Performance of banyan networks with inhomogeneous traffic flow

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Abstract: To date, most research results regarding the performance of banyan networks assumed a uniform traffic model. Sources are assumed to generate connection requests independently with the same rate and, moreover, connection requests are assumed to be independently and equally likely destined to each destination. This assumption, which greatly simplifies analysis, may not be true for real-world systems because the traffic requirements between different source-destination pairs could be quite different in nature. We explore in the paper the performance evaluation of banyan networks under situations of nonuniform traffic requirements. Two types of nonuniform traffic matrices are considered. The results show that the uniform traffic model leads to optimistic performance measures. Moreover, a higher degree of nonuniformity in traffic leads to a more serious performance degradation.

1 Introduction

Banyan networks have attracted increasing interest recently because of their applications in processor-memory interconnections for multiprocessor systems and in constructing the switching fabric of fast packet-switched communication networks. In addition to the nice properties such as regularity for VLSI implementation and easiness of routing, banyan networks were shown to have a far better performance per cost than crossbars in large multiprocessing systems [3]. It has further been proved [5] that flip network, omega network, indirect binary n-cube network, and baseline network are all topologically equivalent to regular SW banyan network with spread and fan-out of 2. Therefore, much recent research work concentrates on discussing the performance of regular banyan networks.

However, almost all of the previous research results regarding the performance of banyan networks assumed a uniform traffic model. Sources are assumed to independently generate connection requests with the same rate, and connection requests are assumed to be independently and equi-probably destined to any destination. The sources and destinations of a $2^n \times 2^n$ banyan network can be numbered, from top to bottom, by 0 to $2^n - 1$. Let t_{ij} represent the probability that a connection request originated at source i is destined to destination j. Then the corresponding traffic matrix $T = [t_{ij}]$ of a $2^n \times 2^n$ banyan network has all its elements equal $(1/2)^n$ under the uniform traffic assumption. This assumption, which greatly simplifies the analysis, may not be true in certain situations because the traffic requirements could be quite different for different source-destination pairs. In reality, as will be seen later, the uniform traffic assumption leads to optimistic performance measures.

It is the purpose of this paper to study the performance of banyan networks under nonuniform traffic requirements. The system we are interested in is slotted along the time axis. The duration of a time slot consists of sending connection requests by sources followed by memory accesses or transmission of data packets by sources whose requests are granted. Two specific types of nonuniform traffic matrices which can better model real-world systems are considered. As usual, the normalised throughput is chosen to be the performance measure of banyan networks. The normalised throughput of a banyan network is defined as the average number of connection requests granted per slot for each source. Recursive formulae will be derived for the probability that a specific connection request will be granted. The normalised throughput of a banyan network is obtained if one multiplies this probability by the offered load of each source.

2 Uniform traffic model

A regular four-stage SW banyan network with spread and fan-out of 2 is illustrated in Fig. 1. Notice that,

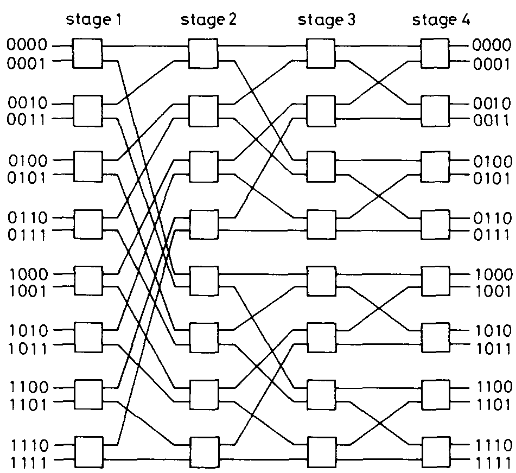

Fig. 1 *Four-stage banyan network*

because of the equivalence relation mentioned before, a baseline network is also referred to as a banyan network in our study. Each 2×2 switching element is called a

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node. For convenience, let the sources and destinations be respectively numbered from 0 to 15 and represented by binary sequences of length four. The leftmost bit is considered to be the most significant bit (MSB). It is interesting to note that the two input links of a node in the ith stage are related to different sources that differ only in the ith bit (counted from right to left) of their representations (or addresses). Consider, for example, the upmost node in stage 3. The upper input link is related to sources {0000, 0001, 0010, 0011} and the lower input link is related to sources {0100, 0101, 0110, 0111}. Such a connection strategy can be easily extended to a $2^n \times 2^n$ banyan network. The sources and destinations are numbered sequentially and are represented by binary sequences of length n with the leftmost bit being the MSB. The MSB is also referred to as the nth bit for a $2^n \times 2^n$ banyan network. Again, the two input links of a node in the ith stage are related to different sources that differ only in the ith bit of their representations. In the following, we will review the iterative algorithm for performance analysis of a banyan network under a uniform traffic assumption.

Consider an n-stage banyan network. Let T represent the traffic matrix, i.e. t_{ij}, the (i, j)th element of the matrix T, is the proportion of connection requests originated at source i that are destined to destination j. Then it is clear that $t_{ij} = (1/2)^n$, $0 \leqslant i, j \leqslant 2^n - 1$ under a uniform traffic assumption. Suppose that the connection requests generated by sources are independent. As a consequence, the two input links of a node in any stage are independent since they are related to different sources. Let p_{k-1} denote the probability that each input link of a node in the kth stage receives an active request. Then we have [7]

$$p_{k+1} = p_k(1 - p_k/4) \quad 0 \leqslant k \leqslant n - 1$$

where p_0 is the offered load of each source, i.e. p_0 is the probability that a connection request is generated by a source at the beginning of a slot. The normalised throughput is then given by p_n. In reality, the normalised throughput can also be computed by a different approach. Consider a specific connection request generated by source 0. Let $R_k = p_k/p_{k-1}$ denote the passing rate of a connection request arriving at an input link of a node in the kth stage. In other words, $R_k = 1 - p_{k-1}/4$ is equal to the probability that the specific request is not blocked in the kth stage, on condition that it is not being blocked in any of the previous stages. The probability that a specific request will be granted, denoted by P_s, is therefore given by

$$P_s = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n} (1 - p_{i-1}/4)$$

Clearly ρP_s equals the normalised throughput of the banyan network, where $\rho = p_0$ is the offered load of each source. Moreover, if P_b denotes the blocking probability of an n-stage banyan network, then P_s equals $1 - P_b$. In the following two Sections, we will derive the recursive formulae for P_s of an n-stage banyan network under two types of nonuniform traffic matrices.

3 Nonuniform traffic matrices: type I

The first type of nonuniform traffic matrix we are interested in looks like:

$$T_n(k) = \frac{1}{S_k} [D_{n-k}(m_1) D_{n-k}(m_2) \cdots D_{n-k}(m_{L(k)})]$$

where m_i's are non-negative numbers (not all zeros), $L(k) = 2^k$, $S_k = \sum_{i=1}^{L(k)} m_i$, and $D_{n-k}(m)$ is a uniform matrix of order $2^n \times 2^{n-k}$ with row sum equal to m, i.e. all the entries of $D_{n-k}(m)$ are equal to $m/(2^{n-k})$. Traffic matrices of this type could occur in quite a few application areas. For example, in a telephone network, it is likely that some destinations are more popular than the others. Similarly, in a computer network, all network nodes may have to report their status to some specific nodes which monitor the condition and manage the operation of the network. The third example is concerned with the multiprocessor system. Since different memory modules contain different variables shared by the processors, they are likely to have different rates of accessing requests. All the above three systems have the so-called hot-spot traffic pattern [12], i.e. one or several destinations receive connection requests more frequently than the others. Consequently, the traffic matrices of these systems belong to the type of nonuniform traffic matrix we will study in this Section. The case when $k = 1$ was studied in previous papers (see References 6 and 13).

The destinations of a banyan network with a traffic matrix of type I can be partitioned into 2^k groups, each group consists of 2^{n-k} destinations. Given the traffic matrix $T_n(k)$, an induced traffic matrix $T'_{n-1}(k-1)$ is defined as

$$T'_{n-1}(k-1) = \frac{1}{S'_{k-1}} [D_{n-k}(m_{L(k)/2+1}) \cdots D_{n-k}(m_{L(k)})]$$

where $S'_{k-1} = S_k - S_{k-1}$. Notice that $T'_{n-1}(k-1)$ is the traffic matrix for connection requests whose destination is located in the lower half. Let $A_k = S_{k-1}/S_k$ and $B_k = 1 - A_k$, i.e. A_k and B_k are the probabilities that the destination of a connection request is located in the upper or the lower half, respectively.

Although the traffic matrix is nonuniform, we still assume that the offered loads of the sources are the same and connection requests originated at sources are independent. Let $P_s(T_n(k), \rho)$, where $T_n(k)$ represents the traffic matrix and ρ is the offered load of each source, denote the probability that a specific connection request will be granted. We assume, as usual, that each input link is selected randomly with equal probability if both input links of a node receive active connection requests destined to the same outgoing link. Notice that $k = 0$ means there is only one group containing all destinations. Therefore, $P_s(T_n(0), \rho)$ equals $P_s^B(n, \rho)$, the probability that a specific request is granted under the uniform traffic model. The boundary condition is defined as $P_s^B(0, \rho) = 1$. A recursive formula for $P_s(T_n(k), \rho)$ is stated in the following theorem:

Theorem 1: The probability $P_s(T_n(k), \rho)$ that a specific request is granted satisfies

$$P_s(T_n(k), \rho) = A_k(1 - \rho A_k/2) P_s(T_{n-1}(k-1), \rho_1) + B_k(1 - \rho B_k/2) P_s(T'_{n-1}(k-1), \rho_2)$$

where $\rho_1 = \rho A_k(2 - \rho A_k)$ and $\rho_2 = \rho B_k(2 - \rho B_k)$.

Proof: Without loss of generality, assume that source 0 sends a connection request. Then, with probability A_k, the destination is located in the upper half. Suppose the destination is indeed located in the upper half. Clearly the request is granted if it is not blocked in any stage. The probability that the specific request is not blocked in the first stage is equal to $1 - \rho A_k/2$. Moreover, after the first stage, the traffic matrix becomes $T_{n-1}(k-1)$ with offered load $\rho_1 = \rho A_k(2 - \rho A_k)$. Thus the request will be granted with probability $(1 - \rho A_k/2)P_s(T_{n-1}(k-1), \rho_1)$. Similarly, the probability of being granted is equal to $(1 - \rho B_k/2)P_s(T'_{n-1}(k-1), \rho_2)$ if the destination of the specific connection request is located in the lower half. Hence theorem 1 is proved.

4 Nonuniform traffic matrices: type II

The second type of nonuniform traffic matrix we will analyse looks like

$$T_n(k) = \frac{1}{S_k}\begin{bmatrix} Q_{n-1}(k-1) & U_{n-1}(m_{k+1}) \\ U_{n-1}(m_{k+1}) & Q_{n-1}(k-1) \end{bmatrix}$$

where m_i's are non-negative numbers (not all zeros), $S_k = \sum_{i=1}^{k+1} m_i$, $U_{n-1}(m_{k+1})$ is a $2^{n-1} \times 2^{n-1}$ matrix with all its elements equal to $m_{k+1}/(2^{n-1})$, and $Q_{n-1}(k-1) = S_{k-1}T_{n-1}(k-1)$. The boundary situations are given as:

$$T_n(0) = \frac{1}{m_1}[D_n(m_1)]$$

and

$$T_1(1) = \frac{1}{m_1 + m_2}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}$$

This type of nonuniform traffic matrix could also appear in several application areas. For example, in a telephone network, different groups of sources and destinations (each group consists of 2^{n-k} sources and destinations) may come from different geographical areas and hence the traffic flow inside a group could very well be different from that outside the group. If both traffic flows inside and outside a group are uniform and all the groups have independent and identical traffic flows, then a nonuniform traffic matrix of type II with an arbitrary m_1 and $m_{i+1} = 2m_i$, $i = 2, \ldots, k$, can be used to describe the traffic pattern. Similarly, in a computer network, a group of nodes may have to interchange their status more frequently than report to other nodes of different groups. A third example is the interconnection of 2^n homogeneous local networks by a bridge node constructed from a banyan network. If the traffic load is uniform, then the traffic matrix of the bridge node is of this type with $m_1 = 0$ and $m_{i+1} = 2m_i$ for $2 \leqslant i \leqslant k$. Notice that $m_1 = 0$ means intra-network messages do not pass through the bridge node. The case when $k = 1$ was also studied in References [6] and [13].

For convenience, the vector $\underline{m} = (m_1, m_2, \ldots, m_{k+1})$ will be called the parameter vector associated with the traffic matrix $T_n(k)$. The case when k equals n needs special care. Given a traffic matrix $T_n(n)$, $n \geqslant 3$, define an induced traffic matrix $T'_{n-2}(n-2)$ as follows. Let $\underline{m} = (m_1, m_2, \ldots, m_{n+1})$ and $\underline{m}' = (m'_1, m'_2, \ldots, m'_{n-1})$ denote the parameter vectors associated with traffic matrices $T_n(n)$ and $T'_{n-2}(n-2)$, respectively. Then the parameters of the induced traffic matrix $T'_{n-2}(n-2)$ satisfy $m'_1 = m_1 + m_2$, $m'_i = m_{i+1}$, $i = 2, 3, \ldots, n-1$. Notice that the induced matrix $T'_{n-2}(n-2)$ describes the traffic flow in the upper half of the $2^n \times 2^n$ banyan network if the first and the last stages are excluded. Again, let $P_s(T_n(k), \rho)$ denote the probability that a specific connection request will be granted given the traffic matrix $T_n(k)$ and the offered load of each source ρ. Then we have

$$P_s(T_n(k), \rho) = A_k P_s^U(T_n(k), \rho) + B_k P_s^L(T_n(k), \rho)$$

where $P_s^U(T_n(k), \rho)$ and $P_s^L(T_n(k), \rho)$ are the probabilities that a specific request will be granted on condition that its destination is located in the upper or the lower half, respectively. Thus all we need to do is to compute $P_s^U(T_n(k), \rho)$ and $P_s^L(T_n(k), \rho)$. Let $p_n^B(\rho)$ denote the probability that an output link of a node in the nth stage of a banyan network receives an active request under a uniform traffic assumption, given the offered load ρ. That is, let $p_n^B(\rho) = p_{n-1}^B(\rho)(1 - p_{n-1}^B(\rho)/4)$, where $p_{n-1}^B(\rho) = p_{n-2}^B(\rho)(1 - p_{n-2}^B(\rho)/4)$, $\ldots$, $p_2^B(\rho) = p_1^B(\rho)(1 - p_1^B(\rho)/4)$, $p_1^B(\rho) = p_0^B(\rho)(1 - p_0^B(\rho)/4)$, and $p_0^B(\rho) = \rho$. Also, let $P_s^B(n, \rho)$ have the same meaning as we defined in the last Section. The recursive formulae for $P_s^U(T_n(k), \rho)$ and $P_s^L(T_n(k), \rho)$ are stated in the following as two lemmas.

Lemma 1: Let $\rho_1 = \rho A_k(2 - \rho A_k)$ and $\rho_2 = \rho B_k(2 - \rho B_k)$. Then the probability $P_s^U(T_n(k), \rho)$ satisfies

(i) $k < n$

$$P_s^U(T_n(k), \rho) = (1 - \rho A_k/2)P_s(T_{n-2}(k-1), \rho_1) \times [1 - p_{n-2}^B(\rho_2)/4]$$

with $P_s(T_n(0), \rho) = P_s^B(n, \rho)$,

(ii) $k = n$

$$P_s^U(T_n(n), \rho) = (1 - \rho A_n/2)P_s(T'_{n-2}(n-2), \rho_1) \times [1 - p_{n-2}^B(\rho_2)/4]$$

(iii) boundary situations

$$P_s^U(T_1(1), \rho) = 1 - \rho B_1/2 = 1 - \rho m_2/[2(m_1 + m_2)]$$

$$P_s^U(T_n(1), \rho) = (1 - \rho A_1/2)P_s^B(n-2, \rho_1) \times [1 - p_{n-2}^B(\rho_2)/4](n \geqslant 2)$$

and

$$P_s^U(T_2(2), \rho) = (1 - \rho A_1/2)(1 - \rho_2/4)$$

Proof: The expressions for boundary situations can be easily verified. Thus we consider only cases (i) and (ii). The specific request will be granted if and only if it is not blocked in any stage. Since the two input links of a node in the kth stage are related to sources which differ only in the kth bit of their representations, the probabilities that the request is not blocked in the first and the last stage are equal to $1 - \rho A_k/2$ and $1 - p_{n-2}^B(\rho_2)/4$, respectively. Moreover, for the intermediate stages, the traffic matrix becomes $T_{n-2}(k-1)$ or $T'_{n-2}(k-2)$ with offered load ρ_1 for $k < n$ and $k = n$, respectively. Thus the formula given in the above lemma is indeed equal to the probability that the specific request will be granted. This proves lemma 1.

We need to define another quantity before stating the second lemma. Suppose the specific connection request is originated at a source located in the upper half while its destination is located in the lower half. Then the sources located in the lower half can only possibly block the specific request at the last (i.e. nth) stage. Let $q_{n-1}(n, k, \rho)$ denote the probability that the lower input link of a node in the last stage receives an active request. Then we have a recursive formula for $P_s^L(T_n(k), \rho)$ stated in lemma 2.

Lemma 2: The probability $P_s^L(T_n(k), \rho)$ satisfies

(i) $k \geqslant 2$

$$P_s^L(T_n(k), \rho) = (1 - \rho B_k/2)P_s^B(n-2, \rho_2) \times [1 - q_{n-1}(n, k, \rho)/4]$$

(ii) boundary situations

$$P_s^L(T_1(1), \rho) = 1 - \rho A_1/2 = 1 - \rho m_1/[2(m_1 + m_2)]$$

and

$$P_s^L(T_n(1), \rho) = (1 - \rho B_1/2)P_s^B(n-2, \rho_2) \times [1 - p_{n-2}^B(\rho_1)/4] \quad (n \geq 2)$$

Proof: Again, the expressions for boundary situations can be checked without any difficulty. Consider the other case when $k \geq 2$. The probability that the specific request is not blocked in the first stage is clearly equal to $1 - \rho B_k/2$. From stage 2 to stage $n-1$, the situation becomes an $(n-2)$-stage banyan network under the uniform traffic model with offered load ρ_2. Hence the probability that the specific request will not be blocked in any of these stages is $P_s^B(n-2, \rho_2)$. Suppose the specific request is not blocked in any of the first $n-1$ stages.

Now consider the last stage. Without loss of generality, assume the addresses of the source and the destination of the specific request are 0 and 2^{n-1}, respectively. Let us consider the $(2^{n-2}+1)$th node in the last stage. With the above assumption, the upper input link of this node receives the specific request. The probability that the lower input link receives an active request from sources located in the lower half is denoted by $q_{n-1}(n, k, \rho)$. Suppose the lower input link does receive an active request. There are two possible situations, namely $k < n$ and $k = n$. When $k < n$, according to the traffic matrix, the destination of the active request received by the lower input link is equally likely to be $2^{n/2}$ or $2^{n/2}+1$ no matter which source actually generates the request. When $k = n$, the same conclusion can be drawn if the address of the source which generates the request is not 2^{n-1} or $2^{n-1}+1$. Suppose the address of the source is either 2^{n-1} or $2^{n-1}+1$. Then, according to the operation principle of switching elements, it is equally likely to be any one of these two, because the offered loads of sources are assumed to be identical. Hence the probability that the request is destined to destination 2^{n-1} is equal to $(1/2)m_1/(m_1+m_2) + (1/2)m_2/(m_1+m_2) = 1/2$. Therefore, combining the above results, we conclude that the destination of the active request, if any, received by the lower input link of the $(2^{n-2}+1)$th node in the last stage is equally likely to be 2^{n-1} or $2^{n-1}+1$. Consequently, the probability that the specific request is not blocked in the last stage is $1 - q_{n-1}(n, k, \rho)/4$. This proves lemma 2.

The result regarding the probability $P_s(T_n(k), \rho)$ of the second type of nonuniform traffic matrix is summarised in the following theorem:

Theorem 2: The probability $P_s(T_n(k), \rho)$ for the second type of nonuniform traffic matrix satisfies

$$P_s(T_n(k), \rho) = A_k(1 - \rho A_k/2)P_s^U(T_n(k), \rho) + B_k(1 - \rho B_k/2)P_s^L(T_n(k), \rho)$$

where $P_s^U(T_n(k), \rho)$ and $P_s^L(T_n(k), \rho)$ are given in the above lemmas.

The remaining work which is not trivial is to compute $q_{n-1}(n, k, \rho)$ for $k \geq 2$. A procedure that can be followed to accomplish this is given in the Appendix. We now proceed to study some examples.

5 Examples

In this Section we shall evaluate the performance of banyan networks for three nonuniform traffic matrices. A traffic matrix of the first type is considered first in Example 1. In Examples 2 and 3, both traffic matrices considered belong to the second type.

Example 1: Let

$$T_n(2) = \frac{1}{S_2}[D_{n-2}(m_1), D_{n-2}(m_2), D_{n-2}(m_3), D_{n-2}(m_4)]$$

an example of the first type of nonuniform traffic matrix. According to theorem 1, we have

$$P_s(T_n(2), \rho) = A_2(1 - \rho A_2/2)P_s(T_{n-1}(1), \rho_1) + B_2(1 - \rho B_2/2)P_s(T'_{n-1}(1), \rho_2)$$

where $S_2 = m_1 + m_2 + m_3 + m_4$, $A_2 = (m_1 + m_2)/S_2$, $B_2 = 1 - A_2$, $\rho_1 = \rho A_2(2 - \rho A_2)$, and $\rho_2 = \rho B_2(2 - \rho B_2)$. The results for $P_s(T_n(2), \rho)$ are shown in Figs. 2(*a*)–2(*c*) for various values of n and ρ. Notice that

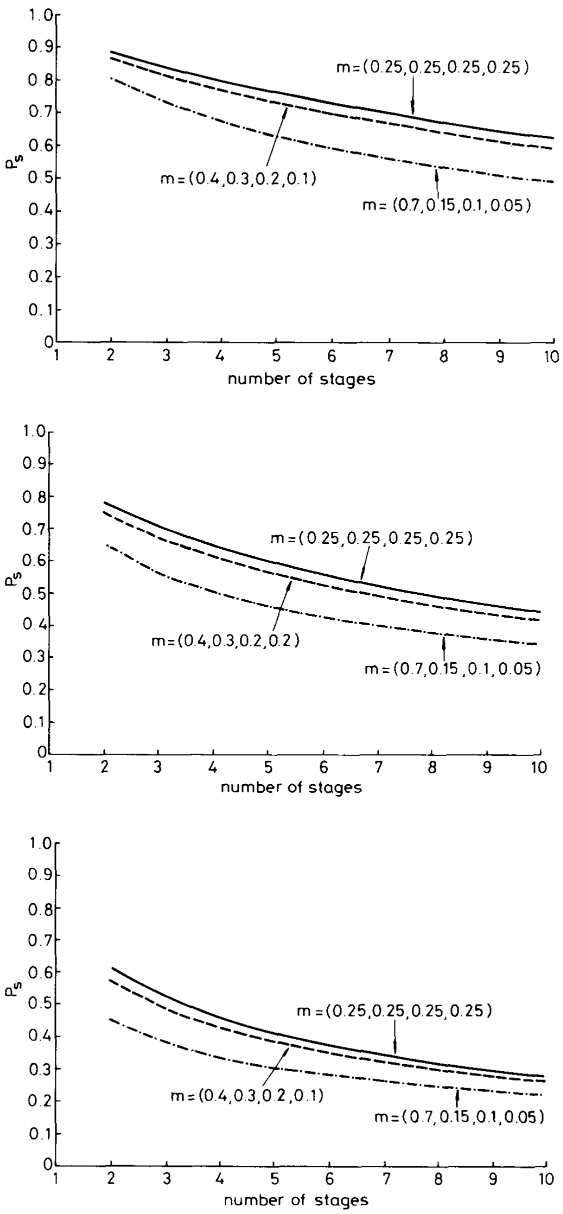

Fig. 2 *P_s for Example 1*
a $P_s(T_n(2), 0.25)$
b $P_s(T_n(2), 0.5)$
c $P_s(T_n(2), 1.0)$

$\rho P_s(T_n(2), \rho)$ is equal to the normalised throughput. It can be seen from the figures that the uniform traffic model leads to an optimistic performance measure and a higher degree of nonuniformity leads to a larger drop in the normalised throughput. Moreover, the effect of nonuniform traffic flow becomes more significant as the number of

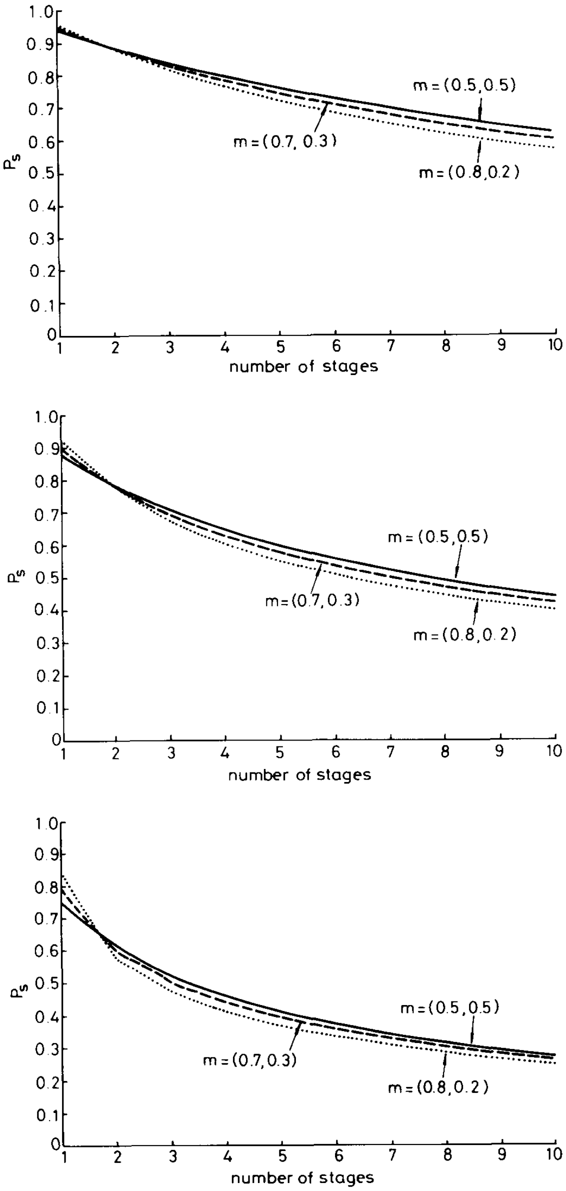

Fig. 3 P_s *for Example 2*
a $P_s(T_n(1), 0.25)$
b $P_s(T_n(1), 0.5)$
c $P_s(T_n(1), 1.0)$

stages increase. For $\underline{m} = (0.4, 0.3, 0.2, 0.1)$ and $n = 10$, the percentage of degradation is about 5.3%, 5.9%, and 5.3% for $\rho = 0.25$, 0.5, and 1.0, respectively. Numerical results reveal that the effect of a nonuniform traffic flow is the most significant for moderate offered loads ($\rho \cong 0.5$). The percentage of degradation for $\underline{m} = (0.7, 0.15, 0.1, 0.05)$ and $n = 10$ is about 22.0%, 23.7%, and 21.6% for $\rho = 0.25$, 0.5, and 1.0, respectively.

Example 2: Let

$$T_n(1) = \frac{1}{S_1}\begin{bmatrix} U_{n-1}(m_1) & U_{n-1}(m_2) \\ U_{n-1}(m_2) & U_{n-1}(m_1) \end{bmatrix}$$

an example of the second type of nonuniform traffic matrix. According to theorem 2, we have, for $n \geqslant 2$,

$$\begin{aligned} P_s(T_n(1), \rho) = {} & A_1(1 - \rho A_1/2)P_s^B(n-2, \rho_1) \\ & \times [1 - p_{n-2}^B(\rho_2)/4] + B_1(1 - \rho B_1/2) \\ & \times P_s^B(n-2, \rho_2)[1 - p_{n-2}^B(\rho_1)/4] \end{aligned}$$

where $S_1 = m_1 + m_2$, $A_1 = m_1/S_2$, $B_2 = m_2/S_2$, $\rho_1 = \rho A_1(2 - \rho A_1)$, and $\rho_2 = \rho B_1(2 - \rho B_1)$. The results are shown in Figs. 3(*a*)–3(*c*). Again, the effect of a nonuniform

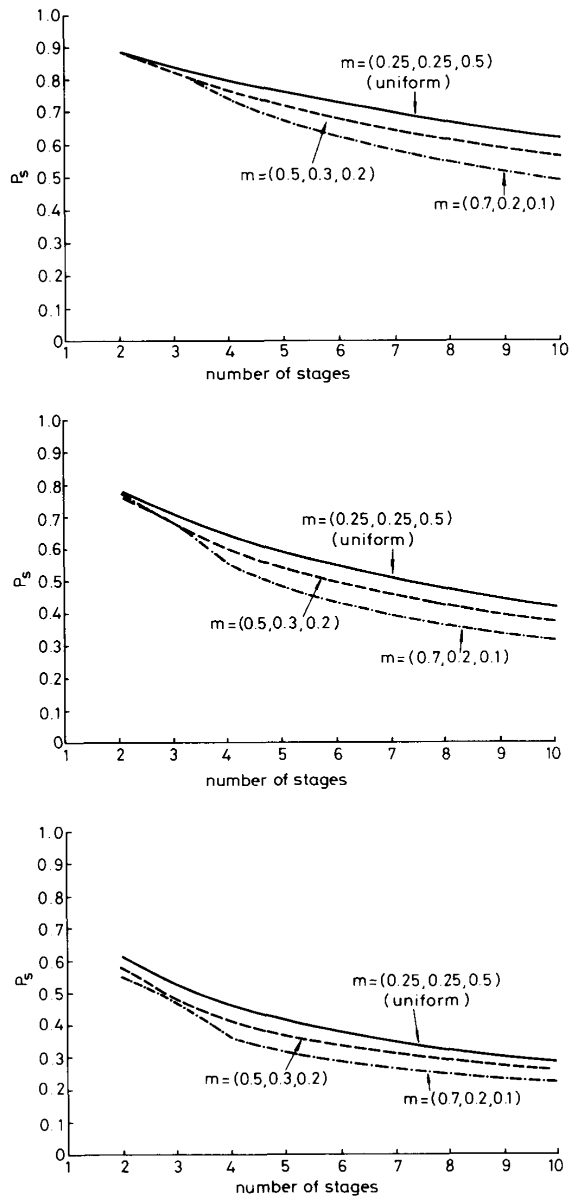

Fig. 4 P_s *for Example 3*
a $P_s(T_n(2), 0.25)$
b $P_s(T_n(2), 0.5)$
c $P_s(T_n(2), 1.0)$

traffic flow becomes more significant as the number of stages increase and the normalised throughput is becoming smaller as the degree of nonuniformity is getting higher. The percentage of degradation is about 3.7%, 4.1%, and 3.5% (or 8.5%, 9.9%, and 9.0%) for $\rho = 0.25$, 0.5, and 1.0, respectively, when $\underline{m} = (0.75, 0.25)$ (or (0.8, 0.2)) and $n = 10$. Moderate offered loads, again, have the most severe degradation.

Example 3: Let

$$T_n(2) = \frac{1}{S_2}\begin{bmatrix} Q_{n-1}(1) & U_{n-1}(m_3) \\ U_{n-1}(m_3) & Q_{n-1}(1) \end{bmatrix}$$

a second example of the second type of nonuniform traffic matrix. According to theorem 2 again, we have

$$P_s(T_n(2), \rho) = A_2 P_s^U(T_n(2), \rho) + B_2 P_s^L(T_n(2), \rho)$$

where $S_2 = m_1 + m_2 + m_3$, $A_2 = (m_1 + m_2)/S_2$, and $B_2 = m_3/S_2$. The values of $P_s^U(T_n(2), \rho)$ and $P_s^L(T_n(2), \rho)$ can be computed by lemmas 1 and 2 and the results are shown in Figs. 4(*a*)–4(*c*). The same conclusions, as were stated in Examples 1 and 2, can be drawn for this example. The percentage of degradation is about 9.5%, 10.8%, and 9.8% (or 21.1%, 24.5%, 24.1%) for $\rho = 0.25$, 0.5, and 1.0, respectively, when $\underline{m} = (0.5, 0.3, 0.2)$ (or (0.7, 0.2, 0.1)) and $n = 10$.

6 Conclusions

We have, in this paper, explored the performance evaluation of banyan networks under certain types of nonuniform traffic requirements. The two types of nonuniform traffic matrices we studied allow more accurate modelling of real-world systems. It is found that, compared to the uniform traffic model, the performance of banyan networks degrades under nonuniform traffic requirements. Moreover, a higher degree of nonuniformity leads to a larger drop in the normalised throughput. This research is a starting point for dealing with the performance of banyan networks with inhomogeneous traffic flow. Considerable further research work can be done in this area.

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8 Appendix

8.1 Computation of $q_{n-1}(n, k, \rho)$

Throughout this appendix, the traffic matrix and the offered load of each source are assumed to be $T_n(k)$ and ρ, respectively. Without loss of generality, we assume the specific request is originated at source 0 and is destined to destination 2^{n-1}. Let $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ be the representation (address) of sources and destinations. Fig. 5

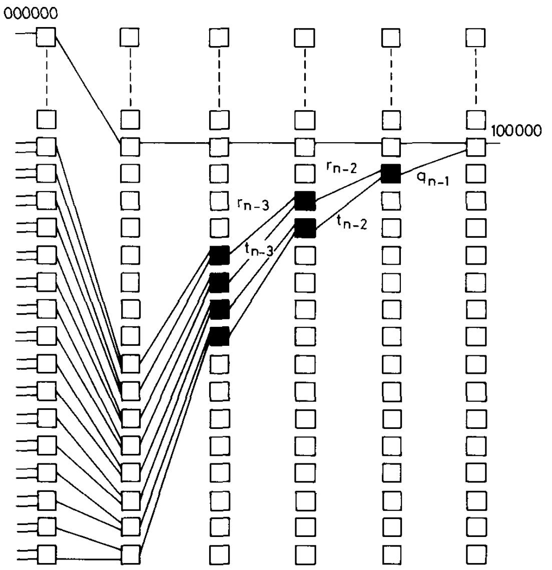

Fig. 5 *64 × 64 banyan network*

shows the route of the specific request and the routes of connection requests originated at sources located in the lower half which can possibly block the specific request for a 64 × 64 banyan network. For convenience, we call those nodes which have inputs that can possibly block the specific request the related nodes. In Fig. 5, the dark switching elements are examples of related nodes. Furthermore, let r_i, $i = 0, 1, \ldots, n-2$, denote the probability that the upper input link of the upmost related node in the ith stage receives an active request, and t_i, the corresponding probability for the lower input link.

Since the destination address of the specific request is assumed to be 2^{n-1}, according to the traffic matrix, the sources located in the lower half can be partitioned into k groups. Group 0 consists of sources whose address satisfies $a_{n-1} = 1$ and $a_{n-2} = \cdots = a_{n-k} = 0$; and group i, $1 \leqslant i \leqslant k-1$, consists of sources whose address satisfies $a_{n-1} = 1$, $a_{n-2} = \cdots = a_{n-k+i} = 0$, and $a_{n-k+i-1} = 1$. Notice that group 0 has 2^{n-k} members and group i, $1 \leqslant i \leqslant k-1$, has $2^{n-k+i-1}$ members. Depending on the relative values of n and k, to evaluate the value of $q_{n-1}(n, k, \rho)$, we need to consider three possible cases separately.

Case 1: $n \geqslant 2k$

Step 1: (Stage 1 to Stage $n-k$)
Let $x_0 = \rho$

Do $i = 1, k$

$$x_i = x_{i-1}A_{k-i+1}(2 - x_{i-1}A_{k-i+1})$$

$$y_i = x_{i-1}B_{k-i+1}(2 - x_{i-1}B_{k-i+1})$$

$$r_{n-k} = p^B_{n-2k}(x_k) \quad \text{and} \quad t_{n-k} = p^B_{n-2k}(y_k)$$

Step 2: (Stage $n - k + 1$ to Stage $n - 2$)
Do $j = 0, k - 3$

$$r_{n-k+1+j} = (1/2)r_{n-k+j} + (1/2)t_{n-k+j} - (1/4)r_{n-k+j}t_{n-k+j}$$

$$t_{n-k+1+j} = p^B_{n-2k+2j+2}(y_{k-j-1})$$

Step 3: (Stage $n - 1$)

$$q_{n-1}(n, k, \rho) = (1/2)r_{n-2} + (1/2)t_{n-2} - (1/4)r_{n-2}t_{n-2}$$

Proof: Remember that the two input links of a node in the ith stage are related to sources which differ only in the ith bit, i.e. bit a_{i-1}, of their representations. Thus the two input links of a node in the first $n - k$ stages are related to sources belonging to the same group. Moreover, according to the traffic matrix, any active request arriving at stage i, $i \geqslant k + 1$, is intended to be routed to the upper or the lower outgoing link with equal probability. Hence it can be seen that $r_{n-k} = p^B_{n-2k}(x_k)$ and $t_{n-k+j} = p^B_{n-2k+2j}(y_{k-j})$ for $j = 0, 1, \ldots, k - 3$, since the upper input link of a node in stage $n - k$ receives requests originated at sources belonging to group 0 and the lower input link of a node in stage $n - k + j$ receives requests originated at sources belonging to group $j + 1$. Besides, the value of $q_{n-1}(n, k, \rho)$ can be obtained after Steps 2 and 3 because, as mentioned above, any active request arriving at stage i, $i \geqslant k + 1$, is intended to be routed to the upper or the lower outgoing link with equal probability and $n - k + 1 \geqslant k + 1$ for $n \geqslant 2k$. This proves the case when $n \geqslant 2k$.

Case 2: $n < 2k$, n even

Step 1: (Stage 1 to Stage $n/2$)
Let $x_0 = \rho$
Do $i = 0, n/2 - 1$

$$x_{i+1} = x_i A_{k-i}(2 - x_i A_{k-i})$$

$$y_{i+1} = x_i B_{k-i}(2 - x_i B_{k-i})$$

$$r_{n/2} = x_{n/2} \quad \text{and} \quad t_{n/2} = y_{n/2}$$

Step 2: (Stage $n/2 + 1$ to Stage k)
Do $j = 0, k - n/2 - 1$

$$r_{n/2+1+j} = (1/2)r_{n/2+j} + (1/2)t_{n/2+j} - (1/4)r_{n/2+j}t_{n/2+j}$$

$$t_{n/2+1+j} = p^B_{2j+2}(y_{n/2-j-1})$$

Step 3: (Stage $k + 1$ to Stage $n - 2$)
Do $j = 0, n - k - 3$

$$r_{k+1+j} = (1/2)r_{k+j} + (1/2)t_{k+j} - (1/4)r_{k+j}t_{k+j}$$

$$t_{k+1+j} = p^B_{2k+2j+2-n}(y_{n-k-j-1})$$

Step 4: (Stage $n - 1$)

$$q_{n-1}(n, k, \rho) = (1/2)r_{n-2} + (1/2)t_{n-2} - (1/4)r_{n-2}t_{n-2}$$

Notice that in practice Step 2 and Step 3 can be merged into one step since the expressions for these two steps are exactly the same. However, they are separately considered because the proofs for them are different. Besides, if $k \geqslant n - 1$, then Steps 3 and 4 are omitted and $q_{n-1}(n, k, \rho) = r_{n-1}$.

Proof: It can be seen that a connection request originated at sources belonging to group i, $i \geqslant 1$, which can possibly block the specific request will enter, from the lower input link, into the upmost related node in stage $n - k + i$. Suppose the lower input link of the upmost related node in stage $n - k + i$ does receive an active request. Clearly the probability that the request is intended to be routed to the upper outgoing link is equal to A_{2k-n-i}, B_{2k-n-i}, or 1/2. For convenience, let $p(i)$ denote this probability. Then we have the following result.

Claim 1: The probability $p(i)$ is given by

$$p(i) = \begin{cases} A_{2k-n-i} & \text{if } i \leqslant k - n/2, \\ 1/2 & \text{if } i \geqslant k - n/2 + 1 \end{cases}$$

The above claim can be proved as follows. According to the traffic matrix, $p(i) = 1/2$ iff (if and only if) the active request has passed at least $k - i + 1$ stages. Therefore, $p(i) = 1/2$ iff $n - k + i > k - i + 1$, or equivalently, iff $i > k - n/2 + 1/2$. Similarly, one can show that $p(i) = A_{2k-n-i}$ iff $i < k - n/2 + 1/2$ and $p(i) = B_{2k-n-i}$ iff $i = k - n/2 + 1/2$. Hence the above claim is true since n is even.

By a similar argument, one can show that if the upper input link of the upmost related node in stage i, $1 \leqslant i \leqslant n/2$, receives an active request, then the probability that the request is intended to be routed to the upper outgoing link is equal to A_{k-i+1}. Hence Step 1 correctly computes $r_{n/2}$ if the following claim is true.

Claim 2: For $i = 0, 1, \ldots, n/2 - 1$, we have $r_i = t_i = x_i$.

The above claim is obviously true for $i = 0, 1, \ldots, n - k - 1$ since both input links of the upmost related node in stages 1 to $n - k$ receive requests originated at sources belonging to the same group, namely group 0. Claim 2 is also true for $i = n - k, n - k + 1, \ldots, n/2 - 1$. Consider the upmost related node in stage $n - k + i$, $i = 1, 2, \ldots, k - n/2$. The upper input link receives requests originated at sources belonging to group 0, group 1, …, or group $i - 1$ and the lower input link receives requests originated at sources belonging to group i. Moreover, for an active request received by either input link of the upmost related node in stage $n - k + i$, it should be routed to the upper outgoing link in any previous stage (except stage 1). Therefore, by claim 1, we know that $r_i = t_i = x_i$ for $i = n - k, n - k + 1, \ldots, n/2 - 1$.

Let us consider stages $n/2 + 1$ to k. With claim 1, it is not hard to see that $t_i = p^B_{2i-n}(y_{n-i})$ for $i \geqslant n/2$. Hence Step 2 correctly computes the values of r_k and t_k if one can show that

$$r_{n/2+1+j} = (1/2)r_{n/2+j} + (1/2)t_{n/2+j} - (1/4)r_{n/2+j}t_{n/2+j}$$

This can be proved stage by stage. Consider, for example, the upmost related node in stage $n/2 + 1$. Suppose the upper input link does receive an active request. The probabilities that this request is intended to be routed to the upper or the lower outgoing link are $A_{k-n/2-1}$ and $B_{k-n/2-1}$, respectively, if the request is originated at sources belonging to group 0, group 1, ..., or group $k - n/2 - 1$. Conversely, the corresponding probabilities are equal to $B_{k-n/2-1}$ and $A_{k-n/2-1}$, respectively, if the request is originated at sources belonging to group $k - n/2$. However, according to the operation principle of switching elements, the probability that the request is originated at group $k - n/2$ is equal to 1/2 since the offered loads of sources are assumed to be identical. Hence the probability that the request is intended to be routed to the upper outgoing link of stage $n/2 + 1$ is equal to $(1/2)A_{k-n/2-1} + (1/2)B_{k-n/2-1} = 1/2$. The same argument can be applied to stages $n/2 + 2$, $n/2 + 3$, ..., and k. Therefore, the value of r_k can be obtained after Step 2.

Finally, any active request arriving at either input link of the upmost related node in stages $k + 1$ to $n - 2$ will be equally likely routed to the upper or the lower outgoing link since it has already passed at least k stages. Hence after Steps 3 and 4, we can obtain the value of $q_{n-1}(n, k, \rho)$. This proves the procedure for Case 2.

Case 3: $n < 2k$, n odd, $n = 2l + 1$

Step 1: (Stage 1 to Stage l)
Let $x_0 = \rho$
Do $j = 0, l - 1$

$$x_{j+1} = x_j A_{k-j}(2 - x_j A_{k-j})$$

$$y_{j+1} = x_j B_{k-j}(2 - x_j B_{k-j})$$

$$r_l = t_l = x_l$$

Step 2: (Stage $l + 1$ to Stage k)

$$r_{l+1} = r_l A_{k-l} + t_l B_{k-l} - r_l t_l A_{k-l} B_{k-l}$$

$$t_{l+1} = p_1^B(y_l)$$

Do $j = 0, k - l - 2$

$$r_{l+2+j} = (1/2)r_{l+1+j} + (1/2)t_{l+1+j} - (1/4)r_{l+1+j}t_{l+1+j}$$

$$t_{l+2+j} = p_{2j+3}^B(y_{l-j-1})$$

Step 3: (Stage $k + 1$ to Stage $n - 2$)
Do $j = 0, n - k - 3$

$$r_{k+1+j} = (1/2)r_{k+j} + (1/2)t_{k+j} - (1/4)r_{k+j}t_{k+j}$$

$$t_{k+1+j} = p_{2k+2j-n+1}^B(y_{n-k-j-1})$$

Step 4: (Stage $n - 1$)

$$q_{n-1}(n, k, \rho) = (1/2)r_{n-2} + (1/2)t_{n-2} - (1/4)r_{n-2}t_{n-2}$$

The proof for this case is similar to that for the second case and hence is omitted. One point worth mentioning is that stage $l + 1$ behaves differently from other stages in Step 2. The active request, if any, arriving at the upper input link of the upmost related node in stage $l + 1$ is intended to be routed to the upper or the lower outgoing link with probability A_{k-l} and B_{k-l}, respectively. However, the corresponding probabilities for the request arriving at the lower input link are B_{k-l} and A_{k-l}, respectively.