國 立 交 通 大 學

應用數學系

碩 士 論 文

圖的有向路徑覆蓋

Covering Graphs with Directed Paths

研 究 生：謝奇璁

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中 華 民 國 九 十 八 年 六 月

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摘 要

在這篇論文裡，我們研究完全路徑雙覆蓋的有向形式。一個圖的有向路徑雙覆蓋是在圖的對稱賦向裡的一個有向路徑集合，其中這個圖的對稱賦向裡的每一個邊都要恰好出現在一個路徑裡，而且對圖裡的每一個點而言都會有唯一一條路徑以此點當作起點以及會有唯一一條路徑以此點當作終點。在這篇論文中，首先我們證明了如果一個圖形沒有包含連通部份為點數3的完全圖且為3退化圖則這個圖就存在有向路徑雙覆蓋。再來我們也找出了完全二分圖$K_{n,n}$與完全多分圖$K_{m(n)}$(n為奇數,m≠3,5)的有向路徑雙覆蓋。

Covering Graphs with Directed Paths

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Abstract

In this thesis we study an oriented version of perfect path double cover (PPDC). An oriented perfect path double cover (OPPDC) of a graph G is a collection of oriented paths in the symmetric orientation $S(G)$ of G such that each edge of $S(G)$ lies in exactly one of the paths and for each vertex $v \in V(G)$ there is a unique path which begins in v (and thus the same holds also for terminal vertices of the paths). First we show that if G has no components which isomorphism to K_3 and G is a 3-degenerate graph, then G has an OPPDC. Next we also construct an OPPDC for complete bipartite graph $K_{n,n}$ and multipartite graph $K_{m(n)}$(n is odd and $m \neq 3, 5$),respectively.

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Contents

1 Introduction

Graph decomposition is one of the most important topics in the study of graph theory.In 1979, P. D. Seymour [20] conjectured that every bridgeless graph has a cycle double cover, which is a collection of cycles such that every edge of G is contained in exactly two cycles of the cycle collection. The cycle double cover conjecture lies in the very heart of the graph theory. It seems that this elementary problem has a deep topological background and only partial results are known. This problem (in the very short time of its existence) also motivated several related conjectures: J. A. Bondy [3] conjectured that every simple bridgeless graph has a small cycle double cover, which is a cycle double cover containing at most $n-1$ cycles on a graph that order n. There are a number of classes of graphs for which the small cycle double cover conjecture has been verified, including complete graphs [3] (excluding K_2), complete bipartite graph [3] (other than $K_{1,m}$), 4-connected planar graphs [18], and simple triangulations of orientable surfaces [3, 17]. A common characteristic of these classes of graphs is that there is some structure to the graphs that allows for assumptions about cycles in the graphs. This seems to be a desirable property, since it is necessary to keep track of the number of cycles when constructing small cycle double covers.

In 1990, Bondy [3] also posed several conjectures about path double covers of graphs. He conjectured that every simple graph admits a path double cover $\mathbf{P}$ such that each vertex occurs exactly twice as an end of a path in $\mathbf{P}$: a perfect path double cover. This conjecture was later provey by H. Li [11]. Bondy also conjectured that every k-regular simple graph admits a path double cover $\mathbf{P}$ such that every path in $\mathbf{P}$ has length k and each vertex of the graph occurs exactly twice as an end of a path in $\mathbf{P}$: a regular perfect path double cover. This conjecture has been proved for $k \leq 3$ [3] and $k = 4$ [8] but is still open for larger values of k. Perfect path double cover for graphs in general is equivalent to small cycle double cover for bridgeless apex graphs (apex graph = graph with a vertex joined to all other vertices). To see this, consider a graph $G\backslash v$ where v is a vertex of degree $n-1$ in a bridgeless graph G. G has an small cycle double cover if and only if $G\backslash v$ has a perfect path double cover.

Also unsolved are oriented versions of these problems. In 1988, Jaeger [9] conjectured that every bridgeless graph has an oriented cycle double cover. No counterexample to the oriented cycle double cover conjecture is presently known. In 1998, J. Maxová [12] show that K_3 and K_5 have no oriented perfect path double cover. In 2001, J. Maxová proved that all 2-connected graph on n vertices with at most $2n-1$ edges have an oriented perfect path double cover(except for K_3). In 2004, J. Maxová conjectured that K_3 and K_5 are the only connected graphs which do not have an oriented perfect path double cover.

In this thesis, the main results are that for every 3-degenerate graph with no components isomorphic to K_3 has an oriented perfect path double. Furthermore, show that for all $n \geq 1$ the complete bipartite graph $K_{n,n}$ has an oriented perfect path double and for $m \neq 3, 5$ and n is odd the multipartite graph $K_{m(n)}$ has an oriented perfect path double.

1.1 Preliminaries

In this section, we first introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West.[23]

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiedges* are edges having the same pair of endpoints. A *simple graph* is a graph without loops or multiedges. In this thesis, all the graphs we consider are simple. The size of the vertex set $V(G)$, $|V(G)|$, is called the *order* of G, and the size of the edge set $E(G)$, $|E(G)|$, is called the *size* of G.

If $e = \{u, v\}$ (uv in short) is an edge of G, then e is said to be *incident* to u and v. We also say that u and v are *adjacent* to each other. For every $v \in V(G)$, $N(v)$ denotes the neighborhood of v, that is, all vertices of $N(v)$ are adjacent to v. The *degree* of v, $deg(v) = |N(v)|$, is the number of neighbors of v.

Let $G = (V; E)$ be a undirected simple graph. A *path* of length k in G is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ of its vertices and edges where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq k$ and $v_1, \ldots, v_{k+1}$ are distinct vertices. A *cycle* of length k is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ of its vertices and edges where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq k$, $v_1 = v_{k+1}$ and $v_1, \ldots, v_k$ are distinct vertices.

The maximum degree is $\triangle(G)$, the minimum degree is $\delta(G)$, and G is *regular* if $\triangle(G) = \delta(G)$. It is *k-regular* if the common degree is k. A *cubic graph* is a graph that is regular of degree 3.

A graph G is *connected* if it has a u, v-path whenever $u, v \in V(G)$ (otherwise, G is *disconnected*). If G has a u, v-path, then u is connected to v in G. The *components* of a graph G are its maximal connected subgraphs. A component (or graph) is *trivial* if it has no edges; otherwise it is *nontrivial*. An *isolated vertex* is a vertex of degree 0.

A *directed graph* (digraph) is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the *tail* of the edge, and the second is the *head*; together they are the endpoints. We say that an edge from its tail to its head.

A digraph is a *path* if it is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering. A *cycle* is defined similarly using an ordering of the vertices on a circle.

In our main results, all graph we consider are simple digraph.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. A graph G is $k-degenerate$ if every subgraph of G has a vertex of degree at most k.

A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of G. A graph G is *m-partite* if $V(G)$ can be expressed as the union of m independent sets. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes s and t, the complete bipartite graph is denoted by $K_{s,t}$. If the sets have the same size n, the complete bipartite graph is called *balanced*, which is denoted by $K_{n,n}$. Similarly, the complete m-partite graph is denoted by $K_{s_1,s_2,\ldots,s_m}$ and the balanced complete m-partite graph is denoted by $K_{m(n)}$ where each partite set has n vertices.

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H.

Let G be a graph of order m with $V(G) = \{g_i : 0 \leq i \leq m-1\}$, and let H ne a graph of order n with $V(H) = \{h_i : 0 \leq i \leq n-1\}$. The *Cartesian product* $G\Box H$ is defined to be the graph with vertex set $\{(g_i, h_j) : 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1\}$ and $(g_i, h_j)(g_s, h_t) \in E(G\Box H)$ if either $g_i = g_s$ and $h_j h_t \in E(H)$ or $h_j = h_t$ and $g_i g_s \in E(G)$.

The *symmetric orientation* of G, denoted by $S(G)$, that is an oriented graph obtained from G by replacing each edge of G by a pair of oppositely directed arcs (i.e. $V(S(G)) = V(G)$ and $E(S(G)) = \{(u,v),(v,u)|(u,v) \in E(G)\}$).

We give some important definitions as followings.

A *cycle double cover* (CDC) of a graph G is a collection of its cycle such that each edge of G lies in exactly two of the cycles. A *small cycle double cover* (SCDC) of a graph on n vertices is a CDC with at most $n-1$ circuits.

A *perfect path double cover* (PPDC) of a graph G is a collection of its paths such that each edge of G lies in exactly two of the paths and each vertex of G appears precisely twice as an endpoint of a path.

A *regular perfect path double cover* (RPPDC) of a k-regular simple graph G is a collection $\mathbb{P}$ of its paths such that every path in $\mathbb{P}$ has length k and each vertex of the graph occurs exactly twice as an end of a path in $\mathbb{P}$.

For a path double cover $\mathbb{P}$ of a graph G, the *associated graph* $A_P(G)$ of $\mathbb{P}$ is defined as a graph having the same vertex set as G, with two vertices x and y adjacent in $A_P(G)$ if and only if there is a path in $\mathbb{P}$ with endpoints x and y.

A PPDC is called an *eulerian perfect path double cover* (EPPDC) if its associated graph is a cycle. If a path double cover is both eulerian and regular, we call it an ERPPDC.

An *oriented perfect path double cover* (OPPDC) of a graph G is a collection of paths on G such that each edge of $S(G)$ lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as an end of a path.

1.2 Known Results

We consider cycle decomposition and path decomposition on undirected graph. The following are some results:

Theorem 1.1. [10] *(1) For all odd integers n and all non-negative integer r satisfying $3r = \frac{n(n-1)}{2}$ there is a decomposition of K_n into r 3-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r satisfying $3r = \frac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r 3-cycles which partitions the edges set of $K_n - F$.*

We can establish the existence of cycle systems not only the 3-cycle system but also the m-cycle system for any m. There are some results below:

Theorem 1.2. [16] *(1) For all odd integers n and all non-negative integer r and m satisfying $mr = \frac{n(n-1)}{2}$ there is a decomposition of K_n into r m-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r and m satisfying $mr = \frac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r m-cycles which partitions the edges set of $K_n - F$.*

Theorem 1.3. [1] *(1) For all odd integers n and all non-negative integer r and s satisfying $3r + 5s = \frac{n(n-1)}{2}$ there is a decomposition of K_n into r 3-cycles and s 5-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r and s satisfying $3r + 5s = \frac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r 3-cycles and s 5-cycles which partitions the edges set of $K_n - F$.*

Theorem 1.4. [7] *(1) For all odd integers n and all non-negative integer r, s and t satisfying $3r + 4s + 6t = \frac{n(n-1)}{2}$ there is a decomposition of K_n into r 3-cycles, s 4-cycles, and t 6-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r, s and t satisfying $3r + 4s + 6t = \frac{n(n-2)}{2}$ there is a decomposition of $K_n - F$ into r 3-cycles, s 4-cycles, and t 6-cycles which partitions the edges set of $K_n - F$.*

Theorem 1.5. [4] *(1) For all odd integers n and all non-negative integer r and s satisfying $4r + 5s = \frac{n(n-1)}{2}$ there is a decomposition of K_n into r 4-cycles and s 5-cycles which partitions the edge set of K_n. (2) For all even integers n and all non-negative integers r*

and s satisfying $4r + 5s = \frac{n(n-2)}{2}$ *there is a decomposition of* $K_n - F$ *into r 4-cycles and s 5-cycles which partitions the edges set of* $K_n - F$.

The following useful contains three different lengths which are $n, n-1, n-2$.

Theorem 1.6. [7] *Let* $S = \{n-2, n-1, n\}$. *If n is odd and* $a(n-2)+b(n-1)+cn = \frac{n(n-1)}{2}$, *then* $K_n = aC_{n-2} + bC_{n-1} + cC_n$. *If n is even and* $a(n-2)+b(n-1)+cn = \frac{n(n-2)}{2}$, *then* $K_n - F = aC_{n-2} + bC_{n-1} + cC_n$.

Alspach Conjecture is also true if the cycles lengths m_i are bounded by some linear function of n and n is sufficiently large.

Theorem 1.7. [2] *Assume n must be larger than* N_2 *which is very large absolute constants. If* $m_1, \ldots, m_t$ *are integers with* $3 \le m_i \le \lfloor \frac{n-112}{120} \rfloor$ *and* $\sum_{i=1}^{t} m_i = \binom{n}{2}$ *(n odd) or* $\binom{n}{2} - \frac{n}{2}$ *(n even), then one can pick* K_n *(n odd) or* $K_n - I$ *(n even) with cycles of length* $m_1, \ldots, m_t$.

Theorem 1.8. [6] *Let n be a n even positive integer. Then* K_n *can be decomposed into* $\frac{n}{2}$ *hamiltonian paths.*

Theorem 1.9. [15] *If n is odd and* $\{a_i : 1 \le i \le r\}$ *is a multiset of r positive integers satisfying* $1 \le a_i \le n-2$ *and* $\sum_{i=1}^{r} a_i = \binom{n}{2}$. *Then* K_n *can be decomposed into* $\{P_{a_i} | 1 \le i \le r\}$.

Theorem 1.10. [21] *Let* $m | \lambda\binom{n}{2}$, *and* $m \le n-1$. *Then* λK_n *caon be decomposed into isomorphic paths of length m.*

Theorem 1.11. [5] *If v is odd. Let* $m_1, m_2, \ldots, m_t$ *be t positive integers such that* $1 \le m_i \le n-2$, $\sum_{i=1}^{t} m_i + k(n-1) = \binom{n}{2}$, *and* $k \in \{1, 2, \frac{n-1}{2}\}$, *then* K_v *can be decomposed into* $t+k$ *paths* $P^1, P^2, \ldots, P^{t+k}$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, \ldots, t$ *and the length of* P^i *is* $n-1$ *for* $i > t$.

Theorem 1.12. [5] *If v is odd. Let* $n-1 \ge m_1 \ge m_2 \ge \cdots \ge m_t \ge 1$ *and* $h \le m_t \le n-h-1$ *such that* $\sum_{i-1}^{t} m_i = \binom{n}{2}$, $m_1 = m_2 = \cdots = m_h = n-1$. *Then* K_v *can be decomposed into t paths* $P^1, P^2, \ldots, P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, \ldots, t$. *Moreover, if there exists a* $h < t' \le t$ *such that* $h \le m_{t'} \le n-h-1$ *or*

$h \leq \sum_{i=t'}^{t} m_i \leq n-h-1$, *then* K_v *can be decomposed into* t *paths* $P^1, P^2, \ldots, P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, \ldots, t$.

Theorem 1.13. [5] *If* v *is odd. Let* $n-1 \geq m_1 \geq m_2 \geq \cdots \geq m_t \geq 1,\ \ m_t < h$, *and* $m_{t-1}m^t \leq n-h-1$ *such that* $\sum_{i=1} tm_i = \binom{n}{2}$, $m_1 = m_2 = \ldots = m_h = n-1$. *Then* K_v *can be decomposed into* t *paths* $P^1, P^2, \ldots, P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, \ldots, t$.

Theorem 1.14. [5] *If* v *is odd. Let* $n-1 \geq m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ *and* $n+h-2 \leq m_t + m_{t-1} \leq 2n-h-3$ *such that* $\sum_{i-1}^{t} m_i = \binom{n}{2}$, $m_1 = m_2 = \ldots = m_h = n-1$. *Then* K_v *can be decomposed into* t *paths* $P^1, P^2, \ldots, P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, \ldots, t$. *Moreover, if there exists a* $h < t' \leq t$ *such that* $n+h-2 \leq \sum_{i=t'}^{t} m_i \leq 2n-h-3$, *then* K_v *can be decomposed into* t *paths* $P^1, P^2, \ldots, P^t$ *such that the length of* P^i *is* m_i *for* $i = 1, 2, \ldots, t$.

Next, we consider some results of RPPDC and EPPDC.

Proposition 1.15. [19] Suppose that G is a graph with an eulerian perfect path double cover. Then for $1 \leq d(x) \leq 3$, $G + x$ has an eulerian perfect double cover.

Proposition 1.16. [19] For any $n \geq 1$, $K_{n,n}$ has an RPPDC. Moreover, if n is odd, then $K_{n,n}$ has an ERPPDC.

Proposition 1.17. [19] For any $m, n \geq 1$, $K_{m,n}$ has an EPPDC.

Proposition 1.18. [19] If G is a k-regular graph, $k \geq 1$, then $L(G)$ has an RPPDC.

Proposition 1.19. [19] Let G be a graph with m edges. Suppose $2G$ has an Euler circuit $e_1, e_2, \ldots, e_{2m}$ such that $S_1 = \{e_1, e_3, \ldots, e_{2m-1}\}$ and $S_2 = \{e_2, e_4, \ldots, e_{2m}\}$ are both the set $E(G)$ of all edges of G. Furthermore, suppose that for each $v \in V(G)$ there is ordering, $C(v)$, of the edges incident to v such that every pair of consecutive edges in$C(v)$ occurs exactly once as a pair of consecutive edges in the Euler circuit. Then $L(G)$ has an EPPDC.

Proposition 1.20. [19] For all $m \geq 2$, $L(K_m)$ has an ERPPDC.

Proposition 1.21. [19] For all $m, n \geq 1$, $L(K_{m,n})$ has an RPPDC. Furthermore, if $gcd(n,m) = 1$ or $gcd(n, n-m+2) = 1$, then $L(K_{m,n})$ has an ERPPDC.

Proposition 1.22. [19] For every positive odd integer n, $L(K_{n,n})$ has an ERPPDC.

Proposition 1.23. [19]

- If G and H have RPPDCs, then $G \Box H$ has an RPPDC.

- If G and H have EPPDCs and $(|G|, |H|) = 1$, then $G \Box H$ has an EPPDC.

Proposition 1.24. [19] If G has an EPPDC, then the Cartesian product $G \Box K_2$ has an EPPDC.

Proposition 1.25. [19] For all $n \geq 0$, the n-cube, Q_n, has an EPPDC.

In our main result we concentrate on oriented version. Now, we consider some results below:

Lemma 1.26. [13] *K_{2n} has an OPPDC.*

Proof. Let $V(K_{2n}) = \{v_0, v_1, \ldots, v_{2n-1}\}$. For $0 \leq i \leq 2n-1$ set $P_i = (v_i, v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+n})$ where all subscripts are read modulo $2n$. It is easy to verity that $\mathbb{P} = \{P_i | 0 \leq i \leq 2n-1\}$ is an OPPDC of K_{2n}. ∎

Lemma 1.26 gives an easy construction of OPPDC for all K_{2n}. Tillson proved in [22] that all K_{2n+1} have an OPPDC for $n \geq 3$.

Example 1.27. [13] K_7 has an OPPDC as follow:

$P_1 = 1263547$ $P_2 = 2731465$

$P_3 = 3742516$ $P_4 = 4536721$

$P_5 = 5764132$ $P_6 = 6175243$

$P_7 = 7156234$

We can check that the collection $\mathbb{P} = \{P_1, \ldots, P_7\}$ is an OPPDC of K_7.

Next, we consider the minimal (i.e. , with minimal number of edges) connected graph G such that $G \neq K_3$, $G \neq K_5$ and G has no OPPDC. J. Maxová had show that G has no vertices of degree $1, 2$.

Lemma 1.28. [13] *Let G_1, G_2 be two graphs which have an OPPDC. Suppose that $G_1 \cap G_2 = \{v\}$. Then the union $G_1 \cup G_2$ has an OPPDC.*

Proof. Denote by $\mathfrak{P}_i$ an OPPDC of G_i, $i = 1, 2$. Let $P_1 \in \mathfrak{P}_1$ be the path that starts in v and $P_2 \in \mathfrak{P}_2$ be the path that ends at v. Then the collection $\mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \{P_1 \cup P_2\} \backslash \{P_1, P_2\}$ is an OPPDC of $G_1 \cup G_2$. ∎

Corollary 1.29. [13] *Let G be a simple graph; $G \neq K_3$, and $v \in V\{G\}$ a vertex of degree 1. If $G \setminus v$ has an OPPDC then G has an OPPDC.*

By applying this corollary, we get that if we add a new vertex of degree 1 to a graph with an OPPDC then the resulting graph also has an OPPDC. Hence every tree has an OPPDC.

Theorem 1.30. [13] *Let G be a simple graph; $G \neq K_3$, and $v \in V\{G\}$ a vertex of degree 2. If $G \setminus v$ has an OPPDC then G has an OPPDC.*

By applying Corollary 1.29 and Theorem 1.30, we get that every 2-degenerate graph has an OPPDC, except K_3. The following are some results

Corollary 1.31. [13] *If G is a union of two arbitrary trees; $G \neq K_3$; then G has an OPPDC.*

Another construct which preserves the property of having an OPPDC is the so-called *arrow construction.*

Definition 1.32. [14] A graph I with two distinguished vertices a, b, $a, b \notin E(I)$, is called an *indicator*. For a given directed graph $D = (V, E)$ and an indicator (I, a, b) we define an (undirected) graph $D * (I, a, b) = (W, F)$ sa follows:

$W = (E \times V(I))/ \sim,$

where the equivalence is generated by the following pairs:

$((x, y), a), ((x, y'), a), \quad ((x, y), b), ((x', y), b), \quad ((x, y), b), ((y, z), a).$

For a pair $(e, x) \in E \times V(I)$ its equivalence class is denoted by $[e, x]$.

We put $\{[e, x], [e', x']\} \in F \iff e = e'$ and $\{x, x'\} \in E(I)$.

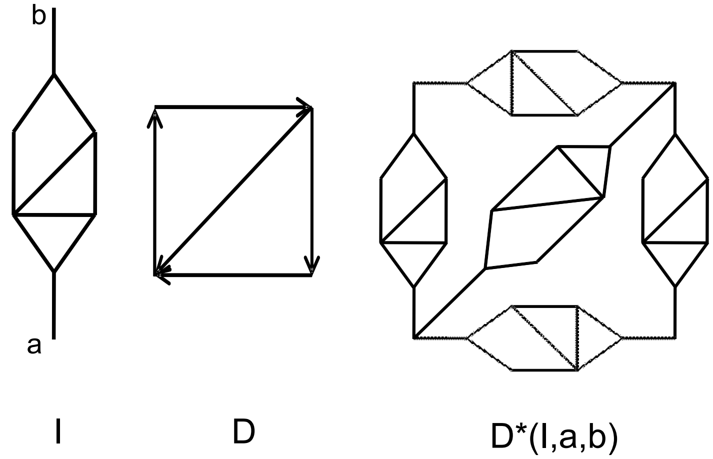


Figure 1: Arrow construction

This arrow construction is schematically indicated in Fig. 1 (One can check that the indicator I in Fig. 1 satisfies the assumptions of Theorem1.33 below.)

Theorem 1.33. [14] *Suppose an indicator* (I, a, b) *has an OPPDC* Π *containing two paths* $P_1, P_2 \in \Pi$ *such that* P_1 *begins in* a *and ends in* b*, and* P_2 *begins in* b *and ends in* a*. Further suppose* G *has an OPPDC. Then for any orientation* D *of* G *the graph* $D * (I, a, b)$ *has an OPPDC.*

Proposition 1.34. [14] If G is a 2-connected graph with $|E(G)| \leq 2n-1$; $G \neq K_3$; then G has an OPPDC.

Conjecture 1.35. [14] K_3 and K_5 are the only connected graphs which do not have an OPPDC.

2 Main Results

In this section, we focus on the minimal degree of a graph G first. We show that if we add a new vertex of degree 3 to a graph with an OPPDC then the resulting graph also has an OPPDC. And we use this theorem to prove that if G is a 3-degenerate graph and G has no components which isomorphism to K_3 then G has an OPPDC. Next, we show that the complete graph $K_{n,n}$ and the multipartite graph $K_{m(n)}$ has an OPPDC by a special construction.

2.1 3-degenerate graph

Theorem 2.1. *Let G be a simple graph; $G \neq K_3$,and $v \in V\{G\}$ a vertex of degree 3. If $G \setminus v$ has an OPPDC then G has an OPPDC.*

Proof. Let $N(v) = \{a, b, c\}$ be the neighbors of the vertex v. Denote by $\mathbb{P}$ an OPPDC of the graph $G \setminus v$. For $u \in V(G \setminus v)$, letP^u (resp. P_u) denote the path of $\mathbb{P}$ beginning (resp. ending) with u. We call P^u (resp. P_u) is the outer (resp. inner) path of u in $G \setminus v$.

Case 1. There exists an outer path P^u,$u \in N(v)$,P^u pass through $N(v)$.

Without loss of generality , we assume P^a pass through b and then c.

Subcase 1-1: $P^c \neq P_b$

Separate P^a into two paths, P_1 and P_2,where P_1 is the path that beginning at a and ending at b along P^a and P_2 is the path that beginning at b and along P^a.

Let $P^{c*} = (c, v) \cup (v, a) \cup P_1$

$P^{a*} = (a, v) \cup (v, b) \cup P_2$

${P_v}^* = P_b \cup (b, v)$

$P^{v*} = (v, c) \cup P^c$

Then the collection $\mathbb{P} \setminus \{P^a, P_b, P^c\} \cup \{{P_v}^*, P^{v*}, P^{c*}, P^{a*}\}$is an OPPDC of G.

Subcase 1-2: $P^c = P_b$

Subcase 1-2-1: $P^b \neq P_a$

Separate P^a into three paths , P_1, P_2, and P_3, where P_1 is the path that beginning at a and ending at b along P^a , P_2 is the path that beginning at b

and ending at c along P^a , P_3 is the path that beginning at c and along P^a.

Let $P^{a*} = P_1 \cup (b, v) \cup (v, c) \cup P_3$

$P^{b^*} = P_2 \cup (c, v) \cup (v, a)$

$P^{v*} = (v, b) \cup P^b$

$P_v{}^* = P_a \cup (a, v)$

Then the collection $\mathbb{P} \setminus \{P_a, P^a, P^b, \} \cup \{P^{a*}, P^{b^*}, P^{v*}, P_v{}^*\}$ is an OPPDC of G.

Subcase 1-2-2: $P^b = P_a$

Let $P^{a*} = (a, v) \cup (v, c) \cup P^c$

$P^{c*} = (c, v) \cup (v, b) \cup P^b$

$P^{v*} = (v, a) \cup P^a$

$P_v{}^* = (b, v)$

Then the collection $\mathbb{P} \setminus \{P^a, P^b, P^c\} \cup \{P^{a*}, P^{c*}, P^{v*}, P_v{}^*\}$ is an OPPDC of G.

Case 2. There is no outer path P^u,$u \in N(v)$,P^u pass through $N(v)$.

Without loss of generality , we assume P^a doesn't pass through c.

Subcase 2-1: P^c doesn't pass through a.

Let $P^{c*} = (c, v) \cup (v, a) \cup P^a$

$P^{a*} = (a, v) \cup (v, c) \cup P^c$

$P^{v*} = (v, b) \cup P^b$

$P_v{}^* = (b, v)$

Then the collection $\mathbb{P} \setminus \{P^a, P^b, P^c\} \cup \{P^{a*}, P^{c*}, P^{v*}, P_v{}^*\}$ is an OPPDC of G.

Subcase 2-2: P^c pass through a.

Since P^c passes through a, it can't pass through b. If P^b doesn't pass through c, it will return to case 2-1. So P^b passes through c. If P^b passes through a, it will return to case 1. So P^b doesn't pass through a.

Let $P^{c*} = (c, v) \cup (v, a) \cup P^a$

$P^{a*} = (a, v) \cup (v, b) \cup P^b$

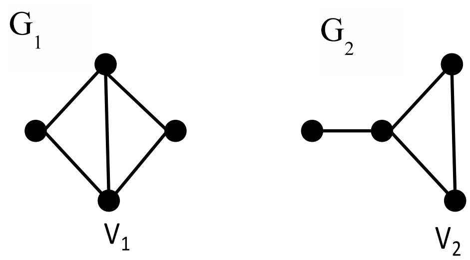
$P^{v*} = (v, c) \cup P^c$

$P_v{}^* = (b, v)$

Then the collection $\mathbb{P} \setminus \{P^a, P^b, P^c\} \cup \{P^{a*}, P^{c*}, P^{v*}, P_v{}^*\}$ is an OPPDC of G.

Thus, we have the prove. ∎

Lemma 2.2. *G_1 and G_2 has an OPPDC.*



Proof. Since $deg_{G_1}(V_1) = 3$ $deg_{G_1}(V_2) = 2$ and $G_1 \backslash V_1$, $G_1 \backslash V_1$ are paths. By Theorem 1.30 and Theorem 2.1 we know that G_1 and G_2 has an OPPDC. ∎

Theorem 2.3. *If G has no components which isomorphism to K_3 and G is a 3-degenerate graph, then G has an OPPDC.*

Proof. We proceed by induction on $n = V|(G)|$. Since G is a 3-degenerate graph, there is a vertex $v \in V(G)$ of degree at most 3. We denote that $G' = G \setminus v$. If G' is isomorphic to K_3 then G is isomorphic to K_4 or one of the graphs G_1, G_2 in Lemma 2.2, that all have an OPPDC. If G' is a disconnected graph with some components which are isomorphic to K_3. Then we choose another vertex v' which in K_3 and let $G' = G \setminus v'$. Since $deg_G(v') \leq 3$ we know that G' applies to the induction hypothesis. If $deg_G(v) = 1$ by induction hypothesis the graph G' has an OPPDC. Then by applying Corollary 1.29 the graph G has an OPPDC. If $deg_G(v) = 2$ by induction hypothesis the graph G' has an OPPDC. Then by applying Theorem 1.30 the graph G has an OPPDC. If $deg_G(v) = 3$ by induction hypothesis the graph G' has an OPPDC. Then by applying Theorem 2.1 the graph G has an OPPDC.

Thus, we have the prove. ∎

Corollary 2.4. *Every cubic graph has an OPPDC.*

Proof. We know that every cubic graph is $3-degenerate$. By Theorem 2.3 we have the prove. ∎

2.2 OPPDC on $K_{n,n}$ and $K_{m(n)}$

Now, we consider a special construction of OPPDC on complete graph $K_{n,n}$ and multipartite graph $K_{m(n)}$.

Lemma 2.5. *For all* $n \geq 1$, $K_{n,n}$ *has an OPPDC.*

Proof. Assume that $K_{n,n}$ has bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. Suppose that n is odd. For each i, $i = 0, 1, \ldots, n-1$, let $P_i = (x_i, y_{n-1+i}, x_{i+1}, y_{n-2+i}, \ldots, y_{(n-3)/2+i}, x_{(n-1)/2+i}, y_{(n-1)/2+i})$. Then $\mathbf{P} = \{P_0, P_1, \ldots, P_{n-1}\}$ is a path decomposition of $K_{n,n}$. Let $P_i{}' = (y_{(n-1)/2+i}, x_{(n-1)/2+i}, y_{(n-3)/2+i}, \ldots, x_{i+1}, y_{n-1+i}, x_i)$, and let $\mathbf{P'} = \{P_0{}', P_1{}', \ldots, P_{n-1}{}'\}$ .The union of the two path decompositions forms an OPPDC of $K_{n,n}$.

If n is even, let $P_i = (x_i, y_{n-1+i}, x_{i+1}, y_{n-2+i}, \ldots, y_{n/2+i}, x_{n/2+i})$. Then $\mathbf{P} = \{P_0, P_1, \ldots, P_{n-1}\}$ is a path decomposition of $K_{n,n}$. Exchanging the $x's$ and $y's$ we obtain a second path decomposition $\mathbf{P'}$ of $K_{n,n}$. The union of these two path decompositions forms an OPPDC of $K_{n,n}$. ∎

Example 2.6. An OPPDC of $K_{5,5}$.

Assume that $K_{5,5}$ has bipartition (X, Y), where $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4\}$.

$$\mathbf{P} = \left\{ \begin{array}{c} (x_0, y_4, x_1, y_3, x_2, y_2) \\ (x_1, y_0, x_2, y_4, x_3, y_3) \\ (x_2, y_1, x_3, y_0, x_4, y_4) \\ (x_3, y_2, x_4, y_1, x_0, y_0) \\ (x_4, y_3, x_0, y_2, x_1, y_1) \end{array} \right\} \quad \mathbf{P'} = \left\{ \begin{array}{c} (y_2, x_2, y_3, x_1, y_4, x_0) \\ (y_3, x_3, y_4, x_2, y_0, x_1) \\ (y_4, x_4, y_0, x_3, y_1, x_2) \\ (y_0, x_0, y_1, x_4, y_2, x_3) \\ (y_1, x_1, y_2, x_0, y_3, x_4) \end{array} \right\}$$

Then $\mathbf{P} \cup \mathbf{P'}$ is an OPPDC of $K_{5,5}$.

Example 2.7. An OPPDC of $K_{4,4}$.

Assume that $K_{5,5}$ has bipartition (X, Y), where $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$.

$$\mathbf{P} = \left\{ \begin{array}{c} (x_0, y_3, x_1, y_2, x_2) \\ (x_1, y_0, x_2, y_3, x_3) \\ (x_2, y_1, x_3, y_0, x_0) \\ (x_3, y_2, x_0, y_1, x_1) \end{array} \right\} \qquad \mathbf{P'} = \left\{ \begin{array}{c} (y_0, x_3, y_1, x_2, y_2) \\ (y_1, x_0, y_2, x_3, y_3) \\ (y_2, x_1, y_3, x_0, y_0) \\ (y_3, x_2, y_0, x_1, y_1) \end{array} \right\}$$

Then $\mathbf{P} \cup \mathbf{P}'$ is an OPPDC of $K_{4,4}$.

Theorem 2.8. *$K_{m(n)}$ has an OPPDC for n is odd, $m \neq 3, 5$.*

Proof. Suppose that m is even. Let $V(K_{m(n)}) = \bigcup_{i=0}^{2k-1} V_i$ where $V_i = \{x_{i,0}, x_{i,1}, \cdots, x_{i,n-1}\}$ and $m = 2k$. By Lemma 1.26 K_{2k} has an OPPDC, and by Lemma 2.5 $K_{n,n}$ has an OPPDC.

Then we let

$$\mathbb{Q} = \bigcup_{i=0}^{2k-1} \begin{pmatrix} & (P^0_{i,i+1} + P^0_{i+1,i-1} + P^0_{i-1,i+2} + \ldots + P^0_{i-k+1,i+k}) \\ \cup & (P^1_{i,i+1} + P^{n-1}_{i+1,i-1} + P^1_{i-1,i+2} + \ldots + P^1_{i-k+1,i+k}) \\ \cup & (P^2_{i,i+1} + P^{n-2}_{i+1,i-1} + P^2_{i-1,i+2} + \ldots + P^2_{i-k+1,i+k}) \\ & \vdots \\ \cup & (P^{n-1}_{i,i+1} + P^1_{i+1,i-1} + P^{n-1}_{i-1,i+2} + \ldots + P^{n-1}_{i-k+1,i+k}) \end{pmatrix}$$

where $P^q_{i,j} = (y_{i,q}, x_{j,n-1+q}, y_{i,q+1}, x_{j,n-2+q}, \ldots, x_{j,\frac{n-3}{2}+q}, y_{i,\frac{n-1}{2}+q}, x_{j,\frac{n-1}{2}+q})$

and $y_{i,q} = f_{j,i}(x_{i,q})$ by

$$f : \begin{cases} y_{i,\frac{n-1}{2}-j} = x_{i,j}, & \text{if } j < \frac{n-1}{2} \\ y_{i,0} = x_{i,j}, & \text{if } j = \frac{n-1}{2} \\ y_{i,\frac{3n-1}{2}-j} = x_{i,j}, & \text{if } j > \frac{n-1}{2}. \end{cases}$$

Then $\mathbb{Q}$ is an OPPDC of $K_{m(n)}$.

Now, we consider m is odd, $m \neq 3, 5$. Let $\mathbb{P} = \{P_0, P_1, \ldots, P_{m-1}\}$ is an OPPDC of K_m and denote that $P_i = (v_i(0), v_i(1), \ldots, v_i(n-1))$, where $v_i(0)$ is the beginning at the path P_i and $v_i(n-1)$ is the end at the path P_i.

Then we let

$$\mathbb{R} = \bigcup_{i=0}^{m-1} \begin{pmatrix} & (P^0_{i(0),i(1)} + P^0_{i(1),i(2)} + P^0_{i(2),i(3)} + \ldots + P^0_{i(n-2),i(n-1)}) \\ \cup & (P^1_{i(0),i(1)} + P^{n-1}_{i(1),i(2)} + P^1_{i(2),i(3)} + \ldots + P^1_{i(n-2),i(n-1)}) \\ \cup & (P^2_{i(0),i(1)} + P^{n-2}_{i(1),i(2)} + P^2_{i(2),i(3)} + \ldots + P^2_{i(n-2),i(n-1)}) \\ & \vdots \\ \cup & (P^{n-1}_{i(0),i(1)} + P^1_{i(1),i(2)} + P^{n-1}_{i(2),i(3)} + \ldots + P^{n-1}_{i(n-2),i(n-1)}) \end{pmatrix}$$

where $P^q_{i,j} = (y_{i,q}, x_{j,n-1+q}, y_{i,q+1}, x_{j,n-2+q}, \ldots, x_{j,\frac{n-3}{2}+q}, y_{i,\frac{n-1}{2}+q}, x_{j,\frac{n-1}{2}+q})$

and $y_{i,q} = f_{j,i}(x_{i,q})$ by

$$f = \begin{cases} x_{i,j} \to y_{i,\frac{n-1}{2}-j}, & \text{if } j < \frac{n-1}{2} \\ x_{i,j} \to y_{i,0}, & \text{if } j = \frac{n-1}{2} \\ x_{i,j} \to y_{i,\frac{3n-1}{2}-j}, & \text{if } j > \frac{n-1}{2}. \end{cases}$$

Then $\mathbb{R}$ is an OPPDC of $K_{m(n)}$. ∎

Example 2.9. An OPPDC of $K_{4(3)}$.

Let $V(K_{4(3)}) = \bigcup_{i=0}^{3} V_i$ where $V_i = \{x_{i,0}, x_{i,1}, x_{i,2}\}$. Let

$$\mathbb{Q} = \bigcup_{i=0}^{3} \left(\begin{array}{cl} & (P^0_{i,i+1} + P^0_{i+1,i-1} + P^0_{i-1,i+2}) \\ \cup & (P^1_{i,i+1} + P^2_{i+1,i-1} + P^1_{i-1,i+2}) \\ \cup & (P^2_{i,i+1} + P^1_{i+1,i-1} + P^2_{i-1,i+2}) \end{array} \right)$$

where $P^q_{i,j} = (y_{i,q}, x_{j,2+q}, y_{i,1+q}, x_{j,1+q})$ and $y_{i,q} = f_{j,i}(x_{i,q})$ by

$$f : \begin{cases} y_{i,1} = x_{i,0}, \\ y_{i,0} = x_{i,1}, \\ y_{i,2} = x_{i,2}. \end{cases}$$

$\Rightarrow$

$$\mathbb{Q} = \left\{ \begin{array}{c} (x_{0,1}, x_{1,2}, x_{0,0}, x_{1,1}, x_{3,2}, x_{1,0}, x_{3,1}, x_{2,2}, x_{3,0}, x_{2,1}) \\ (x_{0,0}, x_{1,0}, x_{0,2}, x_{1,2}, x_{3,1}, x_{1,1}, x_{3,0}, x_{2,0}, x_{3,2}, x_{2,2}) \\ (x_{0,2}, x_{1,1}, x_{0,1}, x_{1,0}, x_{3,0}, x_{1,2}, x_{3,2}, x_{2,1}, x_{3,1}, x_{2,0}) \end{array} \right\}$$

$$\bigcup \left\{ \begin{array}{c} (x_{1,1}, x_{2,2}, x_{1,0}, x_{2,1}, x_{0,2}, x_{2,0}, x_{0,1}, x_{3,2}, x_{0,0}, x_{3,1}) \\ (x_{1,0}, x_{2,0}, x_{1,2}, x_{2,2}, x_{0,1}, x_{2,1}, x_{0,0}, x_{3,0}, x_{0,2}, x_{3,2}) \\ (x_{1,2}, x_{2,1}, x_{1,1}, x_{2,0}, x_{0,0}, x_{2,2}, x_{0,2}, x_{3,1}, x_{0,1}, x_{3,0}) \end{array} \right\}$$

$$\bigcup \left\{ \begin{array}{c} (x_{2,1}, x_{3,2}, x_{2,0}, x_{3,1}, x_{1,2}, x_{3,0}, x_{1,1}, x_{0,2}, x_{1,0}, x_{0,1}) \\ (x_{2,0}, x_{3,0}, x_{2,2}, x_{3,2}, x_{1,1}, x_{3,1}, x_{1,0}, x_{0,0}, x_{1,2}, x_{0,2}) \\ (x_{2,2}, x_{3,1}, x_{2,1}, x_{3,0}, x_{1,0}, x_{3,2}, x_{1,2}, x_{0,1}, x_{1,1}, x_{0,0}) \end{array} \right\}$$

$$\bigcup \left\{ \begin{array}{c} (x_{3,1}, x_{0,2}, x_{3,0}, x_{0,1}, x_{2,2}, x_{0,0}, x_{2,1}, x_{1,2}, x_{2,0}, x_{1,1}) \\ (x_{3,0}, x_{0,0}, x_{3,2}, x_{0,2}, x_{2,1}, x_{0,1}, x_{2,0}, x_{1,0}, x_{2,2}, x_{1,2}) \\ (x_{3,2}, x_{0,1}, x_{3,1}, x_{0,0}, x_{2,0}, x_{0,2}, x_{2,2}, x_{1,1}, x_{2,1}, x_{1,0}) \end{array} \right\}$$

Then $\mathbb{Q}$ is an OPPDC of $K_{4(3)}$.

3 Conclusion

In this thesis, we have obtained the following main results:

1. If G has no components which isomorphism to K_3 and G is a 3-degenerate graph, then G has an OPPDC.

2. For all $n \geq 1$, $K_{n,n}$ has an OPPDC.

3. $K_{m(n)}$ has an OPPDC for n is odd, $m \neq 3, 5$.

But, we are still far from verifying the conjectures(Conjecture 1.35). Hopefully, this task can be done in the near future.

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