Information Processing Letters 34 (1990) 261-263
North-Holland

7 May 1990

THE NECESSARY AND SUFFICIENT CONDITION FOR THE WORST-CASE MALE OPTIMAL STABLE MATCHING

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Communicated by P. Henderson
Received March 1989
Revised September 1989

Keywords: Stable matching problem, worst-case execution, sequential algorithms

1. Introduction

Gale and Shapley [1] introduced and solved the stable matching problem. That problem involves two disjoint sets of equal cardinality n, the men and the women. Each person ranks all members of the opposite sex in order of preference. A stable matching is defined as a complete matching between men and women with the property that no man and woman who are not partners both prefer each other to their actual partners under the matching.

Several stable matching algorithms [1-3, 8-10] were proposed to solve the problem by returning the male optimal stable solution as its answer. Any sequential one of those algorithms will be called a *stable matching algorithm* in this paper.

The purpose of this paper is to consider the worst-case choice for the sequential stable matching problem. This problem has been investigated by some researchers since the early paper of Wilson [11]. Itoga [5] presented some conclusions about the nature of the worst-case situation. Tseng and Lee [10] gave a necessary condition for the worst-case execution of the stable matching problem which takes the maximum number of proposals for the McVitie-Wilson's algorithm [8,9]. Kapur and Krishnamoorthy [6] presented a worst-case choice which takes the maximum number of stages for Gale-Shapley's algorithm [1]. In this paper we give the necessary and sufficient condition for the worst-case execution, which leads the sequential stable matching algorithm to take the maximum number of proposals. We then point out that the probability that the worst-case execution occurs when a sequential stable matching algorithm is employed is extremely small.

2. Definitions and background results

An instance of the stable matching problem consists of a set M of n men and a set W of n women, each member in these two sets has a rank-ordered preference lists of the n people of the opposite sex. For convenience, let $P(m_i)$ and $P(w_j)$ denote the preference lists for any man m_i and any woman w_j respectively, let $m_i[j]$ denote the jth choice of man m_i and let $w_i[j]$ denote the jth choice of woman w_i.

A *matching* μ is a one-to-one mapping of the men and the women, i.e., an invertible function $\mu: M \to W$ such that $\mu(m_i)$ is the woman matched with man m_i and $\mu^{-1}(w_j)$ is the man matched with woman w_j. The pair (m,w) blocks the matching μ if m and w are not matched by μ but prefer each other to their respective partners given in μ. A matching is *stable* if it is not blocked by any pair. A matching that is not stable is called *unstable*. The fundamental theorem [1] is that there is a stable matching for any problem instance. Given any instance $(M,W;P)$ where the pattern P represents the preferences of all members of $M \cup W$, any stable matching algorithm with proposals made by the men will terminate and return the *male optimal* (abbreviated as *M-optimal*) stable matching μ_m as the answer. Similarly, any stable matching algorithm with proposals made by the women will terminate and return the *female optimal* (abbreviated as *W-optimal*) stable matching μ_w as the answer.

The sequential stable matching algorithms for a solution to a stable matching instance [1,8,9] are based on a sequence of proposals from the men to the women. It is shown [8] that the sequence of proposals ends with every woman holding a unique proposal, and that the proposals held constitute a stable matching which is M-optimal. A similar outcome results if the roles of males and females are reversed, in which case the resulting stable matching that is W-optimal may or may not be the same as that obtained from the male proposal sequence.

Two fundamental implications of the male proposal sequence, implicit in [8], are

(i) if m proposes to w, then there is no stable matching in which m has a better partner than w, and

(ii) if w receives a proposal from m, then there is no stable matching in which w has a worse partner than m.

From these observations, it is shown [4] that we should explicitly remove m from w's list, and w from m's, if w receives a proposal from someone she likes better than m. The resulting lists are referred to as the *male-oriented shortlists*, for the given problem instance.

In the context of the male-oriented shortlists, a *male-oriented rotation* [4] exposed in μ is a sequence

$$r = (m_0, w_0), \ldots, (m_{p-1}, w_{p-1})$$

of pairs from μ such that, for each i $(0 \leqslant i \leqslant p-1)$ (i) w_i is first in m_i's shortlist, and (ii) w_{i+1} is second in m_i's shortlist ($i+1$ taken modulo p).

Similarly, we can, by relabeling, also define the *female-oriented rotation* exposed in μ. Note that for a given matching there may be many or there may be no exposed rotations.

It is pointed out in [4] that if the first entries in the male-oriented shortlists do not specify the μ_w, then at least one rotation must be exposed. The chief significance of such a rotation lies in the fact that if, in μ_m, each m_i exchanges his partner w_i for $w_{i+1(\text{mod } p)}$, then the resulting matching is also stable. This process is referred to as *eliminating* a rotation.

3. The necessary and sufficient condition for worst-case execution

We know that the sequential stable matching algorithms [1,8,9] are based on a sequence of proposals from the men to the women. Wilson [11] showed that the maximum number of proposals to obtain the M-optimal μ_m is $n^2 - n + 1$ when their algorithm is used, where n is the problem size.

In this section we present the necessary and sufficient condition for the worst-case choices for the stable matching problem which takes the maximum number of proposals for the sequential stable matching algorithms.

Lemma 1 [10]. *For a worst-case execution of the sequential stable matching algorithms the following two statements are true*:

(i) *There exists one woman w_l who is the last choice of all men in M and the $(n-1)$th choices of all men consists of all the members in $W - \{w_l\}$.*

(ii) *For each woman w_i in $W - \{w_l\}$, the first choice of w_i must be m_j whose $(n-1)$th choice is w_i, i.e., $m_j[n-1] = w_i$.*

The proof is simple, and we will not repeat it. However, the key point in the proof is that there is only one man who will propose to his last choice and all of the other men propose to their $(n-1)$th choice women if the worst case occurs by using the sequential stable matching algorithm.

In order to show our central theorem, we need one more lemma. Given any arbitrary instance $(M,W;P)$, if μ_m is the derivative of the worst-case execution, then there must exist no male-oriented rotation exposed in μ_m, otherwise μ_m will not be the derivative of the worst-case execution. Hence $(M,W;P)$ has only one stable matching and so

Lemma 2. *Given $(M,W;P)$, if μ_m is the derivative of the worst-case execution, then $\mu_w = \mu_m$.*

Now, we are concerned with the necessary and sufficient condition for the worst-case execution of any sequential stable matching algorithm.

Theorem 3. *Given any arbitrary instance $(M,W;P)$ which leads to the worst-case execution of any sequential stable matching algorithm, that returns μ_m as its answer, iff the preference pattern P satisfies the following conditions:*

(i) *Statements* (i)–(ii) *as stated in Lemma* 1.

(ii) *There exists no female-oriented rotation exposed in μ_w.*

Proof. "If" part. We want to show that if P satisfies the given condition, then the worst-case execution will occur. It is easy to derive μ_w from statement (i) of the theorem. Let $\mu_s = \{(w_i[1], w_i) \mid w_i \text{ in } W - \{w_l\}\}$. Then $\mu_w = \mu_s \cup \{(m', m_l)\}$ where m' is the man who is unmatched in μ_s.

It is pointed out in [4], by slight modification, that if there is no female-oriented rotation exposed in μ_w, then the first entries in the female-oriented shortlists should specify the μ_m. That is, $\mu_w = \mu_m$. But statement (i) of the theorem implies that μ_s can also be represented as $\{(m_i, m_i[n-1]) \mid m_i \text{ in } M - \{m'\}\}$. So $\mu_m = \mu_s \cup \{(m', m'[n])\}$ and this implies that the worst-case execution occurs. This completes the proof of the sufficiency of the condition.

"Only if" part. The proof of the necessity of the condition is a direct consequence of Lemmas 1 and 2. Lemma 1 states that statement (i) of the theorem must be true if the worst-case execution occurs. From Lemma 2 we know that if μ_m is the derivative of the worst-case execution, then for the given instance there exists only one stable matching which is both M-optimal and W-optimal. This implies that there must exist no female-oriented rotation exposed in μ_w. This completes the proof of the "only if" part of the theorem.

Herewith, the proof of the theorem has been completed. □

Using the theory of generalized derangements [7] we can show that the probability of the worst case occurring is $n!/[2n^n(n-1)^{2(n-1)}]$. This result indicates that the probability of the worst case occurring is extremely small. For example for $n = 8$ the probability that the worst case occurs is approximately 1.77×10^{-15} and this reduces to 2.95×10^{-42} for $n = 16$.

4. Conclusions

In this paper we have found the necessary and sufficient condition for the worst-case execution of the sequential stable matching algorithms. Moreover, we have pointed out the probability that the worst-case execution occurs when a sequential stable matching algorithm is employed.

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