國立交通大學應用數學系

博士論文

具覆蓋關係動態函數之高維度擾動的拓樸混沌

Topologically chaos for multidimensional perturbations of maps with covering relations

研究生：呂明杰

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摘要

本論文主要研究高維度系統的拓樸動態，其中系統是擾動由 $F(x,y)=(f(x),g(x,y))$ 形式之系統且滿足低維度函數 f 為一連續函數。首先我們會證明如果當低維度函數 f 具有返回擴張固定點，其微小的 C^1 擾動同樣具有返回擴張固定點。

假設函數g具有局部抑制的區域且系統沿著一連續的參數群$\{F_\lambda\}$滿足$F_0=F$。我們會證明如果當低維度函數f為一維度函數且具有正的拓樸熵或f為一高維度函數具有返回擴張固定點，則對於所有夠小的參數λ，F_λ也會具有正的拓樸熵。並且我們證明如果當f為一微分同胚具有 topologically crossing homoclinic point時，則對於參數λ夠接近0時，F_λ具有正的拓樸熵。

更進一步地，我們證明當f具有由轉移矩陣A決定的覆蓋關係時，則F的任意微小C^0擾動系統會存在一緊緻正向的不變集且當系統限定在此不變集上時會拓樸半共軛到由A生成的單邊有限型子轉移。此外，如果覆蓋關係滿足strong Liapunov condition且函數g為一壓縮函數，則我們會證明出F的任意微小C^1擾動同胚會存在一緊緻的不變集且當系統限定在此不變集上時會拓樸共軛到由A生成的雙邊有限型子轉移。

Topologically chaos for multidimensional perturbations of maps with covering relations

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Abstract

In this dissertation, we investigate topological dynamics of high-dimensional systems which are perturbed from a continuous map f of the following form F(x,y) = (f(x),g(x,y)). First, we show that if the lower dimensional map f has a snap-back repeller, then the small C^1 perturbation of f also has a snap-back repeller.

Assume that g is locally trapping and the system is along a one-parameter continuous family $\{F_\lambda\}$ such that $F_0 = F$. We show that if f is a one dimensional map and has positive entropy, or f is a high-dimensional map and has a snap-back repeller then $\{F_\lambda\}$ has a positive topological entropy for all small parameter λ. Also, we show that if f is a C^1 diffeomorphism having a topologically crossing homoclinic point, then $\{F_\lambda\}$ has positive topological entropy for all λ close enough to 0.

Moreover, we show that if f has covering relations determined by a transition matrix A, then any small C^0 perturbed system of F has a compact positively invariant set restricted to which the perturbated system is topologically semi-conjugate to the one-sided subshift of finite type induced by A. In addition, if the covering relations satisfy a strong Liapunov condition and g is a contraction, we show that any small C^1 perturbed homeomorphism of F has a compact invariant set restricted to which the

system is topologically conjugate to the two-sided subshift of finite type induced by A.

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在交大讀博士班六年的時光中，充滿了許許多多的喜怒哀樂。雖然中途在準備英文考試方面有一些挫折，不過最後還是通過考試進而完成學業。首先，感謝指導教授李明佳老師在學業上給了我一些指導與建議，這對我在動態系統的概念上有很大的幫助。由於他的包容與支持，讓我能在迷途中，有了一盞明燈指引方向；從他生活處事與對數學研究態度上，我看到了一個數學家的嚴謹與堅持。另外，他也常常鼓勵我參加國內外的研討會，讓我能增廣見聞及培養我的國際觀，藉以提升數學深度。

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呂明杰 101/07/23 於交大

Contents

1 Introduction

In this dissertation, we mainly study the perturbation from a map f on the lower dimensional phase space, which has some dynamical properties (positive topological entropy, snap-back repeller, topologically crossing homoclinicity, covering relations determined by a transition matrix, etc.) to continuous map G on a high dimensional space such that G is a small perturbation of the singular map F which is one of the following cases:

(i) $F(x) = f(x) \in \mathbb{R}^m$;

(ii) $F(x, y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^n$;

(iii) $F(x, y) = (f(x), g(x, y)) \in \mathbb{R}^m \times \mathbb{R}^n$ and $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$, where $int(S)$ denote the interior of S;

(iv) $F(x, y) = (f(x), g(y)) \in \mathbb{R}^m \times \mathbb{R}^n$, where g is a contraction on the closed unit ball in $\mathbb{R}^n$ and has the unique fixed point in the interior of the unit ball.

The question we discussed is the following.

$$(\#) \text{ Does } G \text{ have chaotic dynamics?}$$

The map G in cases (ii)-(iv) is considered as multidimensional perturbation of f due to bigger dimension of phase space, while G in case (i) is a usual perturbation of f and they have the same phase space. The singular map F in cases (ii)-(iv) can be considered as the skew product $(f(x), q(x, y))$ with different strength on trapping region of $q(x, y)$: vertical contraction $q(x, y) = g(x)$ for case (ii), locally trapping $q(\mathbb{R}^m \times S) \subset int(S)$ for case (iii), and horizontal contraction for $q(x, y) = g(y)$.

In 1975, Li and Yorke [18] introduced the mathematical definition of chaos and established a very simple criterion: "period three implies chaos" for its existence in the real number. This criterion played a key role in predicting and analyzing one-dimensional chaotic dynamical systems. In 1978, Marotto [19] wanted to study chaos for higher dimensional discrete dynamical systems and he proved that "if a differentiable map has a snap-back repeller then it exhibits the sense of Li-Yorke chaos". Based on Marotto's argument, Blanco García [2] showed that a snap-back repeller implied positive topological entropy. Here, in Section 2, we give a definition of snap-back repeller slightly different from Marotto's in [19, 23] so that it is independent of norms and the mentioned results of Marotto and Blanco García still hold obviously. Also we use the implicit function theorem in Banach spaces to prove that any small C^1 perturbation of a (possibly noninvertible) system with a snap-back repeller has a snap-back repeller and exhibits chaos. This establishes one kind of result addressing question (#) in case (i) for snap-back repeller, refer to [13].

In Section 3-4, we focus on the results about topological entropy which is a quantitative measurement of how chaotic a map is. In fact, it is determined by how many different orbits there are for a given map. The methodology we used to study the question (#) is based on the concept of covering relation which was introduced by Zgliczyński in [33, 34], see Section 3 for its background and applications. It allows one to prove the existence of periodic points, the symbolic dynamics and the positive topological entropy without using hyperbolicity. Also, the persistence of covering relation under small perturbation allow one to consider the multidimensional perturbation of systems.

There are several existing literature investigating the question (#) about topological entropy. For the case when f is an interval map and $g = 0$ in a real Banach space, Misiurewicz and Zgliczyński in [8] proved that $\liminf_{\lambda\to 0} h_{\text{top}}(F_\lambda) \geqslant h_{\text{top}}(f)$. For the planar case (ie. $m = n = 1$), Marotto in [21] restricted perturbations to two types: the first one is that

$F_\lambda(x,y) = (\varphi(x,\lambda y),x)$ and $\lambda \in \mathbb{R}$ and the other one that is $F_\lambda(x,y) = (\varphi(x,\lambda_1 y), g(\lambda_2 x, y))$, $\lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2$, and the map $y \mapsto g(0,y)$ has a stable fixed point. Assuming the map $x \mapsto \varphi(x,0)$ is C^1 and has a snap-back repeller, he showed that for all λ near 0, the map F_λ has a transverse homoclinic point. His method relies heavily on the planar structure of the map F_0 and the Birkhoff-Smale transverse homoclinic point theorem. Also, the results from [11, 17] about difference equations can be applied to question (#) for the topological entropy, but these are in fact perturbations of one-dimensional maps.

In subsection 4.1, we establish two kinds of results addressing question (#) in cases (ii) and (iii) for f having positive topological entropy in one dimensional space and snap-back repeller in higher dimensional space, along a one-parameter continuous family $\{F_\lambda\}$ such that $F = F_0$ and $G = F_\lambda$ with small parameter λ. First we show that if f is a one-dimensional map (without any additional assumption) then $\liminf_{\lambda\to 0} h_{\text{top}}(F_\lambda) \geqslant h_{\text{top}}(f)$ (see Theorem 4.1 and 4.2). Second, we allow f to be possibly high-dimensional map and show that if f has a snap-back repeller then $h_{\text{top}}(F_\lambda) > 0$ for all λ near enough 0 (see Theorems 4.9 and 4.10), refer to [16]. Moreover, as a by-product of using covering relation, we give a new proof of Blanco Garcia's result in [2] that the existence of a snap-back repeller implies positive topological entropy (see Proposition 4.8).

Theae results are applicable to a high-dimensional version of the Hénon-like maps. Define a family of maps $H_b(x,y)$ on $\mathbb{R}^m \times \mathbb{R}^n$, with parameter $b \in \mathbb{R}^\ell$, by its components, for $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_n)$,

$$\begin{cases} \bar{x}_i = a_i - x_i^2 + o_i(b)\varphi_i(x,y), & 1 \leqslant i \leqslant m, \\ \bar{y}_j = g_j(x,y), & 1 \leqslant j \leqslant n, \end{cases}$$

where each a_i is a constant, o_i, φ_i, g_j are real-valued continuous functions and $\lim_{b\to 0} o_i(b)/|b| = 0$. If $m = n = 1$, one can reduce H_b to the original Hénon-map $(x,y) \mapsto (a - x^2 + by, x)$ and apply results from this paper as well as from [11, 17, 21]. For the general case when $m \geqslant 1$ and $n \geqslant 1$, we assume that each g_j is either dependent only on x or bounded (hence, the conditions in form (ii) or (iii) are satisfied, respectively). At the singular value $b = 0$, the first m components of H_0, i.e. $\bar{x}_i = a_i - x_i^2$ for $1 \leqslant i \leqslant m$, form a decoupled map from $\mathbb{R}^m$ into itself, and such a map has a positive topological entropy or a snap-back repeller by choosing suitable a_i. By applying the results about topological entropy of multidimensional perturbations with snap-back repellers on lower dimensional map, we get that $h_{\text{top}}(H_b) > 0$ for all b sufficiently near 0.

The idea of a topologically crossing intersection of two submanifolds is from [5, 7, 8] (see subsection 4.2.1 for background). The methodology we use to study the question (#) with f having topologically crossing intersection is based on the construction of topological horseshoe, given by Burns and Weiss in [5], and the concept of covering relations. Topologically crossing homoclinicity guarantees existence of covering relations on which f has both topological contraction and expansion directions. Unlike the discuss in subsection 4.1, the covering relations have only expansion direction for an interval map f with positive topological entropy or a map f with a snap-back repeller. In subsection 4.2.2, we establish the results addressing question (#) in cases (i)-(iii) for f being a C^1 diffeomorphism with a hyperbolic periodic point which has a topologically crossing homoclinic point, along a one-parameter continuous family $\{F_\lambda\}$. We show that F_λ has positive topological entropy for all λ close to 0, refer to [14].

In subsection 4.3.1, we assume that f has covering relations determined by a transition matrix A (see Definition 4.21) and show that for cases (i)-(iii), if G is C^0 close to F, then G has an isolated invariant set to which the restriction G is topologically semi-conjugate to the one-sided subshift of finite type, denote by σ_A^+, and hence the topological entropy of G is greater than the logarithm of the spectral radius of A (see Theorems 4.22-4.24). In addition, in subsection 4.3.2, if the covering relations satisfy the strong Liapunov condition (see Definition 4.28), then we conclude that if a homeomorphism G is C^1 close to F, then G has an isolated invariant set to which the restriction of G is topologically conjugate to the two-side subshift of finite type, denote by σ_A, for the cases (i) and (iv) provided that F is a homeomorphism (see Theorems 4.30 and 4.31), and for the case (ii) provided that G is perturbed from F along a one-parameter continuous family $\{F_\lambda\}$ such that $F = F_0$ and $G = F_\lambda$ with small $|\lambda| \neq 0$ (see Theorem 4.32), refer to [15].

In particular, one can apply the last result to the Hénon-like like family $F_\lambda(x, y) = (f(x) + p(\lambda, x, y), q(\lambda, x, y))$, where f is the logistic map $f(x) = \mu x(1 - x)$ with $\mu > 4$, p and q are C^1 continuous functions of (λ, x, y) such that F_λ is a homeomorphism for $\lambda = 0$, and $h(0, x, y) = 0$ for all (x, y) and $q(0, x, y_1) = q(0, x, y_2)$ for all x, y_1 and y_2. The map f has covering relations which are determined by the 2×2 matrix with all entries one and satisfy the strong Liapunov condition (see Example 4.29). Thus for sufficiently small $|\lambda| \neq 0$, the map F_λ has an isolated invariant set on which F_λ is topologically conjugate to the 2-shift. By setting $p(\lambda, x, y) = \lambda y$ and $q(\lambda, x, y) = x$, the family F_λ becomes the original Hénon family.

2 Snap-back repeller

In this section, we study the snap-back repellers. Recently, Marotto [23] redefined snap-back repeller and stated that his early result in [19]: "a snap-back repeller implies Li-Yorke chaos" is still correct. First, in here, we list the Marotto's definition of snap-back repeller in [23].

Definition 2.1 ([23], Definition 1)**.** *Suppose z is a fixed point of a differentiable map f with all eigenvalues of $Df(z)$ exceeding 1 in magnitude, and suppose there exists a point $x_0 \neq z$ in a repelling neighborhood of z, such that $x_M = z$ and $det(Df(x_k)) \neq 0$ for $1 \leqslant k \leqslant M$, where $x_k = f_k(x_0)$. Then z is called a* snap-back repeller *of f.*

Marotto's definition depend on the norms of the phase space. Now we give our definition of a snap-back repeller which is slightly different form Marotto's definition. It is independent of norms.

Definition 2.2. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a differentiable function. A fixed point w_0 for f is called a* snap-back repeller *if (i) all eigenvalues of $Df(w_0)$ are greater than one in absolute value and (ii) there exists a sequence $\{w_{-n}\}_{n\in\mathbb{N}}$ such that $w_{-1} \neq w_0$, $\lim_{n\to\infty} w_{-n} = w_0$, and for all $n \in \mathbb{N}$, $f(w_{-n}) = w_{-n+1}$ and $det(Df(w_{-n})) \neq 0$.*

Based on Marotto's argument, Blanco García [2] showed a snap-back repeller implies positive topological entropy. The mentioned results of Marotto and Blanco García under our definition till hold. Roughly speaking, a snap-back repeller of a map is a repelling fixed point associated with which there is a transverse homoclinic point. Notice that if there exists a norm $|\cdot|_*$ on $\mathbb{R}^k$ such that for some constants $\delta > 0$ and $\lambda > 1$, one has that $|f(x) - f(y)|_* > \lambda|x - y|_*$ for all $(x, y) \in B(w_0, \delta)$ where $B(w_0, \delta) = \{x \in \mathbb{R}^k : |x - w_0|_* < \delta\}$, then f is one-to-one on $B(w_0, \delta)$ and $f(B(w_0, \delta)) \supset B(w_0, \delta)$; hence item (ii) of Definition 2.2 can be satisfied if there is a point $q \in B(w_0, \delta)$ such that

$f^m(q) = w_0$ and $det(Df^m(q)) \neq 0$ for some positive m. In fact, item (i) implies that such a norm must exist (refer to [29, Theorem V.6.1]). Furthermore, if all eigenvalues of $(Df(w_0))^T Df(w_0)$ are greater than one, then such a norm can be chosen to be the Euclidean norm on $\mathbb{R}^k$ (see [12, Lemma 5]).

2.1 Preliminaries

In this subsection, we recalled the result about "a snap backer repeller implies Li-Yorke Chaos" which was proved by Marotto in [19] and [23] and "a snap-back repeller implies positive topological entropy" which was proved by Blanco García in [2].

First, we describe the mathematical sense of chaos introduced by Li and Yorke in [18]:

Theorem 2.3 ([18], Theorem 1)**.** *Let J be an interval in $\mathbb{R}$ and let $F : J \to J$ be continuous. Assume that there is a point $a \in J$, for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy*

$$d \leqslant a < b < c \text{ (or } d \geqslant a > b > c).$$

Then:

1. *for every $k \in 1, 2, \ldots,$ there is a periodic point in J having period k;*
2. *there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:*

   (a) *for every p, $q \in S$ with $p \neq q$,*

   $$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0$$

   and

   $$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0;$$

(b) *for every* $p \in S$ *and periodic points* $q \in J$,

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

In [19], Marotto studied the Li-Yorke theorem to higher dimensional discrete dynamical systems.

Theorem 2.4 ([19], Theorem 1)**.** *Let* $f : \mathbb{R}^k \to \mathbb{R}^k$ *possess a snap-back repeller. Then* f *exhibits Li-Yorke chaos, that is, there exist*

1. *a positive integer* N *such that if* $m \geqslant N$ *is an integer, the map* f *has a point of period* m*;*

2. *an uncountable set* S *containing no periodic points of* f *such that*

   (a) *if* x, $y \in S$ *with* $x \neq y$, *then*

   $$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0;$$

   (b) *if* $x \in S$ *and* y *is a periodic point for* f, *then*

   $$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0;$$

   (c) $f(S) \subset S$*; and*

3. *an uncountable subset* S_0 *of* S *such that if* x, $y \in S_0$, *then*

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

Next, we review the background of topological entropy. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. For $n \in \mathbb{N}$, the function

$$d_{n,f}(x, y) = \max_{0\leqslant k<n} d(f^k(x), f^k(y))$$

measures the maximum distance between the first n iterates of x and y. For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $S \subset X$ is called (n, ϵ)-separated for f provided

$d_{n,f}(x, y) > \epsilon$ for every pair of points x, $y \in S$ with $x \neq y$. The number of different orbits of length n (as measured by ϵ) is defined by

$$r(n, \epsilon, f) = \max\{\#(S) : S \subset X \text{ is a } (n, \epsilon)\text{-separated set for } f\},$$

where $\#(S)$ is the number (cardinality) of elements in S. In order to measure the growth rate of $r(n, \epsilon, f)$ as n increases, we define

$$h(\epsilon, f) = \limsup_{n\to\infty} \frac{\log(r(n, \epsilon, f))}{n}.$$

Finally, we consider $h(\epsilon, f)$ varies as ϵ goes to 0 and define the topological entropy of f as

$$h_{\text{top}}(f) = \lim_{\epsilon\to 0^+} h(\epsilon, f).$$

Moreover, let $f : X \to X$ be a continuous function where X is a metric space. Here, the topological entropy of f is defined to be the supremum of topological entropies of f restricted to compact invariant sets. Refer to [29] for more background.

Blanco García [2] proved that a snap-back repeller implies positive topological entropy.

Theorem 2.5 ([2], Theorem 1)**.** *Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a differentiable map. If F has a snap-back repeller, then F has positive topological entropy.*

2.2 Persistence of snap-back repeller

In this subsection, we show the persistence of snap-back repeller for small C^1 perturbations by using the implicit function theorem in Banach spaces (refer to Lang's textbook [31, Theorem 6.2.1]). Let k be a positive integer, $|\cdot|_2$ be the Euclidean norm on $\mathbb{R}^k$, and $||\cdot||_2$ be the operator-norm on the space of linear maps on $\mathbb{R}^k$ induced by $|\cdot|_2$.

Theorem 2.6. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1 map on $\mathbb{R}^k$ with a snap-back repeller. If g is a C^1 map on $\mathbb{R}^k$ such that $|f - g|_2 + ||Df - Dg||_2$ is small* *enough, then g has a snap-back repeller, exhibits Li-Yorke chaos, and has positive entropy.*

Proof. Let x_0 be a snap-back repeller of f and $\{x_{-n}\}_{n\in\mathbb{N}}$ be its corresponding homoclinic orbit with $x_{-1} \neq x_0$, $\lim_{n\to\infty} x_{-n} = x_0$, and for all $n \in \mathbb{N}$, $f(x_{-n}) = x_{-n+1}$ and $det(Df(x_{-n})) \neq 0$. Since x_0 is a fixed point of f and all eigenvalues $Df(x_0)$ are greater than one in absolute value, there exists a norm $|\cdot|_*$ on $\mathbb{R}^k$ such that for some constants $\delta_0 > 0$ and $\lambda_0 > 1$, one has that $|f(x) - f(y)|_* > \lambda_0|x-y|_*$ for all x, $y \in B(x_0, \delta_0)$, where $B(x_0,\delta_0) = \{x \in \mathbb{R}^k : |x - x_0|_* < \delta_0\}$. Thus f is one-to-one on $B(x_0, \delta_0)$ and $f(B(x_0, \delta_0)) \supset B(x_0, \delta_0)$. Let $||\cdot||_*$ denote the operator-norm in the space of linear maps on $\mathbb{R}^k$ induced by $|\cdot|_*$. Let λ_1 be a constant with $1 < \lambda_1 < \lambda_0$ and let $U(f, \lambda_0 - \lambda_1)$ denote the set of all C^1 maps g on $\mathbb{R}^k$ with $|f - g|_* + ||Df - Dg||_* < \lambda_0 - \lambda_1$. Then for any $g \in U(f, \lambda_0 - \lambda_1)$ and x, $y \in B(x_0, \delta_0)$, we have that

$$\begin{aligned} |g(x) - g(y)|_* &\geqslant |f(x) - f(y)|_* - |(g-f)(x) - (g-f)(y)|_* \qquad (2.1)\\ &> [\lambda_0 - (\lambda_0 - \lambda_1)]|x-y|_* = \lambda_1|x-y|_*; \end{aligned}$$

hence, g is one-to-one on $B(x_0, \delta_0)$. Let $\delta > \delta_0$ be a constant so that $\{x_{-n}\}_{n\in\mathbb{N}} \subset B(x_0, \delta_0)$. Denote by W the closure of $B(x_0, \delta_0)$. Then W is a compact subset of $\mathbb{R}^k$. Let S be the space of C^1 functions from W to $\mathbb{R}^k$ endowed with the usual C^1 topology d_{C^1} which is induced from the norm $|\cdot|_*$ on $\mathbb{R}^k$. Then S is a Banach space and the restriction of any C^1 map g on $\mathbb{R}^k$ to W, denoted by $g|W$, is in S. Since x_0 is a snap-back repeller of f and all eigenvalues of $Df(x_0)$ are greater than one in absolute value, there exist positive constants λ_2, δ_1 and a positive integer M such that $\lambda_1 < \lambda_2 < \lambda_0$, $\delta_1 < \delta_0$, $x_{-M} \in B(x_0, \delta_1)\backslash\{x_0\}$, $det(Df^M(x_{-M})) \neq 0$, $x_0 \in int(f^M(B(x_0, \delta_1)\backslash\{x_0\}))$ and for all $g \in U(f, \lambda_0 - \lambda_2)$ and $x \in B(x_0, \delta_1)$, all eigenvalues of $Dg(x)$ are greater than one in absolute value. Let λ_3 be a constant such that

$$\max\{\lambda_2, \frac{\lambda_0+\delta_1}{1+\delta_1}\} < \lambda_3 < \lambda_0. \tag{2.2}$$

Then for any $g \in U_W(f, \lambda_0 - \lambda_3)$, we have that g is one-to-one on $B(x_0, \delta_1)$. In addition, if $x \in \mathbb{R}^k$ with $|x - x_0|_* = \delta_1$, by Equation (2.1) with λ_1 replaced by λ_3 and Equation (2.2), we get that

$$|g(x) - x_0|_* \geqslant |f(x) - x_0|_* - |g(x) - f(x)|_* > \lambda_3\delta_1 - (\lambda_0 - \lambda_3) > \delta_1.$$

Moreover, the continuity of g implies that $g(B(x_0, \delta_1)) \supset B(x_0, \delta_1)$. Let $V = B(x_0, \delta_1)\backslash\{x_0\}$ and $U_W(f, \lambda_0 - \lambda_3) = \{g|W : g \in U(f, \lambda_0 - \lambda_3)\}$. For the first desired result, we need to show the existence of a snap-back repeller for any $g \in U_W(f, \lambda_0 - \lambda_3)$ near f. Define $H : U_W(f, \lambda_0 - \lambda_3) \times W \times V \to \mathbb{R}^k \oplus \mathbb{R}^k$ by $H(g, x, y) = (g(x) - x, g^M(y) - x)$. Then $H(f, x_0, x_{-M}) = 0$ and H is C^1 on its domain; refer to [10, Appendix B]. Since all eigenvalues of $Df(x_0)$ are greater than one in absolute value, we have $det(Df(x_0) - I_k) \neq 0$, where I_k denotes the identity matrix of size k; refer to [29, Lemma V.5.7.2]. By the chain rule, $det(Df^M(x_{-M})) = \prod_{i=1}^{M} det(Df(x_{-i})) \neq 0$. Hence, by writing $z = (x, y) \in W \times V$, we have

$$det\left(\frac{\partial H}{\partial z}(g, z)|_{g=f,\ z=(x_0, x_M)}\right) = det\begin{bmatrix} Df(x_0) - I_k & 0 \\ -I_k & Df^M(x_{-M}) \end{bmatrix} \neq 0;$$

refer to [28, Proposition 0.0]. By the implicit function theorem applied to the function H, there exist positive constants λ_4, δ_2, η and a C^1 map $h : U_W(f, \lambda_0 - \lambda_4) \to B(x_0, \delta_2) \times B(x_{-M}, \eta)$ such that $\lambda_3 < \lambda_4 < \lambda_0$, $\delta_2 < \delta_1$, $B(x_{-M}, \eta) \subset V$, $B(x_0, \delta_2) \cap B(x_{-M}, \eta) = \emptyset$, and for every $g \in U_W(f, \lambda_0 - \lambda_4)$, one has that $h(g) \equiv (h_1(g), h_2(g))$ is the unique solution for the system of equations $g(x) = x$ and $g^M(y) = x$ in $B(x_0, \delta_2) \times B(x_{-M}, \eta)$, and $det(Dg^M(h_2(g))) \neq 0$. In particular, $h(f) = (x_0, x_{-M})$.

To conclude that the point $h_1(g)$ is a snap-back repeller of g, it remains to show that $h_2(g)$ has a backward orbit converging to $h_1(g)$. Let $g \in$

$U_W(f, \lambda_0 - \lambda_4)$ and denote $y_{-M+i} = g^i(h_2(g))$ for all $0 \leqslant i \leqslant M-1$. Then $y_{-M} \neq h_1(g)$ and $g^M(y_{-M}) = h_1(g)$. Since g is one-to-one on $B(x_0, \delta_1)$, $g(B(x_0, \delta_1)) \supset B(x_0, \delta_1)$ and $h_2(g) \in B(x_0, \delta_1)$, we can define $y_{-M-i} = \hat{g}^{-i}(h_2(g))$ inductively for $i \geqslant 1$, where $\hat{g}^{-1} = (g|B(x_0, \delta_1))^{-1}$ denotes the inverse of the restriction of g to $B(x_0, \delta_1)$ and $\hat{g}^{-i}$ denotes the ith iterate of $\hat{g}^{-1}$. Then the sequence $\{y_{-i}\}_{i \in \mathbb{N}}$ forms a backward orbit of $h_1(g)$ such that $y_{-n} \in B(x_0, \delta_1)$ for all $n \geqslant M$. From Equation (2.1), we obtain that for any $x, y \in B(x_0, \delta_1)$,

$$|\hat{g}^{-1}(x) - \hat{g}^{-1}(y)|_* < \lambda_1^{-1}|x - y|_* \tag{2.3}$$

By considering inequality (2.3) inductively, we have that for any $i \geqslant 1$,

$$|y_{-M-i} - h_1(g)|_* = |\hat{g}^{-i}(y_{-M}) - \hat{g}^{-i}(h_1(g))|_* < \lambda_1^{-i}|y_{-M} - h_1(g)|_*.$$

This shows that $\lim_{n \to \infty} y_{-n} = h_1(g)$.
Since the norms $|\cdot|_2$ and $|\cdot|_*$ on $\mathbb{R}^k$ are equivalent, the proof of the first desired result is now complete. The second and third assertions immediately follow from Theorem 2.4 and 2.5. □

Notice that from the above proof of Theorem 2.6, it is sufficient to require a smallness of $|f - g|_2 + ||Df - Dg||_2$ locally in a neighborhood of the homoclinic orbit associated to the snap-back repeller, instead of globally in $\mathbb{R}^k$.

As an immediate consequence of the above theorem, we have the following result for a parametrized family.

Corollary 2.7. *Let $f_\mu(x)$ be a one-parameter family of C^1 maps with variable $x \in \mathbb{R}^k$ and parameter $\mu \in \mathbb{R}$. Assume that $f_\mu(x)$ is C^1 as a function jointly of x and μ and that f_{μ_0} has a snap-back repeller. Then for all μ sufficiently close to μ_0, the map f_μ has a snap-back repeller, exhibits Li-Yorke chaos, and has positive topological entropy.*

Next is another application to perturbations of a decoupled system.

Corollary 2.8. *Let $f_\epsilon : \mathbb{R}^k \to \mathbb{R}^k$ be a one-parameter family of C^1 maps with components $(f_\epsilon)_i(x) = h_i(x_i) + \epsilon_i g_i(x)$ for each $1 \leqslant i \leqslant k$; here we denote the variable $x = (x_1, ..., x_k)$ and the parameter $\epsilon = (\epsilon_1, ..., \epsilon_k)$ in $\mathbb{R}^k$. If the number of snap-back repellers for each map h_i is $m_i \geqslant 1$, then for all sufficiently small $|\epsilon|$, the number of snap-back repellers for the map f_ϵ is at least $\prod_{i=1}^M m_i$.*

Gardini et al. [6] studied the double logistic map $T_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_\lambda(x, y) = (1 - \lambda)x + 4\lambda y(1 - y), (1 - \lambda)y + 4\lambda x(1 - x)), \ \lambda \in [0, 1]; \quad (2.4)$$

therein the basins of attraction of the absorbing areas are determined together with their bifurcations. Moreover, it was mentioned that $T_1^2(x, y) = (h^2(x), h^2(y))$, where $h(x) = 4x(1-x)$, has a snap-back repeller at the origin. Therefore, applying Corollary 2.8, we have the following result.

Corollary 2.9. *For all λ near one, the second iterate of system (2.4) has a snap-back repeller, exhibits Li-Yorke chaos, and has positive topological entropy.*

3 Covering relations

In this section, we give the background information about covering relations and list some properties of the local Brouwer degree.

3.1 Background and applications

In this subsection, we introduce the definition and some applications of covering relation. Suppose that $\mathbb{R}^m$ has a norm $|\cdot|$. For $x \in \mathbb{R}^m$ and $r > 0$, we denote $B_m(x, r) = \{z \in \mathbb{R}^m : |z - x| < r\}$, that is, the open ball of radius r centered at the origin 0 in $\mathbb{R}^m$; in short, we write $B_m = B_m(0, 1)$, the open unit ball in $\mathbb{R}^m$. Moreover, for a subset S of $\mathbb{R}^m$, let $\overline{S}$ and ∂S denote the closure and the boundary of S, respectively. It will be always clear from the context which norm is used.

Now, we briefly recall some definitions from [35] concerning covering relations.

Definition 3.1. *[35, Definition 6] An* h-set *in* $\mathbb{R}^m$ *is a quadruple consisting of the following data:*

- *a nonempty compact subset* M *of* $\mathbb{R}^m$*,*
- *a pair of numbers* $u(M), s(M) \in \{0, 1, ..., m\}$ *with* $u(M) + s(M) = m$*,*
- *a homeomorphism* $c_M : \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^{u(M)} \times \mathbb{R}^{s(M)}$ *with* $c_M(M) = \overline{B^{u(M)}} \times \overline{B^{s(M)}}$*, where* $S \times T$ *is the Cartesian product of sets* S *and* T*.*

For simplicity, we will denote such an h-set by M *and call* c_M *the* coordinate chart *of* M*; furthermore, we use the following notations:*

$$M_c = \overline{B^{u(M)}} \times \overline{B^{s(M)}},\ M_c^- = \partial B^{u(M)} \times \overline{B^{s(M)}},\ M_c^+ = \overline{B^{u(M)}} \times \partial B^{s(M)},$$

$$M^- = c_M^{-1}(M_c^-),\ \text{and } M^+ = c_M^{-1}(M_c^+).$$

A covering relation between two h-sets is defined as follow.

Definition 3.2. *[35, Definition 7] Let M, N be h-sets in $\mathbb{R}^m$ with $u(M) = u(N) = u$ and $s(M) = s(N) = s$, $f : M \to \mathbb{R}^m$ be a continuous map, and $f_c = c_N \circ f \circ c_M^{-1} : M_c \to \mathbb{R}^u \times \mathbb{R}^s$. We say M f-*covers *N, and write*

$$M \stackrel{f}{\Longrightarrow} N,$$

if the following conditions are satisfied:

1. *there exists a homotopy $h : [0,1] \times M_c \to \mathbb{R}^u \times \mathbb{R}^s$ such that*

$$\begin{aligned} h(0,x) &= f_c(x) \text{ for } x \in M_c, && (3.1)\\ h([0,1], M_c^-) \cap N_c &= \emptyset, && (3.2)\\ h([0,1], M_c) \cap N_c^+ &= \emptyset; && (3.3) \end{aligned}$$

2. *there exists a map $\varphi : \mathbb{R}^u \to \mathbb{R}^u$ such that*

$$h(1,p,q) = (\varphi(p), 0) \text{ for any } p \in \overline{B^u} \text{ and } q \in \overline{B^s},$$
$$\varphi(\partial B^u) \subset \mathbb{R}^u \backslash \overline{B^u}; \text{ and}$$

3. *there exists a nonzero integer w such that the local Brouwer degree* $\deg(\varphi, B^u, 0)$ *of φ at 0 in B^u is w; refer to [35, Appendix] for its properties.*

Usually, we will be not interested in the values of w among covering relations and we just write $M \stackrel{f}{\Longrightarrow} N$ instead of $M \stackrel{f,w}{\Longrightarrow} N$.

Next, we list two important results derived from the covering relations which is proved by Zgliczyński and Gidea in [35]. The first one is that a closed loop of covering relations implies existence of a periodic point.

Theorem 3.3. *[35, Theorem 9] Let $\{f_i\}_{i=1}^k$ be a collection of continuous maps on $\mathbb{R}^m$ and $\{M_i\}_{i=1}^k$ be a collection of h-sets in $\mathbb{R}^m$ such that $M_{k+1} = M_1$*

and $M_i \overset{f_i}{\Longrightarrow} M_{i+1}$ *for* $1 \leqslant i \leqslant k$. *Then there exists a point* $x \in int(M_1)$ *such that*

$$\begin{aligned} &f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in int(M_{i+1}) \text{ for } i = 1, ...k, \text{ and} \\ &f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \end{aligned}$$

The following one shows that a covering relation is persistent under C^0 small perturbations.

Theorem 3.4. *[35, Theorem 14] Let* M *and* N *be h-sets with* $u(M) = u(N) = u$ *and* $s(M) = s(N) = s$ *and let* $f, g : M \to \mathbb{R}^m$ *be continuous. Assume that* $M \overset{f,w}{\Longrightarrow} N$ *and that the coordinate chart* c_N *satisfies a Lipschitz condition. Then there exists* $\epsilon > 0$ *such that if* $|f(x) - g(x)| < \epsilon$ *for all* $x \in M$ *then* $M \overset{g,w}{\Longrightarrow} N$.

Moreover, the following one shows that a covering relation is persistent under C^0 small perturbations. This result slightly extends theorem 3.4 by dropping the Lipschitz condition of the coordinate chart.

Proposition 3.5. *Let* M_1 *and* M_2 *be h-sets with* $u(M_1) = u(M_2) = u$ *and* $s(M_1) = s(M_2) = s$ *and let* $f,\ g : M_1 \to \mathbb{R}^m$ *be continuous. Assume that*

$$M_1 \overset{f,w}{\Longrightarrow} M_2.$$

Then there exists $\delta > 0$, *such that if* $|f(x) - g(x)| < \delta$ *for all* $x \in M_1$ *then*

$$M_1 \overset{g,w}{\Longrightarrow} M_2.$$

Proof. By using Theorem 3.4, there exists $\epsilon > 0$ such that if $|f_c(x) - g_c(x)| < \epsilon$ for all $x \in M_{1,c}$ then

$$M_1 \overset{g,w}{\Longrightarrow} M_2.$$

Since M_1 is compact, there exists $r > 0$ such that $f(M_1) \subset \overline{B_m(0,r)}$.

If $|f(x) - g(x)| < 1$ for all $x \in M_1$, then $g(M_1) \subset \overline{B_m(0, r+1)}$. By uniform continuity of c_{M_2} on $\overline{B_m(0, r+1)}$, there exists $\delta' > 0$ such that if $z, z' \in \overline{B_m(0, r+1)}$ and $|z - z'| < \delta'$ then $|c_{M_2}(z) - c_{M_2}(z')| < \epsilon$. Let $\delta = \min\{\delta', 1\}$. If $|f(x) - g(x)| < \delta$ for all $x \in M_1$ then

$$\max_{x \in M_{1,c}} |f_c(x) - g_c(x)| = \max_{x \in M_1} |c_{M_2}(f(x)) - c_{M_2}(g(x))| < \epsilon.$$

Thus $M_1 \overset{g,w}{\Longrightarrow} M_2$. □

3.2 Properties of local Brouwer degree

In this subsection, we list some basic properties of local Brouwer degree; refer to [30, Chapter III] for the proof. Let n be a positive integer and $T \subset \mathbb{R}^n$ be an open and bounded set. Let $\varphi : D \to \mathbb{R}^n$ be continuous, $\bar{T} \subset D$ and $q \in \mathbb{R}^n$ with $q \notin \varphi(\partial T)$.

1. Integer property:

$$\deg(\varphi, T, q) \in \mathbb{Z};$$

2. Solution property: If $\deg(\varphi, T, q) \neq 0$, then there exists $x \in T$ such that

$$\varphi(x) = q;$$

3. Invariance under homotopy: Let $H : [0, 1] \times D \to \mathbb{R}^n$ be continuous. Suppose that $p \notin H([0, 1], \partial T)$. Then for all $\lambda \in [0, 1]$,

$$\deg(H_0, T, p) = \deg(H_\lambda, T, p);$$

4. Local constant property: If p and q lie in the same connected component of $\mathbb{R}^n \backslash \varphi(\partial T)$, then

$$\deg(\varphi, T, p) = \deg(\varphi, T, q);$$

5. The excision property: Assume $\varphi^{-1}(q) \cap D \subset T$, then

$$\deg(\varphi, T, q) = \deg(\varphi, D, q);$$

6. Multiplication property: Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mappings and Δ_i be the components of $\mathbb{R}^n \backslash \varphi(\partial T)$. Then

$$\deg(\psi \circ \varphi, T, q) = \sum_{\Delta_i} \deg(\psi, \Delta_i, q) \deg(\varphi, T, \Delta_i);$$

   where $\deg(\varphi, T, \Delta_i) = \deg(\varphi, T, q_i)$ for some $q_i \in \Delta_i$.

7. Addition property: If $T = \bigcup_{i \in I} T_i$, where each T_i is open, $\partial T_i \subset \partial T$, and the family $\{T_i\}_{i \in I}$ are mutually disjoint, then

$$\deg(\varphi, T, q) = \sum_{i \in I} \deg(\varphi, D, q);$$

8. If φ is C^1 and for each $x \in \varphi^{-1}(q) \cap T$ the Jacobian matrix of φ at x, denoted by $D\varphi_x$, is nonsingular, then

$$\deg(\varphi, T, q) = \sum_{x \in \varphi^{-1}(q) \cap T} sgn(\det D\varphi_x),$$

   where sgn represents the sign function.

Form the above properties, we can derive the following proposition which is used later.

Proposition 3.6. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map and $p \in \mathbb{R}^n$ such that $\psi^{-1}(p)$ consists of a single point and lies in a bounded connected component Δ of $\mathbb{R}^n \setminus \varphi(\partial T)$, and $D\psi_{\psi^{-1}(p)}$ is nonsingular. Then*

$$\deg(\psi \circ \varphi, T, p) = sgn(\det D\psi_{\psi^{-1}(p)}) \deg(\varphi, T, v),$$

for any $v \in \Delta$.

4 Topological dynamics for multidimensional perturbations

In this section, the topological dynamics for multidimensional perturbations of maps are studied. We investigate the question (#) with the lower dimensional map, for cases(i)-(iii), which has positive topological entropy, snap-back repeller, or topologically crossing homoclinicity and for cases (i)-(ii) and (iv), which has covering relations determined by a transition matrix.

4.1 Snap-back repellers and one dimensional maps

In this subsection, we state our result about the topological entropy of multidimensional perturbations of a continuous map f on a lower dimensional phase space, say $\mathbb{R}^m$, to a continuous family of maps F_λ on a high-dimensional space, say $\mathbb{R}^m \times \mathbb{R}^n$, where $\lambda \in \mathbb{R}^\ell$ is a parameter, such that at $\lambda = 0$, the singular map F_0 is one of the cases (ii) and (iii) referred to question (#). The case (i) with snap-back repeller on the on a lower dimensional phase space is discussed in section 2.

4.1.1 One dimensional maps

First, we state the results for multidimensional perturbations of a one dimensional maps.

Let f be a continuous map on $\mathbb{R}$. If the singular map F_0 depends only on the phase variable of f (refer to case (ii)), we have the following result.

Theorem 4.1. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R} \times \mathbb{R}^n$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R} \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x))$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}^n$, where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^n$. Then* $\liminf_{\lambda \to 0} h_{top}(F_\lambda) \geqslant h_{top}(f)$.

For the case when the singular map is locally trapping along the normal direction (refer to case (iii)), we have the following.

Theorem 4.2. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}\times\mathbb{R}^n$ such that $F_\lambda(x,y)$ is continuous as a function jointly of $\lambda\in\mathbb{R}^\ell$ and $(x,y)\in\mathbb{R}\times\mathbb{R}^n$. Assume that $F_0(x,y)=(f(x),g(x,y))$ for all $(x,y)\in\mathbb{R}\times\mathbb{R}^n$, where $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, and $g(\mathbb{R}\times S)\subset int(S)$ for some compact set $S\subset\mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda)\geqslant h_{top}(f)$.*

In order to prove the above theorems, we need the following lemma, which can be easy derived from [25]; see also Theorem 3.1 of Misiurewicz and Zgliczyński in [26]. It says that for continuous interval maps, the positive topological entropy is realized by horseshoes.

Lemma 4.3. *Let I be a closed interval in $\mathbb{R}$ and $f:I\to I$ be a continuous map with a positive topological entropy, i.e. $h_{top}(f)>0$. Then there exist sequences $\{s_k\}_{k=1}^\infty$ and $\{t_k\}_{k=1}^\infty$ of positive integers such that for each $k\in\mathbb{N}$ there exist s_k disjoint closed intervals, $N_1,\ldots,N_{s_k}$, which are h-sets in $\mathbb{R}$ and satisfy the covering relations $N_i\overset{f^{t_k},w_{i,j}}{\Longrightarrow}N_j$ with $w_{i,j}\in\{-1,1\}$ for all $1\leqslant i$, $j\leqslant k$; moreover, one has $\lim_{k\to\infty}(log(s_k)/t_k)=h_{top}(f)$.*

Now we are ready to prove the Theorems 4.1 and 4.2.

Proof of Theorem 4.1. We only need to consider the case when f has a positive topological entropy. Let δ be an arbitrary number such that $0<\delta<h_{\mathrm{top}}(f)$. From Lemma 4.3, there exist k, $p\in\mathbb{N}$ such that f^k has p disjoint closed intervals, denoted by $N_i'=[a_{2i},a_{2i+1}]$ for $0\leqslant i\leqslant p-1$ with $a_0<\cdots<a_{2p-1}$, which are h-sets satisfying

$$N_i'\overset{f^{t_k},w_{i,j}}{\Longrightarrow}N_j' \text{ for } 0\leqslant i\leqslant p-1 \text{ and } 0\leqslant j\leqslant p-1,$$

where $w_{i,j}=1$ or -1, and $log(p)/k>\delta$.

Set $N' = \cup_{i=0}^{p-1} N'_i$. Since $g \circ f^{k-1}$ is continuous and N' is compact, there exists $r > 0$ such that $g \circ f^{k-1}(N') \subset B_n(0,r)$. Set $N_i = N'_i \times \overline{B_n(0,r)}$ for $0 \leqslant i \leqslant p-1$ and $N = \cup_{i=0}^{p-1} N_i$. Then every N_i is an h-set for $0 \leqslant i \leqslant p-1$ and N is compact in $\mathbb{R} \times \mathbb{R}^n$. For $\lambda = 0$, we have $F_0^k(x,y) = (f^k(x), g \circ f^{k-1}(x))$. Hence there are covering relations:

$$N_i \overset{F_0^k, w_{i,j}}{\Longrightarrow} N_j \text{ for } 0 \leqslant i \leqslant p-1 \text{ and } 0 \leqslant j \leqslant p-1.$$

Since $F_\lambda^k(z)$ is uniformly continuous on a compact set, say $[-1,1] \times N$, as a function jointly of λ and z, by using Theorem 3.4 for p^2 times while each c_{N_j} is linear and satisfies the Lipschitz condition, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then we have

$$N_i \overset{F_\lambda^k, w_{i,j}}{\Longrightarrow} N_j \text{ for } 0 \leqslant i \leqslant p-1 \text{ and } 0 \leqslant j \leqslant p-1.$$

Let m be a positive integer and $|\lambda| < \lambda_0$. Consider any closed loop

$$N_{\alpha_0} \overset{F_\lambda^k}{\Longrightarrow} N_{\alpha_1} \overset{F_\lambda^k}{\Longrightarrow} \cdots \overset{F_\lambda^k}{\Longrightarrow} N_{\alpha_m},$$

where every $\alpha_i \in \{0, 1, ...p-1\}$ and $\alpha_m = \alpha_0$. By using Theorem 3.3, F_λ^k has a periodic point $x = x(\lambda) \in int(N_{\alpha_0})$ such that $F_\lambda^{km}(x) = x$. Since there are p^m choices of such closed loops, F_λ^k has at least p^m periodic points in N. These periodic points provide a (m,ϵ)-separated set for F_λ^k as long as ϵ is a positive number less than gaps of N'_is, i.e. $0 < \epsilon < min\{a_{2i} - a_{2(i-1)+1} : 1 \leqslant i \leqslant p-1\}$. Since m is arbitrarily chosen, we have $h_{\text{top}}(F_\lambda^k) \geqslant log(p)$ and so $h_{\text{top}}(F_\lambda) \geqslant log(p)/k > \delta$. Therefore, $\liminf_{\lambda \to 0} h_{\text{top}}(F_\lambda) \geqslant h_{\text{top}}(f)$. □

The proof of the second main result is the following.

Proof of Theorem 4.2. Define $G_\lambda = (id, c) \circ F_\lambda \circ (id, c)^{-1}$, where id denotes the identity map on $\mathbb{R}$ and c is a homeomorphism from S to B_n. Then the topological entropies of G_λ and F_λ are equal. By applying the above argument to the family G_λ while the corresponding c_M of a covering relation $N \overset{G_\lambda, w}{\Longrightarrow} M$ is the identity now, we have the desired result. □

4.1.2 Higher dimensional maps

In this subsection, we will study the topological entropy for multidimensional perturbations of a higher dimensional map which has a snap-back repeller.

As the result of Theorem 3.3 and 3.4, we shall construct the a closed loop of covering relations for the map. Throughout this subsection, we assume that $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 map having a snap-back repeller x_0 associated with a transverse homoclinic orbit. We shall construct two closed loops of covering relations for f: the first one is from the snap-back repeller to a homoclinic point then back to the repeller, and the second one consists of just one relation $N_r \overset{f}{\Longrightarrow} N_r$, where N_r is one of the h-sets in the first closed loop. Then we use the covering relations approach to prove that f has a positive topological entropy.

Let L be a linearization of f at x_0, that is, $L(z) = x_0 + Df(x_0)(z - x_0)$ for $z \in \mathbb{R}^m$. Since all eigenvalues of $Df(x_0)$ are greater than one in absolute value, there exist a norm $|\cdot|$ on $\mathbb{R}^m$ and a constant $\rho > 1$ such that

$$|Df(x_0)z| \geqslant \rho|z| \text{ for } z \in \mathbb{R}^m. \tag{4.1}$$

From now on, we keep this norm fixed.

For any $r > 0$ and $x \in \mathbb{R}^m$, we denote the closed ball with the center x and radius r by

$$N(x, r) = \{x\} + \overline{B_m(0, r)}.$$

For any $r > 0$ we define an h-set $N_{x,r}$ in $\mathbb{R}^m$ as follows: we set $N_{x,r} = N(x, r)$, $c_{N_{x,r}}(z) = (z - x)/r$, $u(N_{x,r}) = m$ and $s(N_{x,r}) = 0$. Since the point x_0 is a fixed point for f and will play a distinguished role in the following, we will write N_r instead of $N_{x_0,r}$. Next, we define a homotopy from the map f to L, its linearization at x_0, as follows:

$$f_\mu(z) = (1 - \mu)f(z) + \mu L(z) \text{ for } \mu \in [0, 1] \text{ and } z \in \mathbb{R}^m. \tag{4.2}$$

It is easy to see that $f_0(z) = f(z)$, $f_1(z) = L(z)$ and $Df_\mu(z) = (1-\mu)Df(z) + \mu Df(x_0)$ for all μ and z. This homotopy will be later used in covering relations in the vicinity of the snap-back repeller.

First, we show that the size of the repulsion set for snap-back repeller x_0 can be chosen uniformly for all f_μ for $\mu \in [0,1]$.

Lemma 4.4. *Let $\beta = (\rho+1)/2$. Then there exists $r_0 > 0$ such that for any $\mu \in [0,1]$, $0 < r \leqslant r_0$, $z \in N_r$ with $|z - x_0| = r$, the following holds:*

$$|f_\mu(z) - x_0| > \beta r.$$

Proof. By using Taylor's theorem with an integral remainder, we have

$$f_\mu(z) - x_0 = f_\mu(z) - f_\mu(x_0) = C(z - x_0),$$

where

$$C = C(\mu, z, x_0) = \int_0^1 Df_\mu(x_0 + t(z - x_0))dt.$$

By Equation (4.2), we get that

$$\begin{aligned} C - Df_\mu(x_0) &= \int_0^1 (1-\mu)Df(x_0 + t(z-x_0)) + \mu Df(x_0)dt - Df_\mu(x_0) \\ &= \int_0^1 (1-\mu)[Df(x_0 + t(z-x_0)) - Df(x_0)]dt. \end{aligned} \tag{4.3}$$

Since Df is continuous at x_0 and $\rho > 1$, there exists $r_0 > 0$ such that if $|y - x_0| \leqslant r_0$ then $|Df(y) - Df(x_0)| < (\rho - 1)/2$. Hence, from Equation (4.3), we have that for any $\mu \in [0,1]$ and $z \in \overline{B_m(x_0, r)}$,

$$\begin{aligned} |C - Df_\mu(x_0)| &\leqslant \int_0^1 (1-\mu)|Df(x_0 + t(z-x_0)) - Df(x_0)|dt \\ &< \int_0^1 (1-\mu)\frac{\rho-1}{2}dt \leqslant \frac{\rho-1}{2}. \end{aligned}$$

Therefore, by using Equation (4.1), we have that for any $\mu \in [0,1]$, $0 < r \leqslant r_0$, $z \in N_r$ with $|z - x_0| = r$,

$$\begin{aligned} |f_\mu(z) - x_0| &= |C(z-x_0)| = |(C - Df_\mu(x_0) + Df_\mu(x_0))(z-x_0)| \\ &\geqslant |Df(x_0)(z-x_0)| - |(C - Df_\mu(x_0))(z-x_0)| \\ &> \rho r - \frac{\rho-1}{2}r = \beta r. \end{aligned}$$

$\square$

Throughout the rest of this subsection, we fix the two constants β and r_0 as given in Lemma 4.4. In the following, we establish a covering relation between two h-sets around the snap-back repeller.

Proposition 4.5. *Let r and r_1 be two numbers satisfying $0 < r \leqslant r_0$ and $0 < r_1 \leqslant \beta r$. Then the following covering relation holds:*

$$N_r \overset{f}{\Longrightarrow} N_{r_1}.$$

Proof. Define $h(\mu, z) = c_{N_{r_1}}(f_\mu(c_{N_{r_1}}^{-1}(z))$. We need to check whether all conditions for the covering relation $N_r \overset{f}{\Longrightarrow} N_{r_1}$. are satisfied. First we deal with the conditions in the first item of Definition 3.2. Condition (3.1) is implied by $f_0 = f$, Condition (3.2) follows from Lemma 4.4, and since $N_{r_1}^+ = \emptyset$, Condition (3.3) is also satisfied.

Next, we define a map A on $\mathbb{R}^m$ by $A(z) = (r/r_1)Df(x_0)z$. Then for $z \in \overline{B_m}$, we have

$$h(1, z) = \frac{L(rz + x_0) - x_0}{r_1} = \frac{Df(x_0)(rz)}{r_1} = A(z).$$

Moreover, from Equation (4.1) it follows that for $z \in \overline{B_m}$ with $|z| = 1$,

$$|A(z)| \geqslant \frac{\rho r}{r_1} \geqslant \frac{\rho r}{\beta r} > 1.$$

Since A is linear, from the above equation we have that $\deg(A, B_m, 0) = \pm \det(A) \neq 0$. $\square$

Next, we give a covering relation from the snap-back repeller x_0 to points near x_0, which will be homoclinic points near x_0 as the result is used later.

Lemma 4.6. *Let $r > 0$, $r_1 > 0$ and $z_1 \in \mathbb{R}^m$ near x_0 satisfy that $(|z_1 - x_0| + r_1)/\beta < r < r_0$. Then*

$$N_r \overset{f}{\Longrightarrow} N_{z_1, r_1}.$$

Proof. As in the proof of Proposition 4.5, we set $h(\mu, z) = c_{N_{z_1,r_1}}(f_\mu(c_{N_r}^{-1}(z))$. Again, we need to check all conditions for the covering relation $N_r \overset{f}{\Longrightarrow} N_{z_1,r_1}$.

Condition (3.1) is implied by $f_0 = f$, and since $N_{z_1,r_1}^+ = \emptyset$, Condition (3.3) is also satisfied.

To verify Condition (3.2), observe that it is equivalent to the following one:

$$f_\mu(N_r^-) \cap N_{z_1,r_1} = \emptyset \text{ for } \mu \in [0,1]. \tag{4.4}$$

From Lemma 4.4, it follows that for any $z \in N_r^-$ (hence $|z - x_0| = r$),

$$\begin{aligned} |f_\mu(z) - z_1| &= |f_\mu(z) - x_0 + x_0 - z_1| \geqslant |f_\mu(z) - x_0| - |x_0 - z_1| \\ &\geqslant \beta r - |x_0 - z_1| > |x_0 - z_1| + r_1 - |x_0 - z_1| = r_1. \end{aligned}$$

This proves Equation (4.4).

It remains to investigate $h(1, z)$. Define a map A on $\mathbb{R}^m$ by $A(z) = (rDf(x_0)z + x_0 - z_1)/r_1$. Then A is affine and for $z \in \overline{B_m}$,

$$h(1, z) = \frac{L(rz + x_0) - z_1}{r_1} = \frac{x_0 + Df(x_0)(rz) - z_1}{r_1} = A(z).$$

To prove that $\deg(A, B_m, 0) = det(Df(x_0)) = \pm 1$, it is sufficient to show that the unique solution $\hat{z} = (1/r)Df(x_0)^{-1}(z_1 - x_0)$ of the equation $A(z) = 0$ is in B_m. To this end, observe that from Equation (4.1), we have $|Df(x_0)^{-1}| \leqslant \rho^{-1}$ and hence

$$|\hat{z}| \leqslant \frac{1}{r}|Df(x_0)^{-1}| \cdot |z_1 - x_0| \leqslant \frac{|z_1 - x_0|}{\rho r} < \frac{|z_1 - x_0| + r_1}{\beta r} < 1.$$

□

The following lemma gives a covering relation from a homoclinic point to the snap-back repeller.

Lemma 4.7. *Assume that $z_0 \in \mathbb{R}^m$ such that $f^k(z_0) = x_0$ for some integer $k > 0$ and $det(Df^k(z_0)) \neq 0$. Then there exists $R > 0$ such that if $0 < r < R$ then there is $v \equiv v(r)$ with $0 < v < r_0$ such that for any $0 < r_2 \leqslant v$, we have*

$$N_{z_0,r} \overset{f^k}{\Longrightarrow} N_{r_2}. \tag{4.5}$$

Proof. By continuity of f, there is $R_1 > 0$ such that

$$f^k(\overline{B_m(z_0, R_1)}) \subset B_m(x_0, r_0).$$

Define a homotopy as follows: for $\mu \in [0, 1]$ and $z \in \overline{B_m(z_0, R_1)}$,

$$g_\mu(z) = (1-\mu)f^k(z) + \mu(Df^k(z_0)(z - z_0) + x_0). \tag{4.6}$$

Then $g_\mu(z_0) = x_0$ and $dg_\mu(z) = (1-\mu)Df^k(z) + \mu Df^k(z_0)$ for all μ and z. Since $Df^k(z_0)$ is nonsingular, there is a constant $\alpha > 0$ such that for any $z \in \mathbb{R}^m$,

$$|Df^k(z_0)z| \geqslant \alpha|z|. \tag{4.7}$$

Next, we show that there exists a positive number $R < \min\{R_1, 2r_0/\alpha\}$ such that for all $|z - z_0| < R$ and $\mu \in [0, 1]$, one has

$$|g_\mu(z) - x_0| > \frac{\alpha}{2}|z - z_0|. \tag{4.8}$$

To this end, we have to modify the proof of Lemma 4.4 a bit. By using Taylor's theorem with integral remainder, we have

$$g_\mu(z) - x_0 = g_\mu(z) - g_\mu(z_0) = C(z - z_0),$$

where

$$C = C(\mu, z, z_0) = \int_0^1 Dg_\mu(z_0 + t(z - z_0))dt.$$

By Equation (4.6), we get that

$$\begin{aligned} C - Dg_\mu(z_0) &= \int_0^1 (1-\mu)Df^k(z_0 + t(z - z_0)) + \mu Df^k(z_0)dt - Dg_\mu(z_0) \\ &= \int_0^1 (1-\mu)[Df^k(z_0 + t(z - z_0)) - Df^k(z_0)]dt. \end{aligned} \tag{4.9}$$

Since Df^k is continuous at z_0, there exists $R > 0$ such that if $|y - z_0| < R$ then

$$|Df^k(y) - Df^k(z_0)| < \alpha/2.$$

Hence, from (4.9), we have that for any $\mu \in [0,1]$ and $z \in B_m(z_0, R)$,

$$
\begin{aligned}
|C - Dg_\mu(x_0)| &\leqslant \int_0^1 (1-\mu)|Df^k(z_0 + t(z - z_0)) - Df^k(z_0)|dt \\
&< \int_0^1 (1-\mu)\frac{\alpha}{2}dt \leqslant \frac{\alpha}{2}.
\end{aligned}
$$

Therefore, by using Equation (4.7), we obtain that for any $\mu \in [0,1]$ and $z \in B_m(z_0, R)$,

$$
\begin{aligned}
|g_\mu(z) - x_0| &= |C(z - z_0)| = |(C - Dg_\mu(z_0) + Dg_\mu(z_0))(z - z_0)| \\
&\geqslant |Df^k(z_0)(z - z_0)| - |(C - Dg_\mu(z_0))(z - z_0)| \\
&> \left(\alpha - \frac{\alpha}{2}\right)|z - z_0| = \frac{\alpha}{2}|z - z_0|.
\end{aligned}
$$

Now we are ready to prove the desired covering relation (4.5). Let r be a number with $0 < r < R$ and let $v = \alpha r/2$. Let r_2 be a number with $0 < r_2 \leqslant v$. Since $\alpha > 0$ and $R < 2r_0/\alpha$, we have $0 < v < r_0$. We define a homotopy h_μ by

$$
h_\mu(z) = c_{N_{r_2}}(g_\mu(c^{-1}_{N_{z_0,r}}(z))) \text{ for } \mu \in [0,1] \text{ and } z \in \overline{B_m}.
$$

The conditions from Definition 3.2 requiring the proof are only Condition (3.2) and $\deg(h_1, B_m, 0) \neq 0$ while the others are clear. To verify Condition (3.2), note that it is equivalent to the following one:

$$
g_\mu(N^-_{z_0,r}) \cap N_{r_2} = \emptyset \text{ for } \mu \in [0,1]. \tag{4.10}
$$

From Equation (4.8), it follows that for any $z \in N^-_{z_0,r}$ (hence $|z - z_0| = r$), one has

$$
|g_\mu(z) - x_0| > \frac{\alpha}{2}|z - z_0| > r_2.
$$

This proves Equation (4.10). Finally, since

$$
h_1(z) = \frac{r}{r_2}Df^k(z_0)z,
$$

we obtain that h_1 is a linear isomorphism; therefore

$$\deg(h_1, B_m, 0) = det(Df^k(z_0)) \neq 0.$$

$\square$

The next proposition shows that the existence of a snap-back repeller as defined in Definition 2.2 implies a positive topological entropy. In [2], Blanco Garcia gave the same result based on Marotto's definition of a snap-back repeller and results in [19]. Here, we give a new proof by using covering relations.

Proposition 4.8. *The topological entropy of f is positive.*

Proof. Let β and r_0 be as given in Lemma 4.4. Since x_0 is a snap-back repeller for f, there exists a sequence $\{x_{-i}\}_{i\in\mathbb{N}}$ such that $x_{-1} \neq x_0$, $\lim_{i\to\infty} x_{-i} = x_0$ and for all $i \in \mathbb{N}$, $f(x_{-i}) = x_{-i+1}$ and $det(Df(x_{-i})) \neq 0$. Thus, there is an integer $k > 0$ such that $x_{-k} \in B(x_0, r_0)$. By the chain rule, we have $det(Df^k(x_{-k})) \neq 0$. Furthermore, from Lemma 4.7, there exist positive constants r_k and r_b such that $r_b < r_0$ and

$$\overline{B(x_{-k}, r_k)} \subset B(x_0, r_0), \tag{4.11}$$

$$N_{x_{-k},r_k} \cap N_{r_b} = \emptyset, \tag{4.12}$$

$$N_{x_{-k},r_k} \overset{f^k}{\Longrightarrow} N_{r_b}. \tag{4.13}$$

Since $\beta > 1$, there exists the minimal positive integer a such that $\beta^a r_b > |x_{-k} - x_0| + r_k$. By the minimum of a and Equation (4.11), we have $\beta^{a-1} r_b \leqslant |x_{-k} - x_0| + r_k < r_0$. From Proposition 4.5 and Lemma 4.6, it follows that we have the following chain of covering relations:

$$N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k},r_k}. \tag{4.14}$$

Moreover, from Proposition 4.5, it also follows that

$$N_{r_b} \overset{f}{\Longrightarrow} N_{r_b}. \tag{4.15}$$

These covering relations are enough to produce symbolic dynamics and a positive topological entropy as follows. Let $w = \max(a, k)$. It is sufficient to construct an f^{2w}-invariant set on which f^{2w} can be semi-conjugated onto the shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$, where $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, the one-sided shift space on two symbols with the standard Tikhonov (product) topology. By using Equations (4.13)-(4.15), one can consider the following chains of covering relations, each one of length $2w$ (which is counted by the number of iterates of f):

$$N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b},$$
$$N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k}, r_k},$$
$$N_{x_{-k}, r_k} \overset{f^k}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b},$$
$$N_{x_{-k}, r_k} \overset{f^k}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k}, r_k}.$$

Let us denote $N_0 = N_{r_b}$ and $N_1 = N_{x_{-k}, r_k}$. Then N_0 and N_1 are disjoint due to Equation (4.12). Define Z to be the set of points whose forward orbits under f^{2w} stay in $N_0 \cup N_1$, that is,

$$Z = \{z \in N_0 \cup N_1 : f^{2iw}(z) \in N_0 \cup N_1 \text{ for all } i \in \mathbb{N}\}.$$

Then Z is compact. On Z we define a projection $\pi : Z \to \Sigma_2^+$ by

$$\pi(z)_i = j \text{ if and only if } f^{2iw}(z) \in N_j.$$

It is obvious that the map π is continuous and we have a semiconjugacy: $\pi \circ f^{2w} = \sigma \circ \pi$.

Finally, we shall show that π is onto. This gives us that the topological entropy of f^{2w} on Z is greater than or equal to $\log 2$. Let $\alpha = (\alpha_0, ..., \alpha_{l-1}) \in \{0,1\}^l$ for some positive integer l. By a suitable concatenation of the above listed chains of covering relations and from Theorem 3.3, it follows that there exists a point $x_\alpha \in N_{\alpha_0}$ such that

$$\begin{aligned} f^{2iw}(x_\alpha) &\in N_{\alpha_i} \text{ for } 0 \leqslant i \leqslant l-1, \\ f^{2lw}(x_\alpha) &= x_\alpha. \end{aligned}$$

It is clear that $x_\alpha \in Z$ and $\pi(x_\alpha) = (\alpha, \alpha, ...) \in \Sigma_2^+$. Since α is arbitrarily chosen, the set $\pi(Z)$ contains all repeating sequences. From the density of repeating sequences in Σ_2^+, it follows that $\pi(Z) = \Sigma_2^+$. $\square$

Now, we list our main results about the multidimensional perturbations of a higher dimensional map which has a snap-back repeller. First, if the singular map depends only on the phase variable of a snap-back repeller, we have the following result.

Theorem 4.9. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m \times \mathbb{R}^n$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x))$ for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and has a snap-back repeller and $g : \mathbb{R}^m \to \mathbb{R}^n$. Then F_λ has a positive topological entropy for all λ sufficiently close to* 0.

When the singular map is locally trapping along the normal direction, we have the following.

Theorem 4.10. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m \times \mathbb{R}^n$ such that $F_\lambda(z)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x, y))$ for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and has a snap-back repeller, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, and $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$. Then F_λ has a positive topological entropy for all λ sufficiently close to* 0.

Now, we begin to prove Theorem 4.9 and 4.10.

Proof of Theorem 4.9. From the proof of Proposition 4.8, we have a positive integer a such that the following closed loop of covering relations holds:

$$N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k}, r_k} \overset{f^k}{\Longrightarrow} N_{r_b},$$

By adding the normal direction to the above h-sets and using the persistence of covering relation, we shall construct a closed loop of covering relations for F_λ, similar to the above loop for f. Recall that the singular map F_0 is of the form $F_0(x,y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^n$. Set $N = (\cup_{i=0}^{a-1} N_{\beta^i r_b}) \cup (\cup_{i=0}^{k} f^i(N_{x_{-k},r_k}))$. Since g is continuous and N is compact, there exists $r > 0$ such that $g(N) \subset B_n(0,r)$. Let us define the corresponding h-sets in $\mathbb{R}^m \times \mathbb{R}^n$ as follows. For $i = 0, 1, ..., a-1$, we define h-sets $N'_{\beta^i r_b}$ in $\mathbb{R}^m \times \mathbb{R}^n$ by $N'_{\beta^i r_b} = N_{\beta^i r_b} \times \overline{B_n(0,r)}$, $u(N'_{\beta^i r_b}) = m$, $s(N'_{\beta^i r_b}) = n$ and $c_{N'_{\beta^i r_b}}(x,y) = (c_{N_{\beta^i r_b}}(x), y/r)$. Moreover, we define an h-set N'_{x_{-k},r_k} in $\mathbb{R}^m \times \mathbb{R}^n$ by $N'_{x_{-k},r_k} = N_{x_{-k},r_k} \times \overline{B_n(0,r)}$, $u(N'_{x_{-k},r_k}) = m$, $s(N'_{x_{-k},r_k}) = n$ and $c_{N'_{x_{-k},r_k}}(x,y) = (c_{N_{x_{-k},r_k}}(x), y/r)$.

Observe that we have the following closed loop of covering relations for F_0.

Lemma 4.11. *The following covering relations hold:*

$$N'_{r_b} \overset{F_0}{\Longrightarrow} N'_{r_b} \overset{F_0}{\Longrightarrow} N'_{\beta r_b} \overset{F_0}{\Longrightarrow} \cdots \overset{F_0}{\Longrightarrow} N'_{\beta^{a-1} r_b} \overset{F_0}{\Longrightarrow} N'_{x_{-k},r_k} \overset{F_0^k}{\Longrightarrow} N'_{r_b},$$

Proof of Lemma 4.11. For each covering relation under consideration $N' \overset{F_0^j}{\Longrightarrow} M'$ with $j = 1$ or k, a homotopy $\hat{h} : [0,1] \times \overline{B_m} \times \overline{B_n} \to \mathbb{R}^{m+n}$ by

$$\hat{h}(\mu, x, y) = \left(h(\mu, x), \frac{1-\mu}{r} g \circ f^{j-1}(c_N^{-1}(x)) \right),$$

where h is the homotopy from the corresponding covering relation $N \overset{f^j}{\Longrightarrow} M$. Then we have

$$\begin{aligned} \hat{h}(0,x,y) &= (h(0,x)), \frac{1}{r} g \circ f^{j-1}(c_N^{-1}(x)) \\ &= \left(c_M \circ f^j \circ c_N^{-1}(x), \frac{1}{r} g \circ f^{j-1}(c_N^{-1}(x)) \right) = (F_0^j)_c(x,y). \end{aligned}$$

Since $\hat{h}([0,1], N'^{,-}) \subset h([0,1], N^-) \times \mathbb{R}^n$, we get that Condition (3.2) in Definition 3.2 follows from the analogous Condition for h. Condition (3.3) is satisfied due to

$$\hat{h}([0,1] \times \overline{B_m} \times \overline{B_n}) \subset \mathbb{R}^m \times B_n.$$

Finally, note that

$$\hat{h}(1, x, y) = (h(1, x), 0).$$

Therefore, the other conditions in Definition 3.2 are also satisfied. $\square$

From Theorem 3.4, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then the following chain of covering relations holds for F_λ:

$$N'_{r_b} \overset{F_\lambda}{\Longrightarrow} N'_{r_b} \overset{F_\lambda}{\Longrightarrow} N'_{\beta r_b} \overset{F_\lambda}{\Longrightarrow} \cdots \overset{F_\lambda}{\Longrightarrow} N'_{\beta^{a-1} r_b} \overset{F_\lambda}{\Longrightarrow} N'_{x_{-k}, r_k} \overset{F_\lambda^k}{\Longrightarrow} N'_{r_b}, \tag{4.16}$$

Similar to the proof of Proposition 4.8, covering relations listed in (4.16) are sufficient to produce the symbolic dynamics and a positive topological entropy for F_λ with $|\lambda| < \lambda_0$. This completes the proof of Theorem 4.9. $\square$

Proof of Theorem 4.10. Define $G_\lambda = (id, c) \circ F_\lambda \circ (id, c)^{-1}$, where id denotes the identity map on $\mathbb{R}^k$ and c is a homeomorphism from S to $\overline{B_n}$. Then the conclusion follows from the above argument applied to G_λ. $\square$

4.2 Topologically crossing homoclinicity

In this subsection, we discuss the topological entropy for multidimensional perturbations of topologically crossing homoclinicity.

4.2.1 Background

First, we introduce some definition and results. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism with a hyperbolic periodic point p at which the stable and unstable subspaces have dimensions u and s, respectively. Let $|\cdot|$ be a norm on $\mathbb{R}^m$. The stable and unstable manifolds of p are defined to be $W^s(p) = \{x \in \mathbb{R}^m : |f^n(x) - f^n(p)| \to 0 \text{ as } n \to \infty\}$ and $W^u(p) = \{x \in \mathbb{R}^m : |f^n(x) - f^n(p)| \to 0 \text{ as } n \to -\infty\}$, respectively. The *deleted* stable and unstable manifold of p are given by $\hat{W}^s(p) = W^s(p) \backslash \{p\}$ and $\hat{W}^u(p) = W^u(p) \backslash \{p\}$, respectively. An intersection of $\hat{W}^s(p)$ and $\hat{W}^u(p)$ is called a *homoclinic point* of p. For nonempty subsets A, B of $\mathbb{R}^m$, we denote $d(A, B) = \inf\{|x - y| : x \in A \text{ and } y \in B\}$. Here, we are mainly concern the case when $\hat{W}^s(p)$ and $\hat{W}^u(p)$ has a topologically crossing intersection which is defined as follows.

Definition 4.12. *[8, Definition 3] Consider $\mathbb{R}^m$ as an m-dimensional oriented manifold, and let W^u and W^s be two oriented C^1 submanifolds of $\mathbb{R}^m$ with dimensions u and s, respectively, such that $u + s = m$. We say that W^u and W^s have a* topologically crossing intersection *if there are compact embedded C^1 submanifold V^u of W^u and V^s of W^s with dimensions u and s and with boundaries ∂V^u and ∂V^s (with respect to W^u and W^s), respectively, such that*

1. *$\partial V^u \cap V^s = V^u \cap \partial V^s = \emptyset$;*
2. *For every $0 < \varepsilon < \min\{d(\partial V^u, V^s), d(V^u, \partial V^s)\}$, there exists a homotopy $h : [0, 1] \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying the following:*
   - *(a) $h(0, x) = x$ for all $x \in \mathbb{R}^m$ and the map $x \mapsto h(1, x)$ is an embedding;*
   - *(b) $|h(t, x) - x| < \varepsilon$ for all $x \in V^u \cup V^s$ and all $t \in [0, 1]$;*
   - *(c) $h(1, V^u)$ and V^s are transverse submanifolds; and*
   - *(d) the* oriented intersection number *of $h(1, V^u)$ and V^s, denoted by $I(h(1, V^u), V^s)$, is nonzero, where $I(A, B)$ for two oriented submanifolds A and B of $\mathbb{R}^m$ with $\dim A + \dim B = m$ is defined by*
   $$I(A, B) = \sum_{x \in A \cap B} I_x(A, B),$$
   and $I_x(A, B)$ is $+1$ or -1 depending on whether the orientation induced on $T_xA \oplus T_xB$ agrees or not with the orientation on $T_x\mathbb{R}^m$, respectively.

In this case, the submanifolds V^u and V^s will be referred as a good pair *for the topological crossing between W^u and W^s.*

There is a relation between a topological crossing and the local Brouwer degree. Let V^u and V^s be a good pair for a topological crossing intersection between oriented submanifolds W^u and W^s of an oriented manifold W with $\dim(W) = m$, $\dim(W^u) = u$, $\dim(W^s) = s$, and $u + s = m$. Assume that there exist a closed neighborhood U of $V^u \cap V^s$ in W and local coordinates (x, y) on U such that $V^u \subset U$, $V^s \subset U$, $U = \overline{B_u} \times \overline{B_s}$, $V^s = \{(x, y) \in \overline{B_u} \times \overline{B_s} : x = 0\}$, and $V^u = \{\psi(x) \in \overline{B_u} \times \overline{B_s} : x \in \overline{B_u}\}$, where ψ is a C^1 parametrization of V^u. Let $\pi_u : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u$ be the projection given by $\pi_u(x, y) = x$. The following lemma says the local Brouwer degree and the oriented intersection number are identical.

Lemma 4.13. *[8, Lemma 3] Under the above assumptions and notations, we have that (i) in $\mathbb{R}^u$, the origin $0 \notin \pi_u(\psi(\partial B_u))$; (ii) $\deg(\pi_u \circ \psi, B_u, 0)$ is well defined; and (iii) $\deg(\pi_u \circ \psi, B_u, 0) = I(h(1, V^u), V^s)$, where $I(h(1, V^u), V^s)$ is the oriented intersection number of $h(1, V^u)$ and V^s for any homotopy h as given in Definition 4.12.*

4.2.2 Results

In this subsection, we state our results about the positive topological entropy derived from the topologically crossing homoclinicity. First, we see the result about perturbations of a map.

Theorem 4.14. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m$ such that $F_\lambda(x)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $x \in \mathbb{R}^m$, where λ is a parameter. Assume that $F_0(x) = f(x)$ for all $x \in \mathbb{R}^m$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 diffeomorphism with a hyperbolic periodic point which has a topologically crossing homoclinic point. Then there exist an integer*

$N > 0$ and a number $\lambda_0 > 0$ such that both f and F_λ with $|\lambda| < \lambda_0$ have topological entropies at least $\log(2)/N$.

Next, if the singular map F_0 depends only on the phase variable of f, we have the following result.

Theorem 4.15. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m \times \mathbb{R}^k$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where λ is a parameter. Assume that $F_0(x, y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 diffeomorphism with a hyperbolic periodic point which has a topologically crossing homoclinic point, and $g : \mathbb{R}^m \to \mathbb{R}^k$ is a continuous function. Then there exist an integer $N > 0$ and a number $\lambda_0 > 0$ such that both f and F_λ with $|\lambda| < \lambda_0$ have topological entropies at least* $\log(2)/N$.

For the case when the singular map is a skew product map locally trapping along the second variable, we have the following.

Theorem 4.16. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m \times \mathbb{R}^k$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where λ is a parameter. Assume that $F_0(x, y) = (f(x), g(x, y)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 diffeomorphism with a hyperbolic periodic point which has a topologically crossing homoclinic point, and $g : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous on $\mathbb{R}^m \times S$ and $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^k$ homeomorphic to the closed unit ball in $\mathbb{R}^k$. Then there exist an integer $N > 0$ and a number $\lambda_0 > 0$ such that both f and F_λ with $|\lambda| < \lambda_0$ have topological entropies at least* $\log(2)/N$.

Denote by p the hyperbolic periodic point of f. Without loss of generality, we may assume that p is a fixed point. Set $u = \dim W^u(p)$ and $s = \dim W^s(p)$. Since $\hat{W}^u(p)$ and $\hat{W}^s(p)$ have a topologically crossing intersection, we have $u + s = m$. Let us fix a basis of $\mathbb{R}^m$ such that the Jacobian matrix Df_p of f at p preserves the splitting $\mathbb{R}^m = \mathbb{R}^u \oplus \mathbb{R}^s$. By the Hartman-Grobman Theorem, there exist a closed neighborhood U of p and a homeomorphism φ of U into $\mathbb{R}^m$ such that $\varphi(p) = (0, 0)$ and $\varphi(f(z)) = Df_p(\varphi(z))$ for $z \in U$. In order to simply our notation, we assume $p = (0, 0)$ and $\varphi = id$, the identity map on $\mathbb{R}^m$. Thus f is a linear map on U. Write $f(x, y) = (L_u x, L_s y)$ for $(x, y) \in U$, where L_u is a $u \times u$ matrix with all eigenvalues greater than one in absolute value and L_s is an $s \times s$ matrix with all eigenvalues less than one in absolute value. There exist norms $|\cdot|_u$ and $|\cdot|_s$ on $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively, and constants $\rho_1 > 1$ and $0 < \rho_2 < 1$ such that

$$|L_u x|_u \geqslant \rho_1 |x|_u \text{ and } |L_s y|_s \leqslant \rho_2 |y|_s \text{ for } x \in \mathbb{R}^u \text{ and } y \in \mathbb{R}^s. \tag{4.17}$$

Since all norms on $\mathbb{R}^m$ are equivalent, we may assume $U = \overline{B_u} \times \overline{B_s}$ and define the norm $|\cdot|$ on $\mathbb{R}^m$ to be the maximum norm of the norms $|\cdot|_u$ and $|\cdot|_s$ on $\mathbb{R}^u$ and $\mathbb{R}^s$. Notice that later we still need local coordinates while verifying h-sets in U as required in Definition 3.1. In order to prove the main results we need some lemmas. First, we recall the following lemma in [5]; for readers' convenience, we repeat their proof below.

Lemma 4.17. *[5, Lemma 1.4] Let V be a compact subset of $W^u(p) \cap int(U)$. Suppose we are given positive constants ρ and ε satisfying $0 < \rho < 1$ and $0 < \varepsilon < d(V, \partial U)$. Then for any large enough $n \in \mathbb{N}$ the following hold:*

1. *$f^{-n}(V) \subset \overline{B_u(0, \rho)} \times \{0\}$; and*
2. *if $(x, 0) \in f^{-n}(V)$ then $f^n(\{x\} \times \overline{B_s})$ is in U and has diameter less than ε, where the diameter of a bounded set $E \subset \mathbb{R}^m$ is defined to be $\sup\{|x - y| : x, y \in E\}$.*

Proof. Since $V \subset W^u(p)$, there exists a positive integer n_1 such that $f^{-n_1}(V) \subset \overline{B_u} \times \{0\}$. Since f is a C^1 diffeomorphism , we can take a constant K such that $K > \sup\{\|Df_z^{n_1}\| : z \in U\}$ and $\varepsilon/(2K) < 1$. Let n_2 be an arbitrary positive integer such that

$$n_2 \geqslant \max\{\log(\rho^{-1})/\log(\rho_1), \log(\varepsilon/(2K))/\log(\rho_2)\}. \tag{4.18}$$

Since f is linear and preserves the splitting $\mathbb{R}^u \times \mathbb{R}^s$ on U, Equations (4.17) and (4.18) imply $f^{-n_1-n_2}(V) \subset f^{-n_2}(\overline{B_u} \times \{0\}) \subset \overline{B_u(0,\rho)} \times \{0\}$. This concludes item 1 of the desired result by considering $n = n_1 + n_2$. For item 2, let $(x, 0) \in f^{-n_1-n_2}(V)$ and $y \in \overline{B_s}$. Again Equations (4.17) and (4.18) imply $f^{n_2}(x,y) = (L_u^{n_2}(x), L_s^{n_2}(y)) \in \{L_u^{n_2}(x)\} \times \overline{B_s(0, \varepsilon/(2K))} \subset U$. Take any two points in $\{x\} \times \overline{B_s}$, say $w = (x, y_1)$ and $v = (x, y_2)$. Then $f^{n_2}(w), f^{n_2}(v) \in U$ and $|f^{n_2}(w) - f^{n_2}(v)| = |L_s^{n_2}(y_1) - L_s^{n_2}(y_2)| \leqslant \varepsilon/K$. By the choice of K, we get that $|f^{n_1+n_2}(w) - f^{n_1+n_2}(v)| < K|f^{n_2}(w) - f^{n_2}(v)| \leqslant \varepsilon$. By considering $n = n_1 + n_2$, we have the desired result. $\square$

Since the submanifolds $\hat{W}^u(p)$ and $\hat{W}^s(p)$ have a topological crossing intersection, there exist a point $q \neq p$ and two compact embedded submanifolds V^u of $\hat{W}^u(p)$ and V^s of $\hat{W}^s(p)$ such that V^u and V^s form a good pair, and $q \in V^u \cap V^s$. We may assume that both sets V^u and V^s are in $int(U)$, and V^u has no intersection with the subspace $\mathbb{R}^u \times \{0\}$, based on the following lemma.

Lemma 4.18. *For any sufficiently integer $n \in \mathbb{N}$, there exist submanifolds V_n^u of $\hat{W}^u(p)$ and V_n^u of $\hat{W}^s(p)$ such that V_n^u and V_n^s form a good pair with the same oriented intersection number as good pair V^u and V^s, $V_n^u \subset f^n(V^u)$, $V_n^s \subset f^n(V^s)$, and $V_n^u \cup V_n^s \subset int(U)$.*

Proof. First, we show that $f^n(V^u)$ and $f^n(V^s)$ form a good pair for $n \in \mathbb{N}$. Since f is a C^1 diffeomorphism, $f^n(V^u)$ and $f^n(V^s)$ are compact embedded C^1 submanifolds of $\hat{W}^u(p)$ and $\hat{W}^s(p)$, respectively. Since $V^u \subset \hat{W}^u(p)$, $V^s \subset \hat{W}^s(p)$, and $\partial V^u \cap V^s = V^u \cap \partial V^s = \emptyset$, we also have $\partial f^n(V^u) \cap f^n(V^s) = f^n(V^u) \cap \partial f^n(V^s) = \emptyset$.

Let δ be a constant such that

$$0 < \delta < \min\{d(\partial f^n(V^u), f^n(V^s)), d(f^n(V^u), \partial f^n(V^s))\}.$$

Since f^n is continuous on the compact set $V^u \cup V^s$, there exists a constant ε such that $0 < \varepsilon < \min\{d(\partial V^u, V^s), d(V^u, \partial V^s)\}$ and if $x,\ y \in V^u \cup V^s$ with $|x - y| < \varepsilon$ then $|f^n(x) - f^n(y)| < \delta$. Since V^u and V^s form a good pair, for such an ε, there exists a homotopy h_0 satisfying item (2a)-(2d) of Definition 4.12. Define a homotopy $h_n = f^n \circ h_0 \circ f^{-n}$. It is obviously true that $h_n(0, \cdot) = id$ and $h_n(1, \cdot)$ is an embedding. By item (2b) of Definition 4.12, for $z \in f^n(V^u) \cup f^n(V^s)$ and $t \in [0, 1]$, we have

$$|h_n(t, z) - z| = |f^n(h_0(t, f^{-n}(z))) - f^n(f^{-n}(z))| < \delta.$$

Moreover, $h_n(1, f^n(V^u))$ and $f^n(V^s)$ are transverse submanifolds and the oriented intersection number $I(h_n(1, f^n(V^u)), f^n(V^s)) = I(h_0(1, V^u), V^s)$ is nonzero. Thus, $f^n(V^u) \subset \hat{W}^u(p)$ and $f^n(V^s) \subset \hat{W}^s(p)$ form a good pair.

If $f^n(V^u) \cup f^n(V^s) \subset int(U)$, then we are done by taking $V_n^u = f^n(V^u)$ and $V_n^s = f^n(V^s)$. Otherwise, since p is a hyperbolic fixed point with topologically crossing homoclinic point(s) in $V^u \cap V^s$ which has nonzero oriented intersection number, by letting n large enough, there exists $q \in V^u \cap V^s$ such that if we denote by V_n^u and V_n^s the connected components of $f^n(V^u) \cap (\overline{B_u(0, 4/5)} \times \overline{B_s(0, 4/5)})$ and $f^n(V^s) \cap (\overline{B_u(0, 4/5)} \times \overline{B_s(0, 4/5)})$ containing the point $f^n(q)$, respectively, $\partial V_n^u \cap V_n^s = V_n^u \cap \partial V_n^s = \emptyset$ and $I(h_n(1, V_n^u), V_n^s) = I(h_n(1, f^n(V^u)); f^n(V^s))$. Repeating the above argument, we have that $V_n^u \subset \hat{W}^u(p)$ and $V_n^s \subset \hat{W}^s(p)$ form a good pair with the same oriented intersection number as the good pair V^u and V^s. We have finished the proof of the desired result. $\square$

Set $V_2 = V^u$. Since $\pi_u(q) = 0$, there is a constant η such that $0 < \eta < 1$ and $\overline{B_u(0, \eta)} \subset \pi_u(V_2) \backslash \pi_u(\partial V_2)$. Denote $V_1 = \overline{B_u(0, \eta)} \times \{0\}$.

We shall construct two disjoint h-sets. Let ρ be a constant such that $0 < \rho < \eta$. Denote $R = \overline{B_u(0,\rho)} \times \overline{B_s}$. Let $\varepsilon > 0$ be so small that the closed neighborhoods of V_1 and V_2 are disjoint and contain in $int(U)$ and that the closed ε-neighborhoods of ∂V^u and $\partial\overline{B_u(0,\eta)} \times \{0\}$ are contained in $int(U) \setminus R$. Then

$$\varepsilon < \min\{d(V_1, \partial U), d(V_2, \partial U)\}.$$

By applying Lemma 4.17 to V_1 and V_2, we can pick a common integer N such that $f^{-N}(V_1) \cup f^{-N}(V_2) \subset B_u(0,\rho) \times \{0\}$ and if $(x,0) \in f^{-N}(V_1) \cup f^{-N}(V_2)$ then $f^N(\{x\} \times \overline{B_s})$ is in U and has diameter less than ε. Write $f^{-N}(q) = (q_0, 0)$. Since f is a diffeomorphism, $f^{-N}(V_1)$ and $f^{-N}(V_2)$ are disjoint. Moreover, since f is C^1, V_2 is a C^1 submanifold of $\hat{W}^u(p)$ and hence there exists a C^1 diffeomorphism ζ from $\mathbb{R}^u$ to $\mathbb{R}^u$ such that $\zeta(\pi_u(f^{-N}(\psi(x)))) = x$ for all $x \in \overline{B_u}$, where ψ is a C^1 parametrization of V_2 on $\overline{B_u}$ such that $V_2 = \psi(\overline{B_u})$ and $\psi(0) = q$ (mentioned in Lemma 4.13). Since $f(x,y) = (L_u x, L_s y)$ for $(x,y) \in U$ under the Hartman-Grobman linearization setting at the beginning of this subsection, we have $f^{-N}(V_1) = L_u^{-N}(\overline{B_u(0,\eta)}) \times \{0\}$. Define $M_1 = \pi_u(f^{-N}(V_1)) \times \overline{B_s}$ and $M_2 = \pi_u(f^{-N}(V_2)) \times \overline{B_s}$; or equivalently define $M_1 = L_u^{-N}(\overline{B_u(0,\eta)}) \times \overline{B_s}$ and $M_2 = \pi_u(f^{-N}(\psi(\overline{B_u}))) \times \overline{B_s}$. Then M_1 and M_2 are disjoint h-sets with $u(M_1) = u(M_2) = u$, $s(M_1) = s(M_2) = s$, and $c_{M_1}(x,y) = (L_u^N x/\eta, y)$ and $c_{M_2}(x,y) = (\zeta(x), y)$ for all $(x,y) \in \mathbb{R}^u \times \mathbb{R}^s$.

Next, we show that there are covering relations among M_1 and M_2.

Lemma 4.19. *The following covering relations hold:*

$$M_i \overset{f^N}{\Longrightarrow} M_j \text{ for } i, j \in \{1,2\}.$$

Proof. Define a homotopy H on $\mathbb{R}^m$ from $f^N(x,y)$ to $\pi_u \circ f^N(x,0)$ by

$$H(t,x,y) = (1-t)f^N(x,y) + t(\pi_u(f^N(x,0)), 0),$$

for $(x,y) \in \mathbb{R}^u \times \mathbb{R}^s$ and $t \in [0,1]$. For i, $j \in \{1,2\}$, we set a homotopy h_i^j induced from H by $h_i^j(t,x,y) = c_{M_j}(H(t, c_{M_i}^{-1}(x,y)))$ for $(x,y) \in \mathbb{R}^u \times \mathbb{R}^s$ and

$t \in [0,1]$, and define $A_i^j(x) = \pi_u(h_i^j(1,x,0))$ for $x \in \mathbb{R}^u$. Since $h_i^j(1,x,y)$ is independent of y and lies on the subspace $\mathbb{R}^u \times \{0\}$, we get that $h_i^j(1,x,y) = (A_i^j(x), 0)$ for $x \in \mathbb{R}^u$. Moreover, by the choice of N, we have

$$H(t, M_i^-) \cap M_j = \emptyset \text{ and } H(t, M_i) \cap M_j^+ = \emptyset \text{ for } t \in [0,1].$$

It follows that Condition 1 and 2 of Definition 3.2 are satisfied with $h = h_i^j$ and $\varphi = A_i^j$.

For Condition 3 of Definition 3.2, we first show that $\deg(A_i^j, B_u, 0) \neq 0$ for $i = 1$ and $j \in \{1,2\}$. By the definition of homeomorphisms c_{M_i}, we get that $f^N \circ c_{M_1}^{-1}(U) \subset U$ and $f^N \circ c_{M_2}^{-1}(U) \subset U$. Hence on $\overline{B_u}$, the map A_1^1 is linear and the map A_1^2 is C^1, in fact, they are of the following forms: for $x \in \overline{B_u}$,

$$A_1^1(x) = \pi_u(h_1^1(1,x,0)) = L_u^N x,$$
$$A_1^2(x) = \pi_u(h_1^2(1,x,0)) = \zeta(\eta x).$$

Since L_u is a $u \times u$ matrix with all eigenvalues greater than one in absolute value, by item 8 of the properties of local Brouwer degree listed in subsection 3.2, we get

$$\deg(A_1^1, B_u, 0) = sgn(\det(L_u^N)) \neq 0.$$

The choice of N implies $\pi_u(f^{-N}(V_2)) \subset B_u(0,\rho) \subset B_u(0,\eta)$ and hence the equation $\zeta(x) = 0$ has a unique solution, namely q_0, and $q_0 \in B_u(0,\eta)$. Since $\pi_u(f^{-N}(V_2))$ is a u-dimensional C^1 submanifold, 0 is a regular value for ζ and $sgn(\det D\zeta_{q_0}) \neq 0$. It follows from items 8 again and Proposition 3.6 that

$$\deg(A_1^2, B_u, 0) = sgn(\det D\zeta_{q_0}) \cdot 1 \neq 0.$$

Next, we shall show that $\deg(A_i^j, B_u, 0) \neq 0$ for $i = 2$ and $j \in \{1,2\}$ as applications of Lemma 4.13. By the definitions of the homeomorphisms

c_{M_i} and the linearization of f on U, we get that for $x \in \overline{B_u}$,

$$A_2^1(x) = \varphi \circ \pi_u \circ \psi(x),$$
$$A_2^2(x) = \zeta \circ \pi_u \circ \psi(x),$$

where φ is a map on $\mathbb{R}^u$ defined by $\varphi(x) = L_u^N x/\eta$. By the choice of η, there exists a bounded connected component, namely Δ, of $\mathbb{R}^u \setminus \pi_u(\psi(\partial B_u)$ such that $0 = \varphi^{-1}(0) \in \Delta$. Since φ is linear, Proposition 3.6 implies that

$$\deg(A_2^1, B_u, 0) = \deg(\varphi \circ \pi_u \circ \psi, B_u, 0) = sgn(\det(L_u^N/\eta)) \deg(\pi_u \circ \psi, B_u, 0)$$

Note that ψ is a parametrization of V_2. Since V_2 and V^s form a good pair with the oriented orientation number not zero, by Lemma 4.13, we have $\deg(\pi_u \circ \psi, B_u, 0) \neq 0$. Therefore, $\deg(A_2^1, B_u, 0) \neq 0$.

Similarly, by the choices of η and N, we get $\zeta^{-1}(0) = q_0 \in \Delta$. Since ζ is C^1, Proposition 3.6 gives us that

$$\deg(A_2^2, B_u, 0) = \deg(\zeta \circ \pi_u \circ \psi, B_u, 0) = sgn(\det D\zeta_{q_0}) \deg(\pi_u \circ \psi, B_u, 0).$$

It follows that $\deg(A_2^2, B_u, 0) \neq 0$.

We have finished the proof of the desired result. $\square$

Finally, we are in position to prove our theorems.

Proof of Theorem 4.14. By applying Lemma 4.19 and Proposition 3.5, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then

$$M_i \overset{F_\lambda^N}{\Longrightarrow} M_j \text{ for for } i, j \in \{1, 2\}.$$

Let θ be a positive integer and $|\lambda| < \lambda_0$. Consider any closed loop

$$M_{i_0} \overset{F_\lambda^N}{\Longrightarrow} M_{i_1} \overset{F_\lambda^N}{\Longrightarrow} \cdots \overset{F_\lambda^N}{\Longrightarrow} M_{i_\theta},$$

with each $i_\alpha \in \{1, 2\}$ and $i_\theta = i_0$. By using Theorem 3.3, F_λ^N has a periodic point $x = x(\lambda) \in int(M_{i_0})$ such that $F_\lambda^{N\theta}(x) = x$. Since there are 2^θ choices of such closed loops, F_λ^N has at least 2^θ periodic points in $M_1 \cup M_2$. These periodic points provide a (θ, δ)-separated set for F_λ^N as long as δ is a positive number less than the distance of M_1 and M_2. Since θ is arbitrarily chosen, we have $h_{\text{top}}(F_\lambda^N) \geqslant \log(2)$ and so $h_{\text{top}}(F_\lambda) \geqslant \log(2)/N > 0$. $\square$

Proof of Theorem 4.15. Since g is continuous on $M_1 \cup M_2$ and $M_1 \cup M_2$ is compact, there exists a positive constant r such that $g(M_1 \cup M_2) \subset B_k(0, r)$. Let us define the corresponding h-sets in $\mathbb{R}^m \times \mathbb{R}^k$ as follows. For $i = 1, 2$, we define h-sets M_i' in $\mathbb{R}^m \times \mathbb{R}^k$ by $M_i' = M_i \times \overline{B_k(0, r)}$ with $u(M_i') = u$, $s(M_i') = s + k$, and $c_{M_i'}(x, y, z) = (c_{M_i}(x, y), z/r)$ for $x \in \mathbb{R}^u$, $y \in \mathbb{R}^s$, and $z \in \mathbb{R}^k$. $\square$

Lemma 4.20. *The following covering relations hold:*

$$M_i' \overset{F_0^N}{\Longrightarrow} M_j' \text{ for } i, j \in \{1, 2\}.$$

Proof. Let $i, j \in \{1, 2\}$ be arbitrary. We define a homotopy

$$\hat{h}_i^j(t, x, y, z) = (h_i^j(t, x, y), \frac{1-t}{r} g \circ f^{N-1}(c_{M_i}^{-1}(x, y))),$$

where h_i^j is the homotopy for the covering relation $M_i \overset{f^N}{\Longrightarrow} M_j$. Then we have

$$\begin{aligned}
\hat{h}_i^j(0, x, y, z) &= (h_i^j(0, x, y), \frac{1}{r} g \circ f^{N-1}(c_{M_i}^{-1}(x, y))) \\
&= (c_{M_j} \circ f^N \circ c_{M_i}^{-1}(x, y), \frac{1}{r} g \circ f^{N-1}(c_{M_i}^{-1}(x, y))) \\
&= (F_0^N)_c(x, y, z).
\end{aligned}$$

Since $\hat{h}_i^j([0, 1], M_i'^{,-}) \subset h_i^j([0, 1], M_i^-) \times \mathbb{R}^k$ and $\hat{h}_i^j([0, 1] \times \overline{B_m} \times \overline{B_k}) \subset \mathbb{R}^m \times B_k$, Condition 1 and 2 of Definition 3.2 are satisfied follows from the analogous properties for h_i^j stated in the proof of Theorem 4.14. Finally, notice that

$$\hat{h}_i^j(1, x, y, z) = (h_i^j(1, x, y), 0).$$

Therefore, Condition 3 of Definition 3.2 is satisfied.

By applying Lemma 4.20 and Proposition 3.5, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then the following covering relations hold for F_λ^N:

$$M_i' \overset{F_\lambda^N}{\Longrightarrow} M_j' \text{ for } i, j \in \{1, 2\}. \tag{4.19}$$

As in the proof of Theorem 4.14, the covering relations listed in Equation (4.19) implies the $h_{\text{top}}(F_\lambda) \geqslant \log(2)/N > 0$ with $|\lambda| < \lambda_0$. $\square$

Proof of Theorem 4.16. Define $G_\lambda = (id, c) \circ F_\lambda \circ (id, c)^{-1}$, where c is a homeomorphism from S to $\overline{B_k}$. Then the topological entropies of G_λ and F_λ are equal. By applying the above argument as in the proof of Theorem 4.15 to the family G_λ while the corresponding c_M of a covering relation $N \overset{G_\lambda}{\Longrightarrow} M$ is the identity map, we have the desired result. $\square$

4.3 Liapunov condition

In this subsection, we study the topological dynamics for multidimensional perturbations of high-dimensional systems with covering relation determined by a transition matrix or satisfy a strong Liapunov condition in addition on the lower dimensional phase space.

4.3.1 Covering relations determined by a transition matrix

Here, we will state the definition of the covering relations determined by a transition matrix and list the related main results of the topological dynamics for multidimensional perturbations of high-dimensional systems.

First, we introduce the transition matrix. By a *transition matrix*, it means that a square matrix satisfies (i) all entries are either zero or one, and (ii) all row sums and column sums are are greater than or equal to one. For a transition matrix A, let $\rho(A)$ denote the spectral radius of A. Then $\rho(A) \geqslant 1$ and moreover, if A is irreducible and not a permutation, then $\rho(A) > 1$. Let

Σ_A^+ (resp. Σ_A) be the space of all allowable one-sided (resp. two sided) sequences for the matrix A with a usual metric, and let $\sigma_A^+ : \Sigma_A^+ \to \Sigma_A^+$ (resp. $\sigma_A : \Sigma_A \to \Sigma_A$) be the one-sided (resp. two sided) subshift of finite type for A. Then $h_{\text{top}}(\sigma_A^+) = h_{\text{top}}(\sigma_A) = \log(\rho(A))$. Refer to [29] for more background.

Next, we define covering relation determined by a transition matrix.

Definition 4.21. *Let $A = [a_{ij}]_{1\leqslant i,j\leqslant\gamma}$ be a transition matrix and f be a continuous map on $\mathbb{R}^m$. We say that f has* covering relations determined by A *if the following conditions are satisfied:*

1. *there are γ pairwise disjoint h-sets $\{M_i\}_{i=1}^{\gamma}$ in $\mathbb{R}^m$;*

2. *if $a_{ij} = 1$ then the covering relation $M_i \overset{f}{\Longrightarrow} M_j$ holds.*

It is easy to see that the logistic maps $f(x) = \mu x(1-x)$ with $\mu > 4$ has covering relations determined by the 2×2 matrix with all entries one on intervals $[-\epsilon, 1/2-\delta]$ and $[1/2+\delta, 1+\epsilon]$ as h-sets, where $0 < \epsilon < \mu/4 - 1$ and $0 < \delta < [(\mu/4 - 1 - \epsilon)\mu^{-1}]^{1/2}$.

Now, we begin to state the main theorems of covering relation determined by a transition matrix.

Theorem 4.22. *Let f be a continuous map on $\mathbb{R}^m$ having covering relations determined by a transition matrix A. If g is a continuous map on $\mathbb{R}^m$ with $|g-f|$ small enough, then $h_{top}(g) \geqslant \log(\rho(A))$.*

If the singular map F depends only on the phase variable of f, we have the following result about multidimensional perturbations.

Theorem 4.23. *Let $F(x,y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous map having covering relations determined by a transition matrix A, and $g : \mathbb{R}^m \to \mathbb{R}^k$ is a continuous function. If G is a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ with $|G-F|$ small enough, then $h_{top}(G) \geqslant \log(\rho(A))$.*

For the case when the singular map is a skew product locally trapping along the second variable, we have the following.

Theorem 4.24. *Let $F(x, y) = (f(x), g(x, y)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous map having covering relations determined by a transition matrix A, and $g : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function such that $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^k$ homeomorphic to the closed unit ball in $\mathbb{R}^k$. If G is a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ with $|G - F|$ small enough, then $h_{top}(G) \geqslant \log(\rho(A))$.*

In order to prove the above results, we need a proposition which is described that a continuous map having covering relations determined by a transition matrix is topologically semi-conjugate to a one-sided subshift of finite type. A variant version of the this result was stated without proof in [7, Corollary 5.9].

Proposition 4.25. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map which has covering relations determined by a transition matrix A. Then there exists a compact subset Λ of $\mathbb{R}^m$ such that Λ is maximal positive invariant for f in the union of the h-sets (with respect to A) and $f|\Lambda$ is topologically semi-conjugate to σ_A^+.*

Proof. For convenience, we denote by $\{M_i\}_{i=1}^{\eta}$ the h-sets with covering relations for f determined by A as in Definition 4.21, and write we write $\underline{s} = (s_0, s_1, \ldots)$ for $\underline{s} \in \Sigma_A^+$. Define

$$\Lambda_n = \bigcup_{\underline{s} \in \Sigma_A^+} \left(\bigcap_{i=0}^{n} f^{-i}(M_{s_i}) \right) \text{ for } n \geqslant 0, \text{ and } \Lambda = \bigcap_{n \geqslant 0} \Lambda_n.$$

Then Λ is the set of all points whose forward orbits, following allowable sequences in Σ_A^+, stay in $\cup_{i=1}^{\eta} M_i$. Thus Λ is maximal positive invariant set for f in $\cup_{i=1}^{\eta} M_i$ with respect to A. Since each M_i is compact and f is continuous, the set $\cap_{i=0}^{n} f^{-i}(M_{s_i})$ is compact for all $n \geqslant 0$ and $\underline{s} \in \Sigma_A^+$. Since the number of sets M_i's is η and the intersection $\cap_{i=0}^{n} f^{-i}(M_{s_i})$ involves only the first $n+1$ digits of $\underline{s} \in \Sigma_A^+$, that is, $(s_0, s_1, \ldots, s_n)$, there are at most η^{n+1} sets $\cap_{i=0}^{n} f^{-i}(M_{s_i})$ for all $\underline{s} \in \Sigma_A^+$, although the set Σ_A^+ itself might be uncountable. Thus the set Λ_n is a union of finitely many compact sets and hence is compact for all $n \geqslant 0$. Therefore, Λ is compact.

For semi-conjugacy, we define $h : \Lambda \to \Sigma_A^+$ by $h(z) = \underline{s}$ for $z \in \Lambda$, where $f^n(z) \in M_{s_n}$ for all $n \geqslant 0$. By the pairwise disjointness of M_i's and the definition of Λ, the map h is well defined. It is easy to show that $\sigma_A \circ h = h \circ f$. Next, we show that h is continuous on Λ. Let $z \in \Lambda$, $h(z) = \underline{s}$ and $\{z_n\}_{n=1}^{\infty}$ be a sequence in Λ such that $z_n \to z$ as $n \to \infty$. Since M_i's are pairwise disjoint and compact, there exists $n_0 \in \mathbb{N}$ such that $z_n \in M_{s_0}$ for all $n \geqslant n_0$. By the continuity of f, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geqslant n_0$ and $f(z_n) \in M_{s_1}$ for all $n \geqslant n_1$. By using the same process inductively, we get that for each $i \geqslant 0$, there exist there exists $n_i \in \mathbb{N}$ such that $f^j(z_n) \in M_{s_j}$ for all $n \geqslant n_i$ and $0 \leqslant j \leqslant i$. This proves that $h(z_n) \to \underline{s}$ as $n \to \infty$. Therefore, h is continuous on Λ.

To prove that h is onto, we need the following lemma.

Lemma 4.26. *For any $\underline{s} \in \Sigma_A$, the intersection $\cap_{n \geqslant 0} f^{-n}(M_{s_n})$ is nonempty.*

Proof. Let $\underline{s} \in \Sigma_A$. First, we prove that the intersection $\cap_{i=0}^{n} f^{-i}(M_{s_i})$ is nonempty for all $n \geqslant 0$ by applying Theorem 3.3 to a closed loop of covering relations. Let $n \geqslant 0$. Then we have the loop of covering relations $M_{s_0} \overset{f}{\Longrightarrow} M_{s_1} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} M_{s_n}$. The loop becomes closed by adding a covering relation $M_{s_n} \overset{g}{\Longrightarrow} M_{s_0}$ with a homotopy $h : [0,1] \times M_{s_n,c} \to \mathbb{R}^u \times \mathbb{R}^s$, where $u = u(M_{s_n})$, $s = s(M_{s_n})$, $g_c : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u \times \mathbb{R}^s$ is defined by $g_c(p,q) = (2p, 0)$ for all $(p,q) \in \mathbb{R}^u \times \mathbb{R}^s$, $g = c_{M_{s_0}}^{-1} \circ g_c \circ c_{M_{s_n}}$, and $h(t,p,q) = g_c(p,q)$ for $t \in [0,1]$ and $(p,q) \in M_{s_n,c}$. It follows from Theorem 3.3 that there exists $z \in int(M_{s_0})$ such that $f^i(z) \in int(M_{s_i})$ for $0 \leqslant i \leqslant n$. Thus $z \in \cap_{i=0}^{n} f^{-i}(M_{s_i})$. Therefore, the intersection $\cap_{i=0}^{n} f^{-i}(M_{s_i})$ is nonempty for all $n \geqslant 0$. Since

$\{\cap_{i=0}^{n} f^{-i}(M_{s_i})\}_{n=0}^{\infty}$ is a nested sequence of nonempty compact subsets of $\mathbb{R}^m$, the set $\cap_{n\geqslant 0} f^{-n}(M_{s_n})$ is nonempty. $\square$

Finally, we show that h is onto. Let $\underline{s} \in \Sigma_A$. Then there exists $z \in \cap_{n\geqslant 0} f^{-n}(M_{s_n})$ from Lemma 4.26. By the definitions of Λ and h, we have that $z \in \Lambda$ and $h(z) = \underline{s}$. This proves that h is onto. We have finished the proof of Proposition 4.25. $\square$

Now, we begin to prove the results for covering relation determined by a transition matrix. In the following, we write $A = [a_{ij}]_{1\leqslant i,j\leqslant \eta}$ and denote by $\{M_i\}_{i=1}^{\eta}$ the pairwise disjoint h-sets with covering relations for f determined by A.

Proof of Theorem 4.22. Since the dimension of A is η, there are at most η^2 choices of the covering relations $M_i \stackrel{f}{\Longrightarrow} M_j$. By Proposition 3.5, if $g : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous map with $|g - f|$ small enough, then g has covering relations on h-sets $\{M_i\}_{i=1}^{\eta}$ determined by A. By applying Proposition 4.25 to the map g, there exists a compact subset Λ_g of $\mathbb{R}^m$ such that Λ_g is positive invariant for g and $g|\Lambda_g$ is topologically semi-conjugate to the one-sided subshift of finite type σ_A^+. Therefore, $h_{\text{top}}(g) \geqslant h_{\text{top}}(g|\Lambda_g) \geqslant h_{\text{top}}(\sigma_A^+) = \log(\rho(A))$. $\square$

Proof of Theorem 4.23. Let $M = \cup_{i=1}^{\eta} M_i$. Since g is continuous and M is compact, there exists $r > 0$ such that $g(M) \subset B_k(0, r)$. For $i \in \{1, \dots, \eta\}$, we define h-sets M_i' in $\mathbb{R}^m \times \mathbb{R}^k$ by $M_i' = M_i \times \overline{B_k(0, r)}$, with $u(M_i') = u(M_i)$, $s(M_i') = s(M_i) + n$ and $c_{M_i'}(x, y) = (c_{M_i}(x), y/r)$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$. Suppose $a_{ij} = 1$. Then $M_i \stackrel{f}{\Longrightarrow} M_j$ implies $M_i' \stackrel{F}{\Longrightarrow} M_j'$ by defining a homotopy $H : [0,1] \times \overline{B_m} \times \overline{B_k} \to \mathbb{R}^{m+k}$ as follows

$$H(t, x, y) = \left(h(t, x), \frac{1-t}{r} g(c_{M_i}^{-1}(x)) \right),$$

where h is the homotopy for the covering relation $M_i \stackrel{f}{\Longrightarrow} M_j$. This shows that F has covering relations on $\{M_i'\}_{i=1}^{\eta}$ determined by A. By applying

Theorem 4.22 to the map F on $\mathbb{R}^m \times \mathbb{R}^k$, we get that if G is a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ with $|G - F|$ small enough, then there exists a compact subset Λ_G of $\mathbb{R}^{m+k}$ such that Λ_G is positively invariant for g and $g|\Lambda g$ is topologically semi-conjugate to the one-sided subshift of finite type σ_A^+. $\square$

Proof of Theorem 4.24. Define $\hat{F} = (id, c) \circ F \circ (id, c)^{-1}$, where id denotes the identity map on $\mathbb{R}^m$ and c is a homeomorphism form S to $\overline{B_k}$. Then the conclusion follows from the above argument. $\square$

4.3.2 Liapunov and strong Liapunov condition

In this subsection, we will introduce covering relations with the the Liapunov and strong Liapunov conditions determined by a transition matrix and list the related main results of the topological dynamics for multidimensional perturbations of high-dimensional systems. Here, we will let $|\cdot|$ denote the Euclidean norm and $||\cdot||$ denote the operator norm on the space of linear maps induced by $|\cdot|$.

In the following, we slightly modify the cone condition for a covering relation given by Zgliczyński in [36, Definition 11] and furthermore, we define the strong Liapunov condition. First, We define a quadratic form on a h-set K in $\mathbb{R}^m$ to be of the form

$$Q_K(x, y) = P_K(x) - Q_K(y) \text{ for all } (x, y) \in \mathbb{R}^{u(K)} \times \mathbb{R}^{s(K)}, \tag{4.20}$$

where $P_K : \mathbb{R}^{u(K)} \to \mathbb{R}$ and $Q_K : \mathbb{R}^{s(K)} \to \mathbb{R}$ are positive definite quadratic forms. Note that a quadratic form on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector z in $\mathbb{R}^n$ can be computed by an expression of the form $Q(z) = z^T S z$, where S is an $n \times n$ symmetric matrix and z^T denotes the transpose of z; refer to [27].

Definition 4.27. *Let Q_M and Q_N be quadratic forms on h-sets M and N, respectively, as in Equation (4.20). We say that a covering relation $M \stackrel{f}{\Longrightarrow} N$*

satisfies the Liapunov condition *(resp. the* strong Liapunov condition*) with respect to the pair* (Q_M, Q_N) *if there exists* $\theta \geqslant 0$ *(resp.* $\theta > 0$*) such that for any* $u, v \in M_c$ *with* $u \neq v$*,*

$$Q_N(f_c(u) - f_c(v)) - Q_M(u - v) > \theta \left| u - v \right|^2 . \tag{4.21}$$

As a Liapunov function, a sequence of quadratic forms has scalar values strictly monotone along the difference of two orbits. More precisely, consider covering relations $M_i \overset{f}{\Longrightarrow} M_{i+1}$ satisfying the Liapunov condition with respect to the pair $(Q_{M_i}, Q_{M_{i+1}})$ of quadratic forms for all $i \geqslant 0$. If u, v are two points such that $f^i(u), f^i(v) \in M_{i,c}$ and $f^i(u) \neq f^i(v)$ for all $i \geqslant 0$, then the sequence $\{Q_{M_i}(f^i(u) - f^i(v)\}_{i=0}^{\infty}$ is strictly increasing. This property will play an import role while we prove conjugacy results.

Definition 4.28. *Let* $A = [a_{ij}]_{1 \leqslant i,j \leqslant \eta}$ *be a transition matrix and* f *be a continuous map on* $\mathbb{R}^m$*. We say that* f *has* covering relations with the Liapunov conditions (resp. the strong Liapunov condition) determined by A *if the following conditions are satisfied;*

1. *there are* η *pairwise disjoint h-sets* $\{M_i\}_{i=1}^{\eta}$ *in* $\mathbb{R}^m$*; on each* M_i *there exists a quadratic form* Q_{M_i} *as in Equation (4.20).*

2. *if* $a_{ij} = 1$ *then the covering relation* $M_i \overset{f}{\Longrightarrow} M_j$ *holds and satisfies the Liapunov condition (resp. the strong Liapunov condition) with respect to the pair* (Q_{M_i}, Q_{M_j})*; and*

3. *if* $a_{ij} = 1$ *then the coordinate chart* c_{M_i} *and* c_{M_j} *is a* C^1 *diffeomorphisms.*

The Liapunov condition is for detection of chaos (see Proposition 4.33 below), while the strong Liapunov condition is for stability of chaos under small C^1 perturbations as follows.

Next, we use the logistic map once again as an example of map has covering relations with the Liapunov conditions (resp. the strong Liapunov condition) determined by a transition matrix.

Example 4.29. *Let us show that the logistic map $f(x) = \mu x(1-x)$ with $\mu > 4$ has covering relations with the strong Liapunov condition determined by the 2×2 matrix with all entries one. Set (i) h-sets $M_1 = [-\epsilon, 1/2-\delta]$ and $M_2 = [1/2+\delta, 1+\epsilon]$, where $0 < 2\epsilon < \mu/4-1$ and $0 < \delta < [(\mu/4-1-\epsilon)\mu^{-1}]^{1/2}$; (ii) the coordinate charts $\bar{u} = c_{M_1}(u) = \alpha^{-1}(\int_{-\epsilon}^{u}\rho(t)dt - \int_{u}^{1/2-\delta}\rho(t)dt)$ and $\bar{u} = c_{M_2}(u) = \alpha^{-1}(\int_{1/2+\delta}^{u}\rho(t)dt - \int_{u}^{1+\epsilon}\rho(t)dt)$, where $\rho(t) = [(t+2\epsilon)(1-2\epsilon-t)]^{-1}$ for $t \in (-2\epsilon, 1+2\epsilon)$, and $\alpha = \int_{-\epsilon}^{1/2-\delta}\rho(t)dt = \int_{1/2+\delta}^{1+\epsilon}\rho(t)dt$; and (iii) quadratic forms $Q_{M_1}(\bar{u}) = Q_{M_2}(\bar{u}) = \bar{u}^2$. With a help of the Schwarz lemma and the idea of the Poincaré norm, in Proposition 4.10 of [29], it is shown that there exists $\lambda > 1$ such that if u, $f(u) \in M_1 \cup M_2$, then $\rho(f(u))|f'(u)| \geqslant \lambda\rho(u)$. Let C_1 be a positive constant such that $\rho(t) \geqslant C_1$ for all $t \in M_1 \cup M_2$. Then for any $u, v \in M_1 \cup M_2$ we have $|\int_v^u \rho(t)dt| \geqslant C_1|u-v|$. Since $c'_{M_i}(u) = 2\alpha^{-1}\rho(u)$, there exists $C_2 > 0$ such that $|c'_{M_i}(u)| \leqslant C_2$ for all $u \in M_i$ and $i = 1, 2$. Therefore, the strong Liapunov condition holds*

$$
\begin{aligned}
& Q_{M_i}(f_c(\bar{u}) - f_c(\bar{v})) - Q_{M_j}(\bar{u}-\bar{v}) \\
= \; & (c_{M_i} \circ f(u) - c_{M_i} \circ f(v))^2 - (c_{M_j}(u) - c_{M_j}(v))^2 \\
= \; & \left(2\alpha^{-1}\int_{f(v)}^{f(u)} \rho(t)dt\right)^2 - \left(2\alpha^{-1}\int_v^u \rho(t)dt\right)^2 \\
\geqslant \; & 4\alpha^{-2}(\lambda^2-1)\left(\int_v^u \rho(t)dt\right)^2 \\
\geqslant \; & 4\alpha^{-2}(\lambda^2-1)C_1|u-v|^2 \geqslant 4\alpha^{-2}(\lambda^2-1)C_1C_2^{-2}|u-v|^2.
\end{aligned}
$$

In the followings, we list our results of covering relations with the the Liapunov and strong Liapunov conditions.

Theorem 4.30. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1 homeomorphism having covering relations with the strong Liapunov condition determined by a transition* *matrix A. If g is a C^1 homeomorphism on $\mathbb{R}^m$ with $|g-f| + \|Dg - Df\|$ small enough, then there exists a compact subset Λ_g of $\mathbb{R}^m$ such that Λ_g is invariant for g and $g|\Lambda_g$ is topologically conjugate to σ_A.*

For small C^1 perturbations of a direct product contracting along the second variable, we have the following result.

Theorem 4.31. *Let $F(x,y) = (f(x), g(y)) \in \mathbb{R}^m \times \mathbb{R}^k$ be a C^1 homeomorphism for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ has covering relations with the strong Liapunov condition determined by a transition matrix A, and $g : \mathbb{R}^k \to \mathbb{R}^k$ is a contraction on the closed unit ball $\overline{B_k}$ such that $g(\overline{B_k}) \subset B_k$. If G is a C^1 homeomorphism on $\mathbb{R}^{m+k}$ with $|G-F| + \|DG - DF\|$ small enough, then there exists a compact subset Λ_G of $\mathbb{R}^{m+k}$ such that Λ_G is invariant for G and $G|\Lambda_G$ is topologically conjugate to σ_A.*

Finally, for a one-parameter family of maps with the singular map F depends only on the phase variable of f, we have the following result.

Theorem 4.32. *Let F_λ be a one-parameter family of maps on $\mathbb{R}^m \times \mathbb{R}^k$ satisfying (i) $F_\lambda(x,y)$ is C^1 continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where λ is a parameter; (ii) F_λ is a homeomorphism on $\mathbb{R}^m \times \mathbb{R}^k$ provided $\lambda \neq 0$; and (iii) $F_0(x,y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ has covering relations with the strong Liapunov condition determined by a transition matrix A, and $g : \mathbb{R}^m \to \mathbb{R}^k$. Then for each λ sufficiently close to 0, there exists a compact subset Λ_λ of $\mathbb{R}^{m+k}$ such that if $\lambda \neq 0$ then Λ_λ is invariant for F_λ and $F_\lambda|\Lambda_\lambda$ is topologically conjugate to σ_A, while Λ_0 is positively invariant for F_0 and $F_0|\Lambda_0$ is topologically semi-conjugate to σ_A^+.*

In order to prove the main results, we need the following proposition which is stated that a homeomorphism having covering relations with the Liapunov condition determined by a transition matrix is topologically conjugate to a two-sided subshift of finite type.

Proposition 4.33. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism which has covering relations with the Liapunov condition determined by a transition matrix A. Then there exists a compact subset Λ of $\mathbb{R}^m$ such that Λ is maximal invariant for f in the interior of the union of the h-sets (with respect to A) and $f|\Lambda$ is topologically conjugate to σ_A.*

Proof. We denote by $\{M_i\}_{i=1}^{\eta}$ the h-sets with covering relations and the Liapunov condition for f determined by A as in Definition 4.28, and write $\underline{s} = (\dots, s_{-1}, s_0, s_1, \dots)$ for $\underline{s} \in \Sigma_A$. Define

$$\Lambda_n = \bigcup_{\underline{s}\in\Sigma_A} \left(\bigcap_{i=-n}^{n} f^{-i}(M_{s_i}) \right) \text{ for } n \geqslant 0, \text{ and } \Lambda = \bigcap_{n\geqslant 0} \Lambda_n.$$

Define $h : \Lambda \to \Sigma_A$ by $h(z) = \underline{s}$ for $z \in \Lambda$, where $f^n(z) \in M_{s_n}$ for all $n \in \mathbb{Z}$. By using the same argument as in the proof of Proposition 4.25, we have that Λ is a maximal compact invariant set for f in $\cup_{i=1}^{\eta} M_i$ with respect to A and h is a topological semi-conjugacy. Moreover, the covering relations for f on h-sets implies that any boundary point of a h-set can not have a full orbit staying in h-sets. Therefore, Λ is maximal invariant for f in $\cup_{i=1}^{\eta} int(M_i)$ with respect to A.

To prove that h is one to one, we need the following lemma, which is guaranteed by the Liapunov condition.

Lemma 4.34. *For any $\underline{s} \in \Sigma_A$, the intersection $\cap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ consists of a single point.*

Proof. Let $\underline{s} \in \Sigma_A$. Then, similar to the proof of Lemma 4.26, we have that the intersection $\cap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ is nonempty. Next, we show the uniqueness of the intersection by contradiction. Assume that $u, v \in \cap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ with $u \neq v$. Since f is a homeomorphism, $f^n(u)$ and $f^n(v)$ are different points lying in the same h-set M_{s_n} for all $n \in \mathbb{Z}$. By the covering relation with the

Liapunov condition, we have that for all $n \in \mathbb{Z}$,

$$Q_{M_{s_{n+1}}}(c_{M_{s_{n+1}}} \circ f^{n+1}(u) - c_{M_{s_{n+1}}} \circ f^{n+1}(v)) > Q_{M_{s_n}}(c_{M_{s_n}} \circ f^n(u) - c_{M_{s_n}} \circ f^n(v)). \tag{4.22}$$

That is, the value of $Q_{M_{s_n}}$ at the point $c_{M_{s_n}} \circ f^n(u) - c_{M_{s_n}} \circ f^n(v)$ is strictly increasing as $n \in \mathbb{Z}$ increases. It follows that there exits $j \in \mathbb{Z}$ such that $Q_{M_{s_j}}(c_{M_{s_j}} \circ f^j(u) - c_{M_{s_j}} \circ f^j(v)) \neq 0$.

First, we consider the case when

$$Q_{M_{s_j}}(c_{M_{s_j}} \circ f^j(u) - c_{M_{s_j}} \circ f^j(v)) > 0. \tag{4.23}$$

By using the compactness of the union $\cup_{i=1}^{\eta} M_i$, sequentially twice for two sequences, both sequences $\{f^{n+j}(u)\}_{n=0}^{\infty}$ and $\{f^{n+j}(v)\}_{n=0}^{\infty}$ have convergent subsequences, say $\{f^{n(i)+j}(u)\}_{i=0}^{\infty}$ and $\{f^{n(i)+j}(v)\}_{i=0}^{\infty}$, with the limits, say $\bar{u}$ and $\bar{v}$ in $\cup_{i=1}^{\eta} M_i$, respectively. By the fact that M_i's are pairwise disjoint and compact, and $f^n(u), f^n(v) \in M_{s_n}$ for all $n \in \mathbb{Z}$, there exists $\alpha \in \mathbb{N}$ such that $f^{n(i)+j}(u), f^{n(i)+j}(v)$, $\bar{u}$ and $\bar{v}$ are all in the same h-set, namely $M_{s_{n(\alpha)+j}}$ for all $i \geqslant \alpha$. By the continuity of f, the points $f(\bar{u})$ and $f(\bar{v})$ are limits of sequences $\{f^{n(i)+j+1}(u)\}_{i=0}^{\infty}$ and $\{f^{n(i)+j+1}(v)\}_{i=0}^{\infty}$, respectively. Again by the same fact as above, there exists a integer $\beta \geqslant \alpha$ such that $f^{n(i)+j+1}(u), f^{n(i)+j+1}(v)$, $f(\bar{u})$ and $f(\bar{v})$ are all in the same h-set, namely $M_{s_{n(\beta)+j+1}}$ for all $i \geqslant \beta$. For convenience, we denote $N_0 = M_{s_{n(\alpha)+j}}$ and $N_1 = M_{s_{n(\beta)+j+1}}$.

By Equation (4.22), we get that for all $i \geqslant \beta$,

$$Q_{N_0}(c_{N_0} \circ f^{n(i)+j}(u) - c_{N_0} \circ f^{n(i)+j}(v)) > Q_{M_{s_j}}(c_{M_{s_j}} \circ f^j(u) - c_{M_{s_j}} \circ f^j(v))$$

By letting $i \to \infty$, it follows from the continuity of Q_{N_0} and c_{N_0} that

$$Q_{N_0}(c_{N_0}(\bar{u}) - c_{N_0}(\bar{v})) \geqslant Q_{M_{s_j}}(c_{M_{s_j}} \circ f^j(u) - c_{M_{s_j}} \circ f^j(v)).$$

Thus from Equation (4.23), we have $Q_{N_0}(c_{N_0}(\bar{u}) - c_{N_0}(\bar{v})) > 0$ and hence $\bar{u} \neq \bar{v}$. Since $f(\bar{u}), f(\bar{v}) \in N_1$, the Liapunov condition implies that

$$Q_{N_1}(c_{N_1} \circ f(\bar{u}) - c_{N_1} \circ f(\bar{v})) > Q_{N_0}(c_{N_0}(\bar{u}) - c_{N_0}(\bar{v})). \tag{4.24}$$

Because that $f^{n(i)+j+1}(u)$ and $f^{n(i)+j+1}(v)$ converge to $f(\bar{u})$ and $f(\bar{v})$, respectively, and both Q_{N_1} and c_{N_1} are continuous, we obtain that for some large γ,

$$Q_{N_1}(c_{N_1} \circ f^{n(\gamma)+j+1}(u) - c_{N_1} \circ f^{n(\gamma)+j+1}(v)) > Q_{N_0}(c_{N_0}(\bar{u}) - c_{N_0}(\bar{v})).$$

By using Equation (4.22), we get that for all $i > \gamma + 1$,

$$Q_{N_0}(c_{N_0} \circ f^{n(i)+j}(u) - c_{N_0} \circ f^{n(i)+j}(v)) > Q_{N_1}(c_{N_1} \circ f^{n(\gamma)+j+1}(u) - c_{N_1} \circ f^{n(\gamma)+j+1}(v))$$

Letting $i \to \infty$, it follows from the continuity of Q_{N_0} and c_{N_0} that

$$Q_{N_0}(c_{N_0}(\bar{u}) - c_{N_0}(\bar{v})) \geqslant Q_{N_1}(c_{N_1} \circ f^{n(\gamma)+j+1}(u) - c_{N_1} \circ f^{n(\gamma)+j+1}(v)).$$

Together with Equation (4.24), this leads to a contradiction.

For the case when $Q_{M_{s_j}}(c_{M_{s_j}} \circ f^j(u) - c_{M_{s_j}} \circ f^j(v)) < 0$, by working on the backward orbits of u and v, that is, replacing n and $n(i)$ by $-n$ and $-n(i)$ in the above argument, it leads to a contradiction.

Therefore, the intersection $\cap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ consisting of a single point. We have done the proof of the desired result. $\square$

By using Lemma 4.34, we can easily prove that h is one to one. Indeed, let $\underline{s} \in \Sigma_A$ and $h(z_1) = h(z_2) = \underline{s}$ for $z_1, z_2 \in \Lambda$. Then $z_1, z_2 \in \cap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ and hence $z_1 = z_2$.

Because that the sets Λ and Σ_A are compact and h is a continuous and one to one function, it follows that h is a homeomorphism. This completes the proof of Proposition 4.33. $\square$

Now, we begin to prove the main results for with covering relation with the Liapunov condition determined by a transition matrix. In the following, we write $A = [a_{ij}]_{1\leqslant i,j\leqslant \eta}$ and denote by $\{M_i\}_{i=1}^{\eta}$ the pairwise disjoint h-sets with covering relations for f determined by A. For each h-set M_i, let Q_{M_i} be the quadratic form for the strong cone condition of f.

Proof of Theorem 4.30. Suppose $a_{ij} = 1$. Then $M_i \overset{f}{\Longrightarrow} M_j$ holds. By Proposition 3.5, if $|g - f|$ is small enough, then $M_i \overset{g}{\Longrightarrow} M_j$ holds. Assume that such a map g is C^1. Before proving that $M_i \overset{g}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition, let us have some observations. Since $M_i \overset{f}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition, there exists $\theta_{i,j} > 0$ such that for x, $y \in M_{i,c}$ with $x \neq y$,

$$Q_{M_j}(f_c(x) - f_c(y)) > Q_{M_i}(x - y) + \theta_{i,j} |x - y|^2 . \tag{4.25}$$

For $\alpha = i, j$, let S_α be the $m \times m$ symmetric matrix such that $Q_{M_\alpha}(z) = z^T S_\alpha z$ for $z \in \mathbb{R}^m$. Since f, g and c_{M_i} are C^1, for x, $y \in M_{i,c}$, we can define

$$E_{x,y} = \int_0^1 Df_c(y + t(x - y))dt \text{ and } C_{x,y} = \int_0^1 Dg_c(y + t(x - y))dt.$$

Then $|E_{x,y} - C_{x,y}| \leqslant \|Df_c - Dg_c\|$. Since both f_c and g_c are C^1 on the compact set $M_{i,c}$, there exists $\beta_i > 0$ such that $|E_{x,y}| + |C_{x,y}| < \beta_i$ for all x, $y \in M_{i,c}$. Thus

$$\begin{aligned} & |E_{x,y}^T S_j E_{x,y} - C_{x,y}^T S_j C_{x,y}| \\ \leqslant\ & |E_{x,y}^T S_j E_{x,y} - C_{x,y}^T S_j E_{x,y}| + |C_{x,y}^T S_j E_{x,y} - C_{x,y}^T S_j C_{x,y}| \\ \leqslant\ & \beta_i \|S_j\| \|Df_c - Dg_c\| . \end{aligned} \tag{4.26}$$

Now we check the strong Liapunov condition for $M_i \overset{g}{\Longrightarrow} M_j$. Let u, $v \in M_{i,c}$ with $u \neq v$. By the mean value theorem for integrals, we have that $f_c(u) - f_c(v) = E_{u,v}(u - v)$ and $g_c(u) - g_c(v) = C_{u,v}(u - v)$. Thus,

$$\begin{aligned} & Q_{M_j}(f_c(u) - f_c(v)) - Q_{M_j}(g_c(u) - g_c(v)) \\ =\ & (u - v)^T (E_{u,v}^T S_j E_{u,v} - C_{u,v}^T S_j C_{u,v})(u - v). \end{aligned}$$

From Equation (4.26), we obtain that

$$\begin{aligned} & |Q_{M_j}(f_c(u) - f_c(v)) - Q_{M_j}(g_c(u) - g_c(v))| \\ \leqslant\ & \beta_i \|S_j\| \|Df_c - Dg_c\| |u - v|^2 . \end{aligned}$$

Imposing Equation (4.25), we get that

$$
\begin{aligned}
& Q_{M_j}(g_c(u) - g_c(v)) \\
\geqslant\ & Q_{M_j}(f_c(u) - f_c(v)) - |Q_{M_j}(f_c(u) - f_c(v)) - Q_{M_j}(g_c(u) - g_c(v))| \\
>\ & Q_{M_i}(u - v) + \theta_{i,j} |u - v|^2 - \beta_i \|S_j\| \|Df_c - Dg_c\| |u - v|^2 \\
=\ & Q_{M_i}(u - v) + (\theta_{i,j} - \beta_i \|S_j\| \|Df_c - Dg_c\|) |u - v|^2 .
\end{aligned}
$$

Finally, we denote

$$\hat{\theta}_{i,j} = \theta_{i,j} - \beta_i \|S_j\| \|Df_c - Dg_c\| .$$

Then $\hat{\theta}_{i,j}$ is independent of $u, v \in M_{i,c}$. Since c_{M_α} is C^1 diffeomorphism and M_α is compact for $\alpha = i, j$, we have that $\|Df_c - Dg_c\|$ approaches to zero as $\|Df - Dg\|$ tends to zero. Therefore, if $\|Df - Dg\|$ is small enough, then $\hat{\theta}_{i,j} > 0$ and hence $M_i \overset{g}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition.

Since there are at most η^2 choices of pairs (i, j), from the above, we get that if g is a C^1 continuous map with $|g - f| + \|Dg - Df\|$ small enough, then g has covering relations with the strong Liapunov condition determined by A. In addition, if such maps g are C^1 homeomorphisms, then we have the desired result, by applying Proposition 4.33 and the fact that the strong Liapunov condition implies the Liapunov condition. □

Proof of Theorem 4.31. Suppose $a_{ij} = 1$. Then the covering relation $M_i \overset{f}{\Longrightarrow} M_j$ holds. First, we show that there is a corresponding covering relation for F on h-sets. For $\alpha = i, j$, let $M'_\alpha = M_\alpha \times \overline{B_k}$. Then each M'_α is an h-set with $c_{M'_\alpha}(x, y) = (c_{M_\alpha}(x), y)$, $u(M'_\alpha) = u(M_\alpha)$, and $s(M'_\alpha) = s(M_\alpha) + k$. Define a homotopy $H : [0, 1] \times \overline{B_m} \times \overline{B_k} \to \mathbb{R}^{m+k}$ by

$$H(t, x, y) = (h(t, x), (1 - t)g(y)),$$

where h is the homotopy for the covering relation $M_i \overset{f}{\Longrightarrow} M_j$. Then for all $x \in \overline{B_m}$ and $y \in \overline{B_k}$, we have

$$H(0, x, y) = (h(0, x), g(y)) = (c_{M_j} \circ f \circ c_{M_i}^{-1}(x), g(y)) = F_c(x, y), \text{ and}$$

$$H(1,x,y) = (h(1,x),0).$$

Thus we have that $M_i' \overset{F}{\Longrightarrow} M_j'$ follows from $M_i \overset{f}{\Longrightarrow} M_j$.

Next, we show that the strong Liapunov condition is satisfied for $M_i' \overset{F}{\Longrightarrow} M_j'$. For $\alpha = i$, j, define the quadratic form $Q_{M_\alpha'}(x,y) = Q_{M_\alpha}(x) - |y|^2$. Let (x_1,y_1), $(x_2,y_2) \in M_{i,c}'$ with $(x_1,y_1) \neq (x_2,y_2)$. Since $M_i \overset{f}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition, there exists $\theta_{i,j} > 0$ such that $Q_{M_j}(f_c(x_1) - f_c(x_2)) > Q_{M_i}(x_1 - x_2) + \theta_{i,j} |x_1 - x_2|^2$ if $x_1 \neq x_2$. Since g is a contraction on $\overline{B_k}$, there exists $0 < \gamma < 1$ such that

$$|g(y_1) - g(y_2)| \leqslant \gamma |y_1 - y_2|.$$

Thus no matter what x_1 is equal to x_2 or not, we get that

$$\begin{aligned}
& Q_{M_j'}(F_c((x_1,y_1)) - F_c((x_2,y_2))) - Q_{M_i'}((x_1,y_1) - (x_2,y_2)) \\
= \ & Q_{M_j'}((f_c(x_1) - f_c(x_2), g(y_1) - g(y_2)) - Q_{M_i'}((x_1 - x_2, y_1 - y_2)) \\
= \ & Q_{M_j}(f_c(x_1) - f_c(x_2)) - |g(y_1) - g(y_2)|^2 - Q_{M_i}(x_1 - x_2) \\
& + |y_1 - y_2|^2 \\
\geqslant \ & \theta_{i,j} |x_1 - x_2|^2 + (1 - \gamma^2) |y_1 - y_2|^2 \\
> \ & \hat{\theta}_{i,j} |(x_1,y_1) - (x_2,y_2)|^2,
\end{aligned}$$

where $\hat{\theta}_{i,j} = \min\{\theta_{i,j}, 1 - \gamma^2\}/2 > 0$. Thus $M_i' \overset{F}{\Longrightarrow} M_j'$ satisfies the strong cone condition. Since the number of pairs (i,j) is finite, F has covering relations with the strong Liapunov condition determined by A. From Theorem 4.30, the desired result follows. □

Proof of Theorem 4.32. By the continuity of g on the compact union $\cup_{i=1}^{\eta} M_i$, there exists $r > 0$ such that $g(\cup_{i=1}^{\eta} M_i) \subset B_k(r)$. For each $\alpha \in \{1,2,\ldots,\eta\}$, since g and $c_{M_\alpha}^{-1}$ are C^1, the composition $g \circ c_{M_\alpha}^{-1}$ satisfies the Lipschitz condition on the compact set $M_{\alpha,c}$, i.e., there exists $L_\alpha > 0$ such that for all x_1, $x_2 \in M_{\alpha,c}$,

$$\left| g \circ c_{M_\alpha}^{-1}(x_1) - g \circ c_{M_\alpha}^{-1}(x_2) \right| \leqslant L_\alpha |x_1 - x_2|.$$

For $i,\ j \in \{1, 2, \ldots, \eta\}$ with $a_{ij} = 1$, we have that $M_i \overset{f}{\Longrightarrow} M_j$ holds and satisfies the strong Liapunov condition. Thus there exists $\theta_{i,j} > 0$ such that $Q_{M_j}(f_c(x_1) - f_c(x_2)) > Q_{M_i}(x_1 - x_2) + \theta_{i,j}\,|x_1 - x_2|^2$ if $x_1,\ x_2 \in M_{i,c}$ with $x_1 \neq x_2$. Take a real number $\hat{\theta}$ such that $0 < \hat{\theta} < \min\{\theta_{i,j}/(1 + L_i^2/r^2) : i,\ j \in \{1, 2, \ldots, \eta\},\ a_{ij} = 1\}$.

Suppose $a_{ij} = 1$. For $\alpha \in \{i,\ j\}$, let $M'_\alpha = M_\alpha \times \overline{B_k(r)}$. Then each M'_α is an h-set with $c_{M'_\alpha}(x, y) = (c_{M_\alpha}(x), y/r)$, $u(M'_\alpha) = u(M_\alpha)$, and $s(M'_\alpha) = s(M_\alpha) + k$. Define a quadratic form on M'_α by $Q_{M'_\alpha}(x, y) = Q_{M_\alpha}(x) - \hat{\theta}\,|y|^2$. Then $M'_i \overset{F_0}{\Longrightarrow} M'_j$ holds for a homotopy $H : [0, 1] \times \overline{B_m} \times \overline{B_k} \to \mathbb{R}^{m+k}$ defined by

$$H(t, x, y) = (h(t, x), \frac{1 - t}{r} g(c_{M_i}^{-1}(x))),$$

where h is the homotopy for the covering relation $M_i \overset{f}{\Longrightarrow} M_j$. Furthermore, we check the strong Liapunov condition. Let $(x_1, y_1),\ (x_2, y_2) \in M'_{i,c}$ with $(x_1, y_1) \neq (x_2, y_2)$. Then

Therefore, $M'_i \overset{F_0}{\Longrightarrow} M'_j$ satisfies the strong Liapunov condition.

By the finiteness of the pair (i, j), F_0 has covering relations with the strong Liapunov condition determined by A. By Proposition 4.25, there exists a compact subset Λ_0 of $\mathbb{R}^{m+k}$ such that Λ_0 is positively invariant for F_0 and $F_0|\Lambda_0$ is topologically semi-conjugate to σ_A^+. Since $F_\lambda(x, y)$ is C^1 in the triple (λ, x, y) of variables, by using the same argument as in the proof of Theorem 4.30, there exists $\lambda_0 > 0$ such that for all λ with $|\lambda| < \lambda_0$, the map F_λ has covering relation with the strong Liapunov condition determined by A. Since F_λ is a homeomorphism on $\mathbb{R}^{m+k}$ provided $\lambda \neq 0$, by Proposition 4.33, if $0 < |\lambda| < \lambda_0$ then there exists a compact subset Λ_λ of $\mathbb{R}^{m+k}$ such that Λ_λ is invariant for F_λ and $F_\lambda|\Lambda_\lambda$ is topologically conjugate to σ_A. We have finished the proof of the theorem. $\square$

5 Conclusion

Conclude from this dissertation, we mention some possible future works.

- As the construction of the covering relation, it's interesting to consider the chaotic dynamics for some nonuniformly hyperbolic systems.

  Barreira and Valls [3] consider sequences of Lipschitz maps $A_m + f_m$ such that the linear parts A_m admit a nonuniform exponential dichotomy, and establish the existence of a unique sequence of topological conjugacies between the maps $A_m + f_m$. Also, in [4], they study the relation between nonuniform exponential dichotomy and strict Lyapunov sequences. Given such a sequence, they obtain the stable and unstable subspaces from the intersection of images and preimages of the cones defined by each element of the sequence. Use the ideas of nonuniform exponential dichotomy and strict Lyapunov sequences, we want to construct the covering relations with strong Liapunov condition for the nonuniformly hyperbolic systems.

- It is possible to use the fixed point index theorem to extend the results to the Banach space.

  Misiurewicz and Zgliczyński [8] use the covering relation in real banach space and the fixed point index theorem to give the result to rigorous estimate topological entropy in case of a one dimensional model (i.e. f is in one dimensional space) where the full system is in infinite dimensional real Banach space. As the construction of covering relations in subsection 4.1.2 for map which has a snap-back repeller, we want to extend the result for the compact map which has a snap-back repeller in the real Banach space. Moreover, we want to apply the result to some differential equations.

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