_0 \dot{\alpha}(s) ds d\eta$ in stability analysis. The time derivative of the Lyapunov-Krasovskii function is given by

$$\dot{V}_g = 2\alpha^T P_0 (A_0 \alpha + A_d \alpha_d) + \alpha^T F \alpha - (1 - \dot{d}) \alpha_d^T F \alpha_d + \tau \dot{\alpha}^T P_0 \dot{\alpha} - \int_{t-\tau}^{t} \dot{\alpha}^T(s) P_0 \dot{\alpha}(s) ds.$$

By using (4.110) and the Newton-Leibniz formula $\alpha - \alpha_d - \int_{t-d}^{t} \dot{\alpha}(s) ds = 0$, we have

$$\dot{V}_g = 2\alpha^T P_0 (A_0 \alpha + A_d \alpha_d) + \alpha^T F \alpha - (1 - \dot{d}) \alpha_d^T F \alpha_d + \tau (A_0 \alpha + A_d \alpha_d)^T P_0 (A_0 \alpha + A_d \alpha_d)$$

$$- \int_{t-\tau}^{t} \dot{\alpha}^T(s) P_0 \dot{\alpha}(s) ds + 2(\alpha^T X^T + \alpha_d^T Z^T)(\alpha - \alpha_d - \int_{t-\tau}^{t} \dot{\alpha}(s) ds)$$

where $X = \sum \beta_i X_i$ and $Z = \sum \beta_i Z_i$. By using the inequality $2x^T y \le x^T H x + y^T H^{-1} Y$, where x and y are any vectors with appropriate dimensions and $H > 0$, we can obtain

$$2[\alpha^T(t)X^T + \alpha_d^T(t)Z^T]\int_{t-\tau}^{t}\dot{\alpha}(s)ds \le \tau[\alpha^T(t)X^T + \alpha_d^T(t)Z^T]P_0^{-1}[X\alpha(t) + Z\alpha_d(t)]$$

$$+\int_{t-\tau}^{t}\dot{\alpha}^T(s)P_0\dot{\alpha}(s)ds$$

which leads to

$$\dot{V}_g \le 2\alpha^T(P_0A_0\alpha + P_0A_d\alpha_d) + \alpha^T F\alpha - (1-d_m)\alpha_d^T F\alpha_d + \tau[\alpha^T X^T + \alpha_d^T Z^T]P_0^{-1}[X\alpha + Z\alpha_d]$$
$$+2(\alpha^T X^T + \alpha_d^T Z^T)(\alpha - \alpha_d) + \tau(P_0A_0\alpha + P_0A_d\alpha_d)^T P_0^{-1}(P_0A_0\alpha + P_0A_d\alpha_d).$$

By applying the Schur complement formula [48] to (4.111), we can obtain

$$\begin{bmatrix} \mathrm{N}_{11} & * \\ \mathrm{N}_{21} & \mathrm{N}_{22} \end{bmatrix} + \tau \begin{bmatrix} X_i^T \\ Z_i^T \end{bmatrix} P_0^{-1} \begin{bmatrix} X_i^T \\ Z_i^T \end{bmatrix}^T + \tau \begin{bmatrix} \Lambda^T Y(A_i + \mathrm{T}_i\Pi_i(t))^T\Lambda \\ \Lambda^T Y(A_{\tau i} + \mathrm{T}_i\Pi_i(t))^T\Lambda \end{bmatrix} P_o^{-1} \begin{bmatrix} \Lambda^T Y(A_i + \mathrm{T}_i\Pi_i(t))^T\Lambda \\ \Lambda^T Y(A_{\tau i} + \mathrm{T}_i\Pi_i(t))^T\Lambda \end{bmatrix}^T < 0. \tag{4.113}$$

This implies that $\dot{V}_g \le -\mu(\|\alpha\|^2 + \|\alpha_d\|^2)$ for some $\mu > 0$. After all, we can conclude that the sliding mode dynamics (4.110) is stable.

After the switching surface parameter matrix S is designed so that the reduced-order sliding mode dynamics has a desired response, the next step of the SMC design procedure is to design a switching feedback control law for the reaching mode such that the reachability condition is met [33]. If the switching feedback control law satisfies the reachability condition, it drives the state trajectory to the switching surface $\sigma = Sx = 0$ and maintains it there for all subsequent time. In this section, we design an adaptive fuzzy control law guaranteeing that σ converges to zero. We will use the following nonlinear sliding switching feedback control law as the local controller:

Control Rule i: IF θ_1 is μ_{i1} and ... and θ_s is μ_{is}, THEN

$$u(t) = -\phi_i\sigma - S(A_i + \mathrm{T}_i\Pi_i(t))x - S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d - \hat{\kappa}_i\frac{\sigma}{\|\sigma\|} \tag{4.114}$$

where $\hat{\kappa}_i = \varepsilon_i + \sum_{k=0}^{p}\hat{\rho}_{dk}\|x\|^k + \sum_{k=0}^{q}\hat{\delta}_{dk}\|x_d\|^k$, $\dot{\hat{\rho}}_{dk} = \nu_k\|\sigma\|\cdot\|x\|^k$, $\dot{\hat{\delta}}_{dk} = \chi_k\|\sigma\|\cdot\|x_d\|^k$ (4.115)

and $\phi_i > 0, \varepsilon_i > 0, \nu_k > 0, \chi_k > 0$. The final controller inferred as the weighted average of the each local controller is given by

$$u(t) = -\sum_{i=1}^{r} \beta_i(\theta)\left(\phi_i\sigma + S(A_i + \mathrm{T}_i\Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d + \hat{\kappa}_i \frac{\sigma}{\|\sigma\|} \right) \tag{4.116}$$

and we can establish the following theorem.

Theorem 4.8 Consider the closed-loop control system of the uncertain system (4.107) with control (4.116). Suppose that the LMI (4.111) is feasible and the sliding surface is given by (4.112). Then, the switching feedback control law (4.116) induces an ideal sliding motion on the sliding surface $\sigma = o$ in finite time and the state converges to zero.

Proof: Since Theorem 4.7 implies that the sliding mode dynamics restricted to $\sigma = Sx = 0$ is stable, we only have to show that σ converges to zero. Define a Lyapunov function as $E_g(t) = 0.5\sigma^T\sigma + 0.5\sum_{k=0}^{p}\tilde{\rho}_{dk}^2 + 0.5\sum_{k=0}^{q}\tilde{\delta}_{dk}^2$ where $\tilde{\rho}_{dk} = \hat{\rho}_{dk} - \rho_{dk}$ and $\tilde{\delta}_{dk} = \hat{\delta}_{dk} - \delta_{dk}$. The time derivative of $E_g(t)$ is $\dot{E}_g = \sigma^T\dot{\sigma} + \|\sigma\|\sum_{k=0}^{p}\tilde{\rho}_{dk}\|x\|^k + \|\sigma\| \sum_{k=0}^{q}\tilde{\delta}_{dk}\|x_d\|^k$. From $SB = I$, the assumption A2 and (4.116), we can obtain

$$\sigma^T\dot{\sigma} = \sigma^T\sum_{i=1}^{r}\beta_i(S(A_i + \mathrm{T}_i\Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d + h_i) + \sigma^T u$$

$$\le -\sum_{i=1}^{r}\beta_i\phi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i\varepsilon_i\|\sigma\| - \|\sigma\|\sum_{k=0}^{p}\tilde{\rho}_{dk}\|x\|^k - \|\sigma\|\sum_{k=0}^{q}\tilde{\delta}_{dk}\|x_d\|^k.$$

This implies that $\dot{E}_g \le -\sum_{i=1}^{r}\beta_i\phi_i\|\sigma\|^2 - \sum_{i=1}^{r}\beta_i\varepsilon_i\|\sigma\| \le 0$ which indicates that $E_g \in L_2 \cap L_\infty, \dot{E}_g \in L_\infty$. Finally, by using Barbalat's lemma, we can conclude that σ converges to zero.

Remark 4.4 Theorem 4.7 and 4.8 can be summarized in the form of the following LMI-based design algorithm.

Step 1: Check that $(A_i + A_{\tau i}, B)$ is stabilization. If not, exit.

Step 2: Find a full-rank matrix $\Lambda \in R^{n\times(n-m)}$ such that $B^T\Lambda = 0, \Lambda^T\Lambda = I$.

Step 3: Find a solution vector (Y, c_1, c_2, η) to LMI (4.111).

Step 4: Compute the sliding surface parameter matrix S by using the formula of (4.112).

Step 5: The controller is given by (4.116).

4.4.3 Numerical Examples

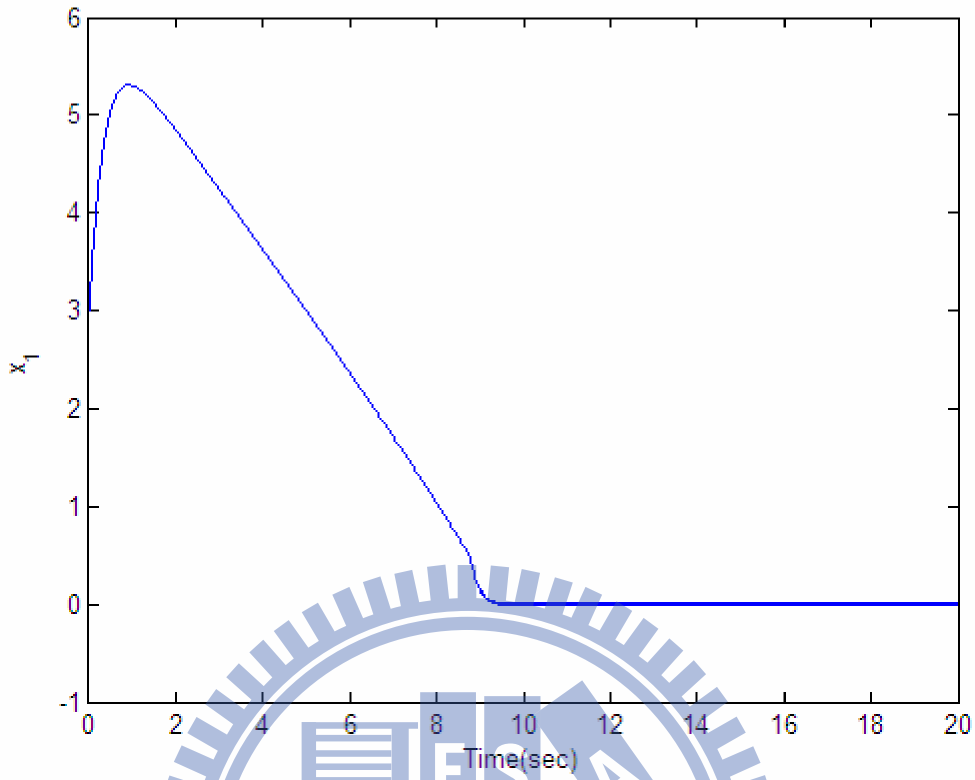
Example 4.7 To illustrate the performance of the proposed adaptive fuzzy control design method, Consider the following T-S fuzzy time-delay model [62] without mismatched parameter uncertainties and external disturbances.

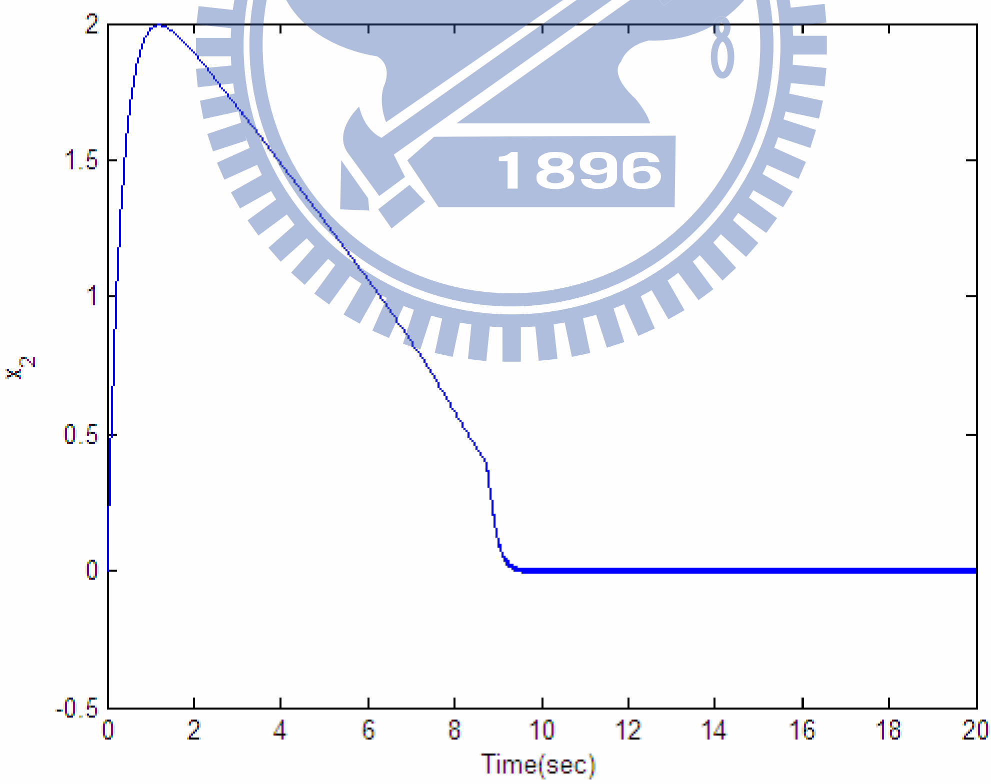
$$\dot{x}(t) = \sum_{i=1}^{2} \beta_i(\theta)[A_i x(t) + A_{\tau i} x_d(t)] + Bu(t) \tag{4.117}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ and $A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1 \end{bmatrix}$, $A_{\tau 1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$,

$A_{\tau 2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\beta_1 = \dfrac{1}{1 + e^{-2x_1(t)}}$, $\beta_2 = 1 - \beta_1$.

We assume that $d(t) = \tau = 0.4, h_i = 0, \phi_i = 0.05$ and $\varepsilon_i = 1$. Figure 4.10 shows the control results for system (4.117) via the proposed controller (4.116) under the initial condition $\psi(t) = [3\ 0]^T$. In Figure 4.10, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system.


x1
Time(sec)


x2
Time(sec)

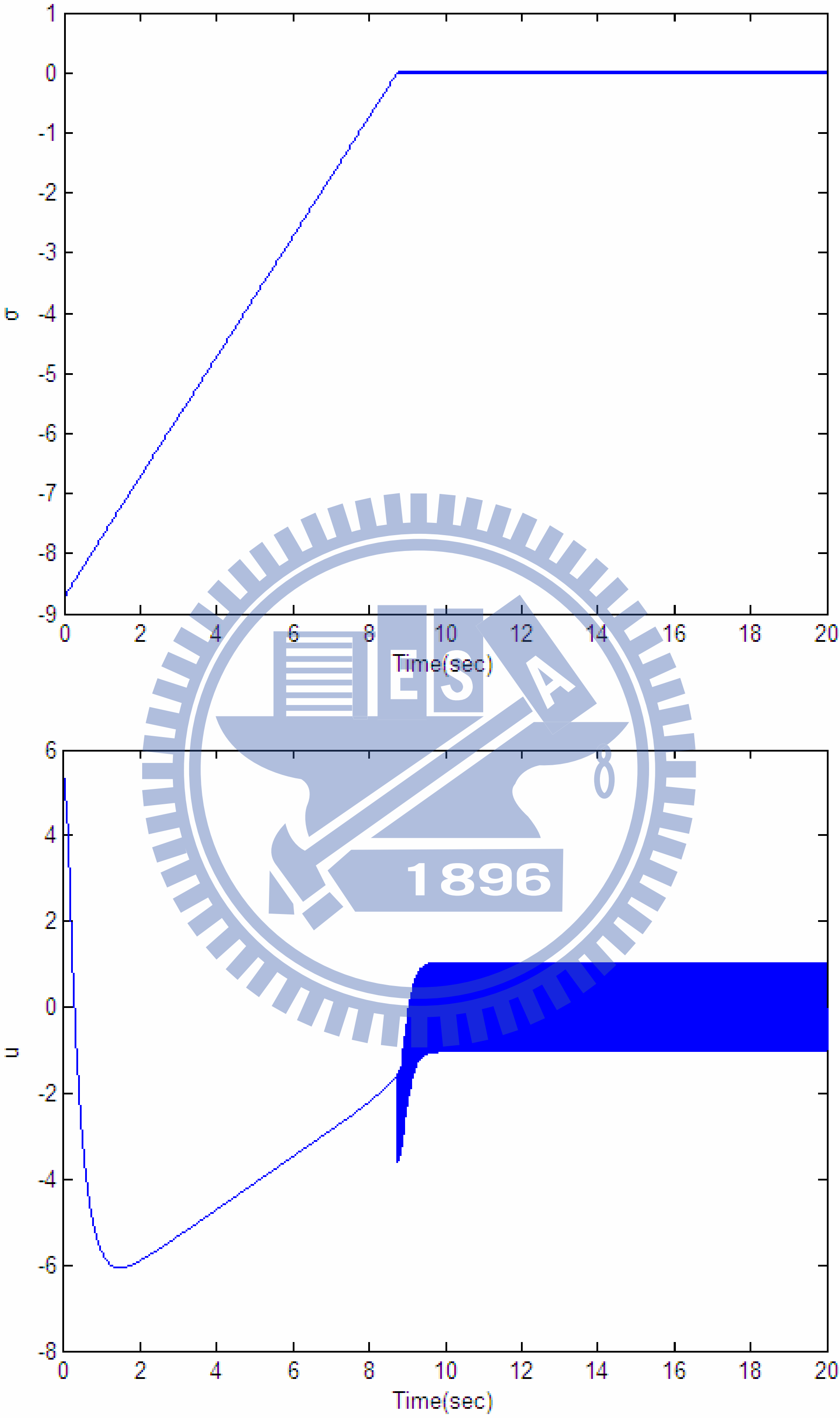


Figure 4.10 Control results for the system (4.117).

Example 4.8 Consider a well-studied example of a continuous-time truck-trailer with time-delay proposed in [63]. The time-delay model is given by

$$\dot{x}_1(t) = -a\frac{vT}{Lt_0}x_1(t) - (1-a)\frac{vT}{Lt_0}x_1(t-d) + \frac{vT}{lt_0}[u(t)+h(t)],$$

$$\dot{x}_2(t) = a\frac{vT}{Lt_0}x_1(t) + (1-a)\frac{vT}{Lt_0}x_1(t-d),$$

$$\dot{x}_3(t) = \frac{vT}{t_0}\sin\left[x_2(t) + a\frac{vT}{2L}x_1(t) + (1-a)\frac{vT}{2L}x_1(t-d)\right] \quad (4.118)$$

where $x_1(t)$ is the angle difference between truck and trailer (in radians), $x_2(t)$ is the angle of trailer (in radians), $x_3(t)$ is the vertical position of rear of trailer (in meters), $u(t)$ is the steering angle (in radians), $T=2.0, l=2.8$, $L=5.5, v=-1.0$ and $t_0=0.5$. The constant parameter a is the retarded coefficient satisfying $a\in[0,1]$. The limits 1 and 0 correspond to a no-delay term and to a completed-delay term. We assume that the disturbance input $h(t)$ is unknown but bounded as $|h(t)|\le 1$. By using the fact that $\sin(x)\approx x$ if $x\approx 0$, we can represent the above model as the following two-rule T-S fuzzy model, including parameter uncertainties and external disturbances:

Plant Rule 1: IF $\theta(t)$ is about 0, THEN

$$\dot{x} = (A_1 + \mathrm{T}_1\Pi_1(t))x + (A_{\tau 1} + \mathrm{T}_1\Pi_1(t))x_d + Bu + Bh_1$$

Plant Rule 2: IF $\theta(t)$ is about $\pm\pi$, THEN

$$\dot{x} = (A_2 + \mathrm{T}_2\Pi_2(t))x + (A_{\tau 2} + \mathrm{T}_2\Pi_2(t))x_d + Bu + Bh_2$$

where $\theta(t) = x_2(t) + avT\,x_1(t)/2L + (1-a)vT\,x_1(t-d)/2L$

$$A_1 = \begin{bmatrix} -a\dfrac{vT}{Lt_0} & 0 & 0 \\ a\dfrac{vT}{Lt_0} & 0 & 0 \\ a\dfrac{v^2T^2}{2Lt_0} & \dfrac{vT}{t_0} & 0 \end{bmatrix}, \quad A_{\tau 1} = \begin{bmatrix} -(1-a)\dfrac{vT}{Lt_0} & 0 & 0 \\ (1-a)\dfrac{vT}{Lt_0} & 0 & 0 \\ (1-a)\dfrac{v^2T^2}{2Lt_0} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -a\dfrac{vT}{Lt_0} & 0 & 0 \\ a\dfrac{vT}{Lt_0} & 0 & 0 \\ a\dfrac{10v^2T^2}{2L\pi} & \dfrac{10vT}{\pi} & 0 \end{bmatrix}, \quad A_{\tau 2} = \begin{bmatrix} -(1-a)\dfrac{vT}{Lt_0} & 0 & 0 \\ (1-a)\dfrac{vT}{Lt_0} & 0 & 0 \\ (1-a)\dfrac{10v^2T^2}{2L\pi} & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{vT}{lt_0} \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{T}_1 = \mathrm{T}_2 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad \Pi_1(t) = \Pi_2(t) = \begin{bmatrix} \sin t & 0 & 0 \end{bmatrix},$$

$$\beta_1 = \frac{1 - 1/(1+e^{-2(\theta-0.5\pi)})}{1+e^{-2(\theta+0.5\pi)}}, \; \beta_2 = 1-\beta_1, \qquad h_1 = h_2 = h(t). \tag{4.119}$$

We assume that $d(t) = \tau = 0.1$. Considering LMI optimization with the data (4.119), $a = 0, \tau = 0.1$ and $d_m = 0$, we can obtain the sliding surface $\sigma = Sx$. By setting $\phi_i = 0.05, \varepsilon_1 = 0.01, \varepsilon_2 = 1, \nu_k = 0.1, \chi_k = 0.1, r = 1, p = 1, q = 1$, and $t_{sampling} = 0.01\sec$, we can obtain the following fuzzy controller:

Control Rule 1: IF $\theta(t)$ is about 0, THEN

$$u(t) = -0.05\sigma - S(A_1 + \mathrm{T}_1\Pi_1(t))x - S(A_{\tau 1} + \mathrm{T}_1\Pi_1(t))x_d - \hat{\kappa}_1 \operatorname{sgn}(\sigma).$$

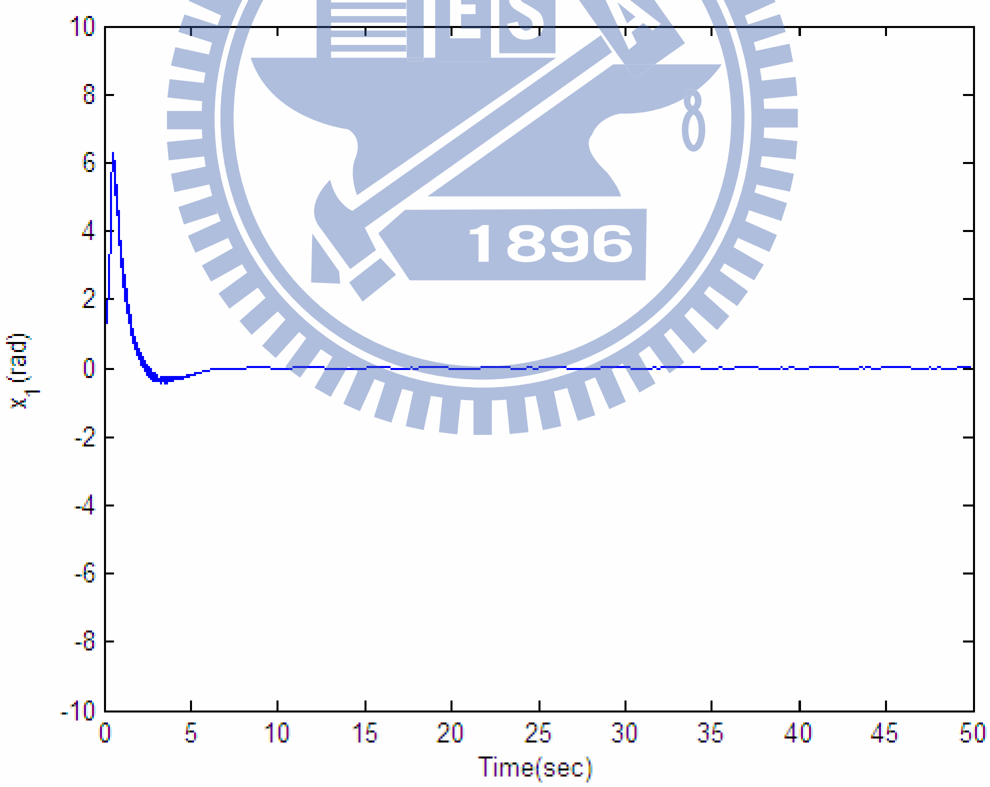
Control Rule 2: IF $\theta(t)$ is about $\pm\pi$, THEN

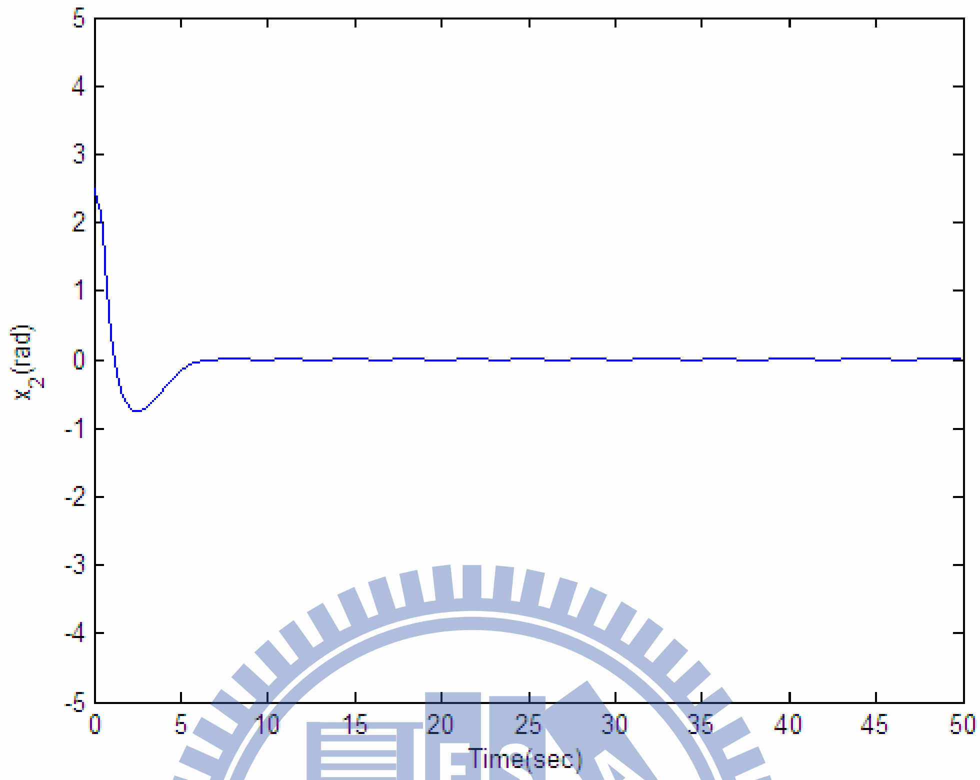
$$u(t) = -0.05\sigma - S(A_2 + \mathrm{T}_2\Pi_2(t))x - S(A_{\tau 2} + \mathrm{T}_2\Pi_2(t))x_d - \hat{\kappa}_2 \operatorname{sgn}(\sigma).$$

The final controller inferred as the weighted average of each local controller is given by

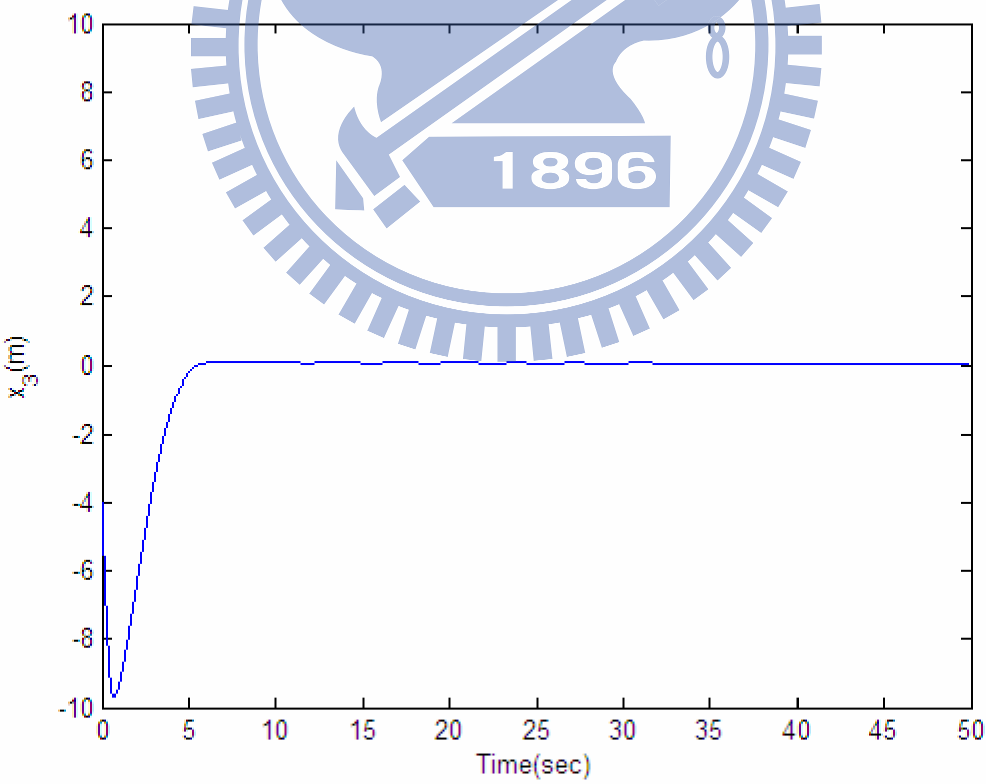
$$u(t) = -\sum_{i=1}^{2} \beta_i [-0.05\sigma - S(A_i + \mathrm{T}_i\Pi_i(t))x + S(A_{\tau i} + \mathrm{T}_i\Pi_i(t))x_d + \hat{\kappa}_i \operatorname{sgn}(\sigma)]. \tag{4.120}$$

To demonstrate the controller ability, we apple the above fuzzy controller (4.120) to the system model (4.119) with $h(t,x,x_d) = x_1 \sin 2\pi t + x_{2d} \cos 2\pi t - 0.5\,\mathrm{sgn}(x_3)$ and $d(t) = \tau = 0.1$. Figure 4.11 shows the closed-loop system responses of (4.119) and the proposed controller (4.120) with the initial condition $\psi(t) = [0.4\pi, 0.8\pi, -4]^T$. In Figure 4.11, it should be noted that since it is impossible to switch the input u instantaneously, oscillations always occur in the sliding mode of a SMC system. From Figure 4.11, the proposed controller stabilizes uncertain fuzzy time-delay systems with mismatched parameter uncertainties in the state matrix and unknown norm-bounded external disturbances. The control performances of the two-rule T-S fuzzy model (4.119) are satisfactory.

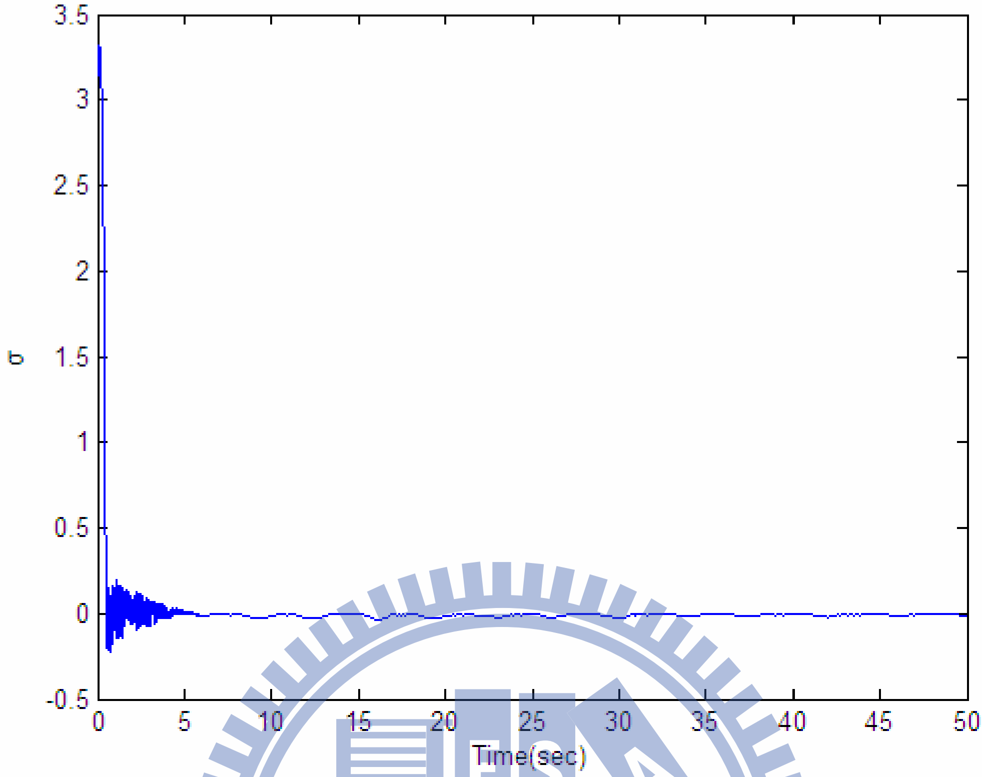


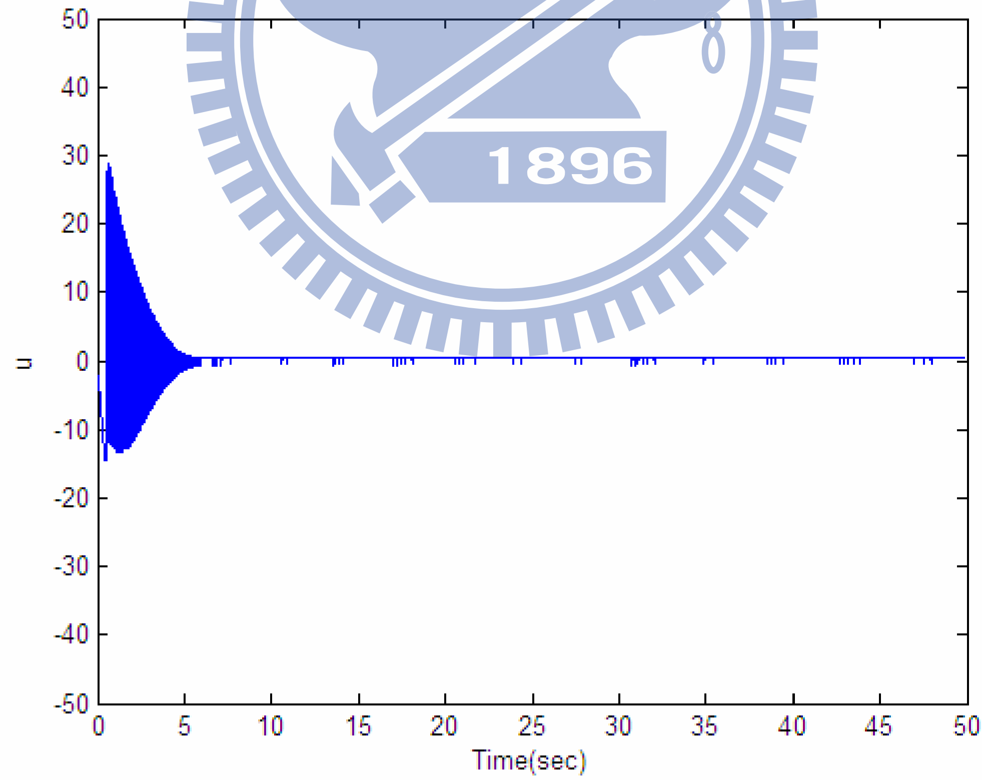


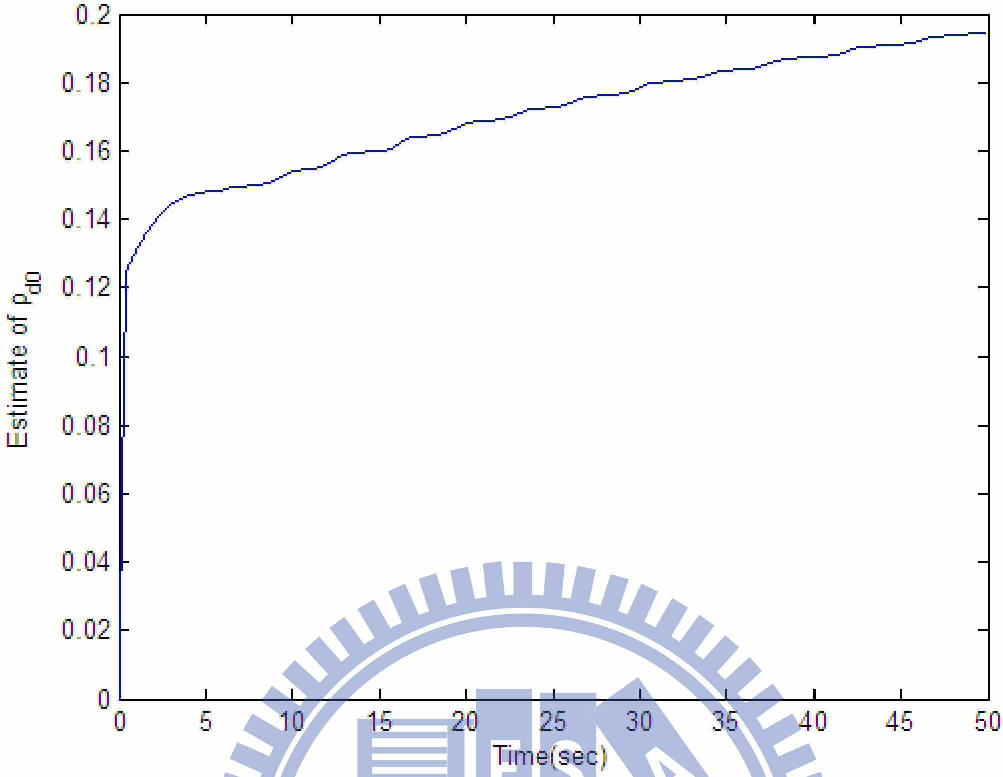
x_2(rad)
Time(sec)

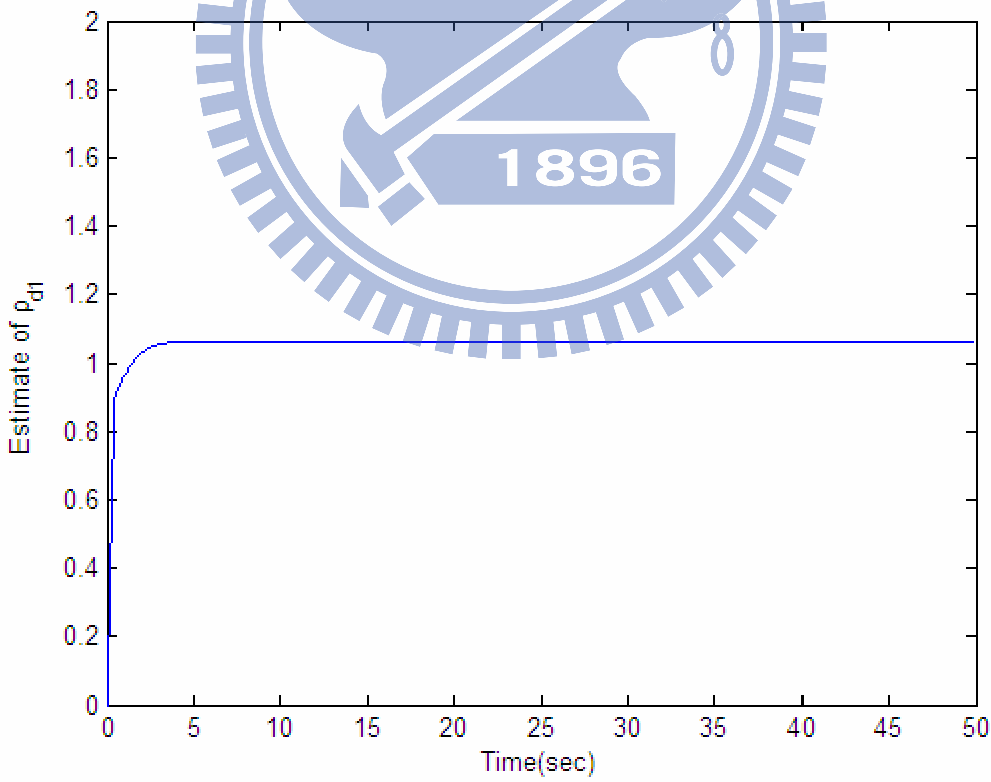


x_3(m)
Time(sec)


σ
Time(sec)


u
Time(sec)


Estimate of ρ_{d0}
Time(sec)


Estimate of ρ_{d1}
Time(sec)

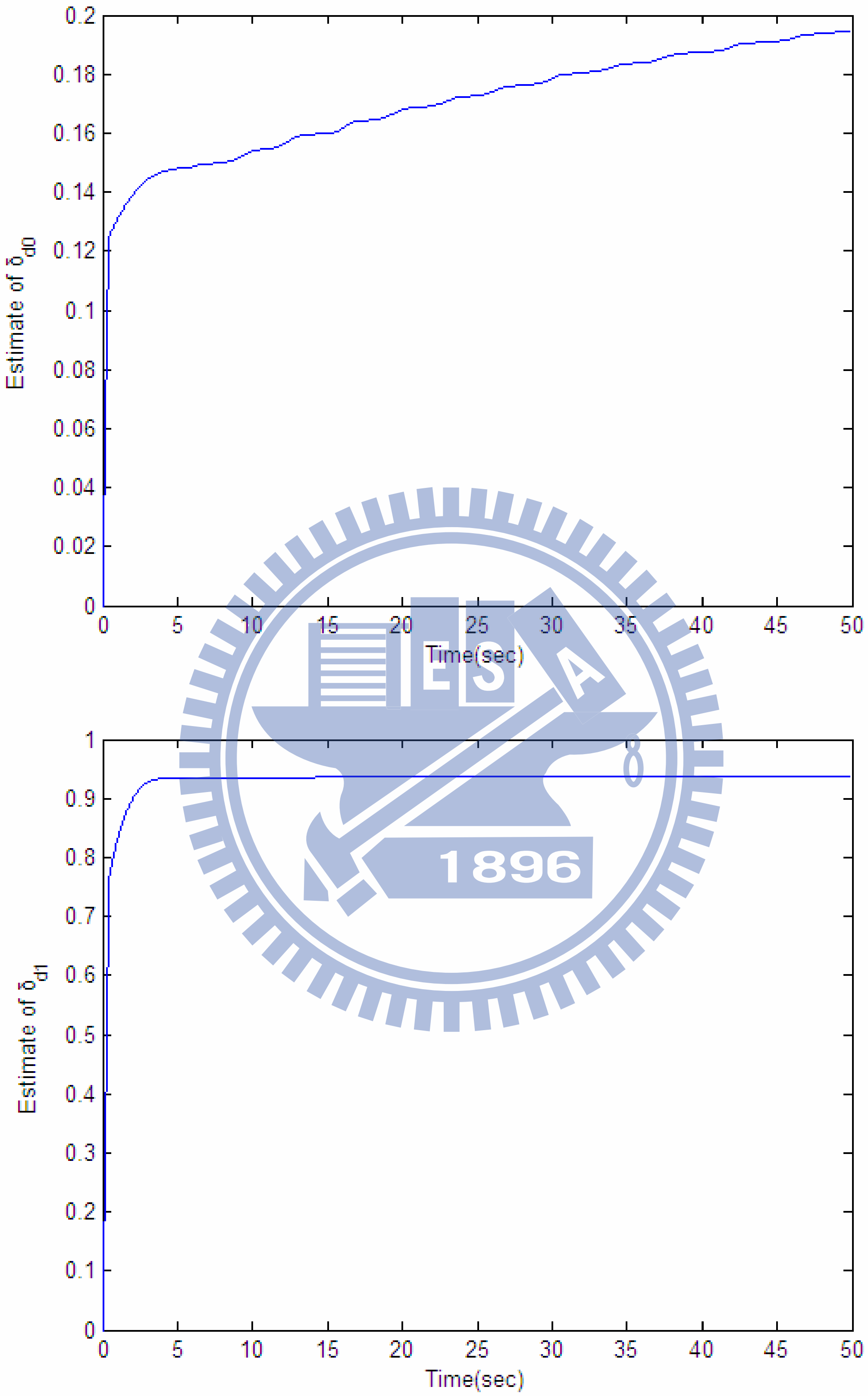


Figure 4.11 Simulation results with the proposed method on the model (4.119).

Chapter 5

Conclusion

The objective of this dissertation is to provide a stable and robust means for uncertain nonlinear systems using T-S fuzzy models/time-delay models by applying two kinds of LMI-based adaptive sliding control, including sliding control methods and adaptive control methods. This dissertation proposes a complete approach to fulfill the objective. This chapter summarizes the contributions of sliding control methods and adaptive control methods in this dissertation and gives suggestions for future work.

5.1 Contributions

Based on adaptive sliding control methods and robust stability criteria, the following objectives are achieved in this dissertation.

1. LMI-based robust sliding control:

   Firstly, a robust sliding control method is proposed for uncertain T-S fuzzy models with matched parameter uncertainties and external disturbances. In the VSS, the control design of the plant is intentionally changed by using a high-speed switching feedback control to obtain a desired system response, from which the VSS arises in finite time. The VSS drives the trajectory of the system onto a specified surface, which is called the sliding surface or the switching surface, and maintains the trajectory on this sliding surface for all subsequent time. The closed-loop response obtained from using a VSS control law comprises two separate modes. The first is the reaching mode in which the trajectory starting from anywhere on the state space is being driven towards the switching surface. The second is the sliding mode in which the trajectory asymptotically tends to the origin. The central feature of the VSS is the sliding mode on the sliding surface on which the system remains insensitive to internal parameter variations and external disturbance. In sliding mode, the order of the system dynamics is reduced. We have relaxed the restrictive assumption that each nominal local system model shares the same input channel, which is required in the traditional VSS-based fuzzy control design methods.

Secondly, two sliding control methods are developed for distinct uncertain T-S fuzzy models, respectively, under different assumptions. The uncertain fuzzy systems under consideration have mismatched parameter uncertainties in the state matrix and external disturbances.

Thirdly, a robust sliding control method is presented for uncertain T-S fuzzy time-delay models with mismatched parameter uncertainties and external disturbances.

Finally, some examples are used to illustrate the effectiveness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection.

2. LMI-based robust adaptive control:

Firstly, a robust adaptive control method is proposed for uncertain T-S fuzzy models with matched parameter uncertainties and external disturbances which are bounded by unknown upper norm bounds. We have presented an adaptation law to estimate the upper norm bounds. Moreover, we have relaxed the restrictive assumption that each nominal local system model shares the same input channel, which is required in previous VSC-based fuzzy control methods.

Secondly, two adaptive control methods are developed for distinct uncertain

T-S fuzzy models, respectively, under different assumptions. The uncertain fuzzy systems under consideration have mismatched parameter uncertainties in the state matrix and external disturbances which are bounded by unknown upper norm bounds. We have presented an adaptation law to estimate the upper norm bounds.

Thirdly, a robust adaptive control method is presented for uncertain T-S fuzzy time-delay models with mismatched parameter uncertainties and external disturbances which are bounded by unknown upper norm bounds. We have proposed an adaptation law to estimate the upper norm bounds.

Finally, some examples are used to illustrate the effectiveness of the proposed methods for distinct uncertain T-S fuzzy models and to compare with the existing methods in each final subsection.

As shown in simulation results, the proposed adaptive sliding control methods can not only deal with different conditions of uncertain T-S fuzzy models but also stabilize mismatched uncertain T-S fuzzy time-delay models. Besides, the control performances of four systems are satisfactory in this dissertation.

5.2 Suggestions for Future Work

The objective of this dissertation is to provide a stable and robust means for uncertain nonlinear systems using T-S fuzzy models/time-delay models by applying two kinds of adaptive sliding control, including sliding control methods and adaptive control methods. In the future, we can develop two kinds of adaptive sliding control for mismatched uncertain T-S fuzzy delay-time models, assuming that $\Delta A_i(t)$ and $\Delta A_{\tau i}(t)$ are of the form $T_i \Pi_i(t)$ where $\Pi_i(t)$ is an unknown time-varying matrix, and each nominal local system model of the uncertain system under consideration may not share the same input channel. Moreover, the proposed approach may be further applied to other control system. Power control systems, robot control systems, motor control systems, and filter design systems are the suggestions for future work.

References

[1] R. Babuka, *Fuzzy Modeling for Control*. Boston: Kluwer Academic, 1998.

[2] M. Sugeno and K. Tanaka, "Successive identification of a fuzzy model and its application to prediction of a complex system," *Fuzzy Sets and Systems*, vol. 42, no. 3, pp. 315-334, 1991.

[3] A. Peigat, *Fuzzy Modeling and Control*. New York: Physica-Verlag, 2001.

[4] B. Friedland, *Advanced Control System Design*. New Jersey: Prentice-Hall, 1996.

[5] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics,* vol. 15, no. 1, pp. 116–132, 1985.

[6] M. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy sets and Systems*, vol. 28, no.1, pp. 15-33, 1988.

[7] S. G. Cao, N. W. Rees, and G. Feng, "Fuzzy control of nonlinear continuous-time systems," *Proc. 35th IEEE Conf. Decision and Control*, pp. 592-597, Kobe, Japan, 1996.

[8] H. Ying, "Sufficient conditions on uniform approximation of multivariate functions by general Takagi-Sugeno fuzzy systems with linear rule consequence," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 28, no. 4, pp. 515-521, 1998.

[9] W. Pedrycz and M. Reformat, "Rule-based modeling of nonlinear relationships," *IEEE Transactions on Fuzzy Systems*, vol.5, no.2, pp. 256-269, 1997.

[10] R. Yager and D. Filev, *Essentials of Fuzzy Modeling and Control*. New York: John Wiley and Sons, 1994.

[11] C.Fantuzzi and R. Rovatti, "On the approximation capacities of the homogeneous Takagi-Sugeno model," *Proc. FUZZ-IEEE'96*, pp. 1067-1072, 1996.

[12] K. Tanaka and H. O. Wang, *Fuzzy Control System Design and Analysis: A linear Matrix Inequality Approach*. New York: Wiley, 2001.

[13] M. Sugeno and G. T. Kang, "Fuzzy Modeling and Control of Multilayer Incinerator," *Fuzy Sets and Systems*, vol. 18, no. 3, pp. 329-346, 1986.

[14] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 135-156, 1992.

[15] H. O. Wang, K. Tanaka, and M. F. Griffin, "Parallel distributed compensation of nonlinear systems by Takagi-Sugeno Fuzzy Model," *Proc. FUZZ-IEEE/IFES'95*, pp. 531-538, 1995.

[16] H. O. Wang, K. Tanaka, and M. F. Griffin, " An analytical Framework of fuzzy modeling and control of nonlinear systems: Stability and design issues," *Proc. 1995 American Control Conference*, Seattle, pp. 2272-2276, 1995.

[17] H. N. Wu and H. X. Li, "New approach to delay-dependent stability analysis and stabilization for continuous-time fuzzy systems with time-varying delay," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 3, pp. 482-493, 2007.

[18] C. Lin, Q. G. Wang, and T. H. Lee, "Stability and stabilization of a class of fuzzy time-delay descriptor systems," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 4, pp. 542-551, 2006.

[19] H. Gassara, A. El Hajjaji, and M. Chaabane, "Delay-dependent stabilization conditions of T-S fuzzy systems with time-varying delay," *17^{th} Mediterranean Conference on Control and Automation, IEEE-MED'09*, 24-26 June, Thessaloniki, Greece, pp. 19-24.

[20] Y. C. Chang, S. S. Chen, S. F. Su, and T. T. Lee, "Static output feedback stabilization for nonlinear interval time-delay systems via fuzzy control approach," *Fuzzy Sets and Systems*, vol. 148, no. 3, pp. 395-410, 2004.

[21] B. Chen and X. Liu, "Delay-dependent robust H_∞ control for T-S fuzzy systems with time delay," *IEEE Transactions on Fuzzy Systems*, vol. 13, no. 4, pp. 544-556, 2005.

[22] X. P. Guan and C. L. Chen, "Delay-dependent guaranteed cost control for T-S fuzzy systems with time delays," *IEEE Transactions on Fuzzy Systems*, vol. 12, no. 2, pp. 236-249, 2004.

[23] K, Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability H_∞ control theory, and linear matrix inequalities," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 1, pp.1-13, 1996.

[24] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: Stability and design issues," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 1, pp. 14-23, 1996.

[25] X. J. Ma, Z. Q. Sun, and Y. Y. He, "Analysis and design of fuzzy controller and

fuzzy observer," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 1, pp. 41-51, 1998.

[26] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 2, pp. 250-265, 1998.

[27] P. Korba, R. Babuska, H. B. Verbruggen, and P. M. Frank, "Fuzzy gain scheduling: Controller and observer design based on Lyapunov method and convex optimization," *IEEE Transactions on Fuzzy Systems*, vol. 11, no.3, pp. 285-298, 2003.

[28] S. K. Nguang and P. Shi, "H_∞ fuzzy output feedback control design for nonlinear systems: An LMI approach," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 3, pp. 331-340, 2003.

[29] F. P. Da and S. T. He, "Exponential stability analysis and controller design of fuzzy systems with time-delay," *Journal of the Franklin Institute*, vol. 348, no. 5, pp. 865-883, 2011.

[30] F. Zheng, Q. G. Wang, and T. H. Lee, "Output tracking control of MIMO fuzzy nonlinear systems using variable structure control approach," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 6, pp. 686-697, 2002.

[31] H. H. Choi, "Variable structure output feedback control design for a class of uncertain dynamic systems," *Automatica*, vol. 38, no. 2, pp. 335-341, 2002.

[32] R. A. DeCarlo, "Variable structure control of nonlinear multi-variable systems: A tutorial," *Proc. IEEE*, vol. 76, no. 3, pp. 212–232, 1988.

[33] V. I. Utkin, "Variable structure systems with sliding modes," *IEEE Transactions on Automatic Control,* vol. 22, no. 2, pp. 212–222, 1977.

[34] B. L. Walcott and S. H. Zak, "Combined observer-controller synthesis for uncertain dynamical systems with applications," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 18, no.1, pp. 88–104, 1988.

[35] H. H. Choi, "An LMI-based switching surface design for a class of mismatched uncertain systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1634–1638, 2003.

[36] C. C. Cheng and S. H. Chien, "Adaptive sliding mode controller design based on T-S fuzzy system models," *Automatica*, vol. 42, no. 6, pp. 1005–1010, 2006.

[37] R. J. Wai, C. M. Lin, and C. F. Hsu, "Adaptive fuzzy sliding-mode control for

electrical servo drive," *Fuzzy Sets and Systems*, vol. 143, no. 2, pp.295-310, 2004.

[38] S. Tong and H. X. Li, "Fuzzy adaptive sliding-mode control for MIMO nonlinear systems," *IEEE Transactions on Fuzzy Systems,* vol. 11, no. 3, pp. 354–360, 2003.

[39] W. S. Lin and C. S. Chen, "Robust adaptive sliding mode control using fuzzy modeling for a class of uncertain MIMO nonlinear systems," *IEE Proceedings - Control Theory and Applications*, vol. 149, no. 3, pp. 193-201, 2002.

[40] J. P. Su, T. M. Chen, and C. C. Wang, "Adaptive fuzzy sliding mode control with GA-based reaching laws," *Fuzzy Sets and Systems*, vol. 120, no. 1, pp. 145-158, 2001.

[41] E. M. Jafarov, "Robust sliding mode controllers design techniques for stabilization of multivariable time-delay systems with parameter perturbations and external disturbances," *International, Journal of Systems Science*, vol. 36, no. 7, pp. 433–444, 2005.

[42] C. Peng, Y. C. Tian, and E. Tian, "Improved delay-dependent robust stabilization conditions of uncertain T–S fuzzy systems with time-varying delay," *Fuzzy Sets and Systems*, vol. 159, no. 20, pp. 2713–2729, 2008.

[43] B. Zhang, J. Lam, S. Xu, and Z. Shu, "Robust stabilization of uncertain T–S fuzzy time-delay systems with exponential estimates," *Fuzzy Sets and Systems*, vol. 160, no. 12, pp. 1720–1737, 2009.

[44] C. Lin, Q. G. Wang, and T. H. Lee, "Stabilization of uncertain fuzzy time-delay systems via variable structure control approach," *IEEE Transactions on Fuzzy Systems*, vol. 13, mo. 6, pp. 787–798, 2005.

[45] H. H. Choi, "On the existence of linear sliding surfaces for a class of uncertain dynamic systems with mismatched uncertainties," *Automatica*, vol. 35 , no. 10, pp. 1707-1715, 1999.

[46] F. Gouaisbaut, M. Dambrine, and J. P. Richard, "Robust control of delay systems: A sliding mode control design via LMI," *Syst. Control Lett*., vol. 46, no. 4, pp. 219–230, 2002.

[47] F. Lin, *Robust control design: An optimal control approach*. Wiley, England, 2007.

[48] S. Boyd, L. ElGhaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

[49] M. C. M. Teixeira and S. H. Zak, "Stabilizing controller design for uncertain nonlinear systems using fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 2, pp. 133–142, 1999.

[50] A. Weinmann, *Uncertain Models and Robust Control*. New York: Springer-Verlag, 1991.

[51] W. J. Cao and J. X. Xu, "Nonlinear integral-type sliding surface for both matched and unmatched uncertain systems, "*IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1355–1360, 2004.

[52] J. Hauser, S. Sastry, and P. Kokotovic, "Nonlinear control via approximate input–output linearization: The ball and beam example," *IEEE Transactions on Automatic Control*, vol. 37, no. 3, pp. 392–398, 1992.

[53] H. H. Choi, "LMI-Based Sliding Surface Design for Integral Sliding Mode Control of Mismatched Uncertain Systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 4, pp. 736-742, 2007.

[54] H. H. Choi "Robust Stabilization of Uncertain Fuzzy Systems Using Variable Structure System Approach," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 3, pp. 715-724, 2008.

[55] L. H. Keel, S. P. Bhattacharyya, and J. W. Howze. "Robust control with structured perturbations," *IEEE Transactions on Automatic Control*, vol. 33, no. 1, pp. 68-78, 1988.

[56] Y. Xia and Y. Jia, "Robust sliding-mode control for uncertain time-delay systems: An LMI approach," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 1086–1092, 2003.

[57] H. H. Choi, "Sliding-mode output feedback control design," *IEEE Transactions on Industrial Electronics*, vol. 55, no. 11, pp. 4047–4054, 2008.

[58] R. El-Khazali, "Variable structure robust control of uncertain time-delay systems," *Automatica*, vol. 34, no. 3, pp. 327–332, 1998.

[59] S. Oucheriah, "Dynamic compensation of uncertain time-delay systems using variable structure approach," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 466–469, 1995.

[60] S. Oucheriah, "Exponential stabilization of linear delayed systems using sliding-mode controllers," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 55, no. 6, pp. 826–830, 2003.

[61] R. A. DeCarlo, S. H. Zak, and G. P. Matthews, "Variable structure control of nonlinear multivariable systems: A tutorial," *Proceedings of the IEEE*, vol. 76, no. 3, pp. 212–232, 1988.

[62] H. N. Wu and H. X. Li, "New approach to delay-dependent stability analysis and stabilization for continuous-time fuzzy systems with time-varying delay," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 3, pp. 482-493, 2007.

[63] B. Chen, X. Liu, C. Lin, and K. Liu, "Robust H∞ control of Takagi–Sugeno fuzzy systems with state and input time delays," *Fuzzy Sets and Systems*, vol. 160, no. 4, pp. 403–422, 2009.

[64] H. H. Choi, "Adaptive controller design for uncertain fuzzy systems using variable structure control approach," *Automatica*, vol. 45, no. 11, pp. 2646-2650, 2009.

Vita

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Publication List

Accepted Journal Papers:

[1] S.C. Liu and S.F. Lin, "LMI-Based Robust Sliding Control for Mismatched Uncertain Nonlinear Systems Using Fuzzy Models," *International Journal of Robust and Nonlinear Control,* vol. 22, no. 16, pp. 1827-1836, 2012. (SCI and EI)

[2] S.C. Liu and S.F. Lin, "Robust Adaptive Controller Design for Uncertain Fuzzy Time-Delay Systems with Mismatched Uncertainties," *Advances in Differential Equations and Control Processes*, vol. 9, no. 2, pp. 123-139, May 2012. (EI)

[3] S.C. Liu and S.F. Lin, "Robust Adaptive Controller Design for Uncertain Fuzzy Systems Using Linear Matrix Inequality Approach," accepted for publication in *International Journal of Modelling, Identification and Control*. (EI)

[4] S.C. Liu and S.F. Lin, "Robust Sliding Control for Mismatched Uncertain Fuzzy Time-Delay Systems Using Linear Matrix Inequality Approach," accepted for publication in *Journal of the Chinese Institute of Engineers*. (SCI and EI)

Submitted Journal Papers:

[1] S.C. Liu and S.F. Lin, "Robust Sliding Controller Design for Uncertain Fuzzy Systems Using Linear Matrix Inequality Approach," *Control and Cybernetics*. (SCI, submitted)

[2] S.C. Liu and S.F. Lin, "Robust Adaptive Control for Uncertain Fuzzy Systems with Mismatched Uncertainties," *Intelligent Automation and Soft Computing*. (SCI, revised)

Conference Papers:

[1] S.C. Liu and S.F. Lin, "LMI-Based Robust Adaptive Control for Mismatched Uncertain Nonlinear Time-Delay Systems Using Fuzzy Models," *International Symposium on Computer, Consumer and Control*, Taichung, Taiwan, pp. 552-555, June 4-6, 2012.