Homogenization of Elliptic Equations in Random Media

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Abstract

In the most general sense, a heterogeneous material is one that is composed of domains of different materials (or phases), such as a composite, or the same material in different states, such as a polycrystal. In many instances, the microstructures can be characterized only statistically, and therefore are referred to as random heterogeneous materials(or random media), the chief of this study.

Consider an elliptic equation :

$$\begin{cases} -div(\mathcal{A}(\varepsilon^{-1}x,\omega)\nabla u^{\varepsilon}(x,\omega)) = f(x) & on\ Q, \\ u^{\varepsilon}(x,\omega)|_{\partial\Omega} = 0 & on\ \partial Q; \end{cases}$$

where $\mathcal{A}$, f, *and* u are in suitable function spaces , $\omega \in \Omega$ and (Ω, Σ, μ) is a suitable probability space. In this study we introduce the ergodic dynamical systems on the probability space to describe the random media; we show the matrix $\mathcal{A}(x,\omega)$ above admits homogenization(see Definition.4.2) and the homogenized matrix is independent of $\omega \in \Omega$.

We give definitions, examples, and proofs about ergodic dynamical systems in section two. Section three is about definition of realizations, and the ergodic theorem. In section four, we recall the definition of homogenization of elliptic equations for individual cases and statistical cases, and use the auxiliary equations to define the homogenized matrix, and prove the main convergence theorem through the div-curl lemma. In section five, we define the random sets of the percolation, consider the existence of the effective conductivity, and state the theorem of the existence of the effective conductivity of such random media.

Contents

List of Notations

Notation	**Definition**
(Ω, Σ, μ)	a probability space
T_x	m-dimensional dynamical system $\{T_x : \Omega \to \Omega \mid x \in \mathbb{R}^m\}$
$\langle f \rangle$	mean value of a function on Ω
$f(T_x(\omega))$	realizations for a μ-measurable function f
$M\{f(T_x(\omega))\}$	mean value of a realization of a function on Ω
∇u	gradient of a function u
$div_x \boldsymbol{p}$	divergence of a vector field $\boldsymbol{p}$
$curl_x \boldsymbol{v}$	curl of a vector field $\boldsymbol{v}$
$\boldsymbol{L}^2(\Omega)$	$(L^2(\Omega))^m$
$\boldsymbol{L}^2_{sol}(\Omega)$	$\{\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) : div_x(\boldsymbol{f}(T_x(\omega))) = 0 \quad in\ \mathbb{R}^m\}$
$\boldsymbol{L}^2_{pot}(\Omega)$	$\{\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) : curl_x(\boldsymbol{f}(T_x(\omega))) = 0 \quad in\ \mathbb{R}^m\}$
$\boldsymbol{V}^2_{sol}(\Omega)$	$\{\boldsymbol{f} \in \boldsymbol{L}^2_{sol}(\Omega) : \langle \boldsymbol{f} \rangle = 0\}$
$\boldsymbol{V}^2_{pot}(\Omega)$	$\{\boldsymbol{f} \in \boldsymbol{L}^2_{pot}(\Omega) : \langle \boldsymbol{f} \rangle = 0\}$
$H^1(Q)$	the Sobolev space $\{f \in L^2(Q) : \nabla f \in \boldsymbol{L}^2(Q) = (L^2(Q))^m\}$
$H^1_0(Q)$	the closure of the set $\mathcal{C}_0^\infty(Q)\, in\, H^1(Q)$
$H^{-1}(Q)$	the set of all continuous linear functionals on $H^1_0(Q)$

1 Introduction

In the most general sense, a heterogeneous material is one that is composed of domains of different materials (or phases), such as a composite , or the same material in different states, such as a polycrystal. When those components are intimately mixed, the parameters (such as conductivity, elasticity, $\cdots$) oscillate very rapidly and the microscopic structure becomes complicated.

In a composite material the heterogeneities are small compared to its global dimension, so we have two scales characterize the material: the microscopic one and the macroscopic one. The microscopic one describes the heterogeneity while the macroscopic one describes the global behaviour of the composite.

The aim of homogenization is finding a systematic approach to find the macroscopic properties by considering the properties of the microscopic structures. In mathematical models, microscopically heterogeneous media are usually described by functions of the form $\mathcal{A}(\varepsilon^{-1}x)$. We want to get a good approximation of the macroscopic behaviour of a heterogeneous material by letting the parameters, which describe the fineness of the microscopic structure, tend to zero in the equations describing phenomena such as heat conduction and elasticity.

For example, a model for the study of the physical behaviour of a heterogeneous body with a fine periodic structure, e.g. in electrostatics, or stationary heat diffusion is given by :

$$\begin{cases} -div(\mathcal{A}(\varepsilon^{-1}x)\nabla u^{\varepsilon}(x)) = f(x) & on\ Q, \\ u^{\varepsilon}(x)|_{\partial\Omega} = 0 & on\ \partial Q; \end{cases}$$

where $Q \subset \mathbb{R}^m$ is a bounded domain which will be considered as a piece of the heterogeneous material. The function $u(x)$ can be considered as the electric potential, the magnetic potential, or the temperature, respectively. $\mathcal{A}(\varepsilon^{-1}\mathrm{x})$ is a matrix (satisfies the elliptic condition (4.1)) with periodic entries and describes the physical properties of the different materials constituting the body . $f(x)$ is the source term.

When the period of the structure is very small, a direct numerical approximation of the solution may be very heavy or impossible. Then homogenization provides an alternative way of approximating such solutions by means of a function $u^0(x)$ which solves the homogenized problem :

$$\begin{cases} -div(\mathcal{A}^0 \nabla u^0(x)) = f(x) & on\ Q, \\ u^0(x)|_{\partial\Omega} = 0 & on\ \partial Q; \end{cases}$$

where $\mathcal{A}^0$ is a constant matrix which just describes as the physical parameter of a homogeneous medium.

The above is an example of an individual periodic medium . In many other instances, the microstructures can be characterized only statistically, and therefore are referred to as random heterogeneous materials(or random media).

The following is the equation we consider in this study :

$$\begin{cases} -div(\mathcal{A}(\varepsilon^{-1}x, \omega)\nabla u^\varepsilon(x, \omega)) = f(x) & on\ Q, \\ u^\varepsilon(x, \omega)|_{\partial\Omega} = 0 & on\ \partial Q; \end{cases}$$

which the coefficient matrix and the solution both depend not only on the spacial variant but also on a suitable probability space (Ω, Σ, μ). Then fix an $\omega \in \Omega$ we have an individual problem.

We study the equation with a dynamical system and prove a result under the assumptions such as ergodicity of a dynamical system and the ellipticity of the coefficient matrix. The main result is the convergence of a homogenized matrix and the matrix is independent of $\omega \in \Omega$, so we don't have to do homogenization on every individual problem. However, we can just prove the convergence under these limited assumptions and the theorem does not tell us how to do a numerical analysis on the homogenized matrix.

The following is the stages how we prove the convergence.

In our study we use the ergodic dynamical system to describe the randomness of random media. We will give definitions and examples of ergodic dynamical systems in section two. We then prove the ergodicity of these examples as well.
In section three, we give the definition of realizations, and state the ergodic theorem. In section four, we recall the definition of homogenization of elliptic equations for individual cases and statistical cases. We study an auxiliary equation to define the homogenized matrix, and state the main convergence theorem through the method of compensated compactness. In section five, we use the random sets to describe the media in percolation, and use the homogenization theorem to prove the effective conductivity of such random media.

2 Ergodic dynamical systems

In this section we introduce the m-dimensional dynamical system on a probability space. And also we define what an ergodic dynamical system is and give some examples of ergodic dynamical systems with proofs.

2.1 Definitions and notations

Definition 2.1 (m-dimensional Dynamical system). Let (Ω, Σ, μ) be a probability space. We define an *m-dimensional dynamical system* as a family of mappings $\{T_x : \Omega \to \Omega\}_{x \in \mathbb{R}^m}$ such that :

(1) $T_0 = I$, where I is the identity mapping,
$T_{x+y} = T_x \circ T_y$ *for all* $x, y \in \mathbb{R}^m$ *(the group property).*
Note that from above we have : $T_x^{-1} = T_{-x}$.

(2) *For all* $x \in \mathbb{R}^m$, and all μ-measurable set $\mathcal{F} \subset \Omega$ we have:
$T_x(\mathcal{F})$ *is* $\mu-$ *measurale*, and $\mu(T_x^{-1}(\mathcal{F})) = \mu(\mathcal{F})$. *(measure preserving)*;

(3) $f \circ T_x$ is measurable on $\Omega \times \mathbb{R}^m$ for any μ-measurable function f on Ω where $\mathbb{R}^m$ is endowed with the Lebesgue measure.

Definition 2.2 (Invariant function). A μ-measurable function f on Ω is said to be ***invariant*** with respect to T_x if

$$f(T_x(\omega)) = f(\omega), \ \textit{for all given } x \in \mathbb{R}^m \textit{ and a.e in } \Omega.$$

Definition 2.3 (Ergodic dynamical system). A dynamical system T_x is said to be ***ergodic*** if **every invariant function is constant a.e in** Ω. We also say that the measure μ is ergodic with respect to T_x.

Definition 2.4 (Realization). Let f be a μ-measurable function on Ω. For a fixed $\omega \in \Omega$, $f(T_x(\omega))$ can be regarded as a function of argument $x \in \mathbb{R}^m$, we call $f(T_x(\omega))$ a ***realization*** of *f*.

2.2 Examples and justifications

Now we give some examples of dynamical systems and prove their ergodicity.

Example 2.1 (Periodic case). Let (Ω, Σ, μ) be a probability space where $\Omega = \square_m = \{\omega \in \mathbb{R}^m, 0 \leq \omega_j < 1, j = 1, \cdots, m\}$, Σ is a suitable σ-algebra, and μ is the probability measure. Then $T_x(\omega) = \omega + x$ $(mod\ 1)$ defines a dynamical system on Ω.

Proof.

Step 1. First, we show that T_x is a dynamical system :

(1) $T_0(\omega) = 0 + \omega \pmod 1 = \omega \pmod 1$ *(the identity mapping)*

$$\begin{aligned} T_x \circ T_y(\omega) &= T_x\big(y + \omega \pmod 1\big) \\ &= x + y + \omega \pmod 1 \\ &= T_{x+y}(\omega) \qquad (\textit{the group property}) \end{aligned}$$

(2) Since the Lebesgue measure is translation invariant , then for all $x \in \mathbb{R}^m$ and every μ-measurable set $\mathcal{F} \in \Omega$, we have:

$$T_x(\mathcal{F}) \ is\ \mu - measurale,\ and\ \mu(T_x^{-1}(\mathcal{F})) = \mu(\mathcal{F})$$

Step 2. Now, we show that T_x is ergodic :
Let $f(\omega)$ be $\mu - measurable$ on Ω and invariant :

$$f(\omega) = f(T_x(\omega)) = f(x + \omega \pmod 1)\ for\ all\ x \in \mathbb{R}^m\ a.e.\ in\ \Omega.$$

Fix an $\omega \in \Omega$, and choose $x_\omega \in \mathbb{R}^m$ *by* $x_\omega = -\omega$. Then for this ω, we have

$$f(\omega) = f(T_{x_\omega}(\omega)) = f(0)$$

Since $\omega \in \Omega$ is arbitrary, then

$$f(\omega) = f(0) = constant, \text{ for all } \omega \in \Omega.$$

Thus, every invariant function is constant, so T_x is ergodic.

□

Example 2.2 (**Quasiperiodic case).** Let (Ω, Σ, μ) be a probability space where $\Omega = \square_m = \{\omega \in \mathbb{R}^m, 0 \leq \omega_j < 1, j = 1, \cdots, m\}$, Σ is a suitable σ-algebra, and μ is the probability measure. For $x \in \mathbb{R}^m$, set

$$T_x(\omega) = \omega + \boldsymbol{\lambda} x \pmod 1,$$
$$where\ \boldsymbol{\lambda} = \lambda_{ij}\ is\ an\ m \times\ m\ invertible\ matrix.$$

Proof.

Step 1. First, we show that T_x is a dynamical system :

(1) $T_0(\omega) = \omega + \boldsymbol{\lambda} 0 \pmod 1 = \omega \pmod 1$ *(the identity mapping)*

$$\begin{aligned}
T_x \circ T_y(\omega) &= T_x(\omega + \boldsymbol{\lambda} y \pmod 1) \\
&= T_x\Bigg(\begin{pmatrix} \omega_1 + \sum_{j=1}^m \lambda_{1j} y_j \\ \vdots \\ \omega_m + \sum_{j=1}^m \lambda_{mj} y_j \end{pmatrix} \pmod 1 \Bigg) \\
&= \Bigg(\begin{pmatrix} \omega_1 + \sum_{j=1}^m \lambda_{1j} y_j \\ \vdots \\ \omega_m + \sum_{j=1}^m \lambda_{mj} y_j \end{pmatrix} \pmod 1 \Bigg) + \\
&+ \begin{pmatrix} \sum_{j=1}^m \lambda_{1j} x_j \\ \vdots \\ \sum_{j=1}^m \lambda_{mj} x_j \end{pmatrix} \pmod 1 \\
&= \begin{pmatrix} \omega_1 + \sum_{j=1}^m \lambda_{1j} (x+y)_j \\ \vdots \\ \omega_m + \sum_{j=1}^m \lambda_{mj} (x+y)_j \end{pmatrix} \pmod 1 \\
&= T_{x+y}(\omega) \qquad \textit{(the group property)}
\end{aligned}$$

(2) Since the Lebesgue measure is translation invariant , then *for all* $x \in \mathbb{R}^m$ and every μ-measurable set $\mathcal{F} \in \Omega$, we have:

$$T_x(\mathcal{F}) \ \textit{is} \ \mu - \textit{measurale}, \ \textit{and} \ \mu(T_x(\mathcal{F})) = \mu(\mathcal{F})$$

Step 2. Now, we show that T_x is ergodic :
Suppose that f is a $\mu - \textit{measurable}$ function defined on Ω and is invariant :

$$f(\omega) = f(T_x(\omega)) = f(\omega + \boldsymbol{\lambda} x \pmod 1) \ \textit{for all} \ x \in \mathbb{R}^m \ \textit{a.e. in} \ \Omega.$$

Fix an $\omega \in \Omega$, and choose $x_\omega \in \mathbb{R}^m$ *by* $x_\omega = -\boldsymbol{\lambda}^{-1}\omega$($\boldsymbol{\lambda}$ *is invertible*). Then for this ω, we have

$$f(\omega) = f(T_{x_\omega}(\omega)) = f(0)$$

Since $\omega \in \Omega$ is arbitrary, then

$$f(\omega) = f(0) = \textit{constant}, \ \text{for all } \omega \in \Omega.$$

Thus, every invariant function is constant, so T_x is ergodic.

$\square$

Example 2.3 ([7]**Tile-based random media case).** In this case we establish a dynamical system to describe the so called tile-based random media. Loosely speaking, these are structures obtained by randomly arranging tiles. A simple example is a checkerboard structure where the colors of tiles are painted randomly.

Step 1. The probability space: Let Y be the set of all possible outcomes for a single tile and consider the probability space $(Y, \mathcal{F}_Y, \mu_Y)$, where $\mathcal{F}_Y$ is an appropriate σ-algebra and μ_Y is a probability measure. Define the product space $(S, \mathcal{F}_S, \mu_S) = \Pi_{\mathbb{Z}^m}(Y, \mathcal{F}_Y, \mu_Y)$, where $\mathcal{F}_S$ is the product σ-algebra and μ_S is the product measure. Finally, to account for translations, define the overall probability space

$$(\Omega, \mathcal{F}, \mu) = (S, \mathcal{F}_S, \mu_S) \otimes (\mathbf{Tor}^m, \mathbb{B}(\mathbf{Tor}^m), L^m)),$$

where $\mathbf{Tor}^m = [0, 1)^m$,$\mathbb{B}(\mathbf{Tor}^m)$ is the Borel σ-algebra on $\mathbf{Tor}^m$ and L^m is the Lebesgue measure.

Step 2 (**The dynamical system).** An element $s \in S$ has the form

$$s = \{y_{\boldsymbol{j}}\}_{\boldsymbol{j} \in \mathbb{Z}^m}, \qquad y_{\boldsymbol{j}} \in Y,$$

and an element $\omega \in \Omega$ has the form

$$\omega = (s, \tau) \qquad s \in S,\ \tau \in \mathbf{Tor}^m.$$

First we define the dynamical system $\{\hat{T}_{\boldsymbol{z}}\}_{\boldsymbol{z} \in \mathbb{Z}^m}$ on S by

$$\hat{T}_{\boldsymbol{z}}(\{y_{\boldsymbol{j}}\}_{\boldsymbol{j} \in \mathbb{Z}^m}) = \{y_{\boldsymbol{j}+\boldsymbol{z}}\}_{\boldsymbol{j} \in \mathbb{Z}^m}.$$

Define the projection operators $P_1 : \mathbb{R}^m \to \mathbb{Z}^m$ and $P_2 : \mathbb{R}^m \to \mathbf{Tor}^m$ by

$$P_1(\boldsymbol{x}) = [\boldsymbol{x}], \quad \boldsymbol{x} \in \mathbb{R}^m, \quad P_2(\boldsymbol{x}) = \boldsymbol{x} - [\boldsymbol{x}] \quad \boldsymbol{x} \in \mathbb{R}^m.$$

Here $[\boldsymbol{x}]$ is the vector whose elements are greatest integers less or equal to the corresponding elements in $\boldsymbol{x}$. Note that each $\boldsymbol{x} \in \mathbb{R}^m$ has the unique decomposition

$$\boldsymbol{x} = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}).$$

Next, define the dynamical system $\{\hat{R}_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathbb{R}^m}$ on $\mathbf{Tor}^m$ by

$$\hat{R}_{\boldsymbol{x}}(\tau) = P_2(\boldsymbol{x} + \tau) = (\boldsymbol{x} + \tau) - [\boldsymbol{x} + \tau] = \boldsymbol{x} + \tau \pmod 1, \quad \tau \in \mathbf{Tor}^m.$$

Note here $\hat{R}_{\boldsymbol{x}}(\tau)$ is just the same dynamical system as in Example 2.1, and so $\hat{R}_{\boldsymbol{x}}(\tau)$ is ergodic. It can be shown that the dynamical system $\{\hat{T}_{\boldsymbol{z}}\}_{\boldsymbol{z} \in \mathbb{Z}^m}$ is ergodic([6, Ch.9]). Finally, we define the dynamical system $\{T_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathbb{R}^m}$ on Ω by

$$T_{\boldsymbol{x}}(\omega) = T_{\boldsymbol{x}}(s, \tau) = (\hat{T}_{P_1(\boldsymbol{x}+\tau)}(s), \hat{R}_{\boldsymbol{x}}(\tau)), \qquad s \in S, \tau \in \mathbf{Tor}^m.$$

Now we prove the following:

Proposition 2.1. $\{T_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{R}^m}$ *defined above is ergodic.*

Proof. Let f be μ-measurable on Ω which is invariant under T,that is,

$$f(T_{\boldsymbol{x}}(\omega)) = f(\omega),\ \textit{for all}\ x \in \mathbb{R}^m, a.e.\ \Omega. \tag{2.1}$$

Recall that $\omega \in \Omega$ has the form $\omega = (s, \tau)$ and $\tau \in \mathbf{Tor}^m$. Now by (2.1) we have also $f(T_{\boldsymbol{z}}(\omega)) = f(\omega)$ *for all* $z \in \mathbb{Z}^m$. Then, using $\hat{R}_{\boldsymbol{z}}(\tau) = \tau$ *for all* $z \in \mathbb{Z}^m$, this gives :

$$f(s,\tau) = f(T_{\boldsymbol{z}}(s,\tau)) = f(\hat{T}_{\boldsymbol{z}}(s), \hat{R}_{\boldsymbol{z}}(\tau)) = f(\hat{T}_{\boldsymbol{z}}(s), \tau).$$

Define $f^\tau(s) = f(s,\tau)$, this takes the form

$$f^\tau(\hat{T}_{\boldsymbol{z}}(s)) = f^\tau(s)\ \textit{for all}\ z \in \mathbb{Z}^m. \tag{2.2}$$

We know $f^\tau(s)$ is measurable on S. Since $\{\hat{T}_{\boldsymbol{z}}\}_{\boldsymbol{z}\in\mathbb{Z}^m}$ is ergodic, then (2.2) implies that f^τ is constant a.e. for each τ. Therefore,

$$f(s,\tau) = \phi(\tau) \quad s \in S, \tau \in \mathbf{Tor}^m. \tag{2.3}$$

Next, using (2.1) again we have

$$f(T_{\boldsymbol{t}}(\omega)) = f(\omega)\ \textit{for all}\ \boldsymbol{t} \in \mathbf{Tor}^m. \tag{2.4}$$

Now, using (2.3) we have

$$f(T_{\boldsymbol{t}}(\omega)) = f(\hat{T}_{P_1(\boldsymbol{t}+\tau)}(s), \hat{R}_{\boldsymbol{t}}(\tau)) = \phi(\hat{R}_{\boldsymbol{t}}(\tau));$$

also, $f(\omega) = f(s,\tau) = \phi(\tau)$. Therefore, we have that

$$\phi(\hat{R}_{\boldsymbol{t}}(\tau))) = f(T_{\boldsymbol{t}}(\omega)) = f(\omega) = f(s,\tau) = \phi(\tau)\ \textit{for all}\ \boldsymbol{t} \in \mathbf{Tor}^m,\ a.e. \tag{2.5}$$

Finally, recalling ergodicity of the dynamical system $\{\hat{R}_{\boldsymbol{t}}\}_{\boldsymbol{t}\in\mathbf{Tor}^m}$, we get that $\phi(\tau) \equiv$ constant a.e. and hence, $f \equiv$ *constant* a.e.
Thus we have proved that every invariant function f is constant, then $\{T_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{R}^m}$ is ergodic. □

Example 2.4 (**Percolations).** There is one more example about percolations on random media, which we will state in section 5 later.

In what follows the dynamical system $T_x(\omega)$ is assumed to be ergodic, Ω is a compact metric space, μ is a Borel measure, and the mapping

$$\mathbb{R}^m \times \Omega \to \Omega,\ \textit{where}\ (x,\omega) \to T_x(\omega)$$

is continuous.

3 Some facts about ergodicity

In this section, we introduce the ***mean value of a realization*** of $f \circ T$ and give some ergodicity related theorems.

Definition 3.1 (**Mean value of functions on** (Ω, Σ, μ) **).** Let $f \in \boldsymbol{L}^{\alpha}(\Omega)$, then we define the mean value of f as the following:

$$\langle f \rangle \stackrel{def}{\equiv} \int_{\Omega} f(\omega) d\mu$$

Definition 3.2 (**Mean value of realizations).** Let $f(x) \in \boldsymbol{L}^1_{loc}(\mathbb{R}^m)$. A number M$\{f\}$ is called the ***mean value*** of f if

$$\lim_{\varepsilon \to 0} \int_K f(\varepsilon^{-1}x)dx = |K| M\{f\}$$

for any Lebesgue measurable bounded set $K \subset \mathbb{R}^m$ (here $|K|$ stands for the Lebesgue measure of K). Under additional assumptions on $f(x)$ the definition of the mean value can be expressed in terms of weak convergence. For instance, let the family of functions $f(x)$ be bounded in $\boldsymbol{L}^{\alpha}_{loc}(\mathbb{R}^m)$ for some $\alpha \geq 1$. Since linear combinations of the characteristic functions of the sets K are dense in $\boldsymbol{L}^{\alpha'}_{loc}(\mathbb{R}^m)$, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, we can replace the definition above by

$$f(\varepsilon^{-1}x) \rightharpoonup M\{f\} \qquad in \qquad \boldsymbol{L}^{\alpha}_{loc}(\mathbb{R}^m).$$

Theorem 3.1 (**The Birkhoff Ergodic Theorem).** [1, 1994,p.225]*Let* $f \in \boldsymbol{L}^{\alpha}(\Omega), \alpha \geq 1$. *Then for almost* $\omega \in \Omega$ *the realization* $f(T_x(\omega))$ *possesses a mean value in the sense*

$$f(T_{\varepsilon^{-1}x}(\omega)) \rightharpoonup M\{f(T_x(\omega))\} \qquad in \qquad \boldsymbol{L}^{\alpha}_{loc}(\mathbb{R}^m).$$

Moreover, the mean value $M\{f(T_x(\omega))\}$*, considered as a function of* $\omega \in \Omega$*, is invariant, and*

$$\langle f \rangle \stackrel{def}{\equiv} \int_{\Omega} f(\omega) d\mu = \int_{\Omega} M\{f(T_x(\omega))\} d\mu.$$

In particular, if the system T_x *is ergodic, then*

$$M\{f(T_x(\omega))\} = \langle f \rangle \qquad for\ almost\ all\ \omega \in \Omega.$$

Definition 3.3 (**Sobolev spaces).** Let $Q \subset \mathbb{R}^m$ be a bounded domain, then we define the following:

(1) A vector $\boldsymbol{v}(x) = (v_1(x), \cdots, v_m(x)), v_i(x) \in L^1(Q)$, is said to be the ***gradient*** of a function $u(x) \in L^1(Q)$, if *for all* $\varphi(x) \in \mathcal{C}_0^{\infty}(Q)$:

$$\int_Q u(x) \frac{\partial \varphi(x)}{\partial x_i} dx = -\int_Q v_i(x) \varphi(x) dx, \quad i = 1, \cdots, m.$$

The gradient of $u(x)$ is denoted by $\nabla u(x)$.

(2) Denote by $H^1(Q)$ the Sobolev space :

$$H^1(Q) = \{f(x) \in L^2(Q) : \nabla f(x) \in \boldsymbol{L}^2(Q) = (L^2(Q))^m\}.$$

Equipped with the scalar product

$$(u, v) = \int_Q u(x)v(x)dx + \int_Q \left(\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}\right)dx$$

$H^1(Q)$ becomes a Hilbert space(here the summation over repeated indices is assumed.) The norm corresponding to the above scalar product is

$$\|u(x)\|^2_{H^1(Q)} = \|u(x)\|^2_0 + \|u(x)\|^2_1,\ where$$
$$\|u(x)\|^2_0 = \int_Q u^2(x)dx,\ and\ \|u(x)\|^2_1 = \int_Q |\nabla u(x)|^2 dx.$$

(3) $H^1_0(Q) \overset{def}{\equiv}$ the closure of the set $\mathcal{C}^\infty_0(Q)\ in\ H^1(Q)$.

(4) $H^{-1}(Q) \overset{def}{\equiv}$ the set of all continuous linear functionals on $H^1_0(Q)$.

(5) A sequence of functions $f_n(x) \in H^1_0(Q)$ is said to converge weakly to $f(x) \in H^1_0(Q)$ if :

$$\lim_{n\to\infty} (f_n(x), g(x)) = (f(x), g(x)) \quad \text{for all } g(x) \in H^1_0(Q)$$

We denote the weak convergence by :

$$f_n(x) \rightharpoonup f(x) \qquad in\ H^1_0(Q).$$

Similarly for a sequence of functions $f_n(x) \in L^2(Q)$, we say that

$$f_n(x) \rightharpoonup f(x) \qquad in\ L^2(Q)$$

$$if \qquad \lim_{n\to\infty} \int_Q f_n(x)\varphi(x)dx = \int_Q f(x)\varphi(x)dx \quad \text{for all } \varphi \in L^2(Q).$$

For the *-weak convergence of $f_n(x) \in L^1(Q)$ we define as following :

$$f_n(x) \overset{*}{\rightharpoonup} f(x) \qquad in\ L^1(Q)$$

$$if \qquad \lim_{n\to\infty} \int_Q f_n(x)\varphi(x)dx = \int_Q f(x)\varphi(x)dx \quad \text{for all } \varphi \in C^\infty_0(Q).$$

(6) For any vector field $\boldsymbol{p}(x) \in \boldsymbol{L}^2(Q)$, the ***divergence*** is defined to be an element of the space $H^{-1}(Q)$ defined by

$$(div_x \boldsymbol{p}(x), \varphi(x)) = -\int_Q \boldsymbol{p}(x) \cdot \nabla\varphi(x)dx,\ for\ all\ \varphi(x) \in H^1_0(Q),$$

where "$\cdot$" denotes the scalar product of two vectors.

(7) For any vector field $\boldsymbol{v} \in \boldsymbol{L}^2(Q)$, the relations

$$(\boldsymbol{curl}_x \boldsymbol{v}(x), \varphi(x))_{ij} = -\int_Q \left(v_i(x) \frac{\partial \varphi(x)}{\partial x_j} - v_j(x) \frac{\partial \varphi(x)}{\partial x_i} \right) dx,$$
$$\textit{for all } \varphi(x) \in H_0^1(Q), \textit{ where } i, j = 1, \cdots, m,$$

define a skew-symmetric matrix $\boldsymbol{curl}_x \boldsymbol{v}(x)$, whose elements belong to the space $H^{-1}(Q)$.

Definition 3.4 (**solenoidal vector field).** Let $Q \subset \mathbb{R}^m$ be a bounded domain. A vector field $\boldsymbol{f}(x) = (f_1(x), f_2(x), \cdots, f_m(x)), f_i(x) \in L^2(Q)$ is said to be ***solenoidal*** if

$$\int_{\mathbb{R}^m} f_i(x) \frac{\partial \varphi(x)}{\partial x_i} dx = 0, \textit{ for all } \varphi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^m),$$

i.e, $div_x(\boldsymbol{f}(x)) = 0$ *in* $\mathbb{R}^m$.

Definition 3.5 (**potential vector field).** Let $Q \subset \mathbb{R}^m$ be a bounded domain. A vector field $\boldsymbol{f}(x) = (f_1(x), f_2(x), \cdots, f_m(x)), f_i(x) \in L^2(Q)$ is said to be ***vortex-free*** if

$$\int_{\mathbb{R}^m} \left(f_i(x) \frac{\partial \varphi(x)}{\partial x_i} - f_j(x) \frac{\partial \varphi(x)}{\partial x_i} \right) dx = 0, \textit{ for all } \varphi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^m),$$

i.e, $\boldsymbol{curl}_x(\boldsymbol{f}(x)) = 0$ *in* $\mathbb{R}^m$. Moreover, if $Q \subset \mathbb{R}^m$ is simply connected, then we also say that $\boldsymbol{f}(x)$ is ***potential***:

$$\boldsymbol{f}(x) = \nabla g(x), \textit{ where } g(x) \in H^1(Q).$$

Definition 3.6. A vector field $\boldsymbol{f}(\omega) = (f_1(\omega), f_2(\omega), \cdots, f_m(\omega)), f_i(\omega) \in L^2(\Omega)$ is say to be **potential**(resp., **solenoidal**), if almost all its realizations $f(T_x(\omega))$ are **potential**(resp., **solenoidal**). Define the following :

$$\begin{aligned}
\boldsymbol{L}^2_{sol}(\Omega) &= \{\boldsymbol{f}(\omega) \in \boldsymbol{L}^2(\Omega) : div_x(\boldsymbol{f}(T_x(\omega))) = 0 \quad in\ \mathbb{R}^m\}; \\
\boldsymbol{L}^2_{pot}(\Omega) &= \{\boldsymbol{f}(\omega) \in \boldsymbol{L}^2(\Omega) : curl_x(\boldsymbol{f}(T_x(\omega))) = 0 \quad in\ \mathbb{R}^m\}; \\
\boldsymbol{V}^2_{sol}(\Omega) &= \{\boldsymbol{f}(\omega) \in \boldsymbol{L}^2_{sol}(\Omega) : \langle \boldsymbol{f} \rangle = 0\}; \\
\boldsymbol{V}^2_{pot}(\Omega) &= \{\boldsymbol{f} \in \boldsymbol{L}^2_{pot}(\Omega) : \langle \boldsymbol{f} \rangle = 0\}; \\
\boldsymbol{L}^2(\Omega) &= (L^2(\Omega))^m.
\end{aligned}$$

Theorem 3.2 (**Weyl's Decomposition).** [1, 1994,p.228] *The following orthogonal decompositions are valid:*

$$\boldsymbol{L}^2(\Omega) = \boldsymbol{V}^2_{pot}(\Omega) \oplus \boldsymbol{V}^2_{sol}(\Omega) \oplus \mathbb{R}^m = \boldsymbol{V}^2_{pot}(\Omega) \oplus \boldsymbol{L}^2_{sol}(\Omega) \tag{3.1}$$

Lemma 3.1 (Compensated compactness(the Div-Curl Lemma)). [1, p.138] *Let $\boldsymbol{p}^\varepsilon(x)$, $\boldsymbol{p}^0(x)$, $\boldsymbol{v}^\varepsilon(x)$, and $\boldsymbol{v}^0(x)$ be vector fields in $\boldsymbol{L}^2(Q)$ such that*

$$\boldsymbol{p}^\varepsilon(x) \rightharpoonup \boldsymbol{p}^0(x), \boldsymbol{v}^\varepsilon(x) \rightharpoonup \boldsymbol{v}^0(x) \quad in \quad \boldsymbol{L}^2(Q) \quad as \quad \varepsilon \to 0.$$

If, in addition, $div_x \boldsymbol{p}^\varepsilon(x)$ and $(curl_x \boldsymbol{v}^\varepsilon(x))_{ij}$ are compact sequences in $H^{-1}(Q)$, then

$$\boldsymbol{p}^\varepsilon(x) \cdot \boldsymbol{v}^\varepsilon(x) \overset{*}{\rightharpoonup} \boldsymbol{p}^0(x) \cdot \boldsymbol{v}^0(x) \ in \ L^1(Q).$$

4 Homogenization

Now we recall the setting of the homogenization problem on individual cases and statistical cases.

Definition 4.1 (Individual homogenization problem). Let $\mathcal{A}$(x)=a_{ij}(x), $a_{ij}(x) \in L^\infty(\mathbb{R}^m)$, be a given matrix satisfying the condition of ellipticity :

$$a_{ij}(x)\xi_i\xi_j \geq \nu_1|\xi|^2, \ for \ all \ x, \xi \in \mathbb{R}^m, \ where \ \nu_1 > 0. \tag{4.1}$$

We say that the matrix $\mathcal{A}(x)$ ***admits homogenization*** if there exists a ***constant elliptic matrix*** $\mathcal{A}^0$ such that for any bounded domain $Q \subset \mathbb{R}^m$ and any $f \in H^{-1}(Q)$ the solutions u^ε of the Dirichlet problems

$$div_x(\mathcal{A}(\varepsilon^{-1}x)\nabla u^\varepsilon(x)) = f(x), \quad u^\varepsilon(x) \in H_0^1(Q), \tag{4.2}$$

possess the following properties of convergence :

$$\begin{cases} u^\varepsilon(x) \rightharpoonup u^0(x) & in \quad H_0^1(Q), \\ \mathcal{A}(\varepsilon^{-1}x)\nabla u^\varepsilon(x) \rightharpoonup \mathcal{A}^0\nabla u^0(x) & in \quad \boldsymbol{L}^2(Q), \end{cases}$$

where u^0 is the solution of the homogenized Dirichlet problem

$$div_x(\mathcal{A}^0(x)\nabla u^0(x)) = f(x), \quad u^0(x) \in H_0^1(Q). \tag{4.3}$$

The above homogenization problem concerns an individual matrix $\mathcal{A}(x)$ and is therefore referred to as the problem of individual homogenization.

The theory of operators with random coefficients deals with a matrix $\mathcal{A}(\omega)$ defined on Ω :

Definition 4.2 (Random coefficient homogenization problem). Let $\mathcal{A}(\omega) = a_{ij}(\omega)$, $a_{ij}(\omega) \in L^\infty(\Omega)$ and satisfying the following ellipticity condition

$$a_{ij}(\omega)\xi_i\xi_j \geq \nu_1|\xi|^2, \ for \ all \ \xi \in \mathbb{R}^m, \ where \ \nu_1 > 0 \tag{4.4}$$

for almost all $\omega \in \Omega$.

Consider the realizations $\mathcal{A}(T_x(\omega))$ and the problem consists in describing the homogenization for almost all $\omega \in \Omega$. We say that the matrix $\mathcal{A}(\omega)$ ***admits homogenization*** if there exists a constant elliptic matrix $\mathcal{A}^0$ such that for any bounded domain $Q \subset \mathbb{R}^m$ and any $f \in H^{-1}(Q)$ the solutions $u^\varepsilon(x,\omega)$ of the Dirichlet problems

$$div_x(\mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^\varepsilon(x,\omega)) = f(x), \quad u^\varepsilon(x) \in H_0^1(Q), \tag{4.5}$$

possess the following properties of convergence

$$\begin{cases} u^\varepsilon(x,\omega) \rightharpoonup u^0(x) & in \quad H_0^1(Q), \\ \mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^\varepsilon(x,\omega) \rightharpoonup \mathcal{A}^0\nabla u^0(x) & in \quad \boldsymbol{L}^2(Q), \end{cases}$$

where $u^0(x)$ is the solution of the Dirichlet problem

$$div_x(\mathcal{A}^0\nabla u^0(x)) = f(x), \quad u^0(x) \in H_0^1(Q). \tag{4.6}$$

4.1 Auxiliary Equations

For each $\xi \in \mathbb{R}^m$ consider the following problem

$$\langle \boldsymbol{\varphi}(\omega) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle = 0, \ \textit{for all}\ \boldsymbol{\varphi} \in \boldsymbol{V}_{pot}^2(\Omega), \boldsymbol{v}_\xi(\omega) \in \boldsymbol{V}_{pot}^2(\Omega). \tag{4.7}$$

The existence of a solution for this problem follows from the Lax-Milgram Lemma [1, p.7] and the estimate

$$\langle \boldsymbol{v}_\xi(\omega) \cdot \mathcal{A}(\omega)\boldsymbol{v}_\xi(\omega)\rangle \geq \nu_1 \|\boldsymbol{v}_\xi(\omega)\|_{\boldsymbol{L}^2(\Omega)}^2.$$

Equation (4.7) can be written in the concise form:

$$\boldsymbol{v}_\xi(\omega) \in \boldsymbol{V}_{pot}^2(\Omega), \quad \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega)) \in \boldsymbol{L}_{sol}^2(\Omega). \tag{4.8}$$

Hence, it is easy to see that for a typical realization equation (4.7) is reduced to an elliptic equation in $\mathbb{R}^m$. Indeed, let $\nabla u_\xi(x,\omega) = \boldsymbol{v}_\xi(T_x(\omega))$ then

$$div_x\bigg(\mathcal{A}(T_x(\omega))(\xi + \nabla u_\xi(x,\omega))\bigg) = 0. \tag{4.9}$$

The solution $\boldsymbol{v}_\xi(\omega)$ of problem (4.7) depends linearly on $\xi \in \mathbb{R}^m$. Therefore $\langle \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle$ is a linear form with respect to ξ. We then define the homogenized matrix $\mathcal{A}^0$ by

$$\mathcal{A}^0\xi \overset{def}{\equiv} \langle \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle. \tag{4.10}$$

In conjunction with problem (4.7) consider a similar problem for the conjugate operator, which can be written in the form

$$\langle \boldsymbol{w}_\lambda(\omega)\rangle = \lambda, \quad \boldsymbol{w}_\lambda(\omega) \in \boldsymbol{L}^2_{pot}(\Omega), \quad \boldsymbol{w}_\lambda(\omega)\mathcal{A}(\omega) \in \boldsymbol{L}^2_{sol}(\Omega). \tag{4.11}$$

Here the dependence of the solution $\boldsymbol{w}_\lambda(\omega)$ on $\lambda \in \mathbb{R}^m$ is also linear, and therefore $\langle \boldsymbol{w}_\lambda(\omega)\mathcal{A}(\omega)\rangle = \lambda\mathcal{C}^0$, where $\mathcal{C}^0$ is a constant matrix.

Claim 4.1. $\mathcal{A}^0 = \mathcal{C}^0$**.**

Proof. Indeed, the orthogonality properties

$$\mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega)) \perp (\boldsymbol{w}_\lambda(\omega) - \lambda), \quad \boldsymbol{w}_\lambda(\omega)\mathcal{A}(\omega) \perp \boldsymbol{v}_\xi(\omega)$$

imply that

$$\begin{aligned} \lambda \cdot \mathcal{A}^0\xi &= \langle \lambda \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\ &= \langle \boldsymbol{w}_\lambda \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\ &= \langle \boldsymbol{w}_\lambda \cdot \mathcal{A}(\omega)(\xi)\rangle \\ &= \lambda \cdot \mathcal{C}^0\xi \end{aligned}$$

□

It follows that for a given symmetric matrix $\mathcal{A}(\omega)$ the homogenized matrix $\mathcal{A}^0$ will also be symmetric.

Claim 4.2. $\mathcal{A}^0$ *satisfies the condition of ellipticity.*

Proof. From (4.8) we have $\mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega)) \perp \boldsymbol{v}_\xi(\omega)$, and then

$$\begin{aligned} \xi \cdot \mathcal{A}^0\xi &= \langle \xi \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\ &= \langle (\xi + \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\ &\geq \nu_1 \langle |\xi + \boldsymbol{v}_\xi|^2\rangle \\ &\geq \nu_1 |\langle \xi + \boldsymbol{v}_\xi\rangle|^2 \\ &= \nu_1 |\xi|^2 \end{aligned}$$

□

Claim 4.3. *If* $\mathcal{A}(\omega)$ *is symmetric, then :*

$$\xi \cdot \mathcal{A}^0\xi = \inf_{\boldsymbol{v}(\omega) \in \boldsymbol{V}^2_{pot}(\Omega)} \langle (\xi + \boldsymbol{v}(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}(\omega))\rangle. \tag{4.12}$$

Proof.

Step 1. For any $\boldsymbol{v}(\omega) \in \boldsymbol{V}^2_{pot}(\Omega)$, we have:

$$
\begin{aligned}
&\langle(\xi + \boldsymbol{v}(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}(\omega))\rangle \\
&\quad = \langle(((\xi + \boldsymbol{v}_\xi(\omega)) + (\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega))) \cdot \mathcal{A}(\omega)((\xi + \boldsymbol{v}_\xi(\omega)) + (\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega)))\rangle \\
&\quad = \langle(\xi + \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\
&\qquad + \langle(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega))\rangle \\
&\qquad + 2\langle(\boldsymbol{v} - \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\
&\quad \geq \xi \cdot \mathcal{A}^0\xi
\end{aligned}
$$

Step 2. Check:$\langle(\xi + \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle = \xi \cdot \mathcal{A}^0\xi$.
From (4.8) and (4.10), we have :

$$
\begin{aligned}
&\langle(\xi + \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\
&\quad = \langle\xi \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle + \langle(\boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle \\
&\quad = \xi \cdot \mathcal{A}^0\xi
\end{aligned}
$$

Step 3. Check: $\langle(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega))\rangle \geq \nu_1|(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega)|^2$.
Just follow the ellipticity condition of $\mathcal{A}(\omega)$.

Step 4. Check:$\langle(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega)) \cdot \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega))\rangle = 0$.

$$(\boldsymbol{v}(\omega) - \boldsymbol{v}_\xi(\omega)) \in \boldsymbol{V}^2_{pot}(\Omega), \; \textit{and} \; \mathcal{A}(\omega)(\xi + \boldsymbol{v}_\xi(\omega)) \in \boldsymbol{L}^2_{sol}(\Omega).$$

□

4.2 Convergence

Theorem 4.1. [1, Theorem 7.4,p.230] *Let $\mathcal{A}(\omega)$ be a matrix defined on a probability space (Ω, Σ, μ), $\mathcal{A}(\omega)$=$a_{ij}(\omega)$, $a_{ij}(\omega) \in \boldsymbol{L}^\infty(\Omega)$; and let $\mathcal{A}(\omega)$ satisfy the condition of ellipticity* (4.4). *Then for almost all $\omega \in \Omega$ the matrix $\mathcal{A}(T_x(\omega))$ admits homogenization, and the homogenized matrix $\mathcal{A}^0$ is independent of ω.*

Proof.

Step 1. The sequence $u^\varepsilon(x, \omega)$ of the solutions of the Dirichlet problems (4.5) is bounded in $H^1_0(Q)$: Multiply both sides of (4.5) by $\varphi \in C^\infty_0(Q)$ and use the integration by parts we have

$$\begin{aligned}
&\mid \int_Q \nabla\varphi(x)\mathcal{A}(T_{\varepsilon^{-1}}(\omega))\nabla u^{\varepsilon}(x,\omega)dx \mid \\
&\qquad =\mid \int_Q f(x)\varphi(x)dx \mid \\
&\qquad \leq \int_Q \mid f(x)\varphi(x) \mid dx \\
&\qquad \leq \|f(x)\|_{L^2(Q)}\|\varphi(x)\|_{L^2(Q)} \\
&\qquad \leq c_0\|f(x)\|_{L^2(Q)}\|\nabla\varphi(x)\|_{L^2(Q)} \qquad (4.13)
\end{aligned}$$

Replace $\varphi(x)$ with $u^{\varepsilon}(x,\omega) \in H_0^1(Q)$ in (4.13), we then have :

$$\begin{aligned}
&\nu_1|\nabla u^{\varepsilon}(x,\omega)|^2 \\
&\quad \leq\mid \int_Q \nabla u^{\varepsilon}(x,\omega)\mathcal{A}(T_{\varepsilon^{-1}}(\omega))\nabla u^{\varepsilon}(x,\omega)dx \mid \\
&\quad \leq c_0\|f(x)\|_{L^2(Q)}\|\nabla u^{\varepsilon}(x,\omega)\|_{L^2(Q)},\ \textit{where } c_0 \textit{ is a constant.} \qquad (4.14)
\end{aligned}$$

The inequality (4.14) implies that :

$$\begin{cases} \|\nabla u^{\varepsilon}(x,\omega)\|_{L^2(Q)} & \leq c_1\|f(x)\|_{L^2(Q)} \\ \|u^{\varepsilon}(x,\omega)\|_{L^2(Q)} & \leq c_2\|f(x)\|_{L^2(Q)} \end{cases}$$

, where c_1 and c_2 are constant. Then we have $u^{\varepsilon}(x,\omega)$ is bounded in $L^2(Q)$, thus

$$u^{\varepsilon}(x,\omega) \rightharpoonup u^0(x,\omega)\ in\ H_0^1(Q).$$

Since

$$\begin{aligned}
\|\mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^{\varepsilon}(x,\omega)\|_{L^2(Q)} &\leq \|\mathcal{A}(\omega)\|_{L^{\infty}(Q)}\|\nabla u^{\varepsilon}(x,\omega)\| \\
&\leq c_3\|f(x)\|_{L^2(Q)}
\end{aligned}$$

then the sequence of the flows $\boldsymbol{p}^{\varepsilon}(x,\omega) = \mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^{\varepsilon}(x,\omega)$ is bounded in $\boldsymbol{L}^2(Q)$, we then have :

$$\boldsymbol{p}^{\varepsilon}(x,\omega) \rightharpoonup \boldsymbol{p}^0(x,\omega)\ in\ \boldsymbol{L}^2(Q).$$

Step 2. Consider the auxiliary problem (4.11) and set

$$\begin{aligned}
\boldsymbol{q}(x,\omega) &= \boldsymbol{w}_{\lambda}(T_x(\omega))\mathcal{A}(T_x(\omega)), \\
\boldsymbol{q}^{\varepsilon}(x,\omega) &= \boldsymbol{w}_{\lambda}(T_{\varepsilon^{-1}x}(\omega))\mathcal{A}(T_{\varepsilon^{-1}x}(\omega)),
\end{aligned}$$

we have: $curl_x \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega)) = 0$, and $div_x \boldsymbol{q}^\varepsilon(x,\omega) = 0$ in $\mathbb{R}^m$,and the ergodic theorem yields

$$\begin{cases} \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega)) \rightharpoonup \langle \boldsymbol{w}_\lambda(\omega)\rangle = \lambda, \\ \boldsymbol{q}^\varepsilon(x,\omega) \rightharpoonup \langle \boldsymbol{w}_\lambda(\omega)\mathcal{A}(\omega)\rangle = \lambda\mathcal{A}^0 \quad in \quad \boldsymbol{L}^2(Q). \end{cases}$$

Step 3. Since

$$\begin{aligned} &\boldsymbol{p}^\varepsilon(x,\omega)\cdot \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega)) \\ &= \mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^\varepsilon(x,\omega)\cdot \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega)) \\ &= \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega))\mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^\varepsilon(x,\omega) \\ &= \boldsymbol{q}^\varepsilon(x,\omega)\cdot\nabla u^\varepsilon(x,\omega) \end{aligned}$$

Now, together with :

$$\begin{cases} curl_x \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega)) = 0 \\ div_x \boldsymbol{q}^\varepsilon(x,\omega) = 0 \\ curl_x \nabla u^\varepsilon(x,\omega) = 0 \\ div_x \boldsymbol{p}^\varepsilon(x,w) = 0 \end{cases}$$

and the Div-Curl lemma, we have

$$\begin{cases} \boldsymbol{p}^\varepsilon(x,\omega)\cdot \boldsymbol{w}_\lambda(T_{\varepsilon^{-1}x}(\omega)) \overset{*}{\rightharpoonup} \boldsymbol{p}^0(x,\omega)\cdot\lambda \\ \boldsymbol{q}^\varepsilon(x,\omega)\cdot\nabla u^\varepsilon(x,\omega) \overset{*}{\rightharpoonup} \lambda\mathcal{A}^0\nabla u^0(x,\omega) \end{cases}$$

, so $\boldsymbol{p}^0(x,\omega)\cdot\lambda = \lambda\cdot\mathcal{A}^0\nabla u^0(x,\omega)$. It follows that $\boldsymbol{p}^0(x,\omega) = \mathcal{A}^0\nabla u^0(x,\omega)$. Since $div_x\boldsymbol{p}^0(x,\omega) = f(x)$, therefore $u^0(x,\omega)$ is a solution of

$$div_x \mathcal{A}^0\nabla u^0(x,\omega) = f(x).$$

Since $\mathcal{A}^0$ is independent of ω, then we can rewrite the above equation as :

$$div_x \mathcal{A}^0\nabla u^0(x) = f(x).$$

Then the homogenization has been proved.

$\square$

4.3 Conclusion

We have proved that for the elliptic equation :

$$\begin{cases} a_{ij}(\omega)\xi_i\xi_j \geq \nu_1|\xi|^2, \ for\ all\ \xi \in \mathbb{R}^m, \ where\ \nu_1 > 0 \\ div_x(\mathcal{A}(T_{\varepsilon^{-1}x}(\omega))\nabla u^\varepsilon(x,\omega)) = f(x), u^\varepsilon(x) \in H_0^1(Q), \ f \in H^{-1}. \end{cases}$$

the matrix $\mathcal{A}(\omega)$ admit the homogenization and is independent of ω. Note here we only prove the case that the dynamical systems is ergodic. If the dynamical system is not ergodic or the random media can not be described by the dynamical system, then our study can not apply on the case. So the theorem that we have proved here is limited, but somehow useful in some particular cases.

5 Application on percolation

The phenomenon of percolation can be modeled by a random structure of chess-board type. For example, a structure of this kind is obtained if we split the plane into squares, painting each square, independently, black or white with probability p or $1-p$, respectively, where $0 \leq p \leq 1$. Then the union of all black squares forms one kind of random sets $\mathcal{F}$(see Definition5.1 below).

Let us assume that the set $\mathcal{F}$ be a perfect dielectric(i.e., a material with zero conductivity) and the set $\mathbb{R}^m \setminus \mathcal{F}$ be a conductor whose conductivity whose conductivity tensor is the identity matrix $\mathcal{I}$. Hence we can consider the homogenization of percolation in such a random medium.

The above is a specific kind of random set, and in this section, we use the homogenization theory above to give one version of the theorem of existence of the effective conductivity for random sets. Again we can show the existence of the homogenized matrix. However, if we need some more informations of the homogenized matrix, then we must do the numerical analysis for the matrix to find the rate of convergence. Now we define the random sets on the random media:

Definition 5.1 (Random set). Let (Ω, Σ, μ) be a probability space with an ergodic dynamical system $\{T_x\}_{x\in\mathbb{R}^m}$. Fix a μ-measurable set $\mathcal{F} \subset \Omega$. The set $\mathcal{F}_\omega \in \mathbb{R}^m$ obtained from $\mathcal{F}$ by

$$\mathcal{F}_\omega \stackrel{def}{\equiv} \{x \in \mathbb{R}^m, \quad T_x(\omega) \in \mathcal{F}\}$$

is called a *random stationary* or simply, *random*, *set*.

As we can see that the random set $\mathcal{F}_\omega$ obtained from $\mathcal{F}$ is dependent not only on the set $\mathcal{F}$ but also some given point $\omega \in \Omega$. In the following theorem we can see that for any random set $\mathcal{F}_\omega$, the homogenized matrix exists and does not depend on $\omega \in \Omega$.

Theorem 5.1. [1, p.300] *Let $\mathcal{F}_\omega$ be an arbitrary random stationary set in $\mathbb{R}^m$. Then there exists a non-negative matrix $\mathcal{A}^0$ such that for almost every $\omega \in \Omega$ and any bounded domain $Q \in \mathbb{R}^m$ we have*

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\mid Q_t \mid} \inf_{u\in\lambda\cdot x+\mathcal{C}_0^\infty(Q_t)} \int_{Q_t\setminus\mathcal{F}_\omega} |\nabla u|^2 dx \\ &= \lim_{\varepsilon\to 0} \frac{1}{\mid Q \mid} \inf_{u\in\lambda\cdot x+\mathcal{C}_0^\infty(Q_t)} \int_{Q\setminus\mathcal{F}_\omega^\varepsilon} |\nabla u|^2 dx \\ &= \lambda\cdot\mathcal{A}^0\lambda \end{aligned} \tag{5.1}$$

where $Q_t = \{tx, x \in Q\}$ is the homothetic dilatation with ratio t¿0 of the domain Q and $\mathcal{F}_\omega^\varepsilon = \{x \in \mathbb{R}^m, \quad \varepsilon^{-1}x \in \mathcal{F}_\omega\}$. The matrix $\mathcal{A}^0$, called effective conductivity, coincides with the formally homogenize matrix:

$$\lambda\cdot\mathcal{A}^0\lambda = \inf_{\boldsymbol{v}(\omega)\in\boldsymbol{V}_{pot}^2(\Omega)} \int_{\Omega\setminus\mathcal{F}} |\lambda + \boldsymbol{v}(\omega)|^2 d\mu \tag{5.2}$$

where $\boldsymbol{V}_{pot}^2(\Omega) = \{\boldsymbol{f} \in \boldsymbol{L}_{pot}^2(\Omega) : \langle\boldsymbol{f}\rangle = 0\}$.

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