國立交通大學

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博 士 論 文

邊著色圖中的混色子圖

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中 華 民 國 九 十 九 年 七 月

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Abstract

A subgraph in an *edge-colored* graph is *multicolored* if all its edges receive distinct colors. In this dissertation, we first prove that a complete graph of order $2m$ ($m \neq 2$) can be properly edge-colored with $2m-1$ colors in such a way that the edges of K_{2m} can be partitioned into m isomorphic multicolored spanning trees. Then, for the complete graph on $2m+1$ vertices, we give a proper edge-coloring with $2m+1$ colors such that the edges of K_{2m+1} can be partitioned into m multicolored *Hamiltonian cycles*.

In the second part, we first prove that if K_{2m} admits a proper $(2m-1)$-edge-coloring such that any two colors induce a 2-factor with each component a 4-cycle, then K_{2m} can be partitioned into m isomorphic multicolored spanning trees. As a consequence, we show the existence of three isomorphic multicolored spanning trees whenever $m \geq 14$. As to the complete graph of odd order, two multicolored isomorphic *unicyclic* spanning subgraphs can be found in an arbitrary proper $(2m+1)$-edge-coloring of K_{2m+1}.

If we drop the condition "isomorphic", we prove that there exist $\Omega(\sqrt{m})$ mutually edge-disjoint multicolored spanning trees in any proper $(2m-1)$-edge-colored K_{2m} by applying a recursive construction. Using an analogous strategy, we can also find $\Omega(\sqrt{m})$ mutually edge-disjoint multicolored unicyclic spanning subgraphs in any proper $(2m-1)$-edge-colored K_{2m-1}.

Finally, we consider the problem of how to forbid a specific multicolored subgraph in a properly edge-colored complete bipartite graph. We (1) prove that for any integer $k \geq 2$, if $n \geq 5k-6$, then any properly n-edge-colored $K_{k,n}$ contains a multicolored C_{2k}, and (2) determine the order of the properly edge-colored complete bipartite graphs which forbid multicolored 6-cycles.

摘要

在一個邊著色的圖中（以下的邊著色須滿足相接的兩條邊必為不同顏色），如果有一個子圖它每個邊的顏色皆不相同，則稱這種子圖為一個混色圖。在這篇論文中，首先我們證明點數為$2m$的完全圖（$m \neq 2$），存在一個$2m-1$個顏色的邊著色，可以將K_{2m}分解成m個互相同構的混色懸掛樹。而對點數為$2m+1$的完全圖，我們也證明其邊適當地著$2m+1$個顏色後，K_{2m+1}將可分解成m個混色的哈米爾頓圈。

第二部分，我們證明對於$2m$個點的完全圖，如果有一種$2m-1$個顏色的邊著色使得任兩種顏色均會形成一組C_4的分割，則這種著色的完全圖也可以分解成m個互相同構的混色懸掛樹。由這個結果，我們可以證明在K_{2m}中（$m \geq 14$），任意給定一種$2m-1$個顏色的邊著色，一定會存在三個同構的混色懸掛樹。至於對於點數為$2m+1$的完全圖，在任意的$2m+1$個顏色邊著色下，也一定存在兩個同構的混色子圖，其中這兩個子圖是懸掛單圈圖。

若捨棄掉「同構」這個限制，我們利用一種遞迴的建構方法，可以證明出在K_{2m}中，任意給定一種$2m-1$個顏色的邊著色，存在約$\Omega(\sqrt{m})$個邊互斥的混色懸掛樹。利用相同的策略－遞迴建構法，在K_{2m-1}中，任意給定一種$2m-1$個顏色的邊著色，我們也可找出約$\Omega(\sqrt{m})$個邊互斥的混色懸掛單圈圖。

最後，我們討論如何在一個邊著色的完全二部圖中避免某些特定的混色子圖的出現。我們的貢獻有下列兩部分： (1) 對任意的$k \geq 2$，如果$n \geq 5k-6$，則任意n著色的完全二部圖$K_{k,n}$中一定找得到混色的C_{2k}； (2) 刻劃出所有可避免混色C_6的完全二部圖。

Acknowledgement

“I hate the word ‘potential’—potential means you haven’t gotten it done.”—Alex Rodriguez

這是 MLB 球星 A-Rod 在棒球場上說過的一句話，在做學問及成長的道路上，何嘗不是如此。學生生涯已結束，面對接下來的挑戰，我想用這句話來勉勵自己、鞭策自己，將「潛力」這兩字拋諸於腦後，努力完成每件可能做到的事。

來到新竹恰好十年，在這條學習的路上，首先最感謝的是我的指導教授－傅恆霖老師。從研究所到博士班，謝謝有老師您的不厭其煩的教誨及包容，讓我能順利的突破重重的難關。尤其在最後這一年，為了讓我能順利畢業，不辭辛勞地幫我驗證和修改期刊及博士論文，一次又一次，感動的心真的不知如何描述。除了在課業上的幫忙，您對於做人處世的態度和堅持也是讓我收獲良多。而在運動方面，也永遠是我最敬重的教練。

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聊天哈拉講八掛，都會覺得心情變得特別好。真的很謝謝妳們這些年來的照顧，永遠不會忘記當年喊出的「一日工讀生，終生工讀生」。

同是傅家子弟的志弘、嘉芬、明輝、志銘、賓賓、robin、惠蘭、裴、敏筠、智懷，研究的路上多虧了你們的幫忙，讓我收獲良多，得以完成我的博士論文；特別是賓賓學長，除了在研究方面是我的榜樣外，他也常提供很多寶貴的個人經驗給我參考，不謹改變我很多生活態度，也開闊了我對數學的視野。另外，小培、啟賢是我生活上的好朋友；小巴、川和和我是 AM93 留下攻讀博班的同學，有了你們的相陪，讓我不會那麼孤單。

接下來要感謝的是系排的戰友們，在交大打滾了十年，排球一直是我生活的重心之一。帶我進入排球世界的 robin、cool、god，國家級教練傅老師，早期隊友致仁、拳頭、kert、劉砰砰、許老、明淇、阿立，中期的小傑、小馬、大樹、小假死、理理人、蛤仔，一直到最近的瑞毅、軒軒、企鵝、亥派、阿陰、小育等等，一起南爭北討、揮撒汗水，贏得冠軍的喜悅、輸掉比賽的眼淚，每一個珍貴的畫面，都烙印在我心中。再來要對這些年來的同學、好麻吉們說聲感謝：拳頭、小偉、樂咖、假死、小龍、小哈，有了你們相挺，讓我這十年過得更加精采！

最後，要感謝的是我親愛的家人和芃瑀，謝謝你們的支持，讓我可以堅持到最後；謝謝你們的體諒，讓我可以不顧一切地勇往直前。最後的最後，謹用此論文獻給你們，表達我的感恩之心。

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Chapter 1

Introduction and Preliminaries

1.1 Motivation

Graph decomposition and graph coloring are two of the most important topics in the study of graph theory. Graph decomposition deals with the partition of the edge set of a graph G into subsets each induces a graph in the list of prescribed subgraphs of G, and graph coloring studies the assignments of colors onto the vertex set of G or the edge set of G or both or some well-understood areas. Either one of them has made a strong impact to make graph theory more interesting and useful through the years.

The research on combining these two topics together starts at observing a subgraph in an edge-colored graph which has many colors. A subgraph whose edges are of distinct colors is known as a *rainbow* (or *multicolored, heterochromatic*) subgraph, see [36] for reference. In 1991, Alon, Brualdi and Shader [3] first showed that in any edge-coloring of K_n such that each color class forms a complete bipartite graph, there is a spanning tree of K_n with distinct colors. Some years later, in 1996, Brualdi and Hollingsworth [10] proved the existence of two edge-disjoint multicolored spanning trees in any proper edge-coloring of K_{2n}. Then, they conjectured that a full partition into multicolored spanning trees is always possible. This conjecture encouraged many scholars to devote themselves to studying this kind of decomposition problem. In 2000, J. Krussel, S. Marshal and H. Verral [32] showed the existence of three edge-disjoint multicolored spanning trees about above conjecture, and it stopped for a while.

How about adding a condition that these spanning trees are isomorphic mutually? In 2002, G. M. Constantine [14] inserted a parallel concept into this problem. He proposed two conjectures. One of them is that any proper $(2n-1)$-edge-coloring of K_{2n} allows a partition of the edges into multicolored isomorphic spanning trees. The other one is a weaker version of above by giving an edge-coloring ourselves and partitioning $E(K_{2n})$. Moreover, Constantine proved the latter conjecture on some specific orders.

It is not a coincidence that decomposing the complete graph with even order into spanning trees, because it is easy to decompose K_{2n} into n Hamiltonian paths. Analogous to the complete graph of even order, how about that of odd order? Due to the chromatic index, it is natural to partition the graph into either *unicyclic* subgraphs or *Hamiltonian* cycles. In 2005, Constantine [15] partitioned K_{2n+1} into n multicolored Hamiltonian cycles by a given proper $(2n+1)$-edge-coloring if n is a prime. Furthermore, he proposed a new conjecture that for any proper $(2n+1)$-edge-coloring of K_{2n+1}, the edges can be partition into multicolored isomorphic spanning unicyclic subgraphs.

The above problems motivate us the study of this thesis.

1.2 Graph Terms

In this section, we first introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West.[35]

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates each edge with two vertices called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiedges* are edges having the same pair of endpoints. A *simple graph* is a graph without loops and multiedges. In this thesis, all the graphs we consider are simple. The size of the vertex set $V(G)$, $|V(G)|$, is called the *order* of G, and the size of the edge set $E(G)$, $|E(G)|$, is called the *size* of G.

If $e = \{u, v\}$ (uv in short) is an edge of G, then e is said to be *incident* to u and v.

We also say that u and v are *adjacent* to each other. For every $v \in V(G)$, $N(v)$ denotes the neighborhood of v; that is, all vertices of $N(v)$ are adjacent to v. The *degree* of v in a graph G, written $d_G(v)$ or $d(G)$, is the number of neighbors of v in G. The maximum degree is $\Delta(G)$, and the minimum degree is $\delta(G)$. Moreover, G is *regular* if $\Delta(G) = \delta(G)$, and it is said to be *k-regular* if the common degree is k.

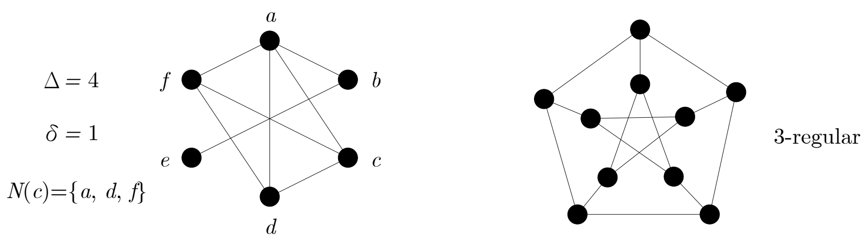


Figure 1.1: Degree, neighborhood and regular

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. Given S be a subset of vertex set $V(G)$, the *induced subgraph determined by* S, denoted by $\langle S \rangle_G$, is a subgraph of G such that for any $u, v \in S$, u is adjacent to v in $\langle S \rangle_G$ if u is adjacent to v in G.

A *spanning subgraph* (or *factor*) of G is a subgraph with vertex set $V(G)$. A spanning subgraph is said to be *k-factor* if it is k-regular.

A *matching* of size k in G is a set of k pairwise disjoint edges. If a matching covers all vertices of G, then it is a *perfect matching*. Accordingly, a perfect matching and an 1-factor are almost the same thing. In Figure 1.2, the edge set $\{af, bg, ch, di, ej\}$ is a perfect matching of G and it induces an 1-factor.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is denoted by C_n. A *Hamiltonian*

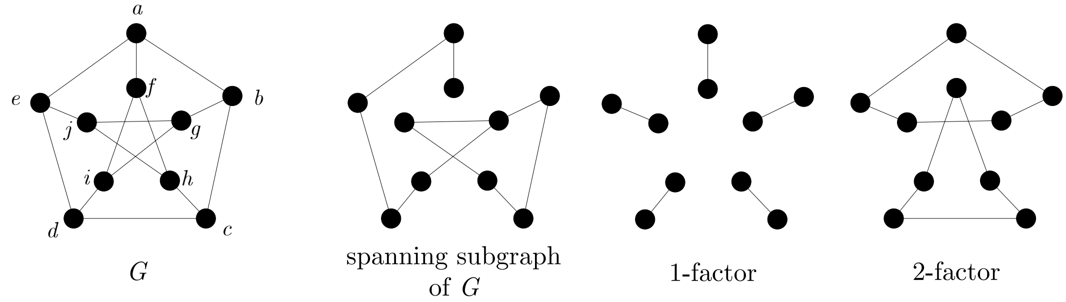


Figure 1.2: spanning, factor and matching

graph is a graph with a spanning cycle, also called a *Hamiltonian cycle*. A graph with exactly one cycle is *unicyclic*; therefore, a hamiltonian cycle in a hamiltonian graph is a unicyclic subgraph.

In contrast, a graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph. A *leaf* (or *pendant vertex*) in a tree is a vertex of degree 1. A *star* is a tree consisting of one vertex adjacent to all the others, and the particular vertex is said to be the *root* (or *center*) of the star. Let S_x denote a star with center x.

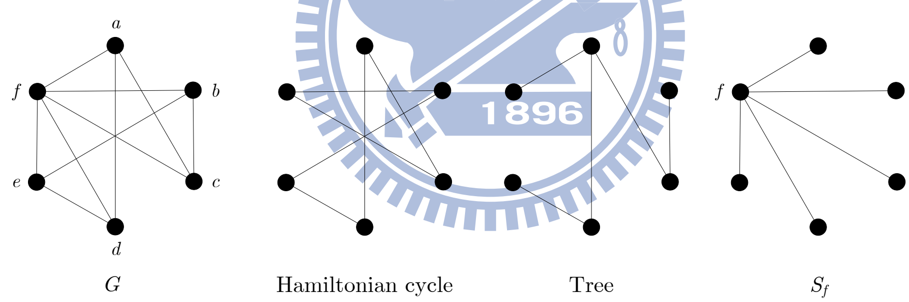


Figure 1.3: Hamiltonian cycle, tree and star

A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise nonadjacent vertices.

A *complete graph* is a simple graph whose vertices are pairwise adjacent, and the complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets, called *partite sets* of G. A graph G is *m-partite* if $V(G)$ can be expressed as the union of m independent sets. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes s and t, the complete bipartite graph is denoted by $K_{s,t}$. If the sets have the same size n, the complete bipartite graph is said to be *balanced*, denoted by $K_{n,n}$. Similarly, the complete m-partite graph is denoted by $K_{s_1,s_2,\ldots,s_m}$ where s_i is the size of the i-th partite set, and the balanced complete m-partite graph is denoted by $K_{m(n)}$ where each partite set has n vertices.

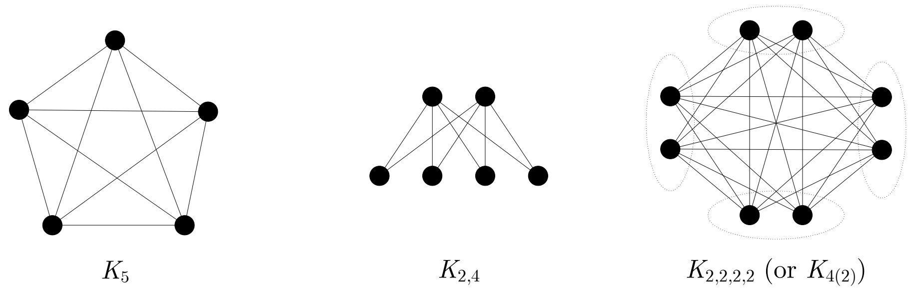


Figure 1.4: Complete graph, complete bipartite and multipartite graph

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is *isomorphic to* H", written $G \cong H$, if there is an isomorphism from G to H.

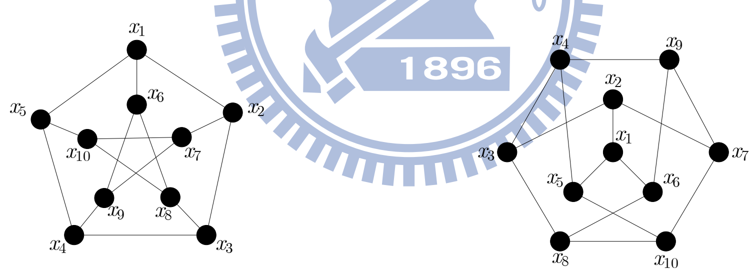
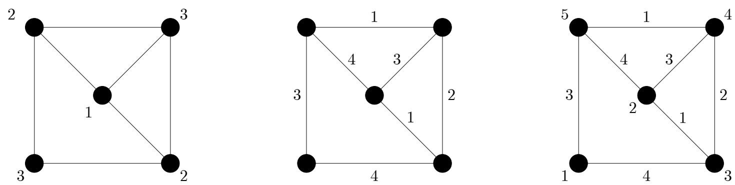


Figure 1.5: Two isomorphic graphs

1.3 Edge-coloring

A *k-coloring* of a graph G is a mapping from $V(G)$ into a set of colors $\{1, 2, \ldots, k\}$, referred as a *color set*. The vertices of one color form a *color class*. A k-coloring is *proper* if adjacent vertices have different colors. A graph is *k-colorable* if it has a proper k-coloring; furthermore, name the least k such that G is k-colorable be the *chromatic number* of G, written $\chi(G)$.

Analogous to k-coloring, a *k-edge-coloring*, *proper k-edge-coloring* and *k-edge-colorable* can be defined by replacing $V(G)$ with $E(G)$, and let the *chromatic index* $\chi'(G)$ be the least k such that G is k-edge-colorable. Combining these two kinds of colorings, an (*proper*) *k-total-coloring* of a graph G is a mapping from $V(G) \cup E(G)$ into a set of colors $\{1, 2, \ldots, k\}$ such that (i) adjacent vertices in G receive distinct colors, (ii) incident edges in G receive distinct colors, and (iii) any vertex and its incident edges receive distinct colors.



3-coloring   4-edge-coloring   5-total-coloring

Figure 1.6: Three types of proper coloring

Figure 1.6 shows the three types of proper coloring: (vertex-)coloring, edge-coloring and total-coloring. Note here we usually use Arabic numerals to denote the colors; however, in same chapters we take symbols such as $c_1, c_2, \ldots$ or $(0,0), (0,1), \ldots$ to denote colors. No matter what they are, different symbols indicate different colors. Here are some famous results about colorings, edge-colorings, and total-colorings.

Theorem 1.3.1. (Brooks [9]) *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Theorem 1.3.2. (Vizing [34]) *If G is simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Theorem 1.3.3. [37] *If n is an odd positive integer, then K_n has an n-total-coloring.*

According to Vizing's theorem, for simple graphs, there are only two possibilities for χ'. A simple graph G is of **Class 1** if $\chi'(G) = \Delta(G)$, while it is of **Class 2** if $\chi'(G) = \Delta(G)+1$. It is not hard to check that K_{2m} is Class 1 and K_{2m+1} is Class 2.

In this thesis, we mainly focus on proper edge-coloring. Let φ be a proper $(2m-1)$-edge-coloring of K_{2m} and C be the color set. For each $x \in V(K_{2m})$, define φ_x as the mapping from $V(K_{2m}) \setminus \{x\}$ to C by $\varphi_x(y) = c$ if $\varphi(xy) = c$. Clearly, φ_x is bijective. For each vertex x, let $\varphi_x^{-1}(c)$ be the vertex adjacent to x with the edge colored c. For convenience, we use $v\langle c\rangle$ to denote the edge incident to v with color c.

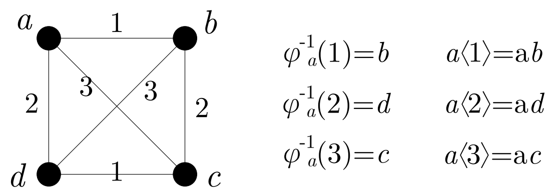


Figure 1.7: φ^{-1} and $v\langle c\rangle$ notations

A subgraph in an edge-colored graph is said to be *multicolored* (or *rainbow, heterochromatic*) if no two edges have the same color. Suppose T is a multicolored spanning tree of K_{2m} with two leaves x_1 and x_2. Let the edges in T incident to x_1 and x_2 be e_1 and e_2 respectively, and $\varphi(e_1) = c_1$, $\varphi(e_2) = c_2$. Then let $T[x_1, x_2]$ be the tree obtained from T by removing the edges e_1, e_2 and adding the edges $x_1\langle c_2\rangle, x_2\langle c_1\rangle$.

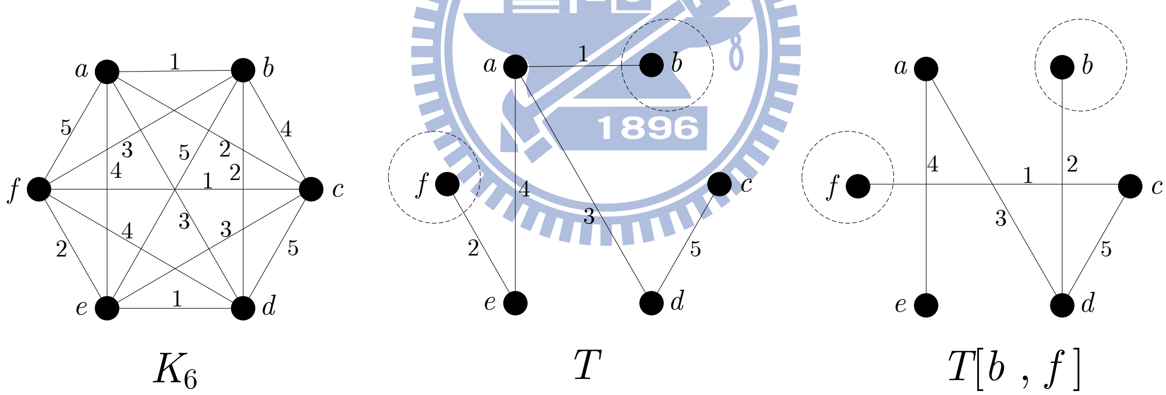


Figure 1.8: T and $T[b, f]$

Figure 1.8 provides a properly 5-edge-colored K_6 and one of its multicolored spanning tree T. Given b and f be two leaves in T. It is easy to see that the tree $T[b, f]$ is still multicolored and spanning.

1.4 Basic Algebra

Definition 1.4.1. A *group* $\langle G, * \rangle$ is a nonempty set G with a binary operation $*$ such that:

(1) $a, b \in G$ implies that $a * b \in G$.

(2) For all $a, b, c \in G$, we have $a * (b * c) = (a * b) * c$.

(3) There is an element $e \in G$, say *identity*, such that $a * e = e * a = a$ for any $a \in G$.

(4) For every $a \in G$ there exists an element $b \in G$ such that $a * b = b * a = e$.

A group $\langle G, * \rangle$ is said to be *abelian* if $a * b = b * a$ for all $a, b \in G$. If the set G has an finite number of elements, we say $\langle G, * \rangle$ is a finite group.

For each positive integer n, we can partition $\mathbb{Z}^+$, all positive integers, into n subsets according to whenever the remainders of two positive integers divided by n is the same. These subsets are called the *residue classes modulo n* in $\mathbb{Z}^+$. If a and b have the same remainder divided by n, then we write $a \equiv b \pmod{n}$, read, "a is *congruent to b modulo n*." For convenient, we use $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ to denote the set of residue classes modulo n. It is easy to see that $\mathbb{Z}_n$, $n \in \mathbb{Z}^+$, is a finite group under the usual addition modulo n. Table 1.1 presents the structure of the group $\langle \mathbb{Z}_7, + \rangle$.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Table 1.1: The group table of $\langle \mathbb{Z}_7, + \rangle$

Definition 1.4.2. A *field* $\langle F, +, \cdot \rangle$ is a nonempty set F with two binary operations $+$ and $\cdot$, as well as two particular elements 0 and 1 such that:

(1) $\langle F, +\rangle$ is an abelian group with identity 0.

(2) $\langle F^*, \cdot\rangle$ is an abelian group with identity 1, where $F^* = F \setminus \{0\}$.

(3) For all $a, b, c \in F$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

Given a prime p, it is not hard to check that $\mathbb{Z}_p$ is a field under usual addition and multiplication modulo p. Table 1.2 presents the structure of the field $\langle \mathbb{Z}_7, +, \cdot\rangle$.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Table 1.2: The group table of $\langle \mathbb{Z}_7, +\rangle$ and $\langle \mathbb{Z}_7^*, \cdot\rangle$

The group $\mathbb{Z}_n$, $n \in \mathbb{Z}^+$, and the filed $\mathbb{Z}_p$, $p \in \mathbb{Z}^+$ a prime, play two important roles in the description of proofs to our results. For more information about algebra, we refer to [19] and [27].

1.5 Latin Square

Let S be an n-set. A *latin square* of order n based on S is an $n \times n$ array such that each element of S occurs in each row and each column exactly once. For example,

0	1
1	0

is a latin square of order 2 based on $\{0, 1\} = \mathbb{Z}_2$. Since this latin square corresponds to a group table of $\langle \mathbb{Z}_2, +\rangle$, the latin square is also known as a 2-group latin square.

For convenience, we denote a latin square of order n based on S by $L = [\ l_{i,j}\ ]$ where $l_{i,j} \in S$ and $i, j \in \mathbb{Z}_n$. Let $L = [\ l_{i,j}\ ]$ and $M = [\ m_{i,j}\ ]$ be two latin squares of order n based on S. Then $L = [\ l_{i,j}\ ]$ and $M = [\ m_{i,j}\ ]$ are a pair of *orthogonal latin squares*, denoted by $L \perp M$, if and only if $\{(l_{i,j}, m_{i,j}) | \ 1 \le i, j \le n\} = S \times S$.

Let $L = [\ l_{i,j}\ ]$ and $M = [\ m_{i,j}\ ]$ be two latin squares of order l based on S and m based on T, respectively. Then the direct product of L and M, $L \times M = [\ h_{i,j}\ ]$, is a latin

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

1	2	0
2	0	1
0	1	2

1	2	0
0	1	2
2	0	1

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

Figure 1.9: Mutually orthogonal latin squares of order 3 and 4

square of order $l \cdot m$ based on $S \times T$, where $h_{x,y} = (\ l_{a,b}, m_{c,d}\ )$ provided that $x = ma + c$ and $y = mb + d$. For example, let L be the 2-group latin square, then $L \times L$ (or L^2) is a latin square of order 4 based on $\mathbb{Z}_2 \times \mathbb{Z}_2$ as in Figure 1.10.

	0	1	2	3
0	(0,0)	(0,1)	(1,0)	(1,1)
1	(0,1)	(0,0)	(1,1)	(1,0)
2	(1,0)	(1,1)	(0,0)	(0,1)
3	(1,1)	(1,0)	(0,1)	(0,0)

Figure 1.10: 2-group latin square of order 4

A *transversal* of a latin square of order n is a set of n entries from each column and each row such that these n entries are all distinct. For example, in Figure 1.10, $\{h_{0,0}, h_{1,2}, h_{2,3}, h_{3,1}\}$ is a transversal. It is not difficult to see $L \times L$ does have 4 disjoint transversals. Clearly, if a latin square of order n has n disjoint transversals, then it has an orthogonal latin square mate.

A latin square $L = [l_{i,j}]$ is *commutative* if $l_{i,j} = l_{j,i}$ for each pair of distinct i and j, and L is *idempotent* if $l_{i,i} = i$, $i \in [n]$. Furthermore, L is *circulant* if $l_{i,j} = l_{i-1,j+1}$ where the indices i, j are taken modulo n.

Let $L = [l_{i,j}]$ be an idempotent commutative latin square of order n, n is odd. There is a corresponding relationship between L and a properly n-edge-colored K_n. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and the edge v_iv_j is colored with $l_{i,j}$ for each $1 \leq i \neq j \leq m$, then we have a proper n-edge-coloring of K_n, and vice versa.

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

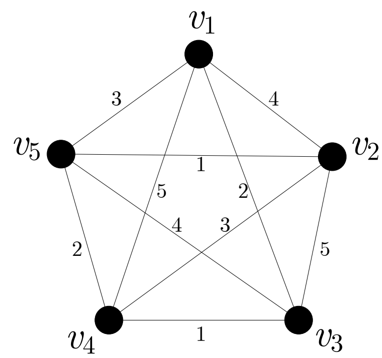


Figure 1.11: Idempotent commutative LS and corresponding edge-coloring

A similar idea shows that a latin square of order n corresponds to an n-edge-coloring of the complete bipartite graph $K_{n,n}$. Let $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ be the two partite sets of $K_{n,n}$ and the edge u_iv_j be colored with $l_{i,j}$ where $L = [l_{i,j}]$ is a latin square , we have a proper n-edge-coloring of $K_{n,n}$. Therefore, a transversal of a latin square of order n is corresponded to a multicolored perfect matching in a properly n-edge-colored $K_{n,n}$.

For more information on latin squares, we refer to [16].

1.6 Parallelism Concept

The notion of *parallelism* has always played an important role in mathematics. Euclid's famous "parallel postulate" asserted that, given any line and any point in the plane, the given point lies on a unique line parallel to the given line.

In a graph $G = (V, E)$ we may consider each edge e as a set $\{x, y\}$ consisting of the two vertices incident to e. Then, two edges e, e' are called *parallel* (or *independent* in this case) if they are disjoint, i.e., $e \cap e' = \phi$. As an extension, two subgraphs are said to be parallel if they use no common edges. Furthermore, if all edges of a graph G can be covered by copies of a subgraph H, then we say the set of these copies is a parallelism of

H's. Therefore, an 1-factorization can be considered as a parallelism of 1-factors.

We mainly consider two aspects of parallelism in a complete graphs. Firstly, given a proper $\chi'(K_n)$-edge-coloring of K_n. Then, the set of edges in a color class is parallel to another set of edges induced by a distinct color. Since each color class is a matching, a proper $\chi'(K_n)$-edge-coloring of K_n is a typical parallelism of matchings.

The second parallelism we will mention is parallelism of isomorphic spanning trees (respectively spanning unicyclic subgraphs) in a complete graph of even order (respectively odd order). Given a complete graph of even order and a partition of all edges into isomorphic spanning trees, it provides a parallelism of spanning trees. Furthermore, if the complete graph K_{2m} is properly $(2m-1)$-edge-colored and the edges of $E(K_{2m})$ can be decomposed into m isomorphic multicolored spanning trees, then we have a *double parallelism* of isomorphic spanning trees, or parallelism of isomorphic spanning trees for short. Subsequently, when it comes to a complete graph of odd order, we have a double parallelism of isomorphic spanning unicyclic subgraphs.

Harary [26] proposed several examples of a hierarchy of parallel structures in a graph in 1993. For more information about parallelism concept, see [11] for an introduction of a parallelism of complete designs. It is worth of mention here that the parallel concept plays important roles in applications. An application of parallelisms of complete designs to population genetics data can be found in [7]. Parallelisms are also useful in partitioning consecutive positive integers into sets of equal size with equal power sums [30]. In addition, the generating function of the multicolored spanning trees in any edge colored graph can be expressed as a sum of formal determinants, in [5] and [6]. These results have been used in constructing parallelisms of multicolored trees in complete graphs on a small number of vertices.

1.7 Known Results

We first consider the proper edge-coloring and total-coloring of a complete graph.

Lemma 1.7.1. [35] *$\forall m \in \mathbb{N}$, $\chi'(K_{2m}) = 2m - 1$ and $\chi'(K_{2m+1}) = 2m + 1$.*

Base on Lemma 1.7.1 and the fact that K_{2m} can be partitioned into paths, Brualdi and Hollingsworth first made the following conjecture in 1996.

Conjecture 1.7.2. [10] If K_{2m} is properly $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m multicolored spanning trees except when $m = 2$.

Meanwhile, they also proved the following theorem.

Theorem 1.7.3. [10] *If the complete graph K_{2m}, $m > 2$, is properly $(2m-1)$-edge-colored, then there exist two edge-disjoint multicolored spanning trees.*

Krussel, Marshall and Verall [32] extend Theorem 1.7.3 to three multicolored spanning trees.

Theorem 1.7.4. [32] *If $m > 2$, then in any proper edge-coloring of K_{2m} with $2m-1$ colors, there exist three edge-disjoint multicolored spanning trees.*

It will be more difficult if the desired multicolored spanning trees are mutually isomorphic. Here is an example of a 5-edge-colored K_6.

Example 1.7.5. In K_6, let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the vertex set and $\{c_1, c_2, \ldots, c_5\}$ be the color set. The following table shows an assignment of colors and a partition of the edge set. The ith row denotes the edges which are colored with c_i for $1 \leq i \leq 5$; and, the jth column denotes the edges contained in the jth multicolored spanning tree for $1 \leq j \leq 3$.

It is not difficult to see that we have a double parallelism of isomorphic spanning trees of K_6. Formally, we say that the complete graph K_{2m} admits a *multicolored tree parallelism* (*MTP*), if there exists a proper $(2m-1)$-edge-coloring of K_{2m} such that the edges can be partitioned into m isomorphic multicolored spanning trees. The following result shown by Constatine [14] provides an infinite number of complete graphs which admit MTP.

	T_1	T_2	T_3
c_1	x_3x_5	x_4x_6	x_1x_2
c_2	x_2x_4	x_1x_5	x_3x_6
c_3	x_2x_5	x_3x_4	x_1x_6
c_4	x_2x_6	x_1x_3	x_4x_5
c_5	x_1x_4	x_2x_3	x_5x_6

Table 1.3: Three multicolored isomorphic spanning trees

Theorem 1.7.6. [14] *The graph K_n admits an MTP whenever $n = 2^k$, $k > 2$, or $n = 5 \cdot 2^k$, $k \geq 1$.*

He also posed the following two conjectures.

Conjecture 1.7.7. (Weak version) [14] K_{2m} can be properly edge-colored with $2m - 1$ colors in such a way that the edges can be partitioned into m multicolored isomorphic spanning trees whenever $m > 2$.

Conjecture 1.7.8. (Strong version) [14] If K_{2m} is properly $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m multicolored isomorphic spanning trees except when $m = 2$.

On the other direction, we can also consider the complete graph of odd order. Since $\chi'(K_{2m+1}) = 2m + 1$, the maximal size of a multicolored subgraph of a properly $(2m+1)$-edge-colored K_{2m+1} is $2m + 1$. So, it is natural to ask if there also exists a partition of the edges of a properly $(2m+1)$-edge-colored K_{2m+1} into multicolored subgraphs of size $2m + 1$. Constatine gave the following result.

Theorem 1.7.9. [15] *If n is an odd prime, then there exists a proper n-edge-coloring of K_n such that the edges can be partitioned into multicolored Hamiltonian cycles.*

In fact, Constantine proposed two conjectures relative to this topic.

Conjecture 1.7.10. (Weak version) [15] For any odd integer $n \geq 3$, there exists a proper n-edge-coloring of K_n such that all edges can be partitioned into multicolored Hamiltonian cycles.

Conjecture 1.7.11. (Strong version) [15] Any proper coloring of the edges of a complete graph on an odd number of vertices allows a partition of the edges into multicolored isomorphic *unicyclic subgraphs*.

In addition, there are results relevant to the existence of a multicolored subgraph in an edge-colored graph. Here we list a couple of them.

Theorem 1.7.12. [36] *For $m \geq 3$, every properly $(2m-1)$-edge-colored K_{2m} has a multicolored perfect matching.*

Theorem 1.7.13. [28] *For any proper n-edge-coloring in $K_{n,n}$, there exists a multicolored matching with size at least $n - (11.053)log^2 n$.*

The rest of this thesis is organized as follows. In Chapter 2 and Chapter 3, we deal with the decomposition of properly edge-colored complete graphs (assigned colorings) of even and odd order into multicolored isomorphic spanning trees and multicolored Hamiltonian cycles, respectively. In the next two chapters, all colorings we consider are given. First, in Chapter 4, we prove the existence of three edge-disjoint multicolored isomorphic spanning trees in a properly $(2m-1)$-edge-colored K_{2m} whenever $m \geq 14$, and about $\sqrt{m}-1$ edge-disjoint multicolored spanning trees in K_{2m}. In Chapter 5, we tackle the cases on K_{2m+1}. Finally, in Chapter 6, the forbidden type problem is concerned. Mainly, we determine the order of those properly edge-colored complete bipartite graphs which forbid multicolored C_6. Certain general results are also mentioned.

Chapter 2

Multicolored Tree Parallelism

2.1 Known Results

Definition 2.1.1. We say that the complete graph K_{2m} admits a multicolored tree parallelism (MTP) if there exists a proper $(2m-1)$-edge-coloring of K_{2m} for which all edges can be partitioned into m isomorphic multicolored spanning trees.

It is clear that the complete graph K_4 does not admit an MTP. We note here that such a partition of the edges of K_{2m} can be viewed as a double parallelism of K_{2m} as defined in Section 1.6. In fact, finding a partition as obtained above corresponds to an arrangement of the edges of K_{2m} into an array of $2m-1$ rows and m columns such that each row contains the edges with the same color which form a perfect matching and the edges in each column form a multicolored spanning tree of K_{2m}; moreover, all the m spanning trees are isomorphic.

Example 2.1.2. The complete graph K_6 admits an MTP. To see this, consider the complete graph K_6 with the vertex set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. Table 2.1 gives a proper edge-coloring of K_6 with the colors c_1, c_2, c_3, c_4, c_5 as well as an MTP for it. The ith row of this table is the set of all edges with color c_i. Each column denotes the edges of a multicolored spanning tree. Figure 2.1 shows that the spanning trees T_1, T_2, T_3 are isomorphic.

	T_1	T_2	T_3
c_1	x_3x_5	x_4x_6	x_1x_2
c_2	x_2x_4	x_1x_5	x_3x_6
c_3	x_2x_5	x_3x_4	x_1x_6
c_4	x_2x_6	x_1x_3	x_4x_5
c_5	x_1x_4	x_2x_3	x_5x_6

Table 2.1: Color assignment of K_6

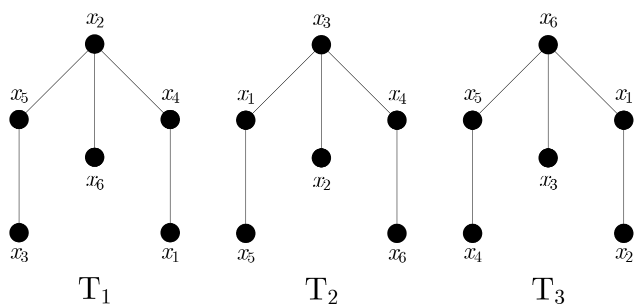


Figure 2.1: K_6 admits an MTP.

The following result has been proved in [14].

Theorem 2.1.3. [14] *If $m \neq 1, 3$ and K_{2m} admits an MTP, then $K_{2^r m}$ admits an MTP, for all $r \geq 1$.*

The mail goal of this chapter is to prove Conjecture 1.7.7, which states that K_{2m} admits an MTP for $m > 2$.

2.2 Main Results

P. Cameron [11] found a decomposition of $K_{6,6}$ into six isomorphic multicolored graphs $K_{1,3} \cup 3K_2 \cup 2K_1$ by using the software Gap. In the next lemma, we use Cameron's decomposition to find an MTP for K_{12}.

Lemma 2.2.1. *The complete graph K_{12} admits an MTP.*

Proof. Consider the complete graph K_{12} with the vertex set $\{u_1, \ldots, u_6, v_1, \ldots, v_6\}$. Table 2.2 gives a proper edge coloring of K_{12} with colors $c_1, \ldots, c_{11}$ as well as an MTP for it. The ith row of this table is the set of all edges with color c_i. Each column denotes the edges of a multicolored spanning tree. Note that the first six rows of the table determine a decomposition of $K_{6,6}$ into six multicolored subgraphs to $K_{1,3} \cup 3K_2 \cup 2K_1$. ∎

	T_1	T_2	T_3	T_4	T_5	T_6
c_1	u_2v_5	u_1v_6	u_6v_1	u_3v_2	u_4v_3	u_5v_4
c_2	u_2v_3	u_5v_2	u_6v_6	u_4v_5	u_3v_4	u_1v_1
c_3	u_4v_1	u_3v_3	u_6v_4	u_1v_2	u_5v_5	u_2v_6
c_4	u_1v_4	u_3v_5	u_5v_3	u_6v_2	u_2v_1	u_4v_6
c_5	u_2v_2	u_4v_4	u_1v_5	u_5v_1	u_6v_3	u_3v_6
c_6	u_5v_6	u_3v_1	u_4v_2	u_2v_4	u_1v_3	u_6v_5
c_7	u_3u_5	u_4u_6	u_1u_2	v_3v_5	v_4v_6	v_1v_2
c_8	u_2u_4	u_1u_5	u_3u_6	v_2v_4	v_1v_5	v_3v_6
c_9	u_2u_5	u_3u_4	u_1u_6	v_2v_5	v_3v_4	v_1v_6
c_{10}	u_2u_6	u_1u_3	u_4u_5	v_2v_6	v_1v_3	v_4v_5
c_{11}	u_1u_4	u_2u_3	u_5u_6	v_1v_4	v_2v_3	v_5v_6

Table 2.2: Color assignment of K_{12}

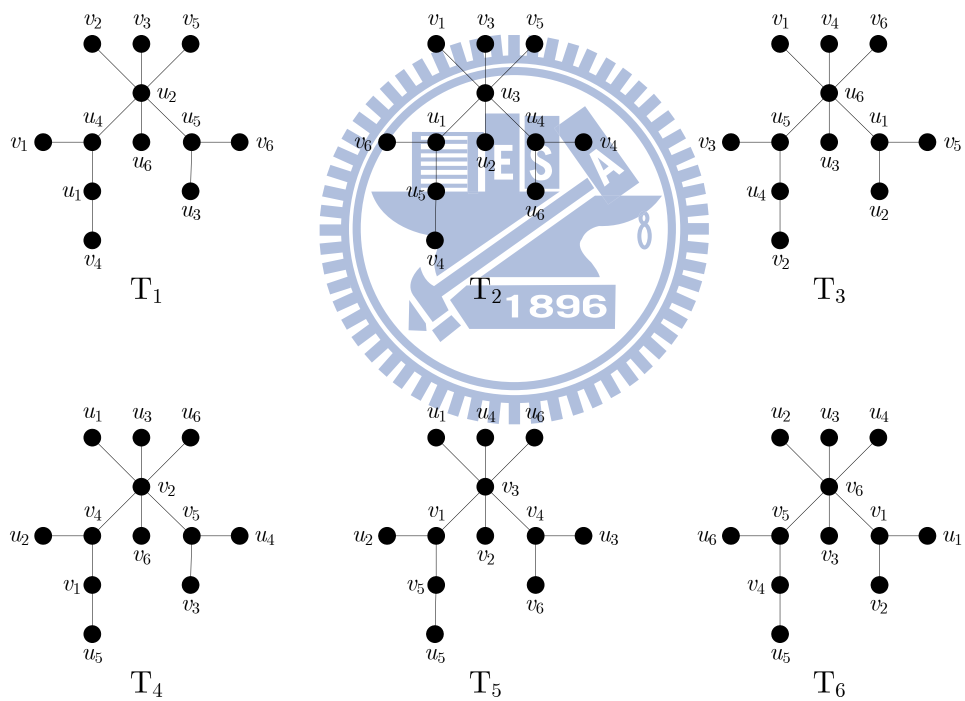


Figure 2.2: K_{12} admits an MTP.

Now, we are ready to prove our main result.

Theorem 2.2.2. *For $m \neq 2$, K_{2m} admits an MTP.*

Proof. First, suppose that m is an odd integer. Consider the complete graph K_{2m} defined on the set $A \cup B$ where $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots b_m\}$. For convenience, let G and H be the complete graphs on the sets A and B, respectively. Since m is odd, G has a total coloring π which uses m colors, $1, \ldots, m$. Now, define a proper edge-coloring φ of K_{2m} as follows:

(a) For each edge $a_j a_k \in E(G)$, let $\varphi(a_j a_k) = \pi(a_j a_k)$;

(b) For each edge $b_j b_k \in E(H)$, let $\varphi(b_j b_k) = \pi(a_j a_k)$;

(c) For each edge $a_i b_i$, $1 \leq i \leq m$, let $\varphi(a_i b_i) = \pi(a_i)$; and

(d) For each edge $a_j b_k, j \neq k$, let $\varphi(a_j b_k) = m + t$ where $t \equiv k - j \pmod{m}$ and $t \in \{1, 2, \cdots, m-1\}$.

Clearly, φ is a proper $(2m-1)$-edge-coloring of K_{2m}. It is left to decompose K_{2m} into m multicolored isomorphic spanning trees. First, for each $i \in \{1, 2, 3, \cdots, m\}$, let T_i be defined on the set $A \cup B$ and $E(T_i) = \{a_i a_{i+2t \ (mod\ m)}, b_i b_{i+2t-1 \ (mod\ m)}, b_i a_{i+2t-1 \ (mod\ m)}, a_{i+1} b_{i+2t \ (mod\ m)} \mid t = 1, 2, \cdots, \frac{m-1}{2}\} \cup \{a_i b_i\}$. Then, it is easy to check that each T_i is a multicolored spanning tree of K_{2m}, and all the T_i's are isomorphic.

Now, if m is not an odd integer, then $2m = 2^t \cdot m'$ where $t \geq 2$ and m' is odd. In case where $m' = 1$, t must be at least 3. Then it is direct consequence of Theorem 1.7.6. Assume $m' \geq 3$. Thus, $K_{2^t m'}$ admits an MTP by Theorem 2.1.3 except when $m' = 3$ and $t = 2$. Since this case can be handled by Lemma 2.2.1, we conclude the proof. ∎

We note here that the above theorem proves Conjecture 1.7.7 and the result has been included in a paper written jointly with S. Akbari, A. Alipour and H. L. Fu [2].

Chapter 3

Multicolored Hamiltonian Cycle Parallelism

To extend the study in Chapter 2 of parallelism to the other graph, K_{2m+1} deserves to be considered first. Since $\chi'(K_{2m+1}) = 2m + 1$, the multicolored subgraph we consider has $2m + 1$ edges. Thus, a multicolored Hamiltonian cycle in K_{2m+1} is the best candidate for the subgraphs. In this chapter, we shall prove that for each positive integer m, there exists a proper $(2m+1)$-edge-coloring of K_{2m+1} for which all edges can be partitioned into multicolored Hamiltonian cycles. Obviously, any two Hamiltonian cycles are isomorphic and therefore we have another parallelism if exists.

3.1 Known Results

Definition 3.1.1. We say that the complete graph K_{2m+1} admits a multicolored Hamiltonian cycle parallelism (MHCP) if there exists a proper $(2m+1)$-edge-coloring of K_{2m+1} for which all edges can be partitioned into m multicolored Hamiltonian cycles.

Review that a latin square $L = [\ell_{i,j}]$ is commutative if $\ell_{i,j} = \ell_{j,i}$ for each pair of distinct i and j in $\mathbb{Z}_n$, and L is idempotent if $\ell_{i,i} = i$ for $i \in \mathbb{Z}_n$. It is well-known that an idempotent commutative latin square of order n exists if and only if n is odd. For the convenience in the proof of our main result, we shall use a special latin square $M = [m_{i,j}]$ of odd order n which is a circulant latin square with 1st row $(0, \frac{n+1}{2}, 1, \frac{n+3}{2}, 2, \cdots, \frac{n+n-2}{2}, \frac{n-1}{2})$. Figure 3.1 is such a latin square of order 7.

0	4	1	5	2	6	3
4	1	5	2	6	3	0
1	5	2	6	3	0	4
5	2	6	3	0	4	1
2	6	3	0	4	1	5
6	3	0	4	1	5	2
3	0	4	1	5	2	6

Figure 3.1: Circulant latin square of order 7

A similar idea shows that a latin square of order n corresponds to a proper n-edge-coloring of the complete bipartite graph $K_{n,n}$. Let $\{u_0, u_1, \cdots, u_{n-1}\}$ and $\{v_0, v_1, \cdots, v_{n-1}\}$ be the two partite sets of $K_{n,n}$ and let $M = [m_{i,j}]$ be a circulant latin square of order n with the first row as described in the preceding paragraph. Color edge $u_i v_j$ of $K_{n,n}$ with color $m_{i,j}$ and observe that the result is a proper n-edge-coloring of $K_{n,n}$ with the extra property that for $0 \leq j \leq n-1$, the perfect matching $\{u_0v_j, u_1v_{j+1}, u_2v_{j+2}, \cdots, u_{n-1}v_{j+n-1}\}$, where the indices of v_i are taken modulo n with $i \in \mathbb{Z}_n$, is multicolored. We note here that if we permute the entries of M, we obtain another proper n-edge-coloring of $K_{n,n}$ which has the same property as above.

The following result by Constantine appears in [15].

Theorem 3.1.2. [15] *If n is an odd prime, then there exists a proper n-edge-coloring of K_n such that all edges can be partitioned into multicolored Hamiltonian cycles.*

Note that this result can be obtained by using a circulant latin square of order n to color the edges of K_n and the Hamiltonian cycles are corresponding to 1st, 2nd, $\cdots$, $(\frac{n-1}{2})$-th sub-diagonals respectively.

Example 3.1.3. In K_7, the edges are colored by using Figure 3.1, and the three cycles are induced by $\{x_0x_{i+1}, x_1x_{i+2}, \cdots, x_6x_i\}$ where $V(K_7) = \{x_0, x_1, \cdots, x_6\}$, $i = 0, 1, 2$, where the sub-indices are in $[n]$. See Table 3.1.

In what follows, we extend Theorem 3.1.2 to the case when n is an odd integer.

	C_1	C_2	C_3
0	x_3x_4	x_6x_1	x_2x_5
1	x_4x_5	x_0x_2	x_3x_6
2	x_5x_6	x_1x_3	x_4x_0
3	x_6x_0	x_2x_4	x_5x_1
4	x_0x_1	x_3x_5	x_6x_2
5	x_1x_2	x_4x_6	x_0x_3
6	x_2x_3	x_5x_0	x_1x_4

Table 3.1: Color assignment of K_7

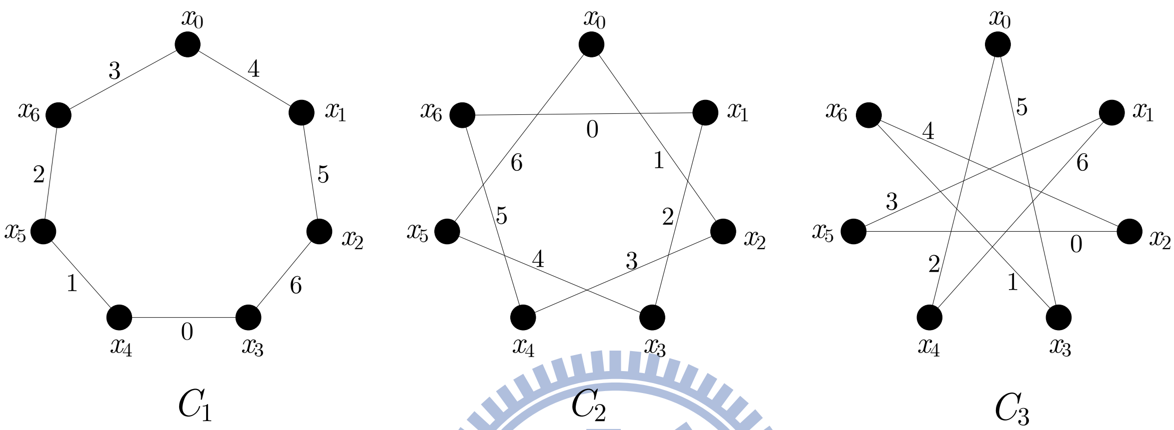


Figure 3.2: K_7 admits an MHCP.

3.2 Main Results

We begin this section with some notations. Let $K_{m(n)}$ be the complete m-partite graph in which each partite set is of size n. In what follows, we will let $\mathbb{Z}_k = \{1, 2, \ldots, k\}$ with the usual addition modulo k. For convenience, let $V(K_{m(n)}) = \bigcup_{i=0}^{m-1} V_i$ where $V_i = \{x_{i,0}, x_{i,1}, \cdots, x_{i,n-1}\}$. The graph $C_{m(n)}$ is a spanning subgraph of $V(K_{m(n)})$ where $x_{i,j}$ is adjacent to $x_{i+1,k}$ for all $j, k \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_m$ (mod m). Clearly, if K_m can be decomposed into $\frac{m-1}{2}$ Hamiltonian cycles (m is odd), then $K_{m(n)}$ can be decomposed into $\frac{m-1}{2}$ subgraphs, each of which is isomorphic to $C_{m(n)}$.

In order to prove the main theorem, we need the following two lemmas.

Lemma 3.2.1. *Let p be an odd prime and m be a positive odd integer with $p \leq m$. Let $t \in \{1, 2, \ldots, p-1\}$. Then, there exists a set $\{S_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,m-1}) |\ 0 \leq i \leq p-1\}$ of m-tuples such that*

(1) $S_0 = (0, 0, \ldots, 0, t)$;

(2) $\{a_{i,j} \mid 0 \le i \le p-1\} = \{0, 1, 2, \ldots, p-1\}$ j *with* $0 \le j \le m-1$; *and*

(3) $p \nmid w_i$ *where* $w_i = \sum_{j=0}^{m-1} a_{i,j}$ *for each* i *with* $0 \le i \le p-1$.

Proof. The proof follows by direct constructions depending on the choice of t where $1 \le t \le p-1$. First, we let $S_0 = (0, 0, \ldots, 0, 1)$, $S_1 = (1, 1, \ldots, 1, 2)$, $\cdots$, and $S_{p-1} = (p-1, p-1, \ldots, p-1, 0)$ be the p m-tuples. For each i with $0 \le i \le p-1$, let $w_i = \sum_{j=0}^{m-1} a_{i,j}$ where $S_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,m-1})$. If for each $0 \le i \le p-1$, $p \nmid w_i$, we do nothing. Otherwise, assume that $p \mid w_j$ for some $j \in \{1, 2, \ldots, p-1\}$, and note that such j is unique. Now, if $j \in \{1, 2, \ldots, p-2\}$, replace S_j and S_{j+1} with $(j, j, \ldots, j, j+1, j+1)$ and $(j+1, j+1, \ldots, j+1, j, j+2)$ respectively. Else, if $j = p-1$, then replace S_{p-2} and S_{p-1} with $(p-2, p-2, \ldots, p-2, p-1, p-1, p-1)$ and $(p-1, p-1, \ldots, p-1, p-2, p-2, 0)$ respectively.

When $t = 1$, clearly, these p m-tuples above satisfies all the four properties. So, in what follows, we consider $2 \le t \le p-1$. Note that we initially use the same m-tuples constructed in the case $t = 1$ and consider that j causing us to adjust entries above.

Case 1. No such j exists.

First, interchange $a_{0,m-1}$ with $a_{t-1,m-1}$. If $w_{t-1} \not\equiv 0 \pmod{p}$, then we are done. On the other hand, suppose $w_{t-1} \equiv 0 \pmod{p}$. If $w_t \not\equiv 1 \pmod{p}$, then replace S_{t-1} and S_t with $(t-1, t-1, \ldots, t-1, t, 1)$ and $(t, t, \ldots, t, t-1, t+1)$ respectively. Otherwise, replace S_{t-1} and S_t with $(t-1, t-1, \ldots, t-1, t-1, t+1)$ and $(t, t, \ldots, t, t, 1)$ respectively.

Case 2. $j \in \{1, 2, \ldots, p-2\}$.

First, interchange $a_{0,m-1}$ with $a_{t-1,m-1}$. If $w_{t-1} \not\equiv 0 \pmod{p}$, then we are done. On the other hand, suppose $w_{t-1} \equiv 0 \pmod{p}$. If $t = j+2$, then replace S_j and S_{j+1}

with $(j, j, \ldots, j, j+1, j+1, j+1)$ and $(j+1, j+1, \ldots, j+1, j, j, 1)$ respectively. Otherwise, interchange $a_{t-1,m-2}$ with $a_{t,m-2}$.

Case 3. $j = p - 1$.

Interchange $a_{0,m-1}$ with $a_{t-1,m-1}$.

Thus we can construct the desired p m-tuples. ∎

Example 3.2.2. Take $p = 5, m = 7$. This implies that $j = 2$. Table 3.2 shows the structure of $\{S_0, S_1, S_2, S_3, S_4\}$ for $t = 1, 2, 3$, and 4.

	$t=1$	$t=2$	$t=3$	$t=4$
S_0	$(0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,2)$	$(0,0,0,0,0,0,3)$	$(0,0,0,0,0,0,4)$
S_1	$(1,1,1,1,1,1,2)$	$(1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,2)$	$(1,1,1,1,1,1,2)$
S_2	$(2,2,2,2,2,3,3)$	$(2,2,2,2,2,3,3)$	$(2,2,2,2,2,2,1)$	$(2,2,2,2,2,3,3)$
S_3	$(3,3,3,3,3,2,4)$	$(3,3,3,3,3,2,4)$	$(3,3,3,3,3,3,4)$	$(3,3,3,3,3,2,1)$
S_4	$(4,4,4,4,4,4,0)$	$(4,4,4,4,4,4,0)$	$(4,4,4,4,4,4,0)$	$(4,4,4,4,4,4,0)$

Table 3.2: Circulating sequences for $p = 5$ and $m = 7$

Lemma 3.2.3. *Let v be a composite odd integer and p be the smallest prime with $p|v$. Assume $v = mp$. If K_m admits an MHCP, then $K_{m(p)}$ has a proper mp-edge-coloring that admits an MHCP.*

Proof. We prove the lemma by giving a proper mp-edge-coloring φ. Since K_m defined on $\{x_i \mid i \in \mathbb{Z}_m\}$ admits an $MHCP$, let μ be such a proper edge-coloring using the colors $0, 1, \cdots, m-1$. Let $V(K_{m(p)}) = \bigcup_{i=0}^{m-1} V_i$ where $V_i = \{x_{i,j} \mid j \in \mathbb{Z}_p\}$ and $L = [\ell_{h,k}]$ be a circulant latin square of order p as defined before Figure 3.1. Now, we have a proper mp-edge-coloring of $K_{m(p)}$ by letting $\varphi(x_{a,b}x_{c,d}) = \mu(x_a x_c) \cdot p + \ell_{b,d}$, where $a, c \in \mathbb{Z}_m$ and $b, d \in \mathbb{Z}_p$. Therefore, the edges in $K_{m(p)}$ joining a vertex of V_a to a vertex of V_c, denoted (V_a, V_c), are colored with the entries in $\mu(x_a x_c) \cdot p + L$. It is not difficult to see that φ is a proper edge-coloring of $K_{m(p)}$. Now, it is left to show that the edges of $K_{m(p)}$ can be partitioned into multicolored Hamiltonian cycles.

Let $C = (x_{i_0}, x_{i_1}, \cdots, x_{i_{m-1}})$ be a multicolored Hamiltonian cycle in K_m obtained from the $MHCP$ of K_m. Define $C_{m(p)}$ to be the subgraph induced by the set of edges in (V_{i_0}, V_{i_1}), $(V_{i_1}, V_{i_2}), \ldots, (V_{i_{m-2}}, V_{i_{m-1}})$, $(V_{i_{m-1}}, V_{i_0})$. Then, let $S(r_0, r_1, \cdots, r_{m-1})$, where $r_j \in \{0, 1, \ldots, p-1\}$ for $0 \leq j \leq m-1$, be the set of perfect matchings in (V_{i_0}, V_{i_1}), $(V_{i_1}, V_{i_2}), \ldots$, $(V_{i_{m-2}}, V_{i_{m-1}})$ and $(V_{i_{m-1}}, V_{i_0})$, respectively, where the perfect matching in $(V_{i_j}, V_{i_{j+1}})$ is the set of edges $x_{i_j,a}x_{i_{j+1},b}$ with $b - a \equiv r_j \pmod p$ for each $j \in \mathbb{Z}_m$. Since these perfect matchings of $(V_{i_j}, V_{i_{j+1}})$ are multicolored, we have that $S(r_0, r_1, \ldots, r_{m-1})$ is a multicolored 2-factor of $C_{m(n)}$. Hence, we can partition the edges of $C_{m(p)}$ into p multicolored 2-factors due to the fact that $r_i \in \{0, 1, \ldots, p-1\}$. Note that $S(r_0, r_1, \cdots, r_{m-1})$ and $S(r'_0, r'_1, \cdots, r'_{m-1})$ are edge-disjoint 2-factors if and only if $r_i \neq r'_i$ for each $i \in \mathbb{Z}_m$.

The proof follows by selecting $(r_0, r_1, \cdots, r_{m-1}) \in \mathbb{Z}_p^m$ properly in order that each 2-factor $S(r_0, r_1, \cdots, r_{m-1})$ of $C_{m(p)}$ is a Hamiltonian cycle. Observe that if $\sum_{i=0}^{m-1} r_i$ is not a multiple of p (odd prime), then $S(r_0, r_1, \cdots, r_{m-1})$ is a Hamiltonian cycle. From Lemma 3.2.1, let $SS_0, SS_1, \cdots, SS_{p-1}$ be the 2-factors of $C_{m(p)}$. This implies that we have a partition of the edges of $C_{m(p)}$ into p edge-disjoint multicolored Hamiltonian cycles. Moreover, since $K_{m(p)}$ can be partitioned into $\frac{m-1}{2}$ copies of $C_{m(p)}$ where each $C_{m(p)}$ arises from a multicolored Hamiltonian cycle in K_m, we have a partition of the edges of $K_{m(p)}$ into $\frac{m-1}{2} \cdot p$ multicolored Hamiltonian cycles. ∎

Example 3.2.4. If $m = p = 3$, then the three multicolored Hamiltonian cycles are $S(0,0,1) = (x_{0,0}, x_{1,0}, x_{2,0}, x_{0,1}, x_{1,1}, x_{2,1}, x_{0,2}, x_{1,2}, x_{2,2})$, $S(1,1,2) = (x_{0,0}, x_{1,1}, x_{2,2}, x_{0,1}, x_{1,2}, x_{2,0}, x_{0,2}, x_{1,0}, x_{2,1})$, $S(2,2,0) = (x_{0,0}, x_{1,2}, x_{2,1}, x_{0,2}, x_{1,1}, x_{2,0}, x_{0,1}, x_{1,0}, x_{2,2})$. In case that $m = 5$ and $p = 3$, then we have 6 multicolored Hamiltonian cycles. For each $C_{5(3)}$, we have three multicolored Hamiltonian cycles of type $S(0,0,0,0,1)$, $S(1,1,1,2,2)$, and $S(2,2,2,1,0)$.

Following above example, in order to partition the edges of a 9-edge-colored K_9 into 4 Hamiltonian cycles, we combine $S(0,0,1)$ with the three cliques (K_3) induced by the three partite sets V_0, V_1 and V_2, to obtain a 4-factor. Since these K_3's can be edge-colored with $\{3,4,5\}, \{6,7,8\}$ and $\{0,1,2\}$ respectively, we have a proper edge-colored 4-factor with each color occurs exactly twice. Thus, if this 4-factor can be partitioned into two multicolored Hamiltonian cycles, then we conclude that K_9 admits an $MHCP$. Figure 3.3 shows how this can be done.

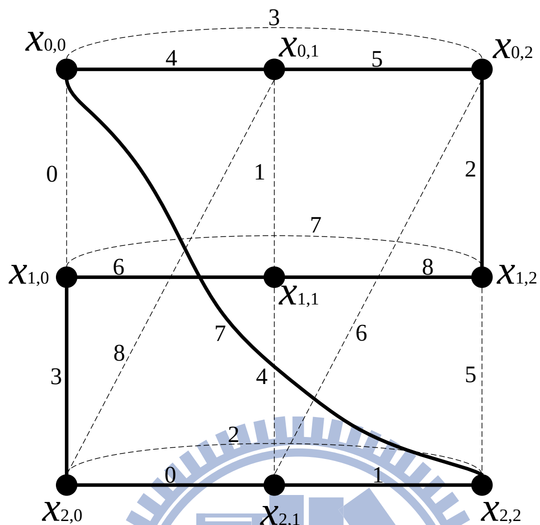


Figure 3.3: Two multicolored Hamiltonian cycles in 9-edge-colored K_9

Notice that in the induced subgraphs $< V_0 >, < V_1 >$ and $< V_2 >$ we have exactly one edge from each graph which is not included in the cycle with solid edges. Therefore, we may first color the edges in $< V_0 >, < V_1 >$ and $< V_2 >$ respectively and then adjust the colors in (V_0, V_1), (V_1, V_2) and (V_2, V_0) respectively in order to obtain a multicolored Hamiltonian cycle. For example, if the color of $x_{0,0}x_{0,2}$ is 4 instead of 3, then we permute (or interchange) the two entries in

3	5	4
5	4	3
4	3	5

, and thus the latin square used to color (V_1, V_2) becomes

4	5	3
5	3	4
3	4	5

. This is an essential trick we shall use when p is a larger prime.

Before the following theorem, we introduce one useful notation. Let μ be a k-edge-coloring of a graph G. If K is a subgraph of G, for convenience, we use $\mu|_K$ to denote the edge-coloring of K induced by μ, i.e., $\mu|_K(e) = \mu(e)$ for each $e \in E(K)$.

Theorem 3.2.5. *For each odd integer $v \geq 3$, K_v admits an MHCP.*

Proof. The proof is by induction on v. By Theorem 3.1.2, the assertion is true for v is a prime. Therefore, we assume that v is a composite odd integer and the assertion is true for each odd order $u < v$. Let p be the smallest prime such that $v = p \cdot m$ and $V(K_v) = \bigcup_{i=0}^{m-1} V_i$ where $V_i = \{x_{i,j} \mid j \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_m$. By induction, K_m admits an $MHCP$ and hence $K_{m(p)}$ can be partitioned into $\frac{m-1}{2}$ $C_{m(p)}$'s each of which admits an $MHCP$. Moreover, by Lemma 3.2.3, each $MHCP$ of $C_{m(p)}$ contains a multicolored Hamiltonian cycle $S(0, 0, \cdots, 0, 1)$. Here, the proper edge-coloring φ of $K_{m(p)}$ is induced by the proper edge-coloring μ of K_m defined as in Lemma 3.2.3. That is, if $v_i v_j$ is an edge of K_m with color $\mu(v_i v_j) = t \in \mathbb{Z}_m$, then the colors of the edges in (V_i, V_j) are assigned by using $M + tp$ where M is a circulant latin square of order p as defined before Figure 3.1. We note here again that permuting the entries of a latin square $M + tp$ gives another edge-coloring, but the edge-coloring is still proper.

So, in order to obtain an $MHCP$ of K_v, we first give a proper v-edge-coloring of K_v and then adjust the coloring if it is necessary. Since $K_{m(p)}$ has a proper mp-edge-coloring φ, the proper edge-coloring π of K_v can be defined as follows: (a) $\pi|_{K_{m(p)}} = \varphi$ and (b) $\pi|_{<V_i>} = \psi_i, i = 0, 1, \cdots, m-1$, where ψ_i is a proper p-edge-coloring of K_p such that K_p can be partitioned into $\frac{p-1}{2}$ multicolored Hamiltonian cycles. Moreover, the images of ψ_i are $tp, tp+1, \cdots, tp+p-1$ where $t \in \mathbb{Z}_m$ and t is the color not occurring in the edges incident to $v_i \in V(K_m)$. (Here, the colors used to color the edges of K_m are $0, 1, 2, \cdots, m-1$.)

It is not difficult to check that π is a proper v-edge-coloring of K_v. We shall revise π by permuting the colors in (V_i, V_{i+1}) for some i and finally obtain the edge-coloring we need.

Let the edges of the K_p induced by V_0 be partitioned into $\frac{p-1}{2}$ multicolored Hamiltonian cycles $D^{(1)}, D^{(2)}, \cdots, D^{(\frac{p-1}{2})}$, and x_{0,t_i} is the neighbor with the larger index t_i of $x_{0,0}$ in $D^{(i)}$. Hence, the m copies of K_p each induces by V_i can be partitioned into m copies of $D^{(1)}, D^{(2)}, \cdots$, and $D^{(\frac{p-1}{2})}$. For convenience, denote them as $mD^{(i)}$, $i = 1, 2, \cdots, \frac{p-1}{2}$. Now, let the edges of $K_{m(p)}$ be partitioned into $C_{m(p)}^{(1)}, C_{m(p)}^{(2)}, \cdots, C_{m(p)}^{(\frac{m-1}{2})}$. By Lemma 3.2.1, we can let each of $C_{m(p)}^{(1)}, C_{m(p)}^{(2)}, \cdots, C_{m(p)}^{(\frac{p-1}{2})}$ contains a multicolored Hamiltonian cycle $E^{(1)}, E^{(2)}, \cdots, E^{(\frac{p-1}{2})}$ of type $S(0, 0, \cdots, 0, p+1-t_i)$. Since $m \geq p$, we consider the 4-factors $E^{(i)} \cup mD^{(i)}$ where $i = 1, 2, \cdots, \frac{p-1}{2}$. Starting from $i = 1$, we shall partition the edges of $E^{(1)} \cup mD^{(1)}$ into two Hamiltonian cycles such that both of them are multicolored. By the idea explained in Figure 3.3, we first obtain two Hamiltonian cycles from $E^{(1)} \cup mD^{(1)}$ by a similar way, see Figure 3.4 for example. For the purpose of obtaining multicolored Hamiltonian cycles, we adjust the colors by permuting them in the latin square for (V_i, V_{i+1}) to make sure the first cycle does contain each color exactly once. Then, the second one is clearly multicolored. Now, following the same process, we partition the edges of $E^{(2)} \cup mD^{(2)}, \cdots$, and $E^{(\frac{p-1}{2})} \cup mD^{(\frac{p-1}{2})}$ into two multicolored Hamiltonian cycles respectively. We remark here that if permuting entries of a latin square is necessary, then we can keep doing the same trick since $C_{m(p)}^{(1)}, C_{m(p)}^{(2)}, \cdots, C_{m(p)}^{(\frac{m-1}{2})}$ are edge-disjoint subgraphs of $K_{m(p)}$. (The permutations are carried out independently.) This implies that after all the permutations are done, we obtain a proper v-edge-coloring of K_v such that K_v can be partitioned into $\frac{v-1}{2}$ multicolored Hamiltonian cycles. ∎

In conclusion, we use Figure 3.4 and Figure 3.5 to explain how our idea works. In Figure 3.4, $t_1 = 4$. The edge xy was colored with 25 originally by using the circulant latin square of order 5 mentioned before Figure 3.2. But, 25 occurs in the Hamiltonian cycle with solid edges already. Therefore, we use $(25, 29)$ to permute the square to obtain the proper edge-coloring we would like to have. After adjusting the colors of zw, $z'w'$ and ab respectively, we have two multicolored Hamiltonian cycles as desired. In Figure 3.5, $t_2 = 3$. For convenience, we reset $V_0, V_2, V_4, V_6, V_1, V_3, V_5$ from top to down. Following the same process, we also have two multicolored Hamiltonian cycles.

We note here that the above theorem proves the weaker conjecture of Constantine and the result has been included in a paper written jointly with H. L. Fu [20].

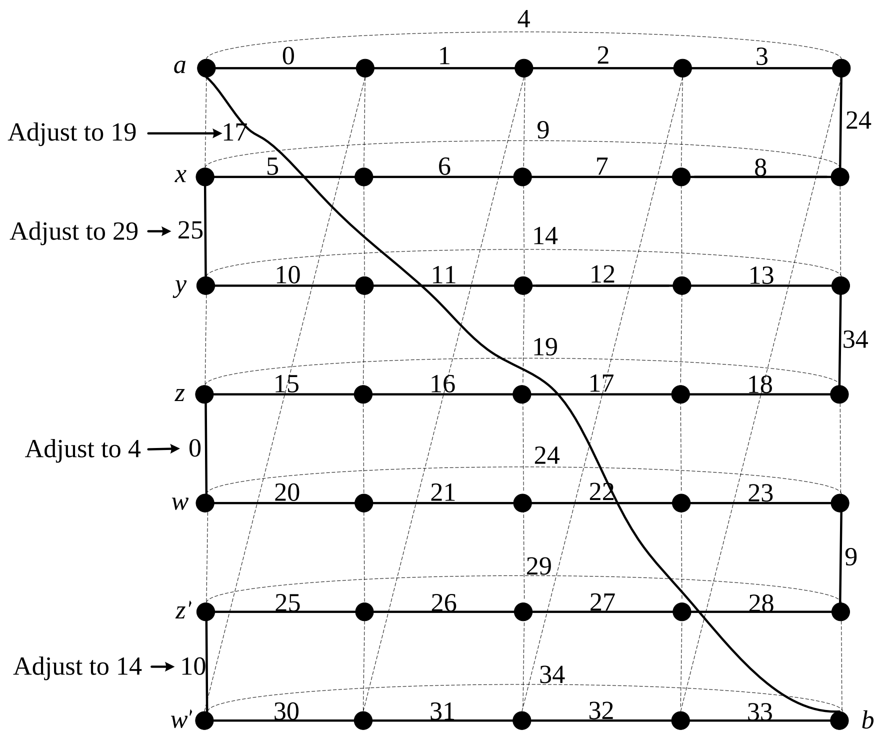


Figure 3.4: $E^{(1)} \cup 7D^{(1)}$ in K_{35}.

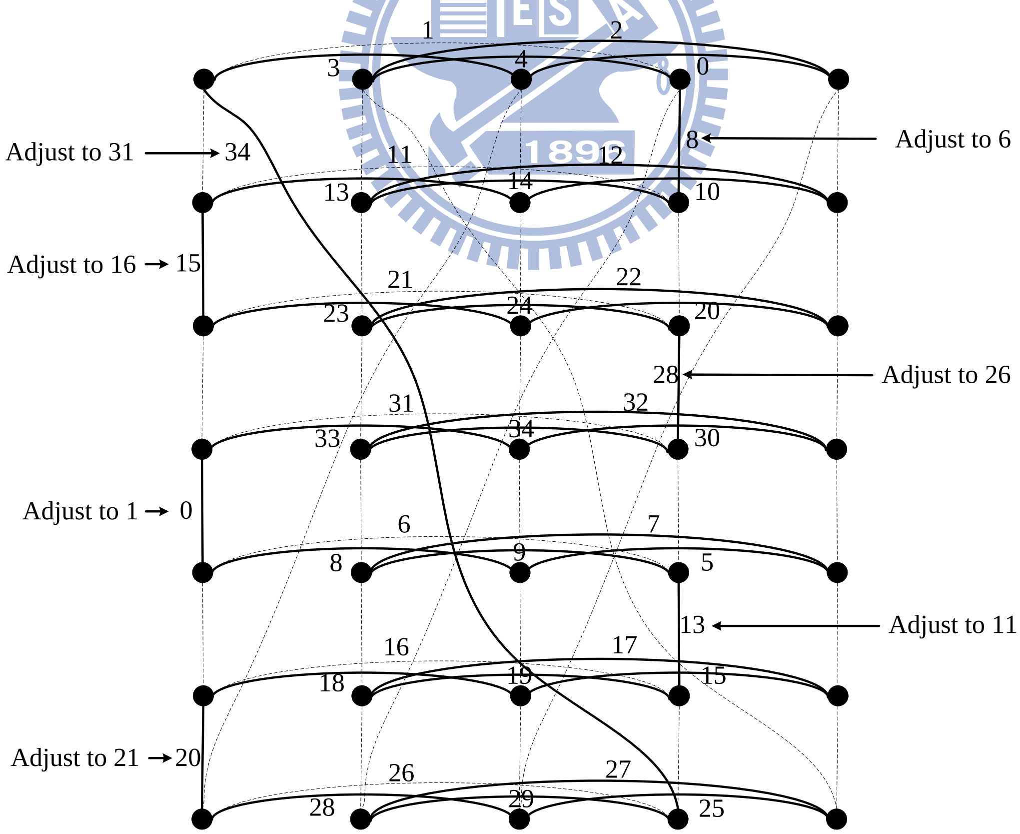


Figure 3.5: $E^{(2)} \cup 7D^{(2)}$ in K_{35}.

Chapter 4

Multicolored Spanning Trees in Edge-Colored Complete Graphs

In this chapter, we consider Conjecture 1.7.2 and Conjecture 1.7.8.

4.1 Isomorphic Multicolored Spanning Trees

Conjecture 1.7.8 states that for any arbitrary proper $(2m-1)$-edge-coloring of K_{2m}, it admits an MTP. We first consider a special proper edge-coloring of K_{2m} with $2m-1$ colors such that for any two colors form an C_4-factor. This kind of edge-coloring is referred to as a C_4-*factor edge-coloring*.

4.1.1 MST for C_4-factor edge-colored K_{2m}

Let L be the 2-group latin square defined earlier in Chapter 1.5. In what follows, we show that $L^n = L \times L \times \cdots \times L$ based on ${\mathbb{Z}_2}^n$ has 2^n disjoint transversals for each $n \geq 2$.

Proposition 4.1.1. L^n has 2^n disjoint transversals for each $n \geq 2$.

Proof. The proof is by induction on n. By Figure 4.1, $n = 2$ is true.

Assume that the assertion is true for each $k \geq 2$. Let $L^k = [{l_{a,b}}^{(k)}]$ and $L^{k+1} = \begin{array}{|c|c|}\hline {L_0}^k & {L_1}^k \\ \hline {L_1}^k & {L_0}^k \\ \hline \end{array}$. By definition of direct product, we have ${L_0}^k = [m_{a,b}]$ where $m_{a,b} = (0\ ,\ {l_{a,b}}^{(k)})$ (a $(k+1)$-dim. vector) and ${L_1}^k = [\overline{m}_{a,b}]$ where $\overline{m}_{a,b} = (1\ ,\ {l_{a,b}}^{(k)})$. We shall use the set of 2^k disjoint transversals in L^k to construct 2^{k+1} disjoint transversals in L^{k+1}.

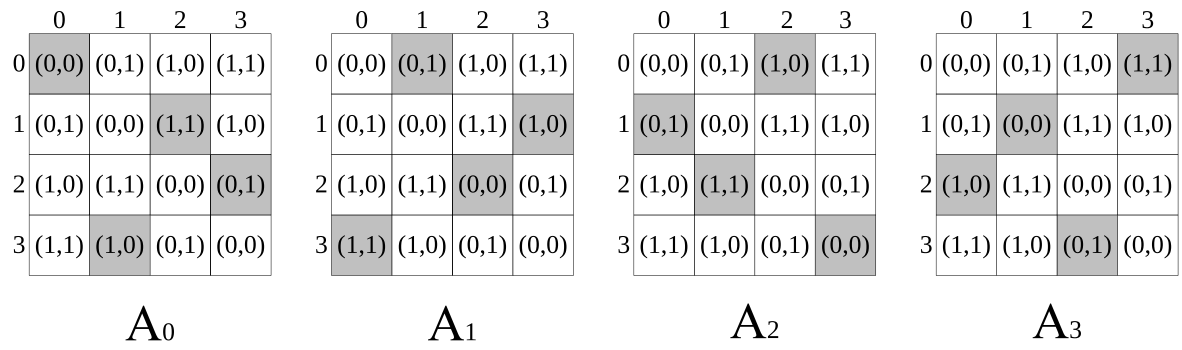


Figure 4.1: 4 transversals in L^2.

Let $\{A_i \mid i = 0, 1, 2, \cdots, 2^k - 1\}$ be the set of disjoint transversals obtained in L^k by induction hypothesis. Without loss of generality, we may let A_i be the transversal which contains the entry ${l_{0,i}}^{(k)}$, $i = 0, 1, 2, \cdots, 2^k - 1$. Now, we shall use A_{2i} and A_{2i+1}, $i = 0, 1, 2, \cdots, 2^{k-1} - 1$, to construct four disjoint transversals in L^{k+1}. For convenience, we explain the construction by using A_0 and A_1.

Since A_0(respectively A_1) is a transversal in L^k, the corresponding entries in ${L_0}^k$ form a transversal, so are the corresponding entries in ${L_1}^k$. Let the corresponding transversals of A_0 in ${L_0}^k$ and ${L_1}^k$ be $\overline{A}_{0,0}$ and $\overline{A}_{1,0}$ respectively. Similarly, let the corresponding transversals of A_1 be $\overline{A}_{0,1}$ and $\overline{A}_{1,1}$ respectively. Note that for $0 \le r, s \le 1$, $\overline{A}_{r,s}$ has 2^k entries, one from each row and from each column. Now, for $0 \le r, s \le 1$, we split $\overline{A}_{r,s}$ into two parts: ${\overline{A}_{r,s}}^{(u)}$ is the set of entries from the first to the 2^{k-1}-th row of $\overline{A}_{r,s}$, and ${\overline{A}_{r,s}}^{(l)}$ is the set of entries of the other half. By defining B_0, B_1, B_2 and B_3 as in Figure 4.2, we have four transversals in L^{k+1} as desired.

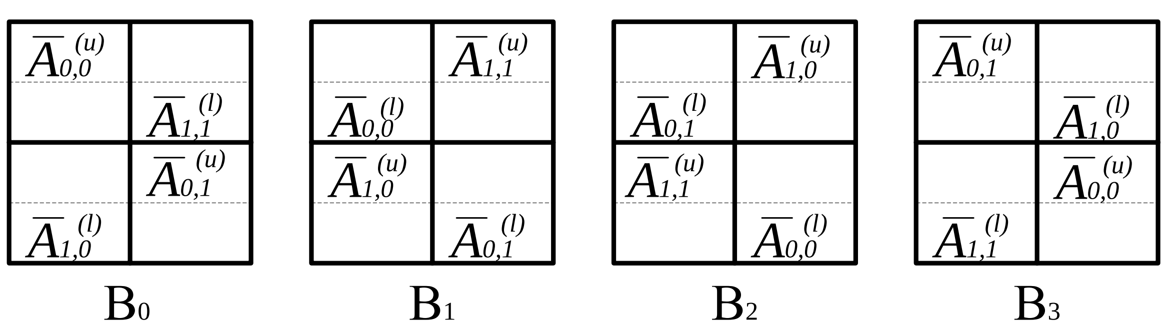


Figure 4.2: 4 transversals in L^{k+1} constructed from A_0 and A_1.

Since for $i = 1, 2, \cdots, 2^{k-1}-1$, $\overline{A}_{2i}$ and $\overline{A}_{2i+1}$ can also be used to construct four transversals in L^{k+1}, we have a set of 2^{k+1} transversals in L^{k+1}. By the reason that $A_0, A_1, \cdots, A_{2^k-1}$ are disjoint transversals, we conclude the proof. ∎

Before the following lemma, we review the notation $\mu|_K$. Let μ be a k-edge-coloring of a graph G. If K is a subgraph of G, for convenience, we use $\mu|_K$ to denote the edge-coloring of K induced by μ, i.e., $\mu|_K(e) = \mu(e)$ for each $e \in E(K)$.

Lemma 4.1.2. *Let μ be a proper $(2m-1)$-edge-coloring of K_{2m}, $m \geq 2$, such that any two colors induce a 2-factor with each component a 4-cycle, then* (a) $2m = 2^n$ *for some* $n \geq 2$ *and* (b) K_{2m} *contains a clique K of order 2^k, $1 \leq k \leq n-1$ such that $\{\mu(e) \mid e \in E(K)\}$ is a (2^k-1)-set, i.e., $\mu|_K$ is a proper (2^k-1)-edge-coloring of K.*

Proof. First, we claim that (b) is true. The proof is by induction on n. Clearly, it is true when $n = 2$. By hypothesis, let H be a clique of order 2^h, $h < k$, and $\mu|_H$ is a proper (2^h-1)-edge-coloring of H. Without loss of generality, let $V(H) = \{x_1, x_2, \cdots, x_{2^h}\}$ and the colors used in H be $\{c_1, c_2, \cdots, c_{2^h-1}\}$. Since μ is a proper $(2m-1)$-edge-coloring of K_{2m}, each color occurs around each vertex. Let c_{2^h} be a color not used in H. Then, we have a set H', $H' \cap H = \phi$, $H' = \{y_1, y_2, \cdots, y_{2^h}\}$ such that $\mu(x_i y_i) = c_{2^h}$ for $i = 1, 2, \cdots, 2^h$. Now, by the reason that any two colors induce a C_4-factor, we conclude that $\mu|_{H'}$ is also a proper (2^h-1)-edge-coloring of H', moreover, $\mu(x_i x_j) = \mu(y_i y_j)$ for $1 \leq i \neq j \leq 2^h$. Therefore, the complete bipartite graph $K_{2^h,2^h} = (H, H')$ has a proper 2^h-edge-coloring following by the same reason. This implies that $\mu|_{H \cup H'}$ is a proper $(2^{h+1}-1)$-edge-coloring of the clique induced by $H \cup H'$. So, we have the proof of (b).

Suppose $2m = 2^r \cdot p$ where p is an odd integer and $p \neq 1$. Using the above argument, we can find the largest clique G of order 2^s which uses $2^s - 1$ colors. Then we partition the vertices of K_{2m} into two sets X and Y where $X = V(G)$, and let $|Y| = q$. Here, we notice that $q < 2^s$. Consider these $2^s - 1$ colors used in coloring the edges of G, there are total $(2^s - 1)(2^{r-1} \cdot p)$ edges which use these colors. But, we have used these colors in G.

Hence, there remains $\frac{1}{2}(2^s-1)(2m-2^s)$ edges to be colored by using these colors. Since the edges between X and Y can not be colored with any of these colors, they have to be in Y. But, since $q < 2^s$ and $2m - 2^s = q$, $\frac{1}{2}(2^s-1)(2m-2^s) > \binom{q}{2}$, a contradiction. This implies that $p = 1$, and we have the proof of (a). ∎

Lemma 4.1.3. [10] *Let μ be a proper 7-edge-coloring of K_8 such that for any two colors form a C_4-factor. Then the edges of K_8 can be partitioned into 4 isomorphic multicolored spanning trees.*

We are ready to tackle the C_4-factor edge-coloring problem.

Theorem 4.1.4. *Let μ be a proper $(2m-1)$-edge-coloring of K_{2m}, $m > 2$, such that any two colors form an C_4-factor, the edges of K_{2m} can be partitioned into m isomorphic multicolored spanning trees.*

Proof. By Lemma 4.1.2, $2m = 2^n$ for some $n > 2$. We prove the theorem by induction on n. By Lemma 4.1.3, $n = 3$ is true.

Assume that the assertion is true for each $k \geq 3$ and consider $K_{2^{k+1}}$.

From the process of the proof of Lemma 4.1.2, there must exist two disjoint cliques of order 2^k with $2^k - 1$ colors in $K_{2^{k+1}}$. Let $V(K_{2^{k+1}}) = A \cup B$ where A, B are the vertex sets of the two cliques. Consider the colors of the edges between A and B. Let $A = \{a_0, a_1, \ldots, a_{2^k-1}\}$, $B = \{b_0, b_1, \ldots, b_{2^k-1}\}$ and $M = [m_{i,j}]$ where $m_{i,j} = \mu(a_i b_j)$. It is clear that M is a latin square; furthermore, $M \cong L^k$. By Proposition 4.1.1, M has 2^k disjoint transversals. This implies that there are 2^k perfect matchings in the complete bipartite graph induced by $A \cup B$. Note that the two cliques induced by A and B respectively have 2^{k-1} multicolored isomorphic spanning trees of order 2^k, respectively. Thus, by assigning a perfect matching to each spanning tree, we obtain 2^k spanning trees of order 2^{k+1}. Moreover, these spanning trees are isomorphic and multicolored. ∎

4.1.2 Main Results

For the presentation of the proof of our main theorem, we review that the notation $T[x_1, x_2]$ is a new tree modified from T, where T is a multicolored spanning tree in a properly edge-colored K_{2m} and x_1, x_2 are two leaves. At first, we show the existence of two disjoint isomorphic multicolored spanning trees.

Lemma 4.1.5. *Let φ be an arbitrary proper $(2m-1)$-edge-coloring of K_{2m}. Then there exist two disjoint isomorphic multicolored spanning trees in K_{2m} for $m \geq 3$.*

Proof. Let $V(K_{2m}) = \{x_i |\ i = 1, 2, \ldots, 2m\}$. We split the proof into two cases.

Case 1. There exists a 4-cycle (x_1, x_2, x_3, x_4) such that $\varphi(x_1x_2) = b$, $\varphi(x_3x_4) = c$, and $\varphi(x_1x_4) = \varphi(x_2x_3) = a$. Let $T_1 = S_{x_1}[x_2, x_4]$ and $T_2 = S_{x_2}[x_1, x_3]$, see Figure 4.3. Clearly, they are the desired spanning trees.

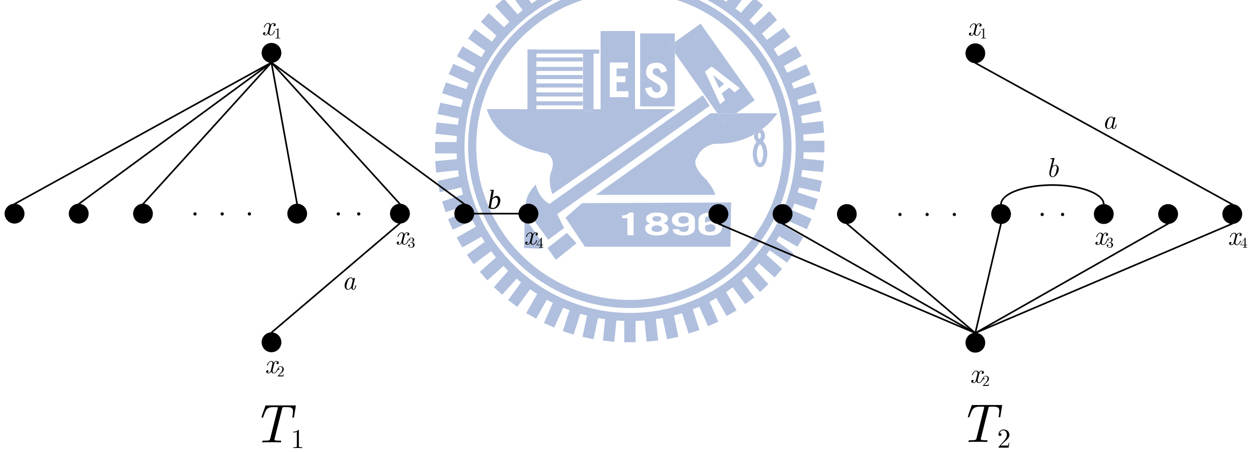


Figure 4.3: Two isomorphic spanning trees of Case 1.

Case 2. If any two colors of this edge-coloring induce a C_4-decomposition of K_{2m}, then we have the proof by Theorem 4.1.4. ∎

Review that if φ is a proper $(2m-1)$-edge-coloring of K_{2m} and C is the color set, φ_x is a bijective mapping from $V(K_{2m})\backslash\{x\}$ to C. Hence, φ_x^{-1} is defined accordingly. For a vertex set $V \in V(K_{2m})$ and a color $c \in C$, in addition, let $[V]_c = V \cup \{u|\ \varphi(uv) = c, v \in V\}$. Now, we are ready for the main result.

Theorem 4.1.6. *Let φ be an arbitrary proper $(2m-1)$-edge-coloring of K_{2m}. Then there exist three disjoint isomorphic multicolored spanning trees in K_{2m} for $m \geq 14$.*

Proof. From the proof of Lemma 4.1.5, we only need to consider the case: there exist two colors which do not induce a 4-cycle factor. Let T_1 and T_2 be the isomorphic multicolored spanning trees obtained in Lemma 4.1.5. Clearly, $K_{2m} - T_1 - T_2$ is disconnected ($\{x_1, x_2\}$ induces a component in this graph). Let $\varphi_{x_3}^{-1}(b) = y_1$, $\varphi_{x_4}^{-1}(b) = y_2$ and $U = V(K_{2m}) - \{x_1, x_2, x_3, x_4, y_1, y_2\}$. Since $m \geq 14$, we can choose a vertex $u \in U$ such that the two colors $\varphi(ux_1)$ and $\varphi(ux_2)$ are different from those colors on the edges of the graph induced by the vertex set $\{x_1, x_2, x_3, x_4\}$. Without loss of generality, let $\varphi(ux_1) = 1$ and $\varphi(ux_2) = 2$. Moreover, let $v_1 \in U \setminus \{u\}$ and $\varphi(x_1v_1) = 3$ such that $\varphi_{v_1}^{-1}(b) \neq \varphi_{x_4}^{-1}(1)$ and the two vertices $\varphi_u^{-1}(3)$ and $\varphi_{v_1}^{-1}(1)$ are elements in $U \setminus \{u\}$. Now, pick $v_2 \in U \setminus \{u, v_1, \varphi_{v_1}^{-1}(b)\}$ and let $\varphi(x_2v_2) = 4$ such that $\varphi_{v_2}^{-1}(b) \neq \varphi_{x_3}^{-1}(2)$ and the two vertices $\varphi_u^{-1}(4)$ and $\varphi_{v_2}^{-1}(2)$ are elements in set $U \setminus \{u\}$. Note that we can always pick v_1 and v_2 consecutively since $m \geq 14$.

Let $T_1' = T_1[u, v_1]$ and $T_2' = T_2[u, v_2]$. Assume that $\varphi_u^{-1}(3) = u_1$ and $\varphi_u^{-1}(4) = u_2$. If $u_1 = \varphi_{v_1}^{-1}(1)$, then adjust T_1' to $T_1'[v_1, x_4]$. Similarly, if $u_2 = \varphi_{v_2}^{-1}(2)$, then adjust T_2' to $T_2'[v_2, x_3]$. Then T_1' and T_2' both have two types. In either case, they are disjoint and isomorphic. Figure 4.4 shows the types of T_1'.

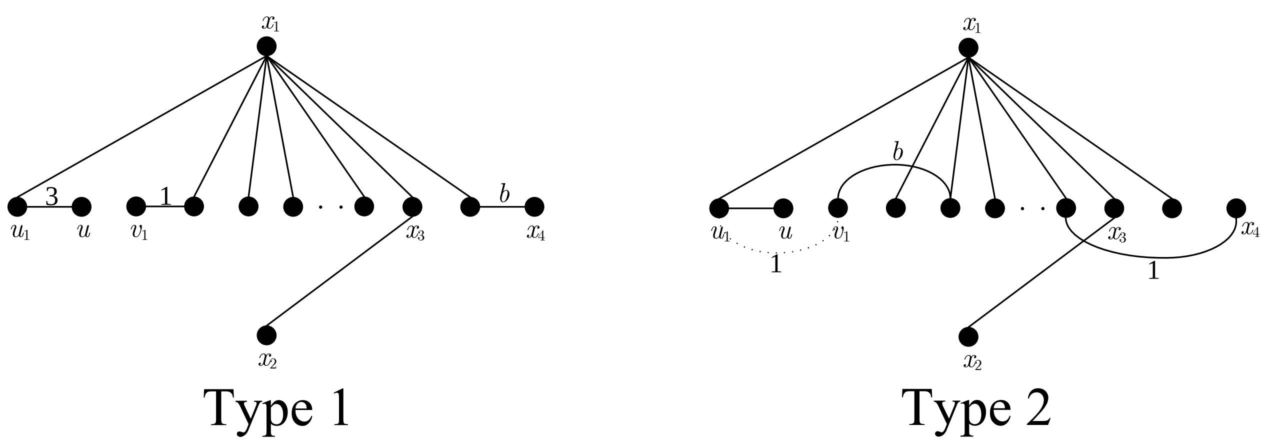


Figure 4.4: Two types of T_1'.

Now, we are ready to construct the third tree. Let T_3 be the graph $S_u[u_1, u_2]$. Then choose one edge w_1w_2 with color 3 in the graph induced by $V(K_{2m})\backslash\{x_1, x_2, u, u_2\}$ and assume $\varphi(uw_1) = c_1$, $\varphi(uw_2) = c_2$. Let $W = \{x_1, x_2, u_1, \varphi_{u_1}^{-1}(4), w_1, w_2\}$. Since $m \geq 14$, there exists one color, c_r, such that $\varphi_{u_2}^{-1}(c_r) \notin W$ and $\varphi_u^{-1}(c_r) \notin [W]_{c_1} \cup [W]_{c_2}$. Let $\varphi_{u_2}^{-1}(c_r) = z_1$ and $\varphi_u^{-1}(c_r) = z_2$. Since $\varphi(z_1z_2)$ may be c_1 or c_2, we assume $\varphi(z_1z_2) \neq c_1$. Finally, let T_3' be obtained from T_3 by removing the edges $u_2\langle 3\rangle, u\langle c_1\rangle, u\langle c_r\rangle$ and then adding the edges $u_2\langle c_r\rangle, w_1\langle 3\rangle, z_2\langle c_1\rangle$. Thus, the third spanning tree is constructed, see Figure 4.5. Since all spanning trees contain exactly four vertices which are of distance 2 from vertices x_1, x_2 and u respectively, they are isomorphic. This concludes the proof. ∎

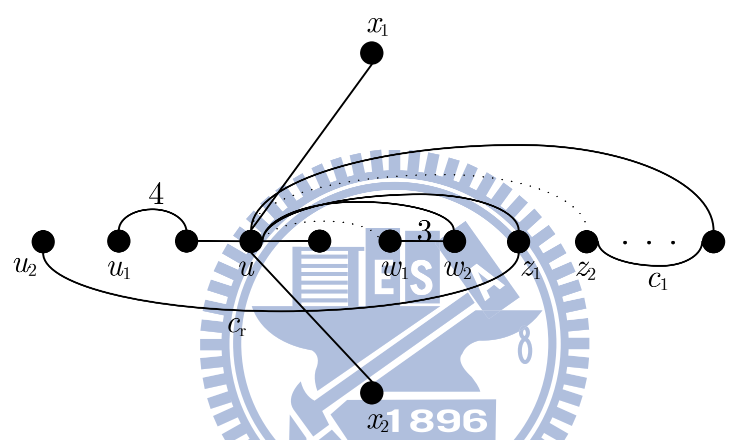


Figure 4.5: T_3'.

We note here that the result obtained (jointly with H. L. Fu) in this section has been included in [22].

4.2 Multicolored Spanning Trees

In this section, we consider Conjecture 1.7.2, the original problem of this topic.

4.2.1 Recursive Construction

We start with notations which will be used throughout this section. Let φ be a proper $(2m-1)$-edge-coloring of K_{2m} and $C = \{c_1, c_2, \cdots, c_{2m-1}\}$ be the color set. Suppose T is a multicolored spanning tree of K_{2m} and x is a root of T. Clearly, if x is incident to two leaves $e_1 = xy_1$ and $e_2 = xy_2$, i.e., the degree of y_1 and y_2 in T, $deg_T(y_1) = deg_T(y_2) = 1$, then $T[x; y_1, y_2; z_1, z_2] = T - e_1 - e_2 + y_1z_1 + y_2z_2$ is a spanning tree of K_{2m} for some vertices z_1 and z_2 (z_1 may be the same as z_2). Furthermore, if $\varphi(e_1) = \varphi(y_2z_2)$ and $\varphi(e_2) = \varphi(y_1z_1)$, then $T[x; y_1, y_2; z_1, z_2]$ is also a multicolored spanning tree of K_{2m} with root x. For convenience, we say $T[x; y_1, y_2; z_1, z_2]$ is obtained from T by using a (y_1, y_2)-*switch operation* on T. We note here that $T[x; y_1, y_2; z_1, z_2]$ and $T[y_1, y_2]$ in Section 4.1 are the same thing.

We shall apply a recursive construction to obtain edge-disjoint multicolored spanning trees in an edge-colored K_{2m}. Since those previously obtained spanning trees will be revised before we find a new one, we use $T_j^{(i)}$ to denote the j^{th} spanning tree which was constructed at round i of the recursive construction. That is to say, in order to construct the $(k+1)^{th}$ tree at $(k+1)^{th}$ round, we first revise the k spanning trees $T_1^{(k)}, T_2^{(k)}, \cdots, T_k^{(k)}$ to obtain $T_1^{(k+1)}, T_2^{(k+1)}, \cdots, T_k^{(k+1)}$ respectively and then define the new one $T_{k+1}^{(k+1)}$ accordingly. As a matter of fact, $T_j^{(k+1)} = T_j^{(k)}[w_j; y', y''; z', z'']$ where w_j is the root of the j^{th} spanning tree and y', y'', z', z'' are suitably chosen to meet the requirements prescribed.

For clearness, we use a properly 27-edge-colored K_{28} as an example to outline the idea of our construction. Let K_{28} be defined on $\{x_i | \ i \in \mathbb{Z}_{28}\}$, φ be a proper 27-edge-coloring of K_{28}, and the entry in ith row and jth column of the 28×28 coloring array be the color of the edge x_ix_j, $\varphi(x_ix_j)$, see Figure 4.6. Let the first spanning tree be the spanning star $T_1^{(1)} = S_{x_1}$ with root x_1. Clearly, $T_1^{(1)}$ is multicolored. In order to achieve a better result and obtain a corresponding corollary in finding edge-disjoint multicolored spanning unicyclic graphs in a properly $(2m-1)$-edge-colored K_{2m-1} (next chapter), we shall enforce x_0 to be a pendent vertex of each spanning tree which is incident to the root. Therefore, x_0 will not be a candidate of roots. For convenience, we let U_i be the set of candidates of roots in constructing the $(i+1)^{th}$ spanning tree.

Now, we are ready to find the second spanning tree. First, we revise $T_1^{(1)}$. As mentioned above, $T_1^{(2)} = T_1^{(1)}[x_1; y, v_1; u_1, v_1']$ for some vertices y, v_1, u_1, v_1' in $\mathbb{Z}_{28} \setminus \{x_0, x_1\}$. At this step, since $U_1 = \mathbb{Z}_{28} \setminus \{x_0, x_1\}$, we let $x_2 = y$ be the root of the second tree. So, it is left to find v_1 for the (x_2, v_1)-switch operation of $T_1^{(1)}$. Notice that we have to make sure that $T_1^{(2)}$ is also a multicolored spanning tree, i.e., after we choose v_1, u_1 and v_1', we have $\varphi(x_1x_2) = \varphi(v_1v_1')$ and $\varphi(x_1v_1) = \varphi(x_2u_1)$. Observe that from the coloring of K_{28} we have $\varphi(x_0x_2) = 2$ and $\varphi(x_1x_2) = 15$. Hence, in the search of v_1, $\varphi(v_1x_1) \neq 2$ and $\varphi(v_1x_0) \neq 15$, for otherwise $deg_{T_1^{(2)}}(x_0) \neq 1$. On the other hand, pick $u \in U_1 \setminus \{x_2\}$. Notice that $T_2^{(2)}$ will be obtained by using an (u, u_1)-switch operation of S_{x_2}. In this case, let $x_3 = u$. Since $\varphi(x_0x_3) = 3$, we have $\varphi(x_1v_1) \neq 3$, for otherwise $deg_{T_2^{(2)}}(x_0) \neq 1$. Furthermore, since $\varphi(x_2x_3) = 16$, $\varphi(x_1v_1) \neq 16$. Finally, if $\varphi(x_0\alpha) = \varphi(x_1\beta) = 16$, then $\varphi(x_1v_1) \neq \varphi(x_2\alpha)$ and $\varphi(x_1v_1) \neq \varphi(x_2\beta)$. This is by the reason that $T_2^{(2)}$ contains neither u_1x_0 nor u_1x_1. Thus, we conclude that v_1 can not be one of $x_3, x_4, x_5, x_{15}, x_{16}$. Therefore, choose $x_6 = v_1$ and then let $T_1^{(2)} = T_1^{(1)}[x_1; x_2, x_6; x_5, x_{24}]$, $T_2^{(2)} = S_{x_2} - x_2x_3 - x_2x_5 + x_3x_4 + x_5x_{27}$. This concludes the 2nd round. Figure 4.7 shows the structure of these two trees.

In the third round, we revise $T_1^{(2)}$ and $T_2^{(2)}$ consecutively and then construct a third tree. Notice that $U_2 = U_1 \setminus (\{x_2\} \cup \{x_3, x_4, x_5, x_6, x_{24}, x_{27}\})$. Precisely, we will first pick $y \in U_2$ as the root of the third tree and revise $T_1^{(3)} = T_1^{(2)}[x_1; y, v_1; u_1, v_1']$, $T_2^{(3)} = T_2^{(2)}[x_2; y, v_2; u_2, v_2']$ consecutively for some suitable vertices $v_1, v_2, u_1, \cdots$. Then, we obtain $T_3^{(3)}$ from S_y by deleting edges yu, yu_1, yu_2 and adding edges uu', u_1u_1', u_2u_2' for some vertices u, u', u_1', u_2' so that $\varphi(uu') = \varphi(yu_2)$, $\varphi(u_1u_1') = \varphi(yu)$ and $\varphi(u_2u_2') = \varphi(yu_1)$. Note here that the two vertices y and u, both in U_2, are assigned at the beginning of this round, namely, $x_7 = y$ and $x_8 = u$. Then, the next step is to find $v_1 \in U_2$ for the (x_7, v_1)-switch operation of $T_1^{(2)}$. From the coloring of K_{28} we have $\varphi(x_0x_7) = 7$, $\varphi(x_0x_8) = 8$, $\varphi(x_1x_7) = 4$, $\varphi(x_2x_7) = 18$ and $\varphi(x_7x_8) = 21$. Then, in the search of v_1, $\varphi(v_1x_0) \neq 4$ and $\varphi(v_1x_1) \neq 7$, for otherwise $deg_{T_1^{(3)}}(x_0) \neq 1$. In addition, since we have to make sure that $T_1^{(3)}$ is edge-disjoint to the other two trees, $\varphi(v_1x_1) \neq 18$ and $\varphi(v_1x_2) \neq 4$. (Though the edge x_2x_7 will disappear in $T_2^{(3)}$, it appears in $T_3^{(3)}$.) On the other hand, the edge x_7u_1 will be dropped away and u_1u_1' will be included in $T_3^{(3)}$

where $\varphi(u_1u_1') = 21$. Since $u \neq u_1$, we have that $\varphi(v_1x_1) \neq 21$. Furthermore, $u_1' = x_0$ yields that $deg_{T_3^{(3)}}(x_0) \neq 1$, and $u_1' \in \{x_1, x_2\}$ implies that $T_3^{(3)}$ is not edge-disjoint to the other two trees. So, if $\varphi(x_0\alpha) = \varphi(x_1\beta) = \varphi(x_2\gamma) = 21$, then $\varphi(x_1v_1) \neq \varphi(x_7\alpha)$, $\varphi(x_1v_1) \neq \varphi(x_7\beta)$ and $\varphi(x_1v_1) \neq \varphi(x_7\gamma)$. Finally, since u_1u_1' can not be an edge in $T_1^{(2)}$ or $T_2^{(2)}$, $\varphi(v_1x_1) \neq \varphi(x_7z)$ where z is an endpoint of an edge with color 21 in these two trees. By the reasons mentioned above, $v_1 \notin \{x_4, x_6, x_7, x_8, x_{13}, x_{14}, x_{19}, x_{20}, x_{27}\}$. Hence, choose $x_9 = v_1$ and then let $T_1^{(3)} = T_1^{(2)}[x_1; x_7, x_9; x_3, x_{26}]$. ($x_3 = u_1$, $x_{26} = v_1'$ and $x_{12} = u_1'$.)

Next, we have to find $v_2 \in U_2$ for the (x_7, v_2)-switch operation of $T_2^{(2)}$. Similarly, we have to restrict the candidates of v_2 in order to achieve our goal. Since $deg_{T_2^{(3)}} = 1$, we have $\varphi(v_2x_0) \neq 18$ and $\varphi(v_2x_2) \neq 7$. From the coloring of K_{28}, $\varphi(x_7x_3) = 5$. In order to make sure that $T_2^{(3)}$ and $T_1^{(3)}$ are disjoint, $\varphi(v_2x_2) \neq 4$ or 5 and $\varphi(v_2x_1) \neq 18$. Now, consider the edges which are going to appear in $T_3^{(3)}$. Notice that u_2, u_2', u' will be fixed once v_2 is chosen. From the construction of $T_3^{(3)}$, $u_2 \neq x_8$ implies that $\varphi(v_2x_2) \neq 21$. Next, $u' = x_0$ or $u_2' = x_0$ yield that $deg_{T_3^{(3)}}(x_0) \neq 1$. Therefore, $\varphi(v_2x_2) \neq 8$ as well as if $\varphi(x_0\alpha) = 5$ for some α, then $\varphi(v_2x_2) \neq \varphi(x_7\alpha)$. In addition, $\varphi(v_2x_2) \neq \varphi(x_1x_8)$ or $\varphi(x_7\beta)$ if $\varphi(x_2\beta) = 5$, for otherwise x_1x_8 or x_2u_2 will be added to $T_3^{(3)}$. Furthermore, u_2u_2' can not be an edge of $T_2^{(3)}$. Hence, $\varphi(v_2x_2) \neq \varphi(x_7\gamma)$ provided that γ is an endpoint incident to an edge in $T_2^{(3)}$ which is colored "5". Finally, we also have to make sure the three edges uu', u_1u_1', u_2u_2' do not form a cycle. Our strategy is to let $u_2 \neq x_{12} = u_1'$, which was fixed after choosing v_1. Thus, $\varphi(v_2x_2) \neq \varphi(x_7x_{12}) = 23$. To sum up, $v_2 \notin \{x_6, x_7, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{17}, x_{18}\}$. Hence, we take $x_{11} = v_2$ and then let $T_2^{(3)} = T_2^{(2)}[x_2; x_7, x_{11}; x_6, x_{25}]$ and $T_3^{(3)} = S_{x_7} - x_7x_8 - x_7x_3 - x_7x_6 + x_3x_{12} + x_6x_4 + x_8x_5$. So, we have three trees now. We illustrate the result of this round by showing the structure of these trees in Figure 4.8.

We may keep going to find the fourth tree as long as the followings are possible:

(1) $U_3 = U_2 \setminus \{x_7, x_8, x_9, x_{11}, x_{12}, x_{25}, x_{26}\}$ has two vertices y (root) and u available.

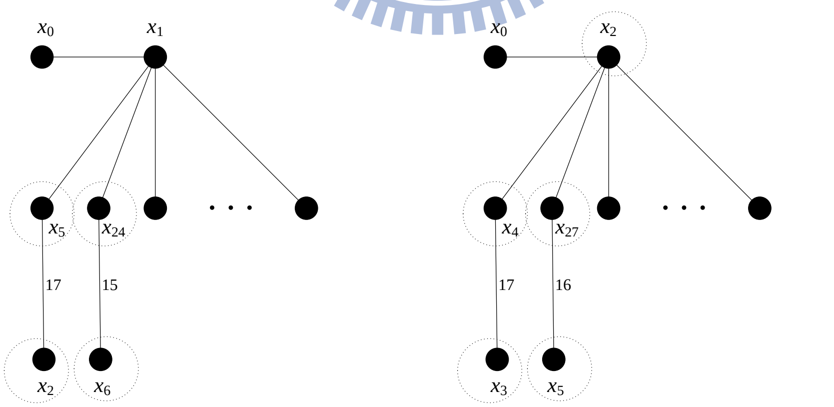
(2) There are suitable vertices $v_1, v_2, v_3, u_1, \cdots$, such that we can revise $T_1^{(3)}$, $T_2^{(3)}$ and $T_3^{(3)}$ into $T_1^{(4)}$, $T_2^{(4)}$ and $T_3^{(4)}$ consecutively and define $T_4^{(4)}$ accordingly.

Indeed, we are able to accomplish the above jobs (see Figure 4.9) by letting $y = x_{10}$, $u = x_{13}$, $v_1 = x_{13}$, $v_2 = x_{13}$ and $v_3 = x_{14}$. Therefore, we have four mutually edge-disjoint multicolored spanning trees in a 27-edge-colored K_{28}.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
x_0		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
x_1			15	2	16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14
x_2				16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1
x_3					17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15
x_4						18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2
x_5							19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16
x_6								20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3
x_7									21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17
x_8										22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4
x_9											23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18
x_{10}												24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18	5
x_{11}													25	12	26	13	27	14	1	15	2	16	3	17	4	18	5	19
x_{12}														26	13	27	14	1	15	2	16	3	17	4	18	5	19	6
x_{13}															27	14	1	15	2	16	3	17	4	18	5	19	6	20
x_{14}																1	15	2	16	3	17	4	18	5	19	6	20	7
x_{15}																	2	16	3	17	4	18	5	19	6	20	7	21
x_{16}																		3	17	4	18	5	19	6	20	7	21	8
x_{17}																			4	18	5	19	6	20	7	21	8	22
x_{18}																				5	19	6	20	7	21	8	22	9
x_{19}																					6	20	7	21	8	22	9	23
x_{20}																						7	21	8	22	9	23	10
x_{21}																							8	22	9	23	10	24
x_{22}																								9	23	10	24	11
x_{23}																									10	24	11	25
x_{24}																										11	25	12
x_{25}																											12	26
x_{26}																												13
x_{27}																												

Figure 4.6: A properly 27-edge-colored K_{28}.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
x_0		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
x_1			15	2	16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14
x_2				16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1
x_3					17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15
x_4						18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2
x_5							19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16
x_6								20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3
x_7									21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17
x_8										22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4
x_9											23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18
x_{10}												24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18	5
x_{11}													25	12	26	13	27	14	1	15	2	16	3	17	4	18	5	19
x_{12}														26	13	27	14	1	15	2	16	3	17	4	18	5	19	6
x_{13}															27	14	1	15	2	16	3	17	4	18	5	19	6	20
x_{14}																1	15	2	16	3	17	4	18	5	19	6	20	7
x_{15}																	2	16	3	17	4	18	5	19	6	20	7	21
x_{16}																		3	17	4	18	5	19	6	20	7	21	8
x_{17}																			4	18	5	19	6	20	7	21	8	22
x_{18}																				5	19	6	20	7	21	8	22	9
x_{19}																					6	20	7	21	8	22	9	23
x_{20}																						7	21	8	22	9	23	10
x_{21}																							8	22	9	23	10	24
x_{22}																								9	23	10	24	11
x_{23}																									10	24	11	25
x_{24}																										11	25	12
x_{25}																											12	26
x_{26}																												13
x_{27}																												

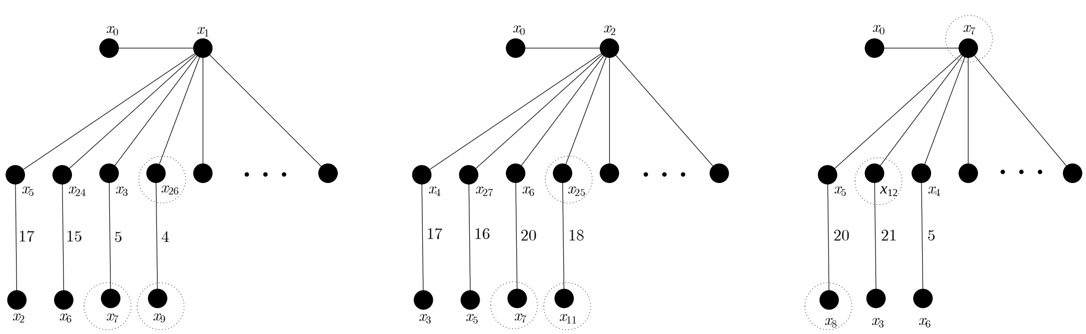


$T_1^{(2)} = T_1^{(1)}[x_1;\ x_2,\ x_6;\ x_5,\ x_{24}]$ $\quad$ $T_2^{(2)} = S_{x_2} - x_2x_3 - x_2x_5 + x_3x_4 + x_5x_{27}$

$$U_2 = U_1 \backslash (\{x_2\} \cup \{x_3, x_4, x_5, x_6, x_{24}, x_{27}\}) = \{x_7, x_8, x_9, x_{10}, \cdots, x_{23}, x_{25}, x_{26}\}$$

Figure 4.7: Two edge-disjoint multicolored spanning trees.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
x_0		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
x_1			15	2	16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14
x_2				16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1
x_3					17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15
x_4						18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2
x_5							19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16
x_6								20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3
x_7									21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17
x_8										22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4
x_9											23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18
x_{10}												24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18	5
x_{11}													25	12	26	13	27	14	1	15	2	16	3	17	4	18	5	19
x_{12}														26	13	27	14	1	15	2	16	3	17	4	18	5	19	6
x_{13}															27	14	1	15	2	16	3	17	4	18	5	19	6	20
x_{14}																1	15	2	16	3	17	4	18	5	19	6	20	7
x_{15}																	2	16	3	17	4	18	5	19	6	20	7	21
x_{16}																		3	17	4	18	5	19	6	20	7	21	8
x_{17}																			4	18	5	19	6	20	7	21	8	22
x_{18}																				5	19	6	20	7	21	8	22	9
x_{19}																					6	20	7	21	8	22	9	23
x_{20}																						7	21	8	22	9	23	10
x_{21}																							8	22	9	23	10	24
x_{22}																								9	23	10	24	11
x_{23}																									10	24	11	25
x_{24}																										11	25	12
x_{25}																											12	26
x_{26}																												13
x_{27}																												



$T_1^{(3)} = T_1^{(2)}[x_1;\ x_7,\ x_9;\ x_3,\ x_{26}]$

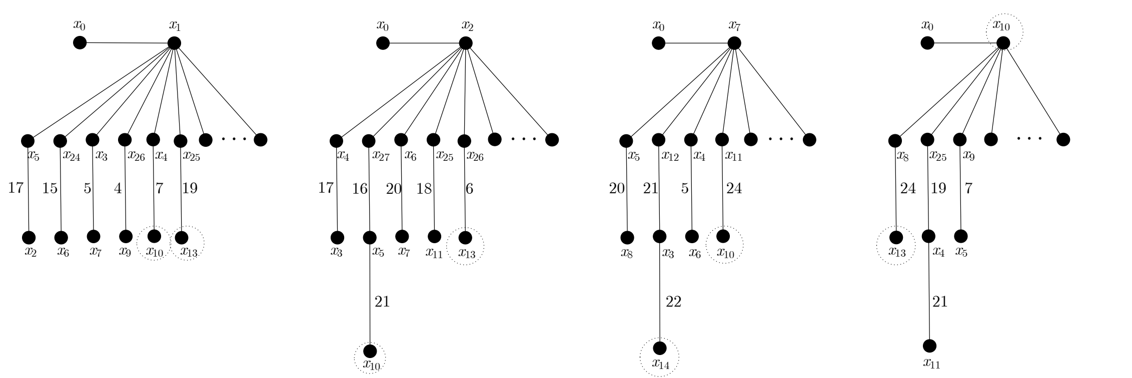
$T_2^{(3)} = T_2^{(2)}[x_2;\ x_7,\ x_{11};\ x_6,\ x_{25}]$

$T_3^{(3)} = S_{x_7} - x_7x_8 - x_7x_3 - x_7x_6 + x_3x_{12} + x_6x_4 + x_8x_5$

$U_3 = U_2 \backslash (\{x_7\} \cup \{x_8, x_9, x_{11}, x_{12}, x_{25}, x_{26}\}) = \{x_{10}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}\}$

Figure 4.8: Three edge-disjoint multicolored spanning trees.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
x_0		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
x_1			15	2	16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14
x_2				16	3	17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1
x_3					17	4	18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15
x_4						18	5	19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2
x_5							19	6	20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16
x_6								20	7	21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3
x_7									21	8	22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17
x_8										22	9	23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4
x_9											23	10	24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18
x_{10}												24	11	25	12	26	13	27	14	1	15	2	16	3	17	4	18	5
x_{11}													25	12	26	13	27	14	1	15	2	16	3	17	4	18	5	19
x_{12}														26	13	27	14	1	15	2	16	3	17	4	18	5	19	6
x_{13}															27	14	1	15	2	16	3	17	4	18	5	19	6	20
x_{14}																1	15	2	16	3	17	4	18	5	19	6	20	7
x_{15}																	2	16	3	17	4	18	5	19	6	20	7	21
x_{16}																		3	17	4	18	5	19	6	20	7	21	8
x_{17}																			4	18	5	19	6	20	7	21	8	22
x_{18}																				5	19	6	20	7	21	8	22	9
x_{19}																					6	20	7	21	8	22	9	23
x_{20}																						7	21	8	22	9	23	10
x_{21}																							8	22	9	23	10	24
x_{22}																								9	23	10	24	11
x_{23}																									10	24	11	25
x_{24}																										11	25	12
x_{25}																											12	26
x_{26}																												13
x_{27}																												



$T_1^{(4)} = T_1^{(4)}[x_1; x_{10}, x_{13}; x_4, x_{25}]$ $T_2^{(4)} = T_2^{(3)}[x_2; x_{10}, x_{13}; x_5, x_{26}]$ $T_3^{(4)} = T_3^{(3)}[x_7; x_{10}, x_{14}; x_{11}, x_3]$ $T_4^{(4)} = S_{x_{10}} - x_{10}x_{13} - x_{10}x_4 - x_{10}x_5 - x_{10}x_{11} + x_4x_{19} + x_5x_9 + x_{11}x_4 + x_{13}x_8$

$U_4 = U_3 \backslash (\{x_{10}\} \cup \{x_{13}, x_{14}, x_{19}\} = \{x_{15}, x_{16}, x_{17}, x_{18}, x_{20}, x_{21}, x_{22}, x_{23}\}$

Figure 4.9: Four edge-disjoint multicolored spanning trees.

Now, we are ready for the recursive construction. Given a proper $(2m-1)$-edge-coloring φ of K_{2m}, we start with $n = 2$, $T_1^{(1)} = S_{x_1}$, and $U_1 = \{x_2, x_3, \cdots, x_{2m-1}\}$, $R_1 = \{x_0, x_1\}$. Note here we use R_n to denote the collection of x_0 and all roots of $T_i^{(n)}$ for $i \in [n]$.

Step 1. (Checking initial value)

If $|U_{n-1}| \geq 9n - 14$, go to Step 2; otherwise, break the recursive construction.

Step 2. (Choosing x_{i_n})

Pick $x_{i_n}, u_0 \in U_n$, where x_{i_n} is the root of nth tree.

Step 3. (Revising $T_1^{(n-1)}, T_2^{(n-1)}, \ldots, T_{n-1}^{(n-1)}$ consecutively)

3.1 Choose $v_1 \in U_{n-1} \backslash \{x_{i_n}\}$ and let $T_1^{(n)} = T_1^{(n-1)}[x_1; x_{i_n}, v_1; u_1, v_1']$ where $\varphi(v_1 v_1') = \varphi(x_1 x_{i_n})$ and $\varphi(u_1 x_{i_n}) = \varphi(x_1 v_1)$ such that (i) $u_1 \notin \{u_0\} \cup (R_{n-1} \setminus \{x_1\})$, (ii) $v_1' \notin R_{n-1} \setminus \{x_1\}$, (iii) $u_1' \notin R_{n-1}$ where $\varphi(u_1' u_1) = \varphi(x_{i_n} u_0)$, and (iv) the edge $u_1 u_1'$ can not appear in $T_i^{(n-1)}$ for all $i \in [n-1]$.

3.2 For $t \leftarrow 2$ to $n-2$, do {

Choose $v_t \in U_{n-1} \backslash \{x_{i_n}\}$ and let $T_t^{(n)} = T_t^{(n-1)}[x_{i_t}; x_{i_n}, v_t; u_t, v_t']$ where $\varphi(v_t v_t') = \varphi(x_{i_t} x_{i_n})$ and $\varphi(u_t x_{i_n}) = \varphi(x_{i_t} v_t)$ such that (i) $u_t \notin \{u_0, u_1, \cdots, u_{t-1}\} \cup (R_{n-1} \setminus \{x_{i_t}\})$, (ii) $v_t' \notin R_{n-1} \setminus \{x_{i_t}\}$, (iii) $u_t' \notin R_{n-1} \setminus \{x_{i_{t-1}}\}$ where $\varphi(u_t' u_t) = \varphi(x_{i_n} u_{t-1})$, (iv) the edge $u_t u_t'$ can not appear in $T_i^{(n-1)}$ for all $i \in [n-1]_{t-1}$, and (v) $u_t \notin \{u_1', \cdots, u_{t-1}'\}$. }

3.3 Choose $v_{n-1} \in U_{n-1} \setminus \{x_{i_n}\}$ and let $T_{n-1}^{(n)} = T_{n-1}^{(n-1)}[x_{i_{n-1}}; x_{i_n}, v_{n-1}; u_{n-1}, v_{n-1}']$ where $\varphi(v_{n-1} v_{n-1}') = \varphi(x_{i_{n-1}} x_{i_n})$ and $\varphi(u_{n-1} x_{i_n}) = \varphi(x_{i_{n-1}} v_{n-1})$ such that (i) $u_{n-1} \notin \{u_0, u_1, \cdots, u_{n-2}\} \cup (R_{n-1} \setminus \{x_{i_{n-1}}\})$, (ii) $v_{n-1}' \notin R_{n-1} \setminus \{x_{i_{n-1}}\}$, (iii) $u_{n-1}' \notin R_{n-1} \setminus \{x_{i_{n-2}}\}$ where $\varphi(u_{n-1}' u_{n-1}) = \varphi(x_{i_n} u_{n-2})$, (iv) the edge $u_{n-1} u_{n-1}'$ can not appear in $T_i^{(n-1)}$ for all $i \in [n-1]_{n-2}$, (v) $u_{n-1} \notin \{u_1', \cdots, u_{n-2}'\}$, (vi)

$u_0' \notin R_{n-1}\backslash\{x_{i_{n-1}}\}$ where $\varphi(u_0u_0') = \varphi(x_{i_{n-1}}v_{n-1})$, and (vii) $u_0' \notin \{u_4, \cdots, u_{n-2}\}$ if $n \geq 6$.

Step 4. (Defining $T_n^{(n)}$)

Let $T_n^{(n)}$ be the tree obtained from $S_{x_{i_n}}$ by removing the edges $x_{i_n}u_0, x_{i_n}u_1, \ldots, x_{i_n}u_{n-1}$ and then adding the edges $u_0u_0', u_1u_1', \ldots, u_{n-1}u_{n-1}'$. Finally, let $R_n = R_{n-1} \cup \{x_{i_n}\}$ and $U_n = U_{n-1} \setminus \left(\{x_{i_n}, u_0, u_0'\} \cup \bigcup_{i=1}^{n-1}\{v_i, v_i', u_i, u_i'\}\right)$ and go back to Step 1 with $n \leftarrow n+1$.

We note here that u, v_i, v_i', u_i, u_i' for all i are just temporary bywords in each round and will be replaced once the vertices fixed which they refer to. More precisely, after doing Step 4 in round k, the bywords u, v_i, v_i', u_i, u_i' drop their references, and then they will carry new vertices in the round $k+1$.

4.2.2 Main Results

Theorem 4.2.1. *Let φ be an arbitrary proper $(2m-1)$-edge-coloring of K_{2m}, $m \geq 3$, and x_0 be an arbitrary vertex. Then there exist at least $\left\lfloor \frac{\sqrt{4m+37}-3}{2} \right\rfloor$ mutually edge-disjoint multicolored spanning trees, each of them contains a pendent vertex x_0.*

Proof. First of all, we show that the recursive construction works for finding the n^{th} tree as long as $|U_{n-1}| \geq 9n-14$. It suffices to show that we can successfully find suitable $v_1, v_2, \cdots, v_n$ consecutively. In the search of v_i, we split the discussion into several parts according to the Step 3 in the construction process. (1) If we want to have $u_t \neq y$, then it is sufficient to ensure $v_t \neq \varphi_{x_{i_t}}^{-1}(c)$, where $c = \varphi(x_{i_n}y)$. (2) If we want to have $v_t' \neq y$, then it suffices to make sure $v_t \neq \varphi_y^{-1}(c)$, where $c = \varphi(x_{i_t}x_{i_n})$. (3) If we want to have $u_t' \neq y$, then it only needs to make sure that $\varphi(v_tx_{i_t}) \neq \varphi(x_{i_n}\alpha)$ whenever $\varphi(y\alpha) = \varphi(x_{i_n}u_{t-1})$. (4) If the edge u_tu_t' can not appear in $T_i^{(n-1)}$, then we only ensure $u_t \neq \alpha$ or β, where $\alpha\beta$ is the edge colored with $\varphi(x_{i_n}u_{t-1})$ in $T_i^{(n-1)}$. (5) Finally, in Step 3.3, $u_0' \neq y$ if $v_t \neq \varphi_{x_{i_t}}^{-1}(c)$ where $c = \varphi(u_0y)$. Applying simple arithmetic, for each v_t , $2 \leq t \leq n-2$, we avoid at most $5n+2t-6$ vertices in choosing v_t, and $9n-16$ vertices in choosing v_n for $n \geq 6$. Since each $v_i \in U_{n-1} \setminus \{x_{i_n}\}$, we conclude that $v_1, v_2, \cdots, v_{n-1}$ can be successfully found if $|U_{n-1}| \geq 9n-14$ for $n \geq 6$.

Secondly, we have to show the revised $T_1^{(n)}$, $T_2^{(n)}, \cdots, T_{n-1}^{(n)}$ are still mutually edge-disjoint multicolored spanning trees. For each $1 \leq t \leq n-1$, since the multicolored and spanning properties hold by the (x_{i_n}, v_t)-switch operation of $T_t^{(n-1)}$, it suffices to show that $T_t^{(n)}$ is edge-disjoint to $T_i^{(n)}$ for every $i < t \leq n-1$. Observe that every vertex in U_{n-1} is adjacent to the root of $T_t^{(n-1)}$ which has degree one. Since $x_{i_n}, v_t \in U_{n-1}$, we need only to check that $v'_t \notin R_{n-1}$ and $u_t \notin \{u_0, u_1, \cdots, u_{t-1}\} \cup (R_{n-1} \setminus \{x_{i_t}\})$. This is a direct consequence of the restriction in Step 3.2.

Next, we claim that $T_n^{(n)}$ is a multicolored spanning tree and edge-disjoint to any other revised tree. The multicolored property is trivial from the definition of $T_n^{(n)}$. Since $u_i \notin \{u'_1, \cdots, u'_{i-1}\}$ for $2 \leq i \leq n-1$ and $u'_0 \notin \{u_4, \cdots, u_{n-2}\}$ if $n \geq 6$, the induced subgraph of the n edges $\{u_0u'_0, u_1u'_1, \cdots, u_{n-1}u'_{n-1}\}$ has no cycles, and thus $T_n^{(n)}$ is a spanning tree. We emphasize here that the second condition can not be dropped, for otherwise $u'_2 = u_0$, $u'_4 = u_2$ and $u'_0 = u_4$ may occur at the same time and thus induce a cycle. Furthermore, the condition (iii), (iv) in Step 3.1, 3.2 and 3.3, as well as the condition (vi) in Step 3.3 guarantee the edge-disjoint property.

In addition, the fact that x_0 is a pendent vertex of each $T_t^{(n)}$, $t \in [n-1]$ can be proved by the conditions: $u_t \neq x_0$ and $v'_t \neq x_0$. Moreover, x_0 is also a pendent vertex of $T_n^{(n)}$ because of $u_i \neq x_0$ and $u'_i \neq x_0$ for $i \in \mathbb{Z}_{n+1}$.

Finally, we evaluate the size of U_{n-1} by Step 4 of the recursive construction: $U_n = U_{n-1} \setminus \left(\{x_{i_n}, u_0, u'_0\} \cup \bigcup_{i=1}^{n-1} \{v_i, v'_i, u_i, u'_i\} \right)$. The worst case is that all the vertices v_i, v'_i, u_i, u'_i and x_{i_n}, u_0, u'_0 are distinct, see Figure 4.10. This implies that we have a recurrence relation: $|U_n| \geq |U_{n-1}| - (4n-1)$ with initial value $U_1 = 2m-2$. Therefore, $|U_{n-1}| \geq 2m - 2n^2 + 3n$. Combining this inequality with the recurrence condition of the construction in the case $n \geq 6$, we obtain $2m - 2n^2 + 3n \geq 9n - 14$. Hence, we can revise $n-1$ mutually edge-disjoint multicolored spanning trees and then find an extra one. This concludes that there exist at least $\left\lfloor \dfrac{\sqrt{4m+37}-3}{2} \right\rfloor$ mutually edge-disjoint multicolored spanning trees, each of them contains a pendent vertex x_0. ∎

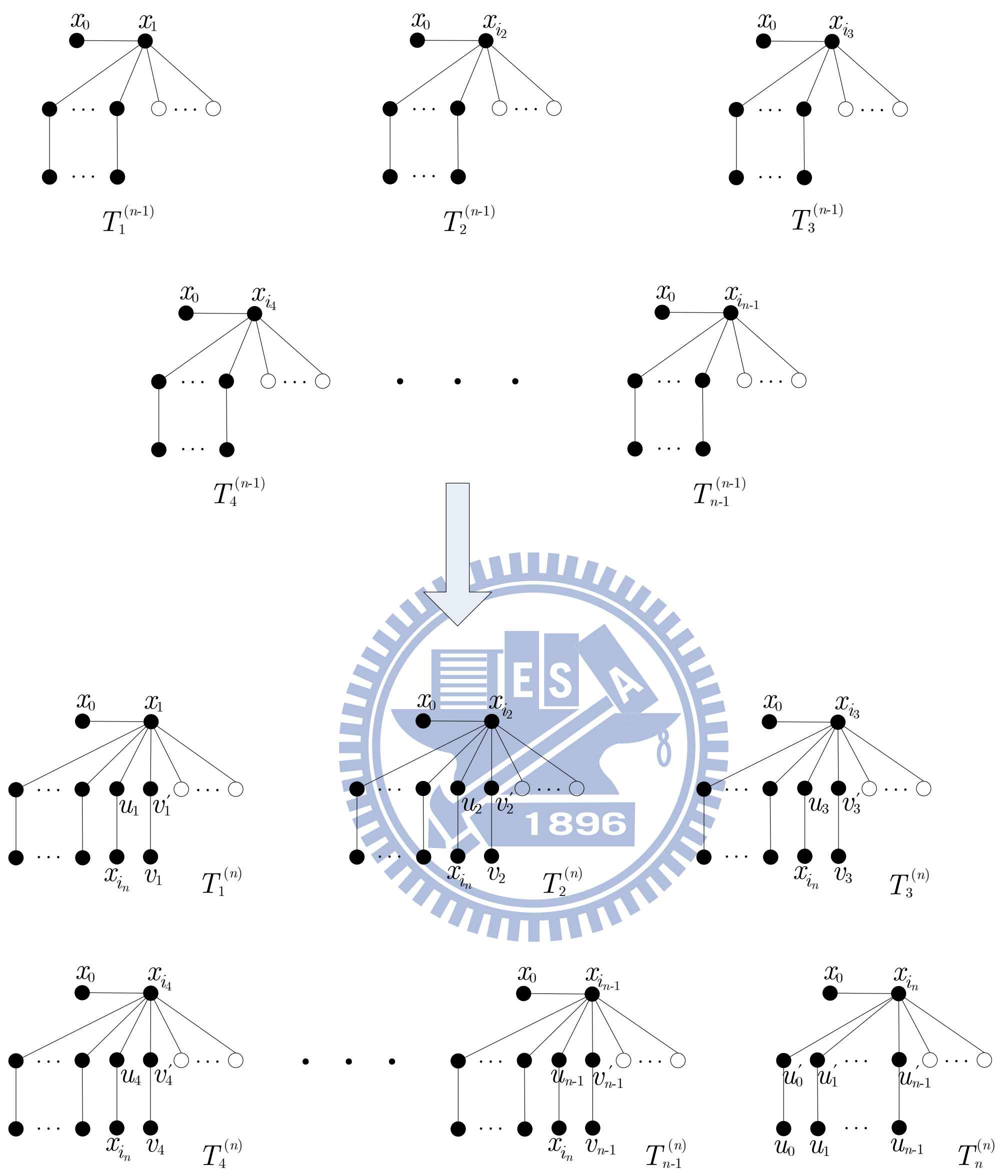


Figure 4.10: Estimate $|U_n|$ from $|U_{n-1}|$.

Since the number trees obtained is around $\sqrt{m}$, we use $\Omega(\sqrt{m})$ to denote its order. We note finally that the above theorem has been included in a paper written jointly with H. L. Fu [23].

Chapter 5

Multicolored Unicyclic Spanning Subgraphs in Edge-Colored Complete Graphs

Recall the statement of Conjecture 1.7.11: any properly edge-colored complete graph of odd order allows a partition of edges into multicolored isomorphic unicyclic spanning subgraphs. In this chapter, we consider a properly n-edge-colored K_n, n is odd.

5.1 Isomorphic Multicolored Unicyclic Spanning Subgraphs

At first, we introduce a special total-coloring in the complete graph of odd order: *symmetric total-coloring*. A symmetric n-total-coloring of K_n, n is odd, is an n-total-coloring μ so that for any three vertices a, b, and c, if $\mu(bc) = \mu(a)$, then $\mu(ab) = \mu(c)$ and $\mu(ac) = \mu(b)$. Then, we have the following result.

Lemma 5.1.1. *Let n be an odd integer and μ is a symmetric n-edge-coloring of K_n, then*

(1) *$n \neq 5$; and*

(2) *if $n = 7$, then all edges can be partitioned into multicolored Hamiltonian cycles.*

Proof. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and the color set be $C = [n]$. For convenience, we can permute the color assignment so that $\mu(x_i) = i$ for every $i \in [n]$. In the case $n = 5$, we can assume that $\mu(x_2x_3) = 1$. Then, $\mu(x_1x_2) = 3$, $\mu(x_1x_3) = 2$ and $\mu(x_4x_5)$ must be

1. This implies that no other edges can be colored with 2, a contradiction to the fact that each color occurs exactly twice on edges. Hence, $m \neq 5$.

In the case $n = 7$, we assume that color 1 appears on the edges x_2x_7, x_3x_6, and x_4x_5. Without loss of generality, let $\mu(x_3x_4) = 2$, then this will imply $\mu(x_5x_6) = 2$ and thus $\mu(x_3x_5) = \mu(x_4x_6) = 7$. By the symmetry of μ, the colors on the other edges are determined, see Figure 5.1. Figure 5.2 shows the existence of three multicolored Hamiltonian cycles under this coloring. ∎

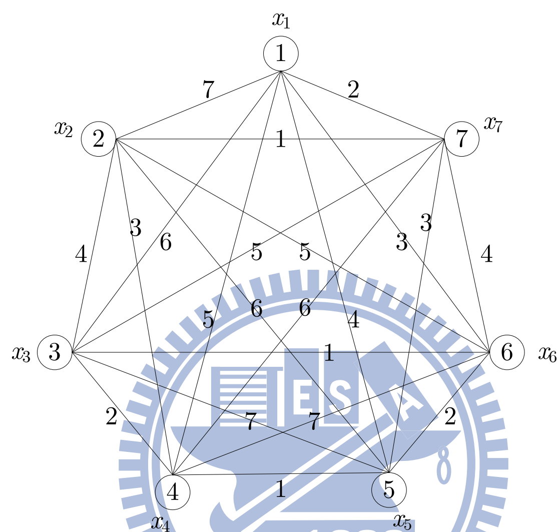


Figure 5.1: Symmetric 7-total-coloring of K_7.

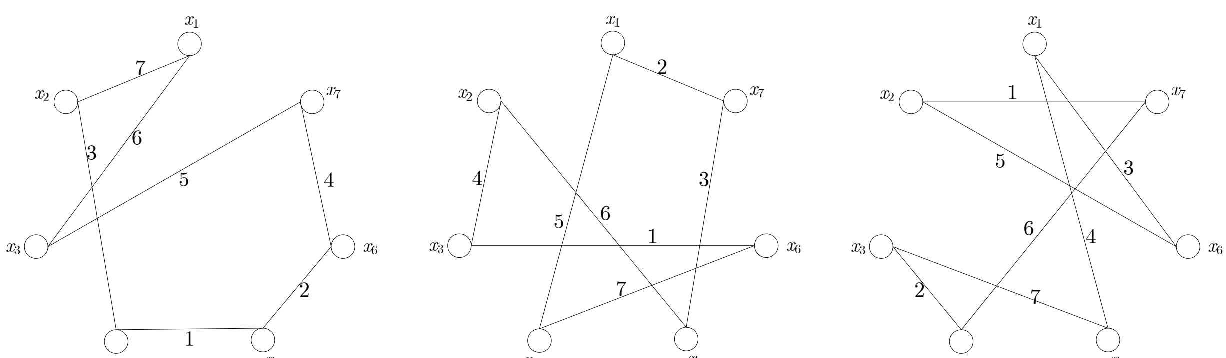


Figure 5.2: Three multicolored Hamiltonian cycles in symmetric 7-total-colored K_7.

Now, we are ready for our main result in this section.

Theorem 5.1.2. *For any positive odd integer $n \geq 5$ and an arbitrary proper n-edge-coloring of K_n, there exists a pair of multicolored isomorphic unicyclic spanning subgraphs of K_n.*

Proof. Let φ be a properly n-edge-colored K_n, we observe that each vertex of K_n is missing exactly once from the color set C, and each color of C occurs exactly $\frac{n-1}{2}$ times. Therefore, the corresponding missing colors of two distinct vertices are distinct. So, without loss of generality, let $V(K_n) = \{x_1, x_2, \cdots, x_n\}$, $C = [n]$, and the missing color at vertex x_i be color i. Note that this edge-coloring can be seen as an n-total-coloring.

We split the proof into two cases.

Case 1. There exists a triangle (x_a, x_b, x_c) such that $\varphi(x_b x_c) = a$ and either $\varphi(x_a x_b) \neq c$ or $\varphi(x_a x_c) \neq b$.

Without loss generality, let $\varphi(x_a x_b) = t \neq c$. Then let G_1 be the graph modified from S_{x_a} by deleting the edge $x_a x_b$ and adding edge $x_b x_c$. Assume $\varphi(x_a x_t) = t'$. Similarly, let G_2 be the graph modified from S_{x_t} by deleting the edge $x_a x_t$ and adding edge $x_a x_b$. Finally, adding edges yy' (colored t) and zz' (colored t') to G_1 and G_2, respectively, yield the desired two isomorphic unicyclic subgraphs. Notice that the two edges yy', zz' can not incident to x_a or x_t, see Figure 5.3.

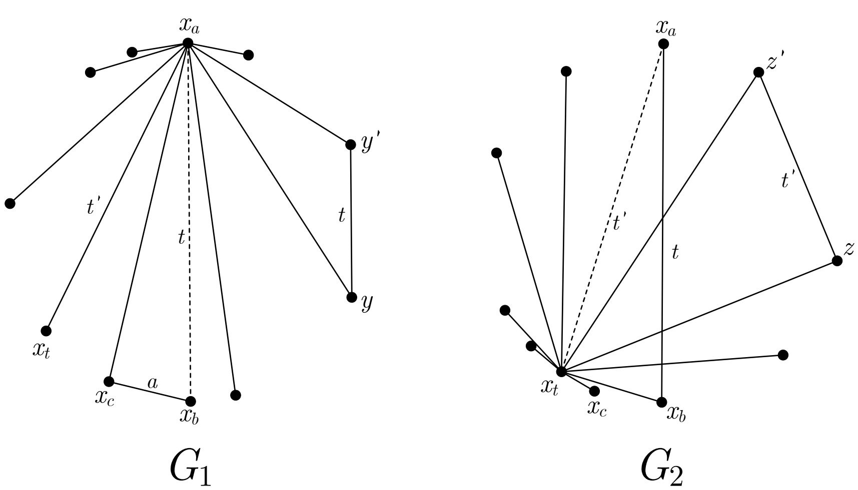


Figure 5.3: (Case 1) Two multicolored isomorphic unicyclic subgraphs.

Case 2. For any triangle (x_a, x_b, x_c), if $\varphi(x_b x_c) = a$, then $\varphi(x_a x_b) = c$ and $\varphi(x_a x_c) = b$.

In this case, we can assume $n \geq 9$ by Lemma 5.1.1. Pick the vertex x_1 and two edges with color 1, say $x_2 x_3$ and $x_4 x_5$. Since $\varphi(x_1 x_3) = 2$ and $\varphi(x_1 x_2) = 3$, we have $\varphi_{x_5}^{-1}(2) \notin \{x_1, x_2, x_3\}$. Since $n \geq 9$, there exists one edge yy' which is colored 4 such that $y, y' \notin \{x_2, x_3, x_5, \varphi_{x_5}^{-1}(2)\}$. Then, let G_1 be the graph modified from S_{x_1} by deleting the two edges $x_1 x_3, x_1 x_5$ and adding the three edges $x_2 x_3, x_5\langle 2\rangle, yy'$. Assume $\varphi_{x_3}^{-1}(5) = x_a$. Analogous to G_1, let G_2 be the graph modified from S_{x_3} by deleting the two edges $x_3 x_2, x_3 x_a$ and adding the three edges $x_2\langle 5\rangle, x_a\langle 3\rangle, x_4 x_5$, see Figure 5.4. Thus, we have two isomorphic multicolored spanning unicyclic subgraphs. ∎

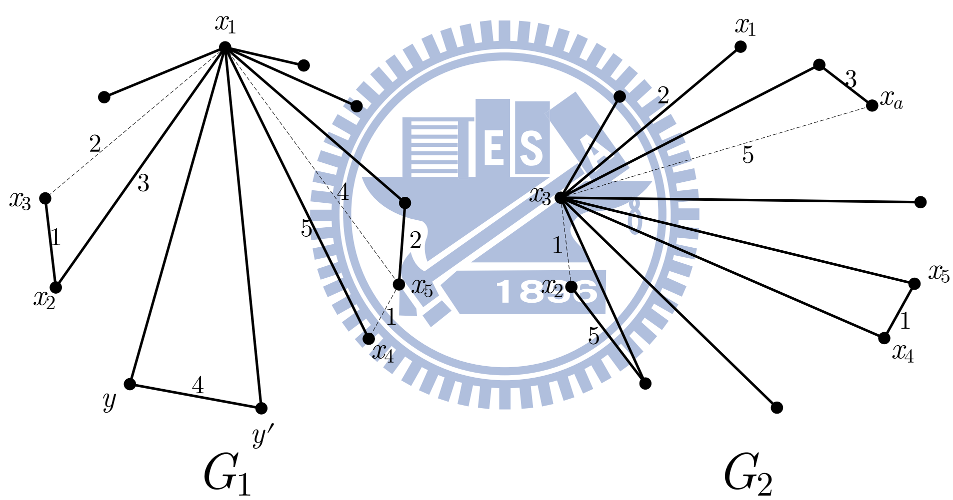


Figure 5.4: (Case 2) Two multicolored isomorphic unicyclic subgraphs.

5.2 Multicolored Unicyclic Spanning Subgraphs

Applying Theorem 4.2.1, we can have the following result.

Theorem 5.2.1. *Let φ be an arbitrary proper $(2m-1)$-edge-coloring of K_{2m-1}, then there exist at least* $\left\lfloor \dfrac{\sqrt{4m+37}-3}{2} \right\rfloor$ *mutually edge-disjoint multicolored spanning unicyclic subgraphs.*

Proof. Let K_{2m} be defined on $V(K_{2m-1}) \cup \{x_0\}$. Then, by the observation in Section 5.1, we obtain a $(2m-1)$-edge-coloring $\widetilde{\varphi}$ of K_{2m} by letting $\widetilde{\varphi}(x_0x_i) = i$ for $i \in [2m-1]$ and $\widetilde{\varphi}(x_ix_j) = \varphi(x_ix_j)$ for $i, j \in [2m-1]$. By Theorem 4.2.1, there exist at least $\left\lfloor \frac{\sqrt{4m+37}-3}{2} \right\rfloor$ mutually edge-disjoint multicolored spanning trees, each of them contains a pendent vertex x_0. Therefore, after deleting the vertex x_0, these trees turn out to be mutually edge-disjoint multicolored spanning trees in K_{2m-1} and each of them misses one color. Assume these trees are $T_1, T_2, \cdots$, and the root of T_i is y_i. Then, let C_i be obtained from T_i by adding an available edge e_i colored with the missing color in T_i; i.e., let $C_i = T_i + e_i$ where $\varphi(e_i) = \varphi(x_0y_i)$. This process always works since the missing colors are distinct and there are $m - 1 - \left\lfloor \frac{\sqrt{4m+37}-3}{2} \right\rfloor$ edges available in each color class. Thus, we have $\left\lfloor \frac{\sqrt{4m+37}-3}{2} \right\rfloor$ mutually edge-disjoint multicolored unicyclic spanning subgraphs in K_{2m-1}. Note that $\widetilde{\varphi}|_{K_{2m-1}} = \varphi$. This concludes the proof. ∎

Chapter 6

Forbidden Multicolored Cycles

From what we have seen in literatures, it is not difficult to see that finding (or proving the existence of) a specific multicolored subgraph such as tree, path or cycle, in an arbitrary properly edge-colored graph is not easy. On the other point of view, avoiding a specific multicolored subgraph is also a tough job. In this chapter, we first introduce some interesting results about the existence of multicolored subgraphs and then focus on the avoiding issue in the posterior part.

6.1 Multicolored Subgraphs in Edge-colored Complete Graphs

We start with some definitions. If the edges of a graph G are colored by r colors $[r]$ which are actually appearing in G, then its *color distribution* $(a_1, a_2, \ldots, a_r)$ means that the number of edges with color i is equal to a_i for every $i \in [r]$. An edge-coloring of a graph G is called an edge-coloring with *complete bipartite decomposition* if each color class forms a complete bipartite subgraph of G. If the edges of G are colored so that no color is appeared in more than k edges, we refer to this as a *k-bounded coloring*. For a vertex v of G, the *color degree* of v, denoted by $deg_{col}(v)$, is the number of colors on the edges which are incident to v.

In this section, some results related to multicolored subgraph in an edge-colored (not necessarily be proper) K_n will be introduced. We split them into the following three categories of multicolored subgraphs.

6.1.1 Multicolored Spanning Tree

Results related to proper edge-coloring have been discussed in previous chapters. In what follows, we consider a general edge-coloring of K_n. Recall the result proved by Brualdi and Hollingsworth [10] that in any proper $(2m-1)$-edge-coloring of the complete graph K_{2m}, $m > 2$, there are two edge-disjoint multicolored spanning trees. In 2006, Akbari and Alipour [1] generalized Brualdi and Hollingsworth's result as follows.

Theorem 6.1.1. [1] *If $(a_1, a_2, \ldots, a_r)$ is a color distribution for the complete graph K_n, $n \geq 5$, such that $2 \leq a_1 \leq \cdots \leq a_r \leq \frac{n+1}{2}$, then there exist two edge-disjoint multicolored spanning trees.*

As early as in 1991, however, Alon, Brualdi and Shader [3] discussed the existence of multicolored spanning trees from the perspective of complete bipartite decomposition.

Theorem 6.1.2. [3] *Every K_n having an edge-coloring (not necessary proper) with complete bipartite decomposition contains a multicolored spanning tree.*

6.1.2 Multicolored Path

Erdős and Gallai [17] first dealt with this type of problems in 1959.

Theorem 6.1.3. [17] *Every r-edge-colored graph G of order n has a multicolored path of length at least $\left\lceil \frac{2r}{n} \right\rceil$.*

In 2005, Broersma, Li, Woeginger and Zhang [8] obtained the following result.

Theorem 6.1.4. [8] *Let G be an edge-colored graph. If $deg_{col}(x) \geq k$ for every vertex x of G, then for every vertex v of G, there exists a multicolored path starting at v and of length at least $\left\lceil \frac{k+1}{2} \right\rceil$.*

Then, Chen and Li [12], [13] improved above theorem.

Theorem 6.1.5. [12] *Let G be an edge-colored graph and $k \geq 1$ be an integer. If $deg_{col}(x) \geq k$ for every vertex x of G, then there exists a multicolored path of length* *at least* $\left\lceil \frac{3k}{5} \right\rceil + 1$. *Moreover, if* $1 \leq k \leq 7$, *there exists a multicolored path of length at least* $k - 1$.

Theorem 6.1.6. [13] *Let* G *be an edge-colored graph and* $k \geq 8$ *be an integer. If* $deg_{col}(x) \geq k$ *for every vertex* x *of* G, *then there exists a multicolored path of length at least* $\left\lceil \frac{2k}{3} \right\rceil + 1$.

Consider the proper edge-coloring of a complete graph, we immediately get the following corollary by Theorem 6.1.6.

Corollary 6.1.7. *In any proper edge-coloring of* K_n, $n \geq 9$, *with* $\chi'(K_n)$ *colors, there exists a multicolored path of length at least* $\left\lceil \frac{2n-2}{3} \right\rceil + 1$.

6.1.3 Multicolored Cycle

When it comes to cycles, it is natural to consider Hamiltonian cycles. The problem to find n which is large enough so that every k-bounded edge-colored K_n, where k is given, contains a multicolored Hamiltonian cycle was mentioned in [18] in 1983. Here are three relative results. We list them in historical order.

Theorem 6.1.8. [29] *There exists a constant number* c *such that if* $n \geq ck^3$, *then every* k*-bounded edge-colored* K_n *has a multicolored Hamiltonian cycle.*

Theorem 6.1.9. [24] *There exists a constant number* c *such that if* n *is sufficiently large and* $k \leq \frac{c}{\ln n}$, *then every* k*-bounded edge-colored* K_n *contains a multicolored Hamiltonian cycle.*

Theorem 6.1.10. [4] *Let* $c < 1/32$. *If* n *is sufficiently lage and* $k \leq \lceil cn \rceil$, *then every* k*-bounded edge-colored* K_n *contains a multicolored Hamiltonian cycle.*

Theorem 6.1.8 was obtained by Hahn and Thomassen [29] in 1986 and implied that k could grow as fast as $n^{1/3}$ to guarantee that a k-bounded edge-colored K_n contains a multicolored Hamiltonian cycle. In 1993, Frieze and Reed [24] made further progress, see

Theorem 6.1.9. Few years later, in 1995, Albert, Frieze and Reed [4] improved Theorem 6.1.9 and proved the growth rate of k could in fact be linear.

Now, we consider general cycles. In an edge-colored K_n, it is clear that there is no multicolored cycle if and only if there is no multicolored C_3. Notice that there exists a cycle somewhere in a subgraph of K_n which has n edges. Montellano-Ballesteros and Neumann-Lara [33] presented the following results.

Theorem 6.1.11. [33] *If the edges of K_n are colored by n or more colors actually appearing, then there is a multicolored K_3 somewhere.*

Above result infers that K_n has an $(n-1)$-edge-coloring which forbids multicolored K_3's. A. Gouge et al. [25], in 2010, not only proved the existence of such colorings but also characterized all such colorings. They defined a $JL(n)$ coloring as an edge-coloring of K_n with exactly $n-1$ colors which forbids multicolored $K_3's$ (and thus multicolored cycles). They also have

Theorem 6.1.12. [25] *Suppose $n \geq 2$. Every $JL(n)$ coloring is obtainable as follows: choose positive integers r, s satisfying $r+s=n$; partition $V(K_n)$ into sets R, S satisfying $|R| = r, |S| = s$. Color all R-to-S edges in K_n with one color-say green. Color $\langle R\rangle_{K_n}$ with a $JL(r)$ coloring and $\langle S\rangle_{K_n}$ with a $JL(s)$ coloring with disjoint sets of colors on the two cliques, and with green not appearing in $\langle R\rangle_{K_n}$ nor $\langle S\rangle_{K_n}$.*

In the same paper, they also considered the edge-coloring, named *equalized*, which the difference of numbers of any two colors is at most 1.

Theorem 6.1.13. [25] *For $n > 1$, there is an equalized t-edge-coloring of K_n which forbids multicolored K_3 if and only if $t \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$.*

6.2 Forbidding Multicolored Cycles in Edge-colored Complete Bipartite Graphs

In this section, motivated by the works in [25] and [33], we consider the proper edge-colorings of $K_{m,n}$, $n \geq m$, which forbid multicolored (even) cycles. Actually, given an integer k, we want to know for what natural numbers n and m, there always exists a multicolored C_{2k} somewhere in any properly n-edge-colored $K_{m,n}$. For $k \geq 2$, we define the *forbidding multicolored 2k-cycles set*, $FMC(2k)$ in short, by the ordered pair $(m, n) \in FMC(2k)$ if there exists a proper n-edge-coloring of $K_{m,n}$ that forbids multicolored $2k$-cycles. Since $m < k$ or $n < 2k$ gives trivial results, we only consider $m \geq k$ and $n \geq 2k$ in the set $FMC(2k)$

Firstly, it is impossible to forbid multicolored 4-cycles in any proper n-edge-coloring of $K_{m,n}$ where $2 \leq m \leq n$ and $n \geq 4$.

Proposition 6.2.1. $FMC(4) = \phi$.

Proof. It suffices to show that there exists a multicolored C_4 in any properly edge colored $K_{2,4}$. Let φ be a proper edge coloring of $K_{2,4}$ and $\{u_1, u_2\}$, $\{v_1, v_2, v_3, v_4\}$ be the two partite sets. For convenience, let $C = \{1, 2, \ldots\}$ be the color set. Without loss of generality, assume $\varphi(u_1v_1) = 1$ and $\varphi(u_2v_1) = 2$. Since φ is proper, there must be one vertex v_i, where $2 \leq i \leq 4$, such that $\varphi(u_1v_i), \varphi(u_2v_i) \notin \{1, 2\}$. Thus $u_1 - v_1 - u_2 - v_i - u_1$ is the desired multicolored C_4. ∎

6.2.1 Forbidding Multicolored $2k$-cycles

We start with some notations. Let S be an n-set. A *latin rectangle* of order $m \times n$, $m \leq n$, based on S is an $m \times n$ array in which every element of S is arranged such that each one occurs at most once in each row and each column. Thus, a latin square of order n based on S, defined in Section 1.5, is a latin rectangle of order $n \times n$. A *partial latin square* of order r, $r < n$, based on S is an $r \times r$ array in which every element of S is arranged such that each one occurs at most once in each row and each column. In this section, we use $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ for the n-set S. For example, $\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array}$ is a latin rectangle of order 2×3 based on $\mathbb{Z}_3$; and $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 0 \\ \hline \end{array}$ is a partial latin square of order 2 based on $\mathbb{Z}_3$. In particular, the size of a partial latin square L, denoted by $|L|$, is the number of elements of S actually appearing in L.

For convenience in presentation, we redefine the method of the product of two latin squares (compare with Section 1.5). Let $L = [l_{i,j}]$ and $M = [m_{i,j}]$ be two latin squares of order s based on $\mathbb{Z}_s$ and t based on $\mathbb{Z}_t$, respectively. Then the direct product of L and M, $L \times M = [h_{i,j}]$,is a latin square of order $s \cdot t$ based on $\mathbb{Z}_{st}$, where $h_{x,y} = t \cdot l_{a,b} + m_{c,d}$ provided that $x = ta + c$ and $y = tb + d$. For instance, let L and M be two latin square of order 2 (based on $\mathbb{Z}_2$) and 3 (based on $\mathbb{Z}_3$) respectively, then $L \times M$ is a latin square of order 6 based on $\mathbb{Z}_6$, as in Figure 6.1.

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 0 & 1 & 5 & 3 & 4 \\ \hline 1 & 2 & 0 & 4 & 5 & 3 \\ \hline 3 & 4 & 5 & 0 & 1 & 2 \\ \hline 5 & 3 & 4 & 2 & 0 & 1 \\ \hline 4 & 5 & 3 & 1 & 2 & 0 \\ \hline \end{array}$$

Figure 6.1: The direct product of L and M

Similar to the definition of transversal in a latin square, the transversal of a partial latin square of order r based on an n-set is set of r cells with exactly one in each row and each column and containing exactly r elements.

Let $L = [l_{i,j}]$ be an $m \times n$ latin rectangle. There is a corresponding relationship between L and a properly n-edge-colored $K_{m,n}$. Let $\{u_0, u_1, \ldots, u_{m-1}\}$ and $\{v_0, v_1, \ldots, v_{n-1}\}$ be two partite sets of $K_{m,n}$, and the edge $u_i v_j$ is colored with $l_{i,j}$ for each $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, then we have a properly n-edge-colored $K_{m,n}$ and vice versa. Now, we have

Theorem 6.2.2. *If k is odd, then $(m, 2k) \in FMC(2k)$ for $k \leq m \leq 2k$.*

Proof. It suffices to find a proper $2k$-edge-coloring of $K_{2k,2k}$ which forbids multicolored C_{2k}. Let L_2 be the latin square of order 2 in Figure 6.1 and M be a latin square of order k. Notice that $L_2 \times M$ is formed by four latin squares of order k, two of them based on $\mathbb{Z}_k$ and other two based on $\mathbb{Z}_{2k} \setminus \mathbb{Z}_k$. For convenience, name the four squares A, B, C and D clockwise from the top-left one, see Figure 6.2.

M based on $\{0, 1, \ldots, k\text{-}1\}$	M based on $\{k, \ldots, 2k\text{-}1\}$
M based on $\{k, \ldots, 2k\text{-}1\}$	M based on $\{0, 1, \ldots, k\text{-}1\}$

A	B
D	C

Figure 6.2: $L_2 \times M$ and the four copies of M

Let φ be the proper $2k$-edge-coloring of $K_{2k,2k}$ obtained by $L_2 \times M$. Suppose it contains a multicolored C_{2k}. Let a, b, c, and d be the numbers of cells in A, B, C, and D, respectively, corresponding to the edges of the multicolored cycle. Then $a + b$ is a sum of the degrees, on the cycle, of some of the vertices on the cycle, so $a + b$ is even. Similarly, $b + c$ is even. Therefore, $a + c$ is even. But since all $2k$ colors $0, 1, \ldots, 2k - 1$ must appear on the edges of the cycle, $a + c = k$, odd. This contradiction completes the proof. ∎

The following result provides an upper bound of the order of complete bipartite graphs to forbid multicolored $2k$-cycles.

Theorem 6.2.3. *For any integer $k \geq 2$, if $n \geq 5k - 6$, then any properly n-edge-colored $K_{k,n}$ contains a multicolored C_{2k}.*

Proof. Let φ be a proper n-edge-coloring of $K_{k,n}$ and the partite sets be $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Let $P = a_1 b_1 a_2 \cdots b_{t-1} a_t$ be the longest multicolored path whose endpoints lie on A. Suppose $t < k$. Assume C is the set of colors which appear on P. Note that $|C| = 2t - 2$. For each $i = 1, \dots, k$, define $S_i \subset B$ by $b \in S_i$ if $\varphi(a_i b) \in C$. Observe that $|S_t \cup S_{t+1} \cup \{b_1, b_2, \dots, b_{t-1}\}| \leq 2(2t-2) + (t-1) - 1 = 5t - 6 < 5k - 6 \leq n$. Therefore, there exists a vertex $b \in \{b_t, b_{t+1}, \dots, b_n\}$ such that $\varphi(a_t b), \varphi(a_{t+1} b) \notin C$. Let $Q = P \cup \{a_t b a_{t+1}\}$, we have $|Q| = 2t > |P|$, a contradiction. Then $t \geq k$. By the fact that a longest path in $K_{k,n}$ with end vertices in A is of length $2k - 2$, we have $t = k$.

We have that $|S_1|, |S_k| \leq 2k-2$ and $b_1 \in S_1, b_{k-1} \in S_k$. Hence, $|S_1 \cup S_k \cup \{b_1, \dots, b_{k-1}\}| \leq 5k - 7$. Since $n \geq 5k - 6$, there exists a vertex $b \in B$ such that $\varphi(a_1 b), \varphi(a_k b) \notin C$. Therefore, a multicolored C_{2k} is found. ∎

6.2.2 Determining $FMC(6)$

By Theorem 6.2.3, if $(m, n) \in FMC(6)$, then we have $3 \leq m \leq n$ and $n = 6, 7, 8$. The case $n = 6$ was done in Theorem 6.2.2, so we consider $n = 7$ and 8 in the following.

Let L be the corresponding latin rectangle of a properly n-edge-colored $K_{m,n}$. If there is a multicolored C_6 somewhere, then there exists a 3×3 partial latin square which contains two disjoint transversals using exactly 6 symbols in L.

Proposition 6.2.4. Let L be a partial latin square of order 3 with $|L| = 7$. Then, there is no multicolored C_6 in its corresponding $K_{3,3}$ if and only if it contains a latin subsquare of order 2.

Proof. It suffices to consider the necessity since the sufficiency is clearly true. Suppose L contains no latin subsquares of order 2. If there is one element appearing 3 times, then the other 6 elements form a multicolored C_6. Therefore, assume that there are two elements, say $1, 2$, appearing twice respectively. Without loss of generality, let the two 1's be arranged at the diagonal in the first two rows. Then at least one of 2's occurs in the third column or the third row. Omitting this cell and one of the cells labeled 1 such that the two cells form a transversal, the 6 of the remaining cells will provide a multicolored C_6, a contradiction. ∎

Proposition 6.2.5. Let L be a partial latin square of order 3 with $|L| = 6$. There does not exist a multicolored C_6 in its corresponding $K_{3,3}$ if one of the following conditions occurs:

(1) There exist 2 columns (or rows) in L using exactly 3 elements.

(2) Some element appears three times in L.

(3) L contains a latin subsquare of order 2.

Proof. Since there are only 6 elements, if there exists a multicolored C_6, all elements should appear in the two disjoint transversals. In case 1, the elements of the third column (or row) can not all appear. In case 2, that element can not appear only once in any two disjoint transversals. In case 3, the argument is similar to the proof of Proposition 6.2.4. ∎

Lemma 6.2.6. *For* $3 \le m \le 8$, $(m, 8) \in FMC(6)$.

Proof. It suffices to prove the claim for $m = 8$. Let L_2 be the latin sqaure of order 2 in Figure 6.1. Let $L = L_2 \times L_2 \times L_2$, a latin square of order 8 based on $\mathbb{Z}_8$. For convenience, name the four copies A, B, C and D of $L_2 \times L_2$ as in Figure 6.3.

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

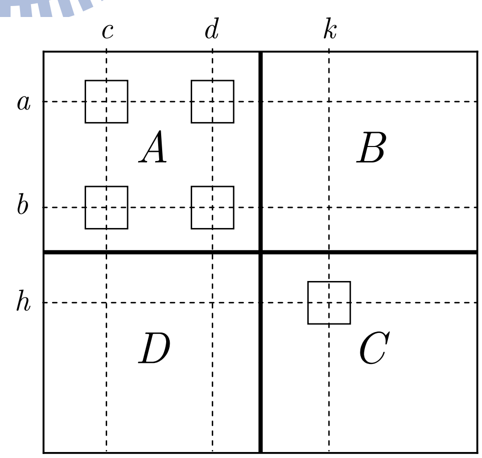


Figure 6.3: $L_2 \times L_2 \times L_2$

Suppose that there are 6 cells whose entries induce a multicolored C_6. Let L' be the 3×3 partial latin square which contains the 6 cells. It is easy to see that any

2×3 partial latin rectangle in $L_2 \times L_2$ (A or B or C or D) contains a latin subsquare of order 2. By Proposition 6.2.4, we can assume that L' traverses all four copies of $L_2 \times L_2$. Without loss of generality, suppose there are 4 cells of L' located in A. Let the 4 cells be $(a, c), (a, d), (b, c), (b, d)$, and the only one cell located on C be (h, k), where $0 \leq a, b, c, d \leq 3$ and $4 \leq h, k \leq 7$ (Figure 6.3). By Proposition 6.2.4 and Proposition 6.2.5, $l_{a,c} \neq l_{b,d}$ or $l_{a,d} \neq l_{b,c}$, and thus the four elements are distinct. Assume that $l_{h,k} = l_{a,c}$. This implies $l_{a,k} = l_{h,c}$. Then we have a copy of L_2, a contradiction. Similarly, if $l_{h,k}$ is any of the $l_{i,j}$, with (i, j) being one of the 4 cells of L' in A, then we have a contradiction. But $l_{h,k}$ must be one of these, since these 4 are distinct elements of $\{0, 1, 2, 3\}$. ∎

Lemma 6.2.7. $(3, 7) \in FMC(6)$. *Furthermore, if $K_{3,7}$ is properly 7-edge-colored such that it forbids multicolored C_6's, there exists an induced $K_{3,3}$ using exactly 3 colors.*

Proof. Firstly, Figure 6.4 gives a 3×7 latin rectangle. It is not difficult to check its corresponding proper 7-edge-coloring of $K_{3,7}$ induces no multicolored C_6 by Proposition 6.2.4 and Proposition 6.2.5.

0	1	2	3	4	5	6
1	0	3	2	6	4	5
2	3	0	1	5	6	4

Figure 6.4: A 3×7 latin rectangle

Secondly, given a proper 7-edge-coloring of $K_{3,7}$ which forbids multicolored 6-cycles, let L be its corresponding latin rectangle. It suffices to show that L contains a latin subsquare of order 3. For convenience, let C^i denote the set of elements in the ith column of L where $i \in \mathbb{Z}_7$.

Claim. There exist i, j such that $C^i \cap C^j = \phi$.

Suppose for any $i \neq j$, $C^i \cap C^j \neq \phi$. Since each element occurs three times, we have $|C^i \cap C^j| = 1$ for all $i \neq j$ under this assertion. Without loss of generality, let $C^0 = \{0, 1, 2\}$

and $C^1 = \{0,3,4\}$. Then 3 and 4 will each occur twice in the remaining five columns. So, there exists a C^t, where $2 \le t \le 6$, such that $C^t \cap \{3,4\} = \phi$. This implies that the three columns C^0, C^1 and C^t create a multicolored C_6 by Proposition 6.2.4, a contradiction.

Thus, assume $C^0 = \{0,1,2\}, C^1 = \{3,4,5\}$ and C^2, C^3, C^4 contain the element 6. Note here that $|C^t \cap C^0| = 2$ or $|C^t \cap C^1| = 2$ for all $t = 2,3,4$; otherwise, C^0, C^1, C^t will create a multicolored C_6 by Proposition 6.2.4. Next, we want to claim $(C^2 \cup C^3 \cup C^4) \setminus \{6\}$ equals C^0 or C^1. Suppose the assertion is not true, without loss of generality, let $|C^2 \cap C^0| = 2, |C^3 \cap C^0| = 2$ and $|C^4 \cap C^1| = 2$. See the left rectangle in Figure 6.5: the elements in cell A are from $\{0,1,2\}$ while the elements in cell B are from $\{3,4,5\}$.

0	3	6	A	B		
1	4	A	6	B		
2	5	A	A	6		

$\longrightarrow$

0	3	6	2	4	5	1
1	4	2	6	3	0	5
2	5	1	0	6	C	C

Figure 6.5: The 3×7 latin rectangle

Proposition 6.2.4 shows that the elements in the cells labelled A and the cells labelled B are uniquely determined; see the right hand side rectangle in Figure 6.5. Meanwhile, the elements in some cells of the last two columns are determined except cells denoted as C, which are filled with 3 and 4. No matter what the elements in C are, C^0, C^4 and C^5 contain a multicolored C_6, a contradiction. Therefore, $(C^2 \cup C^3 \cup C^4) \setminus \{6\}$ equals C^0(or C^1). Hence, combining C^5, C^6 with C^1(or C^0), we have a latin square of order 3. ∎

Lemma 6.2.7 will yield the following result.

Proposition 6.2.8. For any proper 7-edge-coloring of $K_{m,7}$, $4 \le m \le 7$, there exists a multicolored C_6.

Proof.

It's sufficient to consider the case when $m = 4$. Suppose that there exists a properly 7-edge-colored $K_{4,7}$ which forbids multicolored C_6's, then let L be its corresponding 4×7

				4	5	6
				6	4	5
				5	6	4

Figure 6.6: The 4×7 latin rectangle

latin rectangle. By Lemma 6.2.7, there exists a latin square of order 3 in the first three rows of L. Without loss of generality, we put the latin square of order 3 in the last three columns and let the symbols be $\{4, 5, 6\}$, see Figures 6.6. Next, consider the last three rows. It's impossible to find another latin square of order 3. It contradicts Lemma 6.2.7.

∎

To sum up, we have the following conclusion.

Theorem 6.2.9. $FMC(6) = \{(m, 6)|\ 3 \leq m \leq 6\} \cup \{(3, 7)\} \cup \{(m, 8)|\ 3 \leq m \leq 8\}$.

We note here that above result (obtained jointly with H. L. Fu and R. Y. Pei) has been included in [21].

Chapter 7

Conclusion and Remarks

The main focus of this thesis is to find edge-disjoint multicolored subgraphs in a properly edge-colored complete graph. If the complete graph is properly k-edge-colored, then we are aiming to obtain edge-disjoint copies of multicolored subgraphs of size k. This is why we try to find copies of multicolored spanning trees of K_{2m} since it is $(2m-1)$-edge-colorable and find copies of multicolored spanning unicyclic subgraphs of K_{2m+1} since it is $(2m+1)$-edge-colorable.

In case that the proper edge-coloring is of special type or prescribed, then in Chapter 2 and Chapter 3 we have an MTP (multicolored spanning tree parallelism) or an $MHCP$ (multicolored Hamiltonian cycle parallelism) respectively when K_{2m} or K_{2m+1} are considered. However, if the proper edge-colorings are arbitrarily given, then finding copies of multicolored subgraph is going to be very difficult. In fact, except for special graphs such as stars, small paths or small cycles, finding just one copy (multicolored) of a given graph, for example, a multicolored perfect matching in K_{2m}, is difficult enough.

Therefore, we put our effort in searching for edge-disjoint (not necessarily be isomorphic) multicolored spanning trees in a properly $(2m-1)$-edge-colored K_{2m} and multicolored unicyclic spanning subgraphs in a properly $(2m+1)$-edge-colored K_{2m+1} respectively. In Chapter 4 and Chapter 5, by using a recursive construction, we are able to find $\Omega(\sqrt{m})$ edge-disjoint multicolored spanning trees and $\Omega(\sqrt{m})$ edge-disjoint multicolored spanning unicylic subgraphs in K_{2m} and K_{2m+1} respectively. Though this result is the best one obtained so far, it is very far from m spanning trees (conjectured by Brauldi and Hollingsworth) and m unicyclic spanning subgraphs (conjectured by Constantine). Hopefully, we can close the gap in the near future.

In this thesis, we also consider "forbidden" multicolored subgraphs in a properly edge-colored complete bipartite graph. Mainly, we prove that if the two partite sets are large enough, then forbidding a multicolored even cycle of fixed length is not possible. Precisely, we prove that for $f(k) \leq n$, then every properly n-edge-colored $K_{k,n}$ contains a multicolored $2k$-cycle where $f(k) = 5k - 6$. As a consequence, we determine the set of all ordered pairs (m, n), such that multicolored C_6 can be forbidden in $K_{m,n}$. Unfortunately, determining the set of (m, n)'s such that multicolored C_{2k} can be forbidden in $K_{m,n}$ (by giving a proper n-edge-coloring) is still unsolved. We believe that it is close related to find a latin rectangle with special structure which is worth of more study.

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