國立交通大學應用數學系

博 士 論 文

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永續美式勒式選擇權

Two-Sided Optimal Stopping Problems and
The Perpetual American Strangle Options

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摘要

本論文在研究永續美式勒式選擇權在超指數型跳躍擴散模型下的定價問題。利用自由邊界問題的方法，我們解決了所對應之最佳停止時間問題，並且求出永續美式勒式選擇權的合理價格。此外，我們也證明了自由邊界問題再加上平滑銜接條件的解之存在性。

Two-Sided Optimal Stopping Problems and The Perpetual American Strangle Options

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Abstract

This study investigates the problem of pricing perpetual American strangle option under a hyper-exponential jump-diffusion model. By using the free boundary problem approach, we solve the corresponding optimal stopping problem and determine the rational price of the perpetual American strangle options. In particular, we prove the existence of solutions to the free boundary problems with the smooth pasting conditions.

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隨著本篇論文的完稿，也象徵著人生的生涯又要進入了另一階段的考驗。回想著本論文從尋找題目、學習相關領域知識到解決問題的過程，若非仰賴了周遭許多人的協助。單憑我自己一個人，本論文幾乎無法完成。藉此以感謝學術、生活或是其他方面幫忙我的人。

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Contents

1 Introduction

An American option is an option allowing the buyer exercises at any time prior to the maturity. In particular, the American option with the infinite time horizon is called the perpetual American option. For an American call option with a finite expiration time, Merton[16] observed that the price of the American option(written on an underlying stock without dividends) coincides with the price of the corresponding European option. However, the American put option(even without dividends) presents a difficult problem. There are no explicit pricing formulas and the optimal exercise boundaries are also not known. But within the Black-Scholes model, McKean[15] solved the problem of pricing the perpetual American put option. In addtion, in the Lévy-based models, Boyarchenko and Levendorskii[6] derived the closed formula for prices of perpetual American put and call options by the theory of pseudo-differential operators. Using the probabilistic techniques, Mordecki and Salminen[17] obtained explicit formulas under the assumption of mixed-exponentially distributed and arbitrary negative jumps for the call options, and negative mixed-exponentially distributed and arbitrary positive jumps for put options. For related works, see Asmussen et al.[2] and the references therein.

Mathematically, a rational price of the perpetual American instrument is

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) \tag{1}$$

where $r \geq 0$ represents the rate of the discounted factor, $g \geq 0$ is the reward function corresponding to the contract and $\mathcal{T}$ is the family of the stopping times. Therefore, to find the price of the perpetual American instruments is equivalent to solve the optimal stopping problem: to find the value function $V(x)$ and the optimal stopping time τ^*, that is, $V(x) = \mathbb{E}_x\left[e^{-r\tau^*} g(X_{\tau^*})\right]$. It is well-known that under some suit conditions, the value function is a solution to the free boundary(Stephan) problem

$$\begin{cases} (\mathcal{L}_X - r)V(x) = 0 & \text{in } \mathcal{C} \\ V(x) = g(x) & \text{on } \mathcal{D} \end{cases} \tag{2}$$

where $\mathcal{C} = \{x \in \mathbb{R}, V(x) > g(x)\}$ (the continuation region) , $\mathcal{D} = \{x \in \mathbb{R}, V(x) = g(x)\}$ (the stopping region) and $\mathcal{L}_X$ is the infinitesimal operator of X. Also, the stopping time $\tau_{\mathcal{D}} = \inf\{t \geq 0 | X_t \in \mathcal{D}\}$ is an optimal stopping time. For more details and related topics about optimal stopping and free boundary problem, we refer to the monograph of Peskir and Shiryaev[19]. Many authors in the literature suggested that the boundary of the stopping region $\mathcal{D}$ is determined by imposing the smooth pasting fit for the value function. Therefore, to solve the optimal stopping problem (1), it suffieces to solve the above free boundary problem with suitable pasting conditions.

Note that the investigations mentioned as above are American call or put types. The corresponding reward functions are one-sided, that means the support of g is either $\{x \in \mathbb{R} | x \geq K\}$ or $\{x \in \mathbb{R} | x \leq K\}$ for some $K \in \mathbb{R}$. It is proved that the corresponding continuation region is a half line in both cases. For example, see Boyarchenko and Levendorskii[6], McKean[15] and Mordecki[18]. In this study, we consider the pricing problem for the perpetual American strangle(straddle) options whose reward function is a combination of the reward functions of the call and put options. Therefore, the corresponding g is of the two-sided form, that is, the support of g is $[l_1, l_2]$ for some l_1 and $l_2 \in \mathbb{R}$. Boyarchenko[4] conjectured that the corresponding value function shall be the solution the free boundary problem (2) and the continuation region $\mathcal{C}$ is a finite interval whose two boundaries both satisfy the smooth pasting condition. Under the hyper-exponential jump-diffusion Lévy processes, they derived the solution to the

$$\begin{cases} (\mathcal{L}_X - r)V(x) = 0 & \text{in } (h_1, h_2) \\ V(x) = g(x) & \text{on } (h_1, h_2)^c \end{cases} \tag{3}$$

by the Wiener-Hopf decomposition for fixed h_1 and h_2. Also, under suitable parameters, the boundaries h_1 and h_2, required to satisfy the smooth pasting conditions, are obtained by the approximation of the Brownian motion. In this investigation, we prove the optimality and the existence problems posed in Boyarchenko[4]. Inspired by Chen et al.[9], we obtain alternative representation of the solution to Eqs.(3) by the ODE approach. For details, see Chang et al.[7]. Using the representation, we prove the conjecture in Boyarchenko[4] and the existence of the solution to Eqs.(3) with embedding the smooth pasting condition. Moreover, we improve the algorithm in Boyarchenko[4] for computing the value function and the continuation region.

The rest of the paper is organized as follows. Section 2 describes the hyper-exponential jump-diffusion Lévy processes that are considered herein. Also, we derive the solution to Eqs.(3) for fixed (h_1, h_2) by direct calculation, not the Wiener-Hopf decomposition. In fact, the associated integro-differential equation in Eqs.(3) is transformed into a homogeneous ODE of higher order. Theorem 2.6 shows that this ODE is solved in closed form and its solution equals the first passage functional

$$\Phi(x) = \mathbb{E}_x \left[e^{-r\tau_{I^c}} g(X_{\tau_{I^c}}) \right] \tag{4}$$

where $I = (h_1, h_2)$ and $\tau_{I^c} = \inf\{t \geq 0 | X_t \in I^c\}$ is the firs exit time from I. In Section 3, we provide the verification theorems for the optimal stopping problems (1) with a general two-sided reward function. In Section 4, we consider the problem of pricing the perpetual American strangle options when the log-price following the hyper-exponential jump-diffusion processes. We verify that for the reward function g of the perpetual American strangle options, $\Phi(x)$ is the value function when both the boundary points of I satisfy the smooth pasting conditions. The existence of such interval is proved in Section 5. Also, alternative algorithm for solving the value function and the continuation region is given herein.(For the Wiener-Hopf approach, see Boyarchenko[4].) Section 6 presents some numerical results of the perpetual American strangle options and Section 7 concludes this paper.

2 Preliminaries

Throughout this paper, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we assume that X is the hyper-exponential jump-diffusion Lévy process, that is,

$$X_t = ct + \sigma B_t + \sum_{n=1}^{N_t} Y_n \tag{5}$$

where $c \in \mathbb{R}$, $\sigma > 0$, $B = (B_t, t \geq 0)$ is a standard Brownian motion, $(N_t; t \geq 0)$ is a Poisson process with rate $\lambda > 0$. Also, $Y = (Y_n, n \geq 0)$ is a sequence of independent random variables with the identical density function

$$f(x) = \sum_{j=1}^{N^+} p_j \eta_j^+ e^{-\eta_j^+ x} 1_{\{x>0\}} + \sum_{j=1}^{N^-} q_j (-\eta_j^-) e^{-\eta_j^- x} 1_{\{x<0\}} \tag{6}$$

where $\eta_1^- < ... < \eta_{N^-}^- < 0 < \eta_1^+ < ... < \eta_{N^+}^+$, p_j's and q_j's are positive with $\sum_{j=1}^{N^+} p_j + \sum_{j=1}^{N^-} q_j = 1$. Assume further that B, N_t and Y are independent. The jump-diffusion process starting from x is simply defined as $x + X_t$ for $t \geq 0$ and we denote its law by $\mathbb{P}_x$. For convenience, we shall write $\mathbb{P}$ in place of $\mathbb{P}_0$. Also $\mathbb{E}_x$ denotes the expectation with respect to the probability measure $\mathbb{P}_x$. Under these model assumptions, we have $\mathbb{E}(e^{zX_t}) = e^{t\psi(z)}$, $z \in i\mathbb{R}$, where ψ is called the characteristic exponent ψ of X and is given by the formula

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + cz + \lambda \left(\sum_{i=1}^{N^+} \frac{p_i \eta_i^+}{\eta_i^+ - z} + \sum_{j=1}^{N^-} \frac{q_j \eta_j^-}{\eta_j^- - z} \right) - \lambda. \tag{7}$$

Also the infinitesimal generator $\mathcal{L}_X$ of X has a domain containing $C_0^2(\mathbb{R})$ and for $h \in C_0^2(\mathbb{R})$,

$$\mathcal{L}_X h(x) = \frac{1}{2}\sigma^2 h''(x) + ch'(x) + \lambda \int h(x+y) f(y) dy - \lambda h(x). \tag{8}$$

We define $\mathcal{L}_X h(x)$ pointwisely by the expression (8) for all functions h on $\mathbb{R}$ such that h', h'' and the integral in (8) exist at x.

Consider the continuous reward function g given by the formula

$$g(x) = g_1(x)\mathbf{1}_{\{x \leq l_1\}} + g_2(x)\mathbf{1}_{\{x \geq l_2\}} \tag{9}$$

for some $-\infty < l_1 \leq l_2 < \infty$. Here $g_1(x)$ is a strictly positive C^∞-function on $(-\infty, l_1)$ and $g_2(x)$ is a strictly positive C^∞-function on (l_2, ∞). We assume further that g_1 is continuous at l_1 with $g_1(l_1) = 0$, g_2 is continuous at l_2 with $g_2(l_2) = 0$ and $\mathbb{E}_x\left[\sup_{t \geq 0} e^{-rt} g(X_t)\right] < \infty$ for all $x \in \mathbb{R}$. Finally, we denote $\tau_{B^c} = \inf\{t \geq 0 | X_t \notin B\}$ as the first exit time form the Borel set B.

Proposition 2.1. *Assume that g_1 is bounded on $(-\infty, l_1)$ and the function $\int_0^\infty g_2(x+y)f(y)dy, x \geq l_2$, is locally bounded. Consider the interval $I = (h_1, h_2)$ for some $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$. If $\widetilde{\Phi}(x)$ is a solution of the boundary value problem:*

$$\begin{cases} (\mathcal{L}_X - r)\widetilde{\Phi}(x) = 0, & x \in I \\ \widetilde{\Phi}(x) = g(x), & x \in I^c \end{cases} \tag{10}$$

and $\widetilde{\Phi}$ is in $C^2(h_1, h_2) \cap C[h_1, h_2]$, then $\widetilde{\Phi}(x) = \Phi(x)$ for all $x \in \mathbb{R}$ where $\Phi(x)$ is denoted by Eq.(4).

Proof. We follow similar arguments as that in Chen et al.[9]. Fix $x \in (h_1, h_2)$. Pick a sequence of functions $\{\widetilde{\Phi}_n\}$ in $\mathcal{C}_0^2(\mathbb{R})$ such that $\widetilde{\Phi}_n \equiv \widetilde{\Phi}$ on $(h_1+\frac{1}{n}, h_2-\frac{1}{n})$ and $\widetilde{\Phi}_n \to \widetilde{\Phi}$ a.s.. Since g_1 is bounded, we can choose $\{\widetilde{\Phi}_n\}$ such that $\{\widetilde{\Phi}_n\}$ are uniformly bounded on $(-\infty, c]$ for any $c \in \mathbb{R}$, and $\widetilde{\Phi}_n(y) \le 2g_2(y)$ for all n and all $y \ge M$. Here $M > h_2$ is a strictly positive constant(independent of n). Consider any $\epsilon > 0$ such that $x \in I_\epsilon = (h_1+\epsilon, h_2-\epsilon)$. Let n be large enough such that $\frac{1}{n} < \epsilon$. By Dynkin's formula, we have

$$\mathbb{E}_x\left[e^{-r(t\wedge\tau_{I_\epsilon^c})}\widetilde{\Phi}_n(X_{\tau_{I_\epsilon^c}\wedge t})\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_{I_\epsilon^c}} e^{-ru}(\mathcal{L}_X - r)\widetilde{\Phi}_n(X_{u-})du\right] + \widetilde{\Phi}(x). \tag{11}$$

For every $u < t \wedge \tau_{I_\epsilon^c}$, we have $X_u \in I_\epsilon$ and hence $\widetilde{\Phi}_n(X_u) = \widetilde{\Phi}(X_u)$. This gives

$$(\mathcal{L}_X - r)\widetilde{\Phi}(X_u) - (\mathcal{L}_X - r)\widetilde{\Phi}_n(X_u) = \int\left[\widetilde{\Phi}(X_u+y) - \widetilde{\Phi}_n(X_u+y)\right]f(y)dy \tag{12}$$

and hence

$$\begin{aligned} &\left|(\mathcal{L}_X - r)[\widetilde{\Phi}(X_u) - \widetilde{\Phi}_n(X_u)]\right| \\ &\le \sup_{z\le M+|h_1|+|h_2|}\left[|\widetilde{\Phi}(z)| + |\widetilde{\Phi}_n(z)|\right] + \sup_{h_1\le z\le h_2}\int_{M+|h_1|}^{\infty} 3g_2(z+y)f(y)dy \\ &< \infty \quad (\text{ by assumptions }) . \end{aligned} \tag{13}$$

Also, by the dominated convergence theorem and Eq.(12), for all $u < t \wedge \tau_{I_\epsilon}^c$, $(\mathcal{L}_X - r)\widetilde{\Phi}_n(X_u) \to (\mathcal{L}_X - r)\widetilde{\Phi}(X_u)$ as $n \to \infty$. By Eq.(11) and the dominated convergence theorem, we have

$$\lim_{n\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau_{I_\epsilon}^c} e^{-ru}(\mathcal{L}_X - r)\widetilde{\Phi}_n(X_u)du\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_{I_\epsilon}^c} e^{-ru}(\mathcal{L}_X - r)\widetilde{\Phi}(X_u)du\right].$$

Note that $|\widetilde{\Phi}_n(x)| \le \sup_n \sup_{z\le M}|\widetilde{\Phi}_n(z)| + 2g(z)$ and $\mathbb{E}_x\left[\sup_{t\ge0} e^{-rt}g(X_t)\right] < \infty$. Letting $n \to \infty$ for both sides of Eq.(11) together with the dominated convergence theorem gives

$$\mathbb{E}_x\left[e^{-r(t\wedge\tau_{I_\epsilon}^c)}\widetilde{\Phi}(X_{\tau_{I_\epsilon}^c\wedge t})\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_{I_\epsilon}^c} e^{-ru}(\mathcal{L}_X - r)\widetilde{\Phi}(X_u)du\right] + \widetilde{\Phi}(x) = \widetilde{\Phi}(x). \tag{14}$$

Note that the last equality follows from the assumption that $(\mathcal{L}_X - r)\widetilde{\Phi} = 0$ in (h_1, h_2). Observe that

$$\mathbb{E}_x\left[\sup_{t\ge0} e^{-rt}\widetilde{\Phi}(X_t)\right] \le \sup_{y\le h_2}\widetilde{\Phi}(y) + \mathbb{E}_x\left[\sup_{t\ge0} e^{-rt}g(X_t)\right] < \infty.$$

By letting $t \to \infty$ in both sides of the equality in Eq.(14) and the dominated convergence theorem, we obtain $\widetilde{\Phi}(x) = \mathbb{E}_x\left[e^{-r\tau^c_{I_\epsilon}}\widetilde{\Phi}(X_{\tau^c_{I_\epsilon}})\right]$. Note that for any fixed $\omega \in \Omega$, either $X_{\tau^c_{I_\epsilon}}(\omega) \to X_{\tau_I{}^c}(\omega)$ as $\epsilon \to 0$ or $X_{\tau^c_{I_\epsilon}}(\omega) = X_{\tau_I{}^c}(\omega)$ for all sufficiently small ϵ. Since $\widetilde{\Phi}(x)$ is continuous, by the dominated convergence theorem, we obtain

$$\begin{aligned}\widetilde{\Phi}(x) &= \lim_{\epsilon\to 0}\mathbb{E}_x\left[e^{-r\tau^c_{I_\epsilon}}\widetilde{\Phi}(X_{\tau^c_{I_\epsilon}})\right] = \mathbb{E}_x\left[\lim_{\epsilon\to 0}e^{-r\tau^c_{I_\epsilon}}\widetilde{\Phi}(X_{\tau^c_{I_\epsilon}})\right] = \mathbb{E}_x\left[e^{-r\tau^c_I}\widetilde{\Phi}(X_{\tau^c_I})\right]\\ &= \mathbb{E}_x\left[e^{-r\tau_I}g(X_{\tau^c_I})\right]\end{aligned}$$

Therefore, the proof is complete. □

Remark 2.2. *The conclusion of Proposition 2.1 still holds for the general jump-diffusion processes as well as the functions g_1 and g_2 are C^∞(not necessary strictly positive) satisfying the conditions in Proposition 2.1.* □

Note that ψ given by Eq.(7) is an analytic function on $\mathbb{C}$ except at a finite number of poles. Also, the equation $\psi(z) - r = 0$ yields $N^+ + N^- + 2$ distinct real zeros. (If $r = 0$, $c \neq 0$ or $N^+ + N^- > 0$ is further assumed.) Set $N = N^+ + N^-$ and let $\beta_1, \beta_2, \cdots, \beta_{N+2}$ be the distinct real zeros of the equation $\psi(z) - r = 0$.

Let $\mathcal{P}_0(z) = \prod_{j=1}^{N^+}(\eta_j^+ - z)\prod_{j=1}^{N^-}(\eta_j^- - z)$. Now, $\mathcal{P}_1(z) = \mathcal{P}_0(z)(\psi(z) - r)$ is a polynomial whose zeros coincide with those of $\psi(z) - r$. Denote by D the differential operator such that its characteristic polynomial is $\mathcal{P}_1(z)$.

Proposition 2.3. *Suppose a solution $\widetilde{\Phi}$ defined on $\mathbb{R}$ to the boundary value problem (10) exists. Then, on (h_1, h_2), $\widetilde{\Phi}$ is infinitely differentiable and satisfies the ODE,*

$$D\widetilde{\Phi} \equiv 0, \quad on\ (h_1, h_2). \tag{15}$$

Hence, on (h_1, h_2), $\widetilde{\Phi}(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x}$ for some constants C_n.

Proof. This proposition is proved by direct computation. Plugging the density function f, given by Eq.(6), into Eq.(8), yields the generator $\mathcal{L}$ that acts on $\widetilde{\Phi}$:

$$\begin{aligned}\mathcal{L}\widetilde{\Phi}(x) = \quad & \frac{\sigma^2}{2}\widetilde{\Phi}''(x) + c\widetilde{\Phi}'(x) + \lambda\left(\sum_{j=1}^{N^+} p_j\eta_j^+ e^{\eta_j^+ x}\int_x^\infty \widetilde{\Phi}(y)e^{-\eta_j^+ y}dy\right.\\ & \left. + \sum_{j=1}^{N^-} q_j(-\eta_j)e^{\eta_j^- x}\int_{-\infty}^x \widetilde{\Phi}(y)e^{-\eta_j^- y}dy\right) - \lambda\widetilde{\Phi}(x).\end{aligned}$$

From this equation and the fact that $\sigma > 0$ and $(\mathcal{L} - r)\widetilde{\Phi} \equiv 0$, $\widetilde{\Phi}$ is infinitely differentiable on (h_1, h_2), as can be established by induction, as in the work of Chen et al.[9].

Next, $\widetilde{\Phi}$ will be shown to satisfy an ODE. Consider the differentiation rule,

$$\begin{aligned}&\left(\eta_j^+ - \frac{d}{dx}\right) p_j\eta_j^+ e^{\eta_j^+ x}\int_x^\infty \widetilde{\Phi}(y)e^{-\eta_j^+ y}dy\\ &= p_j\eta_j^+\left[-\left(\eta_j^+ e^{\eta_j^+ x}\int_x^\infty \widetilde{\Phi}(y)e^{-\eta_j^+ y}dy - \widetilde{\Phi}(x)\right) + \eta_j^+ e^{\eta_j^+ x}\int_x^\infty \widetilde{\Phi}(y)e^{-\eta_j^+ y}dy\right]\\ &= p_j\eta_j^+\widetilde{\Phi}(x),\end{aligned}$$

and similarly, $\left(\eta_j^- - \frac{d}{dx}\right) q_j(-\eta_j^-)e^{\eta_j^- x}\int_x^\infty \widetilde{\Phi}(y)e^{-\eta_j^- y}dy = q_j\eta_j^-\widetilde{\Phi}(x)$. Since $\widetilde{\Phi}$ is infinitely differentiable on (h_1, h_2) and $(\mathcal{L} - r)\widetilde{\Phi} \equiv 0$ on (h_1, h_2),

$$0 = \prod_{i=1}^{N^+}(\eta_i^+ - D)\prod_{j=1}^{N^-}(\eta_j^- - D)(\frac{1}{2}\sigma^2 D^2 + cD - (\lambda + r))\widetilde{\Phi}(x) + \tag{16}$$

$$\lambda\prod_{i=1}^{N^+}(\eta_i^+ - D)\prod_{j=1}^{N^-}(\eta_j^- - D)(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - D} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - D}))\widetilde{\Phi}(x) \tag{17}$$

Hence, Eq.(17) transforms the integro-differential equation $(\mathcal{L} - r)\widetilde{\Phi} \equiv 0$ into an ODE: $\widetilde{D}\Phi \equiv 0$, where $\widetilde{D}$ is a high order differential operator.

To complete the proof, $\widetilde{D}$ must be shown to coincide with D. (See the definition of D in the paragraph above Proposition 2.3.) To show that the characteristic polynomials of D and $\widetilde{D}$ coincide will suffice. Write $\widetilde{\mathcal{P}}(z)$ as the characteristic polynomial of $\widetilde{D}$. Then, by Eq.(17), $\widetilde{\mathcal{P}}$ is given by

$$\begin{aligned}\widetilde{\mathcal{P}}(z) = &\prod_{j=1}^{N^+}(\eta_j^+ - x)\prod_{j=1}^{N^-}(\eta_j^- - x)\left[\frac{1}{2}\sigma^2x^2 + cx - (\lambda + r) + \lambda(\sum_{j=1}^{N^+}\frac{p_j\eta_j^+}{\eta_j^+ - x} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - x})\right]\\ = &\mathcal{P}_0(z)(\psi(z) - r).\end{aligned}$$

This equation reveals that the characteristic polynomial $\mathcal{P}_1(z)$ of D equals that, $\widetilde{\mathcal{P}}(z)$, of $\widetilde{D}$. The proof is complete. □

Proposition 2.4. *Suppose that $\widetilde{\Phi}$ is a solution to the boundary value problem (10) and, on (h_1, h_2), $\widetilde{\Phi}(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x}$ for some constants C_n. Then the constant vector C satisfies the equation*

$$AC = V_g \tag{18}$$

where

$$A = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2}h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-}e^{\beta_{N+2}h_1} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}e^{\beta_{N+2}h_2} \\ e^{\beta_1 h_1} & \cdots & e^{\beta_{N+2}h_1} \\ e^{\beta_1 h_2} & \cdots & e^{\beta_{N+2}h_2} \end{bmatrix} \tag{19}$$

and V_g is a column vector whose components $V_g(n)$ are given by the formula

$$V_g(n) = \begin{cases} \int_{-\infty}^{h_1} g(y)e^{-\eta_n^-(y-h_1)}dy, & \text{if } 1 \le n \le N^- \\ -\int_{h_2}^{\infty} g(y)e^{-\eta^+_{(n-N^-)}(y-h_2)}dy, & \text{if } N^-+1 \le n \le N \\ g(h_1), & \text{if } n = N+1 \\ g(h_2), & \text{if } n = N+2 \end{cases} \tag{20}$$

Proof. Since $(\mathcal{L}-r)\widetilde{\Phi} = 0$ on (h_1,h_2), for $x \in (h_1,h_2)$,

$$\begin{aligned} 0 = &\ \frac{\sigma^2}{2}\widetilde{\Phi}''(x) + c\widetilde{\Phi}'(x) + \lambda\int\widetilde{\Phi}(x+y)f(y)dy - (\lambda+r)\widetilde{\Phi}(x) \\ = &\ \sum_{n=1}^{N+2} C_n e^{\beta_n x}\left(\frac{\sigma^2}{2}\beta_n^2 + c\beta_n - (\lambda+r)\right) + \lambda\int\widetilde{\Phi}(x+y)f(y)dy. \end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned} &\int\widetilde{\Phi}(x+y)f(y)dy = \left(\int_{-\infty}^{h_1} + \int_{h_2}^{\infty}\right) g(y)f(y-x)dy + \int_{h_1-x}^{h_2-x}\widetilde{\Phi}(x+y)f(y)dy \\ &= \sum_{j=1}^{N^-} q_j e^{\eta_j^- x}\int_{-\infty}^{h_1} g(y)(-\eta_j^-)e^{-\eta_j^- y}dy + \sum_{j=1}^{N^+} p_j e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)\eta_j^+ e^{-\eta_j^+ y}dy \\ &+ \sum_{n=1}^{N+2} C_n e^{\beta_n x}\sum_{j=1}^{N^-} q_j(-\eta_j^-)\int_{h_1-x}^{0} e^{\beta_n y}e^{-\eta_j^- y}dy + \sum_{n=1}^{N+2} C_n e^{\beta_n x}\sum_{j=1}^{N^+} p_j\eta_j^+\int_{0}^{h_2-x} e^{\beta_n y}e^{-\eta_j^+ y}dy \\ &= \sum_{j=1}^{N^+} p_j e^{-\eta_j^+ x}\int_{h_2}^{\infty} g(y)\eta_j^+ e^{-\eta_j^+ y}dy + \sum_{j=1}^{N^-} q_j e^{\eta_j^- x}\int_{-\infty}^{h_1} g(y)(-\eta_j^-)e^{-\eta_j^- y}dy \\ &+ \sum_{n=1}^{N+2} C_n e^{\beta_n x}\sum_{j=1}^{N^-}\frac{-q_j\eta_j^-}{\beta_n-\eta_j^-}\left(1 - e^{(\beta_n-\eta_j^-)(h_1-x)}\right) \\ &+ \sum_{n=1}^{N+2} C_n e^{\beta_n x}\sum_{j=1}^{N^+}\frac{p_j\eta_j^+}{\beta_n-\eta_j^+}\left(e^{(\beta_n-\eta_j^+)(h_2-x)} - 1\right) \end{aligned} \tag{22}$$

Now, by Eq.(21) and Eq.(22) as well as the fact $\psi(\beta_n) - r = 0$ for all n, we have

$$0 = \sum_{j=1}^{N^+} p_j e^{-\eta_j^+ x} \int_{h_2}^{\infty} g(y)\eta_j^+ e^{-\eta_j^+ y} dy + \sum_{j=1}^{N^-} q_j e^{\eta_j^- x} \int_{-\infty}^{h_1} g(y)(-\eta_j^-) e^{-\eta_j^- y} dy$$
$$+ \sum_{j=1}^{N^-} q_j e^{\eta_j^- x} \sum_{n=1}^{N+2} C_n \frac{\eta_j^-}{\beta_n - \eta_j^-} e^{(\beta_n - \eta_j^-)h_1} + \sum_{j=1}^{N^+} p_j e^{-\eta_j x} \sum_{n=1}^{N+2} C_n \frac{\eta_j^+}{\beta_n - \eta_j^+} e^{(\beta_n - \eta_j^+)h_2}.$$

Comparing $e^{-\eta_j^+ x}$ and $e^{\eta_j^- x}$ yields Eq.(18). The proof is complete. □

Lemma 2.5. *For any $h_1 < h_2$, the matrix A given by Eq.(19) is invertible.*

Proof. assume $AC = 0$ for some vector $C = [C_1, C_2, \cdots, C_{N+2}]^T$. Consider the function $\widetilde{V}(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x}$ for $x \in (h_1, h_2)$, and $\widetilde{V}(x) = 0$ otherwise. Since $AC = 0$ and $\widetilde{V}(x)$ is a solution to the boundary value problem (10) with $g(x) \equiv 0$. From the uniqueness of solutions to the boundary value problem (10), $\widetilde{V}(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x} = 0$ for all $x \in (h_1, h_2)$. Now consider the Wronskian

$$W(e^{\beta_1 x}, \cdots, e^{\beta_{N+2} x}) \equiv det \begin{bmatrix} e^{\beta_1 x} & \cdots & e^{\beta_{N+2} x} \\ \beta_1 e^{\beta_1 x} & \cdots & \beta_{m+2} e^{\beta_{N+2} x} \\ \vdots & \ddots & \vdots \\ \beta_1^{N+1} e^{\beta_1 x} & \cdots & \beta_{N+2}^{N+1} e^{\beta_{N+2} x} \end{bmatrix}.$$

Then

$$\begin{aligned} W(e^{\beta_1 x}, \cdots, e^{\beta_{N+2} x}) &= exp\left((\textstyle\sum_{n=1}^{N+2} \beta_n)x\right) det \begin{bmatrix} 1 & \cdots & 1 \\ \beta_1 & \cdots & \beta_{N+2} \\ \vdots & \ddots & \vdots \\ \beta_1^{N+1} & \cdots & \beta_{N+2}^{N+1} \end{bmatrix} \\ &= exp\left((\textstyle\sum_{n=1}^{N+2} \beta_n)x\right) \prod_{1 \le i < j \le N+2} (\beta_i - \beta_j) \neq 0. \end{aligned} \tag{23}$$

(The matrix in Eq.(23) is a Vandermonde matrix.) This inequality implies that $\{e^{\beta_n x} | 1 \le n \le N+2\}$ are linearly independent and so $C = 0$, which implies that A is invertible. □

In the following, for a given function g on $(h_1, h_2)^c$, $C(g) = A^{-1} V_g$ is set where A and V_g are defined as in Eq.(19) and Eq.(20), respectively. Also, $\mathbf{Y} \bullet \mathbf{Z}$ is written for the usual inner product of the vectors $\mathbf{Y}$ and $\mathbf{Z}$ in $\mathbb{R}^{N+2}$ and for every real value x, $\mathbf{e}^{\beta}(x) = [e^{\beta_1 x}, \cdots, e^{\beta_{N+2} x}]$. Our main result is as follows.

Theorem 2.6. *Given a constant $r \geq 0$ and a nonnegative function g satisfying the condition in Proposition 2.1 on $(h_1, h_2)^c$, the function $\Phi(x)$, defined by the formula*

$$\widetilde{\Phi}(x) = \begin{cases} C(g) \bullet \mathbf{e}^{\beta}(x), & \text{if } x \in (h_1, h_2) \\ g(x), & \text{if } x \notin (h_1, h_2) \end{cases}, \tag{24}$$

solves the boundary value problem (10). Additionally, $\widetilde{\Phi}(x) = \mathbb{E}_x \left[e^{-r\tau_{I^c}} g(X_{\tau_{I^c}})\right]$.

Proof. The first statement follows by direct calculation using Eq.(18). The proof of the second statement(concerning the uniqueness of solutions of the boundary value problem (10)) is the same as that of Proposition 4.1 in the work of Chen et al.[9] if $\mathbb{R}_+$ is replaced by $\overline{(h_1, h_2)^c}$. This proof is omitted here. □

3 Verification Theorems

In this section, we introduce the theorems to verify whether the possible candidate function is equal to V given in Eq.(1). Due to the form of the reward function g given in Eq.(9), it conjectures that the possible candidate shall be of the same form as $\Phi(x)$ with some special interval I. To do this, the following verification theorem is required.

Theorem 3.1. *Given $I = (h_1, h_2)$ where $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$. Assume that the function $\Phi(x)$ in Eq.(4) satisfies the following conditions:*

(a) $\Phi(x)$ is the difference of two convex functions.

(b) $\Phi(x)$ is a twice continuously differentiable function except possibly at h_1 and h_2.

(c) The limits $\Phi''(h_i\pm) = \lim_{h\to h_i\pm} \Phi''(h)$, $i = 1, 2$, exist and are finite.

(d) $(\mathcal{L}_X - r)\Phi(x) \leq 0$ for all x except finitely many points.

(e) $\Phi(x) \geq g(x)$ for all $x \in (h_1, h_2)$.

Then $\Phi(x)$ is the value function for the optimal stopping problem (1) with the reward function g given in Eq.(9).

Proof. Let V be the value function for the optimal stopping problem (1). Clearly, we have $\Phi(x) \le V$. It remains to show that $V(x) \le \Phi(x)$. By the Meyer-Itô formula(see, for example, Corollary 1 in Protter[20] ChIV. pp.218-pp.219), we have

$$e^{-rt}\Phi(X_t) - \Phi(x) = -\int_0^t re^{-rs}\Phi(X_s)ds + \int_0^t e^{-rs}\Phi'(X_{s-})dX_s$$
$$+\sum_{0<s\le t} e^{-rs}\left(\Phi(X_s) - \Phi(X_{s-}) - \Phi'(X_{s-})\Delta X_s\right) + \frac{1}{2}\int_0^t e^{-rs}\Phi''(X_{s-})d[X,X]_s^c$$

where $\Phi'(x)$ is its left derivative and $\Phi''(x)$ is the second derivative in the generalized function sense. By similar arguments as that in Mordecki[18] Sec. 3, we have

$$e^{-rt}\Phi(X_t) - \Phi(x) = \int_0^t e^{-rs}(\mathcal{L}_X - r)\Phi(X_{s-})ds + M_t \tag{25}$$

where $\{M_t\}$ is a local martingale with $M_0 = 0$. Let $T_n \uparrow \infty$ be a sequence of stopping times such that for each n, $\{M_{T_n\wedge t}\}$ is a martingale. Let τ be a stopping time. By the optional stopping theorem, we have $\mathbb{E}_x[M_{T_n\wedge t\wedge\tau}] = \mathbb{E}_x[M_0] = 0$. In addition, by (d), we have $\int_0^{T_n\wedge t\wedge\tau} e^{-rs}(\mathcal{L}_X - r)\Phi(X_{s-})ds \le 0$. By Eq.(25), we observe $\mathbb{E}_x[e^{-r(T_n\wedge t\wedge\tau)}\Phi(X_{T_n\wedge t\wedge\tau})] \le \Phi(x)$. Since $g(x)$ is nonnegative and $\mathbb{E}_x\left[\sup_{t\ge 0} e^{-rt}g(X_t)\right] < \infty$, by Dominated Convergence Theorem and (e), we have

$$\mathbb{E}_x[e^{-r\tau}g(X_\tau)] = \mathbb{E}_x[\lim_{t\to\infty}\lim_{n\to\infty} e^{-r(\tau\wedge t\wedge T_n)}g(X_{(\tau\wedge t\wedge T_n)})] = \lim_{t\to\infty}\lim_{n\to\infty}\mathbb{E}_x[e^{-r(\tau\wedge t\wedge T_n)}g(X_{(\tau\wedge t\wedge T_n)})]$$
$$\le \lim_{t\to\infty}\lim_{n\to\infty}\mathbb{E}_x[e^{-r(\tau\wedge t\wedge T_n)}\Phi(X_{(\tau\wedge t\wedge T_n)})] \le \Phi(x).$$

Because τ is arbitrary, we observe $V(x) = \sup_\tau \mathbb{E}_x[e^{-r\tau}g(X_\tau)] \le \Phi(x)$. The proof is complete. □

To verify condition (e) of Theorem 3.1, we have the following general results for a class of two-sided reward functions g.

Proposition 3.2. *Assume that g_1 and g_1' are bounded on $(-\infty, l_1)$ and the functions $\int_0^\infty g_2(x+y)f(y)dy$ and $\int_0^\infty g_2'(x+y)f(y)dy, x \ge l_2$, are locally bounded. We assume further that $g_1(x) - g_1'(x)$ is positive and increasing on $(-\infty, l_1)$, $g_2(x) - g_2'(x)$ is negative and decreasing on (l_2, ∞) and $\mathbb{E}_x[\sup_{t\ge 0} e^{-rt}|g'(X_t)|] < \infty$ for all x. Let $I = (h_1, h_2)$ for some $-\infty < h_1 < l_1 \le l_2 < h_2 < \infty$ and consider a non-negative function $\widetilde{\Phi}(x)$ on $\mathbb{R}$ that is C^2 on (h_1, h_2) and satisfies the following conditions:*

(a) $(\mathcal{L}_X - r)\widetilde{\Phi}(x) = 0, \forall x \in (h_1, h_2)$,

(b) $\widetilde{\Phi}(x) = g(x), \forall x \in I$.

(c) $\frac{d}{dx}\int \widetilde{\Phi}(x+y)f(y)dy = \int \widetilde{\Phi}'(x+y)f(y)dy, \forall x \in (h_1, h_2)$.

(d) $\widetilde{\Phi}$ *is continuous at* h_1 *and* h_2 *and* $\widetilde{\Phi}'(h_i), i = 1, 2$, *exist and are continuous there.*

Then $\widetilde{\Phi}(x) \geq g(x)$ *for all* $x \in (h_1, h_2)$.

Proof. By Proposition 2.1, we have $\widetilde{\Phi} = \Phi$ for all $x \in \mathbb{R}$. Note that $\widetilde{\Phi}$ is C^∞ on (h_1, h_2) (for a proof, see Chen et al.[9].) and, for $x \in (h_1, h_2)$, we have

$$0 = \frac{d}{dx}(\mathcal{L}_X - r)\widetilde{\Phi}(x) = \frac{1}{2}\sigma^2\widetilde{\Phi}'''(x) + c\widetilde{\Phi}''(x) - (\lambda + r)\widetilde{\Phi}'(x) + \lambda \int \widetilde{\Phi}'(x+y)f(y)dy,$$

which implies that $(\mathcal{L}_X - r)\widetilde{\Phi}'(x) = 0$ for $x \in (h_1, h_2)$. By condition (d), $\widetilde{\Phi}' \in C[h_1, h_2]$ and hence by the Remark 2.2, $\widetilde{\Phi}'(x) = E_x[e^{-r\tau_{I^c}}g'(X_{\tau_{I^c}})]$. This implies that $\widetilde{\Phi}(x)$ satisfies the ODE: $\widetilde{\Phi}'(x) - \widetilde{\Phi}(x) = F(x)$, where $F(x) = \mathbb{E}_x[e^{-r\tau_{I^c}}(g'(X_{\tau_{I^c}}) - g(X_{\tau_{I^c}}))]$. Note that $\widetilde{\Phi}(x) = \Phi(x) \geq 0 = g(x)$ for $l_1 \leq x \leq l_2$. First consider the case that $h_1 \leq x \leq l_1$. By the ODE theory and the boundary conditions, we have $\widetilde{\Phi}(x) = e^x\left(\int_{h_1}^x e^{-t}F(t)dt + g_1(h_1)e^{-h_1}\right)$. Set $H(x) \equiv e^{-x}(\widetilde{\Phi}(x) - g(x))$. Then $H(x) = \int_{h_1}^x e^{-t}F(t)dt + g_1(h_1)e^{-h_1} - g_1(x)e^{-x}$ and

$$\begin{aligned}
H'(x) &= e^{-x}F(x) + g_1(x)e^{-x} - g_1'(x)e^{-x} \\
&= e^{-x}\{\mathbb{E}_x[e^{-r\tau_{I^c}}(g'(X_{\tau_{I^c}}) - g(X_{\tau_{I^c}}))] + g_1(x) - g_1'(x)\} \\
&= e^{-x}\{\mathbb{E}_x[e^{-r\tau_{I^c}^+}(g_2'(X_{\tau_{I^c}}) - g_2(X_{\tau_{I^c}})); \{\tau_{I^c} = \tau_{I^c}^+\}] \\
&\quad + \mathbb{E}_x[e^{-r\tau_{I^c}^-}(g_1'(X_{\tau_{I^c}}) - g_1(X_{\tau_{I^c}})); \{\tau_{I^c} = \tau_{I^c}^-\}] + g_1(x) - g_1'(x)\} \\
&\geq e^{-x}\mathbb{E}_x[e^{-r\tau_{I^c}^+}(g_2'(X_{\tau_{I^c}}) - g_2(X_{\tau_{I^c}})); \{\tau_{I^c} = \tau_{I^c}^+\}] \\
&\quad + e^{-x}(g_1(x) - g_1'(x))(1 - \mathbb{E}_x[e^{-r\tau_I^-}; \{\tau_{I^c} = \tau_{I^c}^-\}]
\end{aligned}$$

where $\tau_{I^c}^+ = \inf\{t \geq 0 | X_t \geq h_2\}$ and $\tau_{I^c}^- = \inf\{t \geq 0 | X_t \leq h_1\}$. For the last inequality, we use the facts that $g_1(x) - g_1'(x)$ is increasing and hence $g_1(X_{\tau_{I^c}^-}) - g_1'(X_{\tau_{I^c}^-}) \leq g_1(h_1) - g_1'(h_1) \leq g_1(x) - g_1'(x)$. Since $g_2(x) - g_2'(x)$ is negative and $g_1(x) - g_1'(x)$ is positive, we obtain $H'(x) \geq 0$ which implies that $H(x)$ is increasing . Therefore $H(x) \geq H(h_1) = 0$ and hence $\widetilde{\Phi} \geq g(x)$. By a similar argument, we get $\widetilde{\Phi}(x) \geq g(x)$ for $l_2 \leq x \leq h_2$. The proof is complete. □

Note that the results mentioned as above do not reply on the property of the hyper-exponential jump-diffusion Lévy processes and hence, are adapted to the general jump-diffusion processes together with the general reward functions.

4 Pricing Perpetual American Strangles and Straddles

A strangle is a financial instrument whose reward function is a combination of a put with the strike price K_1 and a call with the strike price K_2 written on the same security, where $K_1 \leq K_2$. In particular, if $K_1 = K_2$, the strangle becomes a straddle. In addition, if the strangle(straddle) can be exercised at any time and has no maturities, then it is called the perpetual American strangle(straddle). In the remainder, we assume that the price is drawn by $S_t = \exp\{X_t\}$ under the chosen risk-neutral measure. Here X is the hyper-exponential jump-diffusion Lévy process in Eq.(5).

A rational price of the perpetual American strangle is the value function for the optimal stopping problem (1) with the reward function g given by the formula

$$g(x) = (K_1 - e^x)^+ + (e^x - K_2)^+ = g_1(x)1_{\{x \leq l_1\}} + g_2(x)1_{\{x \geq l_2\}}. \tag{26}$$

where $l_1 = \ln K_1, l_2 = \ln K_2, g_1(x) = K_1 - e^x$ and $g_2(x) = e^x - K_2$.

In the following, we show that the value function of the perpetual American strangle is $\Phi(x)$ for $I = (h_1, h_2)$ satisfying the smooth pasting condition. To do this, we need some further properties for the coefficients C_n's of $\Phi(x)$. We consider the following conditions on the model :

$$\eta_j^+ > 1 \text{ for } j = 1, 2, ..., N^+ \tag{27}$$

and

$$\frac{1}{2}\sigma^2 + c - (\lambda + r) + \lambda \left(\sum_{j=1}^{N^+} \frac{p_j \eta_j^+}{\eta_j^+ - 1} + \sum_{j=1}^{N^-} \frac{q_j \eta_j^-}{\eta_j^- - 1} \right) < 0 \tag{28}$$

(Note that Eq.(27) implies that $\mathbb{E}[e^{X_1}] < \infty$ and Eq.(28) guarantees $\mathbb{E}[e^{X_1}] < e^r$(hence the underlying asset pays dividends continuously). If $\mathbb{E}[e^{X_1}] < e^r$ and $0 \leq g(x) \leq A + Be^x$ for some constants A and B, then $\mathbb{E}[\sup_{t \geq 0} e^{-rt} g(X_t)] < \infty$. For details, see Lemma 4.1 of Mordecki and Salminen[17].)

Lemma 4.1. *Under the conditions Eq.(27) and Eq.(28), we have* $\beta_{N^-+2} > 1$.

Proof. First consider the case that $N^+ = 0$. Then β_{N^-+2} is the unique solution to the equation $\phi(x) = 0$ in $(0, \infty)$. Observe that $\lim_{x\to\infty}\phi(x)\lim_{x\to 1}\phi(x) = -\infty$. Our result follows by the intermediate value theorem. Next assume that $N^+ \geq 1$. Then β_{N^-+2} is the unique solution to the equation

$$\phi(x) = \prod_{i=1}^{N^+}(\eta_i^+ - x)\prod_{j=1}^{N^-}(\eta_j^- - x)\left[\frac{1}{2}\sigma^2x^2 + cx - (\lambda + r) + \lambda(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - x} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - x})\right] = 0$$

in $(0, \eta_1^+)$. Also we have

$$\phi(1) = \prod_{i=1}^{N^+}(\eta_i^+ - 1)\prod_{j=1}^{N^-}(\eta_j^- - 1)\left[\frac{1}{2}\sigma^2 + c - (\lambda + r) + \lambda(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - 1} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - 1})\right],$$

and $\phi(\eta_1^+) = \lambda p_1\eta_1^+\prod_{i=2}^{N^+}(\eta_i^+ - \eta_1^+)\prod_{j=1}^{N^-}(\eta_j^- - \eta_1^+)$. By Eq.(27) and Eq.(28), we obtain $\phi(1)\phi(\eta_1^+) < 0$ which implies $\beta_{N^-+2} > 1$. □

Now we consider the system of equations Eq.(18) together with the smooth pasting condition. For the case of the perpetual American strangle option, the system of equations is equal to

$$\sum_{n=1}^{N+2} C_n\frac{e^{\beta_n h_2}}{\beta_n - \eta_k^+} = \frac{1}{1-\eta_k^+}e^{h_2} + \frac{K_2}{\eta_k^+}, \quad k = 1, 2, ..., N^+ \tag{29}$$

$$\sum_{n=1}^{N+2} C_n\frac{e^{\beta_n h_1}}{\beta_n - \eta_k^-} = -\frac{1}{1-\eta_k^-}e^{h_1} - \frac{K_1}{\eta_k^-}, \quad k = 1, 2, ..., N^- \tag{30}$$

$$\sum_{n=1}^{N+2} C_n e^{\beta_n h_2} = e^{h_2} - K_2 \tag{31}$$

$$\sum_{n=1}^{N+2} C_n e^{\beta_n h_1} = K_1 - e^{h_1} \tag{32}$$

$$\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n h_2} = e^{h_2} \tag{33}$$

$$\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n h_1} = -e^{h_1}. \tag{34}$$

Subtract Eq.(31) from Eq.(33) and Eq.(32) from Eq.(34), we have

$$\sum_{n=1}^{N+2} C_n(1-\beta_n)e^{\beta_n h_2} = -K_2 \tag{35}$$

$$\sum_{n=1}^{N+2} C_n(1-\beta_n)e^{\beta_n h_1} = K_1 \tag{36}$$

Using Eq.(36), Eq.(34) and Eq.(30), we have

$$\sum_{n=1}^{N+2} C_n \frac{\beta_n(1-\beta_n)}{\beta_n-\eta_k^-} e^{\beta_n h_1} = 0, \tag{37}$$

for $k = 1, 2, ..., N^-$. Similarly, by Eq.(35), Eq.(33) and Eq.(29), we have

$$\sum_{n=1}^{N+2} C_n \frac{\beta_n(1-\beta_n)}{\beta_n-\eta_k^+} e^{\beta_n h_2} = 0 \tag{38}$$

for $k = 1, 2, ..., N^+$. From equations Eq.(35) and Eq.(36), we also have

$$\sum_{n=1}^{N+2} C_n(1-\beta_n)\left(\frac{1}{K_1}e^{\beta_n h_1} + \frac{1}{K_2}e^{\beta_n h_2}\right) = 0. \tag{39}$$

In addition, by Eq.(33) and Eq.(34), we have

$$\sum_{n=1}^{N+2} C_n\beta_n\left(e^{(\beta_n-1)h_1} + e^{(\beta_n-1)h_2}\right) = 0. \tag{40}$$

To prove the following main result(Theorem 4.5), we need the following three technical lemmas.

Lemma 4.2. *Assume that $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ is a solution of the equations Eq.(29)-Eq.(34). Then $C_j \neq 0$ except for at most one j.*

Proof. Set $\Delta h = h_2 - h_1$ and put $\widehat{C}_n = e^{\beta_n h_1}(1-\beta_n)\beta_n C_n$ for $1 \le n \le N+2$. Then, byEq.(35)-Eq.(38), we have $A\widehat{C} = K$ where

$$A = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N+2}} \\ \frac{1}{\beta_1}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}e^{\beta_{N+2}\Delta h} \end{bmatrix}, \widehat{C} = \begin{bmatrix} \widehat{C}_1 \\ \widehat{C}_2 \\ \vdots \\ \widehat{C}_{N+2} \end{bmatrix} \text{ and } K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K_1 \\ -K_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let $F_1(x) = \sum_{i=1}^{N+2} \frac{\widehat{C}_i}{\beta_i - x}$ and $F_2(x) = \sum_{i=1}^{N+2} \frac{e^{\beta_i \Delta h}\widehat{C}_i}{\beta_i - x}$. Clearly, $F_1(x) = \frac{S_1(x)}{\prod_{i=1}^{N+2}(\beta_i - x)}$ and $F_2(x) = \frac{S_2(x)}{\prod_{i=1}^{N+2}(\beta_i - x)}$, where

$$S_1(x) = \sum_{n=1}^{N+2} \widehat{C}_n \prod_{i=1, i\neq n}^{N+2} (\beta_i - x) \quad \text{and} \quad S_2(x) = \sum_{n=1}^{N+2} e^{\beta_n \Delta h}\widehat{C}_n \prod_{i=1, i\neq n}^{N+2} (\beta_i - x). \tag{41}$$

Then $S_1(x)$ and $S_2(x)$ are polynomials with degree at most $N+1$. Also, by the fact $A\widehat{C} = K$, we have $S_1(0) = K_1 \prod_{i=1}^{N+2} \beta_i$, $S_2(0) = -K_2 \prod_{i=1}^{N+2} \beta_i$, $S_1(\eta_k^-) = 0$ for $1 \leq k \leq N^-$ and $S_2(\eta_k^+) = 0$ for $1 \leq k \leq N^+$. By Eq.(41), we have

$$\widehat{C}_n = \frac{S_1(\beta_n)}{\prod_{i=1, i\neq n}^{N+2}(\beta_i - \beta_n)} = \frac{e^{-\beta_n \Delta h} S_2(\beta_n)}{\prod_{i=1, i\neq n}^{N+2}(\beta_i - \beta_n)} \tag{42}$$

for $1 \leq n \leq N+2$. From this, we have $S_2(\beta_n) = S_1(\beta_n) = 0$ if and only if $S_2(\beta_n) - S_1(\beta_n) = 0$. In addition, we have $\widehat{C}_n = 0$ if and only if $S_2(\beta_n) - S_1(\beta_n) = 0$. Also if $S_1(\beta_k)$ and $S_2(\beta_k)$ are nonzero for some $1 \leq k \leq N+2$, $\frac{S_2(\beta_k)}{S_1(\beta_k)} = e^{\beta_k \Delta h}$. It remains to show that $|\Theta| \leq 1$ where $\Theta = \{\beta_n | S_1(\beta_n) - S_2(\beta_n) = 0, \text{ for } 1 \leq n \leq N+2\}$ and $|\Theta|$ is the cardinality of Θ. To do this, we need the following facts :

(1) If $S_2(x) \neq 0$ on $(\eta_k^-, \beta_{k+1}]$ for some k, $1 \leq k \leq N^-$, then $S_2(x) - S_1(x) = 0$ has a solution in (η_k^-, β_{k+1}).

(2) If $S_2(x) \neq 0$ on $[\beta_k, \eta_k^-)$ for some k, $1 \leq k \leq N^-$, then $S_2(x) - S_1(x) = 0$ has a solution in (β_k, η_k^-) .

(3) If $S_1(x) \neq 0$ on $(\eta_k^+, \beta_{N^-+2+k}]$ for some k, $1 \leq k \leq N^+$, then $S_2(x) - S_1(x) = 0$ has a solution in $(\eta_k^+, \beta_{N^-+2+k})$.

(4) If $S_1(x) \neq 0$ on $[\beta_{N^-+1+k}, \eta_k^+)$ for some k, $1 \leq k \leq N^+$, then $S_2(x) - S_1(x) = 0$ has a solution in $(\beta_{N^-+1+k}, \eta_k^+)$.

(5) If $S_2(x) \neq 0$ on $[\beta_{N^-+1}, 0)$, then $S_1(x)$ has a solution in $(\beta_{N^-+1}, 0)$.

(6) If $S_1(x) \neq 0$ on $(0, \beta_{N^-+2}]$, then $S_2(x)$ has a solution in $(0, \beta_{N^-+2})$.

To prove (1), we assume that $S_2(x) \neq 0$ for all $x \in (\eta_k^-, \beta_{k+1}]$. Let $x^* = \sup\{x \in [\eta_k^-, \beta_{k+1}] | S_1(x) = 0\}$. Note that x^* exists because $S_1(\eta_k^-) = 0$ and $x^* < \beta_{k+1}$. Because

$\frac{S_2(x)}{S_1(x)}$ is continuous on $(x^*, \beta_{k+1}]$, $0 < \frac{S_2(\beta_{k+1})}{S_1(\beta_{k+1})} = e^{\beta_{k+1}\Delta h} < 1$ and $\lim_{x \to x^{*+}} \frac{S_2(x)}{S_1(x)} = \infty$, by the intermediate value theorem, there exists $x_0^* \in (x^*, \beta_{k+1})$ such that $\frac{S_2(x_0^*)}{S_1(x_0^*)} = 1$. This completes the proof of the fact (1) above. Facts (2)-(4) are verified by similar arguments.

Next , we verify the fact (5) and assume that $S_2(x) \neq 0$ for all $x \in [\beta_{N^-+1}, 0)$. Then

$$sgn(S_2(\beta_{N^-+1})S_1(\beta_{N^-+1})) = sgn\left(e^{\beta_{N^-+1}\Delta h}\widehat{C}_n^2 \prod_{i=1, i\neq N^-+1}^{N+2} (\beta_i - \beta_{N^-+1})^2\right) > 0,$$

and $sgn(S_2(0)S_1(0)) = sgn(-K_1K_2\prod_{i=1}^{N+2}\beta_i^2) < 0$, which imply that $S_1(x)$ has a solution in $(\beta_{N^-+1}, 0)$. The proof of the fact (6) is similar.

Let $S(x) = S_2(x) - S_1(x)$. Then $S(x)$ is a polynomial with degree at most $N+1$ and $S(\beta_k) = 0$ whenever $\beta_k \in \Theta$. Let

$$\begin{aligned}\Pi = \quad & \{[\beta_{N^-+1}, 0) | \beta_{N^-+1} \notin \Theta\} \cup \{(0, \beta_{N^-+2}] | \beta_{N^-+2} \notin \Theta\} \\ & \cup\{[\beta_k, \eta_k^-) | \beta_k \notin \Theta, 1 \leq k \leq N^-\} \cup \{(\eta_k^-, \beta_{k+1}] | \beta_{k+1} \notin \Theta, 1 \leq k \leq N^-\} \\ & \cup\{[\beta_{N^-+1+k}, \eta_k^+) | \beta_{N^-+1+k} \notin \Theta, 1 \leq k \leq N^+\} \cup \{(\eta_k^+, \beta_{N^-+2+k}] | \beta_{N^-+2+k} \notin \Theta, 1 \leq k \leq N^+\}.\end{aligned}$$

Note that Π is a collection of intervals and $|\Pi| \equiv$ the number of intervals in $\Pi \geq 2(N+1) - 2|\Theta|$. Let $\widetilde{\Pi} = \{I \in \Pi | S(x) = 0$ has no solution in $I\}$. Since $|\{x | S(x) = 0, x \notin \Theta\}| \leq N + 1 - |\Theta|$, $|\widetilde{\Pi}| \geq 2(N+1) - 2|\Theta| - ((N+1) - |\Theta|) = N + 1 - |\Theta|$. For any $I \in \widetilde{\Pi}$, by facts (1)-(4), we obtain

(a) if $\sup_{x\in I} x \leq \beta_{N^-+1}$, then the equation $S_2(x) = 0$ has solutions in I.

(b) if $\inf_{x\in I} x \geq \beta_{N^-+2}$, then the equation $S_1(x) = 0$ has solutions in I.

Also, by fact (5),$S_1(x)S_2(x) = 0$ for some $x \in [\beta_{N^-+1}, 0)$. Similarly, by fact (6), $S_1(x)S_2(x) = 0$ for some $x \in (0, \beta_{N^-+2}]$. From these observation, combining with the fact that for $I_1, I_2 \in \widetilde{\Pi}$, $I_1 \cap I_2 = \phi$ or $I_1 \cap I_2 \subseteq \Theta^c$, we have

$$\begin{aligned} & |\{x | S_2(x) = 0, x < \beta_{N^-+2}, x \notin \Theta\}| + |\{x | S_1(x) = 0, x > \beta_{N^-+1}, x \notin \Theta\}| \\ \geq \quad & |\widetilde{\Pi}| \geq N + 1 - |\Theta|. \end{aligned} \tag{43}$$

Recall that $S_1(\eta_k^-) = 0$ for $1 \le k \le N^-$ and $S_2(\eta_k^+) = 0$ for $1 \le k \le N^+$. Therefore,

$$\begin{aligned} 2(N+1) &\ge |\{x|S_1(x)=0\}| + |\{x|S_2(x)=0\}| \\ &= \left|\{x|S_1(x)=0, x > \beta^-_{N^-+1}, x \notin \Theta\}\right| + \left|\{x|S_1(x)=0, x \le \beta^-_{N^-+1}, x \notin \Theta\}\right| \\ &+ \left|\{x|S_2(x)=0, x < \beta^+_{N^-+2}, x \notin \Theta\}\right| + \left|\{x|S_2(x)=0, x \ge \beta^+_{N^-+2}, x \notin \Theta\}\right| + 2|\{x|x \in \Theta\}| \\ &\ge N+1-|\Theta| + N^- + N^+ + 2|\Theta| = 2N+1+|\Theta|. \end{aligned} \tag{44}$$

This implies that $|\Theta| \le 1$. The proof is complete. □

Lemma 4.3. *Assume that* $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ *is a solution of the equations Eq.(29)-Eq.(34). Then* $C_n \ge 0$ *for all* n.

Proof. We define S_1, S_2, Θ, $\widetilde{\Pi}$, Π, and $\widehat{C}_n$'s as in the proof of Lemma 4.2. Since $\widehat{C}_n = e^{-\beta_n h_1}(1-\beta_n)\beta_n C_n$ and, by Lemma 4.1, we observe $C_n \ge 0$ if and only if $\widehat{C}_n \le 0$. Besides, by Proposition Eq.(2.1), we obtain $\sum_{n=1}^{N+2} C_n e^{\beta_n x} = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ which is nonnegative for all $x \in (h_1, h_2)$. To prove $C_n \ge 0$ for all n, it suffices to show that the $\widehat{C}_n$'s have the same sign. By Lemma Eq.(4.2), $|\Theta| = 0$ or 1. First, we consider the case that $|\Theta| = 1$, that is, $S_1(\beta_{k_0}) = S_2(\beta_{k_0}) = 0$ for some $1 \le k_0 \le N+2$. Then $|\Pi| \ge 2N$ and by Eq.(43), $|\{x|S_2(x) = 0, x < \beta_{N^-+2}, x \ne \beta_{k_0}\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}, x \ne \beta_{k_0}\}| \ge N$. By Eq.(44), we obtain $|\{x|S_2(x) = 0\}| + |\{x|S_1(x) = 0\}| = 2N+2$. Hence $S_1(x)$ and $S_2(x)$ are polynomials with degree $N+1$ and all roots of $S_1(x)$ and of $S_2(x)$ are simple. In addition

$$\begin{aligned} & 2(N+1) \ge |\{x|S_1(x)=0\}| + |\{x|S_2(x)=0\}| \\ \ge\ & |\{x|S_2(x)=0, x < \beta_{N^-+2}, x \ne \beta_{k_0}\}| + |\{x|S_1(x)=0, x > \beta_{N^-+1}, x \ne \beta_{k_0}\}| \\ & +|\{x|S_2(x)=0, x \ge \beta_{N^-+2}, x \ne \beta_{k_0}\}| + |\{x|S_1(x)=0, x \le \beta_{N^-+1}, x \ne \beta_{k_0}\}| + 2 \\ \ge\ & N + |\{x|S_2(x)=0, x \ge \beta_{N^-+2}, x \ne \beta_{k_0}\}| + |\{x|S_1(x)=0, x \le \beta_{N^-+1}, x \ne \beta_{k_0}\}| + 2 \end{aligned}$$

and hence, $N \ge |\{x|S_2(x) = 0, x \ge \beta_{N^-+2}, x \ne \beta_{k_0}\}| + |\{x|S_1(x) = 0, x \le \beta_{N^-+1}, x \ne \beta_{k_0}\}|$. Since $S_2(\eta_k^+) = 0$ for $1 \le k \le N^+$ and $S_1(\eta_k^-) = 0$ for $1 \le k \le N^-$, we obtain $\{x|S_1(x) = 0, x \le \beta^-_{N^-+1}, x \ne \beta_{k_0}\} = \{\eta_k^-|1 \le k \le N^-\}$ and $\{x|S_2(x) = 0, x \ge \beta_{N^-+2}, x \ne \beta_{k_0}\} = \{\eta_k^+|1 \le k \le N^+\}$. Now we consider the case that $k_0 = 1$, that is $S_1(\beta_1) = S_2(\beta_1) = 0$. Because η_i^- is the unique root for $S_1(x)$ in $[\beta_i, \beta_{i+1}]$, $2 \le i \le N^-$, we obtain $S_1(\beta_i)S_1(\beta_{i+1}) < 0$. By similar arguments, we also have $S_2(\beta_j)S_2(\beta_{j+1}) < 0$ for

$N^- + 2 \leq j \leq N+1$. By Eq.(42), we have $\widehat{C}_{n-1}\widehat{C}_n = \frac{e^{-\beta_{n-1}\Delta h}S_2(\beta_{n-1})}{\prod_{i=1,i\neq n-1}^{N+2}(\beta_i-\beta_{n-1})}\frac{e^{-\beta_n\Delta h}S_2(\beta_n)}{\prod_{i=1,i\neq n}^{N+2}(\beta_i-\beta_n)} = \frac{e^{-(\beta_{n-1}+\beta_n)\Delta h}S_2(\beta_{n-1})S_2(\beta_n)(\beta_n-\beta_{n-1})^{-1}(\beta_{n-1}-\beta_n)^{-1}}{\prod_{i=1}^{n-2}(\beta_i-\beta_{n-1})(\beta_i-\beta_n)\prod_{j=n+1}^{N+2}(\beta_j-\beta_{n-1})(\beta_j-\beta_n)} = \frac{S_1(\beta_{n-1})S_1(\beta_n)(\beta_n-\beta_{n-1})^{-1}(\beta_{n-1}-\beta_n)^{-1}}{\prod_{i=1}^{n-2}(\beta_i-\beta_{n-1})(\beta_i-\beta_n)\prod_{j=n+1}^{N+2}(\beta_j-\beta_{n-1})(\beta_j-\beta_n)}$. Therefore, the elements in $C^- \equiv \{\widehat{C}_n | 2 \leq n \leq N^- + 1\}$ have the same sign and this is also true for elements in $C^+ \equiv \{\widehat{C}_n | N^- + 2 \leq n \leq N+2\}$. Because $A\widehat{C} = K$, if the elements in C^- are positive and the ones in C^+ are negative, then we get the contradiction that $K_1 = \sum_{n=1}^{N+2}\widehat{C}_n\frac{1}{\beta_n} < 0$; if the elements in C^- are negative and the ones in C^+ are positive, then we get another contradiction, i.e., $-K_2 = \sum_{n=1}^{N+2}\widehat{C}_n\frac{e^{\beta_n\Delta h}}{\beta_n} > 0$. Therefore, $\widehat{C}_n$'s must have the same sign. For the case $k_0 = N^- + 1$, the proof is the same. For the case $1 < k_0 < N^- + 1$, by a similar argument as above, we obtain the elements in $C_1^- = \{\widehat{C}_n | 1 \leq n \leq k_0 - 1\}$, $C_2^- = \{\widehat{C}_n | k_0 + 1 \leq n \leq N^- + 1\}$, and $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N+2\}$ have the same sign, respectively. There are eight situations for the signs of C_1^-, C_2^-, and C^+ : (1) $C_1^- < 0$, $C_2^- < 0$, and $C^+ < 0$, (2) $C_1^- > 0$, $C_2^- > 0$, and $C^+ > 0$, (3)$C_1^- < 0$, $C_2^- < 0$, and $C^+ > 0$, (4)$C_1^- > 0$, $C_2^- > 0$, and $C^+ < 0$, (5) $C_1^- < 0$, $C_2^- > 0$, and $C^+ > 0$, (6) $C_1^- > 0$, $C_2^- < 0$, and $C^+ < 0$, (7) $C_1^- < 0$, $C_2^- > 0$, and $C^+ < 0$, (8) $C_1^- > 0$, $C_2^- < 0$, and $C^+ > 0$. (We write $C_i^\pm > (<)0$ if all elements in $C_i^\pm$ are greater(smaller) than zero.) We show that cases (3)-(8) are impossible. The arguments for disproving cases (3) and (4) are the same as for the case $k_0 = 1$. Note that $\beta_1 < \eta_1^- < \beta_2 < \eta_2^- < \cdots < \beta_{k_0} < \eta_{k_0}^- < \beta_{k_0} < \cdots < \beta_{N^-} < \eta_{N^-}^- < \beta_{N^-+1} < 0 < 1 < \beta_{N^-+2} < \eta_1^+ < \cdots < \beta_{N+1} < \eta_{N^+}^+ < \beta_{N+2}$. Because $A\widehat{C} = K$, Comparing with the (k_0-1)-th entries in $A\widehat{C}$ and K, we obtain $\sum_{n=1}^{N+2}\widehat{C}_n\frac{1}{\beta_n - \eta_{k_0-1}^-} = 0$. Therefore, it is impossible for cases (5) and (6). Note that the entries of A satisfy the following:

(a) $A_{i,j} < 0$ for $\{(i,j) | 1 \leq j \leq i \leq N^- + 1\} \cup \{(i,j) | N^- + 2 \leq i \leq N+2, 1 \leq j < i\}$ and $A_{i,j} > 0$, otherwise.

(b) If $A_{i,j}$ and $A_{i+1,j}$ are negative, then $A_{i,j} < A_{i+1,j}$.

(c) If $A_{i,j}$ and $A_{i+1,j}$ are positive, then $A_{i,j} < A_{i+1,j}$.

For the case (7), we get the contradiction $K_1 = (A_{N^-+1} - A_{k_0-1})\widehat{C} < 0$ and for the case (8), we get the contradiction $-K_2 = (A_{N^-+2} - A_{k_0-1})\widehat{C} > 0$ where A_i is the ith row of A. Therefore, we complete the proof for the case that $|\Theta| = 1$ and $1 < k_0 \leq N^- + 1$. The proof for the case that $|\Theta| = 1$ and $N^- + 2 \leq k_0 \leq N+2$ is similar.

Consider the case that $|\Theta| = 0$ which implies that $\widehat{C}_n$'s are nonzero. Then we have $|\Pi| = 2N+2$ and by Eq.(43), $|\{x|S_2(x) = 0, x < \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}\}| \geq |\widetilde{\Pi}| \geq N+1$. Therefore $2(N+1) \geq |\{x|S_1(x) = 0\}| + |\{x|S_2(x) = 0\}| \geq |\{x|S_2(x) = 0, x < \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}\}| + |\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}| + 2|\Theta| \geq N + 1 + |\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}|$, which implies $N + 1 \geq |\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}|$. Because $|\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}| \geq N$, we have $|\{x|x > \beta_{N^-+2}, S_2(x) = 0\}| = N^+$ or $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^-$. First, we consider the case $|\{x|x > \beta_{N^-+2}, S_2(x) = 0\}| = N^+$, or equivalently, $\{x|x \geq \beta_{N^-+2}, S_2(x) = 0\} = \{\eta_1^+ \cdots \eta_{N^+}^+\}$. If $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^-$, then we have $\{x|x \leq \beta_{N^-+1}, S_1(x) = 0\} = \{\eta_1^- \cdots \eta_{N^-}^-\}$. Similar arguments as for the case $|\Theta| = 1$ imply that the elements in $C^- = \{\widehat{C}_n | 1 \leq n \leq N^- + 1\}$ and in $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively, and hence, the sign of $\widehat{C}_n$'s are the same. If $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^- + 1$, then either $S_1(x)$ has a root in $(-\infty, \beta_1)$ or $S_1(x)$ has two roots in $(\beta_{k_0}, \beta_{k_0+1})$ for some $1 \leq k_0 \leq N^-$. For the case $(-\infty, \beta_1)$, we can get as above that the elements in $C^- = \{\widehat{C}_n | 1 \leq n \leq N^- + 1\}$ and in $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively. If $S_1(x)$ has two roots in $(\beta_{k_0}, \beta_{k_0+1})$ for some $1 \leq k_0 \leq N^-$, we also observe that the elements in $C_1^- = \{\widehat{C}_n | 1 \leq n \leq k_0 - 1\}$, $C_2^- = \{\widehat{C}_n | k_0 \leq n \leq N^- + 1\}$, and $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively. By the same argument as for the case $|\Theta| = 1$, we know that the coefficients have the same sign. The proof for the case $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^-$ is similar and hence, we omit it. □

Lemma 4.4. *Assume that $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x_0} = e^{x_0}$ for some $x_0 \in \mathbb{R}$. Then there exists $\epsilon > 0$ such that $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} < e^x$ for all $x \in (x_0 - \epsilon, x_0)$. Also we have $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} \geq e^x$ for all $x \geq x_0$.*

Proof. Let $F(x) = \sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x$. Then $F'(x) = \sum_{n=1}^{N+2} C_n \beta_n^2 e^{\beta_n x} - e^x$. Because $\beta_1 < \beta_2 < \cdots < \beta_{N^-+1} < 0 < 1 < \beta_{N^-+2} < \beta_{N^-+3} < \cdots < \beta_{N+2}$, and by Lemma 4.2 and Lemma 4.3,

$$F'(x_0) = \sum_{n=1}^{N+2} C_n \beta_n^2 e^{\beta_n x_0} - e^{x_0} > \sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x_0} - e^{x_0} = 0, \tag{45}$$

which implies that $F(x)$ is strictly increasing in some neighborhood U_{x_0} of x_0 and hence, we complete the proof of the first part of the lemma. Assume that there exists $x' > x_0$

such that $F(x') < 0$. Let $\widehat{x} = \sup\{x | x_0 \leq x < x', F(x) = 0\}$. Then $\widehat{x} < x', F(\widehat{x}) = 0$ and as shown for Eq.(45), we have $F'(\widehat{x}) > 0$. Therefore, there exists a neighborhood $U_{\widehat{x}}$ of $\widehat{x}$ such that for all $x \in U_{\widehat{x}}$ with $x > \widehat{x}$, $F(x) > F(\widehat{x}) = 0$. This is a contradiction because $F(x) < 0$ for all $x \in (\widehat{x}, x')$ and hence, we complete the proof of the lemma. $\square$

Theorem 4.5. *Let $\{C_1, ..., C_N, h_1, h_2\}$ be a solution of the equations Eq.(29)-Eq.(34). Define the function $\widetilde{V}(x)$ by the formula*

$$\widetilde{V}(x) = \begin{cases} \sum_{n=1}^{N+2} C_n e^{\beta_n x} & \text{if} \quad x \in (h_1, h_2) \\ g(x) & \text{if} \quad x \in (h_1, h_2)^c \end{cases}$$

where g is the function in Eq.(26). Then $\widetilde{V}$ is the value function of the optimal stopping problem (1). Also we have $\widetilde{V}(x) = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ for all $x \in \mathbb{R}$ and hence $\tau_{(h_1,h_2)^c}$ is the optimal stopping time for the optimal stopping problem (1).

Proof. Clearly the function $\widetilde{V}(x)$ satisfies conditions (a)-(c) of Theorem 3.1. Direct computation shows that the function $\widetilde{V}$ is a solution of the boundary value problem Eq.(10). Because C_n's are nonnegative according to Lemma 4.3, thus, $h_1 < l_1 = \ln K_1 \leq \ln K_2 = l_2 < h_2$ by Eq.(31) and Eq.(32). Also functions g_1 and g_2 satisfy the conditions in Proposition 2.1. Therefore we have $\widetilde{V}(x) = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ for all $x \in \mathbb{R}$. Note that functions g_1 and g_2 also satisfy the conditions in Proposition 3.2 and $\widetilde{V}$ satisfies conditions (c) and (d) of Proposition 3.2. Hence by Proposition 3.2, we obtain $\sum_{n=1}^{N+2} C_n e^{\beta_n x} \geq g(x)$ for $x \in (h_1, h_2)$. By Theorem 3.1, it remains to show that $(\mathcal{L}_X - r)\widetilde{V}(x) \leq 0$ for $x \in [h_1, h_2]^c$. Note that, on $x > h_2 > \ln K_2$, direct calculation gives

$$\begin{aligned}
& (\mathcal{L}_X - r)\widetilde{V}(x) \\
= \quad & e^x(\frac{1}{2}\sigma^2 + c + \sum_{i=1}^{N^+} \frac{\lambda p_i}{\eta_i^+ - 1} + \sum_{j=1}^{N^-} \frac{\lambda q_j}{\eta_j^- - 1}) - r(e^x - K_2) \\
& + \lambda \sum_{j=1}^{N^-} q_j e^{\eta_j^- x} \left(\sum_{n=1}^{N+2} \frac{C_n \eta_j^-}{\eta_j^- - \beta_n} e^{(\beta_n - \eta_j^-)h_2} - \frac{\eta_j^-}{\eta_j^- - 1} e^{(1-\eta_j^-)h_2} + K_2 e^{-\eta_j^- h_2} \right) \\
= \quad & e^x(\frac{1}{2}\sigma^2 + c + \sum_{i=1}^{N^+} \frac{\lambda p_i}{\eta_i^+ - 1} + \sum_{j=1}^{N^-} \frac{\lambda q_j}{\eta_j^- - 1}) - r(e^x - K_2) \\
& + \lambda \sum_{j=1}^{N^-} q_j e^{\eta_j^- x} \left(\sum_{n=1}^{N+2} \frac{C_n \beta_n}{\eta_j^- - \beta_n} e^{(\beta_n - \eta_j^-)h_2} - \frac{1}{\eta_j^- - 1} e^{(1-\eta_j^-)h_2} \right)
\end{aligned}$$

(The last equality holds because of Eq.(31).) Let $\Psi_j(x) = \sum_{n=1}^{N+2} \frac{C_n\beta_n}{\eta_j^- - \beta_n} e^{(\beta_n - \eta_j^-)x} - \frac{1}{\eta_j^- - 1} e^{(1-\eta_j^-)x}$ for $1 \le j \le N^-$ and $x \in \mathbb{R}$. First we show that $\Psi_j(h_2) \ge 0$. By Eq.(30) and Eq.(32), we have $\Psi_j(h_1) = \frac{2}{1-\eta_j^-} e^{(1-\eta_j^-)h_1} > 0$. Also, we observe $\Psi_j'(x) = -\sum_{n=1}^{N+2} C_n\beta_n e^{(\beta_n-\eta_j^-)x} + e^{(1-\eta_j^-)x} = -e^{-\eta_j^- x}\left(\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} - e^x\right)$. We need the fact that $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} - e^x \ne 0$ for all $x \in (h_1, h_2)$. (Indeed, if $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n h*} - e^{h*} = 0$ for some $h^* \in (h_1, h_2)$, by Lemma 4.4, $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} - e^x \ge 0$ for all $x \in [h*, h_2]$. Note that by Eq.(33), we have $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n h_2} - e^{h_2} = 0$ and by Lemma 4.4, there exists $\epsilon > 0$ such that $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} - e^x < 0$ for all $x \in (h_2 - \epsilon, h_2]$ which is a contradiction.) Combining this fact with the observation that $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n h_1} - e^{h_1} = -2e^{h_1} < 0$, we obtain $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} - e^x \le 0$ for all $x \in [h_1, h_2]$ and hence, $\Psi_j'(x) \ge 0$ on $[h_1, h_2]$. This implies that $\Psi_j(x)$ is an increasing function and hence $\Psi_j(h_2) \ge \Psi_j(h_1) > 0$. Therefore, on $x > h_2 > \ln K_2$, we observe $\frac{d}{dx}(\mathcal{L}_X - r)\widetilde{V}(x) = (\frac{1}{2}\sigma^2 + c + \sum_{i=1}^{N^+} \frac{\lambda p_i}{\eta_i^+ - 1} + \sum_{j=1}^{N^-} \frac{\lambda q_j}{\eta_j^- - 1} - r)e^x + \lambda \sum_{j=1}^{N^-} q_j \Psi_j(h_2)\eta_j^- e^{\eta_j^- x} \le 0$, which implies that $(\mathcal{L}_X - r)\widetilde{V}(x)$ is a decreasing function and its maximum value is $(\mathcal{L}_X - r)\widetilde{V}(h_2+)$. Because $\widetilde{V}(x)$ satisfies the smooth pasting condition at h_2 and $(\mathcal{L}_X - r)\widetilde{V}(h_2-) = 0$, we obtain that

$$(\mathcal{L}_X - r)\widetilde{V}(h_2+) = (\mathcal{L}_X - r)\widetilde{V}(h_2+) - (\mathcal{L}_X - r)\widetilde{V}(h_2-) = \frac{1}{2}\sigma^2(\widetilde{V}''(h_2+) - V''(h_2-))$$

$$= \frac{1}{2}\sigma^2(e^{h_2} - \sum_{n=1}^{N+2} C_n\beta_n^2 e^{\beta_n h_2}) < \frac{1}{2}\sigma^2(e^{h_2} - \sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n h_2}) = 0$$

Therefore $(\mathcal{L}_X - r)\widetilde{V}(x) \le (\mathcal{L}_X - r)V(h_2^+) < 0$ for all $x > h_2$. By the same procedure, we verify $(\mathcal{L}_X - r)\widetilde{V}(x)$ is an increasing function for $x \le h_1$ and $(\mathcal{L}_X - r)V(h_1-) \le 0$, which implies $(\mathcal{L}_X - r)\widetilde{V}(x) \le 0$ for all $x \le h_1$. The proof is complete. □

5 Solutions to Equations Eq.(29)-Eq.(34)

In this section we prove the existence of solutions to the system of equations Eq.(29)-Eq.(34), that is equivalent to the free boundary problem Eq.(10) with smooth pasting condition. It is worth noting that according to Eq.(36)-Eq.(39), we have $\widetilde{A}DC = \widetilde{K}$ where D is an $(N+2)\times(N+2)$ diagonal matrix with entries $d_{ii} = \beta_i(1-\beta_i)$, $\widetilde{K} = [0, 0, ..., 0, K_1]^T$

is an $(N+2) \times 1$ column vector and

$$\widetilde{A} = \begin{bmatrix} \frac{1}{\beta_1 - \eta_1^-} e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^-} e^{\beta_{N+2} h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N-}^-} e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N-}^-} e^{\beta_{N+2} h_1} \\ \frac{1}{\beta_1 - \eta_1^+} e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} e^{\beta_{N+2} h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N+}^+} e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N+}^+} e^{\beta_{N+2} h_2} \\ \frac{1}{\beta_1}(\frac{1}{K_1} e^{\beta_1 h_1} + \frac{1}{K_2} e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1} e^{\beta_{N+2} h_1} + \frac{1}{K_2} e^{\beta_{N+2} h_2}) \\ \frac{1}{\beta_1} e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}} e^{\beta_{N+2} h_1} \end{bmatrix}.$$

Then, the coefficient vector $C = [C_1, ..., C_{N+2}]^T$ is equal to

$$C = \frac{K_1}{det\widetilde{A}} D^{-1} Y \tag{46}$$

where Y is the last column of the cofactor matrix of $\widetilde{A}$. Thus, if we find out the boundary of the continuation region (h_1, h_2), then we can compute the coefficient vector C by Eq.Eq.(46).(For other approach, see Boyarchenko and Boyarchenko[5].) For finding the optimal boundaries, h_1 and h_2, we need the following proposition (Proposition 5.1), which was obtained earlier by Boyarchenko[4].

Proposition 5.1. *Let $\{C_1, ..., C_{N+2}, h_1, h_2\}$ be a solution of the equations Eq.(29)-Eq.(34). Then $\Delta h = h_2 - h_1$ is a solution of the equation $detB(h) = 0$ where for every $h \in \mathbb{R}$, $B(h)$ is a $(N+2) \times (N+2)$ matrix defined by the formula*

$$B(h) = \begin{bmatrix} \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N-}^-} \\ \frac{1}{\beta_1 - \eta_1^+} e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} e^{\beta_{N+2} h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N+}^+} e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N+}^+} e^{\beta_{N+2} h} \\ \frac{1}{\beta_1}(1 + \frac{K_1}{K_2} e^{\beta_1 h}) & \cdots & \frac{1}{\beta_{N+2}}(1 + \frac{K_1}{K_2} e^{\beta_{N+2} h}) \\ \frac{1}{\beta_1 - 1}(1 + e^{(\beta_1 - 1)h}) & \cdots & \frac{1}{\beta_{N+2} - 1}(1 + e^{(\beta_{N+2} - 1)h}) \end{bmatrix} \tag{47}$$

Moreover, we have

$$h_1 = \log \left(\frac{detA_1}{detA_2} \right) \tag{48}$$

and hence, $h_2 = h_1 + \Delta h$, *where*

$$A_1 = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N+2}-1} \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1K_1} & \cdots & \frac{1}{\beta_{N+2}K_1} \end{bmatrix}$$

Proof. Substitute Eq.(46) into Eq.(40), we have

$$\sum_{n=1}^{N+2} \frac{K_1 y_n}{(1-\beta_n)detA}(e^{(\beta_n-1)h_1} + e^{(\beta_n-1)h_2}) = 0 \tag{49}$$

where y_n is the nth entry of the column vector Y. Eq.(49) is equivalent to

$$det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-}e^{\beta_1h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2}h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2}h_1} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}h_2} \\ \frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1h_1}+\frac{1}{K_2}e^{\beta_1h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2}h_1}+\frac{1}{K_2}e^{\beta_{N+2}h_2}) \\ \frac{1}{1-\beta_1}(e^{(\beta_1-1)h_1}+e^{(\beta_1-1)h_2}) & \cdots & \frac{1}{1-\beta_{N+2}}(e^{(\beta_{N+2}-1)h_1}+e^{(\beta_{N+2}-1)h_2}) \end{bmatrix} = 0.$$

Multiply $e^{-\beta_ih_1}$ to the i-th column for each i and then $-e^{h_1}$ to the last row, we observe that $\Delta h = h_2 - h_1$ is a solution of the equation $detB(h) = 0$. Substitute Eq.(46) into Eq.(34), we have $\frac{K_1}{det(A)}[\beta_1e^{\beta_1h_1}, ..., \beta_ne^{\beta_nh_1}]D^{-1}Y = -e^{h_1}$. Note that

$$[\beta_1e^{\beta_1h_1}, ..., \beta_ne^{\beta_nh_1}]D^{-1}Y = [\beta_1e^{\beta_1h_1}, ..., \beta_ne^{\beta_nh_1}] \begin{bmatrix} \frac{1}{\beta_1(1-\beta_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\beta_2(1-\beta_2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{\beta_{N+2}(1-\beta_{N+2})} \end{bmatrix} Y$$

$$= det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-}e^{\beta_1h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2}h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2}h_1} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}h_2} \\ \frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1h_1}+\frac{1}{K_2}e^{\beta_1h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2}h_1}+\frac{1}{K_2}e^{\beta_{N+2}h_2}) \\ \frac{e^{\beta_1h_1}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2}h_1}}{1-\beta_{N+2}} \end{bmatrix}$$

Therefore

$$
\begin{aligned}
-e^{h_1} &= \frac{K_1}{det(A)}[\beta_1 e^{\beta_1 h_1}, ..., \beta_n e^{\beta_n h_1}]D^{-1}Y \\
&= det \begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2} h_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2} h_1} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h_2} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h_2} \\
\frac{e^{\beta_1 h_1}+\frac{K_1}{K_2}e^{\beta_1 h_2}}{\beta_1} & \cdots & \frac{e^{\beta_{N+2} h_1}+\frac{K_1}{K_2}e^{\beta_{N+2} h_2}}{\beta_{N+2}} \\
\frac{e^{\beta_1 h_1}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} h_1}}{1-\beta_{N+2}}
\end{bmatrix}
det \begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2} h_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2} h_1} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h_2} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h_2} \\
\frac{\frac{1}{K_1}e^{\beta_1 h_1}+\frac{1}{K_2}e^{\beta_1 h_2}}{\beta_1} & \cdots & \frac{\frac{1}{K_1}e^{\beta_{N+2} h_1}+\frac{1}{K_2}e^{\beta_{N+2} h_2}}{\beta_{N+2}} \\
\frac{1}{\beta_1}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}}e^{\beta_{N+2} h_1}
\end{bmatrix}^{-1} \\
&= det \begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{1}{1-\beta_1} & \cdots & \frac{1}{1-\beta_{N+2}}
\end{bmatrix}
det \begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{1}{\beta_1 K_1} & \cdots & \frac{1}{\beta_{N+2} K_1}
\end{bmatrix}^{-1} \\
&= -detA_1/detA_2
\end{aligned}
$$

which verifies Eq.(48). □

Proposition 5.2. *Given any $h \in \mathbb{R}$, define the matrix $B(h)$ as in Eq.(47). There exists a positive solution Δh to the equation $detB(h) = 0$.*

Proof. Note that $detB(h) = 0$ if and only if $det\widehat{B}(h) = 0$ where

$$
\widehat{B}(h) = \begin{bmatrix}
\frac{1}{\beta_1-1}(1+e^{(\beta_1-1)h}) & \cdots & \frac{1}{\beta_{N+2}-1}(1+e^{(\beta_{N+2}-1)h}) \\
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} h}) \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h}
\end{bmatrix}
$$

As $h=0$, $det\widehat{B}(0)=2(1+\frac{K_1}{K_2})detZ^{(N+2)}$ where

$$Z^{(N+2)}=\begin{bmatrix} \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N+2}-1} \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N+2}} \\ \frac{1}{\beta_1-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+} \end{bmatrix}.$$

We show that $detZ^{(N+2)}>0$. For simplicity, we set $\alpha_1=1$, $\alpha_{n+1}=\eta_n^-$ for $1\le n\le N^-$, $\alpha_{N^-+2}=0$, and $\alpha_{N^-+2+m}=\eta_m^+$ for $1\le m\le N^+$. Then the entry $z_{i,j}^{(N+2)}$ of $Z^{(N+2)}$ is equal to $\frac{1}{\beta_j-\alpha_i}$. Note that for $2\le i\le N+2$, $z_{i,j}^{(N+2)}>0$ if $i\le j$ and $z_{i,j}^{(N+2)}<0$ if $i>j$. For $2\le k\le N+1$, let $Z^{(k)}$ be the $k\times k$ matrix with entries $z_{i,j}^{(k)}=z_{N+2-k+i,N+2-k+j}^{(N+2)}=\frac{1}{\beta_{N+2-k+j}-\alpha_{N+2-k+i}}$ for $1\le i,j\le k$. First we show that $detZ^{(k)}>0$ for $2\le k\le N+1$. For $k=2$, we have

$$detZ^{(2)}=det\begin{bmatrix} \frac{1}{\beta_{N+1}-\alpha_{N+1}} & \frac{1}{\beta_{N+2}-\alpha_{N+1}} \\ \frac{1}{\beta_{N+1}-\alpha_{N+2}} & \frac{1}{\beta_{N+2}-\alpha_{N+2}} \end{bmatrix}$$

$$=\frac{1}{(\beta_{N+1}-\alpha_{N+1})(\beta_{N+2}-\alpha_{N+2})}-\frac{1}{(\beta_{N+2}-\alpha_{N+1})(\beta_{N+1}-\alpha_{N+2})}$$

$$=\frac{1}{(\beta_{N+1}-\eta_{N^+-1}^+)(\beta_{N+2}-\eta_{N^+}^+)}-\frac{1}{(\beta_{N+2}-\eta_{N^+-1}^+)(\beta_{N+1}-\eta_{N^+}^+)}>0$$

Before proceeding, we need the fact that $detZ^{(k)}\neq 0$ for $3\le k\le N+2$. Indeed, assume that $Z^{(k)}L=0$ for some column vector $L=(l_1,\cdots,l_k)'$. Let

$$F_k(x)=\sum_{j=1}^{k}\frac{l_j}{\beta_{N+2-k+j}-x}=\frac{G_k(x)}{\prod_{n=1}^{k}(\beta_{N+2-k+n}-x)}, \tag{50}$$

where $G_k(x)=\sum_{j=1}^{k}l_j\prod_{n=1,n\neq j}^{k}(\beta_{N+2-k+n}-x)$ is a polynomial with $deg(G_k(x))\le k-1$. Since $G_k(\alpha_{N+2-k+i})=\prod_{n=1}^{k}(\beta_{N+2-k+n}-\alpha_{N+2-k+i})F_k(\alpha_{N+2-k+i})=0$ for $1\le i\le k$, $G_k(x)$ has at least k distinct roots which implies $G_k(x)=0$. This implies $l_j=0$ for $1\le j\le k$ from Eq.(50) and hence $Z^{(k)}L=0$ has no nontrivial solutions, or equivalently, $detZ^{(k)}\neq 0$. Suppose that $detZ^{(k)}>0$ for some $2\le k\le N$. Consider the system of equations $Z^{(k+1)}\widetilde{L}^{(k+1)}=e_1^{(k+1)}$ where $\widetilde{L}^{(k+1)}$ and $e_1^{(k+1)}\equiv[1,0,\cdots,0]'$ are $(k+1)\times 1$ column vectors. Let $\widetilde{F}_{k+1}(x)=\sum_{j=1}^{k+1}\frac{\widetilde{l}_j^{(k+1)}}{\beta_{N+2-(k+1)+j}-x}=\frac{\widetilde{G}_{k+1}(x)}{\prod_{n=1}^{k+1}(\beta_{N+2-k-1+n}-x)}$. Then

$\widetilde{G}_{k+1}(x) = \sum_{j=1}^{k+1} \widetilde{l}_j^{(k+1)} \prod_{n=1,n\neq j}^{k+1}(\beta_{N+2-(k+1)+n} - x)$ is a polynomial with $deg(\widetilde{G}_{k+1}(x)) \leq k$ and $\widetilde{G}_{k+1}(\alpha_{N+2-(k+1)+i}) = \prod_{n=1}^{k}(\beta_{N+2-(k+1)+n} - \alpha_{N+2-(k+1)+i})\widetilde{F}_{k+1}(\alpha_{N+2-(k+1)+i}) = 0$ for $2 \leq i \leq k+1$. Therefore, we have $\{x|\widetilde{G}_{k+1}(x) = 0\} = \{\alpha_{N+2-(k+1)+i}|2 \leq i \leq k+1\}$ which implies $\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+i})\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+i+1}) < 0$ for $1 \leq i \leq k-1$. Since

$$\begin{aligned}
\widetilde{l}_j\widetilde{l}_{j+1} &= \frac{\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j})\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j+1})}{\prod_{n=1,n\neq j}^{k+1}(\beta_{N+2-(k+1)+n} - \beta_{N+2-(k+1)+j}) \prod_{n=1,i\neq j+1}^{k+1}(\beta_{N+2-(k+1)+n} - \beta_{N+2-(k+1)+j+1})} \\
&= \frac{\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j})\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j+1})}{\prod_{n=1,n\neq j,j+1}^{k+1}(\beta_{N+2-(k+1)+n} - \beta_j)^2(\beta_{N+2-(k+1)+j} - \beta_{N+2-k+j+1})(\beta_{N+2-(k+1)+j+1} - \beta_{N+2-(k+1)+j})} \\
&> 0
\end{aligned}$$

for $1 \leq j \leq k$, $\widetilde{l}_j$'s have the same sign. In addition, because the entries of the first row in $Z^{(k+1)}$ are positive and $\widetilde{F}_{k+1}(\alpha_{N+2-(k+1)+1}) = 1$, we obtain $\widetilde{l}_j > 0$ for all $1 \leq j \leq k+1$. On the other hand, by Cramer's rule, we know that $\widetilde{l}_1 = \frac{detZ^{(k)}}{detZ^{(k+1)}}$. Therefore, $detZ^{(k+1)} > 0$. Since $detZ^{(2)} > 0$, by induction, this implies $detZ^{(n)} > 0$ for $1 \leq n \leq N+1$. Consider the system of equations $Z^{(N+2)}\widetilde{L} = e_1$ where $\widetilde{L}$ and $e_1 = [1, 0, 0, ..., 0]'$ are $(N+2) \times 1$ column vectors. Let $\widetilde{F}_{N+2}(x) = \sum_{j=1}^{N+2} \frac{\widetilde{l}_j}{\beta_j - x} = \frac{\widetilde{G}_{N+2}(x)}{\prod_{n=1}^{N+2}(\beta_n - x)}$. Then $\widetilde{G}_{N+2}(x) = \sum_{j=1}^{N+2} \widetilde{l}_j \prod_{n=1,n\neq j}^{N+2}(\beta_n - x)$ is a polynomial with $deg(\widetilde{G}_{N+2}) \leq N+1$. By the equation $Z^{(N+2)}\widetilde{L} = e_1$, we have $\widetilde{G}_{N+2}(\alpha_n) = 0$ for $2 \leq n \leq N+2$. By similar arguments as above, we know that the entries of $\widetilde{L}$ have the same sign. By Lemma 4.1, $F_{N+2}(x)$ is well-defined on $[0,1]$ and in addition, $F_{N+2}(x) \in C([0,1]) \cap C^1(0,1)$. Besides, $Z^{(N+2)}\widetilde{L} = e_1$ implies $\widetilde{F}_{N+2}(1) = 1$ and $\widetilde{F}_{N+2}(0) = 0$. Therefore, by the mean value theorem, there exists $x_0 \in (0,1)$ such that $1 = \widetilde{F}(1) - \widetilde{F}(0) = \widetilde{F}'(x_0)(1-0) = \sum_{i=1}^{N+2} \frac{\widetilde{l}_i}{(\beta_i - x_0)^2}$ which implies that the entries of $\widetilde{L}$ are positive. On the other hand, by Cramer's rule, we have $\widetilde{l}_1 = \frac{detZ^{(N+1)}}{detZ^{(N+2)}}$. Therefore, $detZ^{(N+2)} > 0$. Consider the determinant

$$detW = det\begin{bmatrix} W^{(1)} & O_{N^-+1,N^++1} \\ O_{N^++1,N^-+1} & W^{(2)} \end{bmatrix} = detW^{(1)}detW^{(2)}$$

where $O_{m,n}$ is the $m \times n$ zero matrix,

$$W^{(1)} = \begin{bmatrix} \frac{1}{\beta_1 - 1} & \cdots & \frac{1}{\beta_{N^-+1} - 1} \\ \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1} - \eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1} - \eta_{N^-}^-} \end{bmatrix} \text{ and } W^{(2)} = \begin{bmatrix} \frac{1}{\beta_{N^-+2}} & \cdots & \frac{1}{\beta_{N+2}} \\ \frac{1}{\beta_{N^-+2} - \eta_1^+} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_{N^-+2} - \eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^+}^+} \end{bmatrix}.$$

We show that $detW < 0$. By similar arguments for the proof of the invertibility of $Z^{(k)}$, we observe that $W^{(1)}$ and $W^{(2)}$ are invertible. Consider the system of equations $W^{(1)}\widehat{L} = e_1$ where $\widehat{L}$ and $e^1 = [1, 0, 0, ..., 0]'$ are $(N^- + 1) \times 1$ column vectors. By the same arguments for $detZ^{(N+1)}$, we observe that the entries of $\widehat{L}$ have the same sign and $det\widetilde{W}^{(1)} > 0$ where

$$\widetilde{W}^{(1)} = \begin{bmatrix} \frac{1}{\beta_2-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_2-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} \end{bmatrix}.$$

Because $w_{1,j}^{(1)} < 0$ for all $1 \le j \le N^-+1$ and $\sum_{j=1}^{N^-+1} \frac{\widehat{l}_j}{\beta_j-1} = 1$, the entries of $\widehat{L}$ are negative. On the other hand, by Cramer's rule, we have $\widehat{l}_1 = \frac{det\widetilde{W}^{(1)}}{detW^{(1)}}$ and hence, $detW^{(1)} < 0$. By a similar argument, we obtain $detW^{(2)} > 0$ and hence $detW < 0$. Since

$$\lim_{h\to\infty} \frac{det\widehat{B}(h)}{e^{\sum_{i=N^-+2}^{N+2}\beta_i h}} = \lim_{h\to\infty} det \begin{bmatrix} \frac{1+e^{(\beta_1-1)h}}{\beta_1-1} & \cdots & \frac{1+e^{(\beta_{N^-+1}-1)h}}{\beta_{N^-+1}-1} & \frac{e^{-\beta_{N^-+2}h}+e^{-h}}{\beta_{N^-+2}-1} & \cdots & \frac{e^{-\beta_{N+2}h}+e^{-h}}{\beta_{N+2}-1} \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} & \frac{e^{-\beta_{N^-+2}h}}{\beta_{N^-+2}-\eta_1^-} & \cdots & \frac{e^{-\beta_{N+2}h}}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} & \frac{e^{-\beta_{N^-+2}h}}{\beta_{N^-+2}-\eta_{N^-}^-} & \cdots & \frac{e^{-\beta_{N+2}h}}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1+\frac{K_1}{K_2}e^{\beta_1 h}}{\beta_1} & \cdots & \frac{1+\frac{K_1}{K_2}e^{\beta_{N^-+1}h}}{\beta_{N^-+1}} & \frac{e^{-\beta_{N^-+2}h}+\frac{K_1}{K_2}}{\beta_{N^-+2}} & \cdots & \frac{e^{-\beta_{N+2}h}+\frac{K_1}{K_2}}{\beta_{N+2}} \\ \frac{e^{\beta_1 h}}{\beta_1-\eta_1^+} & \cdots & \frac{e^{\beta_{N^-+1}h}}{\beta_{N^-+1}-\eta_1^+} & \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\beta_1 h}}{\beta_1-\eta_{N^+}^+} & \cdots & \frac{e^{\beta_{N^-+1}h}}{\beta_{N^-+1}-\eta_{N^+}^+} & \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+} \end{bmatrix}$$

$$= det \begin{bmatrix} \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N^-+1}-1} & 0 & \cdots & 0 \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} & 0 & \cdots & 0 \\ \frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N^-+1}} & \frac{K_1}{K_2}\frac{1}{\beta_{N^-+2}} & \cdots & \frac{K_1}{K_2}\frac{1}{\beta_{N+2}} \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+} \end{bmatrix} = \frac{K_1}{K_2} detW < 0,$$

$det\widehat{B}(h) < 0$ as h large enough. (The last equality is due to the basis

$$\left\{ [\frac{1}{\beta_1-1}, \cdots, \frac{1}{\beta_{N^-+1}-1}], [\frac{1}{\beta_1-\eta_1^-}, \cdots, \frac{1}{\beta_{N^-+1}-\eta_1^-}], \cdots, [\frac{1}{\beta_1-\eta_{N^-}^-}, \cdots, \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-}] \right\}.)$$

In addition, we have $det\widehat{B}(0) = 2(1+\frac{K_1}{K_2})detZ^{(N+2)} > 0$. By the intermediate value theorem, this implies that $det\widehat{B}(h) = 0$ has a positive solution Δh. □

Theorem 5.3. *Let Δh be a positive solution of the equation $detB(h) = 0$ and define h_1 by Eq.(48). Set $h_2 = h_1 + \Delta h$ and compute $\{C_1, \cdots, C_{N+2}\}$ by the formula Eq.(46). Then $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ is a solution of the equations Eq.(29)-Eq.(34).*

Proof. The system of equations Eq.(29)-Eq.(34) is equivalent to $\widetilde{A}DC = \widetilde{K}$ together with the smooth pasting conditions Eq.(33) and Eq.(34). From the proof of Proposition 5.1, we know that $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ satisfies $\widetilde{A}DC = \widetilde{K}$ and Eq.(34). It remains to check that Eq.(33) is satisfied. By Eq.(46), the left hand side of Eq.(33) is

$$\sum_{n=1}^{N+2} \frac{K_1 y_n}{det(\widetilde{A})(1-\beta_n)} e^{\beta_n h_2}$$

$$= det \begin{bmatrix} \frac{e^{\beta_1 h_1}}{\beta_1-\eta_1^-} & \cdots & \frac{e^{\beta_{N+2} h_1}}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{e^{\beta_1 h_1}}{\beta_1-\eta_{N-}^-} & \cdots & \frac{e^{\beta_{N+2} h_1}}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{e^{\beta_1 h_2}}{\beta_1-\eta_1^+} & \cdots & \frac{e^{\beta_{N+2} h_2}}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{e^{\beta_1 h_2}}{\beta_1-\eta_{N+}^+} & \cdots & \frac{e^{\beta_{N+2} h_2}}{\beta_{N+2}-\eta_{N+}^+} \\ \frac{(e^{\beta_1 h_1}+\frac{K_1}{K_2}e^{\beta_1 h_2})}{\beta_1} & \cdots & \frac{(e^{\beta_{N+2} h_1}+\frac{K_1}{K_2}e^{\beta_{N+2} h_2})}{\beta_{N+2}} \\ \frac{e^{\beta_1 h_2}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} h_2}}{1-\beta_{N+2}} \end{bmatrix} det \begin{bmatrix} \frac{e^{\beta_1 h_1}}{\beta_1-\eta_1^-} & \cdots & \frac{e^{\beta_{N+2} h_1}}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{e^{\beta_1 h_1}}{\beta_1-\eta_{N-}^-} & \cdots & \frac{e^{\beta_{N+2} h_1}}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{e^{\beta_1 h_2}}{\beta_1-\eta_1^+} & \cdots & \frac{e^{\beta_{N+2} h_2}}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{e^{\beta_1 h_2}}{\beta_1-\eta_{N+}^+} & \cdots & \frac{e^{\beta_{N+2} h_2}}{\beta_{N+2}-\eta_{N+}^+} \\ \frac{(\frac{1}{K_1}e^{\beta_1 h_1}+\frac{1}{K_2}e^{\beta_1 h_2})}{\beta_1} & \cdots & \frac{(\frac{1}{K_1}e^{\beta_{N+2} h_1}+\frac{1}{K_2}e^{\beta_{N+2} h_2})}{\beta_{N+2}} \\ \frac{e^{\beta_1 h_1}}{\beta_1} & \cdots & \frac{e^{\beta_{N+2} h_1}}{\beta_{N+2}} \end{bmatrix}^{-1}$$

$$= det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\ \frac{e^{\beta_1 \Delta h}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} \Delta h}}{1-\beta_{N+2}} \end{bmatrix} det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\ \frac{1}{\beta_1}(1+\frac{1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{1}{K_2}e^{\beta_{N+2} \Delta h}) \\ \frac{1}{\beta_1 K_1} & \cdots & \frac{1}{\beta_{N+2} K_1} \end{bmatrix}^{-1}$$

$$= det A_2^{-1} det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\ \frac{e^{\beta_1 \Delta h}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} \Delta h}}{1-\beta_{N+2}} \end{bmatrix}$$

Since Δh satisfies $detB(h) = 0$, we have

$$-detA_1 = -det\begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N+2}-1} \end{bmatrix}$$

$$= det\begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1}e^{(\beta_1-1)\Delta h} & \cdots & \frac{1}{\beta_{N+2}-1}e^{(\beta_{N+2}-1)\Delta h} \end{bmatrix} = e^{-\Delta h}det\begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-1}e^{\beta_{N+2}\Delta h} \end{bmatrix}$$

Therefore, the left hand side of Eq.(33) is equal to $detA_2^{-1}detA_1e^{\Delta h} = e^{h_1+\Delta h} = e^{h_2}$. The proof is complete. $\square$

6 Numerical Results

In this section, we solve the system of equations Eq.(29)-Eq.(34) numerically. To do this, we first find the length of the continuation region, Δh, by solving the equation $detB(h) = 0$ where $B(h)$ is the square matrix in Eq.(47). Second, we compute h_1 by Eq.(48) and set $h_2 = h_1 + \Delta h$. Finally, we obtain the coefficient vector C according to Eq.(46) and evaluate the value function $V(x)$ by the formula $V(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x}$ for $x \in (h_1, h_2)$.

Example 1: Consider the case that $N^+ = N^- = 1$. In addition, as in Boyarchenko[4], we take $c = -0.105$, $\sigma = 0.25$, $r = 0.06$, $\eta^+ = \frac{1}{0.4}$, $\eta^- = -\frac{1}{0.7}$, $\lambda = \frac{3}{5}$, $p = q = 0.5$ and the strike prices $K_1 = 50$ and $K_2 = 100$. Then the value function is given by $V(x) = \sum_{n=1}^{4} C_n e^{\beta_n x}$ in (h_1^*, h_2^*) where

$$\begin{aligned} (h_1^*, h_2^*) &= (2.1992, 6.1953) \\ \{\beta_1, \beta_2, \beta_3, \beta_4\} &= \{-3.4812, -0.2322, 1.1995, 6.953\} \\ \{C_1, C_2, C_3, C_4\} &= \{2519.533, 61.2124, 0.2183, 1.4624 \times 10^{-18}\}. \end{aligned}$$

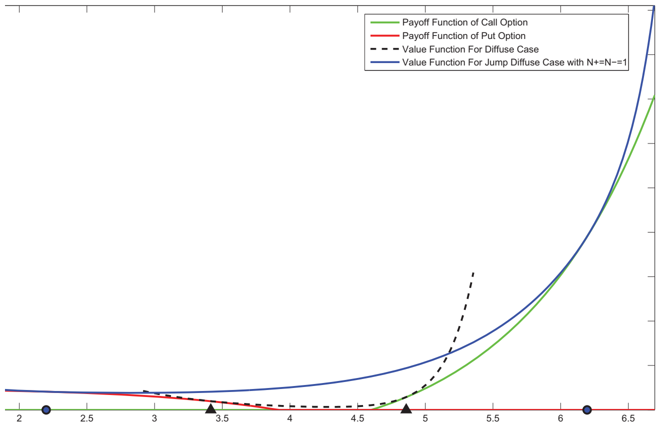

Figure 1: The solid line is the value function $V(x)$ for the jump-diffusion model with $N^+ = N^- = 1$ and the dash line is the one for the diffusion model, that is, $N^+ = N^- = 0$. The optimal boundaries are marked by circles for jump-diffusion model, and by triangles for diffusion model.

Besides, if we take $N^+ = N^- = 0$ which is the diffusion case, then we observe $V(x) = \sum_{n=1}^{2} C_n e^{\beta_n x}$ in (h_1^*, h_2^*) where

$$\begin{aligned} (h_1^*, h_2^*) &= (3.4151, 4.859) \\ \{\beta_1, \beta_2\} &= \{-1.5607, 4.9207\} \\ \{C_1, C_2\} &= \{4037.8534, 1.1088 \times 10^{-9}\}. \end{aligned}$$

It is interesting to note that in the jump-diffusion model, the optimal interval (h_1^*, h_2^*) is much wider than that for the diffusion case. This indeed makes sense because there are more opportunities to earn large gains by the jump occurring and hence it can be expected that the investors will not exercise the options in the jump-diffusion environment earlier than in the diffusion one. Figure 3 shows the graph of the determinant of $B(h)$ as a function of h. It shows that the zero of the determinant (this is Δh) is unique. Besides, the graph descends sharply near the zero of the determinant. This implies that we can get the numerical result for Δh fast and correctly.

Example 2: Consider the jump-diffusion model with $N^- = N^+ = 2$ and let $c = -0.105$, $\sigma = 0.25$, $r = 0.06$, $\eta_1^+ = \frac{1}{0.5}$, $\eta_2^+ = \frac{1}{0.25}$, $\eta_1^- = -\frac{1}{2.4}, \eta_2^- = -7.5$, $\lambda = \frac{3}{5}$, $p_1 = p_2 = q_1 = q_2 = 0.25$ and the strike prices $K_1 = 50$ and $K_2 = 100$. In this model,

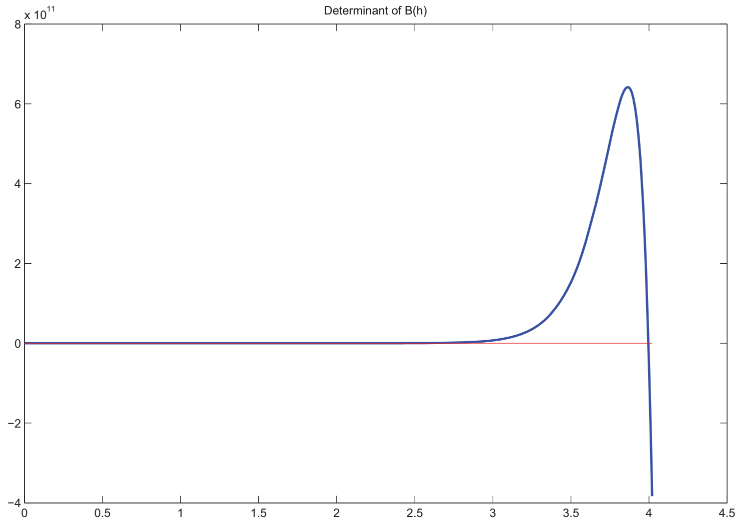

Figure 2: The figure is the graph of the determinant $B(h)$ for finding the length Δh of the optimal interval. It shows that there is only one zero for the determinant.

the expected value $E[e^{X_1}]$ is the same as the one with $N^- = N^+ = 1$ in Example 1. The value function is $V(x) = \sum_{n=1}^{6} C_n e^{\beta_n x}$ in (h_1^*, h_2^*) where

$$\begin{aligned} (h_1^*, h_2^*) &= (2.1153, 6.3801) \\ \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\} &= \{-7.997, -1.9409, -0.1155, 1.1642, 3.2421, 7.0931\} \\ \{C_1, C_2, C_3, C_4, C_5, C_6\} &= \{735200.1029, 240.6048, 44.1297, 0.2679, 8.8413 \times 10^{-9}, \\ &\quad 2.4671 \times 10^{-19}\}. \end{aligned}$$

As noted before, models in Example 1 and Example 2 have the same expected value $E[e^{X_1}]$. However the optimal interval in Example 2 ($N^- = N^+ = 2$) is wider than that for the case $N^- = N^+ = 1$.

7 Conclusion

In this study, we consider the problem of pricing the perpetual American strangle option under the hyper-exponential jump-diffusion Lévy process, which was mentioned in Boyarchenko[4]. Owing to the analytical tractability of the mixture-exponential density function, we derive alternative representation of the two-sided first passage functional by

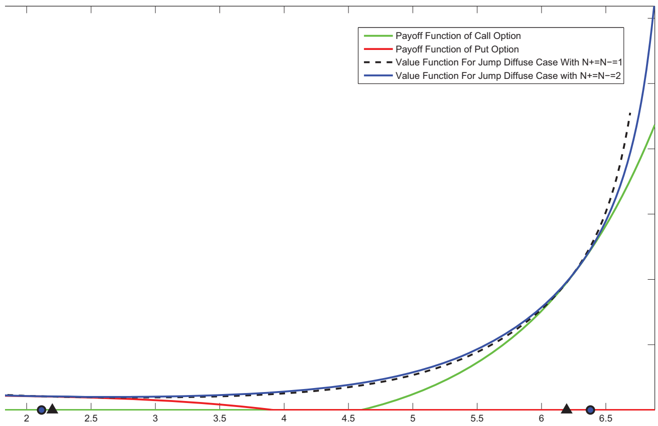

Figure 3: The solid line is the value function $V(x)$ for the jump-diffusion model with $N^+ = N^- = 2$ and the dash line is the one for the model with $N^+ = N^- = 1$. The optimal boundaries for the case $N^+ = N^- = 2$ are marked by circles and by triangles for the case $N^- = N^+ = 1$.

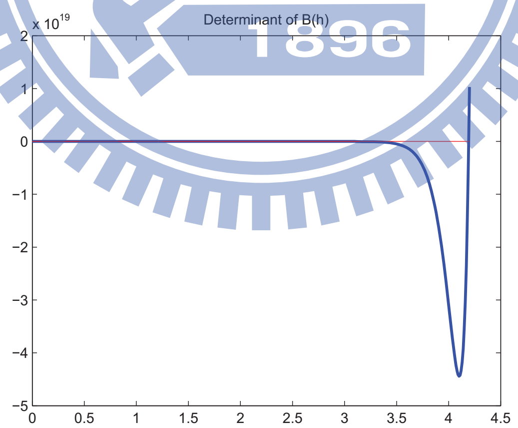

Figure 4: The figure is the graph of the determinant for finding the length of the optimal interval for the case $N^- = N^+ = 2$. The figure has similar properties as for the case $N^- = N^+ = 1$. In particular, there is only one zero for the determinant.

transforming the integro-differential equation in Eq.(3) to higher ODE. Therefore, we obtain that the two-sided first passage functional is a linear combination of the exponential functions. (Using the Winer-Hopf decomposition, Boyarchenko[4] observed the same result.) By Theorem 2.6 and the verification theorems in Section 3, we prove the conjecture in Boyarchenko[4]: the value function of the perpetual American strangle options is a two-sided first passage functional and the continuation region is a finite interval satisfying the smooth pasting condition. Also, we show that the existence of the solution to the free boundary problem with embedding the smooth pasting condition.(This is another open problem posed in Boyarchenko[4].) For calculating the value function and the boundaries of the continuation region, we improve the algorithm in Boyarchenko[4] such that the computing rate is from $O(N^2)$ reduced to $O(N)$ where N is the number of the mixtures in the density function. It is worth noting that the verification theorem can be applied to not only the perpetual American strangle options but also other perpetual American exotic options. These are the further researches.

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