

- (1) (Comparability) For all X and Y are in \mathcal{X} , one of $X \succ Y$, $Y \succ X$ or $X \sim Y$ must be hold.
- (2) (Transitivity) For all X , Y and Z are in \mathcal{X} , if $X \succ Y$ and $Y \succ Z$ then $X \succ Z$. In other words, if X is better than Y , and Y is better than Z , then X must be better than Z .
- (3) (Independence) For all X , Y and Z are in \mathcal{X} and for all $\alpha \in (0, 1)$, if $X \succ Y$ then $\alpha X + (1 - \alpha)Z \succ \alpha Y + (1 - \alpha)Z$ must be true.
- (4) (Continuity) For all X , Y and Z are in \mathcal{X} , if $X \succ Y$ and $Y \succ Z$ then there are α and β in $(0, 1)$ such that $\alpha X + (1 - \alpha)Z \succ Y$ and $Y \succ \beta X + (1 - \beta)Z$.

We can reject or accept above axioms. However, if all of these axioms are accepted, there exists a von Neumann-Morgenstern representation.

Definition 2. (1) If there exists a function $U : \mathcal{X} \rightarrow \mathbb{R}$ such that

$$(2.1) \quad X \succ Y \iff U(X) > U(Y),$$

we call it a numerical representation of preference relation \succ . Moreover such function U is called a utility function.

- (2) Let \mathcal{M} be the collection of all probability distributions on a space (S, s) , and there is $u : S \rightarrow \mathbb{R}$. A von Neumann-Morgenstern representation is a numerical representation of a preference relation \succ , satisfying

$$U(\mu) = \int_S u(x)\mu(dx),$$

for all $\mu \in \mathcal{M}$. Moreover such u is called the von Neumann-Morgenstern utility function.

In addition, if the above function u satisfies the conditions, strictly increasing, strictly concave and continuous on S , (2.1) can be written as

$$X \succ Y \iff E[u(X)] > E[u(Y)],$$

called expected utility representation. Moreover, the conditions of u , strictly concave and strictly increasing, means that the preference relation \succ has the properties, risk averse and monotone, respectively.

Therefore when there are two alternative investments X, Y in \mathcal{X} , and there exists expected utility representation of a preference relation \succ , we will choose X rather than Y if the expected utility of X is greater than the expected utility of Y . In other words, we conclude that

$$X \succ Y \iff E[u(X)] > E[u(Y)]$$

and

$$X \succeq Y \iff E[u(X)] \geq E[u(Y)].$$

This standard of making decision is called expected utility theory. Under expected utility theory our main goal is to find out $X \in \mathcal{X}$ such that $E[u(X)]$ reaches the maximal value.

2.2. Prospect Theory

Expected utility theory has been used popularly for investors to make decision under risk; however it still has some serious shortcomings. For instance, it can not explain the behavior of investor correctly. Many investors would not obey the principle of the expected utility theory, usually. And there exist many counterexamples, the best famous, was called Allais' paradox, which was introduced by French economist Maurice Allais (1953). The example is given as following:

Problem 1: Choose between

$$\begin{array}{l}
 A : \left\{ \begin{array}{ll} 2500 & \text{with probability } 0.33 \\ 2400 & \text{with probability } 0.66 \\ 0 & \text{with probability } 0.01 \end{array} \right. \\
 B : 2400 & \text{with certainty}
 \end{array}$$

Problem 2: Choose between

$$C : \begin{cases} 2500 & \text{with probability } 0.33 \\ 0 & \text{with probability } 0.67 \end{cases}$$

$$D : \begin{cases} 2400 & \text{with probability } 0.34 \\ 0 & \text{with probability } 0.66 \end{cases}$$

The statistic shows that 82 percent of subjects chose B in Problem 1, and 83 percent of subjects chose C in Problem 2. The main result of this example is that in problem 1, investment B is better than investment A, and in problem 2, investment C is better than investment D. The first preference implies

$$0.33u(2500) + 0.66u(2400) + 0.01u(0) < u(2400).$$

Without loss of generality, suppose that $u(0) = 0$ and rearranging above inequality we obtain

$$0.33u(2500) < 0.34u(2400).$$

However, the second preference implies

$$0.33u(2500) > 0.34u(2400).$$

Thus we get a contradiction.

In order to modify the expected utility theory, Kahneman and Tversky (1979) proposed the prospect theory. They found out that investor's attitude toward risk of gains is different from that of losses. Therefore they used value function, which is concave for gains and convex for losses, to instead of utility function that is concave everywhere. The implication of this transformation was that investors are not always risk aversion, they are risk seeking of losses. Moreover, in prospect theory it transfers probability into decision weight, called probability weighting function.

However, decision weights are not probabilities, since it does not keep the axioms of probability measure.

Under prospect theory we use value function v , and probability weighting function w to replace utility function and probability measure in expected utility theory, respectively. And then consider a prospect $(X, P) = (x_1, p_1; x_2, p_2; \dots; x_m, p_m)$ which means that yield outcome x_i with probability p_i , where $p_1 + p_2 + \dots + p_m = 1$. To simplify the notation, we omit null outcomes and use (x, p) to indicate $(x, p; 0, 1-p)$; furthermore, the riskless prospect with outcome x is denoted by (x) . Then the value of this prospect is defined by

$$V(X, P) = w(p_1)v(x_1) + w(p_2)v(x_2) + \dots + w(p_m)v(x_m) = \sum_{i=1}^m w(p_i)v(x_i),$$

where $v(0) = 0$, $w(0) = 0$, and $w(1) = 1$. More precisely, if we consider a prospect $(X, P) = (x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; \dots; x_{-1}, p_{-1}; x_0, p_0; x_1, p_1; \dots; x_n, p_n)$, with $p_{-m} + \dots + p_n = 1$, the value of this prospect is defined by

$$V = V^+ + V^-,$$

where

$$V^+(X, P) = w^+(p_0)v^+(x_0) + w^+(p_1)v^+(x_1) + \dots + w^+(p_n)v^+(x_n)$$

and

$$V^-(X, P) = w^-(p_{-m})v^-(x_{-m}) + w^-(p_{-m+1})v^-(x_{-m+1}) + \dots + w^-(p_{-1})v^-(x_{-1}).$$

Definition 3. For all prospects (X, P) , if (X, P) is indifference of (c) for an investor, then for all $k \in \mathbb{R}^+$, (kX, P) is indifference to (kc) for an investor. we call an investor exhibits preference homogeneity.

The main point of prospect theory is that value function is concave for gains and convex for losses. In other words, investors are risk-averse and risk-seeking when facing gains and losses, respectively.

In the rest of this section we introduce the value function and probability weighting function of prospect theory more precisely.

2.2.1. Value Function. The most important feature of prospect theory is that investors evaluate the value of prospects depend on the change of wealth rather than final wealth. Therefore there are two main viewpoints of value function: takes the current wealth as a reference point, and the size of change from that reference point. In other words, we separate value function into two parts, gains and losses, which above the reference point and below the reference point, respectively.

For many investors the difference in value between a gain of 100 and a gain of 200 becomes more attractive than the difference between a gain of 1000 and a gain of 1100. Hence Kahneman and Tversky (1979) applied this principle to the evaluation of monetary changes. They proposed that the value function is concave above the reference point and convex below it. That is, the value function is concave for gains and convex for losses.

Moreover, the characteristic of attitudes of investors toward the change of wealth is that losses loom larger than gains. In other words, for many investors the degree of miserable in losing a sum of money is greater than the degree of happy in gaining the same amount of money. This is because that most investors are not interesting in symmetric bets with the form $(x, 0.5; -x, 0.5)$. Thus Kahneman and Tversky (1979) proposed that the value function for losses is steeper than that for gains.

Definition 4. Value function v satisfies $v(0) = 0$, strictly increasing and if $|v(-x)| > v(x)$ for $x > 0$, then v has the property, called loss aversion.

In summary, we point out the properties of value function that Kahneman and Tversky (1979) proposed. First, value function is defined on deviations from the reference point. Second, it is concave for gains and convex for losses. Third, value function is steeper for losses than for gains. Furthermore, we can find out the

value function that satisfies above properties is a S-shaped value function which is steepest at the neighborhood of reference point. Furthermore, Kahneman and Tversky suggested that the form of value function is

$$v(x) = \begin{cases} x^\alpha & x \geq 0 \\ -\lambda(-x)^\beta & x < 0 \end{cases}.$$

The estimations of α , β and λ are given by $\alpha = \beta = 0.88$ and $\lambda = 2.25$. In addition, if value function takes the form as above, preference homogeneity must be true for all prospects (X, P) . However, for the purpose of computing easily, we often consider the other useful value function given by

$$v(x) = \begin{cases} 1 - \exp(-\theta x) & x \geq 0 \\ -\lambda(1 - \exp(\theta x)) & x < 0 \end{cases}.$$

It is easy to check that such function satisfies all properties of value function.

2.2.2. Probability Weighting Function. In this subsection we introduce the other major opinion of prospect theory, called probability weighting function.

First, we introduce an example, based on Maurice Allais, that violate the expected utility theory.

Problem 1: Choose between

$$\begin{array}{l} A : \begin{cases} 4000 & \text{with probability } 0.8 \\ 0 & \text{with probability } 0.2 \end{cases} \\ B : 3000 \quad \text{with certainty} \end{array}$$

Problem 2: Choose between

$$C : \begin{cases} 4000 & \text{with probability 0.2} \\ 0 & \text{with probability 0.8} \end{cases}$$

$$D : \begin{cases} 3000 & \text{with probability 0.25} \\ 0 & \text{with probability 0.75} \end{cases}$$

The data shows that 80 percent of subjects choose B in problem 1, and 65 percent of subjects choose C in problem 2. But in fact, we only reduce the probability by equal proportion. Hence we can obtain a result that reducing the probability from 1 to 0.25 makes greater influence than reducing the probability from 0.8 to 0.2. And we call this phenomenon common-ratio effect.

Thus Kahneman and Tversky (1979) explained the common-ratio effect by the method of a nonlinear transformation of probabilities, called "probability weighting function". Next, we illustrate the properties of probability weighting function as follows:

- (1) First, the regressive property, explains the attitudes toward risk. For many investors are risk aversion for gains with large probability and for losses with small probability. In addition, they are risk seeking for gains with small probability and for losses with large probability. Moreover the transformation of probabilities into probability weighting function is overweighting for small probabilities and underweighting for large probabilities.
- (2) Second, changes in probabilities have greater influence on the boundary of probability interval. That is, increasing the probability from 0 to 0.1 have greater effect than increasing the probability form 0.5 to 0.6.

The probability weighting function satisfying above two properties is given by

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}$$

and

$$w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}},$$

where $\gamma = 0.61$ and $\delta = 0.69$ for gains and losses, respectively.

Under prospect theory, probability weighting function for gains and for losses may be different. But if preference homogeneity and loss aversion both hold for all prospects, then the probability weighting function for gains and for losses are the same, i.e., $w^+ = w^-$.

Although prospect theory asserts that we replace utility function with value function which is concave for gains and convex for losses and transform probability into probability weighting function which is nonlinear. There are still some drawbacks. For example, it only works with prospects that have at most two different nonzero outcomes, and stochastic dominance does not still hold. In order to modify the drawbacks of prospect theory Kahneman and Tversky (1992) proposed cumulative prospect theory.

2.3. Cumulative Prospect Theory

Though prospect theory explained the major violations of expected utility theory in decision making under risk, there still exist two problems. First, it does not always satisfies stochastic dominance. Second, it can not be extended to prospects with a large number of outcomes. In order to modify the drawback of prospect theory, Kahneman and Tversky (1992) proposed cumulative prospect theory. The most important element of this theory is that instead of transforming each probability separately, this model transforms the entire cumulative distribution function, called cumulative weighting function or weighting function for short. Further this theory applies the cumulative functional separately to gains and to losses.

We first rearrange the outcomes of each prospect in increasing order, such as

$$(X, P) = (x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; \dots; x_{-1}, p_{-1}; x_0, p_0; x_1, p_1; \dots; x_n, p_n),$$

with $x_{-m} < x_{-m+1} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_n$, and $p_{-m} + \cdots + p_n = 1$. Without loss of generality, we suppose that $x_0 = 0$ and take x_0 as a reference point. We interpret x_{-m}, \dots, x_{-1} as losses and x_1, \dots, x_n as gains. Then cumulative prospect theory asserts that there exist a strictly increasing value function, satisfying $v(0) = 0$, and probability weighting function $w^+(p)$ and $w^-(p)$ such that the value of this prospect is defined by

$$V(X, P) = V^+(X, P) + V^-(X, P),$$

where

$$V^+(X, P) = \pi_0^+ v^+(x_0) + \pi_1^+ v^+(x_1) + \cdots + \pi_n^+ v^+(x_n) = \sum_{i=0}^n \pi_i^+ v^+(x_i)$$

and

$$V^-(X, P) = \pi_{-m}^- v^-(x_{-m}) + \pi_{-m+1}^- v^-(x_{-m+1}) + \cdots + \pi_{-1}^- v^-(x_{-1}) = \sum_{i=-m}^{-1} \pi_i^- v^-(x_i).$$

The decision weights are defined by

$$\pi_i^+ = \begin{cases} w^+(p_n) & , i = n \\ w^+(p_i + \cdots + p_n) - w^+(p_{i+1} + \cdots + p_n) & , 0 \leq i \leq n - 1 \end{cases}$$

and

$$\pi_i^- = \begin{cases} w^-(p_{-m}) & , i = -m \\ w^-(p_{-m} + \cdots + p_i) - w^-(p_{-m} + \cdots + p_{i-1}) & , 1 - m \leq i \leq -1 \end{cases}.$$

where w^+ and w^- are strictly increasing functions from the unit interval to itself satisfying $w^+(0) = w^-(0) = 0$, and $w^+(1) = w^-(1) = 1$.

Since we transform the entire cumulative distribution function rather than transform each probability separately, and consider the rank-dependent models, rearranging the outcomes in increasing order, cumulative prospect theory can extend the original version of prospect theory in several respects. First, it can work with prospects that have infinite outcomes and extend to continuous model. Second, it

allows different decision weights for gains and losses. Furthermore in the version of cumulative prospect theory it no longer violates stochastic dominance. Therefore, cumulative prospect theory has nowadays become one of the most famous versions for investors to make a decision under risk.



CHAPTER 3

Comparison of Optimization in the Sense of Expected Utility Theory and Cumulative Prospect Theory I: A Model without Transaction Cost

Since expected utility theory can not provide an adequate description of individual choice, Kahneman and Tversky proposed cumulative prospect theory to explain the major violations of expected utility theory in choices between risky prospects.

In this chapter, we interest in seeking out optimal strategies, which make us to gain the maximal profit, in the sense of expected utility theory and cumulative prospect theory, respectively. Our main goal is to compare the optimal strategies in the sense of expected utility theory and cumulative prospect theory. Moreover, we also discuss the difference of optimal hedging strategies in these two senses.

3.1. Optimal Trading Strategy in One Period Model

A fundamental problem in the financial mathematics is to find out the optimal trading strategies which can reach the maximal profit. In this section, we are interesting in comparing the optimal trading strategies under two different senses: expected utility theory and cumulative prospect theory.

In the beginning, we set up the market model as follow: Consider a one-period market model in which time points are denoted by 0 and 1, and the market model has one risky asset (stock) and one riskless asset (bond). In time 1 there exist two market states: ω_1 and ω_2 , with probability p_1 and p_2 , respectively. The current bond price B_0 is 1, and the stock price S_0 is s , and in time 1, the stock price denoted by

S_1 is given by $S_1(\omega_1) = l$ and $S_1(\omega_2) = u$ where $l < s < u$.

Suppose that the interest rate at time 1 is 0, and initial wealth is x .

Let $\bar{h} = (h_0, h_1)$ be the trading strategy at time 0, where h_0 and h_1 means the number of shares invested in bond and stock, respectively. And denote the optimal strategy by $\bar{h}^* = (h_0^*, h_1^*)$.

Theorem 1. In the sense of expected utility theory, if the mean of gains is greater than the mean of losses, the optimal investment amount of stock is greater than 0, i.e., $h_1^* > 0$ provided that $p_2(u - s) > p_1(s - l)$. Moreover, the converse is also true. That is, $h_1^* > 0$ if and only if $p_2(u - s) > p_1(s - l)$.

PROOF. Without loss of generality, suppose that the initial wealth is equal to 0. In order to get the optimal trading strategy under expected utility theory, we must to find the strategy $\bar{h} = (h_0, h_1)$ such that $f(h_0, h_1) = p_1U(h_0 + lh_1) + p_2U(h_0 + uh_1)$, reaches the maximal value, subjected to $h_0 + sh_1 = 0$, and where U is a concave function. Thus we can transform $f(h_0, h_1)$ into

$$f(h_1) = p_1U((l - s)h_1) + p_2U(u - s)h_1,$$

then by first order derivative, we have

$$f'(h_1) = p_2(u - s)U'((u - s)h_1) - p_1(s - l)U'((l - s)h_1).$$

Thus, if $f'(h_1) = 0$,

$$\frac{p_2(u - s)}{p_1(s - l)} = \frac{U'((l - s)h_1)}{U'((u - s)h_1)}.$$

Since $p_2(u - s) > p_1(s - l)$, we get

$$\frac{U'((l - s)h_1)}{U'((u - s)h_1)} > 1.$$

Hence $h_1 > 0$ because of the property of concave function U . Moreover by second order derivative we acquire

$$f''(h_1) = p_1(l - s)^2U''((l - s)h_1) + p_2(u - s)^2U''((u - s)h_1) < 0.$$

Therefore the optimal trading strategy is obtained as $h_1^* > 0$.

Next, we want to show that if $h_1^* > 0$ then $p_2(u - s) > p_1(s - l)$. Suppose not, if $p_2(u - s) \leq p_1(s - l)$ and $h_1 > 0$, then

$$f'(h_1) = p_2(u - s)U'((u - s)h_1) - p_1(s - l)U'((l - s)h_1) \leq 0.$$

This implies that $f(h_1)$ is a nonincreasing function, thus $h_1 > 0$ can not be the optimal trading strategy which contradicts to $h_1^* > 0$. Therefore if optimal trading strategy occurs when $h_1^* > 0$ then $p_2(u - s) > p_1(s - l)$. We complete the proof.

If a investor makes the decision under expected utility theory, he prefers risky asset to riskless asset only as the mean of gains is greater than the mean of losses. After that we take cumulative prospect theory into account, and assume that the probability weighting functions of gains and losses are the same with form

$$w^+(p) = w^-(p) = \frac{p^r}{(p^r + (1 - p)^r)^{\frac{1}{r}}}.$$

Theorem 2. In the version of cumulative prospect theory, if

$$(3.1) \quad (w(p_2)(u - s) - \lambda w(p_1)(s - l))(\lambda w(p_1) - w(p_2)) > 0$$

holds, the optimal amount of stock is greater than 0. In fact, the converse is also true, i.e., $h_1^* > 0$ if and only if (3.1) must be hold.

Remark 1. We can separate (3.1) into two cases:

$$(3.2) \quad w(p_2)(u - s) > \lambda w(p_1)(s - l), \text{ where } \frac{\lambda w(p_1)}{w(p_2)} > 1$$

and

$$(3.3) \quad w(p_2)(u - s) < \lambda w(p_1)(s - l), \text{ where } \frac{\lambda w(p_1)}{w(p_2)} < 1.$$

PROOF. (Proof of theorem 2) Without loss of generality, suppose that the initial wealth is equal to 0. In the sense of cumulative prospect theory, if we want to

acquire the optimal trading strategy, we need to find out the strategy $\bar{h} = (h_0, h_1)$ such that

$$f(h_1) = w(p_1)v^-(l-s)h_1 + w(p_2)v^+(u-s)h_1$$

reaches the maximal value, where w is a probability weighting function and v^- , v^+ are the value functions of gains and losses, respectively. That is, v^- is a convex function and v^+ is a concave function. Use the property of value function, we can get

$$f(h_1) = -\lambda w(p_1)v^+(s-l)h_1 + w(p_2)v^+(u-s)h_1.$$

For simplicity, we denote $v^+ = v$. Then by first order derivative, we have

$$f'(h_1) = -\lambda w(p_1)(s-l)v'((s-l)h_1) + w(p_2)(u-s)v'((u-s)h_1),$$

and $f'(h_1) = 0$, which implies

$$(3.4) \quad \frac{\lambda w(p_1)(s-l)}{w(p_2)(u-s)} = \frac{v'((u-s)h_1)}{v'((s-l)h_1)}.$$

If (3.2) holds ,

$$\frac{v'((u-s)h_1)}{v'((s-l)h_1)} < 1.$$

This indicates that $(u-s)h_1 > (s-l)h_1$. Moreover, due to $u-s > s-l$, we can get $h_1 > 0$. Use the same argument we have if (3.3) holds, $h_1 > 0$. Finally, we can check that

$$f''(h_1^*) = -\lambda w(p_1)(s-l)^2v''((s-l)h_1^*) + w(p_2)(u-s)^2v''((u-s)h_1^*) < 0,$$

where h_1^* satisfies the equation (3.4). Thus such h_1^* is the optimal trading strategy.

Conversely, since $h_1^* > 0$ is the optimal strategy, h_1^* must satisfies the equation

$$\frac{\lambda w(p_1)(s-l)}{w(p_2)(u-s)} = \frac{v'((u-s)h_1^*)}{v'((s-l)h_1^*)}.$$

And we separate it into two cases.

Case 1: If $u - s > s - l$ holds, $v'((u - s)h_1^*) < v'((s - l)h_1^*)$ must be true. This implies

$$\lambda w(p_1)(s - l) < w(p_2)(u - s).$$

Case 2: If $u - s < s - l$ holds, $v'((u - s)h_1^*) > v'((s - l)h_1^*)$ must be true. This implies

$$\lambda w(p_1)(s - l) > w(p_2)(u - s).$$

We complete the proof.

In this theorem, we have a result that if an investor makes a decision in the sense of cumulative prospect theory, he is willing to buy the stock only when (3.1) holds. However, it is independent of the form of value function.

Next, we give a one-period model example that was given as above. Our main goal is that find out the optimal strategies in the senses of expected utility theory and cumulative prospect theory. The results of this example support Theorem 1 and Theorem 2, stated before.

Example 1. Consider the market model, set up as before in the beginning of this section. For simplicity we denote it by $b = \begin{pmatrix} 1 \\ s \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ l & u \end{pmatrix}$ where b and D are called price vector and payoff matrix, respectively. Moreover, let $p_1 = p$ and then $p_2 = 1 - p$.

Suppose that the value function which an investor takes is given by

$$v(x) = \begin{cases} 1 - \exp(-\theta x) & x \geq 0 \\ -\lambda(1 - \exp(\theta x)) & x < 0 \end{cases}$$

Furthermore we assume that the probability weighting functions of gains and losses are the same, with form

$$w^+(p) = w^-(p) = \frac{p^r}{(p^r + (1 - p)^r)^{\frac{1}{r}}}$$

In the sense of expected utility theory, we want to find out the strategy $\bar{h} = (h_0, h_1)$ such that

$$f(h_0, h_1) = p(1 - \exp(-\theta(h_0 + lh_1))) + (1 - p)(1 - \exp(-\theta(h_0 + uh_1))),$$

to reach the maximal value that subjects to $h_0 + sh_1 = 0$. It is not difficult to get that

$$h_1 = \frac{1}{\theta(u-l)} \ln \frac{(1-p)(u-s)}{p(s-l)}.$$

By second order derivative test, we have

$$f''(h_1) = -\theta^2(p(s-l)^2 \exp(\theta h_1(s-l)) + (1-p)(u-s)^2 \exp(-\theta h_1(u-s))) < 0.$$

Therefore, the optimal trading strategy is

$$\bar{h}^* = \left(-\frac{s}{\theta(u-l)} \ln \frac{(1-p)(u-s)}{p(s-l)}, \frac{1}{\theta(u-l)} \ln \frac{(1-p)(u-s)}{p(s-l)} \right).$$

And we can find out that if $(1-p)(u-s) > p(s-l)$ holds, the optimal trading strategy $h_1^* > 0$, that is, an investor is willing to buy the stock if and only if the average of gains is greater than that of losses.

Next, we consider the same market model under cumulative prospect theory. Since in the version of cumulative prospect theory, it is defined over gains and losses relative to a specific reference point instead of final wealth. In this case, the optimal trading strategy is $\bar{h} = (h_0, h_1)$ such that

$$f(h_1) = c(p)(1-p)^r(1 - \exp(-\theta(u-s)h_1)) - \lambda c(p)p^r(1 - \exp(\theta(l-s)h_1)),$$

where

$$c(p) = \frac{1}{(p^r + (1-p)^r)^{\frac{1}{r}}},$$

to reach the maximal value that constrains to $h_0 + sh_1 = 0$. By first order derivative, we have

$$f'(h_1) = c(p)\theta((1-p)^r(u-s) \exp(-\theta(u-s)h_1) - \lambda p^r(s-l) \exp(-\theta(s-l)h_1)),$$

and if $f'(h_1) = 0$ implies

$$\ln(1-p)^r(u-s) - \theta(u-s)h_1 = \ln \lambda p^r(s-l) - \theta(s-l)h_1.$$

Thus the optimal trading strategy is

$$\bar{h}^* = \left(-\frac{s}{\theta(u+l-2s)} \ln \frac{(1-p)^r(u-s)}{\lambda p^r(s-l)}, \frac{1}{\theta(u+l-2s)} \ln \frac{(1-p)^r(u-s)}{\lambda p^r(s-l)} \right).$$

And we can find out that if

$$(3.5) \quad ((1-p)^r(u-s) - \lambda p^r(s-l))(\lambda p^r - (1-p)^r) > 0$$

holds, $h_1^* > 0$. In this case, it is easy to check $f''(h_1^*) < 0$.

Remark 2. If (3.5) holds, one of

$$(1-p)^r(u-s) - \lambda p^r(s-l) > 0 \quad \text{where} \quad \frac{\lambda p^r}{(1-p)^r} > 1$$

or

$$(1-p)^r(u-s) - \lambda p^r(s-l) < 0 \quad \text{where} \quad \frac{\lambda p^r}{(1-p)^r} < 1$$

must be true.

According to this example, the results support the above two theorems. For investor who uses the version of expected utility theory to make a decision, he is willing to buy the risky asset if $(1-p)(u-s) > p(s-l)$ holds. In the sense of cumulative prospect theory, if (3.5) holds, the investor is willing to buy the risky asset more than riskless asset. Moreover, if we know the form of value function and probability weighting function that the investor takes, we can compute the optimal trading strategy. Then we want to find out the relation of optimal trading strategies in the sense of expected utility theory and cumulative prospect theory.

Remark 3. Suppose that we take the probability weighting function with the form given by

$$w^+(p) = w^-(p) = \frac{p^r}{(p^r + (1-p)^r)^{\frac{1}{r}}}$$

where $r < 1$, and assume that the sensitivity of losses is about 2.25, which proposed by Kahneman and Tversky. If $\frac{1}{3} < p < \frac{1}{2}$ holds, we have

$$\frac{\lambda w(p)}{w(1-p)} = \lambda \left(\frac{p}{1-p}\right)^r > 1.$$

In this situation, if $w(1-p)(u-s) > \lambda w(p)(s-l)$ holds, $(1-p)(u-s) > p(s-l)$ must hold at the same time.

The main result is that if $w(1-p)(u-s) > \lambda w(p)(s-l)$ where $\frac{1}{3} < p < \frac{1}{2}$, an investor is willing to buy the risky asset in the sense of expected utility theory. That is, an investor is willing to buy the stock in these two senses.

Up to now we only consider the market model with two states space, then we take four states market into account. And give an example as following.

Example 2. Suppose the market model is similar to the above example, that the value function and probability weighting function are the same as above, respectively. But in this example we may assume that there are four states in time 1, called ω_1 , ω_2 , ω_3 and ω_4 , with probability p_1 , p_2 , p_3 and p_4 , respectively. The stock price at time 0 is $S_0 = s$ and price in time 1 is $S_1(\omega_1) = l_1$, $S_1(\omega_2) = l_2$, $S_1(\omega_3) = u_1$ and $S_1(\omega_4) = u_2$, where $l_1 < l_2 < s < u_1 < u_2$.

(1) In the sense of expected utility theory, let

$$\begin{aligned} f(h_0, h_1) = & p_1(1 - \exp(-\theta(h_0 + l_1 h_1))) + p_2(1 - \exp(-\theta(h_0 + l_2 h_1))) \\ & + p_3(1 - \exp(-\theta(h_0 + u_1 h_1))) + p_4(1 - \exp(-\theta(h_0 + u_2 h_1))) \end{aligned}$$

we want to find $\bar{h} = (h_0, h_1)$ such that $f(h_0, h_1)$ reaches the maximum, which subjects to $h_0 + sh_1 = 0$. Then we transfer $f(h_0, h_1)$ into

$$f(h_1) = 1 - \exp(-\theta s h_1) (p_1 \exp(-\theta l_1 h_1) + p_2 \exp(-\theta l_2 h_1) + p_3 \exp(-\theta u_1 h_1) + p_4 \exp(-\theta u_2 h_1)).$$

By first order derivative we have

$$\begin{aligned}
f'(h_1) &= -\theta s e^{\theta s h_1} (p_1 e^{-\theta l_1 h_1} + p_2 e^{-\theta l_2 h_1} + p_3 e^{-\theta u_1 h_1} + p_4 e^{-\theta u_2 h_1}) \\
&\quad + \theta e^{\theta s h_1} (l_1 p_1 e^{-\theta l_1 h_1} + l_2 p_2 e^{-\theta l_2 h_1} + u_1 p_3 e^{-\theta u_1 h_1} + u_2 p_4 e^{-\theta u_2 h_1}) \\
&= \theta e^{\theta s h_1} ((l_1 - s) p_1 e^{-\theta l_1 h_1} + (l_2 - s) p_2 e^{-\theta l_2 h_1} \\
&\quad + (u_1 - s) p_3 e^{-\theta u_1 h_1} + (u_2 - s) p_4 e^{-\theta u_2 h_1}).
\end{aligned}$$

In order to obtain the extreme value, $f'(h_1) = 0$ must hold. In other words, if h_1^* is optimal strategy, it needs to satisfy the equation

$$(u_1 - s) p_3 e^{-\theta u_1 h_1^*} + (u_2 - s) p_4 e^{-\theta u_2 h_1^*} = (s - l_1) p_1 e^{-\theta l_1 h_1^*} + (s - l_2) p_2 e^{-\theta l_2 h_1^*}.$$

Finally, we only need to check $f''(h_1^*) < 0$. Since

$$\begin{aligned}
f''(h_1) &= \theta^2 s e^{\theta s h_1} (-\theta) ((l_1 - s) l_1 p_1 e^{-\theta l_1 h_1} + (l_2 - s) l_2 p_2 e^{-\theta l_2 h_1} \\
&\quad + (u_1 - s) u_1 p_3 e^{-\theta u_1 h_1} + (u_2 - s) u_2 p_4 e^{-\theta u_2 h_1})
\end{aligned}$$

and $l_1 < l_2 < s < u_1 < u_2$, we have $f''(h_1^*) < 0$. Therefore h_1^* which satisfies the equation

$$(u_1 - s) p_3 e^{-\theta u_1 h_1^*} + (u_2 - s) p_4 e^{-\theta u_2 h_1^*} = (s - l_1) p_1 e^{-\theta l_1 h_1^*} + (s - l_2) p_2 e^{-\theta l_2 h_1^*}$$

is the optimal trading strategy.

(2) Under cumulative prospect theory, we add the assumption, $u_1 - s > s - l_1$, to this market model. Our main goal is to find the strategy $\bar{h} = (h_0, h_1)$ such that

$$\begin{aligned}
f(h_1) &= (w(p_1))(-\lambda(1 - e^{\theta(l_1-s)h_1})) + (w(p_1 + p_2) - w(p_1))(-\lambda(1 - e^{\theta(l_2-s)h_1})) \\
&\quad + (w(p_3 + p_4) - w(p_4))(1 - e^{-\theta(u_1-s)h_1}) + (w(p_4))(1 - e^{-\theta(l_1-s)h_1})
\end{aligned}$$

reaches the maximal value, subjected to $h_0 + s h_1 = 0$. For convenience we give some notations as following: $w(p_1) = \bar{p}_1$, $w(p_1 + p_2) - w(p_1) = \bar{p}_2$, $w(p_3 + p_4) - w(p_4) =$

\bar{p}_3 , and $w(p_4) = \bar{p}_4$. By first order derivative, we have

$$\begin{aligned} f'(h_1) &= \lambda \bar{p}_1 \theta (l_1 - s) e^{\theta(l_1-s)h_1} + \lambda \bar{p}_2 \theta (l_2 - s) e^{\theta(l_2-s)h_1} \\ &\quad + \bar{p}_3 \theta (u_1 - s) e^{-\theta(u_1-s)h_1} + \bar{p}_4 \theta (u_2 - s) e^{-\theta(u_2-s)h_1}. \end{aligned}$$

Moreover, $f'(h_1) = 0$ only if h_1 satisfies the equation

$$\begin{aligned} &\bar{p}_3 (u_1 - s) e^{-\theta(u_1-s)h_1} + \bar{p}_4 (u_2 - s) e^{-\theta(u_2-s)h_1} \\ &= \lambda \bar{p}_1 (s - l_1) e^{-\theta(s-l_1)h_1} + \lambda \bar{p}_2 (s - l_2) e^{-\theta(s-l_2)h_1}. \end{aligned}$$

At last, we need to check that if h_1^* such that

$$\begin{aligned} &\bar{p}_3 (u_1 - s) e^{-\theta(u_1-s)h_1^*} + \bar{p}_4 (u_2 - s) e^{-\theta(u_2-s)h_1^*} \\ &= \lambda \bar{p}_1 (s - l_1) e^{-\theta(s-l_1)h_1^*} + \lambda \bar{p}_2 (s - l_2) e^{-\theta(s-l_2)h_1^*} \end{aligned}$$

holds, implies $f''(h_1^*) < 0$. By second order derivative, we have

$$\begin{aligned} f''(h_1) &= \theta^2 (\lambda \bar{p}_1 (s - l_1)^2 e^{-\theta(s-l_1)h_1} + \lambda \bar{p}_2 (s - l_2)^2 e^{-\theta(s-l_2)h_1} \\ &\quad - \bar{p}_3 (u_1 - s)^2 e^{-\theta(u_1-s)h_1} - \bar{p}_4 (u_2 - s)^2 e^{-\theta(u_2-s)h_1}). \end{aligned}$$

Since $u_1 - s > s - l_1$ implies $s - l_2 < s - l_1 < u_1 - s < u_2 - s$, we can get that

$$\begin{aligned} &\bar{p}_3 (u_1 - s)^2 e^{-\theta(u_1-s)h_1} + \bar{p}_4 (u_2 - s)^2 e^{-\theta(u_2-s)h_1} \\ &> \lambda \bar{p}_1 (s - l_1)^2 e^{-\theta(s-l_1)h_1} + \lambda \bar{p}_2 (s - l_2)^2 e^{-\theta(s-l_2)h_1}, \end{aligned}$$

i.e. $f''(h_1^*) < 0$. Therefore h_1^* which satisfies the following equation

$$\begin{aligned} &\bar{p}_3 (u_1 - s) e^{-\theta(u_1-s)h_1^*} + \bar{p}_4 (u_2 - s) e^{-\theta(u_2-s)h_1^*} \\ &= \lambda \bar{p}_1 (s - l_1) e^{-\theta(s-l_1)h_1^*} + \lambda \bar{p}_2 (s - l_2) e^{-\theta(s-l_2)h_1^*} \end{aligned}$$

is the optimal trading strategy.

The main point of this section is that when we are making decision under expected utility theory, the optimal strategy is different from the optimal strategy in the sense of cumulative prospect theory. Under expected utility theory we only concern about the final wealth and it says that investors are risk aversion in choice between risky investments. On the other hand, in the sense of cumulative prospect theory the investor's evaluation of risk depends on gains or losses relative to a reference point; furthermore, the value function is concave for gains and convex for losses, and steeper for losses than for gains. In other words, investors are risk aversion for gains and risk seeking for losses. Further, investors are more sensitive for losses than for gains. Therefore the optimization strategy in the sense of cumulative prospect theory depends on the degree of sensitivity of losses.

3.2. Reference Point Effect

Since the investor evaluates the prospect in the sense of cumulative prospect theory depending on gains and losses relative to a reference point rather than on final wealth; moreover, the investor's attitudes toward risk are different from gains and losses. Therefore the reference point would affect the investor's trading strategy. Furthermore, the reference point is decided by investor's subjective feeling, such as the past experience, and different from people to people. In this section we talk about the influence of reference point on trading strategy.

Consider a one-period market model in which time points are denoted by 0 and 1. Suppose there are two kinds of investors: A and B. Both of them take the same form of value function $v(x)$ which is concave for gains and convex for losses, and x is the wealth change. Assume that before time 0, both A and B made wrong decisions and suffered a loss of w_0 . Furthermore suppose that both of them make decision under cumulative prospect theory. However, they take different attitude toward this prior loss. Investor A only takes the current wealth X_0 into account and takes X_0 as the reference point. But investor B cares about the prior loss and tries to make

up for the prior loss with gains in time 1. Hence investor B takes $X_0 + w_0$ as the reference point.

In such financial market model, there exists a simple lottery, denoted by $(-x, p; y, 1-p)$, where $x > 0$, $y > 0$, and $0 < p < 1$. This means that the lottery has probability p to lose x , and gain y with probability $1 - p$.

Then the value of this lottery for investor A, who takes the current wealth as the reference point is

$$F = w(p)v(-x) + w(1-p)v(y),$$

where $w(p)$ is a probability weighting function. If investor A does not take any action, his wealth change is 0 and thus $v(0) = 0$.

For investor B who takes the prior loss into account, in other words the reference point, he takes, is $X_0 + w_0$. The value of this lottery for investor B is

$$F_{-w_0} = w(p)v(-x - w_0) + w(1-p)v(y - w_0) - v(-w_0) \quad \text{if } y - w_0 \geq 0,$$

or

$$F_{-w_0} = w(p)v(-x - w_0) + (1 - w(p))v(y - w_0) - v(-w_0) \quad \text{if } y - w_0 < 0,$$

where $v(-w_0)$ is the value when investor B does not take any action.

The main point of this section is to compare the optimal amount of lotteries that investor A and investor B are willing to hold, respectively.

Theorem 3. If the gain of lottery is greater than two times of the prior loss, the value of this lottery for investor B is greater than for investor A. That is, if $y > 2w_0$ holds, we can get $F_{-w_0} > F$.

PROOF. Since $y > 2w_0$, we have

$$F_{-w_0} = w(p)v(-x - w_0) + w(1-p)v(y - w_0) - v(-w_0).$$

And because of the property of probability weighting function

$$w(p) + w(1 - p) \leq 1,$$

we can get

$$\begin{aligned} F_{-w_0} &\geq w(p)v(-x - w_0) + w(1 - p)v(y - w_0) - (w(p) + w(1 - p))v(-w_0) \\ &= w(p)(v(-x - w_0) - v(-w_0)) + w(1 - p)(v(y - w_0) - v(-w_0)). \end{aligned}$$

Without loss of generality, let $v(0) = 0$.

If $w_0 \leq x$, we can get

$$v(-x - w_0) - v(-w_0) - v(-x) = v(-x - w_0) - v(-x) - v(-w_0) + v(0).$$

By mean value theorem, there exist c_1, c_2 , where $-x - w_0 < c_1 < -x$ and $-w_0 < c_2 < 0$ such that

$$v(-x - w_0) - v(-x) = (-w_0)v'(c_1)$$

and

$$v(-w_0) - v(0) = (-w_0)v'(c_2).$$

Since $w_0 \leq x$ implies $c_2 > c_1$ and the value function is convex for losses, i.e. $v' > 0$ and $v'' > 0$, we can acquire

$$v(-x - w_0) - v(-x) - v(-w_0) + v(0) = w_0(v'(c_2) - v'(c_1)) > 0.$$

Therefore we have the main result

$$v(-x - w_0) - v(-w_0) > v(-x).$$

If $w_0 > x$, we have

$$v(-x - w_0) - v(-w_0) - v(-x) = v(-x - w_0) - v(-w_0) - v(-x) + v(0).$$

By the same way, we can find c_1^* and c_2^* , where $-x - w_0 < c_1^* < -w_0$ and $-x < c_2^* < 0$, such that

$$v(-x - w_0) - v(-w_0) = (-x)v'(c_1^*)$$

and

$$v(-x) - v(0) = (-x)v'(c_2^*)$$

Since $w_0 > x$ and the value function is convex for losses, we obtain

$$v(-x - w_0) - v(-w_0) > v(-x).$$

Next use the similar argument and loss aversion property, $-v(-w_0) \geq v(w_0)$, we have

$$\begin{aligned} & v(y - w_0) - v(-w_0) - v(y) = v(y - w_0) - v(y) - v(-w_0) \\ & \geq v(y - w_0) - v(y) + v(w_0) = v(y - w_0) - v(y) + v(w_0) - v(0) \\ & = v'(c_3)(-w_0) + v'(c_4)(-w_0) = w_0(v'(c_4) - v'(c_3)), \end{aligned}$$

where $y - w_0 < c_3 < y$ and $0 < c_4 < w_0$. Owing to $y > 2w_0$ we have $c_4 < c_3$, and besides v is concave for gains. Then we can get

$$v(y - w_0) - v(-w_0) > v(y).$$

Therefore

$$\begin{aligned} F_{-w_0} &= w(p)v(-x - w_0) + w(1 - p)v(y - w_0) - v(-w_0) \\ &\geq w(p)v(-x) + w(1 - p)v(y) = F. \end{aligned}$$

We complete the proof.

The main point of this theorem is that if the gain of the lottery is large enough, the value of this lottery for investor B who does not want to accept the reality of the prior loss is greater than for investor A who accepts the reality and takes the current wealth as the reference point.

Then we interest in seeking out the difference of optimal trading strategies for investor A and investor B that takes different reference point into consideration.

Suppose that $h^* > 0$ is the optimal amount of risky assets that investor A is willing to buy, and $h_{-w_0}^*$ is the optimal amount of risky assets that investor B is willing to hold. Our main goal is to compare the number of h^* and $h_{-w_0}^*$. Before this we first give a lemma which describes one property of $h_{-w_0}^*$. The statement is as following.

Lemma 1. If the optimal amount of risky asset that investor B is willing to hold is greater than 0, i.e. $h_{-w_0}^* > 0$, $-w_0 + yh_{-w_0}^* \geq 0$ must hold.

PROOF. Suppose that $-w_0 + yh_{-w_0}^* < 0$ holds, and we define $\Delta > 0$ to be a small unit of asset such that $yh_{-w_0}^* + \Delta y - w_0 < 0$. Then we compare the profit of portfolio $h_{-w_0}^* - \Delta$ with $h_{-w_0}^*$. Since $h_{-w_0}^*$ is the optimal strategy, we have

$$\begin{aligned} & w(p)v(-xh_{-w_0}^* + \Delta x - w_0) + (1 - w(p))v(yh_{-w_0}^* - \Delta y - w_0) \\ \leq & w(p)v(-xh_{-w_0}^* - w_0) + (1 - w(p))v(yh_{-w_0}^* - w_0) \end{aligned}$$

By rearrangement, we get

$$\begin{aligned} & (1 - w(p))(v(yh_{-w_0}^* - w_0) - v(yh_{-w_0}^* - \Delta y - w_0)) \\ \geq & w(p)(v(-xh_{-w_0}^* + \Delta x - w_0) - v(-xh_{-w_0}^* - w_0)). \end{aligned}$$

Because $yh_{-w_0}^* + \Delta y - w_0 < 0$ and $v(x)$ is a convex function defined on $x < 0$, by the property of convex function and mean value theorem we have

$$\begin{aligned} & v(yh_{-w_0}^* - w_0) - v(yh_{-w_0}^* - \Delta y - w_0) \\ \leq & v(yh_{-w_0}^* + \Delta y - w_0) - v(yh_{-w_0}^* - w_0) \end{aligned}$$

Therefore, $h = -1$ is the optimal strategy for the hedger in the sense of expected utility theory.

When the investor evaluates the value of prospects in the sense of expected utility theory, he would accept the prior loss and only cares about the final wealth. Besides, his attitude toward risk is risk aversion. Hence, under expected utility theory the optimal strategy for the hedger is full hedging.

Unlike conventional expected utility theory, cumulative prospect theory replaced the utility function with the value function, $v(x)$. Moreover, it used decision weighting function, $\pi(p)$, instead of probability measure. In the following we discover that under cumulative prospect theory the optimal strategy for the hedger is much more complicated.

Theorem 6. Under cumulative prospect theory, the optimal strategy for the hedger is as following:

(1) In the case $h \leq -1$,

(i) if $E(S_1) > w_0$,

$$h^* = \begin{cases} -1 - \frac{1}{\theta(u-l)} \ln \frac{pw(1-p)}{(1-p)(1-w(1-p))} & \text{when } w(1-p) \geq (1-p) \\ -1 & \text{when } w(1-p) < (1-p) \end{cases}$$

(ii) if $E(S_1) \leq w_0$,

$$h^* = \begin{cases} \frac{l-w_0}{p(u-l)} & \text{when } \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda(1-p)w(p)}{pw(1-p)} \geq 0 \\ -\infty & \text{when } \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda(1-p)w(p)}{pw(1-p)} < 0 \end{cases}.$$

(2) In the case $h \geq -1$,

(i) if $E(S_1) > w_0$,

$$h^* = \begin{cases} -1 + \frac{1}{\theta(u-l)} \ln \frac{(1-p)w(p)}{p(1-w(p))} & \text{when } w(p) \geq p \\ -1 & \text{when } w(p) < p \end{cases}$$

(ii) if $E(S_1) \leq w_0$,

$$h^* = \begin{cases} \frac{w_0 - u}{(1-p)(u-l)} & \text{when } \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda p w (1-p)}{(1-p)w(p)} \geq 0 \\ \infty & \text{when } \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda p w (1-p)}{(1-p)w(p)} < 0 \end{cases}.$$

PROOF. Suppose that $h \leq -1$ and given that $E(S_1) \leq w_0$. When $h \in \{h \mid w_l - w_0 \leq 0\}$,

$$\begin{aligned} F_{-w_0} &= w(p)(-\lambda(1 - \exp(\theta(w_u - w_0)))) + (1 - w(p))(-\lambda(1 - \exp(\theta(w_l - w_0)))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)) \\ &= -\lambda(w(p) + w(p) \exp(\theta(w_u - w_0)) + (1 - w(p))) + \lambda((1 - w(p)) \exp(\theta(w_l - w_0))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)) \end{aligned}$$

By first order derivative, we have

$$\begin{aligned} \frac{dF_{-w_0}}{dh} &= \lambda w(p) \theta (1-p)(u-l) \exp(\theta(w_u - w_0)) + \lambda (1-w(p)) \theta (-p)(u-l) \exp(\theta(w_l - w_0)) \\ &= \lambda \theta (u-l) \exp(-\theta w_0) ((1-p)w(p) \exp(\theta w_u) - p(1-w(p)) \exp(\theta w_l)). \end{aligned}$$

Moreover, in order to guarantee $\frac{dF_{-w_0}}{dh} = 0$, h must satisfies the following equation

$$(1-p)w(p) \exp(\theta(u + h(1-p)(u-l))) = p(1-w(p)) \exp(\theta(l - hp(u-l))).$$

And this implies

$$(3.8) \quad h = -1 - \frac{1}{\theta(u-l)} \ln \frac{(1-p)w(p)}{p(1-w(p))}.$$

However, by second order derivative, we can obtain

$$\frac{d^2 F_{-w_0}}{dh^2} = \lambda \theta (u-l) \exp(-\theta w_0) ((1-p)^2 w(p) \theta (u-l) \exp(\theta w_u) + p^2 (1-w(p)) \theta (u-l) \exp(\theta w_l)) \geq 0.$$

Thus (3.8) is the utility minimizing point. Moreover, we can get that the maximum of F_{-w_0} in the set $\{h \mid w_l - w_0 \leq 0\}$ is reached on the boundary. When $h \in \{h \mid$

$$w_l - w_0 \geq 0\},$$

$$\begin{aligned} F_{-w_0} &= w(p)(-\lambda(1 - \exp(\theta(w_u - w_0)))) + w(1 - p)(1 - \exp(-\theta(w_l - w_0))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)) \\ &= -\lambda w(p)(1 - \exp(\theta(w_u - w_0))) + w(1 - p)(1 - \exp(-\theta(w_l - w_0))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)). \end{aligned}$$

By first order derivative, we get

$$\frac{dF_{-w_0}}{dh} = \theta(u - l)(\lambda(1 - p)w(p) \exp(\theta(w_u - w_0)) - pw(1 - p) \exp(-\theta(w_l - w_0))).$$

Case 1: When

$$(3.9) \quad \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda(1 - p)w(p)}{pw(1 - p)} \geq 0,$$

$\frac{dF_{-w_0}}{dh} \geq 0$, that is, F_{-w_0} is increasing corresponding to h . Thus the maximum of F_{-w_0} in the set $\{h \mid w_l - w_0 \geq 0\}$ is reached on the boundary $w_l - w_0 = 0$. In other words, the optimal hedging strategy is

$$h^* = \frac{l - w_0}{p(u - l)}.$$

Case 2: When

$$(3.10) \quad \theta(-2w_0 + w_u + w_0) + \ln \frac{\lambda(1 - p)w(p)}{pw(1 - p)} < 0,$$

F_{-w_0} is decreasing corresponding to h . Thus the maximum of F_{-w_0} in the set $\{h \mid w_l - w_0 \geq 0\}$ is $h^* = -\infty$.

Given that $E(S_1) \geq w_0$. When $h \in \{h \mid w_u - w_0 \geq 0\}$,

$$\begin{aligned} F_{-w_0} &= w(1-p)(1 - \exp(-\theta(w_l - w_0))) + (1 - w(1-p))(1 - \exp(-\theta(w_u - w_0))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)) \\ &= -w(1-p) \exp(-\theta(w_l - w_0)) + 1 - \exp(-\theta(w_l - w_0)) + w(1-p) \exp(-\theta(w_u - w_0)) \\ &\quad + \lambda(1 - \exp(-\theta w_0)). \end{aligned}$$

By first order derivative, we acquire

$$\frac{dF_{-w_0}}{dh} = \theta(u-l)(-pw(1-p) \exp(-\theta(w_l - w_0)) + (1-p)(1-w(p)) \exp(-\theta(w_u - w_0))).$$

In order to ensure $\frac{dF_{-w_0}}{dh} = 0$, h must satisfies the following equation

$$pw(1-p) \exp(-\theta(l - hp(u-l))) = (1-p)(1-w(1-p)) \exp(-\theta(u + h(1-p)(u-l))).$$

This equation implies

$$h = -1 - \frac{1}{\theta(u-l)} \ln \frac{pw(1-p)}{(1-p)(1-w(1-p))}.$$

In order to ensure such $h \leq -1$, we must add a condition $w(1-p) \geq (1-p)$.

Moreover, by second order derivative we have

$$\begin{aligned} \frac{d^2 F_{-w_0}}{dh^2} &= -\theta(u-l)(p^2 w(1-p)\theta(u-l) \exp(-\theta(w_l - w_0))) \\ &\quad -\theta(u-l)((1-p)^2(1-w(1-p))\theta(u-l) \exp(-\theta(w_u - w_0))) \\ &\leq 0. \end{aligned}$$

Therefore the optimal strategy for the hedger is

$$(3.11) \quad h^* = -1 - \frac{1}{\theta(u-l)} \ln \frac{pw(1-p)}{(1-p)(1-w(1-p))}$$

as $w(1-p) \geq (1-p)$. However, if $w(1-p) < (1-p)$ hold, optimal hedging strategy is reached on the boundary $h \leq -1$. In other words, optimal hedging strategy is

$$h^* = -1.$$

Suppose $h \geq -1$ and given that $E(S_1) \leq w_0$. When $h \in \{h \mid w_u - w_0 \leq 0\}$,

$$\begin{aligned} F_{-w_0} &= w(1-p)(-\lambda(1 - \exp(\theta(w_l - w_0)))) + (1 - w(1-p))(-\lambda(1 - \exp(\theta(w_u - w_0)))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)). \end{aligned}$$

By first order derivative, we obtain

$$\frac{dF_{-w_0}}{dh} = \lambda\theta(u-l)\exp(-\theta w_0)((1-p)(1-w(1-p))\exp(\theta w_u) - pw(1-p)\exp(\theta w_l)),$$

and $\frac{dF_{-w_0}}{dh} = 0$ only if

$$(3.12) \quad h = -1 + \frac{1}{\theta(u-l)} \ln \frac{pw(1-p)}{(1-p)(1-w(1-p))}$$

holds. In addition, by second order derivative we get

$$\frac{d^2 F_{-w_0}}{dh^2} = \lambda\theta^2(u-l)^2((1-p)^2(1-w(1-p))\exp(\theta w_u) + p^2w(1-p)\exp(\theta w_l)) \geq 0.$$

Hence, (3.12) is utility minimizing point. So we can get that the maximum of F_{-w_0} in the set $\{h \mid w_u - w_0 \leq 0\}$ is reached on the boundary, that is, $w_u - w_0 = 0$, which implies that the optimal strategy for the hedger is

$$h^* = \frac{w_0 - u}{(1-p)(u-l)}.$$

When $h \in \{h \mid w_u - w_0 \geq 0\}$,

$$\begin{aligned} F_{-w_0} &= w(1-p)(-\lambda(1 - \exp(\theta(w_l - w_0)))) + w(p)(1 - \exp(-\theta(w_u - w_0))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)). \end{aligned}$$

By first order derivative, we have

$$\frac{dF_{-w_0}}{dh} = \theta(u-l)(w(p)(1-p)\exp(-\theta(w_u - w_0)) - \lambda pw(1-p)\exp(\theta(w_l - w_0))).$$

Case 1. When

$$(3.13) \quad \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda p w(1-p)}{(1-p)w(p)} \geq 0,$$

we know that F_{-w_0} is a decreasing function corresponding to h . Therefore, the maximum of F_{-w_0} in the set $\{h \mid w_u - w_0 \geq 0\}$ is reached on the boundary, which means that

$$h^* = \frac{w_0 - u}{(1-p)(u-l)}$$

is the optimal strategy for the hedger in the futures market.

Case 2. When

$$(3.14) \quad \theta(-2w_0 + w_u + w_l) + \ln \frac{\lambda p w(1-p)}{(1-p)w(p)} < 0,$$

we get that F_{-w_0} is an increasing function corresponding to h . Hence the maximum of F_{-w_0} in the set $\{h \mid w_u - w_0 \geq 0\}$ is $h^* = \infty$.

Given that $E(S_1) \geq w_0$ and $h \in \{h \mid w_l - w_0 \geq 0\}$, then we have

$$\begin{aligned} F_{-w_0} &= w(p)(1 - \exp(-\theta(w_u - w_0))) + (1 - w(p))(1 - \exp(-\theta(w_l - w_0))) \\ &\quad + \lambda(1 - \exp(-\theta w_0)). \end{aligned}$$

By first order derivative, we obtain that $\frac{dF_{-w_0}}{dh} = 0$ implies

$$h = -1 + \frac{1}{\theta(u-l)} \ln \frac{(1-p)w(p)}{p(1-w(p))}.$$

In order to ensure such $h \geq -1$, $w(p) \geq p$ must hold. Then by second order derivative, we get

$$\frac{d^2 F_{-w_0}}{dh^2} = -\theta^2(u-l)^2 \exp(\theta w_0) ((1-p)^2 w(p) \exp(-\theta w_u) + p^2(1-w(p)) \exp(-\theta w_l)) \leq 0.$$

Therefore

$$(3.15) \quad h^* = -1 + \frac{1}{\theta(u-l)} \ln \frac{(1-p)w(p)}{p(1-w(p))}$$

is the optimal strategy for the hedger in the futures market as $w(p) \geq p$. However, if $w(p) < p$ holds, optimal hedging strategy is $h^* = -1$.

According to this theorem, we obtain that under cumulative prospect theory there exists three cases. If the prior losses w_0 are not sufficiently large, $w_0 < ES_1$, the optimal strategy for the hedger is h^* given in (3.11) in the case $h \leq -1$ and $w(1-p) \geq (1-p)$, and in the case $h \geq -1$ and $w(p) \geq p$ the optimal strategy h^* of the form (3.15). If the losses before time 0 satisfy (3.9) or (3.13), the investor will change his hedging strategy. If the prior losses are sufficiently large, and (3.10) or (3.14) holds, the investor will go crazy, and take large positions showing no consideration of risks.

Corollary 1. Optimal strategy for the hedger in the sense of expected utility theory is full hedging. Besides, under expected utility theory the optimal hedging strategy does not depend on the prior losses. On the other hand, in the view of cumulative prospect theory the optimal strategy for the hedger is more complicated and is related to the losses before time 0.

**Comparison of Optimization in the Sense of Expected
Utility Theory and Cumulative Prospect Theory II:
A Model with Transaction Cost**

In this chapter we introduce trading strategy with transaction costs in one period market model, which is derived from the model specified by Kabanov (2002). Suppose that the financial market is one period model, and our portfolio is (h_0, h_1) , which means that the number of shares of assets invested in bond and stock, respectively. Moreover, we assume that the initial wealth is $x_0 = \bar{h}_0 + s\bar{h}_1$, composed of bond and stock.

Suppose that the investor need to pay the transaction costs when they sell the stock, and consider the model with constant proportional transaction costs, denoted by $\bar{\lambda}$. Therefore, we find out that if we buy the stock at time 0, the value of the portfolio at time 0 after trading is $v_0 = h_0 + sh_1$, and in this situation $h_1 > \bar{h}_1$ must hold. If we sell the stock at time 0, the value of the portfolio at time 0 after trading is $v_0^* = h_0 + sh_1 - \bar{\lambda}s(\bar{h}_1 - h_1)$, and in this situation $h_1 < \bar{h}_1$ must hold.

In the following of this section we assume that in this market model our portfolio values are only affected by the asset price fluctuation and transaction costs. Thus we get $v_0 = x_0$ and $v_0^* = x_0 - \bar{\lambda}s(\bar{h}_1 - h_1)$. Our main task is to find out optimal trading strategy with transaction costs in the sense of expected utility theory and cumulative prospect theory, respectively.

Example 3. (Constant absolute risk aversion CARA)

Under expected utility theory, we consider the risk-averse utility function given by

$$U(x) = 1 - \exp(-\theta x),$$

where $\theta > 0$ is absolute risk aversion and be a constant. Moreover, the market model is sat up as the first section in chapter 3.

(1)If we buy the stock at time 0, then the final wealth is

$$w = \begin{cases} w_l = x_0 + h_1(l - s) & \text{with probability } p \\ w_u = x_0 + h_1(u - s) & \text{with probability } 1 - p. \end{cases}$$

Under expected utility theory, we have to maximize the function $f(h_1)$, defined by

$$f(h_1) = pU(x_0 + h_1(l - s)) + (1 - p)U(x_0 + h_1(u - s)),$$

to get the optimal strategy h_1^* . Using first order derivative, we have

$$f'(h_1) = p\theta(l - s) \exp(-\theta(x_0 + h_1(l - s))) + (1 - p)\theta(u - s) \exp(-\theta(x_0 + h_1(u - s))),$$

and let $f'(h_1) = 0$ we get

$$(4.1) \quad h_1 = \frac{1}{\theta(u - l)} \ln \frac{(1 - p)(u - s)}{p(s - l)}.$$

In order to guarantee $h_1 > 0$, we should add the condition, $(1 - p)(u - s) > p(s - l)$, to this market model. Moreover, by second order derivative, we get

$$f''(h_1) = -\theta^2(p(s - l)^2 \exp(-\theta w_l) + (1 - p)(u - s)^2 \exp(-\theta w_u)) \leq 0.$$

Therefore we can get that if we buy the stock at time 0, the optimal trading strategy is

$$h_1^* = \frac{1}{\theta(u - l)} \ln \frac{(1 - p)(u - s)}{p(s - l)} > \bar{h}_1.$$

(2) If we sell the stock at time 0, then the final wealth is

$$w = \begin{cases} w_l = x_0 + h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1) & \text{with probability } p \\ w_u = x_0 + h_1(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1) & \text{with probability } 1 - p \end{cases}$$

Suppose that $l - s + \bar{\lambda}s \leq 0$. Under expected utility theory, we have to maximize the function $f(h_1)$, defined by

$$f(h_1) = pU(x_0 + h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1)) + (1 - p)U(x_0 + h_1(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1)),$$

to acquire the optimal strategy h_1^{**} . Using first order derivative, we have $f'(h_1) = 0$

$$(4.2) \quad h_1 = \frac{1}{\theta(u - l)} \ln \frac{(1 - p)(u - s + \bar{\lambda}s)}{p(s - l - \bar{\lambda}s)}.$$

Moreover, by second order derivative, we get

$$f''(h_1) = -\theta^2(p(s - l - \bar{\lambda}s)^2 \exp(-\theta w_l) + (1 - p)(u - s + \bar{\lambda}s)^2 \exp(-\theta w_u)) \leq 0.$$

Therefore we can get that if we sell the stock at time 0, the optimal trading strategy is

$$h_1^{**} = \frac{1}{\theta(u - l)} \ln \frac{(1 - p)(u - s + \bar{\lambda}s)}{p(s - l - \bar{\lambda}s)} < \bar{h}_1.$$

From (1), (2), we conclude that if an investor who wants to buy the stock at time 0, he may choose the strategy $h_1^* = (4.1)$ at time 0 to reach the maximum profit, and if an investor who wants to sell the stock at time 0 he may choose the strategy $h_1^{**} = (4.2)$ at time 0 to reach the maximum profit.

Remark 4. From above example, we have a result that if $\bar{h}_1 > h_1^*$, the investor will not buy the stock at time 0, and if $\bar{h}_1 < h_1^{**}$, the investor is not willing to sell the stock at time 0. Moreover, because of $(1 - p)(u - s) > p(s - l)$ and

$$\frac{(1 - p)(u - s + \bar{\lambda}s)}{p(s - l - \bar{\lambda}s)} > \frac{(1 - p)(u - s)}{p(s - l)},$$

we have $h_1^{**} > h_1^*$. Thus we can conclude that the investor will neither buy nor sell the stock at time 0 when the number of shares of the stock of their initial wealth is in the interval

$$\left[\frac{1}{\theta(u-l)} \ln \frac{(1-p)(u-s)}{p(s-l)}, \frac{1}{\theta(u-l)} \ln \frac{(1-p)(u-s+\bar{\lambda}s)}{p(s-l-\bar{\lambda}s)} \right],$$

called "no trading" interval.

The main point of this remark is that if the optimal trading strategy in the case of selling the stock at time 0, h_1^{**} , is no less than the optimal strategy in the case of buying the stock at time 0, h_1^* , then there must exist a "no trading" interval.

The following theorem shows that no matter which utility function we choose in the sense of expected utility theory there must exist a "no trading" interval, however there is not a specific form of utility function. In other words, even though the forms of value function which an investor takes are different, there is a interval in which an investor will neither buy nor sell the stock at time 0.

Theorem 7. Assume that $l - (1 - \bar{\lambda})s < 0$. Under expected utility theory, if h_1^* and h_1^{**} exist, the inequality $h_1^{**} > h_1^*$ must be true. Thus there exists a "no trading" interval.

PROOF. Suppose that $h_1^* \geq h_1^{**}$. If we buy the stock at time 0 and the optimal strategy is h_1^* , under expected utility theory, we have

$$\begin{aligned} & pU(x_0 + h_1^*(l-s)) + (1-p)U(x_0 + h_1^*(u-s)) \\ & \geq pU(x_0 + h_1^{**}(l-s)) + (1-p)U(x_0 + h_1^{**}(u-s)), \end{aligned}$$

where U is a concave function. Rearranging the inequality, we have

$$\begin{aligned} & (1-p)(U(x_0 + h_1^*(u-s)) - U(x_0 + h_1^{**}(u-s))) \\ & \geq p(U(x_0 + h_1^{**}(l-s)) - U(x_0 + h_1^*(l-s))). \end{aligned}$$

Because of $h_1^* \geq h_1^{**}$, $h_1^* > \bar{h}_1$ and $h_1^{**} < \bar{h}_1$, we have

$$U(x_0 + h_1^*(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*)) - U(x_0 + h_1^*(u - s)) = U'(c_1)\bar{\lambda}s(h_1^* - \bar{h}_1) > 0,$$

and

$$U(x_0 + h_1^{**}(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**})) - U(x_0 + h_1^{**}(u - s)) = U'(c_2)(-\bar{\lambda}s(\bar{h}_1 - h_1^{**})) < 0,$$

where

$$c_1 \in (x_0 + h_1^*(u - s), x_0 + h_1^*(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*))$$

and

$$c_2 \in (x_0 + h_1^{**}(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**}), x_0 + h_1^{**}(u - s)).$$

Hence we obtain the following inequality

$$\begin{aligned} & U(x_0 + h_1^*(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*)) - U(x_0 + h_1^{**}(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**})) \\ & > U(x_0 + h_1^*(u - s)) - U(x_0 + h_1^{**}(u - s)). \end{aligned}$$

Next, use the similar argument, we have

$$\begin{aligned} & U(x_0 + h_1^{**}(l - s)) - U(x_0 + h_1^*(l - s)) \\ & > U(x_0 + h_1^{**}(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**})) - U(x_0 + h_1^*(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*)). \end{aligned}$$

Therefore, we can get that

$$\begin{aligned} & (1 - p)(U(x_0 + h_1^*(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*)) - U(x_0 + h_1^{**}(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**}))) \\ & > (1 - p)(U(x_0 + h_1^*(u - s)) - U(x_0 + h_1^{**}(u - s))) \\ & \geq p(U(x_0 + h_1^{**}(l - s)) - U(x_0 + h_1^*(l - s))) \\ & > p(U(x_0 + h_1^{**}(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**})) - U(x_0 + h_1^*(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*))). \end{aligned}$$

That is,

$$\begin{aligned} & pU(x_0 + h_1^*(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*)) + (1 - p)U(x_0 + h_1^*(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^*)) \\ & > pU(x_0 + h_1^{**}(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**})) + (1 - p)U(x_0 + h_1^{**}(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1^{**})) \end{aligned}$$

which is contradicted to that h_1^{**} is optimal strategy in the case of selling the stock at time 0. Thus we can obtain $h_1^{**} > h_1^*$.

This theorem shows that if the initial wealth is composed of bond and stock, and investors also take transaction costs into account. For the investor who wants to find out the optimal strategy in the sense of expected utility theory, should consider two cases, buying or selling the stock at time 0. Furthermore, we know that in this situation there exists a "no trading" interval.

Next, we discuss the optimal strategy with transaction costs in the version of cumulative prospect theory. Suppose that the market model is the same as before.

Example 4. We consider the value function given by

$$v(x) = \begin{cases} 1 - \exp(-\theta x) & \text{if } x \geq 0 \\ -\lambda(1 - \exp(\theta x)) & \text{if } x < 0 \end{cases},$$

where $\theta > 0$ and $\lambda \geq 1$.

(1) If we buy the stock at time 0, $h_1 > \bar{h}_1$ and the final wealth is

$$w = \begin{cases} w_l = x_0 + h_1(l - s) & \text{with probability } p \\ w_u = x_0 + h_1(u - s) & \text{with probability } 1 - p \end{cases}.$$

Under cumulative prospect theory, the value function is defined over gains and losses relative to reference point instead of final wealth. In this example we take initial wealth as reference point. Thus when we want to acquire the optimal strategy in the version of cumulative prospect theory, we need to maximize $f(h_1)$ to get the

optimal strategy h_1^* , where

$$f(h_1) = w(p)(-\lambda(1 - \exp(\theta h_1(l - s)))) + w(1 - p)(1 - \exp(-\theta h_1(u - s))).$$

Due to first order derivative we have

$$f'(h_1) = \theta(\lambda w(p)(l - s) \exp(\theta h_1(l - s)) + w(1 - p)(u - s) \exp(-\theta h_1(l - s))),$$

and if $f'(h_1) = 0$ we have

$$h_1^* = \frac{1}{\theta(u + l - 2s)} \ln \frac{w(1 - p)(u - s)}{\lambda w(p)(s - l)}.$$

Moreover, it is easy to check $f''(h_1^*) < 0$. Hence we can guarantee that $h_1^* > \bar{h}_1$ is optimal strategy if we buy the stock at time 0 in the sense of cumulative prospect theory.

(2) If we sell the stock at time 0, $h_1 < \bar{h}_1$ and the final wealth is

$$w = \begin{cases} w_l = x_0 + h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1) & \text{with probability } p \\ w_u = x_0 + h_1(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1) & \text{with probability } 1 - p \end{cases}.$$

Suppose that $l - (1 - \bar{h}_1)s \leq 0$ and we take initial wealth as the reference point. We can transfer the final wealth, w , into

$$w^* = \begin{cases} w_l^* = h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1) & \text{with probability } p \\ w_u^* = h_1(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1) & \text{with probability } 1 - p \end{cases},$$

where w_l^* and w_u^* represent losses and gains, respectively. Thus in the sense of cumulative prospect theory our main goal is to figure out h_1^{**} which maximize the function

$$f(h_1) = pv(h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1)) + (1 - p)v(h_1(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1)).$$

By first order derivative, we have

$$\begin{aligned} f'(h_1) &= \theta(\lambda w(p)(l - s + \bar{\lambda}s) \exp(\theta h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1)) \\ &\quad + w(1 - p)(u - s + \bar{\lambda}s) \exp(-\theta(h_1(u - s) - \bar{\lambda}s(\bar{h}_1 - h_1)))) \end{aligned}$$

and if $f'(h_1) = 0$ we have

$$h_1^{**} = \frac{1}{\theta(u + l - 2(1 - \bar{\lambda})s)} \ln \frac{w(1 - p)(u - s + \bar{\lambda}s)}{\lambda w(p)(s - l - \bar{\lambda}s)} + \frac{2\bar{\lambda}s\bar{h}_1}{u + l - 2(1 - \bar{\lambda})s}.$$

Moreover we can get the inequality, $f''(h_1^{**}) < 0$, thus $h_1^{**} < \bar{h}_1$ is the optimal strategy if we sell the stock at time 0 in the sense of cumulative prospect theory.

By this example we conclude that if the investor wants to buy the stock at time 0 he may choose the strategy

$$h_1^* = \frac{1}{\theta(u + l - 2s)} \ln \frac{w(1 - p)(u - s)}{\lambda w(p)(s - l)}$$

to reach the maximum profit, and if the investor wants to sell the stock at time 0 he may choose the strategy

$$h_1^{**} = \frac{1}{\theta(u + l - 2(1 - \bar{\lambda})s)} \ln \frac{w(1 - p)(u - s + \bar{\lambda}s)}{\lambda w(p)(s - l - \bar{\lambda}s)} + \frac{2\bar{\lambda}s\bar{h}_1}{u + l - 2(1 - \bar{\lambda})s}$$

to reach the maximum profit.

Remark 5. Different from expected utility theory, we discover the fact that under cumulative prospect theory, the optimal strategy with transaction costs in the case of selling the stock at time 0 is relative to \bar{h}_1 .

Remark 6. Due to rearrange the following inequality

$$\frac{1}{\theta(u + l - 2(1 - \bar{\lambda})s)} \ln \frac{w(1 - p)(u - s + \bar{\lambda}s)}{\lambda w(p)(s - l - \bar{\lambda}s)} + \frac{2\bar{\lambda}s\bar{h}_1}{u + l - 2(1 - \bar{\lambda})s} < \bar{h}_1,$$

we have

$$\frac{1}{\theta(u + l - 2(1 - \bar{\lambda})s)} \ln \frac{w(1 - p)(u - s + \bar{\lambda}s)}{\lambda w(p)(s - l - \bar{\lambda}s)} < \frac{u + l - 2s}{u + l - 2(1 - \bar{\lambda})s} \bar{h}_1$$

and implies

$$(4.3) \quad \frac{1}{\theta(u+l-2s)} \ln \frac{w(1-p)(u-s+\bar{\lambda}s)}{\lambda w(p)(s-l-\bar{\lambda}s)} < \bar{h}_1.$$

Thus in above example if (4.3) holds, the investor will sell the stock at time 0 and the optimal strategy is

$$h_1^{**} = \frac{1}{\theta(u+l-2(1-\bar{\lambda})s)} \ln \frac{w(1-p)(u-s+\bar{\lambda}s)}{\lambda w(p)(s-l-\bar{\lambda}s)} + \frac{2\bar{\lambda}s\bar{h}_1}{u+l-2(1-\bar{\lambda})s}.$$

Moreover we get that the investor will neither buy nor sell the stock when \bar{h}_1 is in the interval

$$\left[\frac{1}{\theta(u+l-2s)} \ln \frac{w(1-p)(u-s)}{\lambda w(p)(s-l)}, \frac{1}{\theta(u+l-2s)} \ln \frac{w(1-p)(u-s+\bar{\lambda}s)}{\lambda w(p)(s-l-\bar{\lambda}s)} \right].$$

Following, we discuss the same question. If there exists a "no trading" interval in the sense of cumulative prospect theory, without specific form of value function ?

Theorem 8. Assume that $l - (1 - \bar{\lambda})s < 0$. Under cumulative prospect theory, if h_1^* and h_1^{**} exist, the inequality $h_1^{**} > h_1^*$ must be true. Therefore there exist a "no trading" interval.

PROOF. Suppose that $h_1^* \geq h_1^{**}$. If we buy the stock at time 0 and the optimal strategy is h_1^* , we have

$$\begin{aligned} & w(p)v^-(h_1^*(l-s)) + w(1-p)v^+(h_1^*(u-s)) \\ & \geq w(p)v^-(h_1^{**}(l-s)) + w(1-p)v^+(h_1^{**}(u-s)), \end{aligned}$$

where v^- is a convex function and v^+ is a concave function. Rearranging the above inequality, we get

$$\begin{aligned} & w(1-p)(v^+(h_1^*(u-s)) - v^+(h_1^{**}(u-s))) \\ & \geq w(p)(v^-(h_1^{**}(l-s)) - v^-(h_1^*(l-s))). \end{aligned}$$

Due to $h_1^* \geq h_1^{**}$, $h_1^* > \bar{h}_1$ and $h_1^{**} < \bar{h}_1$, we obtain

$$-\bar{\lambda}s\bar{h}_1 + h_1^*(u - s + \bar{\lambda}s) > h_1^*(u - s)$$

and

$$h_1^{**}(u - s) > -\bar{\lambda}s\bar{h}_1 + h_1^{**}(u - s + \bar{\lambda}s).$$

By mean value theorem, we have

$$\begin{aligned} & v^+(-\bar{\lambda}s\bar{h}_1 + h_1^*(u - s + \bar{\lambda}s)) - v^+(h_1^*(u - s)) \\ & > v^+(-\bar{\lambda}s\bar{h}_1 + h_1^{**}(u - s + \bar{\lambda}s)) - v^+(h_1^{**}(u - s)). \end{aligned}$$

That is,

$$\begin{aligned} & v^+(-\bar{\lambda}s\bar{h}_1 + h_1^*(u - s + \bar{\lambda}s)) - v^+(-\bar{\lambda}s\bar{h}_1 + h_1^{**}(u - s + \bar{\lambda}s)) \\ & > v^+(h_1^*(u - s)) - v^+(h_1^{**}(u - s)). \end{aligned}$$

Similarly, since $l - (1 - \bar{\lambda})s < 0$ we have

$$h_1^{**}(l - s) > -\bar{\lambda}s\bar{h}_1 + h_1^{**}(l - s + \bar{\lambda}s)$$

and

$$-\bar{\lambda}s\bar{h}_1 + h_1^*(l - s + \bar{\lambda}s) > h_1^*(l - s).$$

Using mean value theorem, we get

$$\begin{aligned} & v^-(h_1^{**}(l - s)) - v^-(h_1^*(l - s)) \\ & > v^-(-\bar{\lambda}s\bar{h}_1 + h_1^{**}(l - s + \bar{\lambda}s)) - v^-(-\bar{\lambda}s\bar{h}_1 + h_1^*(l - s + \bar{\lambda}s)). \end{aligned}$$

Combining above inequality, we can obtain

$$\begin{aligned}
& w(1-p)(v^+(-\bar{\lambda}s\bar{h}_1 + h_1^*(u-s+\bar{\lambda}s)) - v^+(-\bar{\lambda}s\bar{h}_1 + h_1^{**}(u-s+\bar{\lambda}s))) \\
> & w(1-p)(v^+(h_1^*(u-s)) - v^+(h_1^{**}(u-s))) \\
\geq & w(p)(v^-(h_1^{**}(l-s)) - v^-(h_1^*(l-s))) \\
> & w(p)(v^-(\bar{\lambda}s\bar{h}_1 + h_1^{**}(l-s+\bar{\lambda}s)) - v^-(\bar{\lambda}s\bar{h}_1 + h_1^*(l-s+\bar{\lambda}s))).
\end{aligned}$$

Thus we have

$$\begin{aligned}
& w(p)v^-(\bar{\lambda}s\bar{h}_1 + h_1^*(l-s+\bar{\lambda}s)) + w(1-p)v^+(-\bar{\lambda}s\bar{h}_1 + h_1^*(u-s+\bar{\lambda}s)) \\
> & w(p)v^-(\bar{\lambda}s\bar{h}_1 + h_1^{**}(l-s+\bar{\lambda}s)) + w(1-p)v^+(-\bar{\lambda}s\bar{h}_1 + h_1^{**}(u-s+\bar{\lambda}s))
\end{aligned}$$

which is contradicted to that h_1^{**} is optimal strategy if we sell the stock at time 0.

Therefore $h_1^{**} > h_1^*$ must hold.

This theorem shows that under cumulative prospect theorem there exists a "no trading" interval, too.

Corollary 2. When we want to find out the optimal strategy with transaction costs in the sense of expected utility theory or cumulative prospect theory, there exists a "no trading" interval, in which we will neither buy nor sell the stock at time 0, no matter which value function we choose.

Next, we talk about the length of "no trading" interval. If we consider the value function given by

$$v(x) = \begin{cases} 1 - \exp(-\theta x) & \text{if } x \geq 0 \\ -\lambda(1 - \exp(\theta x)) & \text{if } x < 0 \end{cases},$$

we are able to contrast two lengths of "no trading" interval in the sense of expected utility theory and cumulative prospect theory, respectively.

Remark 7. The length of the "no trading" interval in the version of cumulative prospect theory is no less than which in the version of expected utility theory.

PROOF. Under expected utility theory, the "no trading" interval is

$$\left[\frac{1}{\theta(u-l)} \ln \frac{(1-p)(u-s)}{p(s-l)}, \frac{1}{\theta(u-l)} \ln \frac{(1-p)(u-s+\bar{\lambda}s)}{p(s-l-\bar{\lambda}s)} \right].$$

Hence the length of "no trading" interval is

$$\frac{1}{\theta(u-l)} \ln \frac{(u-s+\bar{\lambda}s)(s-l)}{(s-l-\bar{\lambda}s)(u-s)}.$$

And under cumulative prospect theory, the "no trading" interval is

$$\left[\frac{1}{\theta(u+l-2s)} \ln \frac{w(1-p)(u-s)}{\lambda w(p)(s-l)}, \frac{1}{\theta(u+l-2s)} \ln \frac{w(1-p)(u-s+\bar{\lambda}s)}{\lambda w(p)(s-l-\bar{\lambda}s)} \right].$$

Thus the length of "no trading" interval is

$$\frac{1}{\theta(u+l-2s)} \ln \frac{(u-s+\bar{\lambda}s)(s-l)}{(s-l-\bar{\lambda}s)(u-s)}.$$

Since $\frac{(u-s+\bar{\lambda}s)(s-l)}{(s-l-\bar{\lambda}s)(u-s)} > 1$ and $\frac{1}{u+l-2s} > \frac{1}{u-l}$, therefore

$$\frac{1}{\theta(u+l-2s)} \ln \frac{(u-s+\bar{\lambda}s)(s-l)}{(s-l-\bar{\lambda}s)(u-s)} > \frac{1}{\theta(u-l)} \ln \frac{(u-s+\bar{\lambda}s)(s-l)}{(s-l-\bar{\lambda}s)(u-s)}.$$

Then we complete the proof.

The main point of this remark is that if we consider the specific form of value function, we are able to find out the "no trading" interval so as to calculate the length of "no trading" interval. Moreover if the form of value function given by

$$v(x) = \begin{cases} 1 - \exp(-\theta x) & \text{if } x \geq 0 \\ -\lambda(1 - \exp(\theta x)) & \text{if } x < 0 \end{cases},$$

the attitude of an investor toward risk bases on cumulative prospect theory is more conservative than which bases on expected utility theory.

Then let us consider another value function, which Kahneman and Tversky suggested, given by

$$v(x) = \begin{cases} x^\alpha & , x \geq 0 \\ -\lambda(-x)^\alpha & , x < 0 \end{cases}.$$

In this case we take constant relative risk aversion (CRRA) function given by

$$U(x) = x^\alpha,$$

as a utility function.

Example 5. Under expected utility theory, suppose that $x_0 + h_1(l - s) - \bar{\lambda}s(\bar{h}_1 - h_1) \geq 0$ and $l - s + \bar{\lambda}s \leq 0$

(1) If we buy the stock at time 0, our task is to compute

$$\max_{h_1} p(x_0 + h_1(l - s))^\alpha + (1 - p)(x_0 + h_1(u - s))^\alpha.$$

By first order derivative, we have

$$h_1^* = \frac{x_0(a - 1)}{u - s + a(s - l)} > \bar{h}_1,$$

where

$$a = \left(\frac{(1 - p)(u - s)}{p(s - l)} \right)^{\frac{1}{1 - \alpha}}.$$

(2) If we sell the stock at time 0, our task is to compute

$$\max_{h_1} p(x_0 - \bar{\lambda}s\bar{h}_1 + h_1(l - s + \bar{\lambda}s))^\alpha + (1 - p)(x_0 - \bar{\lambda}s\bar{h}_1 + h_1(u - s + \bar{\lambda}s))^\alpha.$$

By first order derivative, we have

$$h_1^{**} = \frac{(b - 1)(x_0 - \bar{\lambda}s\bar{h}_1)}{u - s + \bar{\lambda}s + b(s - l - \bar{\lambda}s)} < \bar{h}_1,$$

where

$$b = \left(\frac{(1 - p)(u - s + \bar{\lambda}s)}{p(s - l - \bar{\lambda}s)} \right)^{\frac{1}{1 - \alpha}}.$$

Moreover $h_1^{**} < \bar{h}_1$ implies

$$\frac{(b-1)x_0}{u-s+b(s-l)} < \bar{h}_1.$$

Therefore we have a conclusion that if $\frac{x_0(a-1)}{u-s+a(s-l)} > \bar{h}_1$, the investor is willing to buy the stock at time 0 and the optimal strategy is

$$h_1^* = \frac{x_0(a-1)}{u-s+a(s-l)}.$$

If $\frac{(b-1)x_0}{u-s+b(s-l)} < \bar{h}_1$, the investor is willing to sell the stock at time 0 and the optimal strategy is

$$h_1^{**} = \frac{(b-1)(x_0 - \bar{\lambda}s\bar{h}_1)}{u-s+\bar{\lambda}s+b(s-l-\bar{\lambda}s)}.$$

Moreover, there is a "no trading" interval

$$\left[\frac{x_0(a-1)}{u-s+a(s-l)}, \frac{x_0(b-1)}{u-s+b(s-l)} \right].$$

Under cumulative prospect theory, suppose that the financial market is no lending and $l-s+\bar{\lambda}s \leq 0$.

(1) If we buy the stock at time 0, our task is to compute

$$\max_{h_1} f(h_1) := -\lambda w(p)(h_1(s-l))^\alpha + w(1-p)(h_1(u-s))^\alpha.$$

By first order derivative, we have

$$f'(h_1) = \alpha h_1^{\alpha-1} (-\lambda w(p)(s-l)^\alpha + w(1-p)(u-s)^\alpha).$$

Case 1: If

$$w(1-p)(u-s)^\alpha > \lambda w(p)(s-l)^\alpha,$$

$f(h_1)$ is an increasing function. So the optimal strategy is

$$h_1^* = \bar{h}_1 + \frac{\bar{h}_0}{s}.$$

Case 2: If

$$w(1-p)(u-s)^\alpha < \lambda w(p)(s-l)^\alpha,$$

$f(h_1)$ is a decreasing function. So the optimal strategy is

$$h_1^* = \bar{h}_1.$$

The main point is that if $w(1-p)(u-s)^\alpha > \lambda w(p)(s-l)^\alpha$ holds, transforming all our money into stock makes the maximal profit. On the other hand, if $w(1-p)(u-s)^\alpha < \lambda w(p)(s-l)^\alpha$ holds, we would not buy the stock at time 0.

(2) If we sell the stock at time 0, our task is to compute

$$\max_{h_1} -\lambda w(p)(\bar{\lambda}s\bar{h}_1 + h_1(s-l-\bar{\lambda}s))^\alpha + w(1-p)(\bar{\lambda}s\bar{h}_1 + h_1(u-s+\bar{\lambda}s))^\alpha.$$

By first order derivative, we have

$$h_1^{**} = \frac{\bar{\lambda}s(c+1)\bar{h}_1}{u-s+\bar{\lambda}s-c(s-l-\bar{\lambda}s)} < \bar{h}_1,$$

where

$$(4.4) \quad c = \left(\frac{w(1-p)(u-s+\bar{\lambda}s)}{\lambda w(p)(s-l-\bar{\lambda}s)} \right)^{\frac{1}{1-\alpha}}.$$

Moreover $h_1^{**} < \bar{h}_1$ implies $u-s > c(s-l)$.

Therefore if $w(1-p)(u-s)^\alpha > \lambda w(p)(s-l)^\alpha$ holds, the investor is willing to buy the stock at time 0 and optimal strategy is $h_1^* = \bar{h}_1 + \frac{\bar{h}_0}{s}$. If $u-s > c(s-l)$ holds, the investor is willing to sell the stock at time 0 and the optimal strategy is

$$h_1^{**} = \frac{\bar{\lambda}s(c+1)\bar{h}_1}{u-s+\bar{\lambda}s-c(s-l-\bar{\lambda}s)}.$$

Moreover if

$$\frac{u-s}{s-l} \leq \min\left\{ \left(\frac{\lambda w(p)}{w(1-p)} \right)^{\frac{1}{\alpha}}, \left(\frac{w(1-p)(u-s+\bar{\lambda}s)}{\lambda w(p)(s-l-\bar{\lambda}s)} \right)^{\frac{1}{1-\alpha}} \right\},$$

the investor will neither buy nor sell the stock at time 0.

When investors evaluate investments based on different value functions, the portfolio they choose may be distinct. If the value function is given by

$$v(x) = \begin{cases} 1 - \exp(-\theta x) & \text{if } x \geq 0 \\ -\lambda(1 - \exp(\theta x)) & \text{if } x < 0 \end{cases},$$

the optimal strategy in the case, buying the stock at time 0, is

$$h_1^* = \frac{1}{\theta(u + l - 2s)} \ln \frac{w(1-p)(u-s)}{\lambda w(p)(s-l)},$$

and in the case, selling the stock at time 0, the optimal strategy is

$$h_1^{**} = \frac{1}{\theta(u + l - 2(1 - \bar{\lambda})s)} \ln \frac{w(1-p)(u-s + \bar{\lambda}s)}{\lambda w(p)(s-l - \bar{\lambda}s)} + \frac{2\bar{\lambda}s\bar{h}_1}{u + l - 2(1 - \bar{\lambda})s}.$$

Moreover, there exists a "no trading" interval. If the value function is given by

$$v(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\alpha, & x < 0 \end{cases},$$

the result is much more complicated and there exists boundary condition of market model. Furthermore, if we buy the stock at time 0 and $w(1-p)(u-s)^\alpha > \lambda w(p)(s-l)^\alpha$, the optimal strategy is

$$h_1^* = \bar{h}_1 + \frac{\bar{h}_0}{s}.$$

If we sell the stock at time 0 and $u-s > c(s-l)$ where $c = (4.4)$, the optimal strategy is

$$h_1^{**} = \frac{\bar{\lambda}s(c+1)\bar{h}_1}{u-s + \bar{\lambda}s - c(s-l - \bar{\lambda}s)}.$$

In financial market if we take transaction costs into consideration, we have to talk about two situations, buying the stock at time 0 and selling the stock at time 0. Furthermore, we find out under some restrictions there is a "no trading" interval in which we will neither buy nor sell the stock at time 0.

CHAPTER 5

Conclusion

Cumulative prospect theory modifies the drawbacks of expected utility theory. Under cumulative prospect theory we replace objective probability with probability weighting function which is nonlinear. Moreover, we replace utility function with value function which is concave for gains and convex for losses.

In the sense of expected utility theory if the mean of gains is greater than the mean of losses, optimal amount of risky asset is greater than 0. However, in the sense of cumulative prospect theory optimal strategy depends on the degree of sensitivity in facing loss, denoted by λ . The larger λ is, the more conservative the investors are. If the investor who does not accept the prior loss he will become risk seeking as the gain of lottery is large enough. Furthermore if the prior loss is sufficiently large, the hedger who take the prior loss into account will take very large positions showing no consideration of risks in the sense of cumulative prospect theory. But full hedging is the optimal strategy for a hedger in the sense of expected utility theory.

Finally, we consider the market model with transaction cost. An investor must to pay constant proportional transaction cost when he sells the stock. In the sense of expected utility theory and cumulative prospect theory there exists a "no trading" interval. And the length of the "no trading" interval in the version of cumulative prospect theory is no less than which in the version of expected utility theory. Therefore the attitude of an investor toward risk bases on cumulative prospect theory is more conservative than which bases on expected utility theory.

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