A new interpolative reasoning method in sparse rule-based systems

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Abstract

In [7], Yan et al. analyzed Koczy and Hirota's linear interpolative reasoning method presented in [2, 3] and found that the reasoning consequences by their method sometimes become abnormal fuzzy sets. Thus, they pointed out that a new interpolative reasoning method will be needed which can guarantee that the interpolated conclusion will also be triangular-type for a triangular-type observation. In this paper, we extend the works of [2, 3, 7] to present a new interpolative reasoning method to deal with fuzzy reasoning in sparse rule-based systems. The proposed method can overcome the drawback of Koczy and Hirota's method described in [7]. It can guarantee that the statement "If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" holds. © 1998 Elsevier Science B.V.

Keywords: Linear interpolative reasoning; Fuzzy approximate reasoning; Sparse rule-based systems

1. Introduction

It is obvious that the number of fuzzy rules in fuzzy rule-based systems can significantly affect the performance of rule-based systems. The more sparse the fuzzy rule bases, the faster the rule-based systems in execution. Thus, several approximate reasoning methods based on sparse fuzzy rule bases have been proposed in [1, 2, 3, 6]. In those sparse rule-based systems, the rule bases are incomplete, i.e., there are many empty spaces between membership functions of the antecedents of rules. When the membership function of the observation occurs on empty space, no rule will be fired and no conse-

quence is derived [7]. In order to cope with this problem, in [2, 3] Koczy and Hirota have presented a linear interpolative reasoning method for solving the "tomato classification" problem presented in [6, 8]. We can see that the method presented in [2, 3] is useful in sparse rule-based systems to deal with fuzzy reasoning.

In [7], Yan et al. analyzed Koczy and Hirota's interpolative reasoning method presented in [2, 3], and pointed out that the reasoning consequences of the method sometimes become abnormal fuzzy sets, where they proved that the statement "If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" mentioned in [2, 3] is improper, and they also showed two reasoning conditions in which the method presented in [2, 3] can work normally.

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Furthermore, in [7], Yan et al. also hope that someone can develop a new interpolative reasoning method which can guarantee that the interpolated conclusion will also be triangular-type for a triangular-type observation. From the analytic result of [7], we can see that the reasoning consequence sometimes become abnormal fuzzy set using Koczy and Hirota's interpolative reasoning method presented in [2, 3] due to the fact that their method only interpolated the bottoms of the fuzzy sets but ignored the interpolations of the highest points of the fuzzy set in the interpolative reasoning process.

In this paper, we extend the works of [2, 3, 7] to present a new interpolative reasoning method to deal with fuzzy reasoning in sparse rule-based systems. The proposed method can overcome the drawback of Koczy and Hirota's method described in [7]. The proposed method can guarantee that the statement "If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" holds. Thus, the proposed method is more general than the one presented in [2, 3] because it can overcome the drawback of Koczy and Hirota's interpolative reasoning method described in [7].

The rest of this paper is organized as follows. In Section 2, we briefly review the fuzzy reasoning problem in sparse fuzzy rule bases from [5, 7]. In Section 3, we propose a new interpolative reasoning method based on [2, 3, 7]. In Section 4, we use some examples to illustrate the interpolative reasoning process. The conclusions are discussed in Section 5.

2. Fuzzy reasoning problem in sparse fuzzy rule bases

A typical example for fuzzy reasoning in sparse rule-based systems is the "tomato classification" problem proposed by Mizumoto and Zimmermann [5]. We briefly describe this problem as follows. Assume that the "tomato classifier" decides the degree of ripeness by evaluating the colors of tomatoes, where the membership functions of the fuzzy terms "red", "green", "ripe", "unripe", "yellow", and
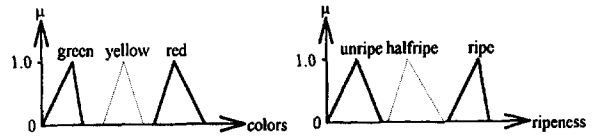

Fig. 1. Fuzzy reasoning assumption of the tomato classification problem.

'halfripe" are shown in Fig. 1. Assume that there is a sparse rule base consisting of only two rules:

Rule 1: If a tomato is red then the tomato is ripe.
Rule 2: If a tomato is green then the tomato is unripe.

We can see that it cannot derive any consequence when the observation "This tomato is yellow" occurs, i.e.,

Rule 1: If a tomato is red then the tomato is ripe.
Rule 2: If a tomato is green then the tomato is unripe.
Observation: This tomato is yellow.

Consequence: ???.

The problem occurred due to the fact that the membership function "yellow" has no overlapping with membership functions "red" or "green". Thus, the conventional fuzzy reasoning schemes cannot fire any rule.

In order to solve the problem mentioned above, Koczy and Hirota presented a linear interpolative fuzzy reasoning method in [2, 3]. In the following, we briefly review some definitions and reasoning conditions on Koczy's interpolative reasoning method in sparse fuzzy rule bases from [2, 3, 7].

Definition 2.1. Let the set of all normal and convex fuzzy sets of the universe X_i be denoted by $P(X_i)$. For $A_1, A_2 \in P(X_i)$, if $\forall \alpha \in (0, 1]$, the following conditions hold:

$$\inf\{A_{1\alpha}\} < \inf\{A_{2\alpha}\}, \quad \sup\{A_{1\alpha}\} < \sup\{A_{2\alpha}\},$$

then A_1 is said to be less than A_2 (i.e., $A_1 < A_2$), where $A_{1\alpha}$ and $A_{2\alpha}$ are α-cuts of A_1 and A_2, respectively, $\inf\{A_{i\alpha}\}$ is the infimum of $A_{i\alpha}$, and $\sup\{A_{i\alpha}\}$ is the supremum of $A_{i\alpha}$ $(i = 1, 2)$ [7].

Definition 2.2. Let $R_<$ be a fuzzy relation, $R_< = \{(A_1, A_2) \mid A_1, A_2 \in P(X),\ A_1 < A_2\}$. If fuzzy sets A_1 and A_2 satisfy $R_<$, then the lower and upper fuzzy distances between A_1 and A_2 are defined as follows [7], by using the resolution principle [2]:

$$d_L(A_1, A_2): R_< \to P([0,1]),$$

$$\mu_{d_L(A_1, A_2)}(\delta) = \sum_{\alpha \in [0,1]} \frac{\alpha}{d(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\})};$$

$$d_U(A_1, A_2): R_< \to P([0,1]),$$

$$\mu_{d_U(A_1, A_2)}(\delta) = \sum_{\alpha \in [0,1]} \frac{\alpha}{d(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\})};$$

where $\delta \in [0,1]$ and d is the Euclidean distance [2,3].

Definition 2.3. Let $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$ be disjoint fuzzy rules on the universe of discourse $X \times Y$, and A_1, A_2 and B_1, B_2 be fuzzy sets on X and Y, respectively. Assume that A^* is the observation of the input universe X. If $A_1 < A^* < A_2$, then the linear fuzzy rule interpolation between R_1 and R_2 is defined as follows [7]:

$$d(A^*, A_1): d(A^*, A_2) = d(B^*, B_1): d(B^*, B_2). \tag{1}$$

Definition 2.4. Let A_1 and A_2 be fuzzy sets on the universe of discourse X with $|X| < \infty$. The lower and the upper distances between α-cuts $A_{1\alpha}$ and $A_{2\alpha}$ are defined as follows [7]:

$$d_L(A_{1\alpha}, A_{2\alpha}) = d(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\}), \tag{2}$$

$$d_U(A_{1\alpha}, A_{2\alpha}) = d(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\}). \tag{3}$$

From the above definitions, (1) can be redefined as

$$d_L(A_\alpha^*, A_{1\alpha}): d_L(A_\alpha^*, A_{2\alpha}) = d_L(B_\alpha^*, B_{1\alpha}): d_L(B_\alpha^*, B_{2\alpha}),$$

$$d_U(A_\alpha^*, A_{1\alpha}): d_U(A_\alpha^*, A_{2\alpha}) = d_U(B_\alpha^*, B_{1\alpha}): d_U(B_\alpha^*, B_{2\alpha}),$$

which can be written as

$$\inf\{B_\alpha^*\}$$

$$= \frac{d_L(A_\alpha^*, A_{1\alpha})\inf\{B_{2\alpha}\} + d_L(A_\alpha^*, A_{2\alpha})\inf\{B_{1\alpha}\}}{d_L(A_\alpha^*, A_{1\alpha}) + d_L(A_\alpha^*, A_{2\alpha})}, \tag{4}$$

$$\sup\{B_\alpha^*\}$$

$$= \frac{d_U(A_\alpha^*, A_{1\alpha})\sup\{B_{2\alpha}\} + d_U(A_\alpha^*, A_{2\alpha})\sup\{B_{1\alpha}\}}{d_U(A_\alpha^*, A_{1\alpha}) + d_U(A_\alpha^*, A_{2\alpha})}. \tag{5}$$

In [7], Yan et al. have shown two reasoning conditions on Koczy and Hirota's interpolative reasoning method which can let the statement "If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" hold. These two conditions are restated as follows:

Condition 1: $\quad \dfrac{d_L(A_\alpha^*, A_{1\alpha})}{d_L(A_\alpha^*, A_{1\alpha}) + d_L(A_\alpha^*, A_{2\alpha})} = \beta,$

where $\beta \in [0, 1].$

Condition 2: $\quad \dfrac{d_L(B_{2\alpha}, B_{1\alpha})}{d_L(A_{2\alpha}, A_{1\alpha})} = \gamma$, where $\gamma > 0.$

From the analytic result of [7], we can see that the reasoning consequence of Koczy and Hirota's interpolative reasoning method sometimes become abnormal fuzzy sets due to the fact that their method only interpolated the bottoms of the fuzzy sets but ignored the interpolations of the highest points of the fuzzy set in the interpolative reasoning process.

3. A new interpolative fuzzy reasoning method

In this section, we present a new interpolative fuzzy reasoning method based on [2,3,7]. The proposed method can overcome the drawback of Koczy and Hirota's method described in [7]. It can guarantee that the statement "If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" holds. The proposed method not only interpolates the bottoms of the fuzzy set, but also interpolates the highest point of the fuzzy set in the interpolative fuzzy reasoning process.

Fig. 2 shows a general case of interpolative fuzzy reasoning schemes with triangular membership functions, where the fuzzy sets A_1, A^*, A_2, B_1, B^*,
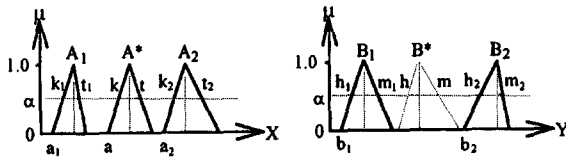
Fig. 2. Interpolation with triangular membership functions.

and B_2 of the rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$ are all defined by the triangular-type membership functions, where

$$a_1 = \inf\{A_{1\alpha}\}, \quad a_2 = \inf\{A_{2\alpha}\}, \quad a = \inf\{A_\alpha^*\},$$

$$b_1 = \inf\{B_{1\alpha}\}, \quad b_2 = \inf\{B_{2\alpha}\},$$

and $k_1, t_1, k, t, k_2, t_2, h_1, m_1, h, m, h_2$, and m_2 represent the slopes of the above triangular-type membership functions as shown in Fig. 2. By formulas (1)–(5), we can get the bottoms of the interpolated fuzzy set B_α^* (i.e., $\inf\{B_\alpha^*\}$ and $\sup\{B_\alpha^*\}$).

The definition of the highest point of a normal fuzzy set is presented as follows.

Definition 3.1. Let A be a normal fuzzy set on the universe of discourse X, then the highest points of the fuzzy set A, denoted by $\mathrm{hst}\{A\}$, can be defined as

$$\mathrm{hst}\{A\} = \{x \mid \mu_A(x) = 1, x \in X\}.$$

The proposed interpolative fuzzy reasoning method is presented as follows:

Step 1: By formulas (1)–(5), we can get the bottoms of the interpolated fuzzy set B_α^* (i.e., $\inf\{B_\alpha^*\}$ and $\sup\{B_\alpha^*\}$).

Step 2: Deriving the highest point $\mathrm{hst}\{B_\alpha^*\}$ of the fuzzy set B_α^*.

Step 3: Based on $\inf\{B_\alpha^*\}$, $\mathrm{hst}\{B_\alpha^*\}$, and $\sup\{B_\alpha^*\}$, the interpolative conclusion B_α^* can be uniquely decided.

In the following, we discuss the interpolation of the highest point $\mathrm{hst}\{B_\alpha^*\}$ (i.e., discuss Step 2 of the proposed method) in order to guarantee that the interpolated consequence will also be triangular-type for a triangular-type observation. The method to derive the highest point $\mathrm{hst}\{B_\alpha^*\}$ of the fuzzy set B_α^* is presented as follows:

Step 2.1: Deciding the slopes h and m of the triangular-type membership function B_α^*, let

$$k = k_1 x + k_2 y, \tag{6}$$

$$t = t_1 x + t_2 y, \tag{7}$$

where x and y are real numbers. If

$$\frac{k_1}{t_1} \neq \frac{k_2}{t_2},$$

then we can uniquely get x and y by solving (6) and (7) simultaneously, and let

$$h = |h_1 x + h_2 y|c, \tag{8}$$

$$m = -|m_1 x + m_2 y|c, \tag{9}$$

where c is a constant. Otherwise, we let

$$h = kc, \tag{10}$$

$$m = tc, \tag{11}$$

where c is a constant.

Step 2.2. Deciding the position of the highest point $\mathrm{hst}\{B_\alpha^*\}$ of the fuzzy set B_α^* by solving the following equation,

$$\frac{1-\alpha}{\mathrm{hst}\{B_\alpha^*\} - \inf\{B_\alpha^*\}} : \frac{\alpha - 1}{\sup\{B_\alpha^*\} - \mathrm{hst}\{B_\alpha^*\}} = h:m,$$

which can be reformulated as

$$\mathrm{hst}\{B_\alpha^*\} = \frac{m(\sup\{B_\alpha^*\}) - h(\inf\{B_\alpha^*\})}{m - h}. \tag{12}$$

4. Examples

In this section, we use the examples shown in [7] to illustrate the interpolative fuzzy reasoning process in sparse fuzzy rule bases.

Example 4.1. Let $\alpha = 0$, $a_1 = 0$, $a_2 = 11$, $a = 7$, $b_1 = 0, b_2 = 10, k_1 = 1/5, k_2 = 1/2, k = 1, h_1 = 1/2$, $h_2 = 1$, $t_1 = -1$, $t_2 = -1$, $t = -1$, $m_1 = -1/2$, and $m_2 = -1/2$. By formulas (4), (5), (8), (9), and (12), we can get

$$\inf\{B_\alpha^*\} = 6.4,$$

$$\sup\{B_\alpha^*\} = 7.4,$$

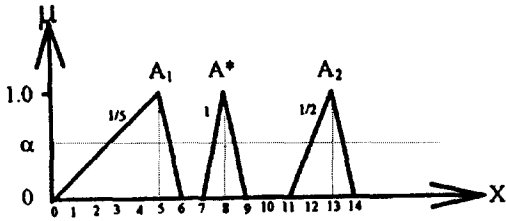$$\mathrm{hst}\{B_\alpha^*\} = 6.6.$$
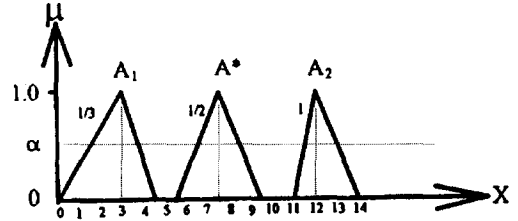
Fig. 3. The reasoning result of Example 4.1.

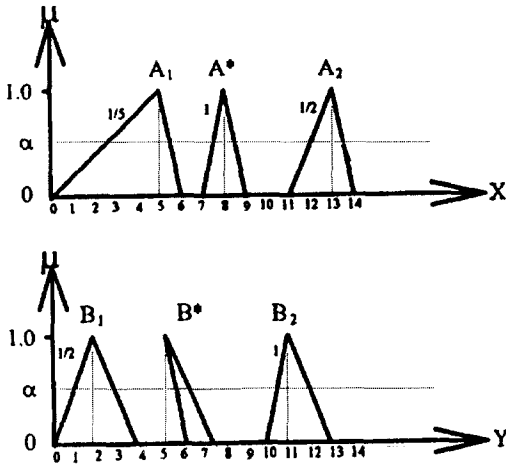Fig. 5. The reasoning result of Example 4.2.

Fig. 4. The reasoning result by Koczy and Hirota's method (adapted from [7]).

The reasoning result is represented by B^* shown in Fig. 3. From [7], we can see that Koczy and Hirota's linear interpolative reasoning method shown in [2,3] will get the abnormal reasoning result represented by the B^* shown in Fig. 4. From Figs. 3 and 4, we can see that the proposed method is better than the one presented in [2,3].

Example 4.2. Let $\alpha = 0$, $a_1 = 0$, $a_2 = 11$, $a = 5.5$, $b_1 = 0, b_2 = 10, k_1 = 1/3, k_2 = 1, k = 1/2, h_1 = 1/2,$

$h_2 = 1$, $t_1 = -2/3$, $t_2 = -1/2$, $t = -1/2$, $m_1 = -1/2$, and $m_2 = -1/2$. By formulas (4), (5), (8), (9), and (12), we can get

$$\inf\{B_\alpha^*\} = 5.0,$$

$$\sup\{B_\alpha^*\} = 8.7,$$

$$\mathrm{hst}\{B_\alpha^*\} = 6.5.$$

The reasoning result is represented by B^* shown in Fig. 5. From Fig. 5, we can see that the reasoning result of the proposed method is very close to the one presented in [7].

Example 4.3. Let $\alpha = 0$, $a_1 = 0$, $a_2 = 9$, $a = 5$, $b_1 = 1$, $b_2 = 10$, $k_1 = 1/3$, $k_2 = 1/4$, $k = 1/2$, $h_1 = 1/2$, $h_2 = 1/3$, $t_1 = -1$, $t_2 = -1$, $t = -1$, $m_1 = -1$, and $m_2 = -1$. By formulas (4), (5), (8), (9), and (12), we can get

$$\inf\{B_\alpha^*\} = 6.0,$$

$$\sup\{B_\alpha^*\} = 8.0,$$

$$\mathrm{hst}\{B_\alpha^*\} = 7.0.$$

The reasoning result is represented by B^* shown in Fig. 6. From Fig. 6, we can see that the reasoning result of the proposed method is very close to the one presented in [7].
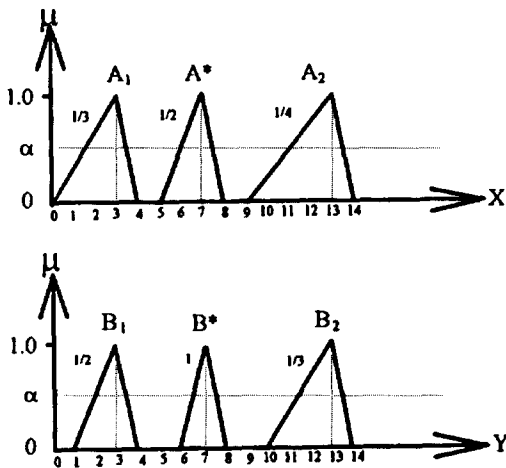
Fig. 6. The reasoning result of Example 4.3.

5. Conclusions

In this paper, we have extended the works of [2, 3, 7] to present a new interpolative reasoning method to deal with fuzzy reasoning in sparse rule-based systems. From the examples shown in Section 4, we can see that the proposed method can overcome the drawback of the one presented in [2, 3]. It can guarantee that the statement "If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" holds. Thus, the proposed method is more general than the one presented in [2, 3] because it can overcome the drawback of Koczy and Hirota's interpolative reasoning method described in [7].

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