國立交通大學

應用數學系

碩士論文

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A Study of Semi-power Dominating Sets

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指導教授：傅恆霖 教授

中華民國九十九年六月

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摘要

用圖的模型來研究一個半電力控制集問題是在圖G上一些點放著測試器，依據我們訂下的規則，若能讓圖G的所有邊都被觀察到，則我們稱這些點所成的集合為圖G的半電力控制集合。在這篇論文中，我們先說明了電力控制集問題與半電力控制集問題的關聯性。接著，我們證明了一些特殊圖的半電力控制集的最少點數，也證明當圖G為連通圖，且G中每個點所連到的邊數至少為兩邊時，半電力控制集的最少點數與回饋點集的最少點數相等。我們也提出一遞迴方法，來建構$P_n \times P_n$、$C_n \times C_n$的半電力控制集；於是，提供了一個半電力控制集最少點數的上界。最後我們證出$P_n \times P_n$的半電力控制集最少點數為F_n或F_n+1，其中$F_n = \left\lceil \frac{n^2-2n+2}{3} \right\rceil$，這結果改善了原來文獻中的最佳結論。

A Study of Semi-power Dominating Sets

Student: Cheng-Hsun Wu

Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30050

Advisor: Hung-Lin Fu

Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30050

Abstract

In a semi-power domination set (SPDS) system, we place measurement units on some vertices of a graph G, and according to the rule we defined, if all the edges of G can be observed, then we say that the vertex set is a semi-power domination set. In this thesis, we first find the relationship between PDS and SPDS, and then we prove that the minimum size of SPDS of a graph G, denoted by $\gamma_{sp}(G)$, is equal to the minimum size of the feedback vertex set of G, provided G is connected and $\delta(G) \geq 2$. In addition, we bring up a recursive idea to produce the SPDS of a graph G. Finally, with the recursive idea, we prove that $\gamma_{sp}(P_n \times P_n)$ is equal to either F_n or $F_n + 1$, where $F_n = \left\lceil \frac{n^2 - 2n + 2}{3} \right\rceil$. This improves known results.

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Contents

1. Introduction and Preliminaries

Domination problems are the most useful ones in real world which are related to graph models. Find an efficient way to build hospitals, power plants and control centers is a typical example. One to the applications with specific needs, the problem has many variations, e.g. k-domination, independent domination, total domination, power domination, etc. This thesis provides a different version, called semi-power domination which has a suitable graph model to fit. We shall explain it in section 1.3. First, some graph notions are necessary.

1.1. Graph Notions

In this section, we first introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West. [18]

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge. Two vertices of an edge are called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multi-edges* are edges having the same pair of endpoints. A *simple graph* is a graph without loops or multi-edges. In this thesis, all the graphs we consider are simple. The size of the vertex set $V(G)$, $|V(G)|$, is called the *order* of G, and the size of the edge set $E(G)$, $|E(G)|$, is called the *size* of G. The neighborhood of v written $N_G(v)$, is the set of vertices adjacent to v. The *degree* of vertex v in a graph G, $\deg(v)$ is the number of edges incident to v, i.e. $\deg(v)=|N_G(v)|$. The maximum degree among all vertices of a graph G is denoted by $\Delta(G)$, the minimum degree is denoted by $\delta(G)$, and G is *regular* if

$\Delta(G)=\delta(G)$. A graph G is *k-regular* if the common degree is *k*. An isolated vertex is a vertex of degree 0.

A *path* is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A path with *n* vertices is denoted by P_n. In the case that path *P* starts at *u* and ends at *v*, we call *P* a (*u*, *v*)-path. A graph *G* is *connected* if it has a (*u*, *v*)-path whenever $u,v\in V(G)$ (otherwise, G is disconnected). The *components* of a graph *G* are its maximal connected subgraphs. A component (or graph) is *trivial* if it has no edges; otherwise it is *nontrivial*.

A *subgraph* of a graph G is a graph H such that $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$ and the assignment of endpoints to edges in *H* is the same as in *G*. A *spanning subgraph* of *G* is a subgraph *H* with $V(H)=V(G)$.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with *n* vertices is denoted by C_n. A graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph. A *spanning tree* is a spanning subgraph that is a tree.

A *complete graph* is a simple graph whose vertices are pair-wise adjacent; the complete graph with *n* vertices is denoted by K_n. A graph *G* is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of *G*. A graph *G* is *m-partite* if $V(G)$ can be expressed as the union of *m* independent sets. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes *s* and *t*, the complete bipartite graph is denoted by $K_{s,t}$. If the sets have the same size *n*, the complete bipartite graph is called *balanced*, which is denoted by $K_{n,n}$. Similarly, the *complete m-partite* graph is denoted by $K_{s_1,s_2,\ldots,s_m}$.

A *star* S_k is the complete bipartite graph $K_{1,k}$, i.e., a tree with one internal node and k leaves. A star with 3 edges is called a claw. Let T be the tree formed from a star by subdividing any number of its edges any number of times; that is, T has at most one vertex of degree 3 or more. We call such a tree T a *spider*. A path, for example, is a special case of a spider.

The *corona* of two graphs G and H, denoted $G \circ H$, is the graph formed from one copy of G and $|V(G)|$ copies of H where the ith vertex of G is adjacent to every vertex in the ith copy of H.

The *diamond* is the graph D obtained from the complete graph K_4 by deleting one edge. For each positive integer k, let D_k be the connected claw-free cubic graph formed from k disjoint copies of D by joining pair-wise $2k$ vertices of degree two. Note that D_1 is just K_4.

Let G be a graph of order m with $V(G) = \{g_i : 0 \le i \le m-1\}$, and let H be a graph of order n with $V(H) = \{h_i : 0 \le i \le n-1\}$. The *Cartesian product* $G \times H$ is defined to be the graph with vertex set $\{(g_i, h_j) : 0 \le i \le m-1 \text{ and } 0 \le j \le n-1\}$ and $(g_i, h_j)(g_s, h_t) \in E(G \times H)$ if either $g_i = g_s$ and $h_j h_t \in E(H)$ or $h_j = h_t$ and $g_i g_s \in E(G)$.

In the following section, we will introduce the power-dominating set problem.

1.2. Power-dominating Sets

Electric power companies monitor the state of their electric power system by placing phasor measurement units (PMUs) in the system. Because of the high cost of a PMU, we want to minimize the number of PMUs to monitor (observe) the entire system. A system is said to be observed if all of the state variables of the system can be determined from a set of measurements.

Let $G = (V, E)$ be a graph representing an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. The problem of locating a smallest set of PMUs to monitor the entire system is a graph model problem closely related to the well-known vertex covering and domination problems. For a thorough study of domination, related subset problems and terminology, the readers may refer to two books [11, 12].

A PMU measures the state variable for the vertex at which it is placed and its incident edges and their endvertices. (These vertices and edges are said to be observed.) The other observation rules are as follows:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of $k > 1$ edges and if $k - 1$ of these edges are observed, then all k of these edges are observed.

For a given vertex set P of representing the nodes where the PMUs are placed, to solve the power system monitoring problem we want to minimize $|P|$. This monitoring problem was introduced and studied in [1, 2, 3 and 17]. We define a set $S \in V(G)$ to be a power dominating set (PDS) in a graph $G = (V, E)$ if every vertex and every edge in G is observed by S. The cardinality of a minimum power dominating set of G is the power domination number $\gamma_p(G)$. A power dominating set of G with minimum cardinality is called a $\gamma_p(G)$-set. In [4, 13], it was proved that to obtain power domination set is NP-complete for planar bipartite graphs, bipartite graphs and chordal graphs, respectively.

In the following section, we will introduce the semi-power domination set problem and some observations.

1.3. Semi-power Dominating Sets

In this thesis, we try to examine an electric power system including edges and vertices in graph model. Then we place some weak measurement units (WMUs) on vertices, and we suppose that all the edges connected to the vertices that has place the WMU can be tested. Furthermore, if there are $n-1$ edges to be tested in n edges connected to a vertex, then all of them must be tested. For economic reason, we minimize the number of WMUs.

Weak measurement units (WMUs) measure the state variable for the vertex at which it is placed and its incident edges and their endpoints. (These vertices and edges are said to be observed.) The other observation rules are as follows:

1. Any vertex that is incident to an observed edge is observed.
2. If a vertex is incident to a total of $k>1$ edges and if $k-1$ of these edges are observed, then all k of these edges are observed.

Note that we delete the second rule of a PDS. For a given vertex set P of representing the nodes where the WMUs are placed, to solve the semi-power system monitoring problem we would try to minimize $|P|$.

A set $S \in V(G)$ is a semi-power dominating set (SPDS) in a graph $G = (V, E)$ if every vertex and every edge in G is observed by S following the rules defined above. The cardinality of a minimum semi-power dominating set of G is the semi-power domination number $\gamma_{sp}(G)$. A semi-power dominating set of G with minimum cardinality is called a $\gamma_{sp}(G)$-set.

We have the following two observations.

Observation 1. For each graph G, $\gamma_{sp}(G) \geq \gamma_p(G) \geq 1$.

Observation 2. G is a graph and H is a subgraph of G, then $\gamma_{sp}(H)$ may be larger than $\gamma_{sp}(G)$.

Example：In Figure 1, H is a subgraph of G, and $\gamma_{sp}(G) = 3 < \gamma_{sp}(H) = 4$.

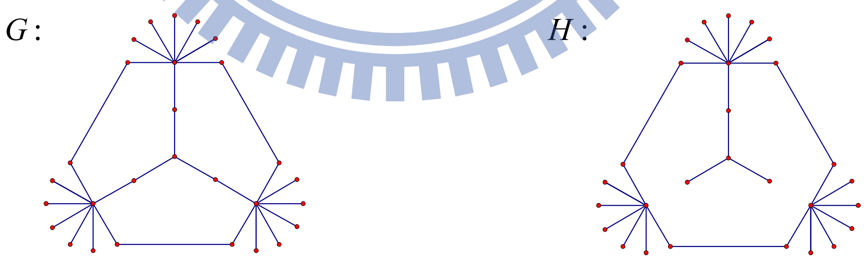

Figure 1:

While studying SPDS problem, in some conditions, we found semi-power domination and feedback vertex sets are quite the same. In the following section, we will introduce feedback vertex sets.

1.4. Feedback Vertex Sets

A feedback vertex set (FVS) of a connected graph $G = (V, E)$ is a subset V' of $V(G)$ such that the graph G' induced by $V \setminus V'$ is a forest. The cardinality of a minimum feedback vertex set (MFVS) in G is the feedback vertex number $\tau(G)$. A feedback vertex set of G with minimum cardinality is called a $\tau(G)$-set.

The problem of finding a minimum feedback vertex set in a graph is one of the classic NP-complete problems [14] and is NP-hard for general graphs [7]. We refer to [10] for a rather complete and recent survey on the feedback vertex set problem.

2. Known Results

Some known results on PDS will be introduced as following.

2.1. On Power-dominating Sets

T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and M. A. Henning had mentioned the following result in [13].

Theorem 2.1.1. [13] *For any tree* T, $\gamma_p(T) = 1$ *if and only if* T *is a spider.*

Theorem 2.1.2. [13] *If* T *is a tree having* k *vertices of degree at least 3, then* $\gamma_p(T) \geq \frac{k+2}{3}$ *and this bound is sharp.*

Theorem 2.1.3. [13] *For any tree* T *of order* $n \geq 3$, $\gamma_p(T) \leq \frac{n}{3}$ *with equality if and only if* T *is the corona* $T \circ K_2$, *where* T *is any tree.*

M. Dorfling and M. A. Henning had mentioned the following results in [8] for the graph $P_n \times P_m$.

Theorem 2.1.4. [8] *If* G *is an* $n \times m$ *grid graph* $P_n \times P_m$ *where* $m \geq n \geq 1$, *then*

$$\gamma_p(G) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil & \text{if } n \equiv 4 \pmod 8; \text{ and} \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases}$$

M. Zhao, L. Kang and G. J. Chang had mentioned the following results in [19].

Let F be the family of graphs obtained from connected graphs H by adding two new vertices v' and v'' to each vertex v of H and new edges vv' and vv'', while $v'v''$ may be added or not.

Theorem 2.1.5. [19] *If* $G=(V,E)$ *is a connected graph of order* $n\geq 3$, *then* $\gamma_p(G)\leq \frac{n}{3}$ *with equality if and only if* $G\in F\cup\{K_{3,3}\}$.

Corollary 2.1.6. [19] *If each component* G_i *of a graph G of order n contains at least three vertices, then* $\gamma_p(G)\leq \frac{n}{3}$ *with equality if and only if each component* $G_i\in F\cup\{K_{3,3}\}$.

Theorem 2.1.7. [19] *If* $G=(V,E)$ *is a connected claw-free cubic graph of order n, then* $\gamma_p(G)\leq \frac{n}{4}$ *with equality if and only if* $G\in A$, *where* $A=\{D_k \mid k\geq 1\}$.

C.C. Chuang had mentioned the following results in [6].

Theorem 2.1.8. [6] $G=K_n\times P_m$, $\gamma_p(G)=1$.

Theorem 2.1.9. [6] $G=K_n\times C_m$, $\gamma_p(G)=\begin{cases}1, & n=1 \text{ or } (n,m)=(2,3).\\ 2, & \text{otherwise.}\end{cases}$

Theorem 2.1.10. [6] $G=K_n\times K_m$, *where* $2\leq n\leq m$, $\gamma_p(G)=n-1$.

Theorem 2.1.11. [6] $G=C_n\times C_m$, $3\leq n\leq m$, *then*

$$\gamma_p(G)=\begin{cases}\left\lceil \frac{n}{2}\right\rceil+1, & \text{if } n\equiv 2 \pmod 4.\\ \left\lceil \frac{n}{2}\right\rceil, & \text{otherwise.}\end{cases}$$

While studying SPDS problem, we will use some results on feedback vertex sets and it will be introduced in the following section.

2.2. On Feedback Vertex Sets

In [15], Luccio proved upper and lower bounds on the sizes of minimal feedback vertex sets in grids. Subsequently, both, Caragiannis, Kaklamanis, Kanellopoulos in [5] and Madelaine, Stewart in [16] improved the upper bounds, respectively.

Theorem 2.2.1. [15] *For all* $n, m \in N$,

$$\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil \le \tau(P_n \times P_m) \le \left\lfloor \frac{mn}{3} + \frac{m+n}{6} + o(m,n) \right\rfloor.$$

Theorem 2.2.2. [5] *For all* $n, m \in N$, $\tau(P_n \times P_m) \le \left\lfloor \frac{mn}{3} + \frac{m+n-5}{6} \right\rfloor$.

Lemma 2.2.3. [16] *For* $n, m \in N$, *let* $a_{n,m}$ *be the upper bound of* $\tau(P_n \times P_m)$,

(i) if n=3k+1, m=2r, $k \ge 1$, $r \ge 2$, *then* $a_{n,m} = F_{n,m}$;

(ii) if n=3k+1, m=2r+1, $k \ge 1$, $r \ge 3$, *then* $a_{n,m} = F_{n,m}$;

(iii) if n=3k, m=3r or *3r+2,* $k \ge 3$, $r \ge 2$ *and r is even, then* $a_{n,m} = F_{n,m} + 1$;

(vi) if n=3k, m=3r, $k \ge 3$, $r \ge 3$ *and r is odd, then* $a_{n,m} = F_{n,m} + 1$;

(v) if n=3k, m=3r+2, $k \ge 3$, $r \ge 3$ *and r is odd, then* $a_{n,m} = F_{n,m} + 2$;

(vi) if n=3k+2, m=3r or *3r+2,* $k \ge 3$, $r \ge 2$ *and r is even, then* $a_{n,m} = F_{n,m} + 1$;

(vii) if n=3k+2, m=3r, $k \ge 3$, $r \ge 3$ *and r is odd, then* $a_{n,m} = F_{n,m} + 2$;

(viii) if n=3k+2, m=3r+2, $k \ge 3$, $r \ge 2$ *and r is odd, then* $a_{n,m} = F_{n,m} + 2$.

Theorem 2.2.4. [16] *If* $(n,m) \notin \{(i,j) \mid i \text{ or } j \in \{2,3,5\} \text{ or } \{i,j\} \subseteq \{6,8\}\}$,

then $\tau(P_n \times P_m) = F_{n,m}$, $F_{n,m} + 1$ *or* $F_{n,m} + 2$, *where* $F_{n,m} = \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

Lemma 2.2.5. [16] *If* $n \in N$, $n \ge 2$, *then* $\tau(P_2 \times P_n) = \left\lceil \frac{n-1}{2} \right\rceil$.

Lemma 2.2.6. [16] *For each* $r \in N$, $r \ge 3$,

(i) $\tau(P_3 \times P_{2r-1}) = \left\lceil \frac{3(r-1)}{2} \right\rceil$.

(ii) $\left\lceil \frac{3(r-1)}{2} \right\rceil \le \tau(P_3 \times P_{2r}) \le \left\lceil \frac{3(r-1)}{2} \right\rceil + 1$.

Lemma 2.2.7. [16] *For all* $p \ge 0$ *and the grid* $P_5 \times P_m$ *with* $m \ge 2$, *we have*

$\tau(P_5 \times P_{8p}) = 11p - 1$; $\tau(P_5 \times P_{8p+1}) = 11p$; $\tau(P_5 \times P_{8p+2}) = 11p + 2$;
$\tau(P_5 \times P_{8p+3}) = 11p + 3$; $\tau(P_5 \times P_{8p+4}) = 11p + 5$; $\tau(P_5 \times P_{8p+5}) = 11p + 6$;
$\tau(P_5 \times P_{8p+6}) = 11p + 8$; $\tau(P_5 \times P_{8p+7}) = 11p + 9$.

Consequently, by Theorem 2.2.4, Lemma 2.2.5, Lemma 2.2.6 and Lemma 2.2.7 we have the following theorem.

Theorem 2.2.8. [16] *There exists a computable function* $f(n,m)$ *such that* $\tau(P_n \times P_m)$ *is equal to one of* $f(n,m)$, $f(n,m)+1$ *or* $f(n,m)+2$, *where* $(n,\ m) \in \{(n,\ m): n \geq 2,\ m \geq 2\}$.

3. Main Results

First, we prove the relationship between SPDS and FVS.

Lemma 3.1. *If* P *is a* $\gamma_{sp}(G)-set$, *then* $G\setminus P$ *has no cycles, i.e.,* P *is a feedback vertex set of* G. *Thus,* $\gamma_{sp}(G)\geq\tau(G)$.

Proof: Suppose not. Then $G\setminus P$ has a cycle and each edge on the cycle is not observed. Hence, P is not a $\gamma_{sp}(G)-set$, a contradiction. ■

Lemma 3.2. *Let* G *be a connected graph with* $\delta(G)\geq 2$. *Then* S *is a semi-power dominating set provided that* S *is a* $\tau(G)-set$. *Thus,* $\gamma_{sp}(G)\leq\tau(G)$.

Proof: Let S be a $\tau(G)-set$. Then, $G\setminus S$ has no cycles and thus, every component in $G\setminus S$ is a tree. Let them be T_1, T_2, ..., T_k. Moreover, let the maximum height of the above k trees be h. We claim that all vertices and edges can be observed after h rounds. In the first round, let $V_1=\{v\mid v\in V(G\setminus S)$ and $\deg_{G\setminus S}(v)$ $=1\}$. Then $\forall\ v\in V_1$, v is adjacent to a vertex of S. Clearly, v is observed thus uv is also observed. Now, consider $v'\in V(G\setminus(S\cup V_1))$ with degree 1 in $G\setminus(S\cup V_1)$. Since v' is observed, $u'v'$ is also observed, where u' is a parent of v'. We continue this step for h times. All of the vertices and edges of G are observed, hence S is also a semi-power dominating set.

■

Theorem 3.3. *If* G *is connected with* $\delta(G)\geq 2$, *then* $\tau(G)=\gamma_{sp}(G)$.

Proof: By Lemma 3.1 and Lemma 3.2 we have the proof. ■

Now, we prove the number $\gamma_{sp}(G)$ for some special graphs *G*.

Theorem 3.4. *For* $n\geq 2$, $\gamma_{sp}(P_n)=1$.

Proof: Let $V(P_n)=\{v_0,v_1,\ldots,v_{n-1}\}$ and $E(P_n)=\{v_iv_{i+1}\mid 0\leq i\leq n-2\}$. It is clear that $\gamma_{sp}(P_n)\geq 1$. Hence, the proof follows by letting $S=\{v_0\}$. ■

Theorem 3.5. *For* $n \geq 3$, $\gamma_{sp}(C_n) = 1$.

Proof: Let $V(C_n) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(C_n) = \{v_i v_j \mid j \equiv i+1 (\bmod n),\ 0 \leq i \leq n-1\}$.

It is clear that $\gamma_{sp}(C_n) \geq 1$. Hence, the proof follows by letting $S = \{v_0\}$. ■

Theorem 3.6. *If* T *is a spider, then* $\gamma_{sp}(T) = 1$.

Proof: Since T is a spider, T has at most one vertex v with $\deg(v) \geq 3$. If T has no vertex v with $\deg(v) \geq 3$, then T is a path. By Theorem 3.4, we have the proof. If T has exactly one vertex v with $\deg(v) \geq 3$, then the proof follows by letting $S = \{v\}$. ■

Theorem 3.7. *For* $n \geq 3$, $\gamma_{sp}(K_n) = n-2$.

Proof: Suppose the size of a $\gamma_{sp}(K_n) - set$ is less than $n-2$. Then $G \setminus \gamma_{sp}(K_n) - set$ contains a K_3 which has a cycle. This is a contradiction. Hence the number $\gamma_{sp}(K_n)$ is at least $n-2$. On the other hand, it is clear that $\gamma_{sp}(K_n) \leq n-2$. Hence, we have the proof. ■

Theorem 3.8. $\gamma_{sp}(K_{n,m}) = n-1$, *where* $2 \leq n \leq m$.

Proof: Let $V(K_{n,m}) = V_1 \cup V_2$ and $E(K_{n,m}) = \{x_i y_j \mid 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$, where $V_1 = \{x_0, x_1, ..., x_{n-1}\}$ and $V_2 = \{y_0, y_1, ..., y_{m-1}\}$. Since two vertices of V_1 and two vertices of V_2 will induce a cycle, $\gamma_{sp}(K_{n,m}) > min.\{m-2,\ n-2\} = n-2$. Hence, the proof follows by letting $S = V_1 \setminus \{x_0\}$. ■

Theorem 3.9. $\gamma_{sp}(K_n \times K_n) = (n-1)^2$, *where* $n \geq 2$.

Proof: Let $V(K_n \times K_n) = \{v_{i,j} \mid 0 \leq i, j \leq n-1\}$ and $E(K_n \times K_n) = \{v_{i,j} v_{k,l} \mid i = k$ or $j = l,\ 0 \leq i,\ j,\ k,\ l \leq n-1\}$. First, we prove the lower bound of $\gamma_{sp}(K_n \times K_n)$. Since $\gamma_{sp}(K_n) = n-2$, $\gamma_{sp}(K_n \times K_n) \leq n(n-2)$. In addition, we know that there are exactly two vertices in each row and each column which has no PMUs. W.L.O.G., let $S=\{v_{i,j} \mid j\equiv i,\ i+1,\ \ldots,\ m+i-3 (\text{mod } m),\ 0\le i\le n-1\}$ with $|S|=n(n-2)$. Then $K_n\times K_n\setminus S$ has a cycle, S is not an SPDS. $\gamma_{sp}(K_n\times K_n)\le n(n-2)+1$. Hence, the proof follows by letting $S'=\{v_{i,j} \mid j\equiv i,\ i+1,\ \ldots,\ m+i-3 (\text{mod } m),\ 0\le i\le n-1\}\cup \{v_{n-2,n-1}\}$ with $|S|=n(n-2)+1$. ■

Theorem 3.10. $\gamma_{sp}(K_n\times K_m)=n(m-2)$, *where* $2\le n<m$.

Proof: Let $V(K_n\times K_m)=\{v_{i,j} \mid 0\le i\le n-1,\ 0\le j\le m-1\}$ and $E(K_n\times K_m)=\{v_{i,j}v_{k,l} \mid i=k \text{ or } j=l,\ 0\le i,\ k\le n-1 \text{ and } 0\le j,\ l\le m-1\}$. First, we find the lower bound of $\gamma_{sp}(K_n\times K_m)$. Since $\gamma_{sp}(K_m)=m-2$, $\gamma_{sp}(K_n\times K_m)\le n(m-2)$. Hence, the proof follows by letting $S=\{v_{i,j} \mid j\equiv i,\ i+1,\ \ldots,\ m+i-3 (\text{mod } m),\ 0\le i\le n-1\}$ with $|S|=n(m-2)$. ■

Theorem 3.11. $\gamma_{sp}(K_n\times P_m)=m(n-2)$, *where* $n,\ m\ge 3$.

Proof: Let $V(K_n\times P_m)=\{v_{i,j} \mid 0\le i\le n-1,\ 0\le j\le m-1\}$ and $E(K_n\times P_m)=\{v_{i,j}v_{k,j} \mid 0\le i,\ k\le n-1 \text{ and } 0\le j\le m-1\}\cup\{v_{i,k}v_{i,k+1} \mid 0\le i\le n-1 \text{ and } 0\le k\le n-2\}$. First, we find the lower bound of $\gamma_{sp}(K_n\times P_m)$. Since $\gamma_{sp}(K_n)=n-2$, $\gamma_{sp}(K_n\times P_m)\le n(m-2)$. Hence, the proof follows by letting $S=\{v_{i,j} \mid i\equiv j,\ j+1,\ \ldots,\ n+j-3 (\text{mod } n),\ 0\le j\le m-1\}$ with $|S|=m(n-2)$. ■

Theorem 3.12. $\gamma_{sp}(K_n\times C_m)=m(n-2)$, where $4\le n\le m$.

Proof: Let $V(K_n\times C_m)=\{v_{i,j} \mid 0\le i\le n-1,\ 0\le j\le m-1\}$ and $E(K_n\times C_m)=\{v_{i,j}v_{k,j} \mid 0\le i,\ k\le n-1 \text{ and } 0\le j\le m-1\}\cup\{v_{i,k}v_{i,l} \mid l\equiv k+1 (\text{mod } n),\ 0\le i\le n-1 \text{ and } 0\le k\le m-1\}$. First, we prove the lower bound of $\gamma_{sp}(K_n\times C_m)$. Since $\gamma_{sp}(K_n)=n-2$, $\gamma_{sp}(K_n\times C_m)\le m(n-2)$. Now, we give an upper bound of $\gamma_{sp}(K_n\times C_m)$.

Case (a). $m\equiv 0 \ (\text{mod } n)$

Let $S=\{v_{i,j} \mid i\equiv j,\ j+1,\ \ldots,\ n+j-3 (\text{mod } n),\ 0\le j\le m-2\}\cup\{v_{i,m-1} \mid 1\le i\le n-2\}$.

Case (b). $m \equiv 1,\ 2 \pmod n$)

Let $S=\{v_{i,j} \mid i \equiv j, j+1, \ldots, n+j-3(\bmod\ n),\ 0 \le j \le m-2\} \cup \{v_{i,m-1} \mid 2 \le i \le n-1\}$.

Case (c). $m \not\equiv 0,\ 1,\ 2 \pmod n$)

Let $S=\{v_{i,j} \mid i \equiv j, j+1, \ldots, n+j-3(\bmod\ \mathrm{n}),\ 0 \le j \le m-1\}$. Then $K_n \times C_m \setminus S$ has no cycles. Thus, $\gamma_{sp}(K_n \times P_m)$ is at most $m(n-2)$. Consequently, $\gamma_{sp}(K_n \times C_m) = m(n-2)$, where $4 \le n \le m$. ■

Theorem 3.13. $\gamma_{sp}(K_n \times C_m) = m(n-2)$, *where* $3 \le m < n$.

Proof: Let $V(K_n \times C_m)=\{v_{i,j} \mid 0 \le i \le n-1,\ 0 \le j \le m-1\}$ and $E(K_n \times C_m)=\{v_{i,j}v_{k,j} \mid 0 \le i,\ k \le n-1 \text{ and } 0 \le j \le m-1\} \cup \{v_{i,k}v_{i,l} \mid l \equiv k+1(\bmod n),\ 0 \le i \le n-1 \text{ and } 0 \le k \le m-1\}$. First, we find the lower bound of $\gamma_{sp}(K_n \times C_m)$. Since $\gamma_{sp}(K_n)=n-2$, $\gamma_{sp}(K_n \times C_m) \le m(n-2)$. Hence, the proof follows by letting $S=\{v_{i,j} \mid i \equiv j, j+1, \ldots, n+j-3(\bmod n),\ 0 \le j \le m-1\}$ with $|S| = m(n-2)$. ■

Now, we use the relation between SPDS and FVS to improve the result of FVS on $P_m \times P_n$.

Lemma 3.14. *Let* G *be a connected graph with* $\delta(G) \ge 2$ *and* $e = xy$ *be an arbitrary edge of* G. *Let* $\tilde{G} = G - e + xz + zy$ *where* $z \notin V(G)$. *Then* $\tau(G) = \tau(\tilde{G})$.

Proof: Let S be a feedback vertex set of G with $|S| = \tau(G)$. Then $G \setminus S$ is a forest. So, it follows that $\tilde{G} \setminus S$ is also a forest, i.e., S is also a feedback vertex set of $\tilde{G}$. Hence, $\tau(G) \ge \tau(\tilde{G})$. On the other direction, let $\tilde{S}$ be a feedback vertex set of $\tilde{G}$ with $|\tilde{S}| = \tau(\tilde{G})$. First, if $\tilde{S} \subseteq V(G)$, then $G \setminus \tilde{S}$ is a forest and thus $\tilde{S}$ is also a feedback vertex set of G. This implies that $\tau(G) \le |\tilde{S}| = \tau(\tilde{G})$. On the other hand, $z \in \tilde{S}$. Now, let $\tilde{S}' = \tilde{S} - z + x$. Clearly, $\tilde{S}'$ is also a feedback vertex set of

$\widetilde{G}$ of size $|\widetilde{S}|$. Since $\widetilde{S}'$ is a feedback vertex set of G, the proof follows by above argument. Therefore, $\tau(G)=\tau(\widetilde{G})$. ■

Let $n \geq 2$. $P_n \times P_n$ is the graph with vertex set $V(P_n \times P_n)$ defined as $\{v_{i,j} : 0 \leq i, j \leq n-1\}$ and edge set $E(P_n \times P_n)$ defined as $\{(v_{i,j}, v_{i+1,j}) : 0 \leq i \leq n-2, 0 \leq j \leq n-1\} \cup \{(v_{i,j}, v_{i,j+1}) : 0 \leq i \leq n-1, 0 \leq j \leq n-2\}$.

Lemma 3.15. $\gamma_{sp}(P_2 \times P_2)=1$, $\gamma_{sp}(P_3 \times P_3)=2$, $\gamma_{sp}(P_4 \times P_4)=4$.

Proof: (i) Since $P_2 \times P_2 = C_4$, $\gamma_{sp}(P_2 \times P_2)=\gamma_{sp}(C_4)=1$.

(ii) Since $\forall v \in V(P_3 \times P_3)$, $P_3 \times P_3 \setminus \{v\}$ always has a cycle, and thus $\gamma_{sp}(P_3 \times P_3) \geq 2$. Let $S=\{v_{0,0}, v_{1,1}\}$. Then $P_3 \times P_3 \setminus S$ has no cycles, S is an SPDS. Hence, $\gamma_{sp}(P_3 \times P_3)=2$.

(iii) Let $V(P_4 \times P_4)=S_1 \cup S_2 \cup S_3 \cup S_4$,

$S_1=\{v_{0,0}, v_{0,1}, v_{1,0}, v_{1,1}\}$, $S_2=\{v_{0,2}, v_{0,3}, v_{1,2}, v_{1,3}\}$, $S_3=\{v_{2,0}, v_{2,1}, v_{3,0}, v_{3,1}\}$, $S_4=\{v_{2,2}, v_{2,3}, v_{3,2}, v_{3,3}\}$. Then each S_i induces a subgraph of G which has a cycle. Hence, $\gamma_{sp}(P_3 \times P_3) \geq 4$. Let $S=\{v_{1,1}, v_{1,3}, v_{2,0}, v_{2,2}\}$. Then $P_4 \times P_4 \setminus S$ has no cycles, S is an SPDS. Hence, $\gamma_{sp}(P_4 \times P_4)=4$. ■

Lemma 3.16. *For* $k, r \geq 1$, $\gamma_{sp}(P_{2k+1} \times P_{2r+1}) \leq kr + \gamma_{sp}(P_{k+1} \times P_{r+1})$.

Proof: Let $X_{2k+1,2r+1}=\{v_{i,j} : i, j \text{ are odd}, 1 \leq i \leq 2k, 1 \leq j \leq 2r\}$. We have the result $\gamma_{sp}(P_{2k+1} \times P_{2r+1} \setminus X_{2k+1,2r+1})=\gamma_{sp}(P_{k+1} \times P_{r+1})$ by using Theorem 3.3 and Lemma 3.14. Hence, we have the proof. ■

Lemma 3.17. *For* $k,\ r \ge 1$, $\gamma_{sp}(P_{2k+2} \times P_{2r+2}) \le (k+1)(r+1) + \gamma_{sp}(P_k \times P_r)$.

Proof: Let $X_{2k+2,2r+2} = A_{2k+2,2r+2} \cup B_{2k+2,2r+2} \cup C_{2k+2,2r+2}$ where

$A_{2k+2,2r+2} = \{v_{i,j} : i, j \text{ are odd and } 1 \le i \le 2k-1,\ 1 \le j \le 2r-1\}$,

$B_{2k+2,2r+2} = \{v_{i,2r} : i \text{ is even },\ 2 \le i \le 2k\} \cup \{v_{2k,j} : j \text{ is even },\ 2 \le j \le 2r\}$ and

$C_{2k+2,2r+2} = \{v_{0,2k}, v_{2r,0}\}$.

We have the result $\gamma_{sp}(P_{2k+2} \times P_{2r+2} \setminus X_{2k+2,2r+2}) = \gamma_{sp}(P_k \times P_r)$ by using Theorem 3.3 and Lemma 3.14. Hence, we have the proof. ■

Theorem 3.18. *For* $n \ge 2$, $\gamma_{sp}(P_n \times P_n) = F_n$ *or* $F_n + 1$, *where* $F_n = \left\lceil \frac{(n-1)^2+1}{3} \right\rceil$.

Proof: By induction on n. Let a_n be an upper bound of $\tau(P_n \times P_n)$. By Lemma 2.2.3 and Theorem 3.3. We know that

(i) $n = 6k+4,\ \ k \ge 0,\ \ a_n = F_n$; (ii) $n = 6k+1,\ \ k \ge 1,\ \ a_n = F_n$;

(iii) $n = 6k,\ \ k \ge 2,\ \ a_n = F_n + 1$; (vi) $n = 6k+3,\ \ k \ge 1,\ \ a_n = F_n + 1$;

(v) $n = 6k+2,\ \ k \ge 2,\ \ a_n = F_n + 1$; (iv) $n = 6k+5,\ \ k \ge 1,\ \ a_n = F_n + 2$.

Now it suffices to prove that $a_n = F_n + t,\ \ t = 0 \text{ or } 1,$ when $n \equiv 5 \pmod 6$.

By Lemma 3.15, Lemma 3.16 and Lemma 3.17 with direct checking, we have $a_2 = 1 = F_2$, $a_3 = 2 = F_3$, $a_5 = 6 = F_5$, $a_6 = 10 = F_6 + 1$, $a_8 = 18 = F_8 + 1$. Hence, the basic cases hold.

By Lemma 3.16.

$$a_{6k+5} \le (3k+2)^2 + a_{3k+3} = (3k+2)^2 + \left\lceil \frac{(3k+3-1)^2+1}{3} \right\rceil + 1$$
$$= \left\lceil \frac{3(3k+2)^2 + (3k+2)^2 + 1}{3} \right\rceil + 1 = \left\lceil \frac{4(3k+2)^2+1}{3} \right\rceil + 1 = \left\lceil \frac{(6k+4)^2+1}{3} \right\rceil + 1.$$

This concludes the proof by induction process. ■

Theorem 3.18. *If* $n, m \equiv 5 \pmod 6$ *then for all* $n, m \geq 11$,

$$\gamma_{sp}(P_n \times P_m) = F_{n,m} \text{ or } F_{n,m} + 1, \quad \textit{where} \quad F_{n,m} = \left\lceil \frac{(n-1)(m-1)+1}{3} \right\rceil.$$

Proof: By Lemma 2.2.3, Theorem 3.3 and Lemma 3.16. Let $a_{n,m}$ be an upper bound of $\gamma_{sp}(P_n \times P_m)$. Then

$$\begin{aligned}
a_{6k+5,6r+5} &\leq (3k+3)(3r+3) + a_{3k+3,3r+3} \\
&= (3k+3)(3r+3) + \left\lceil \frac{(3k+3-1)(3r+3-1)+1}{3} \right\rceil + t \\
&= \left\lceil \frac{3(3k+2)(3r+2) + (3k+2)(3r+2)+1}{3} \right\rceil + t \\
&= \left\lceil \frac{4(3k+2)(3r+2)+1}{3} \right\rceil + t = \left\lceil \frac{((6k+4)(6r+4)+1}{3} \right\rceil + t \\
&= F_{6k+5,6r+5} + t, \quad \text{where } t = 0 \text{ or } 1.
\end{aligned}$$

■

Now, we consider the product of cycles.

Theorem 3.19. *For* $k \geq 2$, $\gamma_{sp}(C_{2k+2} \times C_{2k+2}) \leq (k+1)^2 + \gamma_{sp}(C_{k+1} \times C_{k+1})$.

Proof: Let $X_{2k+2} = \{v_{i,j} : i, j \text{ are odd, } 1 \leq i, j \leq 2k+1\}$. We have the result $\gamma_{sp}(C_{2k+2} \times C_{2k+2} \setminus X_{2k+2}) = \gamma_{sp}(C_{k+1} \times C_{k+1})$ by using Theorem 3.3 and Lemma 3.14. Hence, we have the proof. ■

Theorem 3.20. *For* $k \geq 3$, $\gamma_{sp}(C_{2k+1} \times C_{2k+1}) \leq (k+1)^2 + \gamma_{sp}(P_k \times P_k)$.

Proof: Let $X_{2k+1} = A_{2k+1} \cup B_{2k+1} \cup \{v_{2k,2k}\}$, where $A_{2k+1} = \{v_{i,j} : i, j \text{ are even and } 0 \leq i, j \leq 2k-1\}$ and $B_{2k+1} = \{v_{i,2k} : i \text{ is odd, } 1 \leq i \leq 2k-2\} \cup \{v_{2k,j} : j \text{ is odd, } 1 \leq j \leq 2k-2\}$. We have the result $\gamma_{sp}(C_{2k+1} \times C_{2k+1} \setminus X_{2k+1}) = \gamma_{sp}(P_k \times P_k)$ by using Theorem 3.3 and Lemma 3.14. Hence, we have the proof. ■

4. Concluding Remark

In this thesis, we first introduce a new notion called semi-power dominating set to relax the well-known power dominating set as a graph model in applications. This new SPDS turns out to be exactly the same as the feedback vertex set of a connected graph G with $\delta(G) \geq 2$. Therefore, if the graphs G fit the above conditions which we can find $\gamma_{sp}(G)$, then we also determine $\tau(G)$. Indeed, we have done just that by considering the product of two paths and we are very close to determine $\gamma_{sp}(P_n \times P_m)$. Hopefully, this can be done in the near future.

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