國 立 交 通 大 學

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碩 士 論 文

橢圓函數之理論與運用

The Theory and Applications of the Elliptic Functions

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摘 要

本篇文章主要是研究古典橢圓函數的理論及其在微分方程式上的運用和分析。

在第一章裡我們定義了橢圓函數並分析其性質，接著介紹了 Weierstrass 和 Jacobian 這兩個代表性的橢圓函數。

在第二章裡提供了一些分析相位圖的技巧與方法。

在第三章裡利用 Jacobian 函數解 Sine-Gordon equation 所描繪的理想平面鐘擺運動。接著使用第二章所提供的技巧與方法分析非理想狀態下的平面鐘擺運動。

在第四章裡提供了五個物理問題之數學模式用微分方程式來描繪並運用 Jacodian 函數來求解。

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The Theory and Applications of the Elliptic Functions

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Abstract

In this paper, we study the classical elliptic functions and the applications to the differential equations.

In chapter Ⅰ, we define the elliptic functions and analyze it's properties. And then, we introduce Weierstrass functions and Jacobian functions, the two typical elliptic functions.

In chapter Ⅱ, we analyze phase portraits.

In chapter Ⅲ, we study the Sine-Gordon equation that describes the ideal pendulum motion and use Jacobian functions to represent the solutions. We then use the methods in chapter Ⅱ to analyze pendulum motion with friction.

In chapter Ⅳ, we provide other five physical models described by differential equations and solve them by Jacobian functions.

中 華 民 國 九 十 九 年 一 月

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本篇論文的完成，首先要感謝我的指導老師-李榮耀教授。老師在研究的初期扮演著引導的角色，帶著我一步一步的研究理論內容的深處，在後期則是扮演了敦促與激勵的角色，激發學生在已有的基礎上更加的往前深入與探索。若不是老師一步步的指引就不會有本篇論文的產生。

在口試期間，十分感謝兩位口試委員的幫助，尤其在論文的用詞上給予了十分寶貝的意見，使本篇的論文更佳的完善。

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最後要感謝我的父母親，養育之恩難言，也因為他們的堅持，我才會註冊進到校園裡完成碩士的學業，才會有今天論文的產生。

Contents:

Chapter 1 Elliptic Functions

1.1.1 Introduction:

In this chapter, we study the classical elliptic functions and it mainly follows the references [3, 4, 5, 10]. Some of the tables follow another reference [6]. Moreover, the relation between one period functions and double-period functions are according to the references [4, 7]. Figures inside are drawn by the program Mathematica.

1.1.2 Doubly-Periodic functions:

Let ω_1 ω_2 be any two numbers (real or complex) whose ratio is not purely real.

A function which satisfies the equations

$f(z+2\omega_1)=f(z)$ $\quad$ $f(z+2\omega_2)=f(z)$,

and no further period lies between 0 and ω_1, and 0 and ω_2 respectively,

for all values of z for which $f(z)$ exist, is called a doubly-periodic function of z, with periods $2\omega_1$ $2\omega_2$

1.1.3 Elliptic functions:

1. The singularity (singular point)

If f(z) is not analytic at $z=z_0$ then we call z_0 is the singularity of f(z).

For a singular point z_0 if there exists a neighborhood $N(z_0)$ of z_0 such that the function f(z) is analytic in $N(z_0)/z_0$ then z_0 is called a isolated singularity of f(z).

Moreover, for a isolated singularity of f(z) if there exist an analytic function $g(z)\ N(z) \to C$ such that $g(z) = f(z)$ on $N(z_0)/z_0$ then the point z_0 is called a removable singularity.

2. Pole

If z_0 is isolated singularity and $\exists \min\ k \in N$ such that $(z - z_0)^k f(z)$ is analytic at z_0 then z_0 is a pole of function $f(z)$

3. Elliptic function

A doubly-periodic function which is analytic (except at poles), and which has no singularities other than poles in the finite part of the plane, is called an elliptic function.

Remark 1:

A function defined in real is defined in one dimension. It means we can see all the function if there is one certain period in it. Furthermore, a function defined in complex number is defined in two dimension and a “good” function defined in complex number should have two period that is doubly-periodic function.

1.1.4 Period-parallelograms:

Suppose that in the plane of the variable z for a elliptic function with two primitive periods ω_1 and ω_2 is completely determined in any one of the parallelograms with vertices at z_0, $z_0 + 2\omega_1$, $z_0 + 2\omega_1 + 2\omega_2$, $z_0 + 2\omega_2$, where $z_0 \in C$. For proper choice of z_0, the poles of this elliptic function will not reside on the boundary of any of these parallelograms. Such parallelograms are called the cells.

1.1.5 Some properties of elliptic functions:

1. The number of poles of an elliptic function in any cell is finite.
2. The number of zeros of an elliptic function in any cell is finite.
3. The sum of the residues of an elliptic function at its poles in any cell is zero.
4. Liouvilles theorem

An elliptic function with no poles in a cell is merely a constant.

1.1.6 The order of an elliptic function:

1. The order of an elliptic function

If $f(z)$ be an elliptic function and c be any constant, the number of roots of equation $f(z)=c$ which lie in any cell depends only on $f(z)$, and not on c ; this number is called the order of the elliptic function.

2. Some properties of the order of an elliptic function

a. The number of the elliptic function $f(z)$ is equal to the number of poles of $f(z)$ in the cell.

b. The order of an elliptic function is ≥ 2.

Remark2:

The elliptic functions with order two are classified as two kinds; the Weierstrassian elliptic functions, which have a single double pole in a cell, and the Jacibian elliptic functions, which have two simple poles in a cell.

The importance of the elliptic functions with order two is as indicated by the fact that any elliptic function can be in terms of either of these type. We study the elliptic functions of order two in the next section.

1.1.7 The Weierstrass elliptic function:

The Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\in Z}{}' \left\{ \frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right\}$$

where $\sum'$ denotes that the sum excludes the term when $m = n = 0$ and ω_1, ω_2 satisfy the condition that the ratio is not purely real.

For brevity, we write $\Omega_{m,n}$ in place of $2m\omega_1 + 2n\omega_2$,

so that

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\in Z}{}' \left\{ (z-\Omega_{m,n})^{-2} - \Omega_{m,n}{}^{-2} \right\} \qquad (1.1.1)$$

Remark3:

1. ***If*** $2\omega_1$ ***and*** $2\omega_2$ ***are periods whose ratio is real, then it is not double period for a noconstant elliptic function.***(In reference [2])

a. If $\dfrac{2\omega_1}{2\omega_2} = \dfrac{a}{b}$, where a and b are relatively prime integers, then there exists integers m and n such that $mb + na = 1$

Let $\omega = 2\omega_1 + 2\omega_2$. Then ω is a period and we have the following

$$\omega = \omega_1\left(m + n\frac{\omega_1}{\omega_2}\right) = \omega_1\left(m + n\frac{a}{b}\right) = \frac{\omega_1}{b}(mb + na) = \frac{\omega_1}{b},$$

So $\omega_1 = b\omega$ and $\omega_2 = a\omega$. Thus ω_1 and ω_2 integer multiples of ω.

b. If $\dfrac{\omega_1}{\omega_2} = \lambda$, λ is an irrational number. Given $\varepsilon > 0$, there exist integers p and q such that

$$|p\lambda - q| = \left| p\left(\frac{\omega_1}{\omega_2}\right) - q \right| < \varepsilon \quad \text{or} \quad |2p\,\omega_1 - 2q\,\omega_2| < 2\varepsilon|\omega_2|$$

but then $2p\,\omega_1 - 2q\,\omega_2$ would be a period of arbitrary small modulus, which is impossible.

2. ***The weierstrassian function is a elliptic function.***

a. For ω_1

$$\wp(z + 2\omega_1) = \frac{1}{(z + 2\omega_1)^2} + \sum_{m,\,n\in Z}{}' \left\{ \frac{1}{(z + 2\omega_1 - 2m\,\omega_1 - 2n\,\omega_2)^2} - \frac{1}{(2m\,\omega_1 + 2n\,\omega_2)^2} \right\}$$

$$= \frac{1}{z^2} + \sum_{m,\,n\in Z}{}' \left\{ \frac{1}{(z - 2m\,\omega_1 - 2n\,\omega_2)^2} - \frac{1}{(2m\,\omega_1 + 2n\,\omega_2)^2} \right\}$$

$$= \wp(z)$$

By the same way

$$\wp(z + 2\omega_2) = \wp(z)$$

Hence $\wp(z)$ is a doubly-periodic function of z.

b. The singular points of $\wp(z)$ are $\Omega_{m,n}$ and for any $\Omega_{m,n}$ take $k=2$ then

$(z-\Omega_{m,n})^k \wp(z)$ is analytic on $z=\Omega_{m,n}$.

c. For any finite plane the number of poles are finite.

From a. b. and c. $\wp(z)$ is an elliptic function.

1.1.8 Some properties of Weierstrass elliptic function:

1. $\wp(z)$ is an even elliptic function of z.

2. $\wp(z)' = -2\sum_{m,n\in Z}{}'(z-\Omega_{m,n})^{-3}$ is an odd elliptic function of order three with poles at

   $\{\Omega_{m,n}\}$ and zeros $(2n+1)w_1, (2m+1)w_2$; moreover,

   $\wp'(z+2w_1)=\wp'(z+2w_2)=\wp'(z)$

3. The differential equation satisfied by $\wp(z)$

$$(\wp(z)')^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{1.1.2}$$

where $g_2 = 60\sum_{m,n\in Z}{}'\Omega_{m,n}^{-4}$ $g_3 = 140\sum_{m,n\in Z}{}'\Omega_{m,n}^{-6}$

and the constant g_2, g_3 are called invariants of $\wp(z)$; moreover,

$$\begin{aligned}(\wp'(z))^2 &= 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)\\ &= 4(\wp(z)-\wp(w_1))(\wp(z)-\wp(w_2))(\wp(z)-\wp(w_3))\end{aligned} \tag{1.1.3}$$

Where $\wp(w_1)=e_1$ $\wp(w_2)=e_2$ $\wp(w_3)=e_3$ with $w_3=-w_1-w_2$.

and $e_i \neq e_j$ for $i \neq j$.

4. The intergral formula for $\wp(z)$

$$z = \int_{\wp(z)}^{\infty}\left(4t^3 - g_2 t - g_3\right)^{-\frac{1}{2}}dt \tag{1.1.4}$$

5. The addition-theorem for the function $\wp(z)$

If $u+v+w=0$, then

$$\wp(u)(\wp'(v)-\wp'(w)) - \wp(v)(\wp'(u)-\wp'(w)) + \wp(w)(\wp'(u)-\wp'(v)) = 0 \tag{1.1.5}$$

That is

$$\begin{vmatrix} \wp(u) & \wp(u)' & 1 \\ \wp(v) & \wp(v)' & 1 \\ \wp(w) & \wp(w)' & 1 \end{vmatrix} = 0$$

and $\wp(u)$, $\wp(v)$, $\wp(w)$,are all unequal.

Remark 4:

By (1.1.2)and (1.1.3)

If $\wp(u) = e_1, \wp(v) = e_2, \wp(w) = e_3$ and $u + v + w = 0$

then e_1, e_2, e_3 ,are the roots of the equation $4t^3 - g_2 t - g_3 = 0$

That is $e_1 \neq e_2 \neq e_3$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = \frac{g_2}{4}$$

$$e_1 e_2 e_3 = \frac{g_3}{4}$$

7. Another form of the addition-theorem

$$\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right\}^2 - \wp(z) - \wp(y) \qquad (1.1.6)$$

$$\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z) \text{ unless } 2z \text{ is a period.} \qquad (1.1.7)$$

1.1.9 The Riemann-Zeta function ζ(z):

The function $\zeta(z)$ defined by the equation

$$\frac{d\zeta(z)}{dz} = -\wp(z) \text{ with } \lim_{z \to 0}\{\zeta(z) - \frac{1}{z}\} = 0 . \qquad (1.1.8)$$

The limit condition in (1.1.8) is to assure that $\zeta(z)$ has simple pole at $z = 0$.

and so $\zeta(z) = \frac{1}{z} + \sum_{m,n}' \left\{\frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2}\right\}$

1.1.10 Some properties of The function ζ(z):

$\zeta(z)$ is odd with a simple poles $\Omega_{m,n}$, but is not doubly periodic. In fact, $\zeta(z)$ is quasi-periodic elliptic function,

$$\zeta(z+2\omega_1)=\zeta(z)+2\eta_1$$
$$\zeta(z+2\omega_2)=\zeta(z)+2\eta_2 \tag{1.1.9}$$

where

$$\eta_1=\zeta(\omega_1),\ \eta_2=\zeta(2\omega_1)$$

with

$$\eta_1\omega_2-\eta_2\omega_1=\frac{1}{2}\pi i$$

1.1.11 The sigma function σ(z):

The function $\sigma(z)$ defined by the equation

$$\frac{d}{dz}\log\sigma(z)=\zeta(z) \quad \text{with} \quad \lim_{z\to 0}\left\{\frac{\sigma(z)}{z}\right\}=1 \tag{1.1.10}$$

The limit condition in (1.1.10) is to assure that $\sigma(z)$ has simple zero at $z=0$.

So $\sigma(z)=z\prod'_{m,n}\left\{\left(1-\frac{z}{\Omega_{m,n}}\right)\exp\left(\frac{z}{\Omega_{m,n}}+\frac{z^2}{2\Omega^2_{m,n}}\right)\right\}$

1.1.12 Some properties of The functionσ(z):

The function $\sigma(z)$ is an odd entire function with simple zeros at all the points $\Omega_{m,n}$, and is quasi-periodicity,

$$\sigma(z+2\omega_1)=-\exp\{2\ \zeta_1(z+\omega_1)\}\ \sigma(z)$$
$$\sigma(z+2\omega_2)=-\exp\{2\ \zeta_2(z+\omega_2)\}\ \sigma(z) \tag{1.1.11}$$

1.1.13 Expression of elliptic function:

1. Any elliptic function can express in terms of $\wp(z)$ and $\wp(z)'$
2. Any elliptic function can express in terms of linear combination of Zeta-functions and their derivates.

3. Any elliptic function can express in terms of quotient of Sigma-functions.

1.2.1 Theta functions:

Let τ be a (constant) complex number whose imaginary part is positive; and write $q = e^{\pi i \tau}$, so that $|q| < 1$

Consider the function $\vartheta(z, q)$, defined by the series

$$\vartheta(z, q) = \sum_{n=-\infty}^{\infty} (-1^{n}) q^{n^2} e^{2ni}, \qquad (1.2.1)$$

qua function of the variable z.

It is evident that $\vartheta(z, q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz$

Sand that $\vartheta(z + \pi, q) = \vartheta(z, q)$;

Further $\vartheta(z + \pi\tau, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{2n} e^{2niz}$

$$= -q^{-1} e^{-2iz} \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{(n+1)^2} e^{2(n+1)iz},$$

And so $\vartheta(z + \pi\tau, q) = -q^{-1} e^{-2iz} \vartheta(z, q)$

1.2.2 The four types of Theta functions:

It is customary to write $\vartheta_4(z, q)$ in place of $\vartheta(z, q)$;the other three type of Theta-functions are defined as follows:

$$\vartheta_1(z, q) = -ie^{iz + \frac{1}{4}\pi i \tau} \vartheta_4\left(z + \frac{1}{2}\pi\tau, q\right) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)z \qquad (1.2.2)$$

$$\vartheta_2(z, q) = \vartheta_1\left(z + \frac{1}{2}\pi, q\right) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n+1)z \qquad (1.2.3)$$

$$\vartheta_3(z, q) = \vartheta_4\left(z + \frac{1}{2}\pi, q\right) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz \qquad (1.2.4)$$

$$\vartheta_4(z, q) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz \qquad (1.2.5)$$

Writing down the series at length, we have

$$\vartheta_1(z,q) = 2q^{\frac{1}{4}}\sin z - 2q^{\frac{9}{4}}\sin 3z + 2q^{\frac{2.5}{4}}\sin 5z - ..... \quad (1.2.6)$$

$$\vartheta_2(z,q) = 2q^{\frac{1}{4}}\cos z + 2q^{\frac{9}{4}}\cos 3z + 2q^{\frac{2.5}{4}}\cos 5z + ... \quad (1.2.7)$$

$$\vartheta_3(z,q) = 1 + 2q\cos 2z + 2q^{4}\cos 4z + 2q^{9}\cos 6z + ... \quad (1.2.8)$$

$$\vartheta_4(z,q) = 1 - 2q\cos 2z + 2q^{4}\cos 4z - 2q^{9}\cos 6z + ... \quad (1.2.9)$$

For brevity,

1. The parameter q will usually not be specified, so that $\vartheta_1(z), ...$ will be written for $\vartheta_1(z,q), ...$

2. When it is desired to exhibit the dependence of a Theta-function on the parameter τ, it will be written $\vartheta(z\,|\,\tau)$.

3. $\vartheta_1(0), \vartheta_2(0), \vartheta_3(0), \vartheta_4(0)$ will be replaced by $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ and ϑ_1' will denote the result of making z equal to zero in the derivate of $\vartheta_1(z)$.

1.2.3 Some properties of Theta functions:

1. $\vartheta_1(z,q)$ is an odd function of z and that the other Theta-functions are even functions of z.

2. The relations between the squares of the Theta-functions

$$\vartheta_2^2(z)\vartheta_4^2(z) = \vartheta_4^2(z)\vartheta_2^2 - \vartheta_1^2(z)\,\vartheta_3^2 \quad (1.2.10)$$

$$\vartheta_3^2(z)\vartheta_4^2(z) = \vartheta_4^2(z)\vartheta_3^2 - \vartheta_1^2(z)\vartheta_2^2 \quad (1.2.11)$$

$$\vartheta_1^2(z)\vartheta_4^2(z) = \vartheta_3^2(z)\vartheta_2^2 - \vartheta_2^2(z)\vartheta_3^2 \quad (1.2.12)$$

$$\vartheta_4^2(z)\vartheta_4^2(z) = \vartheta_3^2(z)\vartheta_3^2 - \vartheta_2^2(z)\vartheta_2^2 \quad (1.2.13)$$

Form equation (1.2.13), let z=0 we get fallowing equation.

$$\vartheta_4^4 + \vartheta_2^4 = \vartheta_3^4 . \quad (1.2.14)$$

3. The addition-formulae for the Theta-functions

$$\vartheta_3(z+y)\vartheta_3(z-y) = \vartheta_3^2(y)\vartheta_3^2(z) - \vartheta_1^2(y)\vartheta_1^2(z) \qquad (1.2.15)$$

4. Jacobis expressions for Theta-functions as infinite priducts

$$\vartheta_1(z,q) = 2q^{\frac{1}{4}}\sin z \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1-2q^{2n}\cos 2z + q^{4n}\right) \qquad (1.2.16)$$

$$\vartheta_2(z,q) = 2q^{\frac{1}{4}}\cos z \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+2q^{2n}\cos 2z + q^{4n}\right) \qquad (1.2.17)$$

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+2q^{2n}\cos 2z + q^{4n-2}\right) \qquad (1.2.18)$$

$$\vartheta_4(z,q) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1-2q^{2n}\cos 2z + q^{4n-2}\right) \qquad (1.2.19)$$

5. The differential equation satisfied by Theta-function

$$\frac{\partial \vartheta_3(z/\tau)}{\partial z^2} = -\frac{4}{\pi i}\frac{\partial \vartheta_3(z/\tau)}{\partial \tau} \qquad (1.2.20)$$

6. A relation between Theta-functions of zero argument

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0) \qquad (1.2.21)$$

7. Sigma-function can express in terms of Theta-functions

so any elliptic function can express in terms of Theta-functions

8. Landen's type of transformation

$$\frac{\vartheta_3(z\,|\,\tau)\vartheta_4(z\,|\,\tau)}{\vartheta_4(2z\,|\,2\tau)} = \frac{\vartheta_3(0\,|\,\tau)\vartheta_4(0\,|\,\tau)}{\vartheta_4(0\,|\,2\tau)} \qquad (1.2.22)$$

9. The differential equation satisfied by quotients of Theta-functions

$$\text{a. } \frac{d}{dz}\left\{\frac{\vartheta_1(z)}{\vartheta_4(z)}\right\} = \vartheta_4^{\,2}\,\frac{\vartheta_2(z)}{\vartheta_4(z)}\frac{\vartheta_3(z)}{\vartheta_4(z)} \qquad (1.2.23)$$

$$\text{b. } \frac{d}{dz}\left\{\frac{\vartheta_2(z)}{\vartheta_4(z)}\right\} = -\vartheta_3^{\,2}\,\frac{\vartheta_1(z)}{\vartheta_4(z)}\frac{\vartheta_3(z)}{\vartheta_4(z)} \qquad (1.2.24)$$

$$\text{c. } \frac{d}{dz}\left\{\frac{\vartheta_3(z)}{\vartheta_4(z)}\right\} = -\vartheta_2^{\,2}\,\frac{\vartheta_1(z)}{\vartheta_4(z)}\frac{\vartheta_2(z)}{\vartheta_4(z)} \qquad (1.2.25)$$

From (1.2.23)

We write $\xi \equiv \dfrac{\vartheta_1(z)}{\vartheta_4(z)}$ and use the result of relations between the squares of the Theta-functions.

We see that

$$\left(\frac{d\xi}{dz}\right) = (\vartheta_2^2 - \xi^2\vartheta_3^2)(\vartheta_3^2 - \xi^2\vartheta_2^2) \tag{1.2.26}$$

Write $y = \xi\vartheta_3/\vartheta_2$ $u = z\vartheta_3^2$ $k^{\frac{1}{2}} = \vartheta_2/\vartheta_3$ and k is called **modulus**

We get equation

$$(\frac{dy}{du})^2 = (1 - y^2)(1 - k^2 y^2) \tag{1.2.27}$$

This differential equation has the particular solution

$$y = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u\vartheta_3^{-2})}{\vartheta_4(u\vartheta_3^{-2})} \tag{1.2.28}$$

Rmark5:

1. Let k' be called the **complementary modulus** such that $k^2 + k'^2 = 1$, that is $k' = \sqrt{1 - k^2}$
2. The number u will be called the **argument** and the number $m = k^2$ be called the **parameter** of the functions.
3. The **complementary parameter** is the number $m_1 = 1 - m$, that is $m_1 = k'^2$

1.3.1 Jacobian function:

From (1.2.26) and (1.2.27) we know $y = \dfrac{\vartheta_3}{\vartheta_2}\dfrac{\vartheta_1\left(u\vartheta_3^{-2}\right)}{\vartheta_4\left(u\vartheta_3^{-2}\right)}$ is a particular solution of

differential equation $\left(\dfrac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2)$

We have the integral representation of y is

$$u = \int_0^{y(u)} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt \qquad (1.3.1)$$

so we defined y = sn (u, k) or simply y = sn (u) ,when it is unnecessary to emphasize the modulus k

Clearly, $\operatorname{sn}^{-1}(x,\ k) = \int_0^x \dfrac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt$

Jacobian functions defined as follow

$$\operatorname{sn}(u) = \frac{\vartheta_3\vartheta_1\left(u/\vartheta_3^2\right)}{\vartheta_2\vartheta_4\left(u/\vartheta_3^2\right)} \qquad (1.3.2)$$

$$\operatorname{cn}(u) = \frac{\vartheta_4\vartheta_2\left(u/\vartheta_3^2\right)}{\vartheta_2\vartheta_4\left(u/\vartheta_3^2\right)} \qquad (1.3.3)$$

$$\operatorname{dn}(u) = \frac{\vartheta_4\vartheta_3\left(u/\vartheta_3^2\right)}{\vartheta_3\vartheta_4\left(u/\vartheta_3^2\right)} \qquad (1.3.4)$$

From (1.2.24) (1.2.25)

We get the following integral equations

If $u = \int_{y(u)}^{1} \dfrac{1}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}\,dt$ then $y(u) = \operatorname{cn}(u,\ k)$

and $\operatorname{cn}^{-1}(x,\ k) = \int_x^1 \dfrac{1}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}\,dt$ (1.3.5)

If $u = \int_{y(u)}^{1} \dfrac{1}{\sqrt{(1-t^2)(t^2-k'^2)}}\,dt$ then $y(u) = \operatorname{dn}(u,\ k)$

and $\operatorname{dn}^{-1}(x,\ k) = \int_x^1 \dfrac{1}{\sqrt{(1-t^2)(t^2-k'^2)}}\,dt$ (1.3.6)

This integrals (1.3.1), (1.3.5), (1.3.6) are called ***the elliptic integral of the first kind***.

Glaishers nation for quotients.

A short and convenient notation has been invented by Glaisher to express reciprocals and quotients of the Jacobian elliptic functions

$$\mathrm{ns}(u) = 1/\mathrm{sn}(u) \qquad \mathrm{nc}(u) = 1/\mathrm{cn}(u) \qquad \mathrm{nd}(u) = 1/\mathrm{dn}(u \tag{1.3.7}$$

$$\mathrm{sc}(u) = \mathrm{sn}(u)/\mathrm{cn}(u) \qquad \mathrm{sd}(u) = \mathrm{sn}(u)/\mathrm{dn}(u) \qquad \mathrm{cd}(u) = \mathrm{cn}(u)/\mathrm{dn}(u)$$

$$\mathrm{cs}(u) = \mathrm{cn}(u)/\mathrm{sn}(u) \qquad \mathrm{ds}(u) = \mathrm{dn}(u)/\mathrm{sn}(u) \qquad \mathrm{dc}\,u = \mathrm{dn}(u)/\mathrm{cn}(u) \tag{1.3.8}$$

We have the following results

$$u = \int_0^{\mathrm{sc}\,u} (1+t^2)^{-\frac{1}{2}}(1+k'^2t^2)^{-\frac{1}{2}}\,dt = \int_{\mathrm{cs}\,u}^{\infty} (t^2+1)^{-\frac{1}{2}}(t^2+k'^2)^{-\frac{1}{2}}\,dt$$

$$= \int_0^{\mathrm{sd}\,u} (1-k'^2t^2)^{-\frac{1}{2}}(1+k^2t^2)^{-\frac{1}{2}}\,dt = \int_{\mathrm{ds}\,u}^{\infty} (t^2-k'^2)^{-\frac{1}{2}}(t^2+k^2)^{-\frac{1}{2}}\,dt$$

$$= \int_{\mathrm{cd}\,u}^{1} (1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}}\,dt = \int_{\mathrm{dc}\,u}^{1} (t^2-1)^{-\frac{1}{2}}(t^2-k^2t)^{-\frac{1}{2}}\,dt$$

$$= \int_{\mathrm{ns}\,u}^{\infty} (t^2-1)^{-\frac{1}{2}}(t^2-k^2)^{-\frac{1}{2}}\,dt = \int_{1}^{\mathrm{nc}\,u} (t^2-1)^{-\frac{1}{2}}(k'^2t^2+k^2)^{-\frac{1}{2}}\,dt$$

$$= \int_{1}^{\mathrm{nd}\,u} (t^2-1)^{-\frac{1}{2}}(1-k'^2t^2)^{-\frac{1}{2}}\,dt.$$

1.3.2 Some relation between Jacobian function:

1. $$\frac{d}{du}\mathrm{sn}(u) = \mathrm{cn}(u)\,\mathrm{dn}(u) \tag{1.3. 9}$$

2. $$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1 \tag{1.3.10}$$

3. $$k^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1 \tag{1.3.11}$$

4. $$\mathrm{cn}(0) = \mathrm{dn}(0) = 1 \tag{1.3.12}$$

Differentiate the equation $\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1$ and use relation equation (1.3.9) we get equation (1.3.13)

5. $$\frac{d\,\mathrm{cn}(u)}{du} = -\mathrm{sn}(u)\,\mathrm{dn}(u) \qquad (1.3.13)$$

From equation $k^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1$ and equation (1.3.9) we have equation (1.3.14)

6. $$\frac{d\,\mathrm{dn}(u)}{du} = -k^2\mathrm{sn}(u)\,\mathrm{cn}(u) \qquad (1.3.14)$$

Moreover

$$\frac{d}{du}\mathrm{sn}(u) = \mathrm{cn}(u)\,\mathrm{dn}(u) \quad = \sqrt{(1-k^2\mathrm{sn}^2(u))(1-\mathrm{sn}^2(u))} \qquad (1.3.15)$$

$$\frac{d}{du}\mathrm{cn}(u) = -\mathrm{sn}(u)\,\mathrm{dn}(u) \quad = \sqrt{(k'^2-k^2\mathrm{cn}^2(u))(1-\mathrm{cn}^2(u))} \qquad (1.3.16)$$

$$\frac{d}{du}\mathrm{dn}(u) = -k^2\mathrm{sn}(u)\,\mathrm{cn}(u) \quad = \sqrt{(\mathrm{dn}^2(u)-k'^2)(1-\mathrm{dn}^2(u))} \qquad (1.3.17)$$

And

$$\frac{d}{du}\mathrm{sn}(u) = \mathrm{cn}(u)\,\mathrm{dn}(u)\,, \quad \frac{d}{du}\mathrm{cn}(u) = -\mathrm{sn}(u)\,\mathrm{dn}(u)\,, \qquad (1.3.18)$$

$$\frac{d}{du}\mathrm{dn}(u) = -m\,\mathrm{sn}(u)\,\mathrm{cn}(u) \qquad (1.3.19)$$

$$\frac{d}{du}\mathrm{cs}(u) = -\mathrm{ds}(u)\,\mathrm{ns}(u) \quad \frac{d}{du}\mathrm{ns}(u) = -\mathrm{cs}(u)\,\mathrm{cs}(u) \qquad (1.3.20)$$

$$\frac{d}{du}\mathrm{ds}(u) = -\mathrm{cs}(u)\,\mathrm{ns}(u) \qquad (1.3.21)$$

$$\frac{d}{du}\mathrm{sc}(u) = \mathrm{dc}(u)\,\mathrm{nc}(u) \quad \frac{d}{du}\mathrm{nc}(u) = \mathrm{sc}(u)\,\mathrm{dc}(u) \qquad (1.3.22)$$

$$\frac{d}{du}\mathrm{dc}(u) = m_1\,\mathrm{sc}(u)\,\mathrm{nc}(u) \qquad (1.3.23)$$

$$\frac{d}{du}\mathrm{sd}(u) = \mathrm{sd}(u)\,\mathrm{nd}(u) \quad \frac{d}{du}\mathrm{cd}(u) = -m_1\mathrm{sd}(u)\,\mathrm{nd}(u) \qquad (1.3.24)$$

$$\frac{d}{du}\mathrm{nd}(u) = m\,\mathrm{sd}(u)\,\mathrm{cd}(u) \qquad (1.3.25)$$

7. Relations between the Squares of the Jacobian Functions

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1 \qquad \mathrm{dn}^2(u) + m\,\mathrm{sn}^2(u) = 1 \tag{1.3.26}$$

$$\mathrm{dn}^2(u) - m\,\mathrm{cn}^2(u) = m_1 \tag{1.3.27}$$

$$\mathrm{ns}^2(u) - \mathrm{cs}^2(u) = 1 \qquad \mathrm{ds}^2(u) - \mathrm{cs}^2(u) = m_1 \tag{1.3.28}$$

$$\mathrm{ns}^2(u) - \mathrm{ds}^2(u) = m \tag{1.3.29}$$

$$m\,\mathrm{sd}^2(u) - \mathrm{cd}^2(u) = 1 \qquad \mathrm{nd}^2(u) - m\,\mathrm{sd}^2(u) = 1 \tag{1.3.30}$$

$$m\,\mathrm{cd}^2(u) - m_1\,\mathrm{nd}^2(u) = 1 \tag{1.3.31}$$

$$\mathrm{nc}^2(u) - \mathrm{sc}^2(u) = 1 \qquad \mathrm{dc}^2(u) - m_1\,\mathrm{sc}^2(u) = 1 \tag{1.3.32}$$

$$\mathrm{dc}^2(u) - m_1\,\mathrm{nc}^2(u) = m \tag{1.3.33}$$

with the aid of these identities the square of any function can be expressed in terms of the square of any other. In particular

$$\mathrm{sn}^2(u) = \frac{1}{1+\mathrm{cs}^2(u)} = \frac{1}{m+\mathrm{ds}^2(u)} \tag{1.3.34}$$

$$\mathrm{cn}^2(u) = \frac{1}{1+\mathrm{sc}^2(u)} = \frac{m_1}{\mathrm{ds}^2(u)-m} \tag{1.3.35}$$

$$\mathrm{dn}^2(u) = \frac{1}{1+m\,\mathrm{sd}^2(u)} = \frac{m_1}{1-m\,\mathrm{cd}^2(u)} \tag{1.3.36}$$

1.3.3 Some properties of Jacobian functions:

1. $\mathrm{sn}(u)$ is an odd function of u

   $\mathrm{cn}(u)$ is an even function of u

   $\mathrm{dn}(u)$ is an even function of u

Double and Half Arguments

$$\mathrm{sn}(2u)=\frac{2\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1-k^2\,\mathrm{sn}^4(u)}=\frac{2\mathrm{sn}(u)\mathrm{cn}(u)\,\mathrm{dn}(u)}{\mathrm{cn}^2(u)+\mathrm{sn}^2(u)\,\mathrm{dn}^2(u)} \tag{1.3.37}$$

$$\mathrm{cn}(2u)=\frac{\mathrm{cn}^2(u)-\mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}{1-k^2\,\mathrm{sn}^4(u)}=\frac{\mathrm{cn}^2(u)-\mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}{\mathrm{cn}^2(u)+\mathrm{sn}^2(u)\,\mathrm{dn}^2(u)} \tag{1.3.38}$$

$$\mathrm{dn}(2u)=\frac{\mathrm{dn}^2(u)-k^2\mathrm{sn}^2(u)\,\mathrm{cn}^2(u)}{1-k^2\,\mathrm{sn}^4(u)}=\frac{\mathrm{dn}^2(u)+\mathrm{cn}^2(u)(\mathrm{dn}^2(u)-1)}{\mathrm{dn}^2(u)-\mathrm{cn}^2(u)(\mathrm{dn}^2(u)-1)} \tag{1.3.39}$$

$$\frac{1-\mathrm{cn}(2u)}{1+\mathrm{cn}(2u)}=\frac{\mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}{\mathrm{cn}^2(u)} \tag{1.3.40}$$

$$\frac{1-\mathrm{dn}(2u)}{1+\mathrm{dn}(2u)}=\frac{k^2\mathrm{sn}^2(u)\,\mathrm{cn}^2(u)}{\mathrm{dn}^2(u)} \tag{1.3.41}$$

$$\mathrm{sn}^2\left(\frac{1}{2}u\right)=\frac{1-\mathrm{cn}(2u)}{1+\mathrm{dn}(2u)} \tag{1.3.42}$$

$$\mathrm{cn}^2\left(\frac{1}{2}u\right)=\frac{\mathrm{dn}(u)+\mathrm{cn}(2u)}{1+\mathrm{dn}(2u)} \tag{1.3.43}$$

$$\mathrm{dn}^2\left(\frac{1}{2}u\right)=\frac{k'^2+\mathrm{dn}(u)+k^2\mathrm{cn}(u)}{1+\mathrm{dn}(u)} \tag{1.3.44}$$

3. The addition-theorem for Jacobian function .

$$\mathrm{sn}(u+v)=\frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v)+\mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1-k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \tag{1.3.45}$$

$$\mathrm{cn}(u+v)=\frac{\mathrm{cn}(u)\,\mathrm{cn}(v)-\mathrm{sn}(u)\,\mathrm{sn}(v)}{1-k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \tag{1.3.46}$$

$$\mathrm{dn}(u+v)=\frac{\mathrm{dn}(u)\,\mathrm{dn}(v)-k^2\mathrm{sn}(u)\,\mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{cn}(v)}{1-k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \tag{1.3.47}$$

$$\mathrm{sn}(u+v)\,\mathrm{sn}(u-v)=\frac{\mathrm{sn}^2(u)-\mathrm{sn}^2(v)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{1.3.48}$$

$$\mathrm{sn}(u+v)\,\mathrm{cn}(u-v)=\frac{\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(v)+\mathrm{sn}(v)\,\mathrm{cn}(v)\,\mathrm{dn}(u)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{1.3.49}$$

$$\mathrm{sn}(u+v)\,\mathrm{dn}(u-v)=\frac{\mathrm{sn}(u)\,\mathrm{dn}(u)\,\mathrm{cn}(v)+\mathrm{sn}(v)\,\mathrm{dn}(v)\,\mathrm{cn}(u)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{1.3.50}$$

$$\mathrm{cn}(u+v)\,\mathrm{cn}(u-v)=\frac{\mathrm{cn}^2(u)-\mathrm{sn}^2(v)\,\mathrm{dn}^2(u)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{1.3.51}$$

$$\mathrm{cn}\,(u+v)\,\mathrm{dn}\,(u-v)=\frac{\mathrm{cn}\,(u)\,\mathrm{dn}\,(u)\,\mathrm{cn}(v)\mathrm{dn}\,(v)+k'^2\mathrm{sn}(u)\,\mathrm{sn}\,(v)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{1.3.52}$$

$$\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v)=\frac{\mathrm{dn}^2(u)-k^2\mathrm{cn}^2(u)\,\mathrm{sn}^2(v)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{1.3.53}$$

4. The constant K , K'

a. Symbol K is a function of k such that sn (K, k) = 1

In other words,

$$K\,(k)=\int_0^1(1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}}\,dt \tag{1.3.54}$$

and $\mathrm{sn}\,K = 1$, $\mathrm{cn}\,K = 0$, $\mathrm{dn}\,K = k'$

b. Symbol K' is a function of k'

$$K'(k')=\int_0^1(1-t^2)^{-\frac{1}{2}}(1-k'^2t^2)^{-\frac{1}{2}}\,dt \tag{1.3.55}$$

Remark5:

1. $K\,(m)=K'(1-m)=K'(m')$

2. $K(0)=\dfrac{1}{2}\pi$

   $K'(0)=\infty$

3. Another form of K and K'

$$K\,(k)=\int_0^{\pi/2}(1-k^2\sin^2(\theta))^{-\frac{1}{2}}\,d\theta \tag{1.3.56}$$

$$K'(k)=\int_0^{\pi/2}(1-k'^2\sin^2(\theta))^{-\frac{1}{2}}\,d\theta \tag{1.3.57}$$

5. The periodic properties of the Jacobian elliptic functions

a. associated with K

$$\begin{aligned}\mathrm{sn}\,(u+4K)&=\mathrm{sn}\,(u)\\ \mathrm{cn}\,(u+4K)&=\mathrm{cn}\,(u)\\ \mathrm{dn}\,(u+2K)&=\mathrm{dn}(u)\end{aligned} \tag{1.3.58}$$

b. associated with $K + iK'$

$$\begin{aligned} sn(u + 2K + 2iK') &= -sn(u) \qquad & sn(u + 4K + 4iK') &= sn(u) \\ cn(u + 2K + 2iK') &= cn(u) \qquad & cn(u + 4K + 4iK') &= cn(u) \\ dn(u + 2K + 2iK') &= -dn(u) \qquad & dn(u + 4K + 4iK') &= dn(u) \end{aligned} \tag{1.3.59}$$

c. associated with iK'

$$\begin{aligned} sn(u + 2iK') &= sn(u) \qquad & sn(u + 4iK') &= sn(u) \\ cn(u + 2iK') &= -cn(u) \qquad & cn(u + 4iK') &= cn(u) \\ dn(u + 2iK') &= -dn(u) \qquad & dn(u + 4iK') &= dn(u) \end{aligned} \tag{1.3.60}$$

u	sn (u)	cn (u)	dn (u)
0	0	1	1
$\frac{1}{2}K$	$\frac{1}{(1+m_1^{\frac{1}{2}})^{\frac{1}{2}}}$	$\frac{m_1^{\frac{1}{4}}}{(1+m_1^{\frac{1}{2}})^{\frac{1}{2}}}$	$m_1^{\frac{1}{4}}$
K	1	0	$m_1^{\frac{1}{2}}$
2K	0	-1	1
$\frac{1}{2}iK'$	$im^{-\frac{1}{4}}$	$\frac{(1+m^{\frac{1}{2}})^{\frac{1}{2}}}{m^{\frac{1}{4}}}$	$(1+m^{\frac{1}{2}})^{\frac{1}{2}}$
iK'	∞	∞	∞
$2iK'$	0	-1	-1
$K + iK'$	$m^{-\frac{1}{2}}$	$-i(m_1/m)^{\frac{1}{2}}$	0
$2K + 2iK'$	0	1	-1

Table 1

Special Values of Argument

Remark6:

For any Jacobian elliptic functions pq (u)

$pq\,(u + 4K\,, k) = pq\,(u + 4iK', k) = pq\,(u, k)$

Periods				
$2K\,, 4iK'$	cs (u)	sc (u)	dn (u)	nd (u)
$4K\,, 2iK'$	ns (u)	dc (u)	sn (u)	cd (u)
$4K\,, 2K + 2iK'$	ds (u)	nc (u)	cn (u)	sd (u)

Table 2 Periods

	sn (u)	cn (u)	dn (u)
Zeros	$0,\ 2K$	$K,\ 3K$	$K + iK',\ K + 3iK'$
Poles	$iK', 2K + iK'$	$iK',\ 2K + iK'$	$iK',\ 3iK'$
Periods	$4K,\ 2iK'$	$4K,\ 2K + 2iK'$	$2K,\ 4iK'$

Table 3 Zeros, Poles and Periods

6. Jacobi´s imaginary transformation

$$sn\,(iu\,, k) = i\,sc\,(u,\ k') \quad cn\,(iu\,, k) = nc\,(u,\ k') \quad dn\,(iu\,, k) = dc\,(u\,,\ k') \qquad (1.3.61)$$

If $z = x + i\,y$, the addition theorems the give with

$$s_1 = sn\,(x\,, k)\,,\quad s_2 = sn\,(y\,, k')\,,$$

$$c_1 = cn\,(x\,, k)\,,\quad c_2 = cn\,(y\,, k')\,,$$

$$d_1 = dn\,(x\,, k)\,,\quad d_2 = dn\,(y\,, k')$$

$$sn(z\,, k) = \frac{s_1 d_2 + i\,c_1 d_1 s_2 c_2}{c_2^2 + k^2 s_1^2 s_2^2} \qquad (1.3.62)$$

$$cn(z\,, k) = \frac{c_1 c_2 - i\,s_1 d_1 s_1 d_2}{c_2^2 + k^2 s_1^2 s_2^2} \qquad (1.3.63)$$

$$dn(z\,, k) = \frac{d_1 c_2 d_2 - i\,k^2 s_1 c_1 s_2}{c_2^2 + k^2 s_1^2 s_2^2} \qquad (1.3.64)$$

Argument	sn	cn	dn
u	sn (u)	cn (u)	dn (u)
-u	-sn (u)	cn (u)	dn (u)
$u+K$	cd (u)	$-k'\,\mathrm{sd}\,(u)$	$k'\mathrm{nd}\,(u)$
$u-K$	- cd (u)	$k'\,\mathrm{sd}\,(u)$	$k'\mathrm{nd}\,(u)$
$K-u$	cd (u)	$k'\,\mathrm{sd}\,(u)$	$k'\mathrm{nd}\,(u)$
$u+2K$	- sn (u)	- cn (u)	dn (u)
$u-2K$	- sn (u)	- cn (u)	dn (u)
$2K-u$	sn (u)	- cn (u)	dn (u)
$u+iK'$	$k^{-1}\,\mathrm{ns}\,(u)$	$-i\,k^{-1}\,\mathrm{ds}\,(u)$	$-i\,\mathrm{cs}\,(u)$
$u+2iK'$	sn (u)	- cn (u)	- dn (u)
$u+K+iK'$	$k^{-1}\,\mathrm{dc}\,(u)$	$-ik'k^{-1}\,\mathrm{nc}\,(u)$	$i\,k'\mathrm{sc}\,(u)$
$u+2K+2iK'$	- sn (u)	cn (u)	- dn (u)

Table 4

Chang of Argument

1.3.4 The Jacobian elliptic function and the trigonometric functions:

1. Observation:

Recall the integral representation of $\mathrm{sn}\,(u)$ in (1.3.1),

$$u=\int_0^{\mathrm{sn}\,(u)}\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt \qquad (1.3.65)$$

a. When

$$k=0,$$

(1.3.65) becomes

$$u=\int_0^{\sin\,(u)}\frac{dt}{\sqrt{1-x^2}} \qquad (1.3.66)$$

That is,

$$sn(u) \text{ degenerate to } \sin(u) \text{ as } k \to 0.$$

b. When

$$k = 1,$$

(1.3.65) becomes

$$u = \int_0^{\tanh(u)} \frac{1}{1-\zeta^2} d\xi \qquad (1.3.67)$$

That is,

$$sn(u) \text{ degenerate to } \tanh(u) \text{ as } k \to 1.$$

Similarly,

$$cn(u) \text{ degenerate to } \cos(u) \text{ as } k \to 0,$$

$$dn(u) \text{ degenerate to } 1 \text{ as } k \to 0,$$

and

$$cn(u) \text{ degenerate to } \operatorname{sech}(u) \text{ as } k \to 1,$$

$$dn(u) \text{ degenerate to } \operatorname{sech}(u) \text{ as } k \to 1,$$

2. Exact:

Changing the variable by $t = \sin(\theta)$, the integral (1.3.65) is reduced to ***Legendre's form***,

$$F(\phi, k) = sn^{-1}(\sin\phi, k) = \int_0^{\phi} (1 - k^2 \sin^2\theta)^{-\frac{1}{2}} d\theta \qquad (1.3.68)$$

Then, expanding the integrand in ascending powers of k^2 and integrating term by term, we find that

$$sn^{-1}(\sin\phi, k) = \phi + \frac{1}{4}k^2(\phi - \sin\phi\cos\phi) + ... \quad , \qquad (1.3.69)$$

which is equivalent to

$$u = sn^{-1}(x, k) = \sin^{-1}(x) + \frac{1}{4}k^2(\sin^{-1}(x) - x\sqrt{1-x^2}) + ... \quad . \qquad (1.3.70)$$

where $x = \sin\phi$

This series can now be inverted to expand $\mathrm{sn}(u)$ in powers of k^2, thus:

$$
\begin{aligned}
\mathrm{sn}(u, k) = x &= \sin[u + \frac{1}{4}k^2(x\sqrt{(1-x^2)} - \sin^{-1}(x)) + O(k^4)] \\
&= \sin(u) + \frac{1}{4}k^2(x\sqrt{(1-x^2)} - \sin^{-1}(x))\cos(u) + O(k^4) \\
&= \sin(u) + \frac{1}{4}k^2(\sin(u)\cos(u) - u)\cos(u) + O(k^4)
\end{aligned}
\tag{1.3.71}
$$

Moreover,

1. When the parameter $m = k^2$ is small that its square may be neglected, the following approximations may be used to calculate the elliptic functions in terms of circular functions.

$$\mathrm{sn}(u|m) = \sin(u) - \frac{1}{4}m\cos(u)(u - \sin(u)\cos(u)), \tag{1.3.72}$$

$$\mathrm{cn}(u|m) = \cos(u) + \frac{1}{4}m\sin(u)(u - \sin(u)\cos(u)), \tag{1.3.73}$$

$$\mathrm{dn}(u|m) = 1 - \frac{1}{2}m\sin^2(u), \tag{1.3.74}$$

2. When the parameter $m = k^2$ is so near unity that the square of the complementary parameter $m_1 = 1 - m = 1 - k^2$ may be neglected, the following approximations may be used to calculate the elliptic functions in terms of hyperbolic functions.

$$\mathrm{sn}(u|m) = \tanh(u) + \frac{1}{4}m_1\,\mathrm{sech}^2(u)(\sinh(u)\cosh(u) - u), \tag{1.3.75}$$

$$\mathrm{cn}(u|m) = \mathrm{sech}(u) - \frac{1}{4}m_1\tanh(u)\,\mathrm{sech}(u)(\sinh(u)\cosh(u) - u), \tag{1.3.76}$$

$$\mathrm{dn}(u|m) = \mathrm{sech}(u) + \frac{1}{4}m_1\tanh(u)\,\mathrm{sech}(u)(\sinh(u)\cosh(u) + u), \tag{1.3.77}$$

1.3.5 General form of the elliptic integral of the first kind:

1. integral of the Jacodian function sn (u)

By changing the variable in the integral (1.3.1) using the substitution $t = s/b$, we calculate that

$$sn^{-1}(x, k) = u = \int_0^x \{(1 - t^2)(1 - k^2 t^2)\}^{-\frac{1}{2}} dt \qquad (0 \le x \le 1)$$

$$= a\int_0^{bx} \{(a^2 - s^2)(b^2 - s^2)\}^{-\frac{1}{2}} ds \qquad (1.3.78)$$

where $0 < b < b/k = a$

It now follows that

$$\frac{1}{a} sn^{-1}(\frac{x}{b}, \frac{b}{a}) = \int_0^x \frac{1}{\sqrt{\{(a^2 - t^2)(b^2 - t^2)\}}} dt \qquad (1.3.79)$$

where $0 \le x \le b < a$

2. integral of the Jacodian function cn (u)

In the same way

By changing the variable in the integral (1.3.5) using the substitution,

$t = s/b$, $k = b/\sqrt{(a^2 + b^2)}$

After some manipulation, we arrive at the formula

$$\frac{1}{\sqrt{(a^2 + b^2)}} cn^{-1}[\frac{x}{b}, \frac{b}{\sqrt{(a^2 + b^2)}}] = \int_x^b \frac{1}{\sqrt{\{(a^2 + t^2)(b^2 - t^2)\}}} dt \qquad (0 \le x \le b) \qquad (1.3.80)$$

3. More results below:

$$cd^{-1}[\frac{x}{b}, \frac{b}{a}]$$

$$= a\int_x^b \frac{1}{\sqrt{(a^2 - t^2)(b^2 - t^2)}} dt, \qquad 0 \le x \le b < a, \qquad (1.3.81)$$

$$sd^{-1}[\frac{\sqrt{(a^2 + b^2)}x}{ab}, \frac{b}{\sqrt{(a^2 + b^2)}}]$$

$$= \sqrt{(a^2 + b^2)}\int_0^x \frac{1}{\sqrt{(a^2 + t^2)(b^2 - t^2)}} dt, \qquad 0 \le x \le b, \qquad (1.3.82)$$

$$\mathrm{dc}^{-1}[\frac{x}{b},\frac{b}{a}] = a\int_a^x \frac{1}{\sqrt{(t^2-a^2)(t^2-b^2)}}dt, \quad b < a \le x, \tag{1.3.83}$$

$$\mathrm{ns}^{-1}[\frac{x}{a},\frac{b}{a}] = a\int_x^\infty \frac{1}{\sqrt{(t^2-a^2)(t^2-b^2)}}dt, \quad b < a \le x, \tag{1.3.84}$$

$$\mathrm{nd}^{-1}[\frac{x}{b},\frac{\sqrt{(a^2-b^2)}}{a}] = a\int_b^x \frac{1}{\sqrt{(a^2-t^2)(t^2-b^2)}}dt, \quad b \le x \le a, \tag{1.3.85}$$

$$\mathrm{dn}^{-1}[\frac{x}{a},\frac{\sqrt{(a^2-b^2)}}{a}] = a\int_x^a \frac{1}{\sqrt{(a^2-t^2)(t^2-b^2)}}dt, \quad b \le x \le a, \tag{1.3.86}$$

$$\mathrm{nc}^{-1}[\frac{x}{a},\frac{b}{\sqrt{(a^2+b^2)}}] = \int_a^x \frac{1}{\sqrt{(t^2-a^2)(t^2+b^2)}}dt, \quad a \le x, \tag{1.3.87}$$

$$\mathrm{ds}^{-1}[\frac{x}{\sqrt{(a^2+b^2)}},\frac{b}{\sqrt{(a^2+b^2)}}] = \sqrt{(a^2+b^2)}\int_x^\infty \frac{1}{\sqrt{(t^2-a^2)(t^2+b^2)}}dt, \quad a \le x, \tag{1.3.88}$$

$$\mathrm{sc}^{-1}\left[\frac{x}{b}, \frac{\sqrt{(a^2 - b^2)}}{a}\right]$$

$$= a\int_0^x \frac{1}{\sqrt{(t^2 + a^2)(t^2 + b^2)}}\,dt, \quad 0 < b < a, \quad 0 \le x, , \tag{1.3.89}$$

$$\mathrm{cs}^{-1}\left[\frac{x}{a}, \frac{\sqrt{(a^2 - b^2)}}{a}\right]$$

$$= a\int_x^\infty \frac{1}{\sqrt{(t^2 + a^2)(t^2 + b^2)}}\,dt, \quad 0 < b < a, \quad 0 \le x, \tag{1.3.90}$$

1.3.6 Some graphs of Jacobian functions:

1. Jacobian function sn u

a. sn (u, k)

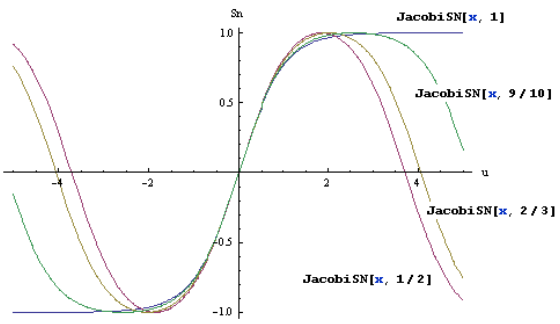


Figure 1.1

b. sn(u,1)

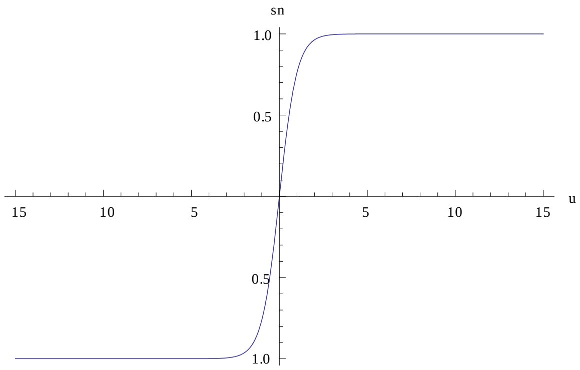


Figure 1.2

c. $\mathrm{sn}(u, \frac{2}{3})$

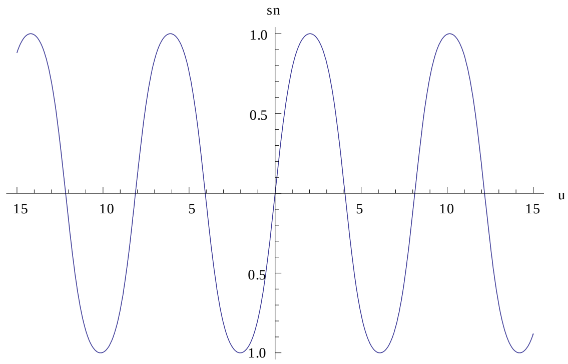


Figure 1.3

d. $\mathrm{sn}(u, \frac{1}{2})$

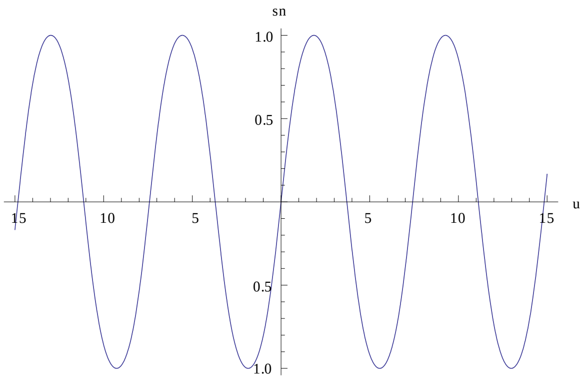


Figure 1.4

Remark 7: .

$sn(u)$ is an odd periodic function of u for u is real. Moreover the period is larger when k is larger.

We only consider that the Jacobian function sn u is defined in real number when we sketch the graphs above. However, the Jacobian function sn (u) is a function from complex number to complex number. Because this function is from two dimension to two dimension, it means that we have to analyze it in a four dimension space. It is difficult for us to do this.

Therefore, we use the method below to analyze the Jacobian function sn (u) defined in complex number.

First, we define two new functions Re(u), Im(u). Re(u) is a function that we take the real part of the Jacobian function sn u and Im(u) is another function we take the imaginary part of the Jacobian function sn (u). That is Re(u) = Re{sn(u)} and Im(u)=Im{sn(u)}.

Example :

1. $sn(3+4i, \frac{1}{2}) = 0.660252 - 0.203738\, i$

   $Re(3+4i) = 0.66025$

   $Im(3+4i) = 0.2037$

2. $sn(6-2i) = -1.47598 + 0.0512946\, i$

   $Re(6-2i) = -1.47598$

   $Im(6-2i) = 0.0512946$

Second, we can use two three-dimensional figures of Re(u) and Im(u) to represent the behavior of the Jacobian function sn (u). It is obvious that the Jacobian function sn (u) is a doubly-periodic function of u and there is a smallest unit parallelogram that can be repeated to form the entire graph.

In the end, we want to examine whether the graphs of Re(u) and Im(u) are right when the domain of the Jacobian function sn (u) is restricted in real number. It is coincident that the range of the Jacobian function sn (u) defined in real umber is also real.

Now, in order to observe the graph or Re(u) clearly, we take the value of x-axis (real part of u) from -5 to 5 and the value of y-axis (imaginary part of u) from 0 to 5. It is easy to discover that the intersection of the plane y = 0 and the graph of Re(u) is the figure of the Jacobian function sn u defined in real number.

In comparison, we find that the three-dimensional figure of the intersection and the two-dimensional graph of the Jacobian function sn (u) defined in real number are the same. In the other hand, the value of Im(u) is zero in the three-dimensional figure when y = 0.

We can use the same way to observe the Jacobian function cn (u) and the Jacobian function dn (u) .

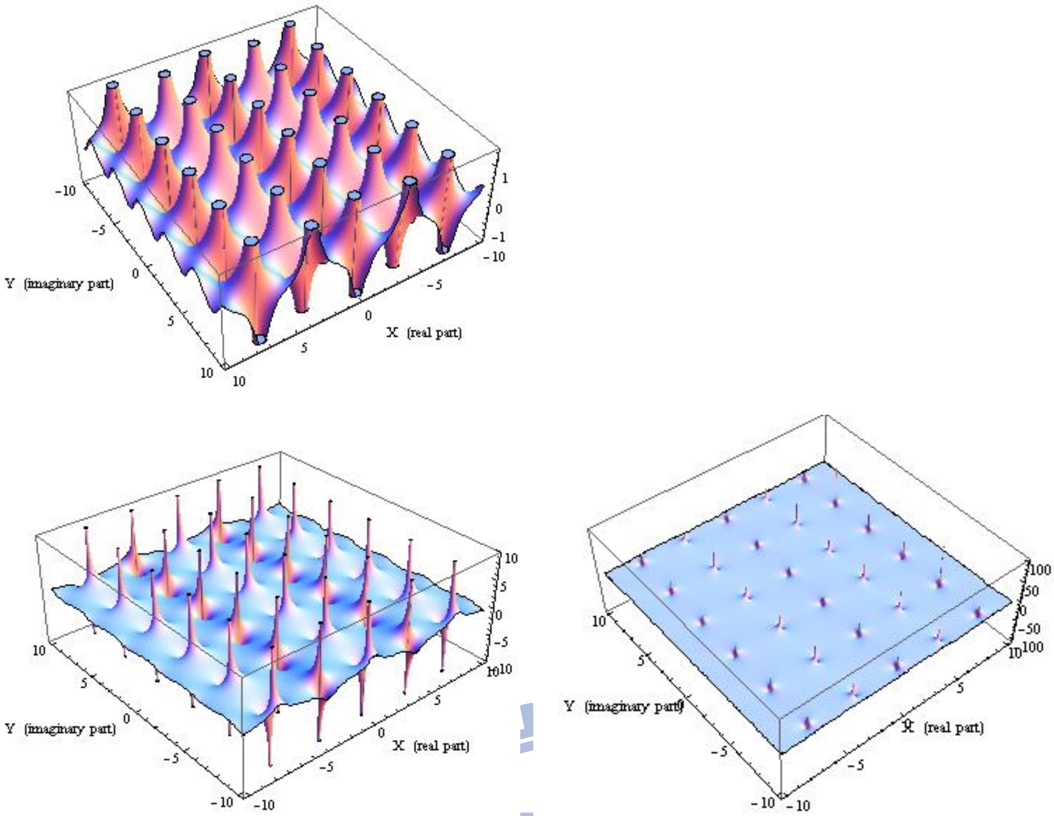
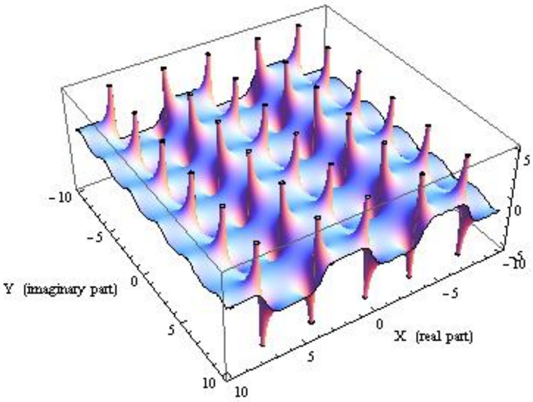


Figure 1.5

The figure Im(u) represents the imaginary part of Jacobian function $\mathrm{sn}(u, \frac{1}{2})$ for complex number u .



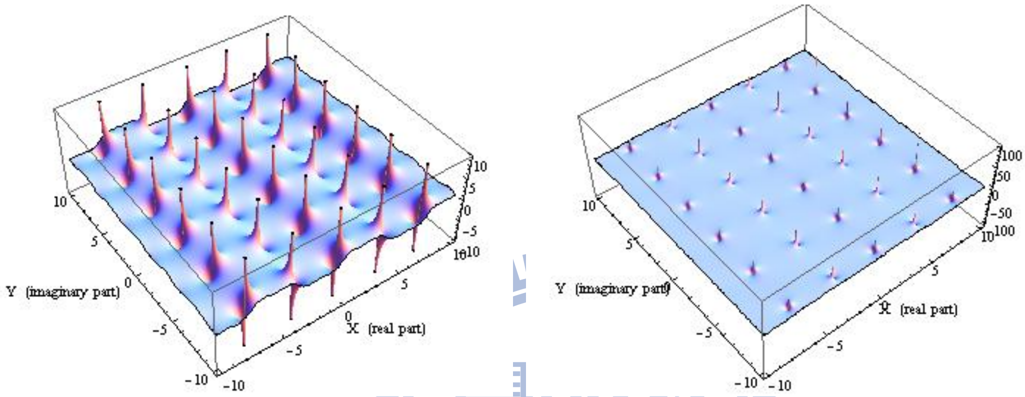


Figure 1.6

The figure Re(u) represent the real part of Jacobian function $sn(u, \frac{1}{2})$ for complex number u .

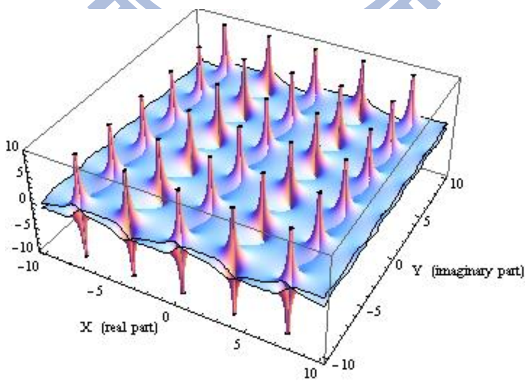


Figure 1.7

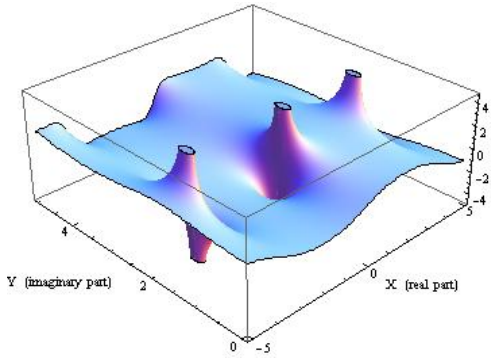


Figure 1.8

The graph of Re(u) for x-axis from -5 to 5 and y-axis from 0 to 5

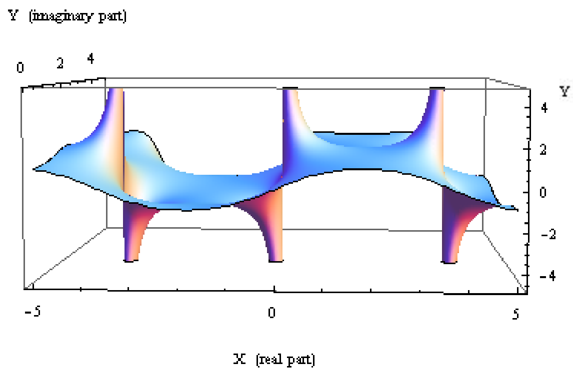


Figure 1.9

We can see the intersection line from this direction.

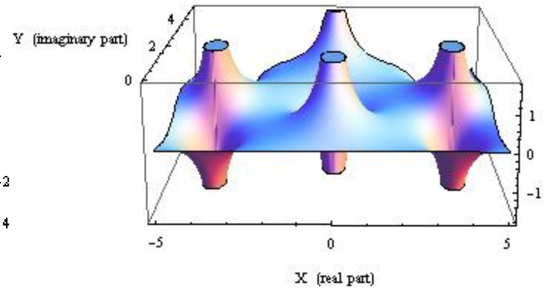


Figure 1.10

The imaginary part of function sn(u).

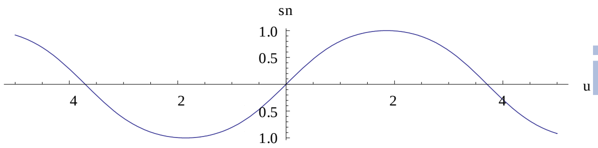


Figure 1.11

The jacobian function sn(u).

2. Jacobian function cn(u)

a. cn(u, k)

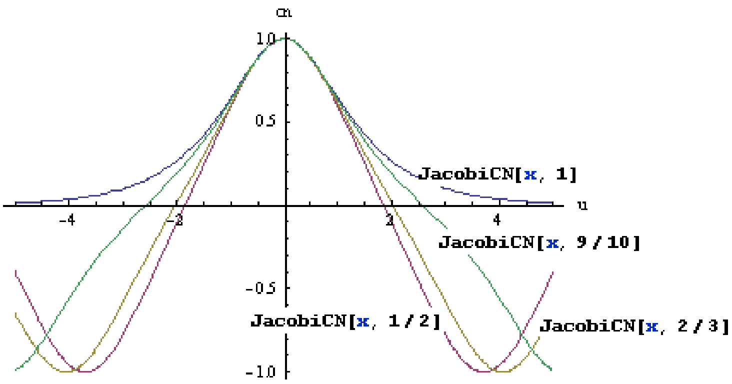


Figure 1.12

b. cn(u,1)

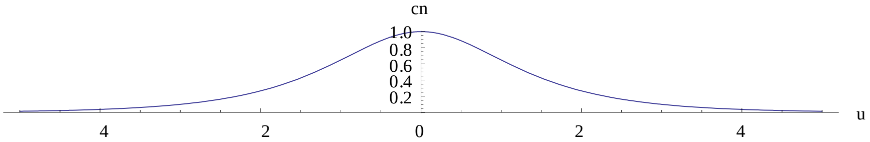


Figure 1.13

c. cn(u, $\frac{2}{3}$)

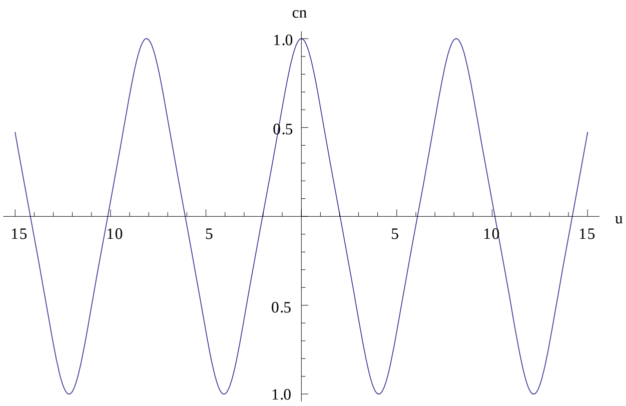


Figure 1.14

d. $cn(u, \frac{1}{2})$

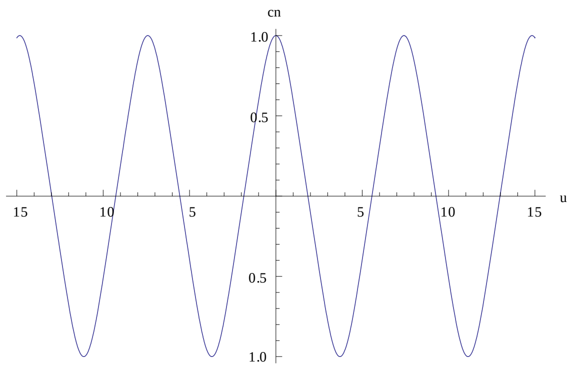


Figure 1.15

Remark 8:

cn(u is an even periodic function of u for u is real. Moreover the period is larger when k is larger.

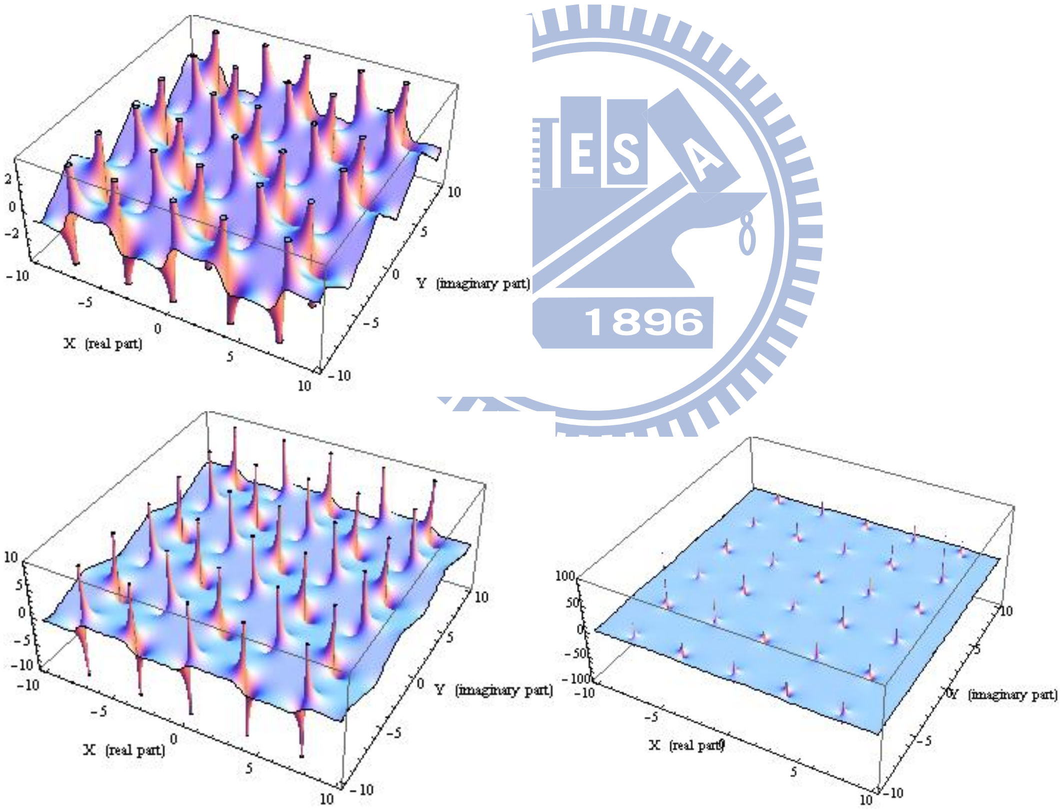


Figure 1.16

The imaginary part of the Jacobian function $cn\left(u, \frac{1}{2}\right)$

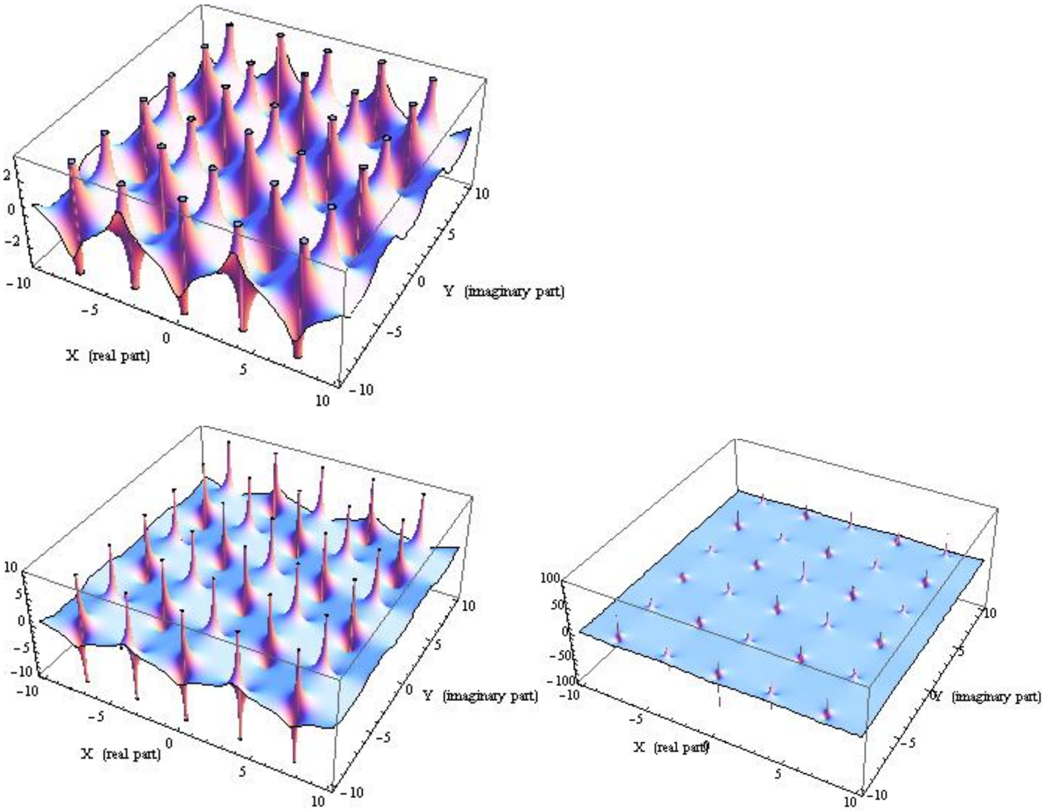


Figure 1.17

The real part of the Jacobian function $\mathrm{cn}\left(u, \frac{1}{2}\right)$

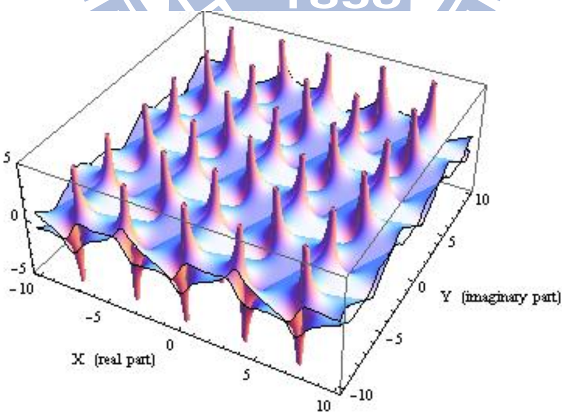


Figure 1.18

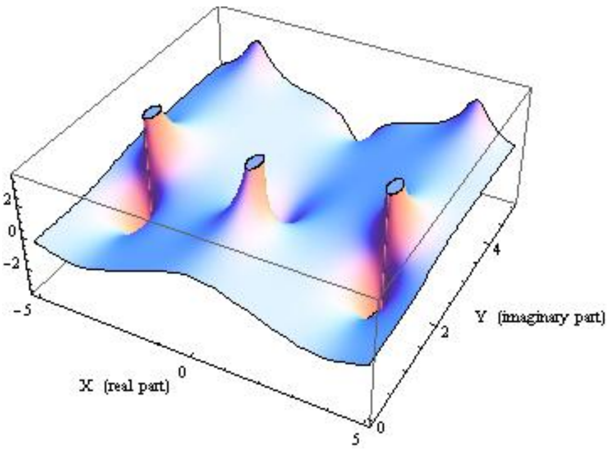


Figure 1.19

The real part of the Jacobian function $\mathrm{cn}\left(u, \frac{1}{2}\right)$ for

x-axis from -5 to 5 and y-axis from 0 to 5

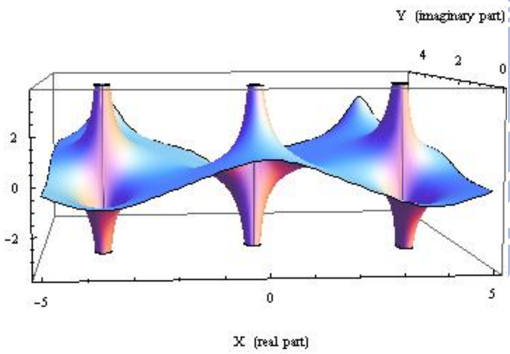


Figure 1.20
We can see the intersection line from this direction.

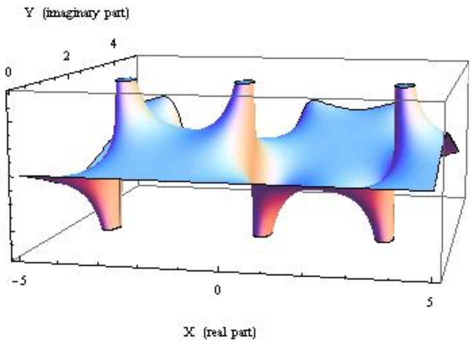


Figure 1.21
The imaginary part of function cn(u).

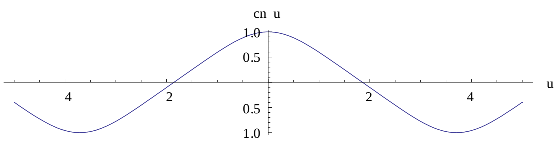


Figure 1.22

The jacobian function cn(u)

3. Jacobian function dn u

a. dn(u, k)

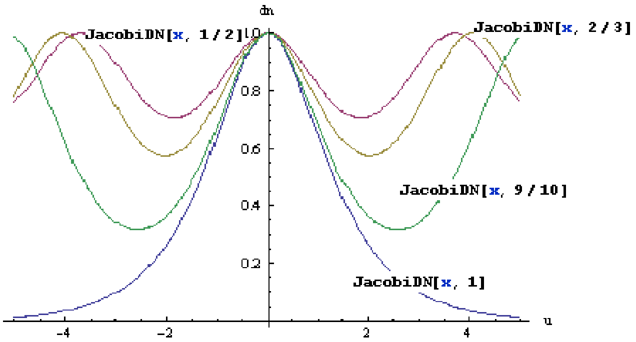


Figure 1.23

b. dn(u,1)

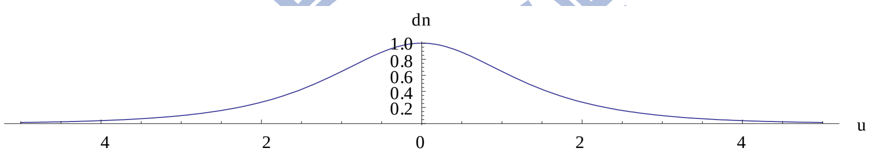


Figure 1.24

Remark 9:

$dn(u,1)=cn(u,1)$

By definition

$$u=\int_{cn(u,k)}^{1}\frac{1}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}dt$$

$$V=\int_{dn(v,k)}^{1}\frac{1}{\sqrt{(1-t^2)(t^2-k'^2)}}dt$$

$$k^2+k'^2=1$$

For k=1 then $k'=0$ and

$$u=\int_{cn(u,1)}^{1}\frac{1}{\sqrt{(1-t^2)(0^2+1^2t^2)}}dt=\int_{cn(u,1)}^{1}\frac{1}{\sqrt{(1-t^2)t^2}}dt=\int_{dn(u,1)}^{1}\frac{1}{\sqrt{(1-t^2)(t^2-0^2)}}dt$$

Therefore, $dn(u,1)=cn(u,1)$.

c. $dn(u, \frac{2}{3})$

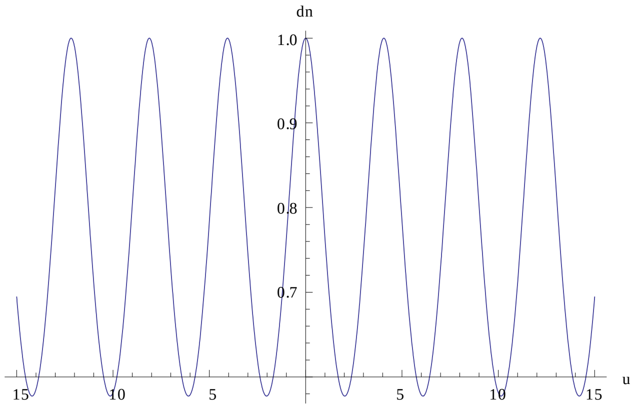


Figure 1.25

d. $dn(u, \frac{1}{2})$

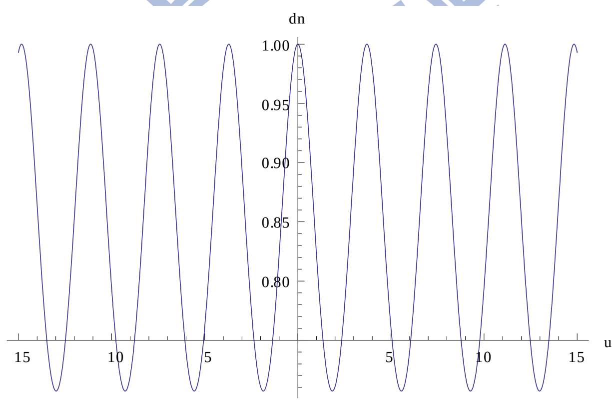


Figure 1.26

Remark 10:

$dn(u$ is an even periodic function of u for u is real. Moreover the period is larger when k is larger.

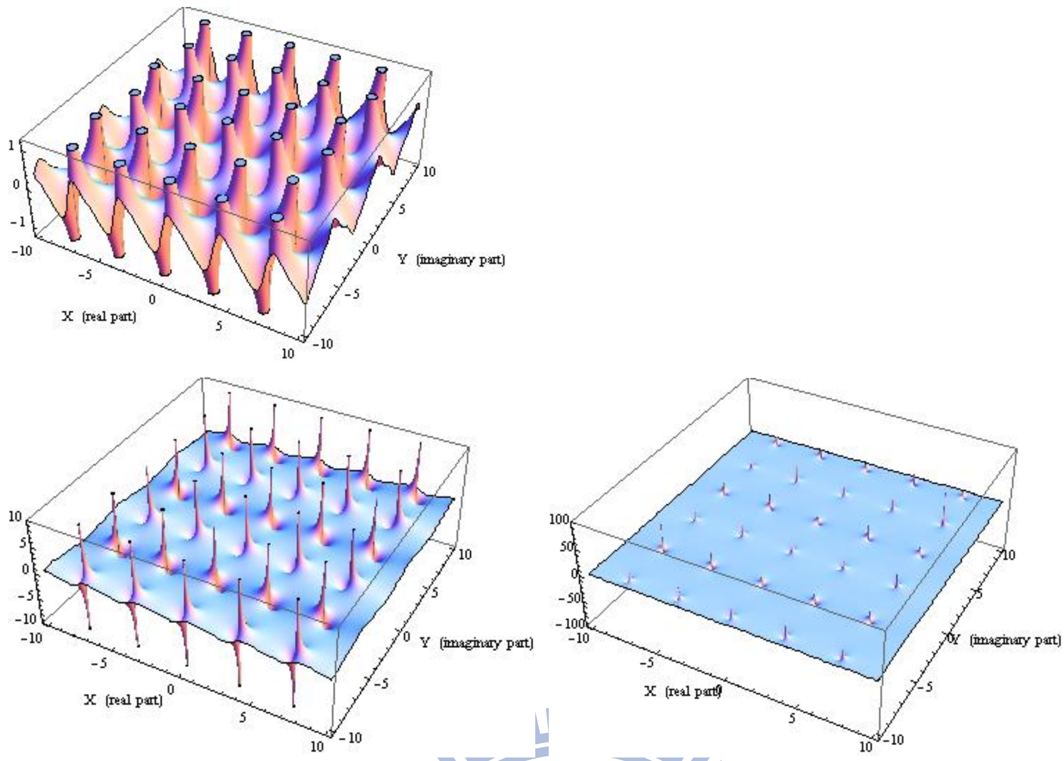


Figure 1.27

The imaginary part of the Jacobian function $\mathrm{dn}\left(u, \frac{1}{2}\right)$

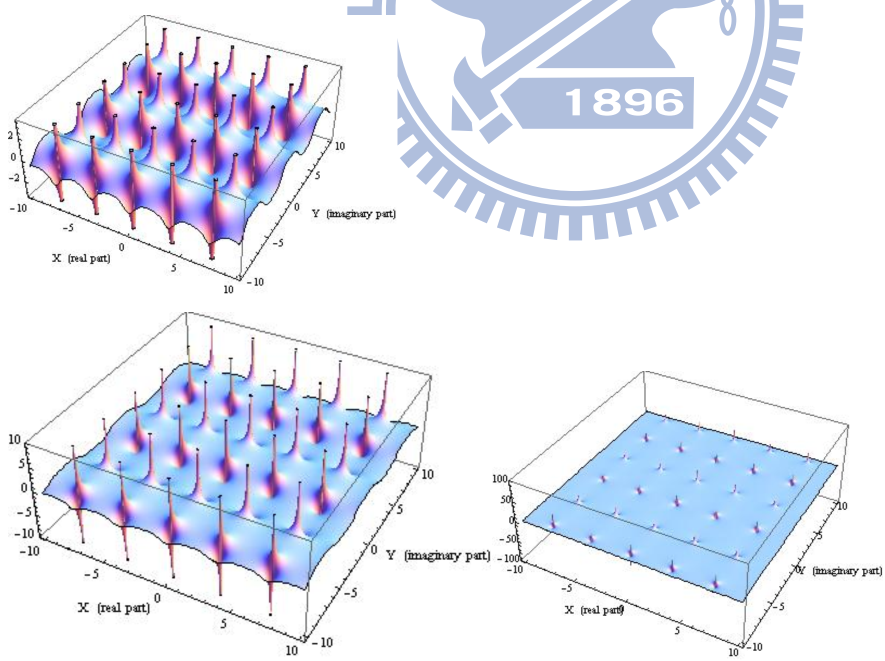


Figure 1.28

The real part of the Jacobian function $\mathrm{dn}\left(u, \frac{1}{2}\right)$

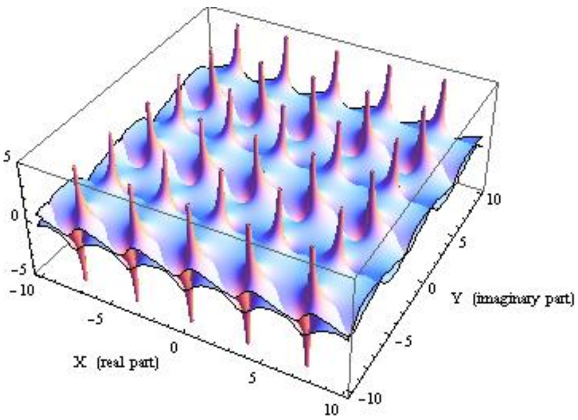


Figure 1.29

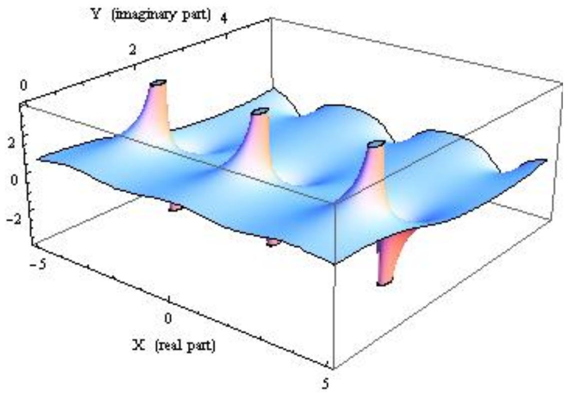


Figure 1.30

The real part of the Jacobian function $\mathrm{dn}\left(u, \frac{1}{2}\right)$ for

x-axis from -5 to 5 and y-axis from 0 to 5

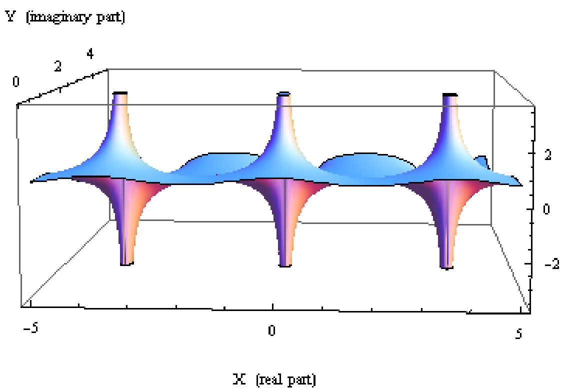


Figure 1.31

We can see the intersection line from this direction.

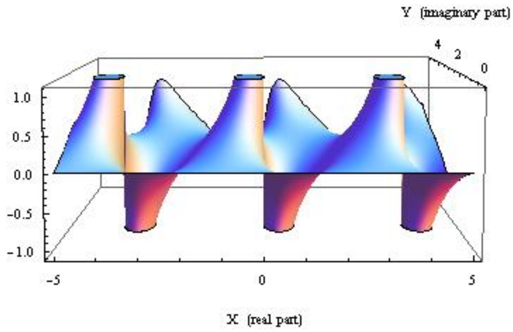


Figure 1.32

The imaginary part of dn(u).

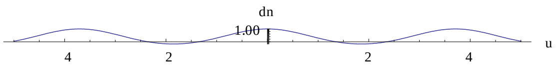


Figure 1.33

The jacobian function dn(u)

4. Jacobian function sc u

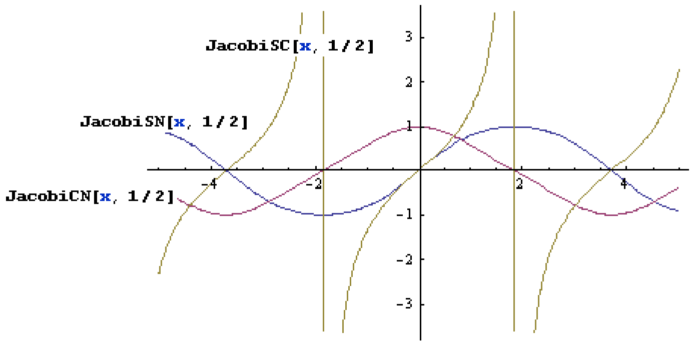


Figure 1.34

Chapter 2 Linearzied , Nullclines , Hamiltonian system and Dissipative system

In this chapter, according to the reference [1], we analyze phase portraits by some ways.

2.1 Linearzied:

1. linearized system

For a general form of a nonlinear system

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

The linearized system at the equilibrium point (x_0, y_0) is

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{Where } u = x - x_0 \quad v = y - y_0$$

And $J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$

is called the Jacobian matrix of the system at (x_0, y_0)

2. Classerify equilibrium points by linearized system

a. If all eigenvalues of J are negative real numbers than (x_0, y_0) is sink.

If all eigenvalues of J are complex numbers with negative real parts than (x_0, y_0) is spiral sink

b. If all eigenvalues of J are positive real numbers than (x_0, y_0) is source

If all eigenvalues of J are complex numbers with positive real parts than (x_0, y_0) is spiral source

c. If J has one positive and one negative eigenvalue than (x_0, y_0) is saddle

2.2 Nullclines:

1. x-nullcline y-nullcline

For the system

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

The x-nullcline is the set of points (x,y) where $f(x, y)$ is zero that is, the level curve where $f(x, y)$ is zero. The y-nullcline is the set of points where $g(x, y)$ is zero.

Example:

2. Some properties of nullclines

a. Along the x-nullcline, the x-component of the vector field is zero, and consequently the vector field is vertical.

Along the y-nullcline, the y-component of the vector field is zero, and consequently the vector field is horizontal.

b. The intersections of the nullclines are the equilibrium points

c. The regions separated by nullclines offer information of vector field

2.3 Hamiltonian system:

1. Conserved quantity

A real-valued function *H*(x,y) of the two variables x and y is a conserved quantity for a system of differential equations if it is constant along all solution curves of system. That is, if (x(t),y(t)) is a solution of the system, then *H*(x(t),y(t))is constant. In other words,

$$H'(x(t), y(t)) = 0 \qquad \text{for } (x(t),y(t)) \text{ is a solution of the system}$$

2. Hamiltonian system

A system of differential equations is called Hamiltonian system if there exists a real-valued function *H*(x,y) such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

for all x and y. The function *H* is called the Hamiltonian function for the system.

3. Relation between conserved quantity and Hamiltonian system

Letting (x(t),y(t)) be any solution of the system then

$$\frac{d}{dt}H(x(t),\quad y(t)) = (\frac{\partial H}{\partial x})(\frac{dx}{dt}) + (\frac{\partial H}{\partial y})(\frac{dy}{dt})$$

$$= (\frac{\partial H}{\partial x})(\frac{dH}{dy}) + (\frac{\partial H}{\partial y})(-\frac{dH}{dx})$$

$$= 0$$

So Hamiltonian system is conserved.

4. Equilibrium points of Hamiltonian system

Suppose (x_0, y_0) is our equilibrium point for the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

The Jacobian matrix at this equilibrium point is given by

$$\begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} \end{pmatrix}$$

where each of these partial derivatives is evaluated at (x_0, y_0).

Since

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$$

The Jacobian matrix assumes the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

where $\alpha = \frac{\partial^2 H}{\partial x \partial y}$ $\beta = \frac{\partial^2 H}{\partial y^2}$ $\gamma = -\frac{\partial^2 H}{\partial x^2}$

The characteristic polynomial of this matrix is

$$(\alpha - \lambda)(-\alpha - \lambda) - \beta\gamma = \lambda^2 - \alpha^2 - \beta\gamma$$

And the eigenvalues are $\lambda = \pm\sqrt{\alpha^2 + \beta\gamma}$ .Thus we see that there are only three possibilities for the eigenvalues:

a. If $\alpha^2 + \beta\gamma > 0$ both eigenvalues are real and have opposite signs.

b. If $\alpha^2 + \beta\gamma < 0$ both eigenvalues are imaginary with real part equal to zero.

c. If $\alpha^2 + \beta\gamma = 0$ then 0 is the only eigenvalue

In case a. we know the equilibrium point must be a saddle

5. Solution of Hamiltonian system

Solution curves of the system lie along the level curves of H . Sketching the phase portrait for Hamiltonian system is the same as sketching the level sets of the Hamiltonian function.

2.4 Dissipative system:

A function L(x,y) is called a Lyapunov function for a system of differential equation if, for every solution (x(t),y(t)) that is not an equilibrium solution of the system ,

$$\frac{d}{dt} L(x(t), y(t)) \le 0$$

For all t with strict inequality except for set of t's.

2.5 Discussion:

There are four methods to analyze the solution of nonlinear systems. By linearized we have some information near equilibrium points. By nullclines we get the trend in the whole phrase plane. In the special case we have the properties of Hamiltonian system and Dissipative system.

Chapter 3 Pendulum

We study the motions of Pendulum in the following paragraphs. The references for this part are [1, 8, 10].

3.0.1 Physic aspect:

Consider a pendulum made of a light rod of length *l* with a ball at one end of mass *m* .The position of the bob at time t is given by an angle U(t),which we choose to measure in the counterclockwise direction with 0 corresponding to the downward vertical axis (see Figure2.1)

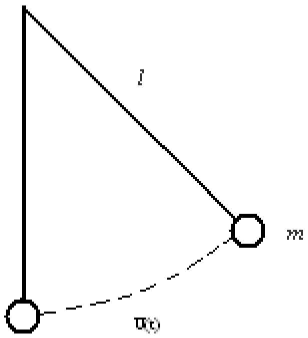


Figure3.1

A pendulum with rod length *l* and angle θ

The speed of the bob is the length of the velocity vector , which is $l\dot{U}(t)$. The component of the acceleration that points along the direction of the motion of the bob is $l\ddot{U}(t)$ We take the force due to friction to be proportional to the velocity , so this force is $-bl\dot{U}(t)$ where b>0 is a parameter that corresponds to the coefficient of damping .

Using Newton's second law, $F = ma$ we obtain the equation of motion

$$-bl\dot{U}(t) - mg\sin U(t) = ml\ddot{U}(t)$$

Which is often written as

$$\ddot{U}(t) + \frac{b}{m}\dot{U}(t) + \frac{g}{l}\sin U(t) = 0 \qquad (3.0.1)$$

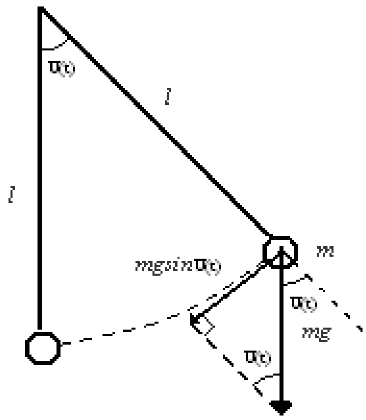


Figure 3.2

3.0.2 Mathematic aspect:

Consider the Sine-Gordon equation is

$$U_{xx}(x, y) - U_{yy}(x, y) + \sin[U(x, y)] = 0 \quad \text{(s-G)} \qquad (3.0.2)$$

Let $t = \kappa x - \omega y$

Then

$$U_x(x, y) = \dot{U}(t)\frac{\partial t}{\partial x} = \kappa \dot{U}(t)$$

$$U_y(x, y) = \dot{U}(t)\frac{\partial t}{\partial y} = -\omega \dot{U}(t)$$

$$U_{xx}(x, y) = \kappa^2 \ddot{U}(t)$$

$$U_{yy}(x, y) = \omega^2 \ddot{U}(t)$$

So we can rewrite equation (3.0.2)

$$\kappa^2 \ddot{U}(t) - \omega^2 \ddot{U}(t) + \sin[U(t)] = 0$$

Let $h^2 - w^2 = 1$

We get $\ddot{U}(t) + \sin[U(t)] = 0$ (3.0.3)

Compare with equation (3.0.1). This is a equation describing ideal pendulum

Multiple $\dot{U}(t)$

$$\dot{U}(t)\ddot{U}(t) + \dot{U}(t)\sin[U(t)] = 0 \qquad (3.0.4)$$

Integrated by t

$$\frac{1}{2}\dot{U}^2(t) - \cos U(t) = E_2 \quad \text{where} \quad E_2 \geq -1 \quad \text{is a constant}$$

Add 1

$$\frac{1}{2}\dot{U}^2(t) + [1 - \cos U(t)] = 1 + E_2 = E_1 \tag{3.0.5}$$

We can see $\frac{1}{2}\dot{U}^2(t)$ as kinetic energy and $[1 - \cos U(t)]$ as potential energy and this system has total energy E_1

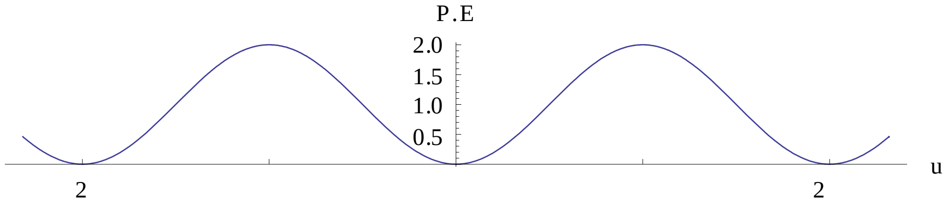


Figure 3.3

The relation between potential energy P.E and angle u

From (3.0.5)

$$\frac{1}{2}\dot{U}^2(t) + [1 - \cos U(t)] = E_1$$

$$\dot{U}(t) = \sqrt{2E_1 - 2[1 - \cos U(t)]} \tag{3.0.6}$$

$$\frac{\dot{U}(t)}{\sqrt{2E_1 - 2[1 - \cos U(t)]}} = 1$$

Integrated

We get

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2E_1 - 2[1 - \cos \zeta]}} d\zeta \tag{3.0.7}$$

3.1.0 Ideal Pendulum:

A system of pendulum with no friction is called ideal pendulum.

When no friction is present, the coefficient b vanish. We get the equation

$$\ddot{U}(t) + \frac{g}{l}\sin U(t) = 0$$

For convenience we suppose $\frac{g}{\iota} = 1$

We can rewrite this equation $\ddot{U}(t) + \sin U(t) = 0$ as first-order system in the usual manner by letting the variable v represent the angular velocity $\dot{U}(t)$ .The corresponding system is

$$\frac{dU}{dt} = V$$

$$\frac{dV}{dt} = -\sin U$$

Equilibrium points of this system are (nπ,0) for $n \in Z$

3.1.1 Apply Linearzied to analysis Ideal pendulum:

For system

$$\frac{dU}{dt} = V$$

$$\frac{dV}{dt} = -\sin U$$

1.The linearized system at the equilibrium point $(n\pi,0)$ for n is odd integer

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The Jacobian matrix of the system is $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and its eigenvalues are ± 1

There are saddle points at the equilibrium point $(n\pi,0)$ for n is odd integer

2. The linearized system at the equilibrium point $(n\pi,0)$ for n is even integer

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The Jacobian matrix of the system is $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and its eigenvalues are ± 1

There are center points at the equilibrium point $(n\pi,0)$ for n is even integer

3.1.2 Apply the Hamiltonian system to analysis Ideal pendulum:

Consider $\ddot{U}(t) + \sin[U(t)] = 0$

Let $H(U,V)=\frac{1}{2}V^2 - \cos U$

Because $\frac{dU}{dt} = V \quad = \frac{\partial H}{\partial V}$

$\frac{dV}{dt} = -\cos U \quad = -\frac{\partial H}{\partial U}$

it is Hamilonian system with Hamiltonian function $H(U,V)=\frac{1}{2}V^2 - \cos U$

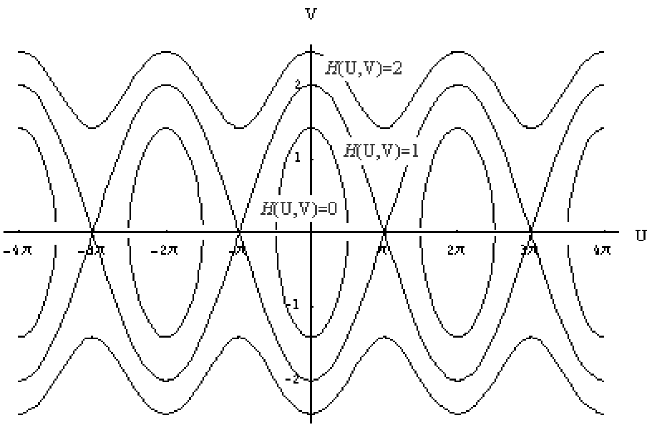


Figure 3.4

Level curve of *H*(U,V)

3.1.3 Apply the Jacobian elliptic function to solve the Ideal pendulum motion:

We want to solve (3.0.5) by Jacobian elliptic functions

Consider $t = \int_0^{U(t)} \frac{1}{\sqrt{2E_1 - 2[1 - \cos\zeta]}} d\zeta$ with $1 + E_2 = E_1$

i.e. $$t = \int_0^{U(t)} \frac{1}{\sqrt{2E_2 + 2\cos\zeta}} d\zeta \qquad (3.1.1)$$

a. If $0 < E_1 < 2$, i.e. $-1 < E_2 < 1$

then there is α such that $-\cos\alpha = E_2$

$$t = \int_0^{U(t)} \frac{1}{\sqrt{-2\cos\alpha + 2\cos\zeta}} d\zeta$$

$$= \int_0^{U(t)} \frac{1}{\sqrt{-2\left(1-2\sin^2\frac{\alpha}{2}\right) + 2\left(1-2\sin^2\frac{\zeta}{2}\right)}} d\zeta$$

$$= \frac{1}{2}\int_0^{U(t)} \frac{1}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\zeta}{2}}} d\zeta$$

Let $0 < k = \sin\frac{\alpha}{2} < 1$ $\quad z = \frac{\sin\frac{\zeta}{2}}{k}$

$$= \frac{1}{2}\int_0^{\frac{\sin\frac{u(t)}{2}}{k}} \frac{1}{\sqrt{k^2 - k^2 z^2}} \frac{2k}{\sqrt{1-k^2 z^2}} dz$$

$$= \int_0^{\frac{\sin\frac{u(t)}{2}}{k}} \frac{1}{\sqrt{(1-z^2)(1-k^2 z^2)}} dz$$

According to Jacobian function

$$sn(t, k) = \frac{1}{k}\sin\frac{U(t)}{2}$$

So $$U(t) = 2\sin^{-1}(k\ sn(t, k)) \quad \text{where} \quad k = \sin\frac{\alpha}{2} \tag{3.1.2}$$

b. If $E_1 = 2$, i.e. $E_2 = 1$

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2 + 2\cos\zeta}} d\zeta$$

$$= \frac{1}{\sqrt{2}}\int_0^{U(t)} \frac{1}{\sqrt{1+\cos\zeta}} d\zeta$$

$$= \frac{1}{\sqrt{2}}\int_0^{U(t)} \frac{1}{\sqrt{1+\cos\zeta}} d\zeta$$

$$= \frac{1}{\sqrt{2}} \int_0^{U(t)} \frac{1}{\sqrt{2 - 2\sin^2 \frac{\zeta}{2}}} d\zeta$$

$$= \frac{1}{2} \int_0^{U(t)} \frac{1}{\sqrt{1 - \sin^2 \frac{\zeta}{2}}} d\zeta$$

Let $x = \sin \frac{\zeta}{2}$

$$= \frac{1}{2} \int_0^{\sin \frac{U(t)}{2}} \frac{1}{\sqrt{1 - x^2}} \frac{2}{\sqrt{1 - x^2}} dx$$

$$= \int_0^{\sin \frac{U(t)}{2}} \frac{1}{1 - x^2} dx$$

$$= \frac{1}{2} \ln \frac{1 + \sin \frac{U(t)}{2}}{1 - \sin \frac{U(t)}{2}} \tag{3.1.3}$$

So $\sin \frac{U(t)}{2} = \tanh(t)$ i.e $U(t) = 2\sin^{-1} \tanh(t)$ (3.1.4)

c. If $E_1 > 2$ i.e. $E_2 > 1$

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2E_2 + 2\cos\zeta}} d\zeta$$

$$= \int_0^{U(t)} \frac{1}{\sqrt{2E_2 + (2 - 4\sin^2 \frac{\zeta}{2})}} d\zeta$$

$$= \int_0^{U(t)} \frac{\frac{1}{\sqrt{2E_2 + 2}}}{\sqrt{1 - \frac{4}{2E_2 + 2} \sin^2 \frac{\zeta}{2})}} d\zeta$$

Compare with $\int \frac{1}{\sqrt{1 - k^2 \sin x}} dx$

Let $k = \frac{2}{\sqrt{2E_2 + 2}}$ $x = \frac{\zeta}{2}$

$$= \int_0^{\frac{U(t)}{2}} \frac{k}{\sqrt{1-k^2\sin^2 x}}\,dx$$

Let $y = \sin x$

$$= k\int_0^{\sin\frac{U(t)}{2}} \frac{1}{\sqrt{1-k^2y^2}}\frac{1}{\sqrt{1-y^2}}\,dy$$

So $$sn(\frac{t}{k},k) = \sin\frac{U(t)}{2}$$

i.e. $$U(t) = \frac{1}{2}\sin^{-1} sn(\frac{t}{k},k) \quad \text{where} \quad k = \frac{2}{\sqrt{2E_2+2}} \qquad (3.1.5)$$

3.1.4 The graph of the Ideal pendulum motion:

1.

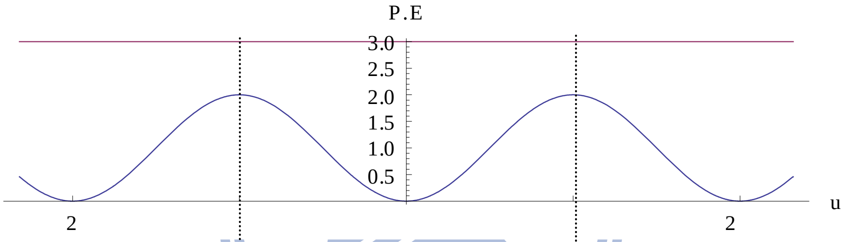


Figure 3.5

The relation between potential energy P.E and angle u with total energy $E_1 = 3$

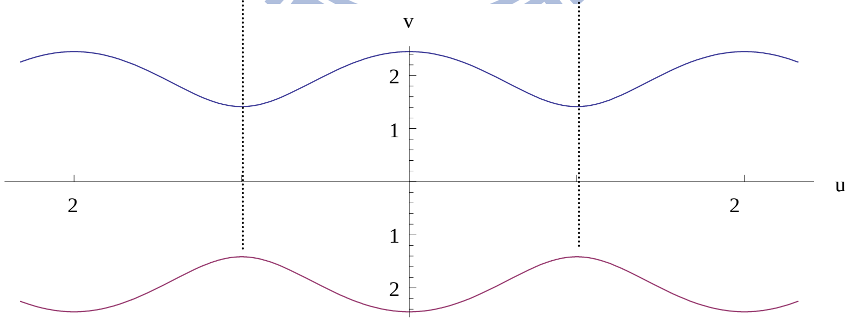


Figure 3.6

The relation between vector v and angle u with $E_1 = 3$

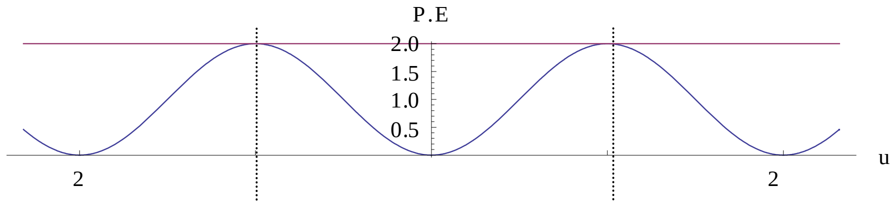


Figure 3.7

The relation between potential energy P.E and angle u with total energy $E_1 = 2$

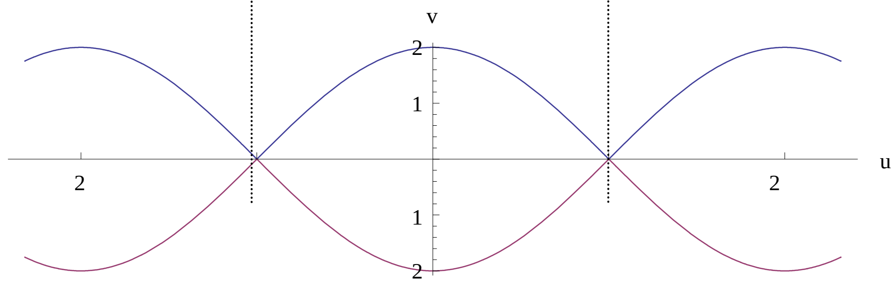


Figure 3.8

The relation between vector v and angle u with $E_1 = 2$

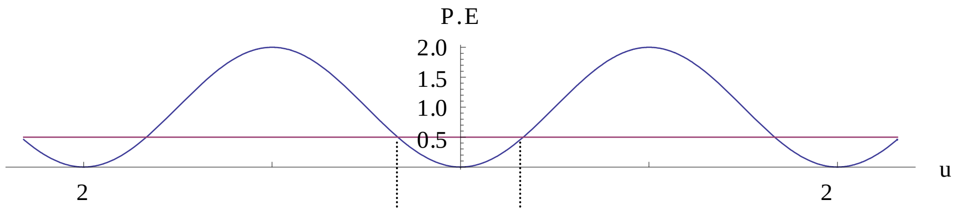


Figure 3.9

The relation between potential energy P.E and angle u with total energy $E_1 = 0.5$

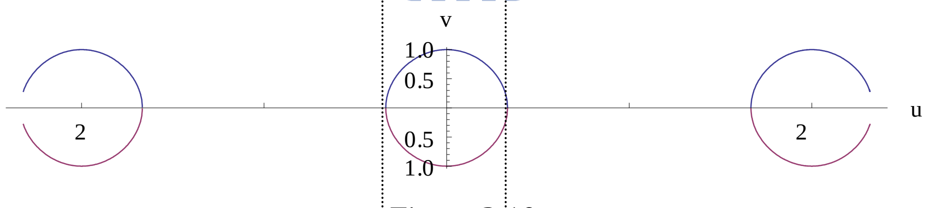


Figure 3.10

The relation between vector v and angle u with $E_1 = 0.5$

2. From (3.0.2)

$\ddot{U}(t) + \sin[U(t)] = 0$ with $t = hx + wy$ and $h^2 - w^2 = 1$

Let $h = 2$ $w = \sqrt{3}$

a. Graph of the ideal pendulum motion with $E_1 = 1 \quad (0 < E_1 < 2)$

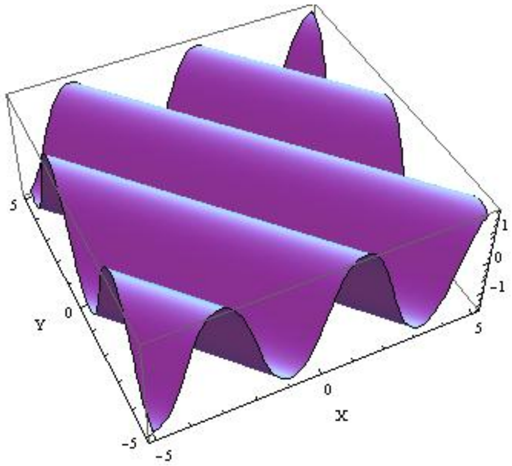


Figure 3.11

b . Graph of the ideal pendulum motion with $E_1 = 2$

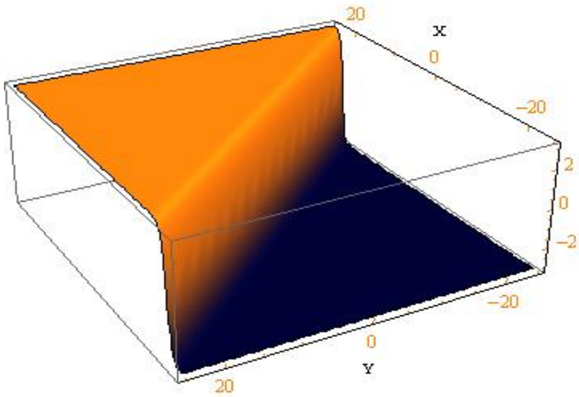


Figure 3.12

c .Graph of the ideal pendulum motion with $E_1 = 3 \ (2 < E_1)$

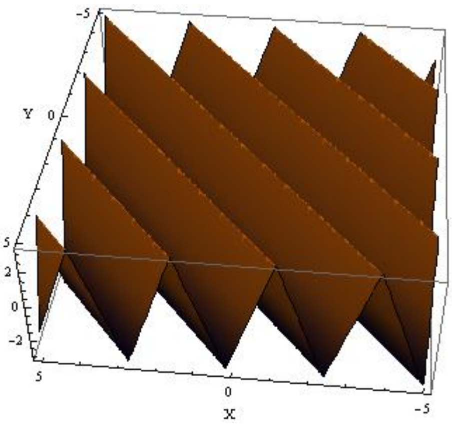


Figure 3.13

3.2.0 Pendulum motion with friction:

Recall that the second-order equation governing the motion of the pendulum is

$$\ddot{U}(t) + \frac{b}{m}\dot{U}(t) + \frac{g}{l}\sin[U(t)] = 0$$

Where b is the coefficient of damping m is the mass of the pendulum bob, g is the acceleration of gravity, and l is the length of the pendulum arm.

For convenies we let $\frac{b}{m} = B$ and $\frac{g}{l} = 1$.And rewrite this equation as first-order system in the usual manner by letting the variable v represent the angular velocity $\dot{U}(\theta)$.

The corresponding system is

$$\frac{dU}{dt} = V$$

$$\frac{dV}{dt} = -BV - \sin U$$

3.2.1 Apply Linearzied to analysis pendulum with friction:

For system

$$\frac{dU}{dt} = V$$

$-BV- \sin U$

1. The linearized system at the equilibrium point $(n\pi, 0)$ for n is odd integer

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -B \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The Jacobian matrix of the system is $J = \begin{pmatrix} 0 & 1 \\ 1 & -B \end{pmatrix}$ and its eigenvalues are

$$\lambda = \frac{-B \pm \sqrt{B^2 + 4}}{2}$$

There are saddle points at the equilibrium point $(n\pi, 0)$ for n is odd integer

2. The linearized system at the equilibrium point $(n\pi, 0)$ for n is even integer

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -B \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The Jacobian matrix of the system is $J = \begin{pmatrix} 0 & 1 \\ -1 & -B \end{pmatrix}$ and its eigenvalues are ± 1

There are center points at the equilibrium point $(n\pi, 0)$ for n is even integer.

3.2.2 Apply the nullclines to analysis pendulum with friction:

For system

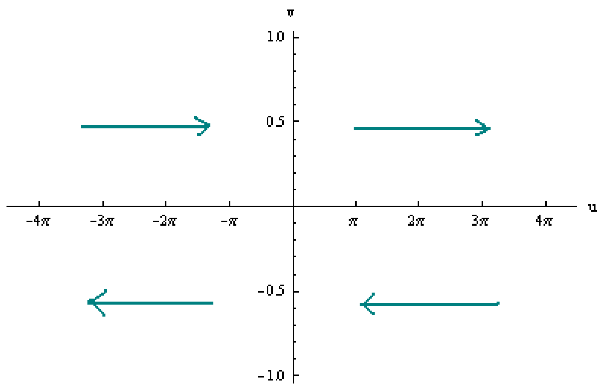
$$\frac{dU}{dt} = V$$

$$\frac{dV}{dt} = -BV - \sin U$$

U-nullcline is $\{(U, V); V = 0\}$

If V>0 then $\frac{dU}{dt} > 0$ This means vector filed “right”

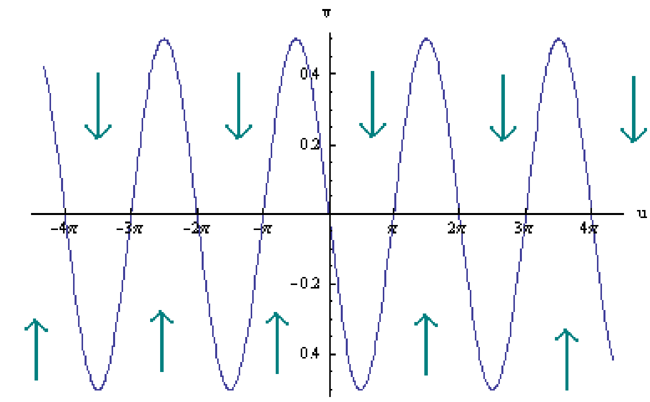
If V<0 then $\frac{dU}{dt} < 0$ This means vector filed “left”



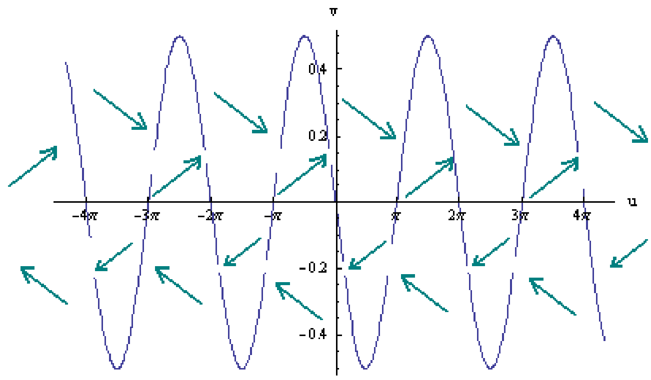
V-nullcline is $\{(U, V); BV + \sin U = 0\}$

If $BV + \sin U < 0$ then $\frac{dV}{dt} > 0$ This means vector filed "up"

If $BV + \sin U > 0$ then $\frac{dV}{dt} < 0$ This means vector filed "down"

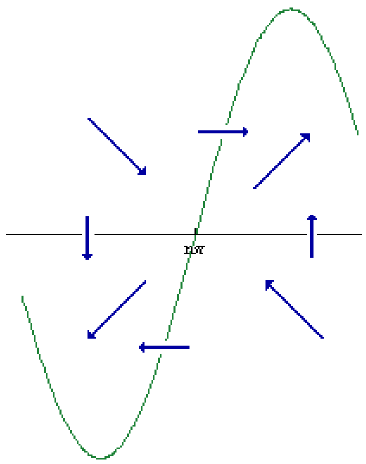


Combine this two graphs

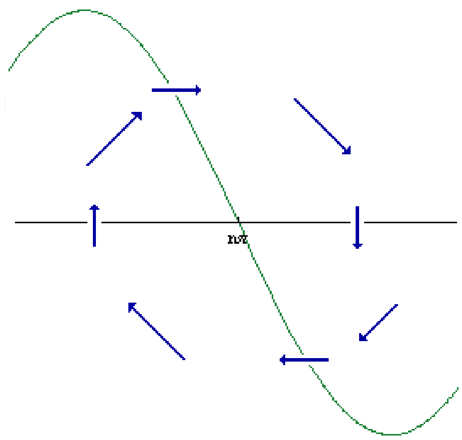


It obvious

1. There are saddle points at the equilibrium point $(n\pi,0)$ for n is odd integer.



2. There are center or spiral sink or spiral source points at the equilibrium point $(n\pi,0)$ for n is even integer.



Chapter 4 Physical Applications of Elliptic functions

In the paragraphs below, we study five physical models' differential systems. An ideal whirling chain, and Duffing's Equation. The other three describe the motions of orbit planets. These parts mainly follow the reference [4].

4.1 Whirling Chain:

Consider a uniform length *l* of rope or chain, whose ends are fixed at point ***0*** and ***A*** and which is set rotating about the axis ***OA*** with constant angular velocity ω .

For ideal case
1. Gravity will be neglected
2. It will be assumed that the chain always lies in a plane through the axis of rotation.

We shall take ***O*** to be the origin of axes ***Ox***, ***Oy***, the x-axis lying along OA, and y-axis lying in the plane of the chain at same instant t (Fig.). Consider the motion of an element $ds = PQ$ of the chain, where P and Q have coordinates (x, y), (x+dx, y+dy) respectively.

The forces acting on this element are tensions T and T+dT at its ends P and Q, and their lines of action are the tangents to the chain at these points; let these tangents make angles ψ, $\psi + d\psi$ respectively with the x-axis.

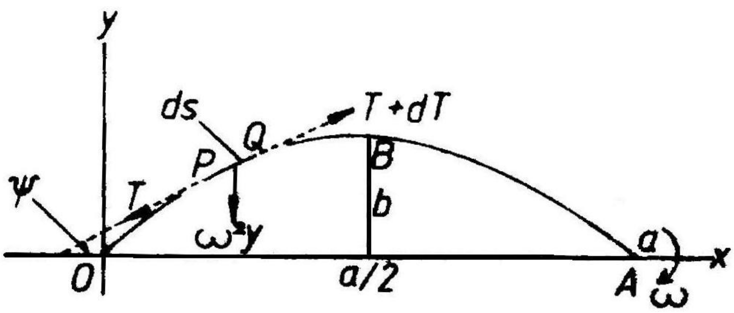


Figure 4.1

Resolving the forces tangentially and normally, we obtain components (dT, Tdψ) respectively. The element moves around a circle of radius y with angular velocity ω and its acceleration is accordingly $\omega^2 y$ directed in the negative sense parallel to the y-axis.

We can now write down the tangential and normal components of the equation of motion thus;

$$dT = -\sigma \, ds \, \omega^2 y \sin(\psi), \qquad Td\psi = -\sigma \, ds \, \omega^2 y \cos(\psi) \qquad (4.1.1)$$

Where σ is the mass per unit length of the chain.

Dividing these equations, we find that

$$dT/T = \tan(\psi)d\psi \quad (4.1.2)$$

Which integrates to give the equation

$$T = T_0 \sec(\psi)\,, \quad (4.1.3)$$

T_0 being the tension at the point ***B*** where $\psi = 0$.

Substituting for T in the second equation (4.1.1), we now deduce that

$$\lambda d\psi/ds = -y\cos^2(\psi) \quad (4.1.4)$$

Where

$$\lambda = T_0/\sigma\omega^2$$

But $dy/ds = \sin(\psi)$ and it therefore follows that

$$\lambda \tan(\psi)\sec(\psi)d\psi = -ydy\ . \quad (4.1.5)$$

This equation integrates to

$$\lambda(\sec(\psi)-1) = \frac{1}{2}(b^2 - y^2)\,, \quad (4.1.6)$$

Where $y = b, \psi = 0$ at ***B***.

Thus,

$$dy/dx = \tan(\psi) = \sqrt{(\sec^2(\psi)-1)} = \frac{1}{2\lambda}\sqrt{\{(b^2-y^2)(b^2+4\lambda-y^2)\}} \quad (4.1.7)$$

and, after integration from , this leads to the equation

$$x = 2\lambda\int_0^y \frac{dy}{\sqrt{\{(b^2-y^2)(c^2-y^2)\}}}\,,$$

where

$$c^2 = b^2 + 4\lambda \quad (4.1.8)$$

Reference to the standard form (1.3.79) now shows that

$$x = \frac{2\lambda}{c}\,\mathrm{sn}^{-1}(y/b)\ ,$$

The modulus being given by

$$k^2 = b^2/c^2 = (1+4\lambda/b^2)^{-1}\ . \quad (4.1.9)$$

We conclude the equation of chain is

$$y = b\,\mathrm{sn}\,(cx/2\lambda)\ .$$

Supposing the end ***A*** to lie at the point $x = a$ on the x-axis, we must have $y = 0$ at $x = a$. Clearly, therefore, it is necessary that

$$ac/2\lambda = 2K \quad (4.1.10)$$

and the equation of the chain can be written

$$y = b\,\mathrm{sn}\,(2Kx/a)$$

By eliminating and between equations (4.1.8), (4.1.9 and (4.1.10), we arrive at the equation

$$\frac{ak}{b(1-k^2)} = K \tag{4.1.11}$$

Since K is a known function of k, this equation determine k and K when a, b are given. λ can then be found from equation (4.1.9) .

For example, if k=0.5, then K=1.6858 and, thus, a/b=2.53 and $\lambda = \frac{3}{4}b^2$.

Instead of b being specified, the length *l* of the chain may be given. This can be related to the other parameters, thus:

$$\begin{aligned} l &= \int_0^a \sqrt{[1 + (dy/dx)^2]}\,dx \\ &= \int_0^a \sqrt{[1 + (2bK/a)^2 \mathrm{cn}^2(2Kx/a)\mathrm{dn}^2(2Kx/a)]}\,dx \\ &= \int_0^a \sqrt{[1 + (4k^2/k')\mathrm{cn}^2(2Kx/a)\mathrm{dn}^2(2Kx/a)]}\,dx \\ &= \int_0^a \left[\left(2/k'^2\right)\mathrm{dn}^2(2Kx/a) - 1\right]dx \\ &= \frac{a}{k'^2 K}\int_0^{2K} \mathrm{dn}^2(u)\,du - a \\ &= \frac{2aE}{k'^2 K} - a \end{aligned} \tag{4.1.12}$$

Where $E = \int_0^K \mathrm{dn}^2(u)du$ .With k=0.5, K=1.6858 as before, we read from the table E=1.4675 and hence, *l*=1.321a.

Remark11:

E (u, k) is a function defined by

$$E(u,\ k) = \int_0^u \mathrm{dn}^2(v,\ k)dv$$

And $E = E(K,\ k) = \int_0^K \mathrm{dn}^2(v,\ k)dv$.

It can be saw a function of k.

4.2 Duffing's Equation:

This is the equation governing the oscillations of mass attached to the end of a spring whose tension (or compression) T is related to its extension x by an equation of the form

$$T = \alpha x + \beta x^3 \tag{4.2.1}$$

α is always positive.

1. If $\beta = 0$, the spring obeys Hooke's law and the oscillations are simple harmonic.
2. If $\beta > 0$, the tension increases with the extension more rapidly than required by Hooke's law and the spring is said to be **hard**.
3. If $\beta < 0$, the tension increases less rapidly than required by the law and the spring is said to be **soft**.

By a suitable time choice of the unit of time, the equation of motion of the mass can be put into the form (***Remark12***)

$$\ddot{x} + x + \varepsilon x^3 = 0 \tag{4.2.2}$$

which is the canonical form of **Duffing's Equation.**

We want to solve this system.

1. The case of a hard spring, for which $\varepsilon > 0$.

Suppose that initially, $t = 0,\ x = a,\ \dot{x} = 0$.

get the system

(4.2.3)
$$\ddot{x} + x + \varepsilon x^3 = 0$$
$$t = 0,\ x = a,\ \dot{x} = 0$$

Since $\ddot{x} = \frac{d}{dx}(\frac{1}{2}\dot{x}^2)$, we can integrate with respect to x to give

$$\tfrac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}\varepsilon x^4 = \frac{1}{2}a^2 + \frac{1}{4}\varepsilon a^4 \tag{4.2.4a}$$

or

$$\dot{x}^2 = (a^2 - x^2)(1 + \frac{1}{2}\varepsilon a^2 + \frac{1}{2}\varepsilon x^2) \tag{4.2.4b}$$

Integrating to obtain t, we find

$$t = \sqrt{\frac{2}{\varepsilon}}\int_x^a \frac{dx}{\sqrt{(a^2 - x^2)(\frac{2}{\varepsilon} + a^2 + x^2)}} \tag{4.2.5}$$

$$= \frac{1}{\sqrt{1 + \varepsilon a^2}}\,\mathrm{cn}^{-1}[\frac{x}{a}, \sqrt{(\frac{\varepsilon a^2}{2 + 2a^2})}], \tag{4.2.6}$$

having referred to the standard form (1.3.80). Inversion now shows that

$$x = a\ \mathrm{cn}\{\sqrt{1 + \varepsilon a^2}\,t\} \tag{4.2.7}$$

with modulus given by

$$k^2 = \frac{\varepsilon a^2}{2 + 2\varepsilon a^2} \tag{4.2.8}$$

The period of oscillation determined by (4.2.7) is given by

$$T = \frac{4K}{\sqrt{(1 + \varepsilon a^2)}} \tag{4.2.9}$$

If ε is very small, we get $k^2 = \frac{1}{2}\varepsilon a^2$. This imply $K = \frac{1}{2}\pi(1 + \frac{1}{8}\varepsilon a^2)$.

Hence, to $\boldsymbol{O(\varepsilon)}$

$$T = 2\pi(1 - \frac{3}{8}\varepsilon a^2), \tag{4.2.10}$$

indicating that, as the amplitude of the oscillation increases, the period decreases and the frequency therefore increases.

2. The case of a soft spring, for which $\varepsilon < 0$.

Let $\varepsilon = -\eta$, $\eta > 0$. Since $x - \eta x^3$ has a maximum at $x = 1/\sqrt{(3\eta)}$

It will be convenient to measure t from an instant when the mass is at the center of oscillation (x=0) and x is increasing. Thus, x=0 at t=0, and the equation for the time is

$$t = \sqrt{\frac{2}{\eta}} \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(\frac{2}{\eta} - a^2 - x^2)}}$$

$$= \sqrt{(\frac{2}{2 - \eta a^2})} sn^{-1}[\frac{x}{a}, \sqrt{(\frac{\eta a^2}{2 - \eta a^2})}],$$

after reference to the standard form (1.3.79) , (to apply this result, we must assume $2/\eta - a^2 > a^2$,i.e., $a < 1/\sqrt{(3\eta)}$ which is guaranteed by $a < 1/\sqrt{(3\eta)}$). Inverting the last equation, we obtain

$$x = a \cdot sn\{\sqrt{(1 - \frac{1}{2}\eta a^2)} t\}, \tag{4.2.11}$$

where the modulus is determined by

$$k^2 = \frac{\eta a^2}{2 - \eta a^2} \tag{4.2.12}$$

Thus, the period of oscillation is given by

$$T = \frac{4K}{\sqrt{(1 - \frac{1}{2}\eta a^2)}} \tag{4.2.13}$$

and, for small η , this reduces to

$$T = 2\pi(1 + \frac{1}{8}\eta a^2) \quad (4.2.14)$$

to $O(\eta)$, in agreement with (4.2.10). For a soft spring, therefore, the frequency of oscillation decreases as the amplitude increases.

Remark12:

By Newton's law

$$T = -ma$$

Where m is the mass and a is the acceleration of the mass
So we get the equation

$$\alpha x + \beta x^3 = T = -ma = -m\ddot{x}$$

Divide m

$$\ddot{x} + \frac{\alpha}{m}x + \frac{\beta}{m}x^3 = 0$$

Let $m = \alpha$ and $\varepsilon = \frac{\beta}{m} = \frac{\beta}{\alpha}$

$$\ddot{x} + x + \varepsilon x^3 = 0$$

We note that $\varepsilon > 0$ if $\beta > 0$ the spring is **hard** and $\varepsilon < 0$ if $\beta < 0$ the spring is **soft**.

4.3 Orbit motion:

4.3.1 Orbits under aμ /r^4 Law of Attraction:

Suppose a particle of unit mass is attracted towards a center ***O*** by a force μ/r^4 , r and θ being its polar coordinates at time t in the plane of motion. Then, since the particle's energy E and angular momentum h about ***O*** will be conserved, we can write down the equations of motion (***Remark13***)

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{3r^3} = E\ , \qquad r^2\dot{\theta} = h \qquad (4.3.1)$$

Putting $r = 1/u$ and eliminating t between these equations, we arrive at the equation

$$\alpha\left(\frac{du}{d\theta}\right)^2 = u^3 - \alpha u^2 + \beta = f(u)\ , \qquad (4.3.2)$$

where $\alpha = 3h^2/2\mu \quad \beta = 3E/\mu$

This equation determines the polar equation of the orbit. Clearly, $\alpha > 0$ (we ignore the case of rectilinear motion), but β may take any real value. We shall always assume the sense of the motion to be such that θ increases (i.e., h>0)

Before solve this equation we should verify that there are five cases to consider:

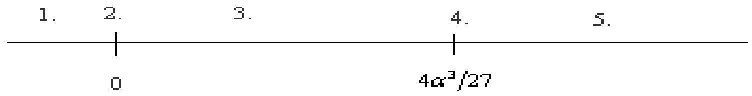


Figure 4.2

1. If , $\beta < 0$,then $f(u)$ has one real zero at greater than α and two complex zeros whose real parts are negative (since the sum of the zeros is α)

For $\alpha = 2 \quad \beta = -1$

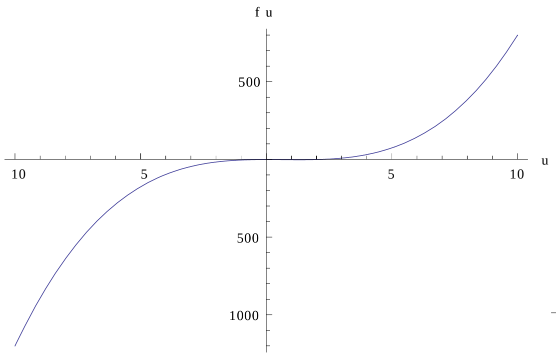


Figure 4.3a

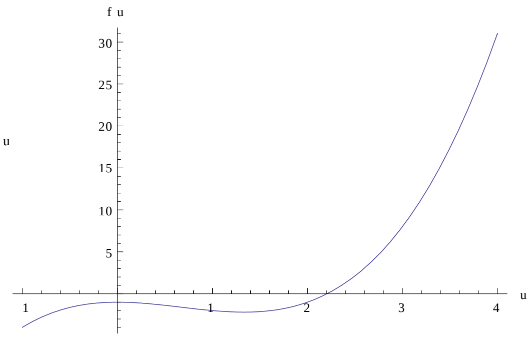


Figure 4.3b

For $\alpha = 2 \quad \beta = -1$

2. If , $\beta = 0$, $f(u)$ has a double zero at u=0 and a simple zero at $u = \alpha$

For $\alpha = 2 \quad \beta = 0$

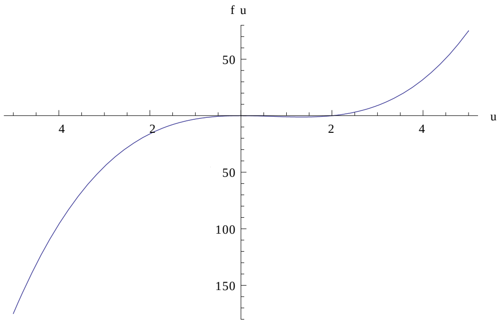


Figure 4.4a

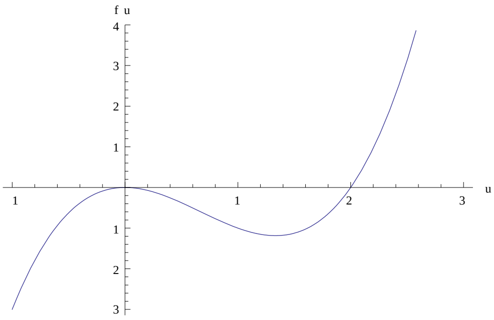


Figure 4.4b

3. If, $0 < \beta < 4\alpha^3/27$, $f(u)$ has three real zeros u_1, u_2, u_3, satisfying $u_1 < 0 < u_2 < 2\alpha/3 < u_3 < \alpha$.

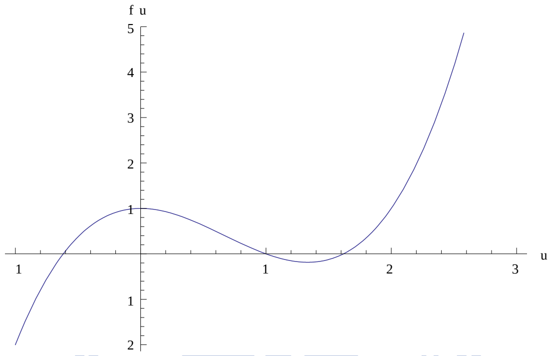


Figure 4.5

For $\alpha = 2 \quad \beta = 1$

4. If $\beta = 4\alpha^3/27$, $f(u)$ has a pair of coincident zeros at $u = 2\alpha/3$ and a simple zero at $u = -\alpha/3$.

For $\alpha = 2 \quad \beta = 32/27$

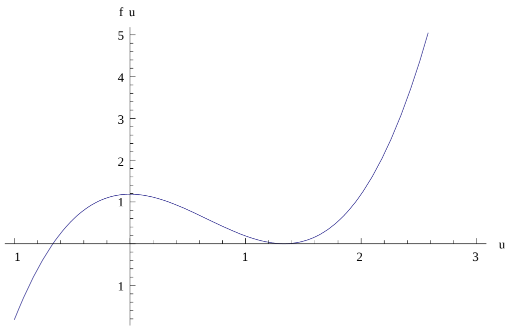


Figure 4.6a

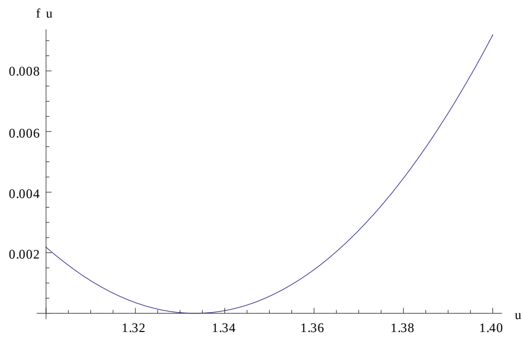


Figure 4.6b

5. If $\beta > 4\alpha^3/27$, f (u) has a real zero with negative u and two complex zeros with positive real parts.

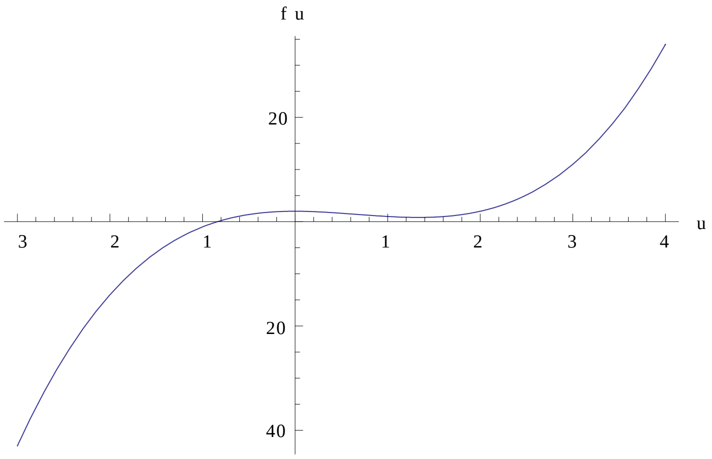


Figure 4.7

For $\alpha = 2 \quad \beta = 2$

Since $\alpha(du/d\theta)^2 > 0$, by consideration of the sign of f(u) over the range of all positives of u, it is possible to establish the character of each of the possible orbits in these cases without further integration.

<u>Case 1.</u>

For $\beta < 0$, $f(u)$ has one real zero at greater than α and two complex zeros whose real parts are negative.

We get $f(u) = (u - a)\{(u + b)^2 + c^2\}$

a, b, c being all positive and $a > \alpha$. Clearly, we need $u \geq a$ to make $f(u)$ positive.

Apply some method

$$f(u) = S_1 S_2 \tag{4.3.3a}$$

$$\text{where} \quad S_1 = \frac{1}{2}(p + q)^{-1}[(u + p)^2 - (u - q)^2] \tag{4.3.3b}$$

$$S_2 = (p + q)^{-1}[(q + b)(u + p)^2 + (p - b)(u - q)^2] \tag{4.3.3c}$$

p and q are positive numbers given by

$$p = \sqrt{\{(a + b)^2 + c^2\}} - a \qquad q = \sqrt{\{(a + b)^2 + c^2\}} + a \tag{4.3.3d}$$

Then, integrating equation (4.3.2) we find

$$\alpha^{-1/2}\theta = \sqrt{2}(p + q)\int \frac{du}{\sqrt{[(u + p)^2 - (u - q)^2][(q + b)(u + p)^2 + (p - b)(u - q)^2]}}, \tag{4.3.4}$$

We now make the substitution

$$x = \frac{u - q}{u + p} \tag{4.3.5}$$

where x increases monotonically for increasing u. This give

$$\alpha^{-1/2}\theta = \sqrt{\frac{2}{p-b}}\int \frac{dx}{\sqrt{(1-x^2)(d+x^2)}}, \tag{4.3.6}$$

where

$$d^2 = \frac{q+b}{p-b}.$$

We have now arrived at a standard from and can make use of the result (1.3.80) to show that

$$\alpha^{-1/2}\theta = -\sqrt{\frac{2}{p-b}}\,\mathrm{cn}^{-1}x, \tag{4.3.7}$$

the modulus being given by

$$k^2 = \frac{p-b}{p+q}. \tag{4.3.8}$$

thus,

$$x = \mathrm{cn}(\gamma\theta), \tag{4.3.9}$$

where

$$\gamma = \sqrt{\{(p+q)/2\alpha\}}. \tag{4.3.10}$$

The polar equation of the orbit now follows in the form

$$r = \frac{1-\mathrm{cn}(\gamma\theta)}{q+p\,\mathrm{cn}(\gamma\theta)} \tag{4.3.11}$$

where $p < q$.

We deduce that, as θ increases form 0, the trajectory spirals outward from the center, the mass being at its maximum distance $2/(q-p) = 1/a$ from ***O*** when $\theta = 2K/\gamma$.

Thereafter, the orbit spirals inward and reaches the center again when $\theta = 4K/\gamma$. As before, negative values of θ yield the mirror image trajectory, which is identical with the original.

Case 2.

If $\beta = 0$, we calculate that

$$\alpha^{-1/2}\theta = \int \frac{du}{u\sqrt{(u-\alpha)}} = \alpha^{-1/2}\cos^{-1}\left(\frac{2\alpha}{u}-1\right). \tag{4.3.12}$$

(Use the substitution $u = 1/v$.)We have ignored the constant of integration, since this can always be eliminated by suitable choice of the line $\theta = 0$. The polar equation of the orbit is now found to be

$$r = \frac{1}{2\alpha}(1+\cos\theta), \tag{4.3.13}$$

Which is a cardioid. Thus, the particle recedes to a maximum distance $1/2\alpha$ from the pole and then falls into the center of attraction.

Case 3.

If, $0 < \beta < 4\alpha^3/27$, $f(u)$ has three real zeros u_1, u_2, u_3, satisfying

$u_1 < 0 < u_2 < 2\alpha/3 < u_3 < \alpha$.

We get $f(u) = (u - u_1)(u - u_2)(u - u_3)$, where $u_1 < 0 < u_2 < u_3$. Hence, either $0 \le u \le u_2$ or $u \ge u_3$ to make $f(u) > 0$, and there are two types of orbit.

Integrating equation (4.3.2). We get

$$\alpha^{-1/2}\theta = \int \frac{du}{\sqrt{\{(u - u_1)(u - u_2)(u - u_3)\}}} . \quad (4.3.14)$$

Changing the variable by the transformation

$$u = u_1 + 1/x^2 \qquad (x > 0) ,$$

we reduce the integral to standard form, thus

$$\alpha^{-1/2}\theta = -\frac{2}{\sqrt{\{(u_2 - u_1)(u_2 - u_1)\}}} \int \frac{dx}{\sqrt{\{(a^2 - x^2)(b^2 - x^2)\}}} , \quad (4.3.15)$$

where

$$a^2 = 1/(u_2 - u_1) , \qquad b^2 = 1/(u_3 - u_1) .$$

Clearly $a > b$.

a. If $u \ge u_3$, then $x \le b$ and the result (1.3.79) may be applied to give

$$\alpha^{-1/2}\theta = -\frac{2}{\sqrt{(u_3 - u_1)}} \mathrm{sn}^{-1}\{\sqrt{(u_3 - u_1)}x\} \quad (4.3.16)$$

with modulus k, where

$$k^2 = \frac{u_2 - u_1}{u_3 - u_1} . \quad (4.3.17)$$

Then, the orbit is found to have equation

$$\frac{1}{r} = u = u_1 + (u_3 - u_1)\mathrm{ns}^2(\gamma\theta) \quad (4.3.18)$$

where

$$\gamma = \frac{1}{2}\sqrt{\{(u_3 - u_1)/\alpha\}} \quad (4.3.19)$$

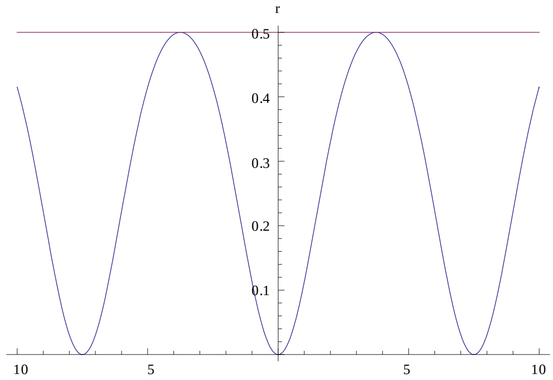


Figure 4.8

$u_1 = -1,\ u_2 = 1,\ u_3 = 2$

This implies, as θ increases from 0, the trajectory spirals outward from the center, achieving maximum distance $1/u_3$, when $\theta = K/\gamma$; it then spirals back into the pole, arriving there when $\theta = 2K/\gamma$.

b. If $0 \le u \le u_2$ then $x \ge a$ and the standard form (1.3.84) is used to yield

$$\alpha^{-1/2}\theta = -\frac{2}{\sqrt{(u_3 - u_1)}}\,\mathrm{ns}^{-1}\{\sqrt{(u_2 - u_1)}\,x\} \tag{4.3.20}$$

whence

$$\frac{1}{r} = u = u_1 + (u_2 - u_1)\,\mathrm{sn}^2(\gamma\theta) \tag{4.3.21}$$

The constant k and γ take the same values as before.

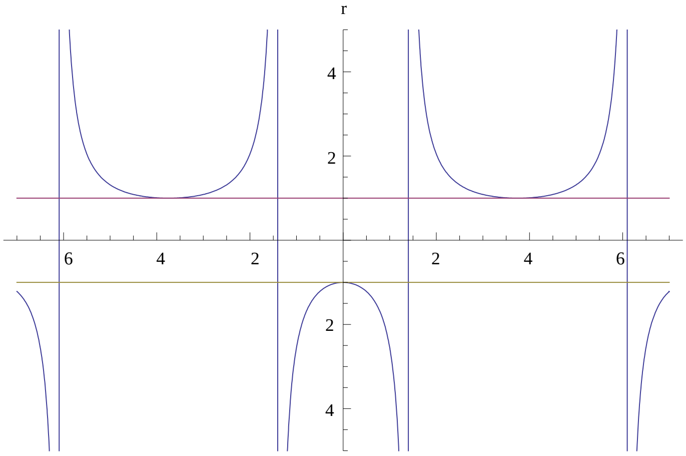


Figure 4.9

$u_1 = -1,\ u_2 = 1,\ u_3 = 2$

For this orbit equation (4.3.21), $\gamma\theta$ must equal or exceed $\omega = \mathrm{sn}^{-1}[\sqrt{\{-u_1/(u_2 - u_1)\}}]$ to give positive values for u and r. When θ has this limit value ω, r is infinite and further increase in θ causes r to decrease to a minimum of $1/u_2$ when $\theta = K/\gamma$. If θ is increased again, r approaches infinity as $\theta \to (2K - \omega)/\gamma$. Thus, the trajectory first approaches the center of attraction from infinity and later recedes again to an infinite distance

Case 4.

If $\beta = 4\alpha^3/27$, then

$$\alpha^{-1/2}\theta = \int \frac{du}{(u - 2\alpha/3)\sqrt{(u + \alpha/3)}} = -\alpha^{-1/2}\cosh^{-1}\left|\frac{u + 4\alpha/3}{u - 2\alpha/3}\right|. \qquad (4.3.25)$$

(Put $u - 2\alpha/3 = 1/v$ if $u > 2\alpha/3$ and $2\alpha/3 - u = 1/v$ if $u < 2\alpha/3$) The equation of the orbit is accordingly

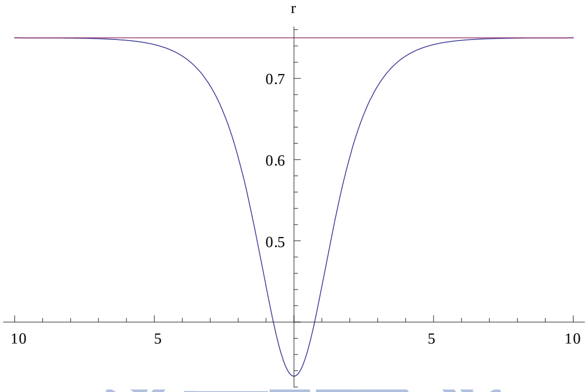


Figure 4.10

$$r = \frac{3}{2\alpha}\frac{\cosh\theta - 1}{\cosh\theta + 2}, \text{ if } u > 2\alpha/3, \qquad (4.3.26a)$$

$$= \frac{3}{2\alpha}\frac{\cosh\theta + 1}{\cosh\theta - 2}, \text{ if } u < 2\alpha/3, \qquad (4.3.26b)$$

In (4.3.26a)

For positive values of θ , the orbit spirals outward from the center of attraction, approaching the circle $r = 3/2\alpha$ asymptotically from inside circle.

For $\alpha = 2$ $\beta = 32/27$

In (4.3.26b)

For $\theta > \cosh^{-1} 2$, the orbit spirals inward from infinity, approaching the circler $r = 3/2\alpha$ asymptotically from outside the circle.

Reflecting these orbits in line $\theta = 0$, we obtain the orbits for negative values of θ , which represent similar trajectories, traversed in the opposite sense, i.e., diverging inward and outward from he circular motion.

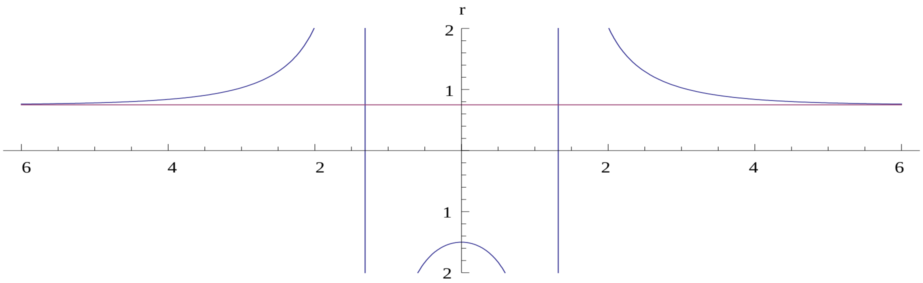


Figure 4.11

Case 5.

In case,

$$f(u) = (u + a)\{(u - b)^2 + c^2\} \qquad (4.3.27)$$

Where a, b, c are all positive; all positive values of u are now admissible. The analysis proceeds as for **Case1.** , the signs of a and b being reversed. Thus

$$p = \sqrt{\{(a + b)^2 + c^2\}} + a \qquad q = \sqrt{\{(a + b)^2 + c^2\}} - a \qquad (4.3.28)$$

and $p > q$.

The orbital equation is as given at (4.3.11)

$$r = \frac{1 - cn\,(\gamma\theta)}{q + p\,cn\,(\gamma\theta)} \qquad (4.3.29)$$

and, as θ increases from 0, r increases from 0 and the orbit spirals outward from the center. However, the equation $cn(\gamma\theta) = -q/p$ now has a real root θ for which r becomes infinite and the trajectory does not return to the center of attraction. Negative values of θ provide a mirror image orbit along which the mass can fall into the center of attraction from an infinite distance.

Remark13:

Total energy E = kinetic energy + potential energy

$$= K + U$$

$$= \frac{1}{2} m\, v^2 + \int \vec{F}\, ds$$

$$= \frac{1}{2} 1 (\sqrt{\dot{r}^2 + r^2\dot{\theta}^2})^2 + \int_\infty^r \frac{\mu}{r^4}\, dr$$

$$= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{3r^3}$$

Angular momentum $h = \vec{r} \times m\,\vec{v} = (r, \theta) \times 1\,(\dot{r}, r\dot{\theta})$

The value of h is

$|h| = r^2\theta$

4.3.2 Orbits under a μ/r^5 Law of Attraction:

If μ/r^5 is the attraction per unit mass, the particle's potential energy in the field is $-\mu/4r^4$ and the equations of energy and angular momentum are

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{4r^4} = E\,, \qquad r^2\dot{\theta} = h\,. \tag{4.3.30}$$

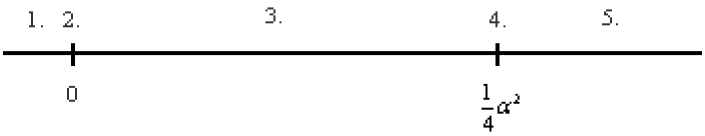
These equations lead to the equation

$$\alpha\left(\frac{du}{d\theta}\right)^2 = u^4 - \alpha u^2 + \beta = g(u^2) = g(v) \tag{4.3.31}$$

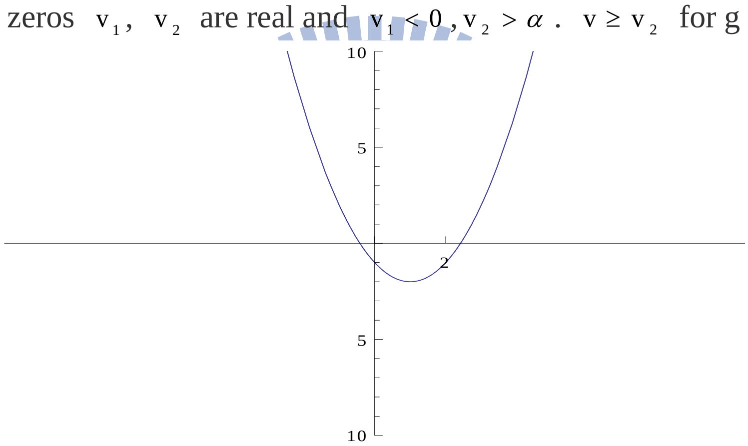
determining the orbits, where

$$\alpha = 2h^2/\mu > 0\,, \qquad \beta = 4E/\mu\,, \qquad v = u^2\,.$$

g(v) $(v = u^2)$ is a quadratic and its zeros distinguish five cases

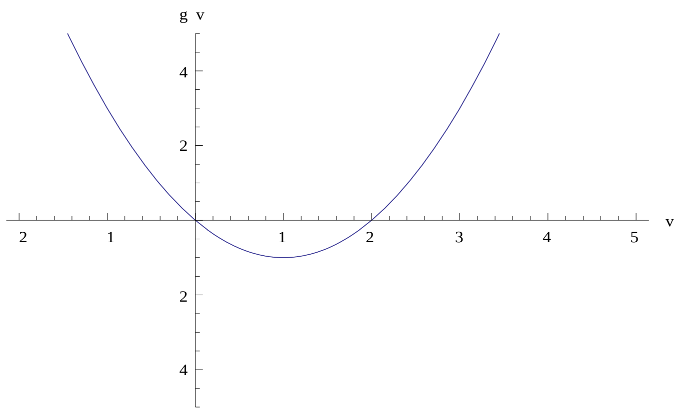


1. $\beta < 0$, both zeros v_1, v_2 are real and $v_1 < 0, v_2 > \alpha$. $v \geq v_2$ for g to be positive.



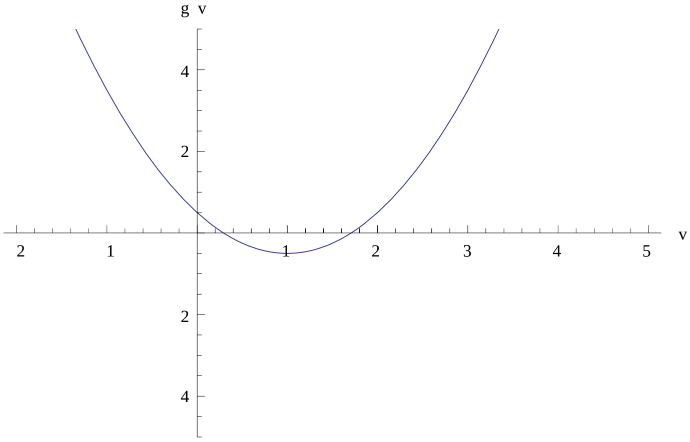
For $\alpha = 2 \quad \beta = -1$

2. $\beta = 0$, zeros are 0 and α. $v \geq \alpha$ on the orbit.



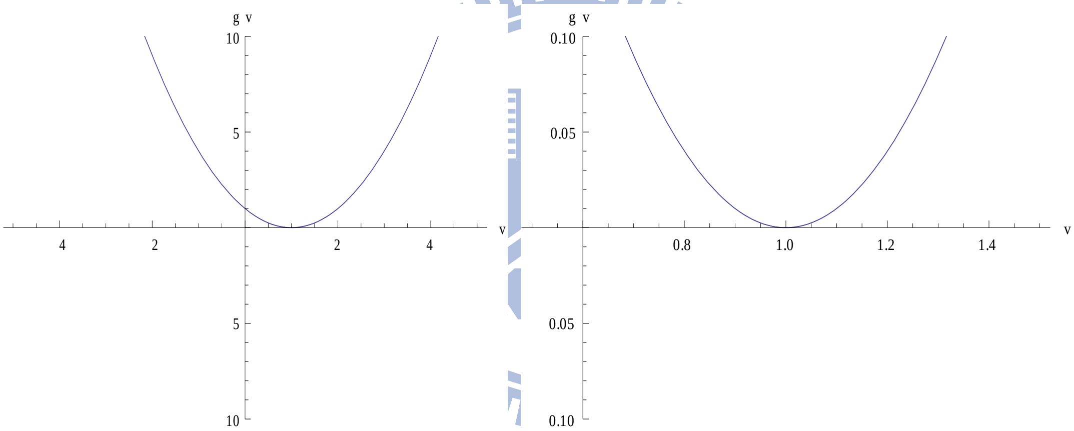
For $\alpha = 2 \quad \beta = 0$

3. $0 < \beta < \frac{1}{4}\alpha^2$, both zeros are real and satisfy $0 < v_1 < v_2 < \alpha$. $v \le v_1$ or $v \ge v_2$ on the orbit.



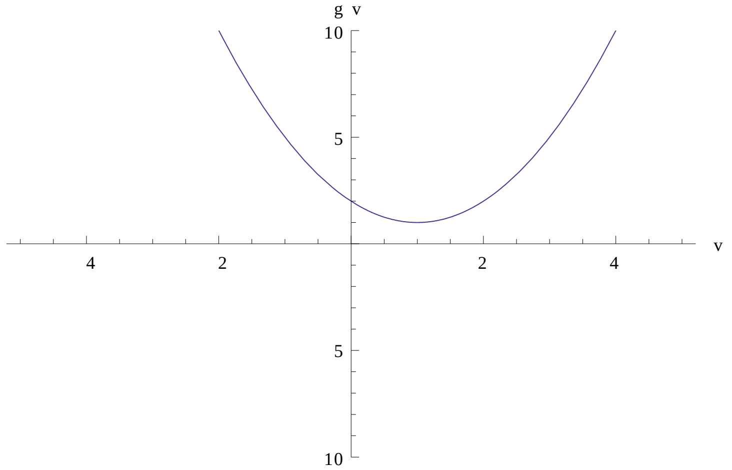
For $\alpha = 2$ $\beta = \frac{1}{2}$

4. $\beta = \frac{1}{4}\alpha^2$, coincident zeros at $v = \frac{1}{2}\alpha$; all positive values of v are admissible.



For $\alpha = 2$ $\beta = 1$

5. $\beta > \frac{1}{4}\alpha^2$, zeros are complex with positive real parts; all positive values of v are admissible.



For $\alpha = 2 \quad \beta = 2$

Suppose θ increases with t (h>0)

Case 1.

$$\alpha\left(\frac{du}{d\theta}\right)^2 = (u^2 - v_1)(u^2 - v_2), \tag{4.3.32}$$

where

$$v_1 = \frac{1}{2}\alpha - \sqrt{\left(\frac{1}{4}\alpha^2 - \beta\right)} < 0, \qquad v_2 = \frac{1}{2}\alpha + \sqrt{\left(\frac{1}{4}\alpha^2 - \beta\right)} > \alpha .$$

We must have $u \geq \sqrt{v_2}$. Integration leads to the orbital equation

$$\alpha^{-1/2}\theta = (v_2 - v_1)^{-1/2}\,\mathrm{nc}^{-1}(u/\sqrt{v_2}), \tag{4.3.33}$$

using the standard integral (1.3.87), the modulus being given by

$$k^2 = -\frac{v_1}{v_2 - v_1}. \tag{4.3.34}$$

We deduce that

$$r = \frac{1}{\sqrt{v_2}}\,\mathrm{cn}\,(\gamma\theta), \tag{4.3.35}$$

where

$$\gamma^2 = (v_2 - v_1)/\alpha . \tag{4.3.36}$$

Thus, as θ increases from $-K/\gamma$ to K/γ, the particle spirals out from the center of attraction to a maximum distance $1/\sqrt{v_2}$ and then falls back into the center along the mirror image spiral.

Case 2.

For $\beta = 0$, integration of (4.3.30) lead to

$$\alpha^{-1/2}\theta = \int \frac{du}{u\sqrt{(u^2 - \alpha)}} = \alpha^{-1/2}\cos^{-1}(\alpha^{1/2} r) \,. \tag{4.3.37}$$

(Put $u = 1/r$.) Clearly $u > \alpha^{\frac{1}{2}}$ and $r < \alpha^{-\frac{1}{2}}$. The polar equation of the orbit is therefore

$$r = \alpha^{-1/2}\cos(\theta)\,, \tag{4.3.38}$$

which is a circle through the center of attraction. Thus, the particle first recedes from ***O*** along one semicircle and then falls back into ***O*** along the remaining semicircle.

Case 3.

In this case, $0 < v_1 < v_2 < \alpha$ and either $u \le \sqrt{v_1}$ or $u \ge \sqrt{v_2}$.

a. We write

$$\alpha\left(\frac{du}{d\theta}\right)^2 = (v_1 - u^2)(v_2 - u^2) \tag{4.3.39}$$

and then apply the standard integral (1.3.79) to give

$$r = \frac{1}{\sqrt{v_1}}\,\mathrm{ns}\,(\gamma\theta)\,, \tag{4.3.40}$$

with

$$k^2 = v_1/v_2 \quad \text{and} \quad \gamma^2 = v_2/\alpha\,. \tag{4.3.41}$$

Thus, as θ increases from 0 to $2K/\gamma$, the particle approaches the center of attraction from infinity to a minimum distance $1/\sqrt{v_2}$ and then recedes again to infinity.

b. If, however, $u \ge v_2$, then we write

$$\alpha\left(\frac{du}{d\theta}\right)^2 = (v_1 - u^2)(v_2 - u^2) \tag{4.3.42}$$

and apply the standard integral (1.3.84). This show that

$$r = \frac{1}{\sqrt{v_2}}\,\mathrm{sn}\,(\gamma\theta) \tag{4.3.43}$$

k and γ being given as before.

Hence, as θ increase form 0 to $2K/\gamma$, the particle leaves the center of attraction and recedes from it to a maximum distance $1/\sqrt{v_2}$; thereafter, it falls back into pole ***O***. This is similar to **Case 1**.

Case 4.

For $\beta = \frac{1}{4}\alpha^2$, we find that

$$\alpha^{-1/2}\theta = \int \frac{du}{\alpha/2 - u^2} = \frac{1}{\sqrt{2\alpha}}\ln\left|\frac{\sqrt{(\alpha/2)} + u}{\sqrt{(\alpha/2)} - u}\right|. \tag{4.3.44}$$

Solving for u, we derive

$$r = 1/u = \sqrt{(2/\alpha)}\tanh(\theta/\sqrt{2}) \quad \text{or} \quad \sqrt{(2/\alpha)}\cosh(\theta/\sqrt{2}) \qquad (4.3.45)$$

as possible equations for the orbit. The first equation corresponds to a trajectory which spirals outward from O, approaching the circle $r = \sqrt{(2/\alpha)}$ asymptotically. The alternative orbit spirals inward from infinity and approaches the same circle asymptotically.

Case 5.

For convenience in later calculations, we shall take the zeros of g(v) to be

$$v_1, v_2 = \frac{1}{2}a^2 - b^2 \pm ia\sqrt{(b^2 - \frac{1}{4}a^2)}, \qquad (4.3.46)$$

where

$$a^2 = \alpha + 2\sqrt{\beta} \qquad b^2 = \sqrt{\beta},$$

taking a and b positive. Note that $b^2 > a^2/4$. Thus

Transforming by

$$u = b\frac{1+t}{1-t}, \qquad (4.3.47)$$

the last equation becomes

$$\alpha(\frac{dt}{d\theta})^2 = (b^2 - \frac{1}{4}a^2)(t^2 + p^2)(t^2 + q^2), \qquad (4.3.48)$$

where

$$p^2 = \frac{2b - a}{2b + a}, \qquad q^2 = \frac{2b + a}{2b - a}.$$

We shall take p, q to be positive and, clearly, $p < q$. Integration, using the standard form (1.3.89) , now yields

$$\alpha^{-1/2}\theta = (b^2 - \frac{1}{4}a^2)^{-1/2}\int\frac{dt}{\sqrt{\{(t^2 + p^2)(t^2 + q^2)\}}}, \qquad (4.3.49)$$

$$= \frac{2}{a + 2b}\mathrm{sc}^{-1}(t/p),$$

where the modulus is given by

$$k = \frac{\sqrt{(8ab)}}{a + 2b} \qquad (4.3.50)$$

The polar equation of the orbit now follows in form

$$r = \frac{1}{b}\cdot\frac{\mathrm{cn}(\gamma\theta) - p\cdot\mathrm{sn}(\gamma\theta)}{\mathrm{cn}(\gamma\theta) + p\cdot\mathrm{sn}(\gamma\theta)}, \qquad (4.3.51)$$

where

$$\gamma = (a + 2b)/(2\alpha^{1/2}).$$

Consideration of equation (4.3.51) reveals that $r = \infty$, when $\theta = -\omega$ and $r = 0$ when $\theta = \omega$, where

$$\omega = \frac{1}{\gamma}\mathrm{sn}^{-1}(\sqrt{\frac{a + 2b}{4b}}). \qquad (4.3.52)$$

(Since $a/b < 2$, ω can be found in the interval (0, K).) We conclude that the particle spirals into the center of attraction form infinity as θ increases from $-\omega$ to ω.

4.3.3 Relativistic Planetary Orbits:

According to the general theory of relativity, a planet falling freely in the gravitational field of a spherically sun behaves as if it were governed by the Newtonian laws and were attracted to the sun by a non-Newtonian gravitational force of

$$\mu\left(\frac{1}{r^2} + \frac{3h^2}{c^2 r^4}\right) \tag{4.3.53}$$

per unit mass, where h is the angular momentum per unit mass of the planet about the center of the sun and c is the velocity of light. Thus, its equations of motion are

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \mu\left(\frac{1}{r} + \frac{h^2}{c^2 r^3}\right) = E, \qquad r^2\dot{\theta} = h. \tag{4.3.54}$$

As in the previous sections, putting , we now arrive at the equation

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{h^2}u - u^2 + \frac{2\mu}{c^2}u^3 + \frac{2E}{h^2} \tag{4.3.55}$$

determining the orbit.

For all planets in the solar system, the term $2\mu \cdot u^3/c^2$ is always very small by comparison with the remaining terms in equation (4.3.55)
For convenience let

$$u = \mu \cdot v/h^2 \tag{4.3.56}$$

and then write equation (4.3.55) in the form

$$\left(\frac{dv}{d\theta}\right)^2 = 2v - v^2 + \alpha v^3 - \beta = f(v) \tag{4.3.57}$$

where

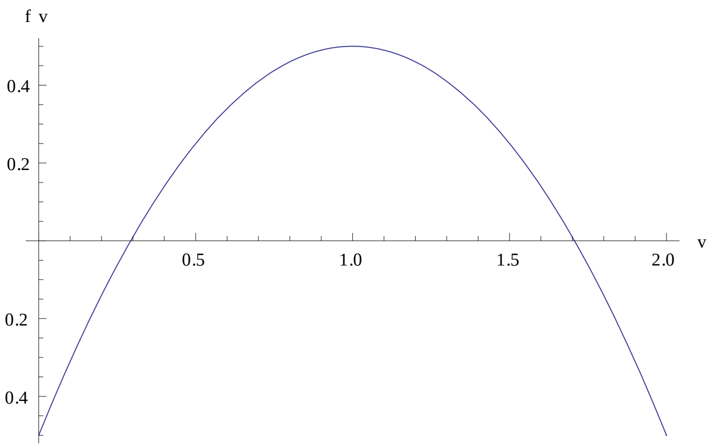
$$\alpha = 2(\mu/ch)^2, \qquad \beta = -2Eh^2/\mu^2. \tag{4.3.58}$$

This circumstance that a planet's energy is insufficient to permit its escape from the sun's field requires that $\beta > 0$. Also $\beta \le 1$, for otherwise $(dv/d\theta)^2 < 0$ in the absence of the relativistic term. α is very small and positive for all planets in the solar system, taking its largest value of 5.09×10^{-8} for Mercury.

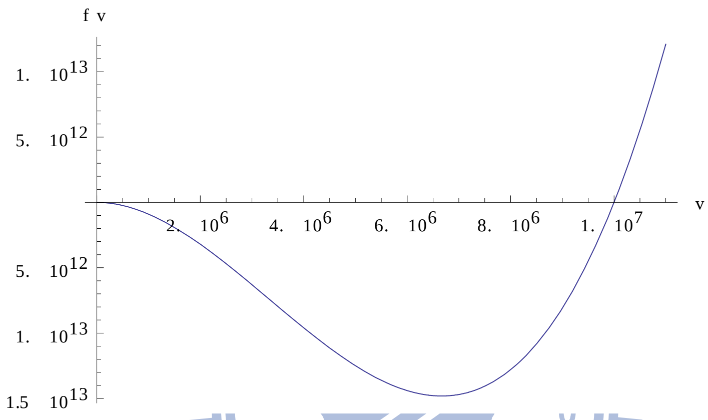
By graphing the function $2v - v^2 + \alpha v^3$, it is easy to establish that the zeros of f(v) (in equation(4.3.57)) are all real and satisfy the inequalities $0 < v_1 < 1 < v_2 < 2 < v_3$, v_3 being very large. Thus

$$f(v) = \alpha(v - v_1)(v_2 - v)(v_3 - v) \tag{4.3.59}$$

and, since $f(v) \ge 0$, v must lie in the interval (v_1, v_2) (***Remark14***)



For $\alpha = 10^{-7}$ $\beta = \frac{1}{2}$



For $\alpha = 10^{-7}$ $\beta = \frac{1}{2}$

Note that v_3 get larger as α become smaller.

α being small, the zeros of f(v) can be expanded in series of ascending powers of , thus

$$v_1 = 1 - e - \frac{\alpha}{2e}(1-e)^3 + O(\alpha^2)$$

$$v_2 = 1 + e + \frac{\alpha}{2e}(1+e)^3 + O(\alpha^2) \qquad (4.3.60)$$

$$v_3 = \frac{1}{\alpha} - 2 + O(\alpha)$$

where

$$e^2 = 1 - \beta$$

Integrating equation (4.3.57), we deduce that

$$\alpha^{1/2}\theta = \int \frac{dv}{\sqrt{(v - v_1)(v_2 - v)(v_3 - v)}} \qquad (4.3.61)$$

Changing the variable in the elliptic integral by $v = v_1 + 1/t^2$, we bring it to standard form, thus:

$$\alpha^{1/2}\theta = -\frac{2}{\sqrt{\{(v_2 - v_1)(v_3 - v_1)\}}}\int\frac{dv}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \tag{4.3.62}$$

where

$$a^2 = 1/(v_2 - v_1), \qquad b^2 = 1/(v_3 - v_1).$$

Use of (1.3.87) now yields the result

$$\alpha^{1/2}\theta = \frac{1}{\sqrt{(v_3 - v_1)}}\,\mathrm{ns}^{-1}\{t\sqrt{(v_2 - v_1)}\} \tag{4.3.63}$$

With modulus given by

$$k^2 = \frac{v_2 - v_1}{v_3 - v_1} \tag{4.3.64}$$

Thus,

$$v = v_1 + (v_2 - v_1)\mathrm{sn}^2\{\frac{1}{2}\sqrt{\alpha(v_3 - v_1)}\theta\} \tag{4.3.65}$$

is the equation of the orbit.

Substituting the expansions (4.3.60), we calculate that

$$\frac{1}{r} = \frac{\mu}{h^2}(A + B\cdot \mathrm{sn}^2(\eta\theta)), \tag{4.3.66}$$

where

$$\begin{aligned} A &= 1 - e - \frac{\alpha}{2e}(1-e)^3 + O(\alpha^2) \\ B &= 2e + \alpha(3e + 1/e) + O(\alpha^2) \\ \eta &= \frac{1}{2} - \frac{1}{4}(3-e)\alpha + O(\alpha^2) \end{aligned} \tag{4.3.67}$$

The modulus id determined by

$$k^2 = 2e\alpha + O(\alpha^2) \tag{4.3.68}$$

If $\alpha = 0$, then A=1-e, B=2e, $\eta = \frac{1}{2}$, $k = 0$, and the orbital equation reduces to

$$\frac{1}{r} = 1 - e\cdot\cos(\theta) \tag{4.3.69}$$

where $l = h^2/\mu$. This represents the classical elliptical orbit with semi-latusrectum l and eccentricity e.

On the relativistic orbit given by equation (4.3.69), perihelion occurs when $\theta = K/\eta$ and, on the next occasion, when $\theta = 3K/\eta$. Thus, θ increases by $2K/\eta$ between two passages through perihelion, instead of the increase of 2π expected from the classical theory. The advance of perihelion per revolution is accordingly

$$\frac{2K}{\eta} - 2\pi = \frac{\pi(1 + k^2/4 + \cdots)}{1/2 - (3-e)\alpha/4 + \cdots} - 2\pi = 3\pi\alpha \tag{4.3.70}$$

For Mercury, $\alpha = 5.09 \times 10^{-8}$ and its period is 88 days. Thus, the advance of perihelion per century predicted by the theory is $43''$; this is exactly the residual advance remaining to be explained at the time the new theory was proposed by Einstein.

Remark14:

$v \geq v_3$ is excluded since this would lead to $v \to \infty$ as $\theta \to \infty$; i.e., the planet would fall into the sun.

Chapter5 Conclusion

The goal that people research differential equations is to describe some phenomena in the real world. However, most of the phenomena are hard to describe and trying to get the solutions of them is another hard work.

In this paper, we introduce one kind of function defined in complex number, which has some 'good' properties, and these functions have powerful usage for getting the solutions of integral equations.

To show the effects of these functions, we choose seven physical examples. By Newton's mechanics, general theory of relativity and some laws of motions, we use differential systems to describe these physical phenomena. And then, we use elliptic Jacobian functions to get the solutions of them. In the end, we compare these solutions with physical phenomena.

The last three examples (Orbits under aμ/r^4 Law of Attraction, Orbits under aμ/r^5 Law of Attraction, Relativistic Planetary Orbits) explain the evolution of the modern theories in Planetary Orbits. The first two examples obey Newton's law, and the last one is the model basing on the general theory of relativity proposed by Einstein. That is the reason for the difference of the estimation of Mercury's period.

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