國立交通大學應用數學系

博 士 論 文

非線性 Klein-Gordon 方程的色散極限

Dispersive Limits of the Nonlinear Klein-Gordon Equations

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中華民國九十九年五月

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博 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Applied Mathematics

May 2010

Hsinchu, Taiwan

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摘要

本論文主要研究非線性 Klein-Gordon 方程的色散極限問題。首先，我們從 Klein-Gordon 方程嚴格數學的推導到可壓縮與不可壓縮的歐拉方程。在極限系統出現奇異點前，非相對論-半古典極限可推導到可壓縮的歐拉方程。假如我們考慮時間的尺度變換，則半古典極限(光速固定)可以得到不可壓縮的歐拉方程。

我們也完成了有關非線性 Klein-Gordon 方程的奇異極限問題，包含了半古典極限、非相對論極限與非相對論-半古典極限。有關半古典極限，我們證明了三次非線性的 Klein-Gordon 方程其波函數收斂到有相對論效應的 wave map 方程，且對應的相函數滿足有相對論效應的線性波方程。另外，非相對論極限的非線性 Klein-Gordon 方程收斂到非線性的薛丁格方程。最後，有關非相對論-半古典極限，我們證明了三次非線性的 Klein-Gordon 方程其波函數收斂到 wave map 方程，且對應的相函數滿足線性波方程。

Dispersive Limits of the Nonlinear Klein-Gordon Equations

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Abstract

This dissertation investigates the dispersive limits of the nonlinear Klein-Gordon equations. First, we perform the mathematical derivation of the compressible and incompressible Euler equations from the modulated nonlinear Klein-Gordon equation. Before the formation of singularities in the limit system, the nonrelativistic-semiclassical limit is shown to be the compressible Euler equations. If we further rescale the time variable, then in the semiclassical limit (the light speed kept fixed), the incompressible Euler equations are recovered.

We also establish the singular limits including semiclassical, nonrelativistic and nonrelativistic-semiclassical limits of the Cauchy problem for the modulated defocusing nonlinear Klein-Gordon equation. For the semiclassical limit, we show that the limit wave function of the modulated defocusing cubic nonlinear Klein-Gordon equation solves the relativistic wave map and the associated phase function satisfies a linear relativistic wave equation. The nonrelativistic limit of the modulated defocusing nonlinear Klein-Gordon equation is the defocusing nonlinear Schrodinger equation. The nonrelativistic-semiclassical limit of the modulated defocusing cubic nonlinear Klein-Gordon equation is the classical wave map for the limit wave function and typical linear wave equation for the associated phase function.

誌　　　謝

本篇論文得以完成，首先最感謝的就是我的指導教授林琦焜老師，在研究方面，老師從如何讀書和讀論文慢慢一步步地教導我，讓我明白做研究的態度和方法，當我迷網不知所措的時候，總是會適時地指引我方向，鼓勵我繼續往前邁進，在老師身上，我看到一個學者在研究上的專業、熱情與堅持，是我將來學習的一個好榜樣。除了研究之外，在做人處世方面，老師也教導我如何應對進退。老師給我的幫助不是簡短的三言兩語所能言盡的，真心感謝老師這些年的教導。

感謝口試委員劉太平教授、李志豪教授、蔡東和教授以及賴明治教授在論文口試時的寶貴意見。其實這四位口試委員也跟我有很深的淵源。從博二開始，我就到中研院跟劉太平老師做 shock wave 的研究，劉老師對問題有非常創意的想法，對我有非常大的啟發，感謝劉老師這些年的照顧。李志豪老師是在中研院認識的老師，對我非常關心，也時常叮嚀我一些生活上的細節。蔡東和老師是我碩士班的指導教授，感謝老師在碩士班為我打下良好的基礎，讓我能在往後的路上更順利。賴明治老師是我本來來交大要跟的指導教授，因為對數值的興趣不大，以致於沒能繼續，感謝賴老師這些年的包容。

感謝在中研院一起打拼的博士後與博士生，大家建立了深厚的感情，讓我在中研院感到非常的溫暖，也讓我覺的在學術的路上，其實我並不孤單。也感謝交大的朋友們，不管在學業上或生活上，都給我非常大的幫助，與你們一起相處的時間非常輕鬆愉快，讓我度過愉快的五年，有了你們我的生活更加多采多姿。

感謝父親吳丁福先生，母親吳楊冬筝女士的養育之恩，從小到大，在各方面我的父母總是盡全力支持我，家是我最好的避風港，不管發生什麼事，都會永遠在那邊守護我，讓我能夠順利完成博士學位。還有佳音、家儀與依玲，都會和我聊聊生活以及學業上的事情，也會鼓勵我繼續努力向前。

最後，有大家一路上的陪伴幫助，我才能順利地博士班畢業，今後我會繼續努力勇敢向前邁進。由衷地感謝大家，謝謝你們。

吳恭儉 2010\5\15　於交大

Contents

Dispersive Limits of the Nonlinear Klein-Gordon Equations

1 Introduction

The relativistic quantum mechanic equation for a free particle can be derived by writing

$$E^2 = c^2p^2 + m^2c^4,$$

where E is energy, p is momentum, m is mass, and c is the speed of light. The quantum mechanical description of a relativistic free particle results from applying the *corresponding principle*, which allows one to replace classical observable by quantum mechanical operators acting on wave functions [21, 24]. Let $\hbar$ denote the Planck constant, then the Schrödinger correspondence principle given by

$$E \to i\hbar\partial_t, \qquad p \to -i\hbar\nabla,$$

will result in the Klein-Gordon equation

$$-\hbar^2\partial_t^2\Psi = -c^2\hbar^2\Delta\Psi + m^2c^4\Psi$$

for wave function Ψ. The Klein-Gordon equation for the complex scalar field is the relativistic version of the Schrödinger equation, which is used to describe spinless particles. It was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. However, this equation was named after the physicists Oskar Klein and Walter Gordon, who in 1927 proposed that it describes relativistic electrons. Although it turned out that the Dirac equation describes the spinning electron, the Klein-Gordon equation correctly describes the spinless pion [21]. The nonlinear Klein-Gordon equation is easily obtained by adding the term $V'(|\Psi|^2)\Psi$, where V is the potential energy density of the fields;

$$\frac{\hbar^2}{2mc^2}\partial_t^2\Psi - \frac{\hbar^2}{2m}\Delta\Psi + \frac{mc^2}{2}\Psi + V'(|\Psi|^2)\Psi = 0\,. \tag{1.1}$$

Since mc^2t and $\hbar$ have the same dimension of action, $[mc^2t] = [\hbar] = [\text{action}]$, and we may consider the modulated wave function [17]

$$\psi(x,t) = \Psi(x,t)\exp(imc^2t/\hbar)\,, \tag{1.2}$$

where the factor $\exp(imc^2t/\hbar)$ describes the oscillations of the wave function, then ψ satisfies the modulated nonlinear Klein-Gordon equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\Delta\psi - V'(|\psi|^2)\psi = \frac{\hbar^2}{2mc^2}\partial_t^2\psi\,. \tag{1.3}$$

The relations between different terms in (1.3) are best seen when the equation is written in terms of dimensionless variables, which will be adorned with carets. The dimensionless independent variables are given by

$$x = L\hat{x}\,, \qquad t = T\hat{t}\,,$$

where L and T denote the reference length and time respectively. We also define the reference velocity by $U = L/T$ and rescale the potential energy as

$$V' = mU^2\hat{V}'\,.$$

Substituting all of these rescaled quantities into the original equation (1.3), and dropping all carets, yields

$$i\varepsilon\partial_t\psi + \frac{1}{2}\varepsilon^2\Delta\psi - V'(|\psi|^2)\psi = \frac{1}{2}\varepsilon^2\nu^2\partial_t^2\psi\,. \tag{1.4}$$

Note that the first important dimensionless parameter ν is given by the ratio of reference velocity and speed of light, $\nu = U/c$, and the scaled Planck constant $\varepsilon = \frac{\hbar}{mUL}$ is the second important dimensionless parameter. The two dimensionless parameters ν and ε show the relativistic and quantum effects respectively.

Over the last twenty years, there has been a vast amount of research concerning the non-relativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation. In particular, in [17] Machihara-Nakanishi-Ozawa proved that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after infinite oscillations in time are removed. The Strichartz estimate plays the most important role to obtain the uniform bound in space and time (see also [20, 22] and references therein). However, to the best of our knowledge, the semiclassical limit $\varepsilon \to 0$ is not well studied and is not clear from (1.4). On the other hand, based on the hydrodynamic structure, the semiclassical limit, $\varepsilon \to 0$, of the defocusing nonlinear Schrödinger equation is quite well understood (see [6] for the review). In [8], Jin-Levermore-McLaughlin applied the inverse scattering to establish the semiclassical limit of the defocusing cubic nonlinear Schrödinger equation; the complete integrability was exploited to obtain the global characterization of the weak limits of the entire cubic NLS hierocracy. Therefore to study the various singular (hydrodynamics) limits of the nonlinear Klein-Gordon equation (1.1), it is better to start from (1.4) because of its analogue to the nonlinear Schrödinger equation. The rest of the thesis is organized as follows:

In chapter 2, we derive the hydrodynamic structure of the modulated nonlinear Klein-Gordon equation and discuss their relation to the compressible and incompressible Euler equations formally.

In chapter 3, we study the hydrodynamic limits of the Klein-Gordon equations, the modulated energy method plays an important role in analysis. It is designed to control the propagation of the charge and current (or momentum) of the Klein-Gordon equation is constituted by the Schrödinger and relativistic parts, thus, the main idea is to show that the relativistic charge and current are small and the main contribution of the nonrelativistic-semiclassical limit comes from the Schrödinger part. In contrast with the Schrödinger equation and its variants, we have to introduce one correction term of the modulated energy which controls the propagation of the relativistic charge and current. In fact, the relativistic parts vanishes as ε tends to zero. Thus we prove the convergence of the charge and the current defined by the modulated nonlinear Klein-Gordon equation towards the solution of the γ-law compressible Euler equations.

Turning to the incompressible limit, we have to rescale the time variable and consider the potential energy designed to represent in the form of pressure instead of the charge (or density). In this case, we show that the current converges to the incompressible Euler equations in the semiclassical limit. Besides the correction term of the modulated energy as discussed in the compressible Euler limit, we have to introduce one more correction term which describes the propagation of the density fluctuation in order to obtain the incompressible limit. This is similar to the zero Mach number limit of the compressible fluid [2, 16, 18]. The convergent result can be improved for $n = 2$ by the standard bootstrap process.

In chapter 4, we study the singular limits of the Klein-Gordon equation directly. First, We investigate the semiclassical limit of the Cauchy problem for the modulated defocusing cubic nonlinear Klein-Gordon Eqs. (4.3)–(4.4). We prove that any finite charge-energy solution converges to the corresponding solution of the relativistic wave map and the scattering sound wave is shown to satisfy a linear relativistic wave equation (see Theorem 4.2 below). Unlike the Schrödinger equation, the charge is not positive definite for Klein-Gordon equation and we have to introduce the charge-energy inequality obtained by combining the conservation laws of charge and energy of the nonlinear Klein-Gordon equation. Besides the linear momentum W of the Schrödinger part, we have to introduce one more term Z, defined by (4.18), of the relativistic part. By rewriting the conservation of charge in terms of W and Z we can prove the convergence to the relativistic wave map by the compactness argument. Shatah [25] has proved the existence of global weak solutions of the wave map. For completeness we also prove the nonrelativistic limit of relativistic wave map in Theorem 4.7.

Second, we employ the same idea to obtain the nonrelativistic limit of the Cauchy problem for the modulated Klein-Gordon equation for general defocusing nonlinearity $V'(|\psi^\nu|^2) = |\psi^\nu|^p$, $p > 0$, and the main result is described in Theorem 4.9 which state that any finite charge-energy solution converges to the corresponding solution of the defocusing nonlinear Schrödinger equation in the energy space. For the sharper Strichartz estimate approach and more complete result the reader is referred to [17]. The main difference is that we combine the charge and energy conservation laws together to obtain the charge-energy inequality. Let us remark that in the case of semiclassical limit, we have $L_t^\infty L_x^2$ bound for $\partial_t\psi^\varepsilon$, but for non-relativistic limit, we only have $L_t^\infty L_x^2$ bound for $\nu\partial_t\psi^\nu$. Thus we need extra argument to obtain the strong convergence for non-relativistic limit.

Finally, we study the nonrelativistic-semiclassical limit of Cauchy problem for the modulated defocusing cubic nonlinear Klein-Gordon equation. We prove that any finite charge-energy solution converges to the corresponding solution of the wave map and the associated phase function is shown to satisfy a linear wave equation, the main result is stated in Theorem 4.13. Moreover, we give a detail proof of Theorem 4.8. The strategy of the proof follows that introduced by Leray in the context of the Navier-Stokes equations, as well as many other existence proofs for weak solutions of other equations.

Notation. In this paper, $L^p(\Omega), (p \geq 1)$ denotes the classical Lebesgue space with norm $\|f\|_p = (\int_\Omega |f|^p dx)^{1/p}$, the Sobolev space of functions with all its k-th partial derivatives in $L^2(\Omega)$ will be denoted by $H^k(\Omega)$, and its dual space is $H^{-k}(\Omega)$. We use $\langle f, g\rangle = \int_\Omega fg dx$ to denote the standard inner product on the Hilbert space $L^2(\Omega)$. Without lost of generality the units of length maybe chosen so that $\int_\Omega dx = 1$. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $p \geq 1$, the space of measurable functions $u = u(t)$ from $[0,T]$ into $\mathbb{X}$ such that $\|u\|_{\mathbb{X}} \in L^p([0,T])$ will be denoted $L^p([0,T];\mathbb{X})$. And $C([0,T];w\text{-}H^k(\Omega))$ will denote the space of continuous function from $[0,T]$ into $w\text{-}H^k(\Omega)$. This means that for every $\varphi \in H^{-k}(\Omega)$, the function $\langle\varphi, u(t)\rangle$ is in $C([0,T])$. Finally, we abbreviate " $\leq C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed parameter.

2 Hydrodynamic Structure

A fluid mechanical interpretation for the linear Schrödinger equation was put forth by Madelung in 1927 and applies to nonlinear Schrödinger equations. Indeed, as shown in [7], the same idea also applied to the modulated nonlinear Klein-Gordon equation (1.4). We introduce the complex wave function, the so-called Madelung transformation,

$$\psi = A\exp(iS/\varepsilon), \tag{2.1}$$

in which both A, the amplitude, and S, the action function, are real-valued function. Plugging (2.1) into modulated nonlinear Klein-Gordon equation (1.4) and separating the real and imagine parts, we obtain

$$\partial_t A + \frac{A}{2}\big(\Delta S - \nu^2\partial_t^2 S\big) + \nabla A\cdot\nabla S - \nu^2\partial_t A\partial_t S = 0. \tag{2.2}$$

$$\partial_t S + \frac{1}{2}|\nabla S|^2 - \frac{1}{2}\nu^2(\partial_t S)^2 + V'(A^2) = \frac{\varepsilon^2}{2}\frac{\Box_\nu A}{A}, \tag{2.3}$$

where the d'Alerbertian $\Box_\nu$ is defined by $\Box_\nu \equiv \Delta - \nu^2\partial_t^2$. Equation (2.2) turns out to be the continuity equation for the *relativistic quantum fluid* and equation (2.3) is the relativistic quantum Hamilton-Jacobi equation. Introducing the new functions

$$\rho = A^2 = |\psi|^2\,, \tag{2.4}$$

$$u = \nabla S = \frac{i\varepsilon}{2}\frac{1}{|\psi|^2}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi)\,, \tag{2.5}$$

$$\tau = \partial_t S = \frac{i\varepsilon}{2}\frac{1}{|\psi|^2}(\psi\partial_t\overline{\psi} - \overline{\psi}\partial_t\psi)\,, \tag{2.6}$$

we can rewrite (2.2)–(2.3) as the dispersive perturbation of the compressible Euler type equations

$$\partial_t\big(\rho(1-\nu^2\tau)\big) + \nabla\cdot\big(\rho u\big) = 0\,, \qquad \partial_t u = \nabla\tau\,, \tag{2.7}$$

$$\begin{aligned}&\partial_t\big(\rho u(1-\nu^2\tau)\big) + \nabla\cdot\big(\rho u\otimes u\big) + \nabla P(\rho)\\ &\quad= \frac{\varepsilon^2}{4}\nabla\cdot\Big(\rho\nabla^2\log\rho\Big) - \frac{\varepsilon^2\nu^2}{4}\partial_t\Big(\rho\nabla\partial_t\log\rho\Big),\end{aligned} \tag{2.8}$$

where $P(\rho) = \rho V'(\rho) - V(\rho)$ is the pressure and ∇^2 denotes the Hessian. Eqs. (2.7)–(2.8) is constituted by the Euler, relativistic and quantum parts. If the "Euler part" of these equations is to be hyperbolic, then the pressure $P(\rho)$ must be a strictly increasing function of ρ; in that case, $P'(\rho) = \rho V''(\rho) > 0$. This means that V must be a strictly convex function of ρ and corresponds to a *defocusing* nonlinear Klein-Gordon equation. The compatibility condition $\partial_t u = \nabla\tau$ also implies that

$$u(x,t) = \nabla\left(\int_0^t \tau(x,\xi)d\xi + S(x,0)\right). \tag{2.9}$$

Defining the Schrödinger part energy density E_S and relativistic part energy density E_K respectively by

$$E_S = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla\rho|^2}{\rho} + V(\rho) = \frac{\varepsilon^2}{2}|\nabla\psi|^2 + V(|\psi|^2), \tag{2.10}$$

$$E_K = \frac{1}{2}\nu^2\rho|\tau|^2 + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho} = \frac{\varepsilon^2\nu^2}{2}|\partial_t\psi|^2, \tag{2.11}$$

we obtain from (2.7)–(2.8) the conservation of energy

$$\partial_t(E_S + E_K) + \nabla\cdot\Big((E_S + P(\rho))u\Big) = \frac{\varepsilon^2}{4}\nabla\cdot\left[u\Delta\rho - \nabla\cdot(\rho u)\frac{\nabla\rho}{\rho}\right]. \tag{2.12}$$

In the formal nonrelativistic limit $\nu \to 0$, one neglects the $O(\nu^2)$ terms appearing in (2.7)–(2.8) and the limit densities ρ, u and P satisfy the quantum hydrodynamic equations

$$\partial_t\rho + \nabla\cdot\big(\rho u\big) = 0, \tag{2.13}$$

$$\partial_t(\rho u) + \nabla\cdot\big(\rho u\otimes u\big) + \nabla P(\rho) = \frac{\varepsilon^2}{4}\nabla\cdot\Big[\rho\nabla^2\log\rho\Big], \tag{2.14}$$

which are exactly the fluid formulation of the defocusing nonlinear Schrödinger equation. In this case the relativistic part energy density E_K vanishes and the limit energy density E will be given by

$$E = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla\rho|^2}{\rho} + V(\rho) \tag{2.15}$$

and will satisfy

$$\partial_t E + \nabla\cdot\Big((E + P(\rho))u\Big) = \frac{\varepsilon^2}{4}\nabla\cdot\left[(\rho u)\frac{\Delta\rho}{\rho} - \nabla\cdot(\rho u)\frac{\nabla\rho}{\rho}\right]. \tag{2.16}$$

Next letting $\nu \to 0$ and $\varepsilon \to 0$ simultaneously, both the relativistic and quantum correction terms in (2.7)–(2.8) vanish and the limit densities ρ, u and P will satisfy the compressible Euler equations

$$\partial_t \rho + \nabla \cdot (\rho u) = 0\,, \tag{2.17}$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = 0\,, \tag{2.18}$$

and the limit energy density E will be given by

$$E = \frac{1}{2}\rho|u|^2 + V(\rho) \tag{2.19}$$

and will satisfy

$$\partial_t E + \nabla \cdot \Big((E + P(\rho))u \Big) = 0 \tag{2.20}$$

hence playing the role of a Lax entropy for the Euler system.

In order to investigate the incompressible limit, we introduce the scaling

$$\widetilde{t} = \varepsilon^\alpha t, \qquad \widetilde{x} = x\,, \qquad \alpha > 0\,.$$

After dropping the tilde, the modulated nonlinear Klein-Gordon equation (1.4) becomes

$$i\varepsilon^{1+\alpha}\partial_t \psi - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2 \psi + \frac{\varepsilon^2}{2}\Delta\psi - V'(|\psi|^2)\psi = 0\,. \tag{2.21}$$

For this model the corresponding fluid dynamics equations (2.7)–(2.8) turn out to be

$$\partial_t\big(\rho(1 - \nu^2\varepsilon^{2\alpha}\tau)\big) + \nabla \cdot (\rho u) = 0\,, \tag{2.22}$$

$$\begin{aligned} \partial_t\Big(\rho u(1 - \nu^2\varepsilon^{2\alpha}\tau) + \frac{\varepsilon^2\nu^2}{4}\rho\nabla\partial_t \log\rho\Big) + \nabla \cdot \Big(\rho u \otimes u\Big) + \frac{1}{\varepsilon^{2\alpha}}\nabla P(\rho) \\ = \frac{\varepsilon^{2-2\alpha}}{4}\nabla \cdot \Big(\rho\nabla^2 \log\rho\Big), \end{aligned} \tag{2.23}$$

and the associated energy equation becomes

$$\partial_t(E_S + E_K) + \nabla \cdot \left(\Big(E_S + \frac{P(\rho)}{\varepsilon^{2\alpha}}\Big)u\right) = \frac{\varepsilon^{2-2\alpha}}{4}\nabla \cdot \left[(\rho u)\frac{\Delta\rho}{\rho} - \nabla \cdot (\rho u)\frac{\nabla\rho}{\rho}\right]. \tag{2.24}$$

where the Schrödinger part energy density E_S and relativistic part energy density E_K are given respectively by

$$E_S = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{1}{\varepsilon^{2\alpha}}V(\rho)\,, \tag{2.25}$$

$$E_K = \frac{\varepsilon^{2\alpha}\nu^2}{2}\rho|\tau|^2 + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho}\,. \tag{2.26}$$

It follows immediately from the energy equation that

$$\int \frac{\varepsilon^{2\alpha}\nu^2}{2}\rho|\tau|^2 + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho} + \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{V(\rho)}{\varepsilon^{2\alpha}}dx \le C \tag{2.27}$$

for all $0 < t < \infty$ if the initial energy is bounded. Assuming the minimum of the convex function $V(\rho)$ occurs at $\rho = 1$ then the energy bound (2.27) implies $\rho \to 1$ as $\varepsilon \to 0$. Formally the density ρ goes to 1, thus we expect that the equation (2.22) yields the limit: $\nabla \cdot u = 0$. Writing $\nabla P(\rho) = \nabla(P(\rho) - P(1))$, we deduce from (2.23) that

$$\partial_t u + \nabla\cdot(u \otimes u) + \nabla\widetilde{P} = 0\,, \tag{2.28}$$

where $\widetilde{P}$ is the limit of $\frac{P(\rho)-P(1)}{\varepsilon^{2\alpha}}$. In other words, we recover the incompressible Euler equations. The reader is referred to [15] for the detail discussion of the incompressible Euler equations.

3 Hydrodynamic Limits

We apply the modulated energy method to study the hydrodynamic limits, i.e., the compressible and incompressible Euler limits of the modulated Klein-Gordon equations. The modulated energy method was introduced by Brenier [1] to prove the convergence of the Vlasov-Poisson system to the incompressible Euler equations. It was immediately extended by Masmoudi in [19] to general initial data allowing the presence of high oscillations in time (see also [9] for the quantum hydrodynamic model of semiconductor). The same idea is also applied to study various singular limits of other equations, for example the Schrödinger-Poisson equation [23], the Gross-Pitaevskii equation [13] and the coupled nonlinear Schrödinger equation [5, 14].

3.1 Compressible Euler Limit

The result we shall prove rigourously in this section is the convergence towards the compressible Euler equations. In fact, we consider the so called nonrelativistic-semiclassical limit, i.e. $\nu \to 0$ and $\varepsilon \to 0$ simultaneously. In order to avoid carrying out a double limits, the parameters ν and ε must be related. For convenience we set $\nu = \varepsilon^\kappa$ for some $\kappa > 0$, $0 < \varepsilon \ll 1$ and assume the potential energy $V'(|\psi^\varepsilon|^2) = |\psi^\varepsilon|^{2(\gamma-1)}$. Indeed we consider the modulated nonlinear Klein-Gordon equation

$$i\varepsilon\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{2+2\kappa}\partial_t^2\psi^\varepsilon + \frac{1}{2}\varepsilon^2\Delta\psi^\varepsilon - |\psi^\varepsilon|^{2(\gamma-1)}\psi^\varepsilon = 0\,, \tag{3.1}$$

supplemented with the initial conditions:

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \Omega\,. \tag{3.2}$$

To avoid the complications at the boundary, we concentrate below on the case where $x \in \Omega = \mathbb{T}^n$, the n-dimensional torus.

Associated with (3.1) are the local conservation laws corresponding to charge, momentum(current) and energy conservation. In fact, we have the hydrodynamic variables: Schrödinger part charge ρ_S^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J_S^ε, relativistic part momentum (current) J_K^ε and energy e^ε given as follows:

$$
\begin{aligned}
&\rho_S^\varepsilon = |\psi^\varepsilon|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\varepsilon^{1+2\kappa}\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big),\\
&J_S^\varepsilon = (J_{S,1}^\varepsilon, J_{S,2}^\varepsilon, ..., J_{S,n}^\varepsilon) = \frac{i}{2}\varepsilon\Big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big),\\
&J_K^\varepsilon = (J_{K,1}^\varepsilon, J_{K,2}^\varepsilon, ..., J_{K,n}^\varepsilon) = \frac{1}{2}\varepsilon^{2+2\kappa}\Big(\partial_t\psi^\varepsilon\nabla\overline{\psi^\varepsilon} + \partial_t\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big),\\
&e^\varepsilon = \frac{1}{2}\varepsilon^{2+2\kappa}|\partial_t\psi^\varepsilon|^2 + \frac{1}{2}\varepsilon^2|\nabla\psi^\varepsilon|^2 + \frac{1}{\gamma}|\psi^\varepsilon|^{2\gamma}.
\end{aligned} \tag{3.3}
$$

The local conservation laws of the modulated Klein-Gordon equation (3.1) are the charge, momentum(current) and energy given below:

(A) Conservation of charge

$$
\frac{\partial}{\partial t}\big(\rho_S^\varepsilon - \rho_K^\varepsilon\big) + \nabla\cdot J_S^\varepsilon = 0\,, \tag{3.4}
$$

(B) Conservation of momentum (current)

$$
\begin{aligned}
&\frac{\partial}{\partial t}\big(J_S^\varepsilon - J_K^\varepsilon\big) + \frac{1}{4}\varepsilon^2\nabla\cdot\Big[2(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon) - \nabla^2(|\psi^\varepsilon|^2)\Big]\\
&+\frac{1}{4}\varepsilon^{2+2\kappa}\nabla\partial_t\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big) + \frac{\gamma-1}{\gamma}\nabla|\psi^\varepsilon|^{2\gamma} = 0\,,
\end{aligned} \tag{3.5}
$$

(C) Conservation of energy

$$
\frac{\partial}{\partial t}e^\varepsilon - \nabla\cdot\Big[\frac{1}{2}\varepsilon^2(\nabla\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon)\Big] = 0\,. \tag{3.6}
$$

They play the crucial role of the hydrodynamic limits. Moreover, we need assume finite initial energy

$$
\int_{\mathbb{T}^n}\frac{1}{2}\varepsilon^{2+2\kappa}|\psi_1^\varepsilon|^2 + \frac{1}{2}\varepsilon^2|\nabla\psi_0^\varepsilon|^2 + \frac{1}{\gamma}|\psi_0^\varepsilon|^{2\gamma}dx \le C\,. \tag{3.7}
$$

The limit equation is the γ-law compressible Euler equations

$$
\begin{cases}
\partial_t\rho + \nabla\cdot(\rho u) = 0\,, \qquad x\in\mathbb{T}^n, \quad t\in[0,T]\,,\\
\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla P(\rho) = 0\,,\\
\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x)\,, \qquad x\in\mathbb{T}^n\,.
\end{cases} \tag{3.8}
$$

where $0 < \rho_0 \in H^s(\mathbb{T}^n)$, $u_0 \in H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 1$, and the equation of states is given by $P(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$.

Motivated by Brenier's pioneer work [1], our result based on the modulated energy. It is easy to see that when the parameter ε is small; the wave function ψ^ε and hydrodynamic variables ρ, u are related according to

$$|\psi^\varepsilon|^2 = \rho_S^\varepsilon \approx \rho\,, \qquad \frac{i\varepsilon}{2}\frac{1}{|\psi^\varepsilon|^2}(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon) \approx u\,.$$

The symbol "$A \approx B$" means that A almost equals B. Moreover, as ε tends to zero, the limiting energy will be $\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma}\rho^\gamma$. Keeping this term in mind and comparing with the energy of the modulated nonlinear Klein-Gordon equation (3.1), we have:

$$\frac{1}{2}\varepsilon^2|\nabla\psi^\varepsilon|^2 \approx \frac{1}{2}\rho|u|^2\,, \quad \frac{1}{2}\varepsilon^{2+2\kappa}|\partial_t\psi^\varepsilon|^2 \approx 0\,, \quad \frac{1}{\gamma}(\rho_S^\varepsilon)^\gamma \approx \frac{1}{\gamma}\rho^\gamma + \rho^{\gamma-1}(\rho_S^\varepsilon - \rho)\,.$$

Thus, we have the relation

$$\begin{aligned}
&\frac{1}{2}\varepsilon^2|\nabla\psi^\varepsilon|^2 - \frac{1}{2}\rho|u|^2 \\
&\approx \frac{\varepsilon^2}{2}\Big(|\nabla\psi^\varepsilon|^2 - 2\varepsilon^{-2}|\psi^\varepsilon|^2|u|^2 + \varepsilon^{-2}|\psi^\varepsilon|^2|u|^2\Big) \\
&\approx \frac{\varepsilon^2}{2}\Big(|\nabla\psi^\varepsilon|^2 - i\varepsilon^{-1}(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon)\cdot u + \varepsilon^{-2}|\psi^\varepsilon|^2|u|^2\Big) \\
&= \frac{\varepsilon^2}{2}|(\nabla - i\varepsilon^{-1}u)\psi^\varepsilon|^2\,.
\end{aligned} \tag{3.9}$$

Therefore we can define the modulated energy of (3.1) as

$$\begin{aligned}
H^\varepsilon(t) = \frac{\varepsilon^2}{2}&\int_{\mathbb{T}^n} |(\nabla - i\varepsilon^{-1}u)\psi^\varepsilon|^2 dx \\
&+\frac{1}{2}\int_{\mathbb{T}^n} |\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon|^2 dx + \int_{\mathbb{T}^n} \Theta(\rho_S^\varepsilon, \rho)dx
\end{aligned} \tag{3.10}$$

where

$$\Theta(\rho_S^\varepsilon, \rho) = \frac{1}{\gamma}\Big((\rho_S^\varepsilon)^\gamma - \rho^\gamma\Big) - \rho^{\gamma-1}\big(\rho_S^\varepsilon - \rho\big) \tag{3.11}$$

is a convex function, minimum occurs at $\rho_S^\varepsilon = \rho$ and satisfies $\Theta(\rho_S^\varepsilon, \rho) \geq 0$. We also assume

$$\begin{aligned} H^\varepsilon(0) = \frac{\varepsilon^2}{2} \int_{\mathbb{T}^n} |(\nabla - i\varepsilon^{-1} u_0)\psi_0^\varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{T}^n} |\varepsilon^{1+\kappa} \psi_1^\varepsilon|^2 dx \\ + \int_{\mathbb{T}^n} \Theta(|\psi_0^\varepsilon|^2, \rho_0) dx = O(\varepsilon^\beta)\,, \quad \text{for some} \quad \beta > 0\,, \end{aligned} \tag{3.12}$$

i.e., we consider the well-prepared initial data. We can rewrite the modulated energy (3.10) in terms of hydrodynamic variables only as

$$\begin{aligned} H^\varepsilon(t) = \int_{\mathbb{T}^n} e^\varepsilon dx - \int_{\mathbb{T}^n} u \cdot J_S^\varepsilon \, dx + \frac{1}{2} \int_{\mathbb{T}^n} \rho_S^\varepsilon |u|^2 dx \\ + \int_{\mathbb{T}^n} \Big(\frac{\gamma - 1}{\gamma} \rho - \rho_S^\varepsilon \Big) \rho^{\gamma-1} dx\,. \end{aligned} \tag{3.13}$$

Therefore to obtain the hydrodynamic limit, we have to show that the modulated energy $H^\varepsilon(t)$ tends to zero as $\varepsilon \to 0$. Indeed, we have the following theorem [12].

Theorem 3.1 *Let $\gamma \geq 2$, $s > \frac{n}{2} + 1$, and ψ^ε be the solution of the modulated nonlinear Klein-Gordon equation* (3.1)–(3.2) *with initial condition $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$ satisfying* (3.7). *Let $(\rho, u) \in C\big([0,T]; H^s(\mathbb{T}^n)\big)$ be the unique local smooth solution of the γ-law compressible Euler equations* (3.8). *If we assume the well-prepared initial condition* (3.12), *and let $\lambda = \min\{1, \kappa, \beta\}$, then there exist $T_* > 0$ such that*

$$H^\varepsilon(t) \leq O(\varepsilon^\lambda) \quad \textit{uniformly in } t \in [0, T_*]\,.$$

Moreover, we have

$$\|(\rho_S^\varepsilon - \rho)(\cdot, t)\|_{L^\gamma(\mathbb{T}^n)} \to 0\,, \qquad \|\rho_K^\varepsilon(\cdot, t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \tag{3.14}$$

$$\|(J_S^\varepsilon - \rho u)(\cdot, t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \qquad \|J_K^\varepsilon(\cdot, t)\|_{L^1(\mathbb{T}^n)} \to 0\,, \tag{3.15}$$

for $t \in [0, T_*)$ as $\varepsilon \downarrow 0$.

Proof. We have to check the evolution of the modulated energy $H^\varepsilon(t)$ given by (3.13). Differentiating the modulate energy H^ε with respect to time variable t and using the conservation of energy (3.6), we obtain

$$\begin{aligned} \frac{d}{dt}H^\varepsilon(t) &= -\frac{d}{dt}\int_{\mathbb{T}^n} u\cdot J_S^\varepsilon\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}\rho_S^\varepsilon|u|^2dx \\ &+ \frac{d}{dt}\int_{\mathbb{T}^n}\Big(\frac{\gamma-1}{\gamma}\rho-\rho_S^\varepsilon\Big)\rho^{\gamma-1}dx\,. \end{aligned} \tag{3.16}$$

We discuss the right hand side of (3.16) separately. Integration by part and using conservation of momentum (3.5), the first term of the right hand side of (3.16) becomes

$$\begin{aligned} &-\frac{d}{dt}\int_{\mathbb{T}^n} u\cdot J_S^\varepsilon dx = -\int_{\mathbb{T}^n}\partial_t u\cdot J_S^\varepsilon dx - \int_{\mathbb{T}^n}\frac{\gamma-1}{\gamma}(\rho_S^\varepsilon)^\gamma\nabla\cdot u dx \\ &-\frac{\varepsilon^2}{4}\int_{\mathbb{T}^n}2(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon}+\nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon):\nabla u+\nabla|\psi^\varepsilon|^2\cdot(\nabla\nabla\cdot u)dx \\ &-\frac{1}{4}\varepsilon^{2+2\kappa}\frac{d}{dt}\int_{\mathbb{T}^n}(\partial_t|\psi^\varepsilon|^2)\nabla\cdot udx - \frac{d}{dt}\int_{\mathbb{T}^n}u\cdot J_K^\varepsilon dx \\ &+\frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}(\partial_t|\psi^\varepsilon|^2)\nabla\cdot\partial_t udx + \int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon dx\,. \end{aligned} \tag{3.17}$$

Next, by conservation of charge (3.4) and integration by part, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}(\rho_S^\varepsilon-\rho_K^\varepsilon)|u|^2dx \\ &= \int_{\mathbb{T}^n}\rho_S^\varepsilon u\cdot\partial_t udx + \frac{1}{2}\int_{\mathbb{T}^n}\nabla|u|^2\cdot J_S^\varepsilon dx - \int_{\mathbb{T}^n}\rho_K^\varepsilon u\cdot\partial_t udx\,. \end{aligned} \tag{3.18}$$

The third term of the right hand side of (3.16) becomes

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{\gamma-1}{\gamma}\rho-\rho_S^\varepsilon\right)\rho^{\gamma-1}dx \\
&=\int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-2}\big(\rho-\rho_S^\varepsilon\big)\partial_t\rho dx-\frac{d}{dt}\int_{\mathbb{T}^n}\rho^{\gamma-1}\rho_K^\varepsilon dx \\
&\quad+\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^\varepsilon-\nabla\rho^{\gamma-1}\cdot J_S^\varepsilon dx\,.
\end{aligned} \tag{3.19}$$

From (3.18)–(3.19) we define the correction term of the modulated energy H^ε as

$$\begin{aligned}
G^\varepsilon(t)=&-\frac{1}{2}\int_{\mathbb{T}^n}|u|^2\rho_K^\varepsilon dx+\int_{\mathbb{T}^n}\rho^{\gamma-1}\rho_K^\varepsilon dx \\
&+\frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}(\partial_t|\psi^\varepsilon|^2)\nabla\cdot udx+\int_{\mathbb{T}^n}u\cdot J_K^\varepsilon dx\,.
\end{aligned} \tag{3.20}$$

It is designed to control the propagation of the relativistic charge and current and will be proved to be small as $\varepsilon\to 0$. Using crucially the limit compressible Euler equations (3.8), we have

$$\begin{aligned}
&\frac{d}{dt}(H^\varepsilon(t)+G^\varepsilon(t)) \\
&=-\frac{\varepsilon^2}{4}\int_{\mathbb{T}^n}2(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon}+\nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon):\nabla u+\nabla|\psi^\varepsilon|^2\cdot(\nabla\nabla\cdot u)dx \\
&+\int_{\mathbb{T}^n}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\cdot\big(u\cdot\nabla u+\nabla\rho^{\gamma-1}\big)dx+\frac{1}{2}\int_{\mathbb{T}^n}J_S^\varepsilon\cdot\nabla|u|^2dx \\
&+\int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-2}\big(\rho_S^\varepsilon-\rho\big)\nabla\cdot\big(\rho u\big)-J_S^\varepsilon\cdot\nabla\rho^{\gamma-1}dx \\
&-\int_{\mathbb{T}^n}\frac{\gamma-1}{\gamma}(\rho_S^\varepsilon)^\gamma\nabla\cdot udx-\int_{\mathbb{T}^n}u\cdot\partial_t u\rho_K^\varepsilon dx+\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^\varepsilon dx \\
&+\frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}(\partial_t|\psi^\varepsilon|^2)\nabla\cdot\partial_t udx+\int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon dx\,.
\end{aligned} \tag{3.21}$$

To deal with the first integral of the right hand side of (3.21), we need the following equality:

$$
\begin{aligned}
&-\mathcal{R}e\Big[\nabla u : \big(\nabla - i\varepsilon^{-1}u\big)\psi^\varepsilon \otimes \overline{\big(\nabla - i\varepsilon^{-1}u\big)\psi^\varepsilon}\Big] \\
&= -\mathcal{R}e \sum_{j,\ell=1}^{n} \big(\partial_\ell u_j\big)\Big[\big(\partial_j - i\varepsilon^{-1}u_j\big)\psi^\varepsilon\overline{\big(\partial_\ell - i\varepsilon^{-1}u_\ell\big)\psi^\varepsilon}\Big] \\
&= -\frac{1}{2}(\nabla\psi^\varepsilon \otimes \nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon} \otimes \nabla\psi^\varepsilon) : \nabla u \\
&\quad +\varepsilon^{-2}\Big\{\big(J_S^\varepsilon - \rho_S^\varepsilon u\big)\cdot\big[(u\cdot\nabla)u\big] + \frac{1}{2}J_S^\varepsilon\cdot\nabla|u|^2\Big\},
\end{aligned}
\tag{3.22}
$$

where $\mathcal{R}e\,(z)$ denotes the real part of the complex number z. The proof of equality (3.22) is simple but lengthy calculation. Therefore the detail is omitted. We deduce from (3.21) and (3.22) that

$$
\begin{aligned}
&\frac{d}{dt}\big(H^\varepsilon(t) + G^\varepsilon(t)\big) \\
&= -\varepsilon^2\int_{\mathbb{T}^n} \mathcal{R}e\Big[\nabla u : \big(\nabla - i\varepsilon^{-1}u\big)\psi^\varepsilon \otimes \overline{\big(\nabla - i\varepsilon^{-1}u\big)\psi^\varepsilon}\Big]dx \\
&-\frac{\varepsilon^2}{4}\int_{\mathbb{T}^n} \nabla|\psi^\varepsilon|^2\cdot(\nabla\nabla\cdot u)dx - \int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)dx \\
&-\int_{\mathbb{T}^n}\Big[\frac{\gamma-1}{\gamma}(\rho_S^\varepsilon)^\gamma - (\gamma-1)\rho^{\gamma-1}\rho_S^\varepsilon\Big]\nabla\cdot u dx - \int_{\mathbb{T}^n} u\cdot\partial_t u\rho_K^\varepsilon dx \\
&+\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^\varepsilon dx + \frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}(\partial_t|\psi^\varepsilon|^2)\nabla\cdot\partial_t u dx + \int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon dx\,.
\end{aligned}
\tag{3.23}
$$

Also using the identity

$$
(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u) = \frac{\gamma-1}{\gamma}(\nabla\rho^\gamma)\cdot u + (\gamma-1)\rho^\gamma\nabla\cdot u\,,
$$

we have

$$
-\int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)dx = -\frac{(\gamma-1)^2}{\gamma}\int_{\mathbb{T}^n}\rho^\gamma\nabla\cdot u dx\,. \tag{3.24}
$$

Employing (3.24), we can rewrite (3.23) as

$$
\begin{aligned}
&\frac{d}{dt}\big(H^\varepsilon(t)+G^\varepsilon(t)\big)\\
&=-\varepsilon^2\int_{\mathbb{T}^n}\mathcal{R}e\Big[\nabla u:\big(\nabla-i\varepsilon^{-1}u\big)\psi^\varepsilon\otimes\overline{\big(\nabla-i\varepsilon^{-1}u\big)\psi^\varepsilon}\Big]dx\\
&-(\gamma-1)\int_{\mathbb{T}^n}\Big[\frac{1}{\gamma}\big((\rho_S^\varepsilon)^\gamma-\rho^\gamma\big)-\rho^{\gamma-1}\big(\rho_S^\varepsilon-\rho\big)\Big]\nabla\cdot u dx \qquad (3.25)\\
&-\frac{\varepsilon^2}{4}\int_{\mathbb{T}^n}\nabla|\psi^\varepsilon|^2\cdot(\nabla\nabla\cdot u)dx-\int_{\mathbb{T}^n}u\cdot\partial_t u\rho_K^\varepsilon dx+\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^\varepsilon dx\\
&+\frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}(\partial_t|\psi^\varepsilon|^2)\nabla\cdot\partial_t u dx+\int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon dx.
\end{aligned}
$$

One can estimate the first term of the right hand side of (3.25) as follows

$$
\begin{aligned}
&\big|\nabla u:\big(\nabla-i\varepsilon^{-1}u\big)\psi^\varepsilon\otimes\overline{\big(\nabla-i\varepsilon^{-1}u\big)\psi^\varepsilon}\big|\\
&\le\|\nabla u\|_{L^\infty(\mathbb{T}^n)}\sum_{j,\ell=1}^n\big|(\partial_j-i\varepsilon^{-1}u_j)\psi^\varepsilon\overline{(\partial_\ell-i\varepsilon^{-1}u_\ell)\psi^\varepsilon}\big| \qquad (3.26)\\
&\le n\|\nabla u\|_{L^\infty(\mathbb{T}^n)}|(\nabla-i\varepsilon^{-1}u)\psi^\varepsilon|^2\,.
\end{aligned}
$$

Furthermore, for $t\in[0,T_*)$, by (3.7), (3.4) and (3.6) we have

$$\|\varepsilon\nabla\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}=\|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}=O(1) \qquad (3.27)$$

and

$$\|\psi^\varepsilon\|_{L^q(\mathbb{T}^n)}=O(1)\,,\qquad 2\leqslant q\leqslant 2\gamma\,. \qquad (3.28)$$

Then by Hölder inequality we have the following estimates

$$
\begin{aligned}
&\varepsilon^2\int_{\mathbb{T}^n}\nabla|\psi^\varepsilon|^2\cdot(\nabla\nabla\cdot u)dx \qquad (3.29)\\
&\le\varepsilon\|\varepsilon\nabla\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}\|\psi^\varepsilon\|_{L^{2\gamma}(\mathbb{T}^n)}\|\nabla\nabla\cdot u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)}\lesssim\varepsilon\|u\|_{H^s(\mathbb{T}^n)}\,,
\end{aligned}
$$

and

$$\begin{aligned}\int_{\mathbb{T}^n} \rho_K^\varepsilon u \cdot \partial_t u dx &\leq \varepsilon^\kappa \|u \cdot \partial_t u\|_{L^\infty(\mathbb{T}^n)} \|\psi^\varepsilon\|_{L^2(\mathbb{T}^n)} \|\varepsilon^{1+\kappa} \partial_t \psi^\varepsilon\|_{L^2(\mathbb{T}^n)} \\ &\lesssim \varepsilon^\kappa \|u \cdot \partial_t u\|_{L^\infty(\mathbb{T}^n)} \lesssim \varepsilon^\kappa \|u\|^2_{H^s(\mathbb{T}^n)} .\end{aligned} \tag{3.30}$$

Similar to (3.29)–(3.30), we also have

$$\int_{\mathbb{T}^n} \partial_t \rho^{\gamma-1} \rho_K^\varepsilon dx \lesssim \varepsilon^\kappa \|\rho\|^{\gamma-1}_{H^s(\mathbb{T}^n)} , \tag{3.31}$$

$$\varepsilon^{2+2\kappa} \int_{\mathbb{T}^n} (\partial_t |\psi^\varepsilon|^2) \nabla \cdot \partial_t u dx \lesssim \varepsilon^{1+\kappa} \|u\|_{H^s(\mathbb{T}^n)} , \tag{3.32}$$

$$\int_{\mathbb{T}^n} \partial_t u \cdot J_K^\varepsilon dx \lesssim \varepsilon^\kappa \|u\|_{H^s(\mathbb{T}^n)} . \tag{3.33}$$

Combing the above estimates we obtain the inequality

$$\begin{aligned}\frac{d}{dt}\big(H^\varepsilon(t) + G^\varepsilon(t)\big) &\lesssim \|\nabla u\|_{L^\infty(\mathbb{T}^n)} H^\varepsilon(t) \\ &+\varepsilon^\delta \Big(\|u\|_{H^s(\mathbb{T}^n)} + \|u\|^2_{H^s(\mathbb{T}^n)} + \|\rho\|^{\gamma-1}_{H^s(\mathbb{T}^n)} \Big)\end{aligned} \tag{3.34}$$

for $t \in [0, T_*)$ and $\delta = \min\{1, \kappa\}$. Integrating (3.34) with respect to time variable t yields

$$H^\varepsilon(t) \leq H^\varepsilon(0) + G^\varepsilon(0) - G^\varepsilon(t) + C_1 \int_0^t H^\varepsilon(\tau) d\tau + C_2 \varepsilon^\delta t. \tag{3.35}$$

Similar to (3.29)–(3.33) one can show that $G^\varepsilon(0) - G^\varepsilon(t) = O(\varepsilon^\kappa)$; and hence

$$H^\varepsilon(t) \leq C_1 \int_0^t H^\varepsilon(\tau) d\tau + H^\varepsilon(0) + C_2 \varepsilon^\delta t + C_3 \varepsilon^\kappa . \tag{3.36}$$

Employing the initial condition $H^\varepsilon(0)$ and the Gronwall inequality we derive

$$H^\varepsilon(t) \leq (C_4 \varepsilon^\beta + C_2 \varepsilon^\delta t + C_3 \varepsilon^\kappa)(1 + C_1 t e^{C_1 t}) . \tag{3.37}$$

This shows $H^\varepsilon(t) \leq O(\varepsilon^\lambda)$ for $t \in [0, T_*)$, where $\lambda = \min\{1, \kappa, \beta\}$.

It is easy to check that the modulated energy can be rewritten as

$$\begin{aligned} H^\varepsilon(t) = \frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} |\nabla\sqrt{\rho_S^\varepsilon}|^2 dx + \frac{1}{2}\int_{\mathbb{T}^n} \Big|\frac{1}{\sqrt{\rho_S^\varepsilon}}(J_S^\varepsilon - \rho_S^\varepsilon u)\Big|^2 dx \\ + \frac{1}{2}\int_{\mathbb{T}^n} |\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon|^2 dx + \int_{\mathbb{T}^n} \Theta(\rho_S^\varepsilon,\rho)\, dx\,. \end{aligned} \tag{3.38}$$

Using (3.38), we have

$$\int_{\mathbb{T}^n} \Big|\frac{1}{\sqrt{\rho_S^\varepsilon}}(J_S^\varepsilon - \rho_S^\varepsilon u)\Big|^2 dx \to 0\,, \qquad \int_{\mathbb{T}^n} \Theta(\rho_S^\varepsilon,\rho) dx \to 0 \tag{3.39}$$

as $\varepsilon \to 0$. Also the elementary computation shows that ([16])

$$\frac{1}{\gamma}\big|\rho_S^\varepsilon - \rho\big|^\gamma \le \Theta(\rho_S^\varepsilon,\rho) \tag{3.40}$$

and hence $\|\rho_S^\varepsilon - \rho\|_{L^\gamma(\mathbb{T}^n)} \to 0$ as $\varepsilon \to 0$. On the other hand, applying the triangle and Hölder inequalities we have

$$\begin{aligned} &\Big\|(J_S^\varepsilon - \rho u)\Big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \\ &\le \|(J_S^\varepsilon - \rho_S^\varepsilon u)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} + \|(\rho_S^\varepsilon - \rho)u\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \\ &\le \|\sqrt{\rho_S^\varepsilon}\|_{L^{2\gamma}(\mathbb{T}^n)} \Big\|\frac{1}{\sqrt{\rho_S^\varepsilon}}(J_S^\varepsilon - \rho_S^\varepsilon u)\Big\|_{L^2(\mathbb{T}^n)} \\ &\quad + \|\rho_S^\varepsilon - \rho\|_{L^\gamma(\mathbb{T}^n)} \|u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)} \end{aligned} \tag{3.41}$$

which converges to zero as $\varepsilon \to 0$ by (3.39)–(3.40). Combing (3.27) and (3.28) we have

$$\|\rho_K^\varepsilon(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \lesssim \varepsilon^\kappa \|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\|_{L^2(\mathbb{T}^n)} \|\psi^\varepsilon\|_{L^{2\gamma}(\mathbb{T}^n)} \to 0 \tag{3.42}$$

and

$$\|J_K^\varepsilon(\cdot,t)\|_{L^1(\mathbb{T}^n)} \lesssim \varepsilon^\kappa \|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\|_{L^2(\mathbb{T}^n)} \|\varepsilon\nabla\psi^\varepsilon\|_{L^2(\mathbb{T}^n)} \to 0 \tag{3.43}$$

as $\varepsilon \to 0$. This completes the proof of Theorem 3.1. ■

3.2 Incompressible Euler Limit

The second result we want to address in this chapter concerns the convergence towards the incompressible Euler equations. We still consider only the n-dimensional torus $\mathbb{T}^n$ as discussed in the previous section. To obtain the incompressible limit, the time variable need to be rescaled, $t \to \varepsilon^\alpha t, \alpha > 0$ and potential energy is given by $V'(|\psi^\varepsilon|^2) = |\psi^\varepsilon|^{2(\gamma-1)} - 1, \gamma \geq 2$. More precisely, we will investigate the time-scaled modulated nonlinear Klein-Gordon equation

$$i\varepsilon^{1+\alpha}\partial_t\psi^\varepsilon - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - (|\psi^\varepsilon|^{2(\gamma-1)} - 1)\psi^\varepsilon = 0\,, \tag{3.44}$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \mathbb{T}^n\,. \tag{3.45}$$

We will consider the limit as the scaled Planck constant $\varepsilon \to 0$ and the parameter ν is kept fixed. To prove the incompressible limit of (3.44) we have to define the hydrodynamic variables; Schrödinger part charge ρ_S^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J_S^ε, relativistic part momentum J_K^ε and energy e^ε as follows:

$$\begin{aligned}
&\rho_S^\varepsilon = |\psi^\varepsilon|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\nu^2\varepsilon^{1+\alpha}\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big)\,,\\
&J_S^\varepsilon = (J_{S,1}^\varepsilon, J_{S,2}^\varepsilon, ..., J_{S,n}^\varepsilon) = \frac{i}{2}\varepsilon^{1-\alpha}\Big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big)\,,\\
&J_K^\varepsilon = (J_{K,1}^\varepsilon, J_{K,2}^\varepsilon, ..., J_{K,n}^\varepsilon) = \frac{\nu^2\varepsilon^2}{2}\Big(\partial_t\psi^\varepsilon\nabla\overline{\psi^\varepsilon} + \partial_t\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big)\,,\\
&e^\varepsilon = \frac{1}{2}\nu^2\varepsilon^2|\partial_t\psi^\varepsilon|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}|\nabla\psi^\varepsilon|^2 + \frac{1}{\varepsilon^{2\alpha}}\Theta(\rho_S^\varepsilon, 1)\,,
\end{aligned} \tag{3.46}$$

where

$$\Theta(\rho_S^\varepsilon, 1) = \frac{1}{\gamma}\big((\rho_S^\varepsilon)^\gamma - 1\big) - \big(\rho_S^\varepsilon - 1\big)\,. \tag{3.47}$$

The local conservation laws associated with the rescaled modulated nonlinear Klein-Gordon equation (3.44) are the charge, momentum and energy given respectively by:

(A) Conservation of charge

$$\frac{\partial}{\partial t}\big(\rho_S^\varepsilon - \rho_K^\varepsilon\big) + \nabla\cdot J_S^\varepsilon = 0\,, \tag{3.48}$$

(B) Conservation of momentum

$$\begin{aligned}&\frac{\partial}{\partial t}\big(J_S^\varepsilon - J_K^\varepsilon\big) + \frac{1}{4}\varepsilon^{2-2\alpha}\nabla\cdot\Big[2(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon) - \nabla^2(|\psi^\varepsilon|^2)\Big]\\&+\frac{1}{4}\nu^2\varepsilon^2\nabla\partial_t\Big(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\Big) + \frac{1}{\varepsilon^{2\alpha}}\frac{\gamma-1}{\gamma}\nabla|\psi^\varepsilon|^{2\gamma} = 0\,,\end{aligned} \tag{3.49}$$

(C) Conservation of energy

$$\frac{\partial}{\partial t}e^\varepsilon - \nabla\cdot\Big[\frac{1}{2}\varepsilon^{2-2\alpha}(\nabla\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon)\Big] = 0\,. \tag{3.50}$$

Moreover, we need assume finite initial energy

$$\int_{\mathbb{T}^n}\frac{1}{2}\nu^2\varepsilon^2|\psi_1^\varepsilon|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}|\nabla\psi_0^\varepsilon|^2 + \frac{1}{\varepsilon^{2\alpha}}\Theta(|\psi_0^\varepsilon|^2,1)dx < C\,. \tag{3.51}$$

The limit equation is the incompressible Euler equations

$$\begin{cases}\partial_t u + (u\cdot\nabla)u + \nabla\pi = 0\,, & \nabla\cdot u = 0\,,\\ u(x,0) = u_0(x)\,, & \nabla\cdot u_0 = 0\,.\end{cases} \tag{3.52}$$

with initial condition $u_0 \in H^s(\mathbb{T}^n)$, $s > 1 + \frac{n}{2}$.

Similar to the previous section, we define the modulated energy

$$\begin{aligned}H^\varepsilon(t) =& \frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\big|(\nabla - i\varepsilon^{\alpha-1}u)\psi^\varepsilon\big|^2dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}|\partial_t\psi^\varepsilon|^2dx\\&+\frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\Theta(\rho_S^\varepsilon,1)dx\,,\end{aligned} \tag{3.53}$$

which satisfies the well-prepared initial condition

$$\begin{aligned}H^\varepsilon(0) =& \frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}|(\nabla - i\varepsilon^{\alpha-1}u_0)\psi_0^\varepsilon|^2dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}|\psi_1^\varepsilon|^2dx\\&+\frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\Theta(|\psi_0^\varepsilon|^2,1)dx = O(\varepsilon^\beta)\,,\end{aligned} \tag{3.54}$$

for some $\beta > 0$. The modulated energy can be further rewritten in terms of the hydrodynamic variables as

$$H^\varepsilon(t) = \int_{\mathbb{T}^n} e^\varepsilon dx - \int_{\mathbb{T}^n} u \cdot J_S^\varepsilon dx + \frac{1}{2}\int_{\mathbb{T}^n} \rho_S^\varepsilon |u|^2 dx\,. \tag{3.55}$$

Therefore to obtain the hydrodynamic limit, we have to show that the modulated energy $H^\varepsilon(t)$ tends to zero as $\varepsilon \to 0$. Indeed, we have the following theorem [12].

Theorem 3.2 *Let $\alpha > 0$, $\gamma \geq 2$, $s > \frac{n}{2} + 1$, and ψ^ε be the solution of the time scale modulated nonlinear Klein-Gordon equation* (3.44)–(3.45) *with initial condition $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$ satisfying* (3.51)*. Let $u \in C\big([0,T]; H^s(\mathbb{T}^n)\big)$ be the unique local smooth solution of the incompressible Euler equations* (3.52)*. If we assume the well-prepared initial condition* (3.54)*, and let $\lambda = \min\{\beta, \delta\}$, where $\delta = 2\alpha/\gamma$, then there exist $T_* > 0$ such that,*

$$H^\varepsilon(t) \leq O(\varepsilon^\lambda)\,, \qquad t \in [0, T_*]\,.$$

Moreover, we have

$$\|(\rho_S^\varepsilon - 1)(\cdot,t)\|_{L^\gamma(\mathbb{T}^n)} \to 0\,, \qquad \|\rho_K^\varepsilon(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \tag{3.56}$$

$$\|(J_S^\varepsilon - \rho_S^\varepsilon u)(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0\,, \qquad \|J_K^\varepsilon(\cdot,t)\|_{L^1(\mathbb{T}^n)} \to 0\,, \tag{3.57}$$

for $t \in [0, T_*)$ as $\varepsilon \downarrow 0$.

Proof. Differentiating the modulated energy (3.55) with respect to t and using conservation of energy (3.50), we obtain

$$\frac{d}{dt}H^\varepsilon(t) = -\frac{d}{dt}\int_{\mathbb{T}^n} u \cdot J_S^\varepsilon dx + \frac{d}{dt}\int_{\mathbb{T}^n} \frac{1}{2}\rho_S^\varepsilon |u|^2 dx \equiv I_1 + I_2\,. \tag{3.58}$$

By conservation of momentum (3.49), integration by part and using the fact that u is divergence free, we obtain

$$\begin{aligned} I_1 = &-\int_{\mathbb{T}^n} \partial_t u \cdot (J_S^\varepsilon - J_K^\varepsilon) dx - \frac{d}{dt}\int_{\mathbb{T}^n} u \cdot J_K^\varepsilon dx \\ &-\frac{\varepsilon^{2-2\alpha}}{4}\int_{\mathbb{T}^n} 2(\nabla\psi^\varepsilon \otimes \nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon} \otimes \nabla\psi^\varepsilon) : \nabla u dx\,. \end{aligned} \tag{3.59}$$

Next employing conservation of charge (3.48) and integration by part, we have

$$
\begin{aligned}
I_2 &= \int_{\mathbb{T}^n} \rho_S^\varepsilon u \cdot \partial_t u dx + \int_{\mathbb{T}^n} \frac{1}{2}\nabla|u|^2 \cdot J_S^\varepsilon dx \\
&+ \frac{d}{dt}\int_{\mathbb{T}^n} \frac{1}{2}\rho_K^\varepsilon |u|^2 dx - \int_{\mathbb{T}^n} \rho_K^\varepsilon u \cdot \partial_t u dx\,.
\end{aligned}
\tag{3.60}
$$

As before we define the relativistic correction term of the modulation energy by

$$
G^\varepsilon(t) = -\frac{1}{2}\int_{\mathbb{T}^n} \rho_K^\varepsilon |u|^2 dx + \int_{\mathbb{T}^n} u \cdot J_K^\varepsilon dx\,, \tag{3.61}
$$

then using crucially the incompressible Euler system (3.52), we have

$$
\begin{aligned}
&\frac{d}{dt}(H^\varepsilon(t) + G^\varepsilon(t)) \\
&= -\frac{\varepsilon^{2-2\alpha}}{4}\int_{\mathbb{T}^n} 2(\nabla\psi^\varepsilon \otimes \nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon} \otimes \nabla\psi^\varepsilon) : \nabla u dx \\
&\quad + \int_{\mathbb{T}^n} \big(J_S^\varepsilon - \rho_S^\varepsilon u\big) \cdot \big(u \cdot \nabla u\big) dx + \int_{\mathbb{T}^n} \frac{1}{2} J_S^\varepsilon \cdot \nabla |u|^2 dx \\
&\quad + \int_{\mathbb{T}^n} \big(J_S^\varepsilon - \rho_S^\varepsilon u\big) \cdot \nabla\pi dx - \int_{\mathbb{T}^n} \rho_K^\varepsilon u \cdot \partial_t u dx + \int_{\mathbb{T}^n} \partial_t u \cdot J_K^\varepsilon dx\,.
\end{aligned}
\tag{3.62}
$$

To deal with the first integral of the right hand side of (3.62), we need the following equality

$$
\begin{aligned}
&-\mathcal{R}e\,\varepsilon^{2-2\alpha}\Big[\nabla u : \big(\nabla - i\varepsilon^{\alpha-1}u\big)\psi^\varepsilon \otimes \overline{\big(\nabla - i\varepsilon^{\alpha-1}u\big)\psi^\varepsilon}\Big] \\
&= -\mathcal{R}e\,\varepsilon^{2-2\alpha}\sum_{j,\ell=1}^{n}\big(\partial_\ell u_j\big)\Big[\big(\partial_j - i\varepsilon^{\alpha-1}u_j\big)\psi^\varepsilon\overline{\big(\partial_\ell - i\varepsilon^{\alpha-1}u_\ell\big)\psi^\varepsilon}\Big] \\
&= -\frac{1}{2}\varepsilon^{2-2\alpha}\big(\nabla\psi^\varepsilon \otimes \nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon} \otimes \nabla\psi^\varepsilon\big) : \nabla u \\
&\qquad + \Big\{\big(J_S^\varepsilon - \rho_S^\varepsilon u\big) \cdot \big[(u \cdot \nabla)u\big] + \frac{1}{2}J_S^\varepsilon \cdot \nabla|u|^2\Big\}\,.
\end{aligned}
\tag{3.63}
$$

Similar to the (3.22) as discussed in previous section, this equality follows by direct computation. Combing (3.62) and (3.63), we have the equality

$$
\begin{aligned}
&\frac{d}{dt}\Big(H^\varepsilon(t)+G^\varepsilon(t)\Big)\\
&=-\varepsilon^{2-2\alpha}\int_{\mathbb{T}^n}\mathcal{R}e\Big[\nabla u:\big(\nabla-i\varepsilon^{\alpha-1}u\big)\psi^\varepsilon\otimes\overline{\big(\nabla-i\varepsilon^{\alpha-1}u\big)\psi^\varepsilon}\Big]dx \\
&\quad+\int_{\mathbb{T}^n}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\cdot\nabla\pi dx-\int_{\mathbb{T}^n}\rho_K^\varepsilon u\cdot\partial_t u dx+\int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon dx\,.
\end{aligned}
\tag{3.64}
$$

Now we will estimate the second, third and fourth integral of right side of (3.64) separately. By (3.51) and (3.50), we have for $t\in[0,T_*)$

$$
\|\varepsilon^{1-\alpha}\nabla\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}=\|\varepsilon\partial_t\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}=O(1)\,. \tag{3.65}
$$

Moreover, from the inequality

$$
\frac{1}{\gamma}\big|\rho_S^\varepsilon-1\big|^\gamma\le\Theta(\rho_S^\varepsilon,1)\,,
$$

we have

$$
\|\rho_S^\varepsilon-1\|_{L^\gamma(\mathbb{T}^n)}=O\big(\varepsilon^{\frac{2\alpha}{\gamma}}\big)\,. \tag{3.66}
$$

Hence by (3.65), (3.66) and Hölder inequality, we arrive at the inequality

$$
\begin{aligned}
&\int_{\mathbb{T}^n}\rho_S^\varepsilon(u\cdot\nabla\pi)dx=\int_{\mathbb{T}^n}(\rho_S^\varepsilon-1)(u\cdot\nabla\pi)+u\cdot\nabla\pi dx\\
&=\int_{\mathbb{T}^n}(\rho_S^\varepsilon-1)(u\cdot\nabla\pi)dx\lesssim\varepsilon^{\frac{2\alpha}{\gamma}}\|u\cdot\nabla\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)}\,.
\end{aligned}
\tag{3.67}
$$

To go further, we need the relation

$$
\begin{aligned}
&\int_{\mathbb{T}^n}J_S^\varepsilon\cdot\nabla\pi dx=\int_{\mathbb{T}^n}\pi\partial_t(\rho_S^\varepsilon-1)-\pi\partial_t\rho_K^\varepsilon dx\\
&=\frac{d}{dt}\int_{\mathbb{T}^n}\pi(\rho_S^\varepsilon-1)-\rho_K^\varepsilon\pi dx-\int_{\mathbb{T}^n}\partial_t\pi(\rho_S^\varepsilon-1)-\rho_K^\varepsilon\partial_t\pi dx\,.
\end{aligned}
\tag{3.68}
$$

The last integral of (3.68) can be estimated by Hölder inequality

$$
\int_{\mathbb{T}^n}\partial_t\pi\big[(\rho_S^\varepsilon-1)-\rho_K^\varepsilon\big]dx\lesssim\varepsilon^{\frac{2\alpha}{\gamma}}\|\partial_t\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)}+\varepsilon^\alpha\|\partial_t\pi\|_{L^\infty(\mathbb{T}^n)}\,, \tag{3.69}
$$

and the estimates of the third and fourth integrals of the right hand side of (3.64) are given respectively by

$$\int_{\mathbb{T}^n} \rho_K^\varepsilon u \cdot \partial_t u dx \lesssim \varepsilon^\alpha \|u \cdot \partial_t u\|_{L^\infty(\mathbb{T}^n)}, \tag{3.70}$$

and

$$\int_{\mathbb{T}^n} \partial_t u \cdot J_K^\varepsilon dx \lesssim \varepsilon^\alpha \|\partial_t u\|_{L^\infty(\mathbb{T}^n)}. \tag{3.71}$$

To obtain the incompressible limit we have to introduce one more correction term of the modulated energy defined by

$$W^\varepsilon(t) = \int_{\mathbb{T}^n} \left[\rho_K^\varepsilon - (\rho_S^\varepsilon - 1)\right] \pi dx. \tag{3.72}$$

The correction term $W^\varepsilon(t)$ can be served as the acoustic part (density fluctuation) of the modulated energy $H^\varepsilon(t)$. It is designed to control the propagation of the acoustic wave. Hence for $t \in [0, T_*)$ we have

$$\begin{aligned} \frac{d}{dt}\Big(H^\varepsilon(t) + G^\varepsilon(t) + W^\varepsilon(t)\Big) &\lesssim \|\nabla u\|_{L^\infty(\mathbb{T}^n)} H^\varepsilon(t) \\ &+\varepsilon^\delta \Big(\|u \cdot \nabla \pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)} + \|\partial_t \pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)} + \|\partial_t \pi\|_{L^\infty(\mathbb{T}^n)} \\ &+ \|u \cdot \partial_t u\|_{L^\infty(\mathbb{T}^n)} + \|\partial_t u\|_{L^\infty(\mathbb{T}^n)}\Big) \end{aligned} \tag{3.73}$$

where $\delta = 2\alpha/\gamma$. Integrating this inequality yields

$$\begin{aligned} H^\varepsilon(t) &\leq H^\varepsilon(0) + G^\varepsilon(0) + W^\varepsilon(0) - G^\varepsilon(t) - W^\varepsilon(t) \\ &+ C_1 \int_0^t H^\varepsilon(\tau) d\tau + C_2 \varepsilon^\delta t. \end{aligned} \tag{3.74}$$

One can show that $G^\varepsilon(0) + W^\varepsilon(0) - G^\varepsilon(t) - W^\varepsilon(t) = O(\varepsilon^\delta)$, and hence

$$H^\varepsilon(t) \leq C_1 \int_0^t H^\varepsilon(\tau) d\tau + H^\varepsilon(0) + C_2 \varepsilon^\delta t + C_3 \varepsilon^\delta. \tag{3.75}$$

Applying the Gronwall inequality and the decay rate of $H^\varepsilon(0)$ we derive the inequality

$$H^\varepsilon(t) \leq (C_4 \varepsilon^\beta + C_2 \varepsilon^\delta t + C_3 \varepsilon^\delta)(1 + C_1 t e^{C_1 t}). \tag{3.76}$$

Thus $H^\varepsilon(t) \le O(\varepsilon^\lambda)$ for $t \in [0, T_*)$, where $\lambda = \min\{\beta, \delta\}$.

It is easy to rewrite the modulated energy (3.53) as

$$\begin{aligned} H^\varepsilon(t) = \frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\left|\nabla\sqrt{\rho_S^\varepsilon}\right|^2 dx + \frac{1}{2}\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho_S^\varepsilon}}(J_S^\varepsilon - \rho_S^\varepsilon u)\right|^2 dx \\ + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}|\partial_t\psi^\varepsilon|^2 dx + \frac{1}{\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\Theta(\rho_S^\varepsilon, 1)dx, \end{aligned} \tag{3.77}$$

then from (3.77) we have

$$\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho_S^\varepsilon}}(J_S^\varepsilon - \rho_S^\varepsilon u)\right|^2 dx \to 0 \tag{3.78}$$

as $\varepsilon \to 0$. We deduce from (3.78) and Hölder inequality that

$$\|(J_S^\varepsilon - \rho_S^\varepsilon u)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \le \left\|\sqrt{\rho_S^\varepsilon}\right\|_{L^{2\gamma}(\mathbb{T}^n)}\left\|\frac{1}{\sqrt{\rho_S^\varepsilon}}(J_S^\varepsilon - \rho_S^\varepsilon u)\right\|_{L^2(\mathbb{T}^n)} \tag{3.79}$$

which converges to zero as $\varepsilon \to 0$. Finally, combing (3.65) and (3.66), we have

$$\|\rho_K^\varepsilon(\cdot, t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \lesssim \varepsilon^\alpha \|\varepsilon\partial_t\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}\|\psi^\varepsilon\|_{L^{2\gamma}(\mathbb{T}^n)} \to 0 \tag{3.80}$$

and

$$\|J_K^\varepsilon(\cdot, t)\|_{L^1(\mathbb{T}^n)} \lesssim \varepsilon^\alpha \|\varepsilon\partial_t\psi^\varepsilon\|_{L^2(\mathbb{T}^n)}\|\varepsilon^{1-\alpha}\nabla\psi^\varepsilon\|_{L^2(\mathbb{T}^n)} \to 0 \tag{3.81}$$

as $\varepsilon \to 0$. This completes the proof of Theorem 3.2. ■

When $\alpha > 1 - \frac{\lambda}{2}$, we deduce from (3.77) that

$$\int_{\mathbb{T}^n}\left|\nabla\sqrt{\rho_S^\varepsilon}\right|^2 dx = \frac{1}{2}\int_{\mathbb{T}^n}\left|\frac{\nabla\rho_S^\varepsilon}{\sqrt{\rho_S^\varepsilon}}\right|^2 dx \to 0$$

as $\varepsilon \to 0$, and

$$\|\nabla(\rho_S^\varepsilon - 1)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \le \left\|\frac{\nabla\rho_S^\varepsilon}{\sqrt{\rho_S^\varepsilon}}\right\|_{L^2(\mathbb{T}^n)}\left\|\sqrt{\rho_S^\varepsilon}\right\|_{L^{2\gamma}(\mathbb{T}^n)}, \tag{3.82}$$

by Hölder inequality. Thus, $\rho_S^\varepsilon \to 1$ strongly in $W^{1,\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)$. Furthermore, by Sobolev inequality we can show that $\rho_S^\varepsilon \to 1$ strongly in $L^{\frac{2n\gamma}{n(\gamma+1)-2\gamma}}(\mathbb{T}^n)$ for $n \ge 2$. In particular $n = 2$, iterating the estimate (3.82) by the so called "*bootstrap process*", we have $\rho_S^\varepsilon \to 1$ in $L^p(\mathbb{T}^2)$ for any $1 \le p < \infty$, and hence we have the following improvement of Theorem 3.2.

Theorem 3.3 *Assume the same hypothesis of Theorem* 3.2. *Let* $\alpha > 1 - \frac{\lambda}{2}$ *and* $n = 2$ *then there exists* $T_* > 0$ *such that for any* $\eta > 0$,

$$\|(\rho_S^\varepsilon - 1)(\cdot, t)\|_{L^{\frac{1}{\eta}}(\mathbb{T}^2)} \to 0\,, \qquad \|\rho_K^\varepsilon(\cdot, t)\|_{L^{2-\eta}(\mathbb{T}^2)} \to 0\,, \tag{3.83}$$

$$\|(J_S^\varepsilon - \rho_S^\varepsilon u)(\cdot, t)\|_{L^{2-\eta}(\mathbb{T}^2)} \to 0\,, \qquad \|J_K^\varepsilon(\cdot, t)\|_{L^1(\mathbb{T}^2)} \to 0\,, \tag{3.84}$$

for $t \in [0, T_*)$ *as* $\varepsilon \to 0$, *where* u *is the unique local smooth solution of the incompressible Euler equations* (3.52).

.

4 Singular Limits

In this chapter we discuss the singular limit of the modulated nonlinear Klein-Gordon equation and the detail of the proof is referred to [11]. The main idea is based on the conservation laws of charge and energy;

$$\frac{\partial}{\partial t}\left[|\psi|^2 + \frac{i}{2}\varepsilon\nu^2(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi})\right] + \nabla\cdot\left[\frac{i}{2}\varepsilon(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi)\right] = 0\,, \qquad (4.1)$$

$$\frac{\partial}{\partial t}\left[\left(\nu^2|\partial_t\psi|^2 + |\nabla\psi|^2\right) + \frac{2}{\varepsilon^2}V\right] - \nabla\cdot\left[(\nabla\psi\partial_t\overline{\psi} + \nabla\overline{\psi}\partial_t\psi)\right] = 0\,. \qquad (4.2)$$

Examining the charge equation (4.1) we see that although $|\psi|^2$, Schrödinger part, is positive-definite but Klein-Gordon part $\frac{i}{2}\varepsilon\nu^2(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi})$ is not. Here we face one of the major difficulties with the Klein-Gordon equation. However, the energy density is positive-definite and can be employed to obtain the estimate of the Schrödinger part charge. Thus we introduce the charge-energy inequality to establish the singular limits. This is consistent with Einstein's relativity of mass-energy equivalent.

4.1 Semiclassical Limit

The specific problem we will consider in this section is the semiclassical limit of the modulated nonlinear Klein-Gordon equation (1.4) with potential function given by $V'(|\psi^\varepsilon|^2) = |\psi^\varepsilon|^2 - 1$. For convenience let us call it the modulated defocusing cubic nonlinear Klein-Gordon equation. After dividing by ε, we relabel it as

$$i\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon\nu^2\partial_t^2\psi^\varepsilon + \frac{\varepsilon}{2}\Delta\psi^\varepsilon - \Big(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\Big)\psi^\varepsilon = 0\,. \qquad (4.3)$$

The initial conditions are supplemented by

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \Omega\,. \qquad (4.4)$$

The superscript ε in the wave function ψ^ε indicates the ε-dependence and ν is assumed to be a fixed number in this section. To avoid the complications at the boundary, we concentrate below on the case where $x \in \Omega = \mathbb{T}^n$, the n-dimensional torus. Notice that the 4th term $\frac{|\psi^\varepsilon|^2-1}{\varepsilon}$ of (4.3) can be served as the density fluctuation of the sound wave which is similar to the acoustic wave as discussed in the low Mach number limit of the compressible fluid [2, 10, 16, 18]. For this model (4.3)–(4.4) we have the following existence result.

Theorem 4.1 *Let $\nu, T > 0$ and $0 < \varepsilon \ll 1$. Given initial data $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ and $\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon} \in L^2(\mathbb{T}^n)$, there exists a function ψ^ε such that*

$$\psi^\varepsilon \in L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big) \cap C\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \tag{4.5}$$

$$\partial_t\psi^\varepsilon \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big) \cap C\big([0,T]; H^{-1}(\mathbb{T}^n)\big)\,, \tag{4.6}$$

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \tag{4.7}$$

and satisfies the weak formulation of (4.3) *given by*

$$\begin{aligned} 0 = i\Big\langle \psi^\varepsilon(\cdot,t_2) - \psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle - \frac{1}{2}\varepsilon\nu^2\Big\langle \partial_t\psi^\varepsilon(\cdot,t_2) - \partial_t\psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle \\ -\frac{\varepsilon}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\varepsilon(\cdot,\tau), \nabla\varphi\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle\Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)\psi^\varepsilon(\cdot,\tau),\varphi\Big\rangle\, d\tau\,, \end{aligned} \tag{4.8}$$

for every $[t_1,t_2] \subset [0,T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Moreover, for all $t \in [0,T]$, it satisfies the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 + \nu^2|\partial_t\psi^\varepsilon|^2 + |\nabla\psi^\varepsilon|^2 + \frac{1}{2}\Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)^2 dx \le 2C_1 + \Big(1 + 2\varepsilon^2\nu^2\Big)C_2\,, \tag{4.9}$$

where

$$\begin{aligned} C_1 &= \int_{\mathbb{T}^n} |\psi_0^\varepsilon|^2 + \frac{i}{2}\varepsilon\nu^2(\psi_1^\varepsilon\overline{\psi_0^\varepsilon} - \overline{\psi_1^\varepsilon}\psi_0^\varepsilon)dx\,, \\ C_2 &= \int_{\mathbb{T}^n} \nu^2|\psi_1^\varepsilon|^2 + |\nabla\psi_0^\varepsilon|^2 + \frac{1}{2}\Big(\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon}\Big)^2 dx\,, \end{aligned} \tag{4.10}$$

are the initial charge and energy respectively.

The charge is constituted by the Schrödinger part (positive definite) and the Klein-Gordon part (not positive definite). However, it can be bounded by the energy. We denote by "∩" the intersection of topological spaces equipped with the relative topology induced by the inclusion maps. Since we are concerned with the semiclassical limit in this chapter, so the proof of this theorem, Theorem 4.8 and Theorem 4.12 of the following two sections will be given in section 4.4.

Now, we state the main theorem of this section.

Theorem 4.2 *Let $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, $|\psi_0^\varepsilon| = 1$ a.e. and $(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0)$ in $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, $|\psi_0| = 1$ a.e., and let ψ^ε be the corresponding weak solution of the modulated defocusing cubic nonlinear Klein-Gordon equation* (4.3)–(4.4). *Then the weak limit ψ, satisfying $|\psi| = 1$ a.e., solves the relativistic wave map*

$$(1+\nu^2)\partial_t^2\psi - \Delta\psi = \Big[|\nabla\psi|^2 - (1+\nu^2)|\partial_t\psi|^2\Big]\psi\,, \quad |\psi| = 1 \quad a.e.$$

$$\psi(x,0) = \psi_0(x), \qquad \partial_t\psi(x,0) = 0, \qquad x \in \mathbb{T}^n, \qquad |\psi_0| = 1 \quad a.e. \ .$$

Moreover, let $\psi = e^{i\theta}$ then the phase function θ satisfies the relativistic wave equation

$$(1+\nu^2)\partial_t^2\theta = \Delta\theta\,, \qquad \theta(x,0) = arg\ \psi_0\,, \qquad \partial_t\theta(x,0) = 0\ .$$

Proof. First we deduce from the charge-energy inequality (4.9) that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big)\,, \tag{4.11}$$

$$\{\partial_t\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \tag{4.12}$$

$$\left\{\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \tag{4.13}$$

then the classical compactness argument shows that there exists a subsequence still denoted by $\{\psi^\varepsilon\}_\varepsilon$ and a function ψ satisfying

$$\psi \in L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \qquad \partial_t\psi \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$$

such that

$$\psi^\varepsilon \ \rightharpoonup \ \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big)\,, \tag{4.14}$$

$$\partial_t\psi^\varepsilon \ \rightharpoonup \ \partial_t\psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,. \tag{4.15}$$

Next, from (4.13), we have

$$|\psi^\varepsilon|^2 \to 1 \quad \text{a.e. and strongly in } L^2(\mathbb{T}^n). \tag{4.16}$$

Note that (4.13) only shows that $\left\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right\}_\varepsilon$ is a weakly relative compact set in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$. Thus to overcome the difficulty caused by nonlinearity, i.e., the 4th term on the right hand side of (4.8), we have to prove $\psi^\varepsilon \to \psi$ strongly in $C\big([0,T];L^2(\mathbb{T}^n)\big)$.

Lemma 4.3 *For all* $0<\varepsilon\ll 1$*, the sequence* $\{\psi^\varepsilon\}_\varepsilon$ *is a relatively compact set in* $C\big([0,T];L^2(\mathbb{T}^n)\big)$ *endowed with its strong topology, i.e., there exists* $\psi\in C\big([0,T];L^2(\mathbb{T}^n)\big)$ *such that*

$$\psi^\varepsilon \;\;\to\;\; \psi \quad \textit{strongly in} \quad C\big([0,T];L^2(\mathbb{T}^n)\big)\,. \tag{4.17}$$

Proof. In this case the compactness requires more than just boundness here because of the strong topology over the time variable t. We appeal to the Arzela-Ascoli theorem which asserts that $\{\psi^\varepsilon\}_\varepsilon$ is a relatively compact set in $C\big([0,T];L^2(\mathbb{T}^n)\big)$ if and only if
(1) $\{\psi^\varepsilon(t)\}_\varepsilon$ is a relatively compact set in $L^2(\mathbb{T}^n)$ for all $t\geq 0$;
(2) $\{\psi^\varepsilon\}_\varepsilon$ is equicontinuous in $C\big([0,T];L^2(\mathbb{T}^n)\big)$.
From (4.9) or (4.11) we know that $\{\psi^\varepsilon(t)\}_\varepsilon$ is a bounded set in $H^1(\mathbb{T}^n)$ and hence is a relatively compact set in $L^2(\mathbb{T}^n)$ by Rellich lemma which states that $H^1(\mathbb{T}^n)\hookrightarrow L^2(\mathbb{T}^n)$ is a compact imbedding.

In order to establish condition (2), we apply the fundamental theorem of calculus and the uniform bound of $\{\partial_t\psi^\varepsilon\}_\varepsilon$ to obtain

$$\|\psi^\varepsilon(t_2)-\psi^\varepsilon(t_1)\|_{L^2(\mathbb{T}^n)} \leq |t_2-t_1|\|\partial_t\psi^\varepsilon(s)\|_{L^2(\mathbb{T}^n)} \lesssim |t_2-t_1|$$

for some $s\in(t_1,t_2)$. This completes the proof of Lemma 4.3. ∎

The quantity $\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon}$ is bounded in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$, and hence it converges weakly $*$ to some function $w\in L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$. To find the explicit form of w, we define two functions $W(\psi^\varepsilon)$ and $Z(\psi^\varepsilon)$ respectively by

$$W(\psi^\varepsilon)=\frac{i}{2}\Big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon}-\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\Big), \quad Z(\psi^\varepsilon)=\frac{i}{2}\nu^2\Big(\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon-\psi^\varepsilon\partial_t\overline{\psi^\varepsilon}\Big)\,. \tag{4.18}$$

We rewrite the conservation of charge (4.1) as

$$\frac{\partial}{\partial t}\left[\frac{|\psi^\varepsilon|^2-1}{\varepsilon}+Z(\psi^\varepsilon)\right]+\operatorname{div} W(\psi^\varepsilon)=0\,, \tag{4.19}$$

then integrating (4.19) with respect to t and using the initial condition $|\psi_0^\varepsilon|^2 = 1$, we have

$$\frac{|\psi^\varepsilon(x,t)|^2 - 1}{\varepsilon} = -Z(\psi^\varepsilon) + Z(\psi^\varepsilon(x,0)) - \int_0^t \text{div } W(\psi^\varepsilon) d\tau \,. \tag{4.20}$$

Thus to obtain the compactness of the sequence $\left\{\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon}\right\}_\varepsilon$, we have to treat the compactness of $\{Z(\psi^\varepsilon)\}_\varepsilon$ and $\{W(\psi^\varepsilon)\}_\varepsilon$ separately. First we have the following lemma.

Lemma 4.4 *Assume the hypothesis of Theorem 4.1, then*

$$\psi^\varepsilon \partial_t \overline{\psi^\varepsilon} \rightharpoonup \psi \partial_t \overline{\psi} \tag{4.21}$$

$$\int_0^t \text{div } \left(\psi^\varepsilon \nabla \overline{\psi^\varepsilon}\right) d\tau \rightharpoonup \int_0^t \text{div } \left(\psi \nabla \overline{\psi}\right) d\tau \tag{4.22}$$

in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$.

Proof. We observe that $\psi^\varepsilon \in C\big([0,T]; L^2(\mathbb{T}^n)\big)$ implies $\psi^\varepsilon \in L^2([0,T] \times \mathbb{T}^n)$ and $\partial_t \psi^\varepsilon \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$ implies $\partial_t \psi^\varepsilon \in L^2([0,T] \times \mathbb{T}^n)$. Also ψ^ε converges strongly to ψ in $L^2([0,T] \times \mathbb{T}^n)$ and $\partial_t \psi^\varepsilon$ converges weakly to $\partial_t \psi$ in $L^2([0,T] \times \mathbb{T}^n)$. Thus for all $\varphi \in C_0^\infty(\mathbb{T}^n)$ we have

$$\lim_{\varepsilon \to 0} \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \psi^\varepsilon(x,t) \partial_t \overline{\psi^\varepsilon}(x,t) \varphi(x) dx dt = \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \psi(x,t) \partial_t \overline{\psi}(x,t) \varphi(x) dx dt \,.$$

Similarly $\nabla \psi^\varepsilon \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$ implies $\nabla \psi^\varepsilon \in L^2([0,T] \times \mathbb{T}^n)$ and $\nabla \psi^\varepsilon$ converges weakly to $\nabla \psi$ in $L^2([0,T] \times \mathbb{T}^n)$, then integration by part then by Fubini theorem and Lebesgue dominated convergence theorem we conclude that

$$\begin{aligned}
&- \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \int_0^t \text{div } \Big[\psi^\varepsilon(x,\tau) \nabla \overline{\psi^\varepsilon}(x,\tau) - \psi(x,\tau) \nabla \overline{\psi}(x,\tau)\Big] d\tau \varphi(x) dx dt \\
&= \int_{t_1}^{t_2} \int_0^t \int_{\mathbb{T}^n} \Big[\psi^\varepsilon(x,\tau) - \psi(x,\tau)\Big] \nabla \overline{\psi^\varepsilon}(x,\tau) \cdot \nabla \varphi(x) dx d\tau dt \\
&+ \int_{t_1}^{t_2} \int_0^t \int_{\mathbb{T}^n} \Big[\nabla \overline{\psi^\varepsilon}(x,\tau) - \nabla \overline{\psi}(x,\tau)\Big] \psi(x,\tau) \cdot \nabla \varphi(x) dx d\tau dt \to 0
\end{aligned}$$

as $\varepsilon \to 0$. This completes the proof of Lemma 4.4. ∎

It follows from Lemma 4.4 that $Z(\psi^\varepsilon) \rightharpoonup Z(\psi)$, $Z(\psi^\varepsilon(x,0)) \rightharpoonup 0$ and

$$\int_0^t \text{div } W(\psi^\varepsilon) d\tau \rightharpoonup \int_0^t \text{div } W(\psi) d\tau$$

in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$, thus

$$\frac{|\psi^\varepsilon(x,t)|^2 - 1}{\varepsilon} \rightharpoonup -Z(\psi) - \int_0^t \text{div } W(\psi) d\tau \tag{4.23}$$

in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$, and the limit function w is given explicitly by

$$w = -Z(\psi) - \int_0^t \text{div } W(\psi) d\tau \,.$$

Passage to the limit ($\varepsilon \to 0$). The uniform boundness of the sequences $\{\psi^\varepsilon\}_\varepsilon$ in $L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big)$ and $\{\partial_t \psi^\varepsilon\}_\varepsilon$ in $L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$ imply

$$\frac{1}{2}\varepsilon\nu^2 \Big\langle \partial_t \psi^\varepsilon(\cdot, t_2), \varphi \Big\rangle \to 0 \,, \qquad \frac{1}{2}\varepsilon\nu^2 \Big\langle \partial_t \psi^\varepsilon(\cdot, t_1), \varphi \Big\rangle \to 0 \,, \tag{4.24}$$

$$\frac{\varepsilon}{2} \int_{t_1}^{t_2} \Big\langle \nabla \psi^\varepsilon(\cdot, \tau), \nabla \varphi \Big\rangle d\tau \to 0 \tag{4.25}$$

as $\varepsilon \to 0$. The strong convergence of ψ^ε in $C\big([0,T]; L^2(\mathbb{T}^n)\big)$ implies

$$\Big\langle \psi^\varepsilon(\cdot, t_2), \varphi \Big\rangle \to \Big\langle \psi(\cdot, t_2), \varphi \Big\rangle \,, \quad \Big\langle \psi^\varepsilon(\cdot, t_1), \varphi \Big\rangle \to \Big\langle \psi(\cdot, t_1), \varphi \Big\rangle \,. \tag{4.26}$$

The convergence of the nonlinear term follows by combing (4.17) and (4.23) together, so that for all $t > 0$

$$\left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \right) \psi^\varepsilon \rightharpoonup -\Big[Z(\psi) + \int_0^t \text{div } W(\psi) d\tau \Big] \psi \tag{4.27}$$

in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$ and hence

$$\begin{aligned} &\int_{t_1}^{t_2} \left\langle \left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \right) \psi^\varepsilon(\cdot, \tau), \varphi \right\rangle d\tau \\ \to &- \int_{t_1}^{t_2} \left\langle \Big(Z(\psi) + \int_0^t \text{div } W(\psi) d\tau \Big) \psi(\cdot, \tau), \varphi \right\rangle d\tau \,. \end{aligned} \tag{4.28}$$

Putting all the above convergent results into the weak formulation (4.8), the limit wave function ψ satisfies

$$i\partial_t\psi + \left[Z(\psi) + \int_0^t \text{div } W(\psi)d\tau\right]\psi = 0 \tag{4.29}$$

in the sense of distribution. Note $|\psi|^2 = 1$, we have $\overline{\psi}\partial_t\psi + \psi\partial_t\overline{\psi} = 0$ and $\overline{\psi}\nabla\psi + \psi\nabla\overline{\psi} = 0$, hence

$$\frac{1}{2}\left(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi}\right) = \overline{\psi}\partial_t\psi = -\psi\partial_t\overline{\psi}\,, \qquad \frac{1}{2}\left(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}\right) = \overline{\psi}\nabla\psi\,.$$

Differentiating (4.29) with respect to t, we have

$$\partial_t^2\psi + \left[\nu^2\partial_t(\overline{\psi}\partial_t\psi) - \text{div }(\overline{\psi}\nabla\psi)\right]\psi - \frac{\partial_t\psi}{\psi}\partial_t\psi = 0\,, \tag{4.30}$$

or

$$\partial_t^2\psi + \left[\nu^2\left(\overline{\psi}\partial_t^2\psi + \partial_t\psi\partial_t\overline{\psi}\right) - \left(\overline{\psi}\Delta\psi + \nabla\psi\cdot\nabla\overline{\psi}\right)\right]\psi + |\partial_t\psi|^2\psi = 0\,. \tag{4.31}$$

Therefore ψ satisfies the relativistic wave map equation

$$(1+\nu^2)\partial_t^2\psi - \Delta\psi = \left[|\nabla\psi|^2 - (1+\nu^2)|\partial_t\psi|^2\right]\psi\,, \quad |\psi| = 1 \quad \text{a.e.} \tag{4.32}$$

supplemented with the initial conditions

$$\psi(x,0) = \psi_0(x), \qquad \partial_t\psi(x,0) = 0, \qquad x\in\mathbb{T}^n, \qquad |\psi_0| = 1 \quad \text{a.e.} \tag{4.33}$$

Using the fact $|\psi| = 1$ and writing $\psi = e^{i\theta}$ shows

$$(1+\nu^2)\partial_t^2\theta = \Delta\theta\,, \qquad \theta(x,0) = \arg\psi_0\,, \qquad \partial_t\theta(x,0) = 0\,, \tag{4.34}$$

i.e., θ is a distribution solution of the linear relativistic wave equation. ∎

For completeness we also discuss the non-relativistic limit of the relativistic wave map equation (4.32)–(4.33). To indicate the ν-dependence of the wave function, we replace ψ by ϕ^ν and rewrite (4.32)–(4.33) as

$$(1+\nu^2)\partial_t^2\phi^\nu - \Delta\phi^\nu = \left[|\nabla\phi^\nu|^2 - (1+\nu^2)|\partial_t\phi^\nu|^2\right]\phi^\nu\,, \tag{4.35}$$

$$\phi^\nu(x,0) = \phi_0^\nu(x)\,, \quad \partial_t\phi^\nu(x,0) = 0, \qquad x \in \mathbb{T}^n\,, \tag{4.36}$$

$|\phi^\nu| = |\phi_0^\nu| = 1$ almost everywhere. Let $\mathcal{R}e\,\phi^\nu$ and $\mathcal{I}m\,\phi^\nu$ denote the real and imaginary parts of ϕ^ν, $\phi^\nu = \mathcal{R}e\,\phi^\nu + i\mathcal{I}m\,\phi^\nu$, and $u^c = (\mathcal{R}e\,\phi^\nu, \mathcal{I}m\,\phi^\nu)^t$ then (4.35)–(4.36) can be rewritten as

$$(1+\nu^2)\partial_t^2 u^\nu - \Delta u^\nu = \Big[|\nabla u^\nu|^2 - (1+\nu^2)|\partial_t u^\nu|^2\Big]u^\nu\,, \tag{4.37}$$

$$u^\nu(x,0) = u_0^\nu(x), \qquad \partial_t u^\nu(x,0) = 0, \qquad x \in \mathbb{T}^n\,, \tag{4.38}$$

where $u_0^\nu(x) = (\mathcal{R}e\,\phi_0^\nu, \mathcal{I}m\,\phi_0^\nu)^t$ and $|u^\nu| = |u_0^\nu| = 1$ almost everywhere. When $\nu = 0$ the necessary and sufficient condition for the existence of weak solutions to (4.37)–(4.38) were proved by Shatah [25] (see also [26]). His result is easily extended to general ν by replacing the Riemann metric $\eta = \mathrm{diag}(1,-1,-1,...,-1)$ by $\eta_\nu = \mathrm{diag}(1+\nu^2,-1,-1,...,-1)$ and $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$ by $\widetilde{\partial}^\alpha = \eta_c^{\alpha\beta}\partial_\beta$.

Lemma 4.5 (Shatah [25]) *If $|u^\nu| = 1$ almost everywhere and satisfies $\nabla u^\nu \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$, $\partial_t u^\nu \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$, then u^ν is a weak solution of (4.37)–(4.38) if and only if $\partial_\alpha(\widetilde{\partial}^\alpha u^\nu \wedge u^\nu) = 0$, where $\wedge$ denotes the wedge product.*

By lemma 4.5, we have the existence of weak solutions of the wave map equation.

Theorem 4.6 (Shatah [25]) *Given initial data $u_0^\nu \in H^1(\mathbb{T}^n)$ and $|u_0^\nu| = 1$, there exists a function u^ν, $|u^\nu| = 1$ a.e., such that*

$$\nabla u^\nu \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \quad \partial_t u^\nu \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big) \tag{4.39}$$

and satisfies the wave map equation

$$(1+\nu^2)\partial_t^2 u^\nu - \Delta u^\nu = \Big[|\nabla u^\nu|^2 - (1+\nu^2)|\partial_t u^\nu|^2\Big]u^\nu \tag{4.40}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Moreover, for all $t \in [0,T]$, it satisfies the energy relation

$$\int_{\mathbb{T}^n} (1+\nu^2)|\partial_t u^\nu|^2 + |\nabla u^\nu|^2 dx \le \int_{\mathbb{T}^n} |\nabla u_0^\nu|^2 dx\,. \tag{4.41}$$

As before we assume $\phi_0^\nu \to \phi_0$ strongly in $H^1(\mathbb{T}^n)$ and $|\phi_0| = 1$ a.e., equivalently if $u_0 = (\mathcal{R}e\,\phi_0, \mathcal{I}m\,\phi_0)^t$, $|u_0| = 1$ a.e., then $u_0^\nu \to u_0$ in $H^1(\mathbb{T}^n)$. We deduce from the energy relation (4.41) and $|u^\nu| = 1$ a.e. that

$$\{u^\nu\}_\nu \quad \text{is bounded in} \quad L^\infty([0,T]; H^1(\mathbb{T}^n))\,, \tag{4.42}$$

$$\{\partial_t u^\nu\}_\nu \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n))\,. \tag{4.43}$$

By classical compactness argument and diagonalization process there exists a subsequence still denoted by $\{u^\nu\}_\nu$ satisfying $u \in L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big)$ and $\partial_t u \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$ such that

$$u^\nu \;\rightharpoonup\; u \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big)\,, \tag{4.44}$$

$$\partial_t u^\nu \;\rightharpoonup\; \partial_t u \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,. \tag{4.45}$$

The same argument as Lemma 4.3, we deduce from (4.42)–(4.43) that

$$u^\nu \;\to\; u \quad \text{strongly in} \quad C\big([0,T]; L^2(\mathbb{T}^n)\big)\,. \tag{4.46}$$

Combing (4.46) and $|u^\nu| = 1$ a.e., we have $|u| = 1$ a.e.. Moreover, using (4.44)–(4.46), we have

$$\partial_\alpha u^\nu \wedge u^\nu \to \partial_\alpha u \wedge u \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{T}^n)\,. \tag{4.47}$$

Note u^ν satisfies $\partial_\alpha(\widetilde{\partial}^\alpha u^\nu \wedge u^\nu) = 0$ in the sense of distribution;

$$\begin{aligned} &(1+\nu^2)\Big\langle \partial_t u^\nu \wedge u^\nu(t_2,\cdot) - \partial_t u^\nu \wedge u^\nu(t_1,\cdot), \varphi \Big\rangle \\ &\quad + \sum_{i=1}^n \int_{t_1}^{t_2} \Big\langle \partial_i u^\nu \wedge u^\nu(\cdot,\tau), \partial_i \varphi \Big\rangle d\tau = 0 \end{aligned} \tag{4.48}$$

for every $[t_1, t_2] \subset [0,T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Letting $\nu \to 0$ in (4.48) and using (4.47), we have shown that u satisfies $\partial_\alpha(\partial^\alpha u \wedge u) = 0$ in the sense of distribution, and by Lemma 4.5 it solves the wave map equation

$$\partial_t^2 u - \Delta u = \big(|\nabla u|^2 - |\partial_t u|^2\big) u \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{T}^n). \tag{4.49}$$

Denote $u = (\alpha, \beta)^t$ and $\phi = \alpha + i\beta$, then we have $\nabla\phi^\nu \to \nabla\phi$ weakly $*$ in $L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$, $\phi^\nu \to \phi$ strongly in $L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$ and $\partial_t\phi^\nu \to \partial_t\phi$ weakly $*$ in $L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$. Moreover, ϕ satisfies the wave map equation

$$\partial_t^2 \phi - \Delta\phi = \big(|\nabla\phi|^2 - |\partial_t\phi|^2\big)\phi\,, \qquad (t,x) \in [0,T] \times \mathbb{T}^n, \tag{4.50}$$

$$\phi(x,0)=\phi_0(x)\,,\qquad \partial_t\phi(x,0)=0,\qquad x\in\mathbb{T}^n, \tag{4.51}$$

in the sense of distribution and $|\phi|=|\phi_0|=1$ almost everywhere.

Theorem 4.7 *Let $\phi_0^\nu,\phi_0\in H^1(\mathbb{T}^n)$, $|\phi_0^\nu|=|\phi_0|=1$ a.e. and $\phi_0^\nu\to\phi_0$ in $H^1(\mathbb{T}^n)$. Let ϕ^ν be the corresponding weak solution of the relativistic wave map* (4.35)–(4.36)*. Then the weak limit ϕ of $\{\phi^\nu\}_\nu$ satisfies $|\phi|=1$ a.e. and solves the wave map* (4.50)–(4.51).

4.2 Nonrelativistic Limit

This section is devoted to the non-relativistic limit of the modulated non-linear Klein-Gordon equation with potential function given by $V'(|\psi^\nu|^2)=|\psi^\nu|^p$, $p>0$,

$$i\varepsilon\partial_t\psi^\nu-\frac{1}{2}\varepsilon^2\nu^2\partial_t^2\psi^\nu+\frac{\varepsilon^2}{2}\Delta\psi^\nu-|\psi^\nu|^p\psi^\nu=0\,. \tag{4.52}$$

As usual, we supplement the system (4.52) with initial conditions

$$\psi^\nu(x,0)=\psi_0^\nu(x)\,,\qquad \partial_t\psi^\nu(x,0)=\psi_1^\nu(x)\,,\qquad x\in\mathbb{T}^n\,. \tag{4.53}$$

Here the Planck constant ε is a fixed positive number and the superscript ν in the wave function ψ^ν indicates the ν-dependence. Similar to the semiclassical limit discussed in the previous section we only discuss the periodic domain $\mathbb{T}^n$ and state the existence theorem of (4.52)–(4.53) first, leaving the proof in the appendix.

Theorem 4.8 *Let $p,\varepsilon,T>0$ and $\nu\ll 1$. Given initial data (ψ_0^ν,ψ_1^ν) in $H^1\cap L^{p+2}(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, there exists a function ψ^ν such that*

$$\psi^\nu\in L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\cap C\big([0,T];L^2(\mathbb{T}^n)\big)\,, \tag{4.54}$$

$$\partial_t\psi^\nu\in L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\cap C\big([0,T];H^{-1}(\mathbb{T}^n)\big)\,, \tag{4.55}$$

$$\psi^\nu\in L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)\,, \tag{4.56}$$

and satisfies the weak formulation of (4.52) *given by*

$$0=-\frac{1}{2}\varepsilon^2\nu^2\Big\langle\partial_t\psi^\nu(\cdot,t_2)-\partial_t\psi^\nu(\cdot,t_1),\varphi\Big\rangle+i\varepsilon\Big\langle\psi^\nu(\cdot,t_2)-\psi^\nu(\cdot,t_1),\varphi\Big\rangle$$
$$-\frac{\varepsilon^2}{2}\int_{t_1}^{t_2}\Big\langle\nabla\psi^\nu(\cdot,\tau),\nabla\varphi\Big\rangle d\tau-\int_{t_1}^{t_2}\Big\langle|\psi^\nu|^p\psi^\nu(\cdot,\tau),\varphi\Big\rangle d\tau\,, \tag{4.57}$$

for every $[t_1,t_2]\subset[0,T]$ *and for all* $\varphi\in C_0^\infty(\mathbb{T}^n)$. *Moreover,* ψ^ν *satisfies the charge-energy inequality*

$$\int_{\mathbb{T}^n}|\psi^\nu|^2+\frac{1}{2}\varepsilon^2\nu^2|\partial_t\psi^\nu|^2+\frac{\varepsilon^2}{2}|\nabla\psi^\nu|^2+\frac{|\psi^\nu|^{p+2}}{p+2}dx\le 2C_1+\Big(1+2\nu^2\Big)C_2\,, \tag{4.58}$$

where C_1 *and* C_2 *are the initial charge and energy given respectively by*

$$\begin{aligned} C_1&=\int_{\mathbb{T}^n}|\psi_0^\nu|^2+\frac{i}{2}\varepsilon\nu^2\big(\psi_1^\nu\overline{\psi_0^\nu}-\overline{\psi_1^\nu}\psi_0^\nu\big)dx\,,\\ C_2&=\int_{\mathbb{T}^n}\frac{1}{2}\varepsilon^2\nu^2|\psi_1^\nu|^2+\frac{\varepsilon^2}{2}|\nabla\psi_0^\nu|^2+\frac{1}{p+2}|\psi_0^\nu|^{p+2}dx\,. \end{aligned} \tag{4.59}$$

Now, we state the main theorem of this section.

Theorem 4.9 *Let* $(\psi_0^\nu,\psi_1^\nu)\in H^1\cap L^{p+2}(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $(\psi_0^\nu,\psi_1^\nu)\to(\psi_0,0)$ *in* $H^1\cap L^{p+2}(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, *and* ψ^ν *be the corresponding weak solution of the modulated defocusing nonlinear Klein-Gordon equation* (4.52)–(4.53). *Then the weak limit* ψ *of* $\{\psi^\nu\}_\nu$ *solves the defocusing nonlinear Schrödinger equation*

$$\begin{aligned} i\varepsilon\partial_t\psi+\frac{\varepsilon^2}{2}\Delta\psi-|\psi|^p\psi=0\,,&\qquad (x,t)\in\mathbb{T}^n\times(0,T)\,,\\ \psi(x,0)=\psi_0(x)\,,&\qquad x\in\mathbb{T}^n\,. \end{aligned}$$

Proof. We deduce from the charge-energy inequality (4.58) that

$$\{\psi^\nu\}_\nu\quad\text{is bounded in}\quad L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\,, \tag{4.60}$$

$$\{\nu\partial_t\psi^\nu\}_\nu\quad\text{is bounded in}\quad L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\,, \tag{4.61}$$

$$\{\psi^\nu\}_\nu\quad\text{is bounded in}\quad L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)\,. \tag{4.62}$$

In the case of semiclassical limit, we have $L_t^\infty L_x^2$ bound for $\partial_t\psi^\varepsilon$, but for non-relativistic limit, we only have $L_t^\infty L_x^2$ bound for $\nu\partial_t\psi^\nu$, so we need further argument to show $\psi^\nu\to\psi$ in $C\big([0,T];L^2(\mathbb{T}^n)\big)$.

Lemma 4.10 *For all* $\nu\ll 1$, *the sequence* $\{\psi^\nu\}_\nu$ *is a relatively compact set in* $C\big([0,T];w\text{-}H^1(\mathbb{T}^n)\big)$, *thus there exists* $\psi\in C\big([0,T];w\text{-}H^1(\mathbb{T}^n)\big)$ *such that*

$$\psi^\nu\to\psi\quad in\quad C\big([0,T];w\text{-}H^1(\mathbb{T}^n)\big)\quad as\quad \nu\to 0.$$

Furthermore, $\{\psi^\nu\}_\nu$ *is a relatively compact set in* $C\big([0,T];L^2(\mathbb{T}^n)\big)$ *endowed with its strong topology and*

$$\psi^\nu\to\psi\quad in\quad C\big([0,T];L^2(\mathbb{T}^n)\big)\quad as\quad \nu\to 0.$$

Proof. As discussed in the previous section, we appeal to the Arzela-Ascoli theorem which states that the sequence $\{\psi^\nu\}_\nu$ is a relatively compact set in $C\big([0,T]; w\text{-}H^1(\mathbb{T}^n)\big)$ if and only if
(1) $\{\psi^\nu(t)\}$ is a relatively compact set in $w\text{-}H^1(\mathbb{T}^n)$ for all $t \geq 0$;
(2) $\{\psi^\nu\}$ is equicontinuous in $C\big([0,T]; w\text{-}H^1(\mathbb{T}^n)\big)$, i.e., for every $\varphi \in H^{-1}(\mathbb{T}^n)$ the sequence $\{\langle \psi^\nu, \varphi\rangle\}_\nu$ is equicontinuous in the space $C([0,T])$.

Since $\{\psi^\nu(t)\}_\nu$ is uniformly bounded in $H^1(\mathbb{T}^n)$, thus $\{\psi^\nu(t)\}_\nu$ is a relatively compact set in $w\text{-}H^1(\mathbb{T}^n)$ for every $t > 0$. In order to establish condition (2), let $A \subset C_c^\infty(\mathbb{T}^n)$ be an enumerable set which is dense in H^{-1}, then for any $\rho \in A$, we have

$$i\varepsilon\Big\langle \psi^\nu(\cdot,t_2) - \psi^\nu(\cdot,t_1), \rho\Big\rangle = \frac{1}{2}\varepsilon^2\nu^2\Big\langle \partial_t\psi^\nu(\cdot,t_2) - \partial_t\psi^\nu(\cdot,t_1), \rho\Big\rangle$$

$$+\frac{\varepsilon^2}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\nu(\cdot,\tau), \nabla\rho\Big\rangle d\tau + \int_{t_1}^{t_2}\Big\langle |\psi^\nu|^p\psi^\nu(\cdot,\tau), \rho\Big\rangle d\tau,$$

hence

$$|\langle \psi^\nu(\cdot,t_2) - \psi^\nu(\cdot,t_1), \rho\rangle| \lesssim \nu\|\rho\|_{L^2(\mathbb{T}^n)} + |t_2 - t_1|\big(\|\rho\|_{H^1(\mathbb{T}^n)} + \|\rho\|_{L^\infty(\mathbb{T}^n)}\big).$$

Thus for any $\epsilon > 0$, we can choose $\delta = \epsilon$ such that if $|t_2 - t_1| < \delta$ and $\nu < \epsilon$, then

$$|\langle \psi^\nu(\cdot,t_2) - \psi^\nu(\cdot,t_1), \rho\rangle| \lesssim \epsilon\,.$$

Moreover, by density argument we can prove

$$|\langle \psi^\nu(\cdot,t_2) - \psi^\nu(\cdot,t_1), \varphi\rangle| \lesssim \epsilon\,, \tag{4.63}$$

for all $\varphi \in H^{-1}(\mathbb{T}^n)$. Thus $\{\psi^\nu\}_\nu$ is equicontinuous in $C\big([0,T]; w\text{-}H^1(\mathbb{T}^n)\big)$ for c larger. The second statement follows immediately by Rellich lemma which states that $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ compactly, i.e., $w\text{-}H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ continuously. This completes the proof of Lemma 4.10. ∎

In order to overcome the difficulty caused by nonlinearity, we need the following lemma.

Lemma 4.11 *Assume the hypothesis of Theorem 4.8. Let ψ^ν be a sequence of weak solution to* (4.52)–(4.53) *then there exists $\psi \in L^\infty\big([0,T]; L^{p+1}(\mathbb{T}^n)\big)$ such that*

$$\psi^\nu \to \psi \quad in \quad L^\infty\big([0,T]; L^{p+1}(\mathbb{T}^n)\big).$$

Proof. The proof is divided into two cases. First, for $0 < p \le 1$, since $L^2(\mathbb{T}^n) \subset L^{p+1}(\mathbb{T}^n)$ for bounded measure $|\mathbb{T}^n| < \infty$, the strong convergence in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$ also implies the strong convergence in $L^\infty\big([0,T];L^{p+1}(\mathbb{T}^n)\big)$.

Second, $p > 1$, the strong convergence in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$ and the weakly $*$ convergence in $L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)$ combined with interpolation argument yields the result. Indeed, $\psi^\nu \rightharpoonup \psi$ weakly $*$ in $L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)$, the sequence $\{\psi^\nu - \psi\}_\nu$ is a norm bounded set in $L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)$, there exists a constant $K > 0$ such that

$$\limsup_{c\to\infty} \|\psi^\nu - \psi\|^{p+2}_{L^\infty([0,T];L^{p+2}(\mathbb{T}^n))} = K < \infty. \tag{4.64}$$

Next, let $\eta > 0$ be arbitrary, and choose $\delta < \eta/K$, the Young inequality gives

$$|\psi^\nu - \psi|^{p+1} = |\psi^\nu - \psi|^{p+1-2/p}|\psi^\nu - \psi|^{2/p} \le \delta|\psi^\nu - \psi|^{p+2} + C|\psi^\nu - \psi|^2. \tag{4.65}$$

Integrating this inequality over $\mathbb{T}^n$, we have

$$\|\psi^\nu - \psi\|^{p+1}_{L^{p+1}(\mathbb{T}^n)} \le \delta\|\psi^\nu - \psi\|^{p+2}_{L^{p+2}(\mathbb{T}^n)} + C\|\psi^\nu - \psi\|^2_{L^2(\mathbb{T}^n)}\,. \tag{4.66}$$

Thus

$$\limsup_{c\to\infty} \|\psi^\nu - \psi\|^{p+1}_{L^\infty([0,T];L^{p+1}(\mathbb{T}^n))} \le K\delta \le \eta\,. \tag{4.67}$$

Because $\eta > 0$ is arbitrary, we have $\psi^\nu \to \psi$ in $L^\infty\big([0,T];L^{p+1}(\mathbb{T}^n)\big)$. ∎

Passage to limit ($\nu \to 0$). The uniform boundness of the sequence $\{\nu\partial_t\psi^\nu\}_\nu$ in $L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$ yields

$$\frac{1}{2}\varepsilon^2\nu^2\Big\langle \partial_t\psi^\nu(\cdot,t_2) - \partial_t\psi^\nu(\cdot,t_1), \varphi\Big\rangle \to 0\,. \tag{4.68}$$

The weak $*$ converge of ψ^ν in $L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)$ and the strong convergence in $C\big([0,T];L^2(\mathbb{T}^n)\big)$ imply

$$\int_{t_1}^{t_2} \Big\langle \nabla\psi^\nu(\cdot,\tau), \nabla\varphi\Big\rangle d\tau \to \int_{t_1}^{t_2} \Big\langle \nabla\psi(\cdot,\tau), \nabla\varphi\Big\rangle d\tau\,, \tag{4.69}$$

$$\Big\langle \psi^\nu(\cdot,t_2) - \psi^\nu(\cdot,t_1), \varphi\Big\rangle \to \Big\langle \psi(\cdot,t_2) - \psi(\cdot,t_1), \varphi\Big\rangle\,. \tag{4.70}$$

For the nonlinear term, we rewrite it as

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\Big[|\psi^\nu|^p\psi^\nu(x,\tau)-|\psi|^p\psi(x,\tau)\Big]\varphi(x)dxd\tau$$

$$=\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\Big[\psi^\nu(x,\tau)-\psi(x,\tau)\Big]|\psi^\nu|^p(x,\tau)\varphi(x)dxd\tau \tag{4.71}$$

$$+\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\Big[|\psi^\nu|^p(x,\tau)-|\psi|^p(x,\tau)\Big]\psi(x,\tau)\varphi(x)dxd\tau\equiv I+II$$

for all $\varphi\in C_0^\infty(\mathbb{T}^n)$. We will estimate the integrals I and II separately. First, by Hölder inequality, we have

$$I\le\|\psi^\nu-\psi\|_{L^{p+1}([t_1,t_2]\times\mathbb{T}^n)}\|\varphi\|_{L^\infty(\mathbb{T}^n)}\|\psi^\nu\|^p_{L^{p+1}([t_1,t_2]\times\mathbb{T}^n)}\to 0\,, \tag{4.72}$$

thus I tends to 0 as $c\to\infty$ by Lemma 4.11. The estimate of II requires higher integrability. Since $|\psi^\nu|^p\rightharpoonup|\psi|^p$ weakly in $L^{\frac{p+2}{p}}([0,T]\times\mathbb{T}^n)$ for $T<\infty$ and ψ is bounded in $L^q([0,T]\times\mathbb{T}^n)$, $1\le q\le p+2$, hence we can choose $q=\frac{p+2}{2}$ such that

$$II=\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\Big[|\psi^\nu|^p(x,\tau)-|\psi|^p(x,\tau)\Big]\psi(x,\tau)\varphi(x)dxd\tau\to 0. \tag{4.73}$$

Combing the above convergent results into the weak formulation (4.57), as $c\to\infty$, we deduce that ψ is a distribution solution of the defocusing nonlinear Schrödinger equation;

$$i\varepsilon\partial_t\psi+\frac{\varepsilon^2}{2}\Delta\psi-|\psi|^p\psi=0\,,\qquad (x,t)\in\mathbb{T}^n\times(0,T)\,, \tag{4.74}$$

$$\psi(x,0)=\psi_0(x)\,,\qquad x\in\mathbb{T}^n\,. \tag{4.75}$$

■

4.3 Nonrelativistic-Semiclassical Limit

In this section we will consider the nonrelativistic-semiclassical limit of the modulated nonlinear Klein-Gordon equation with potential function given by $V'(|\psi|^2)=|\psi|^2-1$. In order to avoid carrying out a double limits the parameters c and ε must be related. For simplicity, we take $\varepsilon = \varepsilon$, $\nu = \varepsilon^\alpha$ for some $\alpha > 0$, $0 < \varepsilon \ll 1$ and rewrite the modulated defocusing cubic nonlinear Klein-Gordon equation as

$$i\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{1+2\alpha}\partial_t^2\psi^\varepsilon + \frac{\varepsilon}{2}\Delta\psi^\varepsilon - \Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)\psi^\varepsilon = 0\,, \tag{4.76}$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x)\,, \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x)\,, \qquad x \in \mathbb{T}^n\,. \tag{4.77}$$

Here the superscript ε in the wave function ψ^ε indicates the ε-dependence. As discussed in section 4.1 and 4.2, we only discuss the periodic domain $\mathbb{T}^n$ and state the following existence theorem.

Theorem 4.12 *Given $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ and $\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon} \in L^2(\mathbb{T}^n)$, there exists a function ψ^ε such that*

$$\psi^\varepsilon \in L^\infty([0,T]; H^1(\mathbb{T}^n)) \cap C([0,T]; L^2(\mathbb{T}^n))\,, \tag{4.78}$$

$$\partial_t\psi^\varepsilon \in L^\infty([0,T]; L^2(\mathbb{T}^n)) \cap C([0,T]; H^{-1}(\mathbb{T}^n))\,, \tag{4.79}$$

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \in L^\infty([0,T]; L^2(\mathbb{T}^n))\,, \tag{4.80}$$

and satisfies the weak formulation of (4.76) given by

$$\begin{aligned} 0 \;=\; & -\frac{1}{2}\varepsilon^{1+2\alpha}\Big\langle \partial_t\psi^\varepsilon(\cdot,t_2) - \partial_t\psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle + i\Big\langle \psi^\varepsilon(\cdot,t_2) - \psi^\varepsilon(\cdot,t_1), \varphi\Big\rangle \\ & - \frac{\varepsilon}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\varepsilon(\cdot,\tau), \nabla\varphi\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle \Big(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\Big)\psi^\varepsilon(\cdot,\tau), \varphi\Big\rangle\, d\tau, \end{aligned} \tag{4.81}$$

for every $[t_1, t_2] \subset [0,T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Moreover, it satisfies the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 + \varepsilon^{2\alpha}|\partial_t\psi^\varepsilon|^2 + |\nabla\psi^\varepsilon|^2 + \frac{1}{2}\bigg(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\bigg)^2 dx \le 2C_1 + (1 + 2\varepsilon^{2+2\alpha})C_2 \tag{4.82}$$

where C_1 and C_2 denote the initial charge and energy given respectively by

$$\begin{aligned} C_1 &= \int_{\mathbb{T}^n} |\psi_0^\varepsilon|^2 + \varepsilon^{1+2\alpha}\frac{i}{2}(\psi_1^\varepsilon\overline{\psi_0^\varepsilon} - \overline{\psi_1^\varepsilon}\psi_0^\varepsilon)dx\,, \\ C_2 &= \int_{\mathbb{T}^n} \varepsilon^{2\alpha}|\psi_1^\varepsilon|^2 + |\nabla\psi_0^\varepsilon|^2 + \frac{1}{2}\bigg(\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon}\bigg)^2 dx\,. \end{aligned} \tag{4.83}$$

The main theorem of this section as follows:

Theorem 4.13 *Let* $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$*,* $|\psi_0^\varepsilon| = 1$*, and* $(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0)$ *in* $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$*,* $|\psi_0| = 1$*, and let* ψ^ε *be the corresponding weak solution of the modulated cubic nonlinear Klein-Gordon equation* (4.76)–(4.77)*. Then the weak limit* ψ *satisfies* $|\psi| = 1$ *a.e. and solves the wave map*

$$\partial_t^2 \psi - \Delta \psi = \big(|\nabla \psi|^2 - |\partial_t \psi|^2\big)\psi\,, \qquad |\psi| = 1 \quad a.e.$$

$$\psi(x,0) = \psi_0(x)\,, \qquad \partial_t \psi(x,0) = 0\,, \qquad x \in \mathbb{T}^n\,.$$

Moreover, let $\psi = e^{i\theta}$ *then the phase function* θ *satisfies the wave equation*

$$\partial_t^2 \theta = \Delta \theta, \qquad \theta(x,0) = arg\ \psi_0\,, \qquad \partial_t \theta(x,0) = 0\,.$$

Proof. It follows immediately from the charge-energy inequality (4.82) that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big)\,, \tag{4.84}$$

$$\{\varepsilon^\alpha \partial_t \psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,, \tag{4.85}$$

$$\left\{\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,. \tag{4.86}$$

We deduce from (4.86) that

$$|\psi^\varepsilon|^2 \to 1 \quad \text{a.e. and strongly in } L^2(\mathbb{T}^n)$$

as ε tends to 0. As discussed above (4.86) only shows that the quantity $\big\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\big\}_\varepsilon$ is a weakly relative compact set in $L^\infty([0,T]; L^2(\mathbb{T}^n))$, then (up to a subsequence) the sequence $\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\}_\varepsilon$ converges weakly $*$ to some function w in $L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$. In order to find w explicitly, we rewrite the conservation of charge as

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} = -Z(\psi^\varepsilon) + Z(\psi^\varepsilon(x,0)) - \int_0^t \text{div } W(\psi^\varepsilon) d\tau\,, \tag{4.87}$$

where $Z(\psi^\varepsilon)$ and $W(\psi^\varepsilon)$ are defined similarly to (4.18). We deduce from (4.84) and (4.85) that $Z(\psi^\varepsilon) \rightharpoonup 0$ in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$, and the same discussion as Lemma 4.4, we can prove

$$\int_0^t \text{div } W(\psi^\varepsilon) d\tau \rightharpoonup \int_0^t \text{div } W(\psi) d\tau$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$, hence

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \rightharpoonup -\int_0^t \operatorname{div} W(\psi)d\tau \tag{4.88}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Therefore

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \rightharpoonup -\int_0^t \operatorname{div} W(\psi)d\tau \tag{4.89}$$

weakly $*$ in $L^\infty([0,T];L^2(\mathbb{T}^n)$, and thus

$$\left(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right)\psi^\varepsilon \rightharpoonup -\psi\int_0^t \operatorname{div} W(\psi)d\tau \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^n)\,. \tag{4.90}$$

By combing the above convergent results, one can pass to the limit in each term of (4.81) and conclude that the limit ψ satisfies $|\psi|=1$ a.e. and

$$i\partial_t\psi + \left(\int_0^t \operatorname{div} W(\psi)d\tau\right)\psi = 0 \tag{4.91}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Similar discussion as the case of semiclassical limit using $|\psi|=|\psi_0|=1$ a.e., we can prove that ψ satisfies the wave map equation

$$\partial_t^2\psi - \Delta\psi = \big(|\nabla\psi|^2-|\partial_t\psi|^2\big)\psi\,, \qquad |\psi|=1 \quad \text{a.e.} \tag{4.92}$$

$$\psi(x,0)=\psi_0(x)\,, \qquad \partial_t\psi(x,0)=0\,, \qquad x\in\mathbb{T}^n\,. \tag{4.93}$$

Using the fact $|\psi|=|\psi_0|=1$ again and writing $\psi=e^{i\theta}$ shows

$$\partial_t^2\theta = \Delta\theta, \qquad \theta(x,0)=\arg\psi_0\,, \qquad \partial_t\theta(x,0)=0\,. \tag{4.94}$$

■

4.4 Existence of Weak Solutions

The goal of this section is a short and direct proof of Theorem 4.8. (The proof of Theorem 4.1 or Theorem 4.1 proceeds along the same lines with modification.) We employ the Fourier-Galerkin method to construct a sequence of approximation solutions, and use the compactness argument to prove the existence of weak solutions, this technique was applied to complex

Ginzburg-Landau equation by Doering-Gibbon-Levermore in [4]. The light speed c and the Planck constant ε are assumed to be fixed numbers (or both equal 1 after proper rescaling) and the proof is decomposed into four steps.

Step 1. *Construction of approximation solutions ψ^δ by Fourier-Galerkin method.* Let P_δ denote the L^2 orthogonal projection onto the span of all Fourier modes of wave vector ξ with $|\xi| \le 1/\delta$. Define $\psi_0^\delta = P_\delta \psi_0$, $\psi_1^\delta = P_\delta \psi_1$ and let $\psi^\delta = \psi^\delta(t)$ be the unique solution of the ODE

$$-\frac{1}{2}\varepsilon^2\nu^2\partial_t^2\psi^\delta + i\varepsilon\partial_t\psi^\delta + \frac{\varepsilon^2}{2}\Delta\psi^\delta - P_\delta(|\psi^\delta|^p\psi^\delta) = 0\,, \tag{4.95}$$

with initial conditions

$$\psi^\delta(x,0) = \psi_0^\delta(x)\,, \qquad \partial_t\psi^\delta(x,0) = \psi_1^\delta(x)\,, \qquad x \in \mathbb{T}^n\,. \tag{4.96}$$

The regularized initial data are chosen such that $(\psi_0^\delta, \psi_1^\delta) \to (\psi_0, \psi_1)$ in $H^1 \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ as δ tends to zero. These solutions will satisfy the regularized version of the weak formulation

$$\begin{aligned} 0 \;&=\; -\frac{1}{2}\varepsilon^2\nu^2\Big\langle \partial_t\psi^\delta(\cdot,t_2) - \partial_t\psi^\delta(\cdot,t_1), \varphi\Big\rangle + i\varepsilon\Big\langle \psi^\delta(\cdot,t_2) - \psi^\delta(\cdot,t_1), \varphi\Big\rangle \\ &-\; \frac{\varepsilon^2}{2}\int_{t_1}^{t_2}\Big\langle \nabla\psi^\delta(\cdot,\tau), \nabla\varphi\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle |\psi^\delta|^p\psi^\delta(\cdot,\tau), \varphi\Big\rangle d\tau \end{aligned}$$

for every $[t_1, t_2] \subset [0, \infty)$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Furthermore the approximate solution $\psi^\delta \equiv P_\delta\psi$ will converge to ψ in C^∞ as δ tends to zero and satisfies the conservation laws of charge and energy given respectively by

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 + \frac{i}{2}\varepsilon\nu^2\Big(\overline{\psi^\delta}\partial_t\psi^\delta - \psi^\delta\partial_t\overline{\psi^\delta}\Big)dx = C_1^\delta\,, \tag{4.97}$$

$$\int_{\mathbb{T}^n} \frac{1}{2}\varepsilon^2\nu^2|\partial_t\psi^\delta|^2 + \frac{\varepsilon^2}{2}|\nabla\psi^\delta|^2 + \frac{1}{p+2}|\psi^\delta|^{p+2}dx = C_2^\delta\,. \tag{4.98}$$

Here C_1^δ and C_2^δ denote the initial charge and initial energy respectively. By

Young's inequality and uniform boundness of the charge and energy we derive

$$\begin{aligned}\int_{\mathbb{T}^n} |\psi^\delta|^2 dx &\le \varepsilon\nu^2 \int_{\mathbb{T}^n} |\partial_t\psi^\delta||\psi^\delta| dx + C_1^\delta \\ &\le \frac{1}{2}\int_{\mathbb{T}^n} |\psi^\delta|^2 + \varepsilon^2\nu^4|\partial_t\psi^\delta|^2 dx + C_1^\delta \\ &\le \frac{1}{2}\int_{\mathbb{T}^n} |\psi^\delta|^2 dx + \nu^2 C_2^\delta + C_1^\delta\,,\end{aligned}$$

i.e.,

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 dx \le 2C_1^\delta + 2\nu^2 C_2^\delta\,. \tag{4.99}$$

Adding (4.98) and (4.99) together, we have shown that the approximate solution ψ^δ satisfies the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 + 2\varepsilon^2\nu^2|\partial_t\psi^\delta|^2 + \frac{\varepsilon^2}{2}|\nabla\psi^\delta|^2 + \frac{|\psi^\delta|^{p+2}}{p+2} dx \le 2C_1^\delta + \Big(1+2\nu^2\Big)C_2^\delta\,. \tag{4.100}$$

Step 2. *Show that $\{\psi^\delta\}$ is a relatively compact set in $C\big([0,T];L^2(\mathbb{T}^n)\big) \cap L^\infty\big([0,T];L^{p+1}(\mathbb{T}^n)\big)$ and $\{\partial_t\psi^\delta\}$ is relatively compact in $C\big([0,T];H^{-1}(\mathbb{T}^n)\big)$.* We deduce from the charge-energy bound (4.100) that

$$\{\psi^\delta\}_\delta \quad \text{is bounded in} \quad L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\,, \tag{4.101}$$

$$\{\partial_t\psi^\delta\}_\delta \quad \text{is bounded in} \quad L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\,, \tag{4.102}$$

$$\{\psi^\delta\}_\delta \quad \text{is bounded in} \quad L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)\,. \tag{4.103}$$

It follows from (4.101)–(4.103) and the classical compactness argument that there exists a subsequence of $\{\psi^\delta\}_\delta$ which we still denote by $\{\psi^\delta\}_\delta$ and $\psi \in L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)$, $\partial_t\psi \in L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)$ such that

$$\psi^\delta \rightharpoonup \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\,, \tag{4.104}$$

$$\partial_t\psi^\delta \rightharpoonup \partial_t\psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T];L^2(\mathbb{T}^n)\big)\,, \tag{4.105}$$

$$\psi^\delta \rightharpoonup \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)\,. \tag{4.106}$$

The same technique as discussed in Lemma 4.3 and Lemma 4.11, we can apply the Arzela-Ascoli theorem and interpolation theorem to conclude

$$\psi^\delta \to \psi \quad \text{in} \quad C\big([0,T];L^2(\mathbb{T}^n)\big)\cap L^\infty\big([0,T];L^{p+1}(\mathbb{T}^n)\big).$$

The convergence of $\partial_t\psi^\delta \to \partial_t\psi$ in $C\big([0,T];w\text{-}L^2(\mathbb{T}^n)\big)$ also follows by the Arzela-Ascoli theorem. First, it is obvious that $\{\partial_t\psi^\delta(t)\}_\delta$ is a relatively compact set in $w\text{-}L^2(\mathbb{T}^n)$ for all $t \ge 0$ by energy bound. To show $\{\partial_t\psi^\delta\}$ is equicontinuous in $C\big([0,T];w\text{-}L^2(\mathbb{T}^n)\big)$, let $A \subset C_0^\infty(\mathbb{T}^n)$ be an enumerable set which is dense in $L^2(\mathbb{T}^n)$, then for any $\rho \in A$, we have

$$\frac{1}{2}\varepsilon^2\nu^2\Big\langle \partial_t\psi^\delta(\cdot,t_2)-\partial_t\psi^\delta(\cdot,t_1),\rho\Big\rangle = i\varepsilon\int_{t_1}^{t_2}\Big\langle\partial_t\psi^\delta(\cdot,\tau),\rho\Big\rangle d\tau$$

$$-\frac{\varepsilon^2}{2}\int_{t_1}^{t_2}\Big\langle\nabla\psi^\delta(\cdot,\tau),\nabla\rho\Big\rangle d\tau - \int_{t_1}^{t_2}\Big\langle|\psi^\delta|^p\psi^\delta(\cdot,\tau),\rho\Big\rangle d\tau,$$

so we derive the estimate

$$|\langle\partial_t\psi^\delta(\cdot,t_2)-\partial_t\psi^\delta(\cdot,t_1),\rho\rangle| \lesssim |t_2-t_1|\big(\|\rho\|_{H^1(\mathbb{T}^n)}+\|\rho\|_{L^\infty(\mathbb{T}^n)}\big).$$

The rest follows by density argument and this proves the equicontinuity of $\{\partial_t\psi^\delta\}$ in $C\big([0,T];w\text{-}L^2(\mathbb{T}^n)\big)$, so $\partial_t\psi^\delta \to \partial_t\psi$ in $C\big([0,T];w\text{-}L^2(\mathbb{T}^n)\big)$. Indeed, we have the strong convergence $\partial_t\psi^\delta \to \partial_t\psi$ in $C\big([0,T];H^{-1}(\mathbb{T}^n)\big)$ by Rellich lemma; $L^2 \hookrightarrow H^{-1}$ is a compact imbedding.

Step 3. *Passage to the limit* ($\delta \to 0$). The weak $*$ convergence of ψ^δ in $L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)$, the strong convergence of ψ^δ in $C\big([0,T];L^2(\mathbb{T}^n)\big)$ and the strong convergence of $\partial_t\psi^\delta$ in $C\big([0,T];H^{-1}(\mathbb{T}^n)\big)$ give the following convergent results;

$$\int_{t_1}^{t_2}\Big\langle\nabla\psi^\delta(\cdot,\tau),\nabla\varphi\Big\rangle d\tau \to \int_{t_1}^{t_2}\Big\langle\nabla\psi(\cdot,\tau),\nabla\varphi\Big\rangle d\tau\,, \tag{4.107}$$

$$\Big\langle\psi^\delta(\cdot,t_2)-\psi^\delta(\cdot,t_1),\varphi\Big\rangle \to \Big\langle\psi(\cdot,t_2)-\psi(\cdot,t_1),\varphi\Big\rangle\,, \tag{4.108}$$

$$\Big\langle\partial_t\psi^\delta(\cdot,t_2)-\partial_t\psi^\delta(\cdot,t_1),\varphi\Big\rangle \to \Big\langle\partial_t\psi(\cdot,t_2)-\partial_t\psi(\cdot,t_1),\varphi\Big\rangle\,. \tag{4.109}$$

Moreover, the same argument as the non-relativistic limit shows $|\psi^\delta|^p\psi^\delta \to |\psi|^p\psi$ in the sense of distribution, i.e.,

$$\int_{t_1}^{t_2}\Big\langle|\psi^\delta|^p\psi^\delta(\cdot,\tau),\varphi\Big\rangle d\tau \to \int_{t_1}^{t_2}\Big\langle|\psi|^p\psi(\cdot,\tau),\varphi\Big\rangle d\tau\,. \tag{4.110}$$

Therefore ψ satisfies the weak formulation of (4.52).

Step 4. *Proof of the charge-energy inequality.* The strong convergence of ψ^δ in $C\big([0,T];L^2(\mathbb{T}^n)\big)$ implies

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 dx \to \int_{\mathbb{T}^n} |\psi|^2 dx\,. \tag{4.111}$$

Next, the weak convergence of ψ^δ in $L^\infty\big([0,T];H^1(\mathbb{T}^n)\big)\cap L^\infty\big([0,T];L^{p+2}(\mathbb{T}^n)\big)$, together with the fact that the norm of the weak limit of a sequence is a lower bound for the limit inferior of the norms, yields

$$\int_{\mathbb{T}^n} |\nabla\psi|^2 dx \le \liminf_{\delta\to 0} \int_{\mathbb{T}^n} |\nabla\psi^\delta|^2 dx\,, \tag{4.112}$$

$$\int_{\mathbb{T}^n} |\psi|^{p+2} dx \le \liminf_{\delta\to 0} \int_{\mathbb{T}^n} |\psi^\delta|^{p+2} dx\,. \tag{4.113}$$

Similarly the weak convergence of $\partial_t\psi^\delta$ in $L^\infty([0,T];L^2(\mathbb{T}^n))$ implies

$$\int_{\mathbb{T}^n} |\partial_t\psi|^2 dx \le \liminf_{\delta\to 0} \int_{\mathbb{T}^n} |\partial_t\psi^\delta|^2 dx\,. \tag{4.114}$$

By combining (4.100) and the above inequalities, we obtain the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi|^2 + \frac{1}{2}\varepsilon^2\nu^2|\partial_t\psi|^2 + \frac{\varepsilon^2}{2}|\nabla\psi|^2 + \frac{|\psi|^{p+2}}{p+2} dx \le 2C_1 + \Big(1+2\nu^2\Big)C_2\,, \tag{4.115}$$

where the two constants

$$\begin{aligned} C_1 &= \int_{\mathbb{T}^n} |\psi_0|^2 + \frac{i}{2}\varepsilon\nu^2(\psi_1\overline{\psi_0} - \overline{\psi_1}\psi_0)dx\,,\\ C_2 &= \int_{\mathbb{T}^n} \frac{1}{2}\varepsilon^2\nu^2|\psi_1|^2 + \frac{\varepsilon^2}{2}|\nabla\psi_0|^2 + \frac{1}{p+2}|\psi_0|^{p+2}dx\,, \end{aligned} \tag{4.116}$$

represent the initial charge and energy respectively. This completes the proof of Theorem 4.8. ∎

5 Concluding Chapter

We conclude this chapter by mentioning some possible future works.

- It is interesting to consider the Klein-Gordon equations with electro magnetic fields.

- To consider the Klein-Gordon equations with non-well prepared initial condition is an challenge problem.

- It is possible to apply modulated energy method to other equations.

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