國 立 交 通 大 學

應用數學系
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弦環式網路的邊刪除直徑不變性質

The Diameter-edge-invariant Property

of Chordal Ring Networks

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中 華 民 國 九 十 八 年 一 月

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The diameter-edge-invariant property of chordal ring networks

合於碩士資格水準，業經本委員會評審認可。

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弦環式網路的邊刪除直徑不變性質

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摘　要

環狀網路是最簡單的網路架構，然而，環狀網路的可靠度低、傳輸延遲高，因此，以環狀網路為基礎的混合式環狀網路相繼被提出，以提高其可靠度與降低傳輸延遲。Arden 和 Lee [1]在 1981 年提出了以環狀網路延伸而成的弦環式網路。弦環式網路是在環狀網路的結構中增加弦，使得其可靠度提高、直徑降低。更具體的來說，一個弦環式網路 $CR(N,w)$ 具有 N 個點(N 為偶數，點的編號為 0 至 $N-1$) 與 $3N/2$ 條邊，邊的連線方式為：

$$i \text{ 連至 } (i+1) \bmod N \text{，} \forall i = 0,1,\ldots,N-1 \text{，及}$$

$$i \text{ 連至 } (i+w) \bmod N \text{，} \forall i = 1,3,\ldots,N-1 \text{，}$$

其中 w 為不大於 $N/2$ 的奇數。弦環式網路為 3-正則圖，它與環狀網路一樣具有漢彌爾頓圈，而且比環狀網路擁有更好的直徑。在 1987 年，Lee 和 Tanoto [14 , 15] 提出了邊刪除直徑不變圖的概念。令 $D(G)$ 為圖 G 的直徑。一個圖 G 為 diameter-edge-invariant 若它滿足 $D(G-e)=D(G)$，$\forall e \in E(G)$。本篇論文之目的在於研究弦環式網路 $CR(N,w)$ 的邊刪除直徑不變性質，特別是在 $w \in \{3,5,7,9\}$ 時，我們判斷出所有的 $CR(N,w)$ 是否為邊刪除直徑不變圖。

關鍵詞：環狀網路、弦環式網路、連接網路、直徑、邊刪除、正則圖。

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The Diameter-edge-invariant Property of Chordal Ring Networks

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Abstract

One of the simplest topologies for interconnection networks is the ring network. However, the ring network has poor reliability (any failure in a node or link destroys the function of the network) and it has high transmission delay (large diameter). As a result, hybrid topologies utilizing the ring network as a basis for synthesizing richer interconnection schemes have been proposed to improve the reliability and reduce the transmission delay. The chordal ring network, proposed by Arden and Lee [1] in 1981, is a commonly used extension for the ring network. The chordal ring network is considered to be obtained by adding chords to a cycle (a ring network) so that the diameter can be reduced and the reliability can be increased. More specifically, a chordal ring network $CR(N, w)$, where N is a positive even integer and w is a positive odd integer such that $w \leq N/2$, is a graph with N nodes $0, 1, \ldots, N-1$ and $3N/2$ links of the form:

$$\begin{aligned} &(i, (i+1) \bmod N), \quad i = 0, 1, 2, \ldots, N-1, \\ &(i, (i+w) \bmod N), \quad i = 1, 3, 5, \ldots, N-1. \end{aligned}$$

The chordal ring network is 3-regular, preserves the Hamiltonian cycle from the ring network, and has a better diameter than the ring network. In 1987, Lee and Tanoto [14, 15] proposed diameter-edge-invariant graphs. Let $\mathcal{D}(G)$ denote the diameter of a graph G. G is diameter-edge-invariant (dei) if $\mathcal{D}(G-e) = \mathcal{D}(G)$ for all $e \in E(G)$. The purpose of this thesis is to study the dei property of chordal ring networks. In particular, we determine if $CR(N, w)$ is dei for all $w \in \{3, 5, 7, 9\}$.

Keywords: ring network, chordal ring network, interconnection network, diameter, edge deletion, regular graph.

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1 Introduction

Our graph terminology and notation are standard; see [3] and [21] except as indicated. In this thesis, a graph is always undirected, without multiple edges and loops. For convenience, vertices of a graph are also called nodes. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* $d_G(u,v)$ between two nodes $u,v \in V(G)$ is the length of a shortest path joining u,v. The *eccentricity* $e(v)$ of a vertex $v \in V(G)$ is the distance between v and a farthest node from v. The *diameter* $\mathcal{D}(G)$ and the *radius* $r(G)$ of G are the maximum and the minimum eccentricities of all nodes of G, respectively. Let $e \in E(G)$ and $v \in V(G)$. $G-e$, $G+e$ and $G-v$ denote the graph obtained by deleting e, adding e, and deleting v, respectively.

Ore [17] first considered graphs G with the property $\mathcal{D}(G) > \mathcal{D}(G+e)$ for all $e \notin E(G)$. Almost at the same time, Glivjak [6] initiated the study of graphs G with the property $\mathcal{D}(G-e) > \mathcal{D}(G)$ for all $e \in E(G)$. G is called *diameter-minimal* if $\mathcal{D}(G-e) > \mathcal{D}(G)$ for all $e \in E(G)$. A lot of researches have been done on the above two concepts; see [7, 8, 9, 10, 18]. In particular, Glivjak showed that every graph can be embedded as an induced subgraph in a diameter-minimal graph with diameter 2 [6]; thus finding a forbidden subgraph characterization for diameter-minimal graphs becomes impossible.

A network can be modeled by a graph. The diameter of a network represents the maximum transmission delay between two nodes. In general, the failure of a link will increase the diameter of the network. It is therefore an interesting problem to design a network such that the diameter is invariant under any link failure. Based on this idea, in [14], Lee proposed the the definition: A graph G is *diameter-edge-invariant* (dei) if $\mathcal{D}(G-e) = \mathcal{D}(G)$ for all $e \in E(G)$. For convenience, in this thesis, if G is not dei, then we simply write G is non-dei.

It is clear that there is no dei graphs with diameter 1. It is also clear that if a graph is dei, then it is 2-edge-connected. However, there exists 2-edge-connected graphs that are non-dei; $K_2 \times C_3$ is an example. In [14], Lee proposed some constructions for dei graphs: the Zykov sum construction, the Sabidussi sum construction, the Cartesian product method and the edge expansion construction. Lee also proved that every connected graph is an induced subgraph of a dei graph with diameter ≥ 2. Thus finding a forbidden subgraph characterization for dei graphs becomes impossible. In [15], Lee and Tanoto constructed three classes of planar dei networks: the young tableau graphs, the young tableau graphs with diagonal crossing and the reverse young tableau graph with diagonal crossing. Recently, in [20], Walikar et al. proved that for any two nonnegative integers n and q, where $0 \leq n \leq q$, there exists a connected graph G having q edges, precisely n of which are diameter-increasing except for some n, q. They proved that every graph can be embedded as an induced subgraph in a diameter-edge-invariant graph. They also provided a characterization for dei graphs that has diameter 2 and a characterization for dei graphs that has radius 1.

The following variations of dei graphs have been discussed in the literatures: critical dei graphs, cocritical dei graphs, radius-edge-invariant graphs, diameter-vertex-invariant graphs and diameter-adding-invariant graphs. More precisely, a graph G is *critical dei* (resp. *cocritical dei*) if G is dei and $G - v$ is non-dei (resp. still dei) for all $v \in V(G)$; see [16]. A graph G is *radius-edge-invariant* if $r(G - e) = r(G)$ for all $e \in E(G)$; see [5, 20]. A graph G is *diameter-vertex-invariant* if $\mathcal{D}(G - v) = \mathcal{D}(G)$ for all $v \in V(G)$ and *diameter-adding-invariant* if $\mathcal{D}(G + e) = \mathcal{D}(e)$ for all edges e of the complement of G; see [19].

The ring network is one of the simplest topologies for interconnection networks. It has many attractive properties such as simplicity, extendibility, low degree, and ease of implementation. It has drawbacks as well: it has poor reliability (any failure in an interface or communication link destroys the function of the network) and it has high transmission delay. As a result, a lot of hybrid topologies utilizing the ring network as a basis for synthesizing richer interconnection schemes have been proposed to improve the reliability and reduce the transmission delay [1, 2, 4, 11, 12, 22]. One example of the commonly used extensions for the ring network is the *multi-loop network*, which was first proposed by Wong and Coppersmith in [22] for organizing multi-module memory services.

Another example of the commonly used extensions for the ring network is the *chordal ring network* and is formally defined below.

The chordal ring network was first proposed by Arden and Lee [1]. It is considered to be obtained by adding chords to a cycle (a ring network) so that the diameter can be decreased and the reliability can be increased. More specifically, a *chordal ring network* $CR(N, w)$, where N is a positive even integer and w is a positive odd integer such that $w \leq N/2$, is a graph with N nodes $0, 1, \ldots, N-1$ and $3N/2$ links of the form:

$$\begin{array}{ll} (i, (i+1) \bmod N), & i = 0, 1, 2, \ldots, N-1, \\ (i, (i+w) \bmod N), & i = 1, 3, 5, \ldots, N-1. \end{array}$$

See Fig. 1 for an example. The chordal ring network is 3-regular. It preserves the Hamiltonian cycle from the ring network and has a better diameter than the ring network.

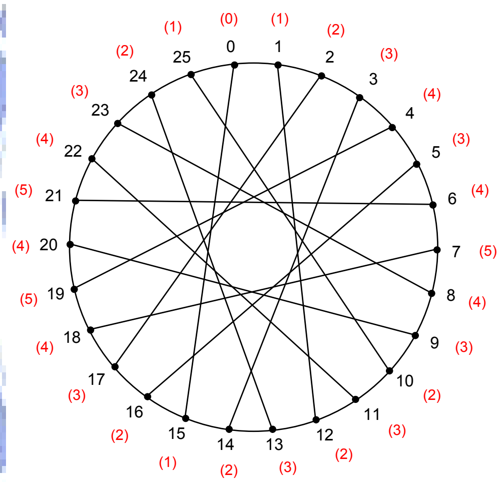

Figure 1: $CR(26, 11)$: the chordal ring with $N = 26$ and $w = 11$. The number inside parentheses is the distance to node 0.

A network is usually been evaluated by its maximum transmission delay (diameter) and by it fault-tolerant capability. Therefore a network with the dei property is preferred than those without the property. The purpose of this thesis is to study the dei property of chordal ring networks $CR(N, w)$. So far as we know, nobody has ever studied the dei property for this kind of networks. In particular, we will determine if $CR(N, w)$ is dei for all $w \in \{3, 5, 7, 9\}$. In [1], Arden and Lee proposed a formula for computing the diameter of $CR(N, w)$. Unfortunately, their formula is not always correct. In this thesis, we will prove that for each $N \geq 26$, there exists a w such that Arden and Lee's formula is not correct. We will also fix the formula for two of the faulty cases.

This thesis is organized as follows. Section 2 gives some preliminaries. Section 3 gives counterexamples for Arden and Lee's diameter formulas. Section 4 contains our dei results for chordal ring networks. The concluding remarks are given in the last section.

2 Preliminaries

Throughout this thesis, nodes in a graph are assumed taken modulo N. As an example, $u+v$ is the node $(u+v) \bmod N$ and $u-v$ is the node $(u-v) \bmod N$. An edge between nodes u and v is denoted by (u,v). The following definitions will be used throughout this thesis. For each odd-numbered node i, we call a chordal traversal from node i to node $i+w$ a *clockwise chordal traversal*. Similarly, for each even-numbered node j, we call a chordal traversal from node j to node $j-w$ a *counterclockwise chordal traversal*. A path from node u to node v is called a *clockwise path* if it consists of clockwise chordal traversals (possibly zero) plus appropriate ring-edge traversals. Similarly, a path from node u to node v is called a *counterclockwise path* if it consists of counterclockwise chordal traversals (possibly zero) plus appropriate ring-edge traversals. Let $\text{dist}_{G,R}(u,v)$ (resp. $\text{dist}_{G,L}(u,v)$) denote the length of a shortest clockwise (resp. counterclockwise) path from u to v in G. In [1], Arden and Lee had proven that

$$d_G(u,v) = \min\{\text{dist}_{G,R}(u,v), \text{dist}_{G,L}(u,v)\}.$$

Take $CR(26,11)$ in Fig. 1 as an example. Then $\text{dist}_{G,R}(0,11) = 3$ and two clockwise paths with such a distance are $0 \to 1 \to 12 \to 11$ and $0 \to 25 \to 10 \to 11$. Also, $\text{dist}_{G,L}(0,11) = 5$; three counterclockwise paths with such a distance is $0 \to 15 \to 14 \to 13 \to 12 \to 11$, $0 \to 25 \to 24 \to 23 \to 22 \to 11$, and $0 \to 25 \to 24 \to 13 \to 12 \to 11$. Form the above, $d_G(0,5) = 3$.

It is obvious that in a chordal ring network, all even-numbered nodes are symmetric; also, all odd-numbered nodes are symmetric. If we flip a chordal ring network vertically, then it is not difficult to see that node 1 is symmetric to node 0, node 2 is symmetric to node $N-1$, node 3 is symmetric to node $N-2$, and so on. Thus a chordal ring network is vertex-transitive and consequently,

$$\mathcal{D}(G) = \max\{d_G(0, v)\}.$$

By using the vertex-transitive property, we obtain the following identities, which will be used in the remaining proof.

- $\mathrm{dist}_{G,R}(u, v) = \mathrm{dist}_{G,L}(v, u)$.
- For u even, $d_G(u, v) = d_G(0, v-u)$.
- For u even, $\mathrm{dist}_{G,R}(u, v) = \mathrm{dist}_{G,R}(0, v-u)$ and $\mathrm{dist}_{G,L}(u, v) = \mathrm{dist}_{G,L}(0, v-u)$.
- For u odd, $d_G(u, v) = d_G(0, u-v)$.
- For u odd, $\mathrm{dist}_{G,R}(u, v) = \mathrm{dist}_{G,L}(0, u-v)$ and $\mathrm{dist}_{G,L}(u, v) = \mathrm{dist}_{G,R}(0, u-v)$

In [1], Arden and Lee provided the following formula for computing $\mathrm{dist}_{G,R}(0, v)$.

Theorem 1. [1] *Suppose $G = CR(N, w)$ and v is a node in G. Let $g = \lceil \frac{v}{w+1} \rceil$, $\triangle_v = v \bmod (w+1)$, and define $\varphi(v) = 0$ if v is even and $\varphi(v) = 1$ if v is odd. Then*

(i) When $g \geq \frac{w-1}{2}$, $\mathit{dist}_{G,R}(0, v) = \begin{cases} 2g-1 & \text{if } \triangle_v = 1, \\ 2g+\varphi(v) & \text{if } \triangle_v = 0 \text{ or } 2 \leq \triangle_v \leq w. \end{cases}$

(ii) When $g < \frac{w-1}{2}$, $\mathrm{dist}_{G,R}(0, v) = \begin{cases} 2(g-1)+\triangle_v & \text{if } 1 \leq \triangle_v \leq \frac{w+1}{2} - g, \\ w+1-\triangle_v & \text{if } \frac{w+3}{2} - g \leq \triangle_v \leq w-2g, \\ 2g+\varphi(v) & \text{if } \triangle_v = 0 \text{ or } w-2g+1 \leq \triangle_v \leq w. \end{cases}$

Arden and Lee did not provide an explicit formula for computing $\mathrm{dist}_{G,L}(0, v)$ and the readers are asked to refer to the dissertation in [13]. For completeness of this thesis, we now provide such a formula.

Theorem 2. *Suppose $G = CR(N, w)$ and v is a node in G. Then*

(i) When $N - w + \frac{w-1}{2} + 1 \leq v \leq N-1$, $\mathrm{dist}_{G,L}(0, v) = N - v$.

(ii) When $N-w \le v \le N-w+\frac{w-1}{2}$, $\mathrm{dist}_{G,L}(0,v) = 1+v-(N-w)$.

(iii) When $v < N-w$, $\mathrm{dist}_{G,L}(0,v) = \min\{v, 1+\mathrm{dist}_{G,R}(0, N-w-v)\}$.

Proof. To calculate $\mathrm{dist}_{G,L}(0,v)$, we divide the nodes in $CR(N,w)$ into several counterclockwise intervals each of size $w+1$ (except possibly the last interval). The first interval contains node $0, N-1, N-2, \ldots, N-w$. Obviously, the length of a shortest counterclockwise path from node 0 to node $N-w$ is 1. The length of a shortest counterclockwise path from node 0 to any node $v \in \{N-1, N-2, \ldots, N-w+\frac{w-1}{2}+1\}$ is $N-v$ and one such path is $0 \to N-1 \to N-2 \cdots \to v$. On the other hand, the length of a shortest counterclockwise path from node 0 to any node $v \in \{N-w+\frac{w-1}{2}, N-w+\frac{w-1}{2}-1, \ldots, N-w\}$ is $1+v-(N-w)$ and one such path is via the chord $(0, N-w)$ and then follows the ring-edges, that is, $0 \to N-w \to N-w+1 \cdots \to N-w+2 \to v$. Since a chordal ring network is vertex-transitive, by Theorem 1 and by the observation that $\mathrm{dist}_{G,L}(u,v) = \mathrm{dist}_{G,R}(0, u-v)$ if u is odd, we have $\mathrm{dist}_{G,L}(N-w, v) = \mathrm{dist}_{G,R}(0, N-w-v)$. When $v < N-w$, node 0 can get to node v by using only ring-edges or by using the chord $(0, N-w)$ and appropriate ring-edges; therefore $\mathrm{dist}_{G,L}(0,v) = \min\{v, 1+\mathrm{dist}_{G,R}(0, N-w-v)\}$. ∎

In [1], Arden and Lee also provided a formula for computing the diameter of a chordal ring network; see the following theorem. For convenience, call this theorem the *Diameter Theorem*.

Theorem 3. [1] **(The Diameter Theorem)**
Suppose $G = CR(N,w)$, $i = \lceil \frac{N}{2(w+1)} \rceil$, *and* $\triangle = \frac{N}{2} \bmod (w+1)$. *Then*

Case 1. *When* $i \ge \frac{w-1}{2}$,
$$\mathcal{D}(G) = \begin{cases} 2i-1 & \text{if } \triangle = 1, \\ 2i & \text{if } 2 \le \triangle \le \frac{w+3}{2}, \\ 2i+1 & \text{if } \triangle = 0 \text{ or } \frac{w+3}{2} \le \triangle \le w. \end{cases}$$

Case 2. *When* $i = \frac{w-3}{2}$,
$$\mathcal{D}(G) = \begin{cases} w-3 & \text{if } 1 \le \triangle \le 2, \\ w-2 & \text{if } \triangle = 0 \text{ or } 3 \le \triangle \le w. \end{cases}$$

Case 3. *When* $i \le \frac{w-5}{2}$,
$$\mathcal{D}(G) = \begin{cases} i+\frac{w-3}{2} & \text{if } 1 \le \triangle \le \frac{w+1}{2}-i \text{ or } \frac{w+5}{2}-i \le \triangle \le w-i, \\ i+\frac{w-1}{2} & \text{if } \triangle = 0 \text{ or } \triangle = \frac{w+3}{2}-i \text{ or } w-i+1 \le \triangle \le w. \end{cases}$$

Before closing this section, we have to mention that Theorems 1 and 2 hold when $2w \leq N \leq 2w+2$ and when $N \geq (\frac{w-1}{4})(w+1)-2$; for $2w+4 \leq N \leq (\frac{w-1}{4})(w+1)-4$, these two theorems may not hold. The incorrectness of the Diameter Theorem might come from the use of Theorem 1 for N such that $2w+4 \leq N \leq (\frac{w-1}{4})(w+1)-4$.

3 Counterexamples for Theorem 3

Let N_i denote the set of nodes in a chordal ring network whose distance to node 0 is i. We now give a counterexample for the Diameter Theorem. Consider $CR(26,11)$ shown in Fig. 1. It is not difficult to see that $CR(26,11)$ has $N_0 = \{0\}$, $N_1 = \{1,15,25\}$, $N_2 = \{2,10,12,14,16,24\}$, $N_3 = \{3,5,9,11,13,17,23\}$, $N_4 = \{4,6,8,18,20,22\}$, and $N_5 = \{7,19,21\}$. Since $\bigcup_{i=1}^{5} N_i = \{0,1,\ldots,25\}$, $\mathcal{D}(CR(26,11)) = 5$. However, the Diameter Theorem obtains $\mathcal{D}(CR(26,11)) = 6$.

We have run computer programs to obtain the diameters of chordal ring networks $CR(N,w)$ for $N = 6,8,\ldots,5000$, and for each N, we test all possible chord lengths w. Our experimental results show that the Diameter Theorem is correct if $w \in \{3,5,7,9\}$, the first fault occurs at $CR(26,11)$, and for each $N \geq 26$ (and N even), there exists a chord length w such that the Diameter Theorem is wrong for $CR(N,w)$. For example, the Diameter Theorem is wrong for $CR(28,11)$, $CR(30,13)$, $CR(32,13)$, $CR(34,15)$, and so on. Moreover, our experimental shows that Cases 1 and 2 of the Diameter Theorem are correct and faults occur in Case 3.

Recall that the Diameter Theorem is correct if $w \in \{3,5,7,9\}$ and the first fault of the Diameter Theorem occurs at $CR(26,11)$. The chordal ring network $CR(26,11)$ satisfies $N = 2w+4$. While we are unable to fix all the faults in the Diameter Theorem, we do fix this theorem for the $N = 2w+4$ and $N = 2w+6$ cases. See the following for details.

For $G = CR(2w+4,w)$, the Diameter Theorem obtains $\mathcal{D}(G) = \frac{w+1}{2}$. However, in Theorem 4, we show that this result is incorrect. For $G = CR(2w+6,w)$, the Diameter Theorem obtains $\mathcal{D}(G) = \frac{w+1}{2}$. In Theorem 5, we show that this result is also incorrect. Do notice that Theorems 4 and 5 together show that for all $N \geq 26$, there exists a w such that the Diameter Theorem is incorrect.

Theorem 4. *Suppose $N = 2w + 4$ and $w \geq 11$. Write w in the form $w = 12t + p$, where t is a positive integer and p is an odd integer such that $-1 \leq p \leq 9$. Then*

$$\mathcal{D}(CR(N, w)) = \begin{cases} \frac{w+3}{2} - 2t & \text{if } -1 \leq p \leq 3 \\ \frac{w+1}{2} - 2t & \text{if } 5 \leq p \leq 9. \end{cases}$$

Proof. Let $G = CR(N, w)$. Then $N_0 = \{0\}$, $N_1 = \{1, w + 4, 2w + 3\}$, $N_2 = \{2, w - 1, w + 1, w + 3, w + 5, 2w + 2\}$, and $N_3 = \{3, 5, w - 2, w, w + 2, w + 6, 2w + 1\}$. Each node in N_3 (except $w + 2$) has at least one neighbor v such that node 0 can not reach node v in three edges. In particular, node 0 can reach node 4 via node 3, reach node 6 via node 5, reach node $w - 3$ via node $w - 2$, reach node $w + 7$ via node $w + 6$, and reach node $2w$ via node w or $2w + 1$. Thus $N_4 = \{4, 6, w - 3, w + 7, 2w - 2, 2w\}$. In general, we can obtain N_i for $i \geq 4 + 4j$ as follows. Let $j \geq 0$. Then

$N_{4+4j} = \{4 + 6j, 6 + 6j, w - 3 - 6j, w + 7 + 6j, 2w - 2 - 6j, 2w - 6j\}$,

$N_{5+4j} = \{7 + 6j, w - 4 - 6j, w + 8 + 6j, w + 10 + 6j, 2w - 1 - 6j, 2w - 3 - 6j\}$,

$N_{6+4j} = \{8 + 6j, w - 7 - 6j, w - 5 - 6j, w + 9 + 6j, w + 11 + 6j, 2w - 4 - 6j\}$, and

$N_{7+4j} = \{9 + 6j, 11 + 6j, w - 8 - 6j, w - 6 - 6j, w + 12 + 6j, 2w - 5 - 6j\}$.

We now calculate $\mathcal{D}(G)$ according to the value of p.

(i) Suppose $p = -1$. Then $w = 12t - 1$. Let $j = \frac{w-11}{12}$. Then

$$N_{4+4j} = \{\tfrac{w-3}{2}, \tfrac{w+1}{2}, \tfrac{w+5}{2}, \tfrac{3w+3}{2}, \tfrac{3w+7}{2}, \tfrac{3w+11}{2}\} \text{ and } N_{5+4j} = \{\tfrac{w+3}{2}, \tfrac{3w+5}{2}, \tfrac{3w+9}{2}\}.$$

Since $N_{6+4j} = \{\frac{w-3}{2}, \frac{w+1}{2}, \frac{w+5}{2}, \frac{3w+7}{2}, \frac{3w+3}{2}\}$, clearly $N_{6+4j} \subseteq N_{4+4j}$ and therefore $\mathcal{D}(G) \leq 5 + 4j$. Since $\{\frac{w+3}{2}, \frac{3w+5}{2}, \frac{3w+9}{2}\} \not\subseteq \bigcup_{i=1}^{4+4j} N_i$, Thus $\mathcal{D}(G) = 5 + 4j = \frac{w+3}{2} - 2t$.

(ii) Suppose $p = 1$. Then $w = 12t + 1$. Let $j = \frac{w-13}{12}$. Then

$$N_{4+4j} = \{\tfrac{w-1}{2}, \tfrac{w-5}{2}, \tfrac{w+7}{2}, \tfrac{3w+1}{2}, \tfrac{3w+9}{2}, \tfrac{3w+13}{2}\}, \quad N_{5+4j} = \{\tfrac{w+1}{2}, \tfrac{w+5}{2}, \tfrac{3w+3}{2}, \tfrac{3w+7}{2}, \tfrac{3w+11}{2}\},$$

$$N_{6+4j} = \{\tfrac{w-1}{2}, \tfrac{w+3}{2}, \tfrac{3w+1}{2}, \tfrac{3w+5}{2}, \tfrac{3w+9}{2}\} \text{ and } N_{7+4j} = \{\tfrac{w-3}{2}, \tfrac{w+1}{2}, \tfrac{w+5}{2}, \tfrac{w+9}{2}, \tfrac{3w+3}{2}, \tfrac{3w+11}{2}\}.$$

Note that $\{\frac{w-3}{2}, \frac{w+9}{2}\} \subseteq N_{7+4j}$ and $\{\frac{w-3}{2}, \frac{w+9}{2}\} \not\subseteq N_{5+4j}$. Since $\{\frac{w-3}{2}, \frac{w+9}{2}\} \subseteq N_{3+4j}$, $d_G(0, \frac{w-3}{2}) = d_G(0, \frac{w+9}{2}) = 3 + 4j$, therefore $\mathcal{D}(G) \leq 6 + 4j$. Since $\{\frac{w+3}{2}, \frac{3w+5}{2}\} \subseteq N_{6+4j}$ and $\{\frac{w+3}{2}, \frac{3w+5}{2}\} \not\subseteq \bigcup_{i=1}^{5+4j} N_i$, Thus $\mathcal{D}(G) = 6 + 4j = \frac{w+3}{2} - 2t$.

(iii) Suppose $p = 3$. Then $w = 12t + 3$. Let $j = \frac{w-15}{12}$. Then

$$N_{5+4j} = \{\tfrac{w-1}{2}, \tfrac{w+7}{2}, \tfrac{3w+1}{2}, \tfrac{3w+5}{2}, \tfrac{3w+9}{2}, \tfrac{3w+13}{2}\}\ ,\ N_{6+4j} = \{\tfrac{w+1}{2}, \tfrac{w+5}{2}, \tfrac{3w+3}{2}, \tfrac{3w+7}{2}\}\ ,$$

$$N_{7+4j} = \{\tfrac{w-1}{2}, \tfrac{w+3}{2}, \tfrac{w+7}{2}, \tfrac{3w+5}{2}, \tfrac{3w+9}{2}\} \text{ and } N_{8+4j} = \{\tfrac{w-3}{2}, \tfrac{w+5}{2}, \tfrac{w+9}{2}, \tfrac{3w-1}{2}, \tfrac{3w+11}{2}, \tfrac{3w+3}{2}\}.$$

Note that $\{\frac{w-3}{2}, \frac{w+9}{2}, \frac{3w-1}{2}, \frac{3w+11}{2}\} \subseteq N_{8+4j}$ and $\{\frac{w-3}{2}, \frac{w+9}{2}, \frac{3w-1}{2}, \frac{3w+11}{2}\} \not\subseteq N_{6+4j}$. Since $\{\frac{w+9}{2}, \frac{w-3}{2}, \frac{3w+11}{2}\} \subseteq N_{4+4j}$, $d_G(0, \frac{w+9}{2}) = d_G(0, \frac{w-3}{2}) = d_G(0, \frac{3w+11}{2}) = d_G(0, \frac{3w-1}{2}) = 4 + 4j$, therefore $\mathcal{D}(G) \leq 7 + 4j$. Since$\frac{w+3}{2} \in N_{7+4j}$ and $\frac{w+3}{2} \notin \bigcup_{i=1}^{6+4j} N_i$, Thus $\mathcal{D}(G) = 7 + 4j = \frac{w+3}{2} - 2t$.

(iv) Suppose $p = 5$. Then $w = 12t + 5$. Let $j = \frac{w-17}{12}$. The proof is similar to case (i). The farthest nodes from node 0 is in N_{7+4j} and $\mathcal{D}(G) = \frac{w+4}{3} = \frac{w+1}{2} - 2t$.

(v) Suppose $p = 7$. Then $w = 12t + 7$. Let $j = \frac{w-19}{12}$. The proof is similar to case (ii). The farthest nodes from node 0 is in N_{4+4j} and $\mathcal{D}(G) = \frac{w+5}{3} = \frac{w+1}{2} - 2t$.

(vi) Suppose $p = 9$. Then $w = 12t + 9$. Let $j = \frac{w-21}{12}$. The proof is similar to case (iii). The farthest nodes from node 0 is in N_{5+4j} and $\mathcal{D}(G) = \frac{w+6}{3} = \frac{w+1}{2} - 2t$. ∎

Theorem 5. *Suppose $N = 2w + 6$ and $w \geq 11$. Write w in the form $w = 8t + p$, where t is a positive integer and p is an odd integer such that $-1 \leq p \leq 5$. Then*

$$\mathcal{D}(CR(N, w)) = \begin{cases} 2t + 2 = \frac{w+9}{4} & \text{if } p = -1 \\ 2t + 3 = \frac{w-p+12}{4} & \text{if } 1 \leq p \leq 5. \end{cases}$$

Proof. Similar to that of Theorem 4. ∎

4 The dei property of chordal ring networks

In this section, we will discuss the dei property of chordal ring networks. Notice that N satisfies Case 1 of Theorem 3 if $N \geq (w - 3)(w + 1) + 2$, Case 2 of Theorem 3 if $(w - 5)(w + 1) + 2 \leq N \leq (w - 3)(w + 1)$, and Case 3 of Theorem 3 if $2w \leq N \leq$

$(w-5)(w+1)$. The discussions in this section are grouped according to the value of the chord length w and the above ranges of N.

The following lemma provides a sufficient condition for a chordal ring network to be dei. For convenience, call this lemma the *DEI Lemma*.

Lemma 6. (The DEI Lemma) *Let $G = CR(N, w)$. If for all $v \in V(G)$, there are always two edge-disjoint paths P_1, P_2 between node 0 and node v such that $|P_i| \leq \mathcal{D}(G)$ for $i = 1, 2$, then G is dei.*

Proof. Since $CR(N, w)$ is vertex-transitive and there are always two edge-disjoint paths P_1, P_2 between node 0 and node v such that $|P_i| \leq \mathcal{D}(G)$ for $i = 1, 2$, deleting any edge will not raise the diameter. Hence we have this lemma. ∎

Let e be an edge of $G = CR(N, w)$ and $H = G - e$. To prove that G is non-dei, it suffices to prove that

$$\mathcal{D}(G - e) \geq d_H(0, x) > \mathcal{D}(G). \tag{1}$$

This inequality is used heavily in the remaining proofs.

Before going further, we introduce the *triangular grid representation* for a chordal ring network $CR(N, w)$. See Fig. 2 for an illustration. Consider the triangular grid on the plane. We associate $CR(N, w)$ a labeling on the cells of the triangular grid as follows (all the labels are assumed taken modulo N). Label 0 to an arbitrary downward triangle. Once a downward triangle is labeled i, label its three neighboring triangles by $i+1$, $i-1$, and $i-w$ in such a way that the triangle to the right of it receives $i+1$, to the left of it receives $i-1$, and above it receives $i-w$. Once a upward triangle is labeled j, label its three neighboring triangles by $j+1$, $j-1$, and $j+w$ in such a way that the triangle to the right of it receives $j+1$, to the left of it receives $j-1$, and below it receives $j+w$. Note that in this thesis, only nodes within distance $\mathcal{D}(CR(N, w))$ will be shown in the triangular grid representation.

For clarity, the remaining part of this section is divided into subsections.

Subsection 4.1 considers $CR(N, w)$ with $w = 3$.

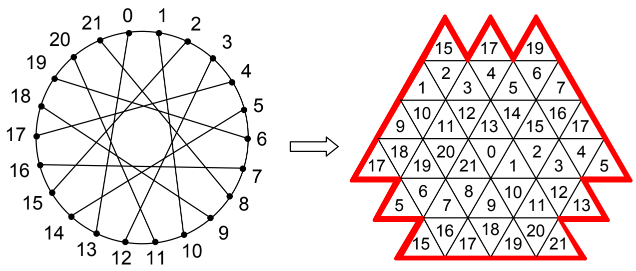

Figure 2: $CR(22,9)$ and its associated triangular grid representation.

Subsection 4.2 considers $CR(N,w)$ with ($N=2w$ or $2w+2$) and $w \geq 5$.

Subsection 4.3 considers $CR(N,w)$ with $2w+4 \leq N \leq (w-5)(w+1)$ and $w=9$.

Subsection 4.4 considers $CR(N,w)$ with $(w-5)(w+1)+2 \leq N \leq (w-3)(w+1)$.

Subsection 4.5 considers $CR(N,w)$ with $N \geq (w-3)(w+1)+2$.

4.1 $CR(N,w)$ with $w=3$

For chordal ring networks with $w=3$, we have the following result.

Theorem 7. *$CR(N,3)$ is non-dei if $N=4k+6$ and dei if $N=4k+8$, where $k \geq 0$.*

Proof. Let $G=CR(N,3)$. First assume that $N=4k+6$. Let e denote the edge $(0,N-3)$ and let $H=G-e$. Consider the node $x=\frac{N}{2}+1$ if k is odd and $x=\frac{N}{2}$ if k is even. Since any shortest clockwise path from 0 to x in G does not use the edge e,

$$\mathrm{dist}_{H,R}(0,x)=\mathrm{dist}_{G,R}(0,x) \overset{\text{by Theorem 1}}{=} k+3.$$

Since any shortest counterclockwise path from 0 to x in H must traverse via node $N-2$,

$$\begin{aligned}\mathrm{dist}_{H,L}(0,x) &= 2+\mathrm{dist}_{H,L}(N-2,x)\\ &= 2+\mathrm{dist}_{G,L}(0,x-(N-2)) \overset{\text{by Theorem 2}}{=} k+3.\end{aligned}$$

Thus $d_H(0,x)=\min\{\mathrm{dist}_{H,R}(0,x),\mathrm{dist}_{H,L}(0,x)\}=k+3$. By Theorem 3, $\mathcal{D}(G)=k+2$. By (1), G is non-dei.

Now assume that $N=4k+8$. Then G can be drawn as two concentric circles in such a way that half of the nodes are on the inner circle and half of them, the outer circle. (See Fig. 3 for an example.) By Theorems 1 and 2, when k is even (resp. odd), the node that is farthest from node 0 is $\frac{N}{2}-1$ (resp. $\frac{N}{2}$). By Theorem 3, $\mathcal{D}(G)=k+3$. We now prove that for all $v \in V(G)$, there are two edge-disjoint paths P_1, P_2 between 0 and v such that $|P_i| \leq \mathcal{D}(G)$ for $i=1,2$. See Fig. 3 for an illustration.

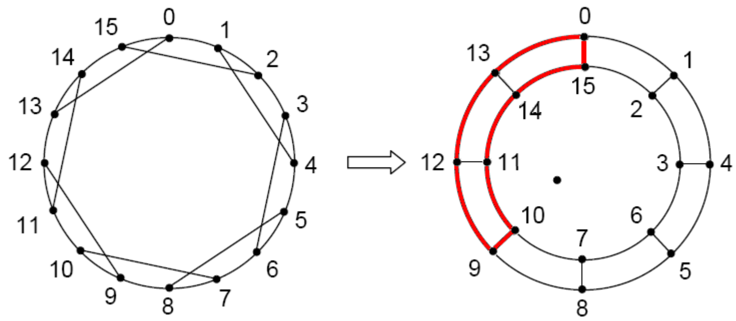

Figure 3: $CR(16,3)$.

Depending on which node is v, there are four possibilities. (i) If $v=\frac{N}{2}-1$ (resp. $\frac{N}{2}$), then let P_1 be path that follows edge $(0, N-1)$ and edges on the inner circle until v is reached; let P_2 be the path that follows edges on the outer circle until $\frac{N}{2}$ (resp. $\frac{N}{2}-1$) is reached, and then, follows edge $(\frac{N}{2}, \frac{N}{2}-1)$. (ii) If $v=\frac{N}{2}$ (resp. $\frac{N}{2}-1$), then the outer circle provides P_1, P_2. (iii) If $v=N-1$, then let C be the circle formed by the four edges $(0,1)$, $(1,2)$, $(2, N-1)$, and $(N-1, 0)$; clearly, C provides P_1, P_2. (iv) If $v \in V(G) - \{\frac{N}{2}-1, \frac{N}{2}, N-1\}$, then let $u=v+1$ if v is odd and let $u=v-1$ if v is even. Let C be the circle formed by the edges $(0, N-1)$, (u, v), and the portions of the inner and the outer circles between $(0, N-1)$ and (u, v); clearly, C provides P_1, P_2 (Fig. 3 shows an example of C with $v=9$). It is not difficult to verify that P_1, P_2 are two edge-disjoint paths between 0 and v and $|P_i| \leq \mathcal{D}(G)$ for $i=1,2$. Hence G is dei. ∎

4.2 $CR(N,w)$ with ($N=2w$ or $2w+2$) and $w \geq 5$

For chordal ring networks with $N=2w$ or $2w+2$, we have the following result.

Theorem 8. *$CR(N,w)$ is non-dei if ($N=2w$ or $2w+2$) and $w \geq 5$.*

Proof. Let $G=CR(N,w)$. First assume that $N=2w$ and $w \geq 5$. Set $e=(0,1)$ and $H=G-e$ for easy writing. Consider the node $x=\frac{w-1}{2}$. Since any shortest clockwise path from 0 to x in H must use the edges $(0, N-1)$ and $(N-1, w-1)$,

$$\mathrm{dist}_{H,R}(0,x) = 2 + \mathrm{dist}_{G,R}(w-1,x) = 2 + \mathrm{dist}_{G,R}(0, x-(w-1)) \overset{\text{by Theorem 1}}{=} 2 + \tfrac{w-1}{2}.$$

Since any shortest counterclockwise path from 0 to x in G does not use the edge e,

$$\mathrm{dist}_{H,L}(0,x) = \mathrm{dist}_{G,L}(0,x) \overset{\text{by Theorem 2}}{=} \tfrac{w+3}{2}.$$

Thus $\mathrm{d}_H(0,x) = \frac{w+3}{2}$. By Theorem 3, $\mathcal{D}(G) = \frac{w+1}{2}$. By (1), G is non-dei.

Now assume that $N = 2w+2$ and $w \geq 5$. Let $e = (0, N-1)$ and $H = G - e$. Consider the node $x = \frac{w+3}{2}$. Since any shortest clockwise path from 0 to x in H must use the edges $(0,1)$ and $(1,2)$ or the edges $(0,1)$ and $(1, 1+w)$,

$$\begin{aligned} \mathrm{dist}_{H,R}(0,x) &= 2 + \min\{\mathrm{dist}_{H,R}(2,x), \mathrm{dist}_{H,R}(1+w,x)\} \\ &= 2 + \min\{\mathrm{dist}_{G,R}(2,x), \mathrm{dist}_{G,R}(1+w,x)\} \overset{\text{by Theorem2}}{=} \tfrac{w+3}{2}. \end{aligned}$$

Since any shortest counterclockwise path from 0 to x in G must use the edge $(0, N-w)$,

$$\mathrm{dist}_{H,L}(0,x) = 1 + \mathrm{dist}_{H,L}(N-w,x) = 1 + \mathrm{dist}_{G,R}(0, N-w-x) \overset{\text{by Theorem 1}}{=} \tfrac{w+3}{2}.$$

Thus $\mathrm{d}_H(0,x) = \frac{w+3}{2}$. By Theorem 3, $\mathcal{D}(G) = \frac{w+1}{2}$. By (1), G is non-dei. ∎

4.3 $CR(N,w)$ with $2w+4 \leq N \leq (w-5)(w+1)$ and $w = 9$

For chordal ring networks with $w = 9$, we have the following result.

Theorem 9. *$CR(N,9)$ is dei if $N \in \{22, 24, 40\}$ and non-dei if $N \in \{26, 28, 30, 32, 34, 36, 38\}$.*

Proof. First assume that $N \in \{22, 24, 40\}$. Let $G = CR(22, 9)$. By Theorem 3, $\mathcal{D}(G) = 5$. All the nodes that can be reached from node 0 in $\mathcal{D}(G)$ edges are shown in the triangular grid representation in Fig. 2. As was shown in this figure, for all $v \in V(G)$, there are always two paths P_1, P_2 satisfying the DEI Lemma. Hence G is dei. Similarly, we can prove that $CR(24, 9)$ and $CR(40, 9)$ are dei.

Now assume $N \in \{26, 28, 30, 32, 34, 36, 38\}$. Let $G = CR(N, 9)$. There are three cases.

Case 1: $N \in \{26, 28, 30, 32, 34\}$. Let $e = (0, N-1)$ and $H = G - e$. Consider the node $x = N - 4$. Since any shortest clockwise path from 0 to x in H must use the edges $(0,1)$ and $(1,2)$ or the edges $(0,1)$ and $(1,10)$,

$$\begin{aligned} \mathrm{dist}_{H,R}(0,x) &= 2 + \min\{\mathrm{dist}_{H,R}(2,x), \mathrm{dist}_{H,R}(10,x)\} \\ &= 2 + \min\{\mathrm{dist}_{G,R}(2,x), \mathrm{dist}_{G,R}(10,x)\} \overset{\text{by Theorem 1}}{=} 6. \end{aligned}$$

Since any shortest counterclockwise path from 0 to x in G must use the edge $(0, N-9)$,

$$\mathrm{dist}_{H,L}(0,x) = 1 + \mathrm{dist}_{H,L}(N-9, x) = 1 + \mathrm{dist}_{G,L}(0,-5) \overset{\text{by Theorem 2}}{=} 6.$$

Thus $d_H(0,x) = 6$. By Theorem 3, $\mathcal{D}(G) = 5$. By (1), $CR(N,9)$ is non-dei if $N \in \{26, 28, 30, 32, 34\}$.

Case 2: $N = 36$. Let $e = (0,1)$ and $H = G - e$. Consider the node $x = 13$. Since any shortest clockwise path from 0 to x in H must use the edges $(0,35)$ and $(35,8)$,

$$\mathrm{dist}_{H,R}(0,x) = 2 + \mathrm{dist}_{G,R}(8,x) = 2 + \mathrm{dist}_{G,R}(0,5) \overset{\text{by Theorem 1}}{=} 7.$$

Since any shortest shortest counterclockwise path from 0 to x in G does not use edge e,

$$\mathrm{dist}_{H,L}(0,x) = \mathrm{dist}_{G,L}(0,x) \overset{\text{by Theorem 2}}{=} 7.$$

Thus $\mathrm{d}_H(0,x) = 7$. By Theorem 3, $\mathcal{D}(G) = 6$. By (1), $CR(36,9)$ is non-dei.

Case 3: $N = 38$. Let $e = (0,37)$ and $H = G - e$. Consider the node $x = 15$. Since any shortest clockwise path from 0 to x in H must use the edges $(0,1)$ and $(1,2)$ or the edges $(0,1)$ and $(1,10)$,

$$\begin{aligned} \mathrm{dist}_{H,R}(0,x) &= 2 + \min\{\mathrm{dist}_{H,R}(2,x), \mathrm{dist}_{H,R}(10,x)\} \\ &= 2 + \min\{\mathrm{dist}_{G,R}(2,x), \mathrm{dist}_{G,R}(10,x)\} \overset{\text{by Theorem 1}}{=} 7. \end{aligned}$$

Since any shortest counterclockwise path from 0 to x in G must use the edge $(0,29)$,

$$\mathrm{dist}_{H,L}(0,x) = 1 + \mathrm{dist}_{H,L}(29,x) = 1 + \mathrm{dist}_{G,L}(0,14) \overset{\text{by Theorem 2}}{=} 7.$$

Thus $\mathrm{d}_H(0,x) = 7$. By Theorem 3, $\mathcal{D}(G) = 6$. By (1), $CR(38,9)$ is non-dei. ∎

4.4 $CR(N, w)$ with $(w-5)(w+1)+2 \le N \le (w-3)(w+1)$

When $(w-5)(w+1)+2 \le N \le (w-3)(w+1)$, Case 2 of the Diameter Theorem occurs and we have the following result.

Theorem 10. *For $(w-5)(w+1)+2 \le N \le (w-3)(w+1)$, we have:*

(A) *$CR(18, 7)$ is non-dei.*

(B) *$CR(N, w)$ is dei if $N = (w-5)(w+1)+2$ and $w \ge 9$.*

(C) *$CR(N, w)$ is non-dei if $N = (w-5)(w+1)+4$ and $w \ge 7$.*

(D) *$CR(N, w)$ is dei if $N = (w-5)(w+1)+2t$, $w \ge 7$ and $3 \le t \le \frac{w+3}{2}$.*

(E) *$CR(N, w)$ is non-dei if $N = (w-5)(w+1)+2(w-t)$, $w \ge 7$ and $0 \le t \le \frac{w-5}{2}$.*

(F) *$CR(N, w)$ is non-dei if $N = (w-3)(w+1)$ and $w \ge 7$.*

Proof. Let $G = CR(18, 7)$, $e = (0, 11)$, and $H = G - e$. By Fig. 4, $d_H(0, 11) = 5$. By Theorem 3, $\mathcal{D}(G) = 4$. By (1), $CR(18, 7)$ is non-dei.

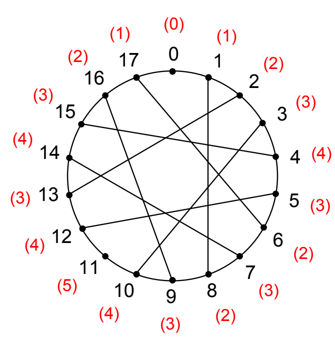

Figure 4: The graph $H = G - e$, where $G = CR(18, 7)$ and $e = (0, 11)$. The number inside parentheses is the distance to node 0.

We now prove (B). Let $G = CR(N, w)$. By Theorem 3, $\mathcal{D}(G) = w - 3$, which is an even number. The triangular grid representation of G is of the form shown in Fig. 5(a). In the following, we will consider the triangular grid representation of G as the combination of several shaded areas. For each shaded area, we will prove that for each node v in that shaded area, there are always two paths P_1, P_2 satisfying the DEI Lemma. See also Fig. 6 for a specific triangular grid representation.

First consider the shaded area shown in Fig. 5(a). Since $\mathcal{D}(G) \geq 6$, for each node v in this shaded area, there are always two paths P_1, P_2 satisfying the DEI Lemma. Now consider the shaded area shown in Fig. 5(b). Obviously, for each node v in in this shaded area, there are always two paths P_1, P_2 satisfy the DEI Lemma.

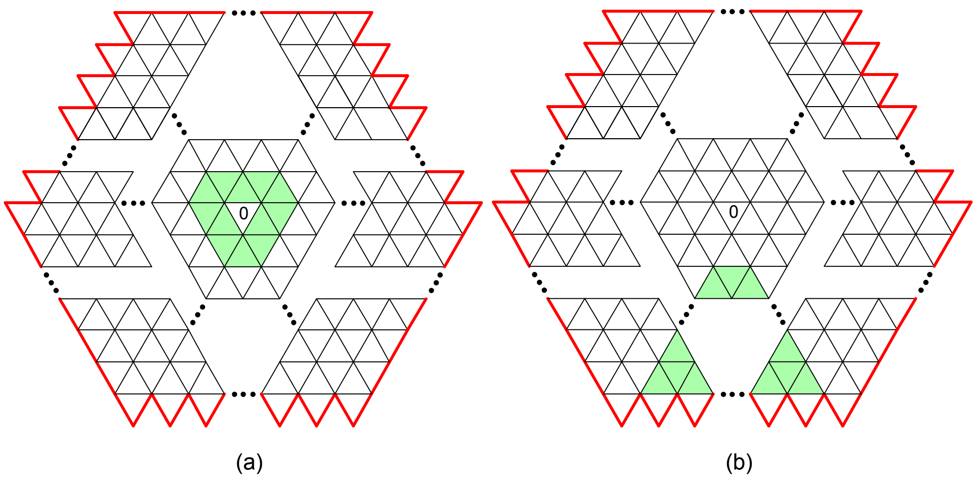

Figure 5: (a) The triangular grid representation and the first shaded area. (b) The second shaded area.

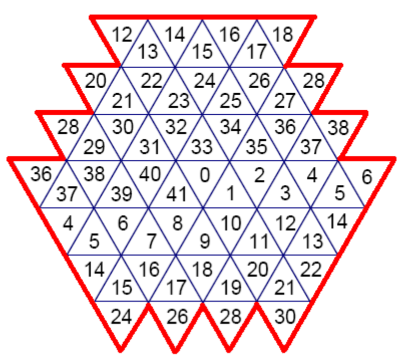

Figure 6: The triangular grid representation of $CR(42, 9)$.

Consider the shaded area shown in Fig. 7(a). For convenience, let a denote the node $w + 1$. Clearly, $d_G(0, a) = 2$. Each node v in this shaded area has $d_G(0, v) = d_G(a, v)$ and the distance from node 0 to any node in this shaded area is at most $\mathcal{D}(G) - 2$.

Consequently, for each node v in in this shaded area, there are always two paths P_1, P_2 satisfy the DEI Lemma. Consider the shaded area shown in Fig. 7(b). For convenience, let b denote the node $N - w - 1$. Clearly, $d_G(0, b) = 2$. Each node v in this shaded area has $d_G(0, v) = d_G(b, v)$ and the distance from node 0 to any node in this shaded area is at most $\mathcal{D}(G) - 2$. Consequently, for each node v in in this shaded area, there are always two paths P_1, P_2 satisfy the DEI Lemma.

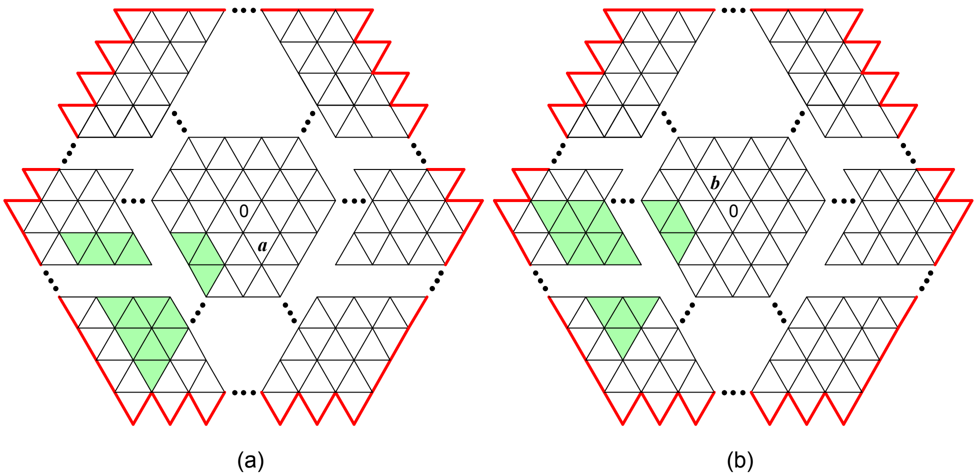

Figure 7: (a) The third shaded area. (b) The fourth shaded area.

By similar arguments, for each node v, except those in the shaded area of Fig. 8, there are always two paths P_1, P_2 satisfying the DEI Lemma. We now consider nodes in the shaded area of Fig. 8. See Figs. 9-13 for an illustration.

First consider Fig. 9 and the cells labeled $i_1, i_2, u_1, u_2, v_1, v_2$, in this figure. The label i_1 is in the leftmost cell of the first row; thus $i_1 = N - (\frac{w+1}{2})(w-3) = \frac{w^2-6w-3}{2}$. Therefore $i_2 = i_1 + 2 = \frac{w^2-6w+1}{2}$. The label u_1 is in the leftmost cell of the last row; thus $u_1 = (\frac{w-1}{2})(w-3) = \frac{w^2-4w+3}{2}$. The label u_2 is in the rightmost node of the reciprocal second row; therefore $u_2 = u_1 - (w+1) = \frac{w^2-6w+1}{2}$. The label v_1 is in the rightmost cell of the last row; thus $v_1 = (\frac{w+1}{2})(w-3) = \frac{w^2-2w-3}{2}$. The label v_2 is in the rightmost node of the reciprocal third row; therefore $v_2 = v_1 - 2(w-1) = \frac{w^2-6w+1}{2}$. From the above,

$$i_2 = u_2 = v_2. \tag{2}$$

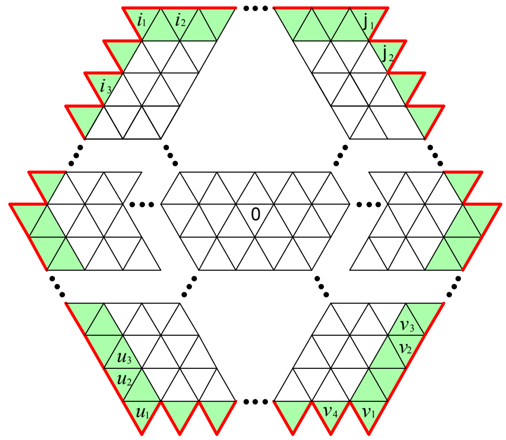

Figure 8: Those shaded boundary nodes need additional discussion.

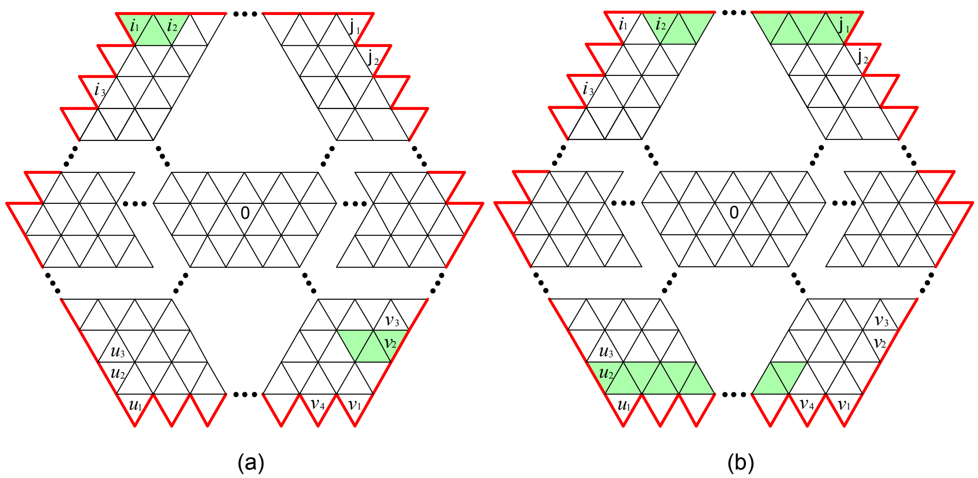

Figure 9: The shaded nodes in (a) are identical; those in (b) are also identical.

Since $i_2 = v_2$, in Fig. 9(a), the set of nodes in the shaded area of the first row is identical to the set of nodes in the shaded area of the reciprocal third row. Since $i_2 = u_2$, in Fig. 9(b), the set of nodes in the shaded area of the first row is identical to the set of nodes in the shaded area of the reciprocal second row. By similar arguments, the sets of nodes in the two shaded areas in Figs. 10(a), 10(b), 11(a) and 11(b) are identical, too.

Now consider Fig. 12 and the cells labeled i_3, j_1, j_2, v_4 in this figure. The label i_3 is in the leftmost cell of the third row; thus $i_3 = i_1 + 2(w-1) = \frac{w^2-2w-7}{2}$. The label j_1 is in the rightmost cell of the first row; thus $j_1 = N - (\frac{w-1}{2})(w-3) = \frac{w^2-4w-9}{2}$. Therefore

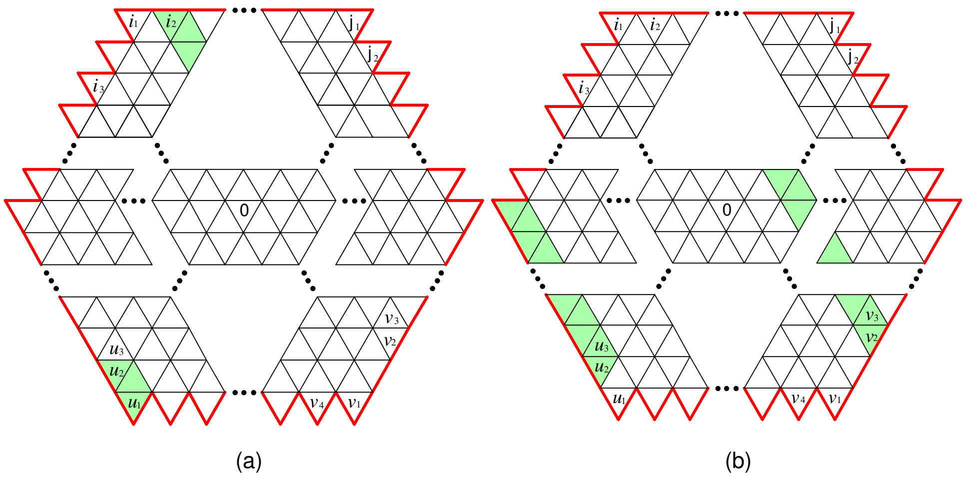

Figure 10: The shaded nodes in (a) are identical; those in (b) are also identical.

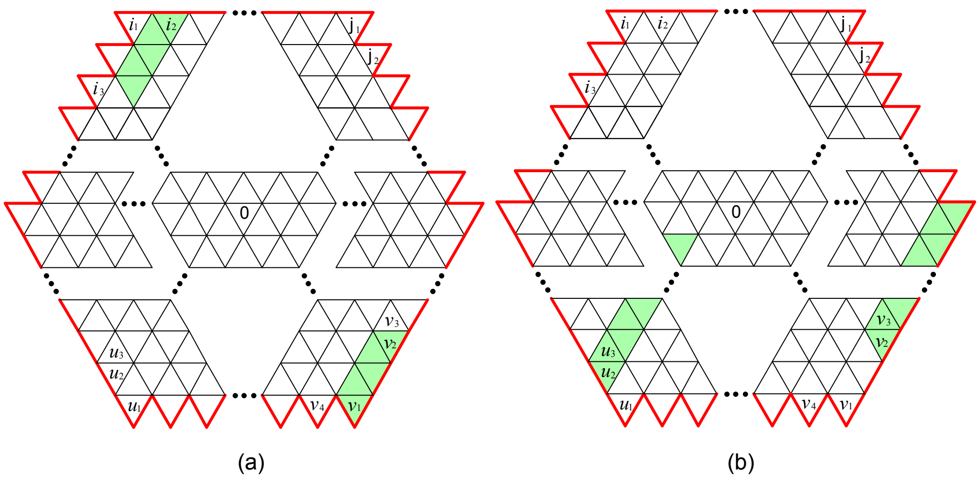

Figure 11: The shaded nodes in (a) are identical; those in (b) are also identical.

$j_2 = j_1 + (w+1) = \frac{w^2-2w-7}{2}$. Since $v_1 = \frac{w^2-2w-3}{2}$, $v_4 = v_1 - 2 = \frac{w^2-2w-7}{2}$. From the above,

$$i_3 = j_2 = v_4. \tag{3}$$

Thus the sets of nodes in the two shaded areas in Figs. 12(a), 12(b), 13(a) and 13(b) are identical, too.

From the above discussion, for each node v in the shaded area of Fig. 8, except the five nodes in $\{i_2-1, i_2, i_2+1, u_3, i_3\}$, there are always two paths P_1, P_2 satisfying the DEI Lemma. For each of the five nodes in the set, we now use (i)-(v) to prove that there are also two paths P_1, P_2 satisfying the DEI Lemma.

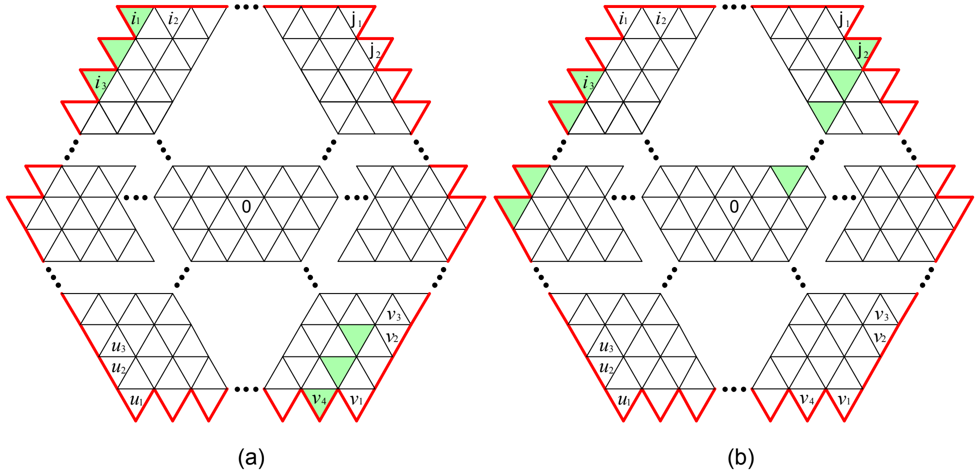

Figure 12: The shaded nodes in (a) are identical; those in (b) are also identical.

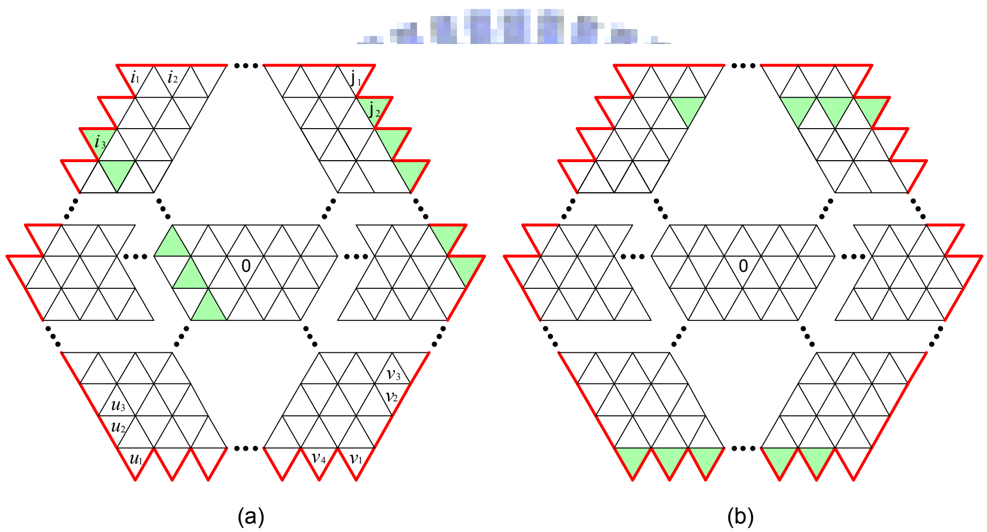

Figure 13: The shaded nodes in (a) are identical; those in (b) are also identical.

(i) Consider node i_2-1. By (2), $i_2-1=v_2-1$. Let $P_1 = 0 \xrightarrow{-w} N-w \xrightarrow{-1} N-(w+1) \xrightarrow{-w} N-(2w+1) \xrightarrow{-1} N-(2w+2) \xrightarrow{-w} N-(3w+2) \xrightarrow{-1} N-(3w+3) \xrightarrow{-w} \cdots \xrightarrow{-1} N-(\frac{w-5}{2})(w+1) \xrightarrow{-w} N-(\frac{w-5}{2})(w+1)-w = i_2-1$. Let $P_2 = 0 \xrightarrow{+1} 1 \xrightarrow{+w} w+1 \xrightarrow{+1} w+2 \xrightarrow{+w} 2w+2 \xrightarrow{+1} 2w+3 \xrightarrow{+w} 3w+3 \xrightarrow{+1} \cdots \xrightarrow{+w} (\frac{w-7}{2})(w+1) \xrightarrow{+1} (\frac{w-7}{2})(w+1)+1 \xrightarrow{+1} (\frac{w-7}{2})(w+1)+2 \xrightarrow{+1} (\frac{w-7}{2})(w+1)+3 = v_2-1$. See Fig. 6 for an illustration. Then $i_2-1=13$, $P_1 = 0 \to 33 \to 32 \to 23 \to 22 \to 13$ and $P_2 = 0 \to 1 \to 10 \to 11 \to 12 \to 13$.

(ii) Consider node i_2. By (2), $i_2 = v_2$. Let $P_1 = 0 \xrightarrow{-w} N-w \xrightarrow{-1} N-(w+1) \xrightarrow{-w}$

$N-(2w+1) \xrightarrow{-1} N-(2w+2) \xrightarrow{-w} N-(3w+2) \xrightarrow{-1} N-(3w+3) \xrightarrow{-w} \cdots \xrightarrow{-1} N-(\frac{w-7}{2})(w+1) \xrightarrow{-w} N-(\frac{w-7}{2})(w+1)-w \xrightarrow{+1} N-(\frac{w-7}{2})(w+1)-w+1 \xrightarrow{-w} N-(\frac{w-7}{2})(w+1)-2w+1 \xrightarrow{-1} N-(\frac{w-7}{2})(w+1)-2w = i_2$. Let $P_2 = 0 \xrightarrow{+1} 1 \xrightarrow{+w} w+1 \xrightarrow{+1} w+2 \xrightarrow{+w} 2w+2 \xrightarrow{+1} 2w+3 \xrightarrow{+w} 3w+3 \xrightarrow{+1} \cdots \xrightarrow{+w} (\frac{w-7}{2})(w+1) \xrightarrow{+1} (\frac{w-7}{2})(w+1)+1 \xrightarrow{+1} (\frac{w-7}{2})(w+1)+2 \xrightarrow{+1} (\frac{w-7}{2})(w+1)+3 \xrightarrow{+1} (\frac{w-7}{2})(w+1)+4 = v_2$. Again, see Fig. 6. Then $i_2 = 14$, $P_1 = 0 \to 33 \to 32 \to 23 \to 24 \to 15 \to 14$ and $P_2 = 0 \to 1 \to 10 \to 11 \to 12 \to 13 \to 14$.

(iii) Consider node i_2+1. By (2), $i_2+1 = u_2+1$. Let $P_1 = 0 \xrightarrow{-w} N-w \xrightarrow{-1} N-(w+1) \xrightarrow{-w} N-(2w+1) \xrightarrow{-1} N-(2w+2) \xrightarrow{-w} N-(3w+2) \xrightarrow{-1} N-(3w+3) \xrightarrow{-w} \cdots \xrightarrow{-1} N-(\frac{w-7}{2})(w+1) \xrightarrow{-w} N-(\frac{w-7}{2})(w+1)-w \xrightarrow{+1} N-(\frac{w-7}{2})(w+1)-w+1 \xrightarrow{-w} N-(\frac{w-7}{2})(w+1)-2w+1 = i_2 = 1$. Let $P_2 = 0 \xrightarrow{-1} N-1 \xrightarrow{+w} w-1 \xrightarrow{-1} w-2 \xrightarrow{+w} 2w-2 \xrightarrow{-1} 2w-3 \xrightarrow{+w} 3w-3 \xrightarrow{-1} \cdots \xrightarrow{+w} (\frac{w-5}{2})(w-1) \xrightarrow{-1} (\frac{w-5}{2})(w-1)-1 = u_2+1$. See Fig.6 for an example. Then $i_2+1 = 15$, $P_1 = 0 \to 33 \to 32 \to 23 \to 24 \to 15$ and $P_2 = 0 \to 41 \to 8 \to 7 \to 16 \to 15$.

(iv) Consider node u_3. By (2), $i_2+1 = u_2+1$; thus $u_3 = v_3$. Let $P_1 = 0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} 5 \xrightarrow{+w} 5+w \xrightarrow{+1} 5+(w+1) \xrightarrow{+w} 5+(2w+1) \xrightarrow{+1} 5+(2w+2) \xrightarrow{+w} 5+(3w+2) \xrightarrow{+1} 5+(3w+3) \xrightarrow{+w} \cdots \xrightarrow{+1} 5+(\frac{w-9}{2})(w+1) = v_3$. Let $P_2 = 0 \xrightarrow{-1} N-1 \xrightarrow{+w} w-1 \xrightarrow{-1} w-2 \xrightarrow{+w} 2w-2 \xrightarrow{-1} 2w-3 \xrightarrow{+w} 3w-3 \xrightarrow{-1} \cdots \xrightarrow{+w} (\frac{w-7}{2})(w-1) \xrightarrow{-1} (\frac{w-7}{2})(w-1)-1 \xrightarrow{-1} (\frac{w-7}{2})(w-1)-2 \xrightarrow{-1} (\frac{w-7}{2})(w-1)-3 = u_3$. Again, see Fig. 6. Then $u_3 = 5$, $P_1 = 0 \to 1 \to 2 \to 3 \to 4 \to 5$ and $P_2 = 0 \to 41 \to 8 \to 7 \to 6 \to 5$.

(v) Consider node i_3. By (3), $i_3 = v_4$. Let $P_1 = 0 \xrightarrow{-w} N-w \xrightarrow{-1} N-(w+1) \xrightarrow{-w} N-(2w+1) \xrightarrow{-1} N-(2w+2) \xrightarrow{-w} N-(3w+2) \xrightarrow{-1} N-(3w+3) \xrightarrow{-w} \cdots \xrightarrow{-1} N-(\frac{w-7}{2})(w+1) \xrightarrow{-1} N-(\frac{w-7}{2})(w+1)-1 \xrightarrow{-1} N-(\frac{w-7}{2})(w+1)-2 \xrightarrow{-1} N-(\frac{w-7}{2})(w+1)-3 \xrightarrow{-1} N-(\frac{w-7}{2})(w+1)-4 = i_3$. Let $P_2 = 0 \xrightarrow{+1} 1 \xrightarrow{+w} w+1 \xrightarrow{+1} w+2 \xrightarrow{+w} 2w+2 \xrightarrow{+1} 2w+3 \xrightarrow{+w} 3w+3 \xrightarrow{+1} \cdots \xrightarrow{+w} (\frac{w-5}{2})(w+1) \xrightarrow{-1} (\frac{w-5}{2})(w+1)-1 \xrightarrow{+w} (\frac{w-5}{2})(w+1)-1+w = v_4$. Take Fig. 6 for an example again. Then $i_3 = 28$, $P_1 = 0 \to 33 \to 32 \to 31 \to 30 \to 29 \to 28$ and $P_2 = 0 \to 1 \to 10 \to 11 \to 20 \to 19 \to 28$.

In (i)-(iii) and (v), P_1 is a counterclockwise path, P_2 is a clockwise path and P_1, P_2 are edge-disjoint. In (iv), P_1 and P_2 are both clockwise paths and they are edge-disjoint. It is not difficult to verify that P_1, P_2 satisfy the DEI Lemma. We now complete the proof of (B).

Consider (C). Let $G = CR(N, w)$, $e = (0, 1)$, and $H = G - e$. Consider the node $x = \frac{N}{2} - 1$. Note that $\mathrm{d}_H(0, x) = \min\{\mathrm{dist}_{H,R}(0, x), \mathrm{dist}_{H,L}(0, x)\}$. Since any shortest counterclockwise path from 0 to x in G will not use the edge e,

$$\mathrm{dist}_{H,L}(0, x) = \mathrm{dist}_{G,L}(0, x) \overset{\text{by Theorem 2}}{=} w - 2.$$

Since any shortest clockwise path from 0 to x in H must use the edge $(0, N - 1)$,

$$\mathrm{dist}_{H,R}(0, x) = 2 + \mathrm{dist}_{G,R}(w - 1, x) = 2 + \mathrm{dist}_{G,R}(0, x - (w - 1)) \overset{\text{by Theorem 1}}{=} 2 + w - 4.$$

Thus $d_H(0, x) = w - 2$. By Theorem 3, $\mathcal{D}(G) = w - 3$. By (1), G is non-dei.

Consider (D). Let $G = CR(N, w)$. By Theorem 3, $\mathcal{D}(G) = w - 2$, which is an odd number. The proof of (D) is similar to that of (B) except that the triangular grid representation is of the form shown in Fig. 14 and the nodes corresponding to those in Fig. 8 are those shown in Fig. 14.

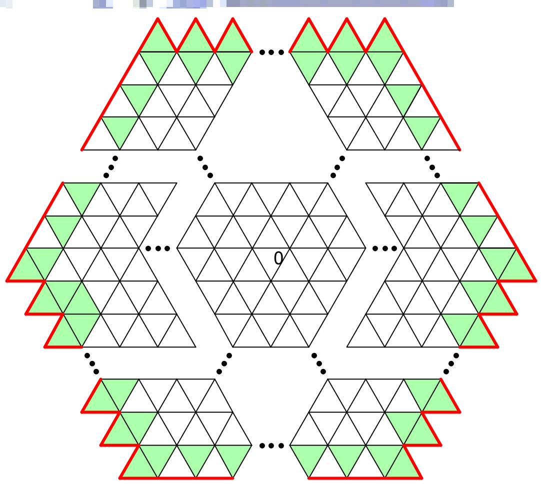

Figure 14: Those shaded boundary nodes need additional discussion.

Consider (E). Let $G = CR(N, w)$, $e = (0, 1)$, and $H = G - e$. Consider the node $x = \frac{N}{2} + t + 2 - w$. Since any shortest counterclockwise path from 0 to x in G will not use the edge e,

$$\mathrm{dist}_{H,L}(0,x) = \mathrm{dist}_{G,L}(0,x) \overset{\text{by Theorem 2}}{=} w-1.$$

Since any shortest clockwise path from 0 to x in H must use the edge $(0, N-1)$,

$$\mathrm{dist}_{H,R}(0,x) = 2 + \mathrm{dist}_{G,R}(w-1,x) = 2 + \mathrm{dist}_{G,R}(0, x-(w-1)) \overset{\text{by Theorem 1}}{=} 2+w-3.$$

Thus $d_H(0,x) = w-1$. By Theorem 3, $\mathcal{D}(G) = w-2$. By (1), $CR(N,w)$ is non-dei.

Consider (F). Let $G = CR(N,w)$, $e = (0, N-w)$, and $H = G-e$. Consider the node $x = \frac{N}{2} - 2$. Since any shortest clockwise path from 0 to x in G will not use the edge e,

$$\mathrm{dist}_{H,R}(0,x) = \mathrm{dist}_{G,R}(0,x) \overset{\text{by Theorem 1}}{=} w+2k.$$

Since any shortest counterclockwise path from 0 to x in H must travel via node $N-2$,

$$\begin{aligned}\mathrm{dist}_{H,L}(0,x) &= 2 + \mathrm{dist}_{H,L}(n-2,x) = 2 + \mathrm{dist}_{H,L}(0, x-(n-2)) \\ &\overset{\text{by Theorem 2}}{=} 2+w-3.\end{aligned}$$

Thus $d_H(0,x) = w-1$. By Theorem 3, $\mathcal{D}(G) = w-2$. By (1), $CR(N,w)$ is non-dei. ∎

4.5 $CR(N,w)$ with $N \geq (w-3)(w+1)+2$

When $N \geq (w-3)(w+1)+2$, Case 1 of the Diameter Theorem occurs and we have the following result.

Theorem 11. *For $N \geq (w-3)(w+1)+2$, we have:*

(a) *$CR(N,w)$ is non-dei if $N = (w-3)(w+1)+2+2k(w+1)$, $w \geq 5$ and $k \geq 0$.*

(b) *$CR(16,5)$ is non-dei.*

(c) *$CR(N,w)$ is dei if $N = (w-3)(w+1)+4+2k(w+1)$ and ($w=5$ and $k \geq 1$) or ($w \geq 7$ and $k \geq 0$).*

(d) *$CR(N,w)$ is non-dei if $N = (w-3)(w+1)+2t+2k(w+1)$, $w \geq 5$, $3 \leq t \leq \frac{w+3}{2}$ and $k \geq 0$.*

(e) *$CR(N,w)$ is dei if $N = (w-3)(w+1)+(w+5)+2k(w+1)$, $w \geq 5$ and $k \geq 0$.*

(f) *$CR(N,w)$ is non-dei if $N=(w-3)(w+1)+2(w-t)+2k(w+1)$, $w\geq 7$, $0\leq t\leq \frac{w-7}{2}$ and $k\geq 0$.*

(g) *$CR(N,w)$ is non-dei if $N=(w-1+2k)(w+1)$, $w\geq 5$ and $k\geq 0$.*

Proof. Consider (a). Let $G=CR(N,w)$, $e=(0,1)$, and $H=G-e$. Consider the node $x=\frac{N}{2}-1$. Since any shortest counterclockwise path from 0 to x in G will not use the edge e,

$$\mathrm{dist}_{H,L}(0,x)=\mathrm{dist}_{G,L}(0,x)\overset{\text{by Theorem 2}}{=}w-1+2k.$$

Since any shortest clockwise path from 0 to x in H must use the edge $(0,N-1)$,

$$\mathrm{dist}_{H,R}(0,x)=2+\mathrm{dist}_{G,R}(w-1,x)=2+\mathrm{dist}_{G,R}(0,x-(w-1))\overset{\text{by Theorem 1}}{=}2+w-3+2k.$$

Thus $d_H(0,x)=w-1+2k$. By Theorem 3, $\mathcal{D}(G)=w-2+2k$. By (1), G is non-dei.

Consider (b). Let $G=CR(16,5)$, $e=(0,1)$, and $H=G-e$. It is not difficult to see from Fig. 15 that $d_H(0,1)=5$. By Theorem 3, $\mathcal{D}(G)=4$. By (1), $CR(16,5)$ is non-dei.

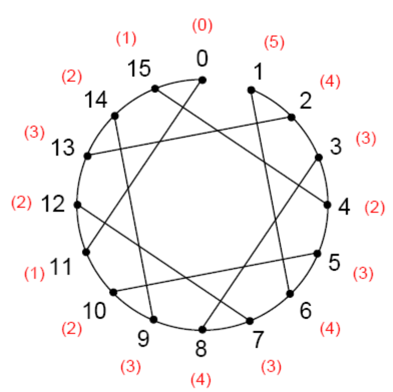

Figure 15: The graph $H=G-e$, where $G=CR(16,5)$ and $e=(0,1)$. The number inside parentheses is the distance to node 0.

The proofs of (c) and (e) are similar to that of Theorem 10(B) and we omit them.

Consider (d). Let $G=CR(N,w)$, $e=(0,N-w)$, and $H=G-e$. Consider the node $x=\frac{N}{2}-t+3$. Since any shortest clockwise path from 0 to x in G does not use edge e,

$$\mathrm{dist}_{H,R}(0,x)=\mathrm{dist}_{G,R}(0,x)\overset{\text{by Theorem 1}}{=}w+2k.$$

Since any shortest counterclockwise path from 0 to x in H must travel via node $N-2$,

$$\begin{aligned}\mathrm{dist}_{H,L}(0,x) &= 2+\mathrm{dist}_{H,L}(N-2,x) = 2+\mathrm{dist}_{H,L}(0,x-(N-2)) \\ &\overset{\text{by Theorem 2}}{=} 2+w-2+2k.\end{aligned}$$

Thus $d_H(0,x) = w+2k$. By Theorem 3, $\mathcal{D}(G) = w-1+2k$. By (1), G is non-dei.

Consider (f). Let $G = CR(N,w)$, $e = (0, N-w)$, and $H = G-e$. Consider the node $x = \frac{N}{2}+t+3$. Since any shortest clockwise path from 0 to x in G will not use the edge e,

$$\mathrm{dist}_{H,R}(0,x) = \mathrm{dist}_{G,R}(0,x) \overset{\text{by Theorem 1}}{=} w+2k.$$

Since any shortest counterclockwise path from 0 to x in H must travel via node $N-2$,

$$\begin{aligned}\mathrm{dist}_{H,L}(0,x) &= 2+\mathrm{dist}_{H,L}(N-2,x) = 2+\mathrm{dist}_{H,L}(0,x-(N-2)) \\ &\overset{\text{by Theorem 2}}{=} 2+w-1+2k.\end{aligned}$$

Thus $d_H(0,x) = w+2k+1$. By Theorem 3, $\mathcal{D}(G) = w+2k$. By (1), G is non-dei.

Consider (g). Let $G = CR(N,w)$, $e = (0, N-w)$, and $H = G-e$. Consider the node $x = \frac{N}{2}+2$. Since any shortest clockwise path from 0 to x in G will not use the edge e,

$$\mathrm{dist}_{H,R}(0,x) = \mathrm{dist}_{G,R}(0,x) \overset{\text{by Theorem 1}}{=} w+2k.$$

Since any shortest counterclockwise path from 0 to x in H must travel via node $N-2$,

$$\begin{aligned}\mathrm{dist}_{H,L}(0,x) &= 2+\mathrm{dist}_{H,L}(N-2,x) = 2+\mathrm{dist}_{H,L}(0,x-(N-2)) \\ &\overset{\text{by Theorem 2}}{=} 2+w-1+2k.\end{aligned}$$

Thus $d_H(0,x) = w+2k+1$. By Theorem 3, $\mathcal{D}(G) = w+2k$. By (1), G is non-dei. ∎

5 Concluding remarks

The purpose of the thesis is to discuss the dei property of chordal ring networks. The chordal ring network is a commonly used extension of the ring network. In [1], Arden and Lee proposed a formula for computing the diameter of a chordal ring network $CR(N, w)$. In this thesis, we have shown that this formula is incorrect. We have successfully determined if $CR(N, w)$ is dei for all $w \in \{3, 5, 7, 9\}$. Arden and Lee's formula actually contains three cases: Cases 1 and 2 are correct and Case 3 is incorrect (see the Diameter Theorem for details). In this thesis, we have successfully determined if $CR(N, w)$ is dei for Cases 1 and 2 and some special cases of Case 3.

Before ending this thesis, we list the dei results for chordal ring networks $CR(N, w)$ with $N \leq 200$ and $w \leq 11$ in the following table. For clarity, we omit the word "non-dei", and "?" means "not determined yet". From this table, it is easy to see that

when $w = 3$ and $N \geq 8$, there is a dei graph whenever N increases $w + 1$;

when $w = 5$ and $N \geq 22$, there is a dei graph whenever N increases $w + 1$; and

when $w \geq 7$ and $N \geq (w - 3)(w + 1) + 4$, there is a dei graph if N increased $w + 1$.

$N \setminus w$	3	5	7	9	11
6	Thm 7				
8	dei Thm 7				
10	Thm 7	Thm 8			
12	dei Thm 7	Thm 8			
14	Thm 7	Thm 11	Thm 8		
16	dei Thm 7	Thm 11	Thm 8		
18	Thm 7	Thm 11	Thm 10	Thm 8	
20	dei Thm 7	Thm 11	Thm 10	Thm 8	
22	Thm 7	dei Thm 11	dei Thm 10	dei Thm 9	Thm 8
24	dei Thm 7	Thm 11	dei Thm 10	dei Thm 9	Thm 8
26	Thm 7	Thm 11	dei Thm 10	Thm 9	?
28	dei Thm 7	dei Thm 11	Thm 10	Thm 9	?
30	Thm 7	Thm 11	Thm 10	Thm 9	?
32	dei Thm 7	Thm 11	Thm 10	Thm 9	?
34	Thm 7	dei Thm 11	Thm 11	Thm 9	?
36	dei Thm 7	Thm 11	dei Thm 11	Thm 9	?
38	Thm 7	Thm 11	Thm 11	Thm 9	?
40	dei Thm 7	dei Thm 11	Thm 11	dei Thm 9	?
42	Thm 7	Thm 11	Thm 11	dei Thm 10	?
44	dei Thm 7	Thm 11	dei Thm 11	Thm 10	?
46	Thm 7	dei Thm 11	Thm 11	dei Thm 10	?
48	dei Thm 7	Thm 11	Thm 11	dei Thm 10	?
50	Thm 7	Thm 11	Thm 11	dei Thm 10	?
52	dei Thm 7	dei Thm 11	dei Thm 11	dei Thm 10	?
54	Thm 7	Thm 11	Thm 11	Thm 10	?
56	dei Thm 7	Thm 11	Thm 11	Thm 10	?
58	Thm 7	dei Thm 11	Thm 11	Thm 10	?
60	dei Thm 7	Thm 11	dei Thm 11	Thm 10	?
62	Thm 7	Thm 11	Thm 11	Thm 11	?
64	dei Thm 7	dei Thm 11	Thm 11	dei Thm 11	?
66	Thm 7	Thm 11	Thm 11	Thm 11	?
68	dei Thm 7	Thm 11	dei Thm 11	Thm 11	?
70	Thm 7	dei Thm 11	Thm 11	Thm 11	?
72	dei Thm 7	Thm 11	Thm 11	Thm 11	?
74	Thm 7	Thm 11	Thm 11	dei Thm 11	dei Thm 10
76	dei Thm 7	dei Thm 11	dei Thm 11	Thm 11	Thm 10
78	Thm 7	Thm 11	Thm 11	Thm 11	dei Thm 10
80	dei Thm 7	Thm 11	Thm 11	Thm 11	dei Thm 10
82	Thm 7	dei Thm 11	Thm 11	Thm 11	dei Thm 10
84	dei Thm 7	Thm 11	dei Thm 11	dei Thm 11	dei Thm 10
86	Thm 7	Thm 11	Thm 11	Thm 11	dei Thm 10
88	dei Thm 7	dei Thm 11	Thm 11	Thm 11	Thm 10
90	Thm 7	Thm 11	Thm 11	Thm 11	Thm 10
92	dei Thm 7	Thm 11	dei Thm 11	Thm 11	Thm 10
94	Thm 7	dei Thm 11	Thm 11	dei Thm 11	Thm 10
96	dei Thm 7	Thm 11	Thm 11	Thm 11	Thm 10
98	Thm 7	Thm 11	Thm 11	Thm 11	Thm 11
100	dei Thm 7	dei Thm 11	dei Thm 11	Thm 11	dei Thm 11
102	Thm 7	Thm 11	Thm 11	Thm 11	Thm 11

$N \setminus w$	3		5		7		9		11	
104	dei	Thm 7		Thm 11		Thm 11	dei	Thm 11		Thm 11
106		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
108	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
110		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
112	dei	Thm 7	dei	Thm 11		Thm 11		Thm 11	dei	Thm 11
114		Thm 7		Thm 11		Thm 11	dei	Thm 11		Thm 11
116	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
118		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
120	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
122		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
124	dei	Thm 7	dei	Thm 11	dei	Thm 11	dei	Thm 11	dei	Thm 11
126		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
128	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
130		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
132	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
134		Thm 7		Thm 11		Thm 11	dei	Thm 11		Thm 11
136	dei	Thm 7	dei	Thm 11		Thm 11		Thm 11	dei	Thm 11
138		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
140	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
142		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
144	dei	Thm 7		Thm 11		Thm 11	dei	Thm 11		Thm 11
146		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
148	dei	Thm 7	dei	Thm 11	dei	Thm 11		Thm 11	dei	Thm 11
150		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
152	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
154		Thm 7	dei	Thm 11		Thm 11	dei	Thm 11		Thm 11
156	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
158		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
160	dei	Thm 7	dei	Thm 11		Thm 11		Thm 11	dei	Thm 11
162		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
164	dei	Thm 7		Thm 11	dei	Thm 11	dei	Thm 11		Thm 11
166		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
168	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
170		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
172	dei	Thm 7	dei	Thm 11	dei	Thm 11		Thm 11	dei	Thm 11
174		Thm 7		Thm 11		Thm 11	dei	Thm 11		Thm 11
176	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
178		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
180	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
182		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
186	dei	Thm 7	dei	Thm 11		Thm 11	dei	Thm 11	dei	Thm 11
186		Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
188	dei	Thm 7		Thm 11	dei	Thm 11		Thm 11		Thm 11
190		Thm 7	dei	Thm 11		Thm 11		Thm 11		Thm 11
192	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
194		Thm 7		Thm 11		Thm 11	dei	Thm 11		Thm 11
196	dei	Thm 7	dei	Thm 11	dei	Thm 11		Thm 11	dei	Thm 11
198	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11
200	dei	Thm 7		Thm 11		Thm 11		Thm 11		Thm 11

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