國立交通大學

運輸科技與管理學系

碩士論文

交通路網設計問題之矛盾現象預測

The Prevalence of Braess' Paradox in Transportation Network Design Problem

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摘　　要

Braess 矛盾現象在運輸規劃和路網設計範疇裡，皆為一重要議題且已被廣泛討論。在從事路網設計時，我們希望藉由新建道路或是提升道路容量來解決擁擠問題，但是 Braess 矛盾現象卻顯示，若沒有全盤作好路網敏感性分析，例如分析需求或是路段成本函數對整個路網使用者的影響，所新建的道路可能會毫無用武之地；更甚之，使整個路網之起迄成本更高。過去學者研究中，皆以最初 Braess 所提出簡單路網為基礎，予以探究發生矛盾現象的原因，並給予建議預防。從以往的研究中，可發現 Braess 矛盾現象發生原因會和路網結構、路段成本函數以及起訖點需求有關。本論文以 Dafermos 與 Nagurney 於 1984 年提出之預測矛盾現象公式為基礎，將其路網假設條件放鬆，其假設條件為路網中，路徑數必須小於等於路段數加上起迄對數，使之模式能應用於大型路網。

關鍵字：Braess 矛盾現象、路網設計、運輸規劃、廣義反矩陣

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ABSTRACT

In transportation planning and network design, Braess paradox problem has been discussed for many decades. Those researches were originated from the simple network illustrated by Braess. Many works devoted to seek efficient methods to avoid the occurrence of paradox problem or find some rules for network designers to refer. Under link-OD/path matrix is full column rank , i.e., the number of paths is less than the number of links plus origin/destination pairs, Dafermos and Nagurney (Dafermos and Nagurney,1984) derived the formulas to determine whether Braess' paradox occurs in the network. Using their formula, transportation planners could foresee occurrence of Braess' paradox before great capital investment in road construction. This study proposes generalized inverse approach to relax the full column rank assumption presented by Dafermos and Nagurney. Then the modified model could be applied to large networks.

Keywords: Braess' paradox , Network design, Transportation planning, Generalized inverse

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1. Introduction

"Network" is commonly used to describe a structure that can be either physical (e.g., streets and intersection) or conceptual (e.g., information lines and people). There are two elements in each of these networks: a set of points and line segments connecting these points. In standard terminology, these points of a network are referred to as *nodes* (or vertices) and the lines of a network as *links* (or edges). Each network link is typically associated with some impedance that affects the flow using it. Impedance can represent electrical resistance, time, costs, utility, or any other measure. When the flow involves people, the term "level of service" is usually used instead of "impedance". The travel impedance, or level of service, associated with the links representing an urban network include many components, reflecting travel time, safety, cost of travel and others. However, the primary component is travel time, which is often used as the sole measure of link impedance. The level of service offered by many transportation systems is a function of the usage of these systems. Because of congestion, travel time on urban streets and intersections is an increasing function of flow. Thus, the *performance function* relates the travel time on each link to the flow traversing the link. Typically, the networks are "connected", so it is possible to get from any node to any other node by following a *path* (or a route) through the network. A path is a sequence of directed links leading form one node to another. A pair of nodes is usually connected by more than one path. (Sheffi, [1])

Given a network, and assume that the number of travelers who wish to travel between a given origin point and a given destination point known. Furthermore, assume that these points are connected by several possible paths. The question of the interest here is how these travelers will be distributed among the possible paths. The problem is known as that of *traffic assignment*. The traffic assignment process is traditionally viewed as the final stage of the four stage process used to model travel demand in transportation planning. After the Trip Generation, Trip Distribution, and Modal Split process, Traffic Assignment is to assign flows by various modes in given links to paths in transportation networks. Through traffic assignment techniques, not only all the link flows in a network can be estimated, but also travel cost between origin-destination (OD) pairs can be provided, which is used by trip distribution or modal split model. There are four traffic assignment techniques as follows (Meyer and Miller, [2]):

(1) All-or-nothing assignment:

This is the simplest approach involving the selection and loading between each origin and destination. Assume road capacity is unlimited, and link cost is fixed. For each OD pair, find the shortest path and assign all the travel demand into it. The method ignores the limitations imposed by restriction on the capacity of the network. Links may be allocated far greater flows than they are capable of carrying.

(2) Equilibrium assignment

The idea of equilibrium in the analysis of transportation networks arises from the dependence of the link travel time on the link flows. In 1952, Wardrop[3] proposed a concept of distributing the travel demand on a transportation network; that is, user equilibrium (UE). User equilibrium is that the costs on all paths used between any given OD pair are equal and not greater than the cost experienced by other travelers in an unused path between them. In practice, user equilibrium is generally considered as the more likely basis for network equilibrium. Initially, the computer power was not available to solve the equilibrium alignment problem and approximate methods were used to obtain equilibrium solutions, including capacity restrain assignment (is also called iterative assignment) and incremental assignment. Beckmann [4] showed that the equilibrium assignment problem could be transformed into an equivalent optimization problem: if the cost on any link is a function of the flow and of no other flows, then the flows satisfying user equilibrium principle are unique and are the same as the following minimize a specified objective function as (1-1):

$$MInimize\ Z = \sum_{ij} \int_0^{f_{ij}} c_{ij}(x)dx \qquad (1\text{-}1)$$

st.

$$\sum_{p \in P_w} h_p = T_w, \quad w \in W \qquad (1.2)$$

$$h_p \geq 0, \ \ p \in P, \qquad (1.3)$$

where i, j represents two endpoint of link, f_{ij} is link(i,j) flows and c_{ij} is link(i,j) travel time function, which is only dependent on its flows. T_w is the OD demand between OD pair $w \in W$.(1.2) is the OD flow conservation constraint and (1.3) a nonnegativity constraint, here $f_{ij} = \sum_{p \in P} f_p \delta_{(i,j),p}$ where $\delta_{(i,j),p}$ =1 if path p uses link(i,j); and 0 otherwise. This could be solved using the Frank-Wolfe algorithm to combine the results of successive all-or-nothing assignment in an iterative manner.

(3) Stochastic assignment:

The second assignment technique is also called deterministic user equilibrium, because it assumes all travelers obtain *perfect information* on travel costs on any given path are perfect, resulting in making rational route choices. However, in real world, traveler can not always obtain the whole network information. This leads to development of stochastic assignment, in which link travel time function is viewed as random variables varying with users' preferences, perception and experience.

(4) Dynamic assignment:

Network flows will not vary with time in the above approaches. In static assignment techniques, these procedures assume that each vehicle is simultaneously located on every link on its chosen path and assign all flow simultaneously to all links on the chosen paths. It is unrealistic assumption obviously, but for many regional transportation planning applications, static assignment assumption is acceptable and can yield useful results. However, the static representation of network performance is not sufficiently accurate. A dynamic representation of route choice behavior and resulting network performance is required in which the movements of vehicles along their chosen paths is explicitly simulated through time. Dynamic assignment models may be either probabilistic in terms of the simulation of users' route choices and/or determination of vehicles' travel times along given links, or they can be deterministic.

This research focus on equilibrium assignment problem, says, user equilibrium. Based on user equilibrium, it turns out that all users traveling with the same origin and destination incur the same travel cost in equilibrium, and is irrelevant to their originally chosen path. It is useful to predict how changes in the travel demand or network geometry will affect traveler costs. The congestion in a network resulting from user equilibrium flows is greater than the congestion that would exist if some central controller could assign all travelers between their origins and destinations. This would involve in that: whether road investment can alleviate traffic congestion or not is an issue cared about by transportation planners. In other words, when getting start to network design problem, the first thing is to define user behaviors, network performance function and budget limit.

However, in 1968, Braess[5] presented an example of equilibrium assignment problem: adding an extra link associated with an OD pair in a network does not benefit travelers; that is, the total travel time may increase. This phenomenon has become known as Braess' paradox and is discussed below. (Nagurney, [6]) Figure 1.1 shows a simple network including one OD pair connecting by four links. The link performance functions for the four links are also presented below.

$$t_1(x_1) = 50 + x_1, \quad t_2(x_2) = 50 + x_2,$$
$$t_3(x_3) = 10x_3, \quad t_4(x_4) = 10x_4. \tag{1.4}$$

Assume there are 6 units of flow traveling between O and D. The user equilibrium flow pattern for the network can be solved by inspection. The link flow pattern would be $x_1 = x_2 = x_3 = x_4 = 3$ flow units. The associated link travel times are $t_1 = 53$, $t_2 = 53$, $t_3 = 30$, $t_4 = 30$ time units and the path times are $c_1 = c_2 = 83$ time units. The total travel time on the network is 498 (flow-time) units.

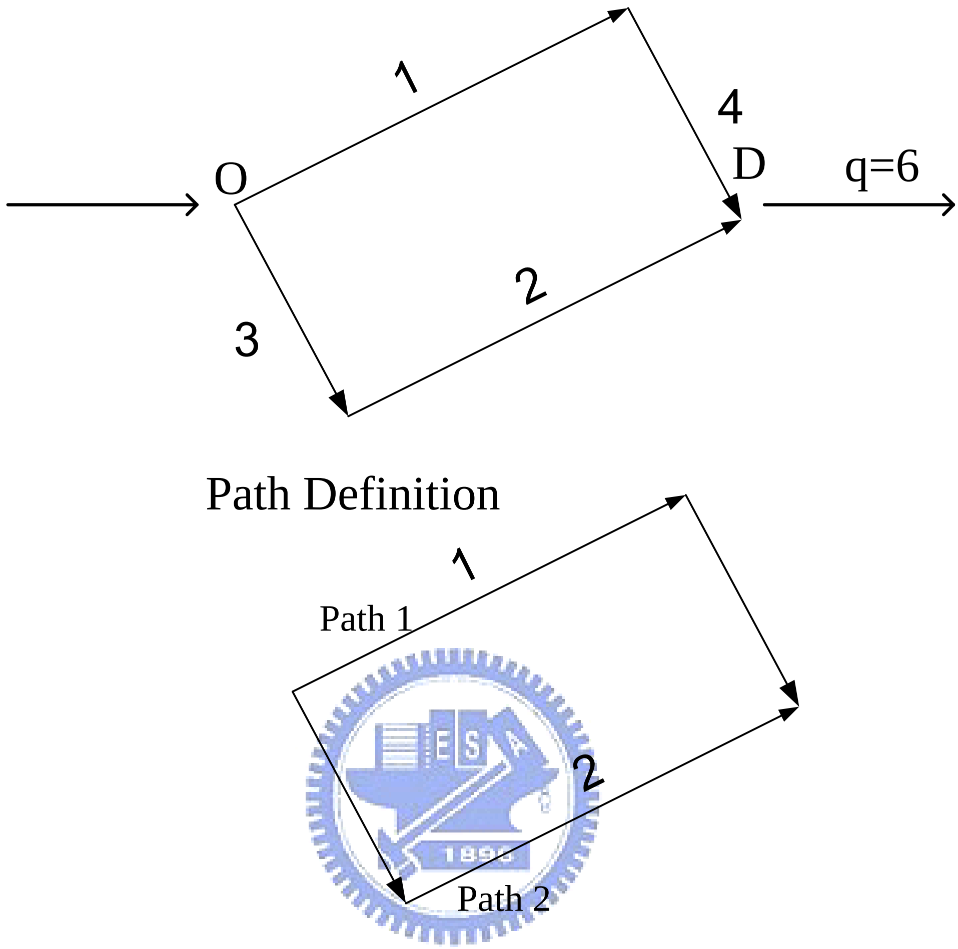
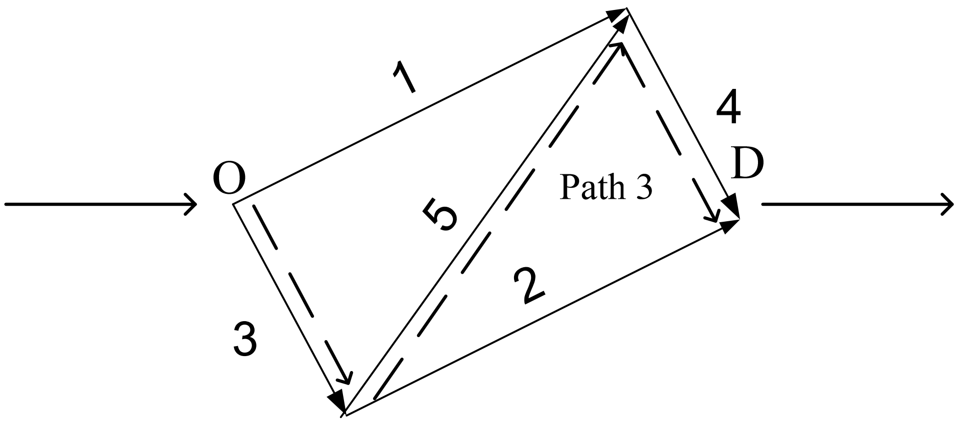


Figure 1.1 Initial Braess' Network

Then add a new link connecting the two intermediate modes to the network. Figure 1.2 shows the expanded network, the performance function for this new link and the new path (number 3) resulting from the addition of the link.



Added link: $t_5 = 10 + x_5$

Figure 1.2 the Expanded Braess' Network

The travel time on the unused path (path 3) is lower than the ones on the two used path so the travel demand need to be allocated again. The equilibrium flow pattern for the expanded network is give by the solution $x_1 = 2$, $x_2 = 2$, $x_3 = 4$, $x_4 = 4$, $x_5 = 2$ flow units and path travel times $c_1 = c_2 = 92$ time units. So the total travel time on the new network is 552 (flow-time) units now. The addition of the new link has therefore made the situation worse. In practice, there is some evidence that this situation may occur: a case in Stuttgart. Major road investment in the city centre, in the vicinity of the Schlossplatz, failed to yield the benefits that had been expected. The benefits were only obtained when a cross street, the lower part of Königstrasse, was withdrawn from use by traffic. (Murchland, [7])

Braess' paradox has given warning to many researchers: when involving this NDP, not every investment will benefit network users. This inspired many researchers dedicated to identifying corresponding causes of Braess' paradox. Some devoted to developing mathematical models to predict the occurrence of Braess' paradox, and some tried to find some rules for engineers when planning road construction. The purpose of this study is to relax the model for predicting Braess' paradox by using generalized inverse approach. The model was proposed by Dafermos and Nagurney [8] in 1984. Their model was constrained to the *rank assumption*. The rank assumption is that number of arcs plus number of OD pairs are larger than number of paths in the network. This is not suitable in real world. Generalized Inverse approach is used to modify limitation in the model such that it can conform to real situation. Without loss of generality, sensitivity analysis to equilibrium network analysis will be described to emphasize the close relationship between Braess' paradox and sensitivity analysis before road investment

The rest of this research is organized as follows. Chapter 2 briefly revisits literatures; chapter 3 introduces corresponding notations which will be used in the mathematical model, and then presents the model for predicting occurrence of Braess' Paradox developed by Dafermos and Nagurney. Next, Chapter 4 shows generalized inverse matrix approach and the modified model, and Chapter 5 is discussions and conclusions along with the future research.

2. Literature Reviews

Literature reviews are divided into five parts. Section 2.1 introduces network design problem according to its class and formulations. From section 2.2 to 2.7, we focus on Braess' paradox basing on its characteristics, methods, purposes: section 2.2 shows Braess' paradox and its extensions under different traffic assignment techniques; section 2.3 mainly illustrates mathematical model for predicting occurrence of network presents; section 2.4 lays attention on routing game about Braess' paradox; section 2.5 introduces sensitivity analysis in Braess' paradox; section 2.6 introduces Braess' paradox in non-transportation networks, such as queuing networks, telecommunication networks; and section 2.7 discusses route guidance system in order to prevent from Braess' paradox and summarizing all sections.

2.1 Network Design Problem

The network design problem has long been recognized to be one of the most difficult and challenging problems in transportation. Magnanti and Wong [9], Friesz [10], Yang and Bell [11], and Friesz and Shan [12] have reviewed models and algorithms for road network design problem. In this section, a brief introduction for network design problem will be presented. This is to emphasize the importance of sensitivity analysis before road investment in order to prevent occurrence of Braess' paradox. Traditionally, network design problems seek an optimal network design in terms of additional facilities or capacity enhancements when the network flow pattern is constrained to be a static equilibrium. For example, Braess' paradox requires static design models satisfy user equilibrium constraints. In detail, there are two kinds of NDP: discrete and continuous. A discrete form deals with the adding new links or roadway segments to an existing road network which is called as the discrete network design problem (DNDP), and a continuous form deals with the optimal capacity expansion of existing links which is called as continuous network design problem (CNDP). No matter which form is, the objective of NDP is to optimize a given system performance measure such as to minimize total system total travel cost, while accounting for the user behaviors. NDP can be represented as a leader-follower game where the transportation planning departments as leaders, and the users who can choose the path freely as followers. It is assumed transportation planning managers can influence but not control user behavior. The interaction between the two players can be represented in the following bi-level programming problem (Friesz [10]; Yang and Bell [11]):

(Upper level) $\underset{\mathbf{u}}{Minimize}\ F(\mathbf{x},\mathbf{u})$

$$st.\ \mathbf{G}(\mathbf{x},\mathbf{u}) \le 0$$

where $\mathbf{x} = \mathbf{x}(\mathbf{u})$ is implicitly defined by

(Lower level) $\underset{\mathbf{x}}{Minimize}\ f(\mathbf{x},\mathbf{u})$

$$st.\ \mathbf{g}(\mathbf{x},\mathbf{u}) \le 0 .$$

The whole bi-level NDP is to find an optimal capacity improvement $\mathbf{u}^*$ such that the system objective function F is optimized subject to a given budget constraint while taking account of the user behavior.

It is worth to emphasizing that the NDP must be solved with the network flow pattern constrained to user equilibrium (the lower level problem). Moreover, addition of a new road segment or capacity enhancement to a congested network without considering the response of the network users may increase system-wide congestion. This well-known phenomenon has been demonstrated by the Braess' paradox. Therefore, it is essential to predict traffic pattern via a sound behavior model for the network design process. The user equilibrium problem with fixed demand can be formulated as (1.1)-(1.3). In addition to deterministic user equilibrium, Chen and Alfa [13] and Davis [14] used the logit-based stochastic user equilibrium assignment approach formulating the lower level problem. The advantages for using stochastic user equilibrium assignment are that the path flows are uniquely determined and their derivatives with respect to the design variables can calculated. The upper level problem can be posed in a discrete or continuous form. Earlier studies have used discrete design variables (Leblanc, [15]; Boyce and Janson, [16]), but CNDP receives much more attention form transportation researchers. Abdulaal and Leblanc [17], and Dantzig et al. [18] assumed decision variables were continuous, which simplified the problems because it removed the combinational aspects and made the problem amenable to a number of nonlinear programming algorithms. Magnanti and Wong [9] presented a unified view of modeling the DNDP, and proposed a unifying framework for describing a number of algorithms such as Lagrange relaxation and dual ascent procedures in providing bounds for the special cases of the DNDP. Other techniques include branch and bound methods and other heuristics. Leblanc [15] applies the branch and bound approach for solving the DNDP with construction cost being a budget constraint.

The following is a general form of DNDP (upper level problem) is written as follows (Magnanti and Wong [9]):

$$Minimize \sum_{k} \sum_{(i,j)\in A} c_{ij}^{k} f_{ij}^{k} + \sum_{(i,j)\in A} F_{ij} y_{ij} \tag{2.1-1}$$

$st.$

$$\sum_{j\in N} f_{ij}^{k} - \sum_{l\in N} f_{li}^{k} = \begin{cases} R_k & if\ i = O(k) \\ -R_k & if\ i = D(k)\ all\ k \in \kappa \\ 0 & otherwise \end{cases} \tag{2.1-2}$$

$$f_{ij} \equiv \sum_{k\in\kappa} f_{ij}^{k} \le K_{ij} y_{ij} \quad all\ (i,j) \in A \tag{2.1-3}$$

$$(\mathbf{f}, \mathbf{y}) \in S \tag{2.1-4}$$

$$f_{ij}^{k} \ge 0,\ y_{ij} = 0\ or\ 1\ \ all\ (i,j) \in A,\ k \in \kappa. \tag{2.1-5}$$

The discrete network design problem is to determine the best improvement to an existing transportation system. Thus, choose an optimal subset from a set of proposed link additions to an existing road network. The objective is to find that network configuration whose user equilibrium flow results in the smallest travel cost. Each proposed link has a cost of construction, ie., a budget is given which limits total expenses incurred. The basic ingredients of the model includes a set of nodes N and a set of links A that are available for designing a network. The model permits multiple commodities: let κ denotes the set of commodities and for each $k \in \kappa$, let R_k denotes demand of commodity k to be shipped from its origin, denoted $O(k)$, to its destination, denoted $D(k)$. Let y_{ij} be a binary variable indicating whether or not link(i,j) is chosen as part of the network's design. Let f_{ij}^{k} denotes the flow of commodity k on link (i,j). Then, $\mathbf{y} \equiv (y_{ij})$ and $\mathbf{f} \equiv (f_{ij}^{k})$ are vectors of design and flow variables. Let c_{ij}^{k} be the per unit routing cost on link(i,j) for commodity k, and F_{ij} denotes the fixed cost of constructing link(i,j). Most frequently, the demand and flows would be assumed unchanged over the lifetime of the network's design and the flow costs is net present values of the per unit routing costs evaluated over the network's lifetime. Constraint (2.1-1) represents the usual network flow conservation equations. Constraint (2.1-2) , the forcing constraint, state that the total flow f_{ij} on link (*i*,*j*) of all commodities cannot exceed the capacity K_{ij} of the link if it is chosen as part of the network design. The set S includes any side constraints imposed either individually or jointly on the flow and design variables. The side constraints might model limitations imposed on resources shared by several links, such as a budget constraint:

$$\sum_{(i,j)\in A} e_{ij} y_{ij} \le B . \qquad (2.1\text{-}6)$$

The coefficient e_{ij} is the cost incurred if link(*i*,*j*) is constructed in the network design. With the budget side constraint (2.1-6) and no fixed costs in the objective function and uncapacitated link, the problem is often called the *budget design problem*. Without side constraints, this unconstrained, linear cost version of the problem is often referred as the *fixed charge design problem*. Several studies have developed solution methods for the fixed charge design problem (Balakeishnan et al.,[19]; Lamar et al.,[20]; Gendron et al., [21]; Holmberg and Hellstrand, [22]; Holmberg and Yuan,[23])

CNDP is to determine the set of link capacity expansions where satisfying user equilibrium. Abdulaal and Leblanc [17] formulated the CNDP under deterministic user equilibrium as a bilevel programming model and proposed the Hooke-Jeeves algorithm to solve CNDP. Marcotte [24] presented heuristics for CNDP on the basis of system optimal approach and obtained good numerical results. However, this is not tested on large-scale networks generally. As for the development of solution methods to CNDP for practical use, Suwansirikul et al. [25] porposed a simple heuristic called Equilibrium Decomposed Optimization (EDO) and performed this heuristic on several example networks. Friesz et al. [26] used a simulated annealing approach to solve the multi-objective equilibrium network design problem as a single level minimization problem; this approach is only suitable for small networks. Since the bilevel program for CNDP is non-convex and non-differentiable, Yang and Bell [27] conducted a survey of recent advances in transportation bilevel programming problems. Recently, Chiou [28] exploited a descent approach via the implementation of gradient-based methods to solve CNDP; Waller et al. [29] formulated CNDP as a linear model based on dynamic traffic assignment model that propagates traffic according to the cell transmission model. A major limitation of the static models is that they can not capture the traffic interaction among adjacent links and they assume steady-state time-invariant OD demean, which is unrealistic during the peak period and leads to suboptimal solutions.

2.2 Braess' Paradox and Others

Since Braess presented the paradox, other researches have been intrigued, appearing frequently in textbooks and the popular science literatures. Smith [31] and Fisk [31] both presented phenomena similar to Braess' paradox in transportation networks. Smith used a simple model where the network and the congestion characteristic are particular and showed the total travel time may be reduced by increasing travel time locally. This result is of particular relevance to towns with a good bypass or an outer ring road. Fisk 's result is like Braess' paradox but its variants are on travel demand: this study investigated the sensitivity of travel costs to change input flows in the user equilibrium problem. The result showed that when increasing input flows, both origin-destination and global travel costs may decrease, contradicting general intuition. The same phenomenon occurred in the two-mode equilibrium problem. The OD travel cost may decrease as a result of an increase in automobile input flows. Fisk and Pallottino [32] also illustrated the phenomenon using the City of Winnipeg network data, proving in real life it may occur. For a corridor with two groups of users, Arnott et al. [33] showed that expanding capacity of an upstream bottleneck raises travel costs when reducing congestion upstream is more than offset by increased congestion downstream. Yang and Bell [34] demonstrated a capacity paradox which may be encountered in road network design. They showed that creating a new link in a road network may actually reduce the potential capacity of the network and this can be avoided by using the concept of network reserve capacity into network capacity improvement plans. The reserve capacity for a road network can be measured by how large a common multiplier can be applied to a given OD matrix subject to the flow on each link not exceeding its capacity when the multiplied OD matrix is allocated to the network which satisfies user equilibrium. Comparing with the capacity paradox, the occurrence of Braess' paradox depends on the level of *demand*, which has been mentioned by Rilett and Van Aerde [35], Pas and Principio [36], and Penchina [37]. In fact, the above paradoxes are a direct consequence of the difference between the user equilibrium and system optimal assign solutions. When doing transportation planning, the objective is system optimal. System optimal means the average journey cost over all paths used is the minimum possible. However, inside rules are user equilibrium. Stewart [38] also showed this phenomenon: the user equilibrium flow does not necessarily minimize total cost. If the investment costs for an existing network can not be recovered, use of part of the network should be restricted or completely suppressed. Therefore if the user equilibrium is a good approximation of reality, there may be some planning strategies existing for a given network which yield improvements both for traffic and the environment.

Paradoxes may not only occur in user equilibrium; Catoni and Pallottino [39] illustrated seeming paradoxes which may occur in different equilibrium models: in addition to the user equilibrium, system optimal 、mixed behavior equilibrium with one non-cooperative player disposing of the whole demand and Cournot-Nash equilibrium with two players are also included. Florian [40] based on multipath stochastic assignment methods, pointing out that a phenomenon similar to Braess' paradox may occur with stochastic choice models that are not based on the logit function. Steinberg and Stone [41] presented a paradox in a congested network: if the congestion effect along a path is increased sufficiently, this can result in abandonment of a different path having the same origin and destination while the original path continued to be used. Akamatsu [42] used dynamic equilibrium assignment with a point queue model. The study analyzed dynamic flow patterns on two symmetrical networks: an evening-rush-hour network with one-to-many origin-destinations and a morning-rush-hour network with many-to-one origin-destinations. Finally, a dynamic version of Braess' paradox is also identified. Nagurney et al. [43] developed an evolutionary variation inequality model with multiple classes of traffic and demonstrate its utility through the formulation and solution of a time-dependent Braess' paradox. We summarize up above contents for reference (See Table 2.2-1).

Table 2.2-1 Summary of Paradoxes in User Equilibrium models

User equilibrium assignment	Network type	Demand pattern	Travel cost function	Results
Smith [30]	Specific	Fixed	Constant	TC (local) ↑ , TTC ↓ .
Fisk [31]	Specific	Fixed	Linear	Q ↑ , TTC ↓ .
Stewart [38]	Specific	Fixed	Linear	Restrict on part of the network, TTC ↓ .
Fisk and Pallottino [32]	The City of Winnipeg	Fixed	BPR form	Q ↑ , TTC ↓ .
Steinberg and Stone [41]	Specific	Fixed	Linear	TC of path ↑ , abandon other path of the same OD.
Arnott et al.[33]	Specific	Fixed	Linear	Expanding capacity of upstream bottleneck, TTC ↑ .
Yang and Bell [34]	Specific	Fixed	Linear	Creating a new link, capacity of network ↓ . (Capacity paradox)

Note. TC: total cost; TTC: Total travel cost; Q: input flows.

2.3 Avoidance of Braess' Paradox

This section is aimed at conditions of Braess' paradox. Frank[44] analyzed the simple network as Figure 1.1, and showed its mathematical characterization. Necessary and sufficient conditions in terms of the link performance functions are obtained for the existence of Braess' paradox. Steinberg and Zangwill [45] considered the network in which the travel cost on every link depends solely on the traffic load in that link and provided a formula expressing how the users' cost associated with a particular OD pair changes with this OD pair is joined by a path. Dafermos and Nagurney [8] is similar to Steinberg and Zangwill [45] but fewer tedious calculations. They also derived formulas under certain conditions and used a specific matrix form to determine occurrence of Braess' paradox when the matrix is positive semidefinite. Hagstrom and Abrams (2001) and Abrams and Hagstrom (2006) presented a natural generalization of Braess' paradox to include multicommodity traffic flows with multiple origins and destinations. They characterized the occurrence of Braess' paradox in terms of the solution of a mathematical program. Braess' paradox occurs if and only if the equilibrium solution is not optimal for the mathematical form. However, when applied to the mathematical form the total number of linear programs that must be solved is no more than the number of nodes in the network, which limits the feasibility. Huang et al. [48] analyzed the equilibrium of a model incorporating a self-interest service provider and studied the performance gap between the user equilibrium and the system optimal in a network with a general topology. They provided a characterization of the user equilibrium of flow rates and routing decisions under the Wardrop assumption that each user is small and a full characterization of the "monopoly equilibrium", i.e., profit-maximizing prices from the viewpoint of service-provider and the resulting allocations. At the monopoly prices, there can never be Braess' paradox, so for-profit incentives appear sufficient to eliminate Braess' paradox.

Milchtaich [49] listed network topologies which may not lead to Braess' paradox. It showed that it is essentially the only kind of network in which Braess' paradox can occur: a necessary and sufficient condition for the existence of some cost function for which the paradox occurs is that the network has embedded Wheatsone network. In networks without this property, so-called series-parallel networks, Braess' paradox cannot occur. Morgan et al. [50] presented theory and experiments to investigate how network architecture affects route-choice behavior. They examined two paradoxes: Pigou-Knight-Down paradox and Braess' paradox and identified two principles: the least congestible principle and the size principle. The former states that improvement should be made on the path least sensitive to congestion and the size principle states that adding costless links reduces travel time when there are a sufficiently large number of travelers on the network.

2.4 Braess' Paradox in Routing Games

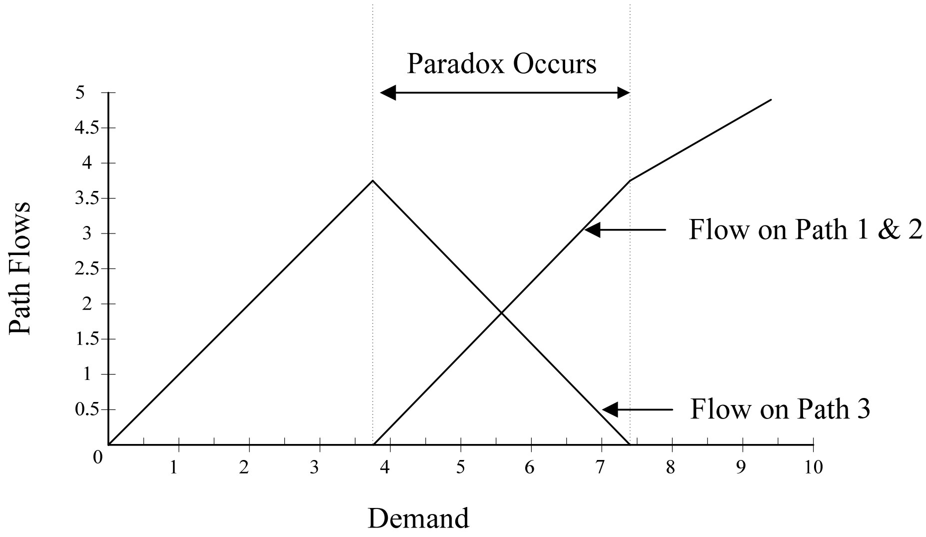
Many studies consider the problem of routing traffic to optimize the performance of a congested network. It may be expensive or impossible to regulate network traffic so as to implement am optimal assignment of routes. Generally assume each network user routes its traffic on the minimum-cost path available to it, such a selfish motivated assignment of traffic to path will not minimize the total system cost (the same concept as user equilibrium principle and *non-cooperative game*). Korilis et al. ([51]-[53]) and Altman et al. [54] studied strategies for adding new links and/or capacity to a network that guarantee to improve network performance. The result showed that capacity across the network rather than on a local scale (for example, single link) and upgrading network should be aimed at direct connections between the origin and destination. Altman et al. also studied routing in the framework of a non-cooperative game with selfish users in loss networks. They provided the mathematical models for both user equilibrium and Nash equilibrium in loss networks and showed non-uniqueness of two situations even under the simplest topology of parallel links.

Kameda [55, 56] , Lin et al. [57] and Roughgarden et al. [58, 59] quantified the degradation in network performance due to selfish routing. They proved that if the link performance function is a liner function of its congestion, then the total travel time of the paths chosen by selfish users is at most 4/3 times the minimum possible OD travel cost. While assume each link performance function is only to be continuous and non-decreasing in the link congestion, the total travel time of the paths chosen by unregulated selfish network users may be arbitrarily larger than the minimum possible OD travel cost but it is no more than the total travel time incurred by optimal routing twice as much traffic. Kameda [55] also compared with Cohen-Kelly-Jeffries networks; Cohen-Kelly-Jeffries networks are ones that contain multiple Braess networks, that is, networks that have multiple OD pairs. Valiant and Roughgarden [60] also showed the probability of occurrence of Braess' paradox in a natural network model under the situation of selfish routing. With high probability as the number of vertices goes to infinity, there is a choice of traffic rate such that the removal of one or more edges can improve the travel time in an equilibrium flow.

2.5 Braess' Paradox and Network Sensitivity

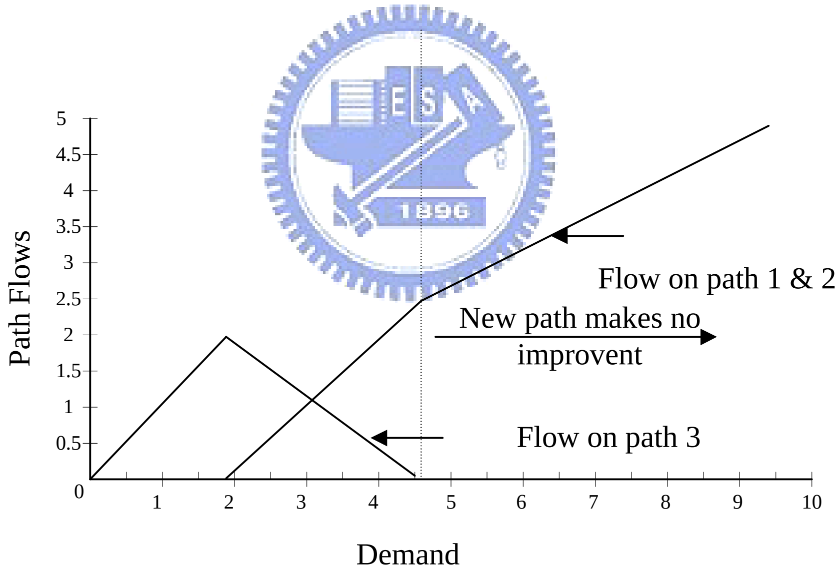
Dafermos and Nagurney [61]、Hallefjoed et al. [62], Pas and Principio [36], Yang [63] and Cho and Lo [64] all investigated that how changes in demand or link cost would affect trajectory for occurrence of Braess' paradox. Dafermos and Nagurney [61] expressed the equilibrium condition (see equation (3.1-2) in section 3.1) as a variational inequality and analyzed how changes in the input data affect traffic equilibrium pattern and the incurred travel cost. They showed even though increases in the demand may result in decrease in travel cost for *some* users of the network, an average travel cost will necessarily increase. On the other hand, they assume the addition of new paths means the new path was in the network all the time just because its cost was so high that no travelers use it. Therefore, they could discuss how changes in travel cost function affect equilibrium load pattern. If the travel cost function satisfies the monotonicity condition (see equation (3.1-1) in section 3.1) and that only one path is improved while others remain unchanged, the travel cost along the path will necessarily decrease while the flow on the path will increase, thus in this situation Braess' paradox cannot occur. Hallefjord et al. [62] tried to clarify what a paradox really is in the case of elastic demand. They chose to view the problem as one of supply and demand of travel and discussed the interpretation of an elastic demand paradox in the case of single OD pair. A (weaker) type of paradox is that the improvement leads to a decrease in social surplus.

Pas and Principio [36] and Cho and Lo [64] exploited travel cost parameters which were based on the initial Braess' network, derived how changes in travel cost or demand resulted in occurrence or disappearance of Braess' paradox. Pas and Principio analyzed the original Braess' network and determined that Braess' paradox occurs only in the total demand range from 2.58 to 8.89 units on the network. It is interesting to note that there is no flow on the new link once the demand reaches the upper limit of the range where Braess' paradox occurs. They mentioned that if the road price is charged as *marginal* cost, Braess' paradox will disappear. Figure 2.5-1 and figure 2.5-2 shows under different cost pricing, Braess' paradox only occur in some demand range basing on parameters of travel cost functions.



From: [36]

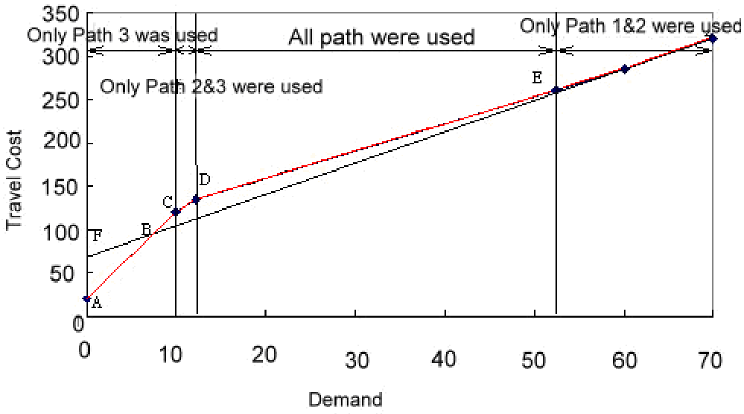
Figure 2.5-1 Average Cost Pricing



From: [36]

Figure 2.5-2 Marginal Cost Pricing

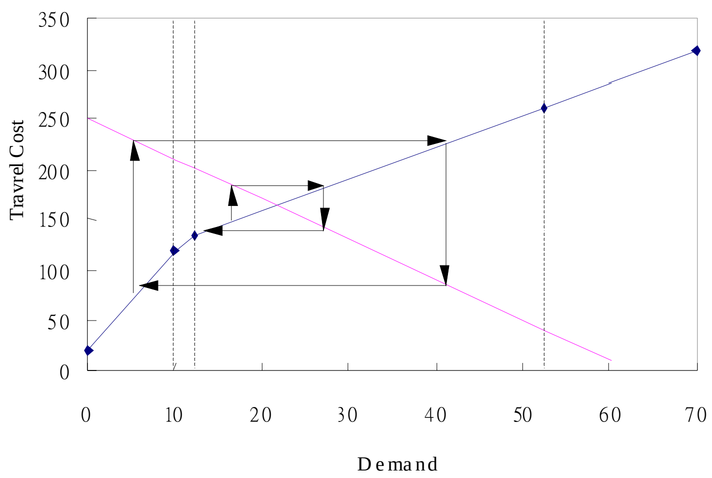
Cho and Lo investigated under different situation: fixed and elastic demand, how changes in demand affect the trajectory for occurrence of Braess' paradox. Under assumption that positive path flows before and after the path addition, figure 2.5-3 shows occurrence of Braess' paradox when demand is fixed. Folded line ABCDE is travel cost function of the expanded Braess' network, and straight line FE is that of initial Braess' network. When demand is more than point E, the new path would be abolished.



From: [64]

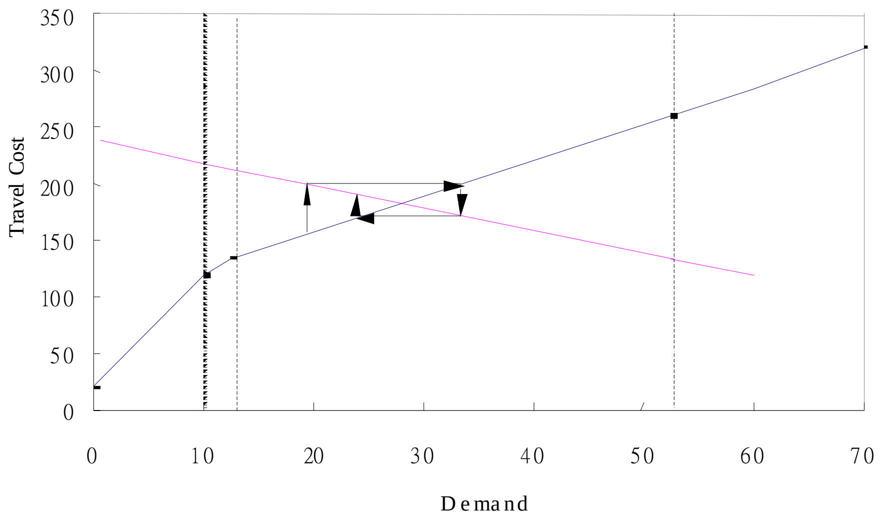
Figure 2.5-3 Travel Cost Changing with the Fixed Demand

When in situation of elastic demand, assume demand in this period depends on the cost of the preceding period and travel cost depends on demand in this period; this is similar to cobweb theory. Figure 2.5-4 and figure 2.5-5 illustrates divergence and convergence of travel cost changing with demand.



From: [64]

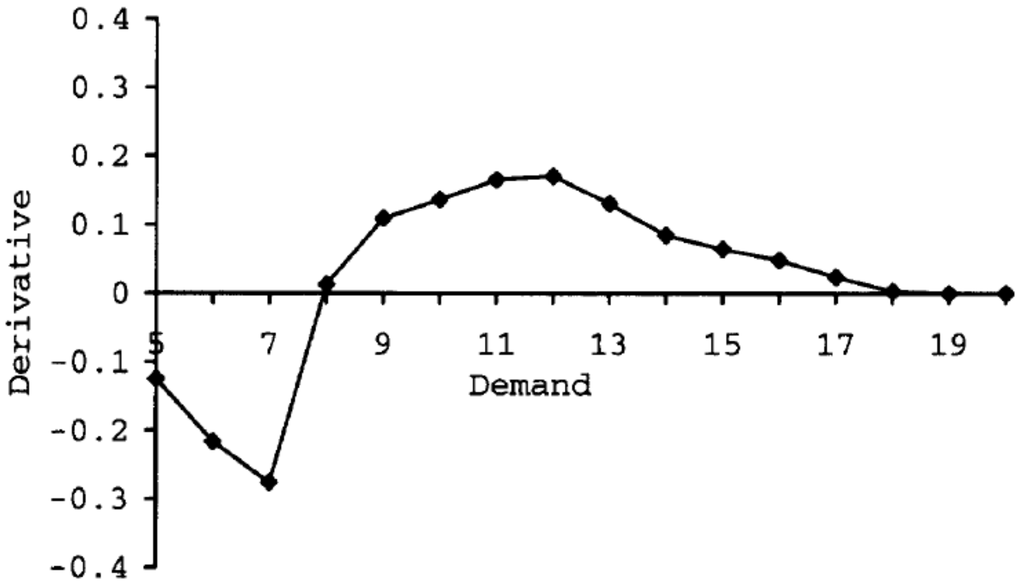
Figure 2.5-4 Travel Cost Changing with the Elastic Demand: Divergence



From: [64]

Figure 2.5-5 Travel Cost Changing with the Elastic Demand: Convergence

Yang [63] presented a general framework for the quantitative analysis of the behavior of equilibrium flows with elastic demand according to the sensitivity analysis method developed by Tobin and Friesz [65]. Usually, the sensitivity analysis methods are designed to calculate the derivatives of decision variables and constraint multipliers with respect to a variety of perturbation parameters. Using the restriction approach proposed by Tobin and Friesz, he examined the effects of changes in link cost given that the link already exists in the networks, the same assumption as Defermos and Nagurney [8]. Figure 2.5-5 presents the derivatives of total cost incurred by all users with respect to the cost of the new link at various levels of travel demand. When demand > 8.0 units, the derivative will become positive; this means Braess' paradox may occur if adding a new link (or the link is improved). Therefore, the selection of links for improvement must be done carefully.



From: [63]

Figure 2.5-6 Derivatives of total cost with respect to capacity of new link at various levels of demand.

2.6 Braess' Paradox in Other networks

Cohen and Kelly [66] gave an example of a queuing network in which added capacity leads to an increase in the mean transit time for everyone. Calvert et al. [67] continued the work of Cohen and Kelly but under a particular state-dependent routing scheme. Bean et al. ([68, 69]) discussed the question of whether Braess' paradox can occur in loss networks (for example, a circuit-switched telephone network) as in queuing networks. Loss networks are used to model many multi-resource access problems where requests for access that cannot be fully met are denied and lost. They consider two important performance measures: acceptance probabilities and surplus values and presented two simple explicitly analyzed examples of the occurrence of Braess' paradox. One is a network operating under fixed routing, and the other concerns a network in which alternative routing is allowed. In Kelly [70], the paper described some examples from various fields including queuing networks and transportation networks. He indicated how analogies with fundamental concepts such as energy and price can provide insights into the design of routing schemes for communication networks.

Cohen and Horowitz [71] proposed a mechanical network analogue of Braess' paradox. They showed for certain combinations of strength of springs, length of string and mass of weight, the weight will rise instead of dropping as could be expected. This behavior is also analogs in electrical, hydraulic and thermal networks. (Penchina et al,[72]) Kameda et al. [73] presented a case where a paradox similar to that of Braess' paradox in a Nash equilibrium (for a large number of users) but does not appear in a user equilibrium (infinitely many users) in the same environment in distributed computer systems. Aashtiani and Poorzahedy [74] showed Braess' paradox in the management of networks where the decision variables may be continuous in nature, such as the distribution or allocation of time or space. For example, the allocation of time to traffic approaching an intersection (time allocation) and the number of lanes to the directions of movement in a street (space allocation). This paper showed that in traffic signals, from a fixed-time to a traffic-sensitive device, may increase the travel cost for users of the network.

2.7 Route Guidance System and Braess' Paradox

Rilett and Van Aerde [35] illustrated how Braess' paradox may arise in Route Guidance System situations, when the addition of a low capacity link to the in-vehicle network data base can lead travelers to take apparent short-cuts, which in reality lead to a net increase in the level of traffic congestion. For certain traffic demand conditions, user-optimizing vehicles will fall in the Braess trap while an enhanced system-optimizing vehicle will avoid the trap. Turner and Wolpert [75] investigated the use of a multi-agent system to control routing of packages in a computer network using the so-called Collective Intelligence (COIN) formalism. They concluded that because agents may try to reduce their individual routing times in a greedy way, then it resulted in increasing global time. Bazzan and Klügl [75] used a learning mechanism to allow drivers to adapt to the changes in the network. They discussed the effects of giving route recommendation to drivers in order to divert them to a situation in which the effects of Braess' paradox are reduced. Bazzan and Klügl's work is different from Turner and Wolpert's ;they use the classical scenario proposed by Braess. Furthermore, the COIN formulism assumes that agents can be aligned with the global objective, and this is only possible in computer network in which router nodes have an aggregate knowledge that drivers in the traffic network do not have.

In summary, some previous studies have shown ways of avoiding paradox. Other works have attempted to develop analysis models based on specific assumptions. Moreover, some studies have tried to find rules for designing network to avoid Braess's paradox. Most of the above studies are based on the illustrated network presented by Braess, namely, a small network with four nodes and five links. Moreover, some studies investigated the degree of harm created by Baress's paradox. These works indicated us that the performance improvement from link removal can be arbitrarily large in large networks. They provided us some ideas related to predicting the occurrence of Braess's paradox in real networks, i.e. large-scale network.

This study investigates the specific assumption demonstrated in the model of Dafermos and Nagureny [8], which was often presented in many corresponding researches: the number of paths of the network should be less than the number of links plus the number of OD pairs (called *the rank assumption*). This avoids general characteristic of large transportation networks. Thus, the above assumption resulted in less application in the theorem of forecasting the occurrence of Braess' paradox presented by Dafermos and Nagureny [8]. Because in real world, paths usually exceed the number of links plus the number of OD pairs. In this paper, the specific assumption illustrated by Dafermos and

Nagureny is relaxed by using generalized inverse matrix method and an illustrative example is presented to demonstrate its feasibility. Notations and definitions are listed in next chapter and Dafermos and Nagurney's model will also be briefly introduced.

3. Dafermos and Nagurney's Model

This chapter will introduce notations and Dafermos and Nagurney's work on predicting occurrence of Braess' paradox, including their assumptions and formula.

3.1 Notations

The real numbers, nonnegative real numbers, and positive real numbers are denoted respectively by R, R_+, R_{++}.

$N \triangleq$ set of nodes of the network.

$i, j \in N \triangleq$ specific nodes in the network.

$A \triangleq$ set of links of the network.

$a \in A \triangleq$ a link in the network; $a = (i, j)$.

$W \triangleq$ set of OD pairs.

$w \in W \triangleq$ an OD pair; $w = (i, j)$.

$P_w \triangleq$ set of paths between OD pair w.

$p \in P_w \triangleq$ path between OD w.

$\mathbf{\Delta} = \lfloor \Delta_{ap} \rfloor \triangleq$ link/path incidence matrix, where Δ_{ap} = 1 if link a is in path p, 0 otherwise.

$\mathbf{\Lambda} = \lfloor \Lambda_{wp} \rfloor \triangleq$ OD/path incidence matrix, where Λ_{wp} = 1 if path $p \in P_w$, 0 otherwise.

$T_w \triangleq$ number of trips between OD pair w.

$\mathbf{T} = [T_w] \in R_{++}^{\omega} \triangleq$ vectors of all trips.

$h_p \triangleq$ flow on path *p*.

h = $[h_p] \in R_+^{\rho} \triangleq$ vector of all path flows.

$f_a \triangleq$ flow on link *a*.

f = $[f_a] \in R_+^{\alpha} \triangleq$ vector of all link flows; note that $\mathbf{f} = \Delta\mathbf{h}$.

$c_a(f) \triangleq$ cost on link *a* as a function of all link flows; the cost functions should satisfy the strong monotonicity condition:

$$(c(f) - c(\bar{f})^T (f - \bar{f})) \geq \alpha \left\| f - \bar{f} \right\|^2 \forall f, \bar{f} \in \text{s}. \qquad (3.1\text{-}1)$$

In common, we assume $c_a(f)$ is affine: $c_a(f) = \sum_{b \in A} g_{ab} f_b + h_a$.[1]

$c(\mathbf{f}) = [c_a(f)] \triangleq$ vector of link cost function.

$\mathbf{G} = [g_{ab}] \triangleq$ the link cost Jacobian matrix; note it is positive definite.

$c_p(h) \triangleq$ cost on path *p* as a function of all path flows.

$c(\mathbf{h}) = [c_p(h)] \triangleq$ vector of path cost function.

$U_w \triangleq$ cost associated with OD pair *w* and a change in U_w denotes U'_w. According to Wardrop's principles, user equilibrium is described as

$$\begin{array}{l} \text{If } h_p > 0, \quad c_p(h) = U_w; \\ \text{if } c_p(h) \geq U_w, \ h_p = 0. \end{array} \qquad (3.1\text{-}2)$$

[1] This does not represent link cost function must be linear. Note that the only assumption for link cost function is strong monotonicity condition.

After path addition, some notations will be changed.

$\hat{\boldsymbol{\Delta}} = \left[\frac{\Delta|\Delta_n}{0|1}\right] \triangleq$ the new link/path incidence matrix after adding a new path connecting an OD pair.

$\hat{\boldsymbol{\Lambda}} = [\Lambda|e] \triangleq$ the new OD pair/path incidence matrix after adding a new path connecting an OD pair.

$\mathbf{h}'' = [\mathbf{h}'|\mathbf{h}_n]^T \triangleq$ the new change in the equilibrium vector of path flows; $\mathbf{h}'$ means change of the original path flow and $\mathbf{h}_n$ is the new path flow.

$\mathbf{f}'' = [\mathbf{f}'|\mathbf{f}_n]^T \triangleq$ the new change in the equilibrium vector of link flows; $\mathbf{f}'$ means change of the original link flow and $\mathbf{f}_n$ is the new link flow.

$c(\mathbf{f})'' = [c(\mathbf{f})'|c(\mathbf{f})_n]^T \triangleq$ the new change in the equilibrium link costs.

$\hat{\mathbf{G}} = \left[\frac{\mathbf{G}|\mathbf{0}}{\hat{\mathbf{g}}}\right] \triangleq$ the new link cost Jacobian matrix after adding a new path connecting an OD pair.

$\hat{c}(\mathbf{h})'' = [\hat{c}(\mathbf{h})'|\hat{c}(\mathbf{h})_n]^T \triangleq$ the new change in the equilibrium path costs; $\hat{c}(\mathbf{h})_n$ is the equilibrium cost of the new path.

3.2 Dafermos and Nagurney’ Model

Consider a network *N*, given the travel demand, and the case of affine cost function. Assume that the flow on each path of the network *N*, including the new path, is positive before and after the path addition. Now add a new path *r* which connects the OD pair w_1, and according to definitions in Section 3.1, we have the system as follows:

$$\hat{\mathbf{\Delta}}\mathbf{h}'' = \mathbf{f}'', \mathbf{\Lambda h}' = -\mathbf{h}_n \mathbf{e}$$
$$\hat{\mathbf{G}}\mathbf{f}'' = c(\mathbf{f})'', \hat{\mathbf{\Delta}}^T c(\mathbf{f})'' = \hat{c}(\mathbf{h})''. \tag{3.2-1}$$

Since the flow in each path is positive before and after adding a new route, this leads to the following:

$$c_p(h)' = U'_w \quad \textit{for every } p \in P_w,\ p \neq r . \tag{3.2-2}$$

By combining (3.2-1) and (3.2-2), the model is constructed as follows:

$$\left[\begin{array}{c|c} \mathbf{\Delta}^T\mathbf{G\Delta} & \mathbf{\Lambda}^T \\ \hline \mathbf{\Lambda} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{h}' \\ \hline -\mathbf{U}' \end{array}\right] = \left[\begin{array}{c} -\boldsymbol{\rho} \\ \hline -\mathbf{e} \end{array}\right]\mathbf{h}_n \tag{3.2-3}$$

where

$$\rho = \mathbf{\Delta}^T\mathbf{G\Delta}_n . \tag{3.2-4}$$

Assume $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ *is an* $m \times n$ *matrix of rank n*, then it is implied that

$$\det\left[\begin{array}{c|c} \mathbf{\Delta}^T\mathbf{G\Delta} & \mathbf{\Lambda}^T \\ \hline \mathbf{\Lambda} & 0 \end{array}\right] \neq 0 .$$

Therefore, we can apply Cramer's rule to solve the system as (3.2-3). Let U'_{wi} be change of the *i*-th OD pair, $\mathbf{L} = \left[\mathbf{A_r}|\mathbf{A_r}|\ldots|\mathbf{A_r}\right]$, $\mathbf{\Lambda}_i$ be the $\mathbf{\Lambda}$ marix with the i-th row removed, and $\mathbf{\Lambda}_1$ be the $\mathbf{\Lambda}$ matrix with the first row removed, The change in each OD pair cost can be obtained through

$$U'_{wi} = (-1)^{i+1}\frac{\det\left[\begin{array}{c|c} \Delta^T G\Delta - \Delta^T GL & \Lambda_i^T \\ \hline \Lambda_1 & 0 \end{array}\right]}{\det\left[\begin{array}{c|c} \Delta^T G\Delta & \Lambda^T \\ \hline \Lambda & 0 \end{array}\right]} h_r \tag{3.2-5}$$

then we can determine whether Braess paradox occurs.

Under the assumption (3.1-1), the rank assumption and positive path flows including the new path, it follows that for $[\Delta^T G\Delta - \Delta^T GL]$ positive semidefinite, Braess' paradox may occur. Corresponding corollary states that joining an OD pair of a network by a new path containing none of the original links of the network will result in a decrease in travel cost for users of the OD pair.

4. The Modified Model

This chapter illustrate using generalized inverse matrix approach relaxes the rank assumption proposed by Dafermos and Nagurney [8]. Section 4.1 introduces the main definitions and theorems of the generalized inverse matrix. Details could be referred in Graybill [76]. Section 4.2 illustrates generalized inverse matrix approach applies to traffic equilibrium models. Section 4.3 derives the modifier model basing on Dafermos and Nagurney's. Final is a numerical example.

4.1 Generalized Inverse Matrix Approach

Definition 4.1

Let $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ be an $m \times n$ matrix. If there exists an $n \times m$ matrix $[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}$, which satisfies the following four conditions,

(i) $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}$ is symmetric;

(ii) $[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ is symmetric;

(iii) $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T = [\mathbf{\Delta}|\mathbf{\Lambda}]^T$;

(iv) $[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-} = [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}$;

then it is a generalized inverse of $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$.

.

Theorem 4.1 *For any given matrix* $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$*, there exists a unique generalized inverse matrix.*

Proof:

Assume that $[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}$ and $[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}$ are two generalized inverse of $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ .This means that both $[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}$ and $[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}$ satisfy Definition 4.1. Multiplying $[\mathbf{\Delta}|\mathbf{\Lambda}]^T = [\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ on the right by $[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}$, we have

$$[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-} = [\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}. \qquad (4.1\text{-}1)$$

By Definition 4.1,both the left-hand side and the right-hand side of (4.1-1) are symmetric. Hence

$$[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-} = \left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\right)^{T}. \qquad (4.1\text{-}2)$$

Then

$$\begin{aligned}
&[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-} = [\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-} \\
&= \left[\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\right)\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\right)\right]^T = \left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\right)^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\right)^T \\
&= \left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\right)\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\right) = [\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}. \qquad (4.1\text{-}3)
\end{aligned}$$

Similarly,

$$[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T = [\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T. \qquad (4.1\text{-}4)$$

By using (4.1-1) and (4.1-2) , we have

$$\begin{aligned}
&[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-} = [\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-} = [\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_1^{T-} \\
&= [\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}\,[\mathbf{\Delta}|\mathbf{\Lambda}]^T\,[\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-} = [\mathbf{\Delta}|\mathbf{\Lambda}]_2^{T-}. \qquad (4.1\text{-}5)
\end{aligned}$$

The proof follows.

Theorem 4.2 If $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ is an $m \times n$ matrix of rank m, then $[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-} = \left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T\right)^{-1}$ and $[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-} = \mathbf{I}$.

Proof:

By Definition 4.1(iii), $[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T=[\mathbf{\Delta}|\mathbf{\Lambda}]^T$, and then transpose each side,

$$\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)\right)^T=\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T, \tag{4.1-6}$$

$$\left([\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T=\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T, \tag{4.1-7}$$

$$[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}=\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\left([\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)^T\right)^{-1}. \tag{4.1-8}$$

From (4.1-8), we can verify $[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}=\mathbf{I}$ easily. Hence the proof is complete.

Theorem 4.3 Let $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\mathbf{y}=\mathbf{g}$ be an $m\times n$ *matrix*, $\mathbf{y}$ *be an* $n\times 1$ *vector and* $\mathbf{g}$ *be an* $m\times 1$ *vector. If* the system of equation $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\mathbf{y}=\mathbf{g}$ has a solution, then for each $n\times 1$ vector $\boldsymbol{k}$, the vector

$$\mathbf{y_0}\triangleq[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\mathbf{g}+\left(\mathbf{I}-[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)\mathbf{k}. \tag{4.1-9}$$

Moreover, each solution to the system can be written in the form of (4.1-9) .

Proof:

By Theorem 4.2, if $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$ is an $m\times n$ matrix of rank *m,* then $[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}=\mathbf{I}$, and hence $[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}g=g$. Since we assume there is a solution to the system, first multiply (4.1-9) on the left by $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$, we have

$$[\mathbf{\Delta}|\mathbf{\Lambda}]^T\mathbf{y_0}=[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\mathbf{g}+[\mathbf{\Delta}|\mathbf{\Lambda}]^T\left(\mathbf{I}-[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)\mathbf{k}. \tag{4.1-10}$$

Since $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\left(\mathbf{I}-[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^T\right)=0$ and $[\mathbf{\Delta}|\mathbf{\Lambda}]^T[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\mathbf{g}=\mathbf{g}$, this reduces $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\mathbf{y_0}=\mathbf{g}$, and hence $\mathbf{y_0}$ is a solution.

Next assume that $\mathbf{y_0}$ is any solution to the system. Since $\mathbf{y_0}$ is a solution, we have $[\mathbf{\Delta}|\mathbf{\Lambda}]^T\mathbf{y_0}=\mathbf{g}$, and multiplying on the left by $[\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}$ gives

$$0 = [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\mathbf{g} - [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\mathbf{y_0}. \tag{4.1-11}$$

Then add $\mathbf{y_0}$ to both sides of (4.1-11), this obtains

$$\mathbf{y_0} = [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\mathbf{y_0} + \mathbf{y_0} - [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\mathbf{y_0} = [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}\mathbf{y_0} + (\mathbf{I} - [\mathbf{\Delta}|\mathbf{\Lambda}]^{T-}[\mathbf{\Delta}|\mathbf{\Lambda}]^{T})\mathbf{y_0}, \tag{4.1-12}$$

which is of the form of (4.1-9) with $\mathbf{k} = \mathbf{y_0}$, and the theorem is proved.

Theorem 4.4 If *A* is an $m \times n$ matrix of rank *m*, $\mathbf{y}$ *be an* $n \times 1$ *vector and* $\mathbf{g}$ *be an* $m \times 1$ *vector, then the system* $[\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\mathbf{y} = \mathbf{g}$ has a solution

$$\mathbf{y_0} = [\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\left([\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\left([\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\right)^{T}\right)^{-1}\mathbf{g} + \left\{\mathbf{I} - [\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\left([\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\left([\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\right)^{T}\right)^{-1}[\mathbf{\Delta}|\mathbf{\Lambda}]^{T}\right\}\mathbf{k}. \tag{4.1-13}$$

Proof:

Theorem 4.4 can be proved by Theorem 4.2 and Theorem 4.3.

4.2 Generalized Inverse Approach Application to Equilibrium Network Problems

As mentioned before, Yang [63] based on the work by Tobin and Friesz [65] to develop his network sensitivity model. In section 4.2.1, we will summarize and introduce contribution by Tobin and Friesz [65] and Yang [63]. Section 4.2.2 extend to relax feasible solution sets by generalized inverse approach, which is also could be done by minimum distance method proposed by Cho and Lin [78].

4.2.1 Sensitivity Analysis to Equilibrium Network Problems

Before going on introducing sensitivity analysis to equilibrium network problem, there are some theorems which should be stated in advance. The following results are from Tobin [79] and are presented without proof. Let $F: R^n \to R^n$ be continuous, let $g: R^n \to R^m$ be differentiable, and $h: R^n \to R^p$ be liner affine. Define

$$K = \{x \in R^n \mid g(x) \geq 0, h(x) = 0\} . \qquad (4.2.1\text{-}1)$$

If we can find $x^* \in K$ such that

$$F(x^*)^T(x - x^*) \geq 0 \quad \forall x \in K . \qquad (4.2.1\text{-}2)$$

Inequality (4.2.1-2) is a variational inequality problem and x^* is a solution.

Theorem 4.5 *Necessary conditions for solution*: If the vector $x^* \in K$ is a solution to the variational inequality (4.2.1-2) and the gradients $\nabla g_i(x^*)$, i such that $g_i(x^*) = 0$, and $\nabla h_i(x^*)$, i=1,…, p, are linearly independent, then there exists $\lambda \in R^m$, $\mu \in R^p$ such that

$$F(x^*) - \nabla g(x^*)^{\mathrm{T}} \lambda - \nabla h(x^*)^T \mu = 0 \qquad (4.2.1\text{-}3)$$

$$\lambda^T g(x^*) = 0 \qquad (4.2.1\text{-}4)$$

$$\lambda \geq 0 \qquad (4.2.1\text{-}5)$$

Theorem 4.6 *Sufficient conditions for solution*: If $g_i(x)$ for i=1,..,m are concave and $\mathbf{x}^* \in K, \lambda \in R^m, \mu \in R^p$ satisfy (4.2.1-3), (4.2.1-4) and (4.2.1-5), then x^* is a solution to the variational inequality (4.2.1-2).

Theorem 4.7 *Sufficient conditions for a locally unique solution*: If the condition of Theorem 4.6 hold, and, in addition, if F are differentiable and

$$y^T \nabla F(x^*) y > 0 \qquad (4.2.1\text{-}6)$$

for all $y \neq 0$ such that

$$\nabla g_i(x^*)y \geq 0 \quad \text{for all I such that} \quad g_i(x^*) = 0 \qquad (4.2.1\text{-}7)$$

$$\nabla g_i(x^*)y = 0 \quad \text{for all I such that} \quad \lambda_i > 0 \qquad (4.2.1\text{-}8)$$

$$\nabla h_i(x^*)y = 0 \quad \text{for i=1,…,p,} \qquad (4.2.1\text{-}9)$$

then x^* is a locally unique solution to variational inequality (4.2.1-2).

Let $F(x,\varepsilon)$ be once continuously differentiable, let $g(x,\varepsilon)$ be concave in x and twice continuously differentiable in (x,ε), and let $h(x,\varepsilon)$ be linear affine in x and once continuously differentiable in ε. Consider the following perturbed variational inequality, denoted as $VI(\varepsilon)$: Find $x^*_\varepsilon \in (\varepsilon)$ such that

$$F(x^*_\varepsilon,\varepsilon)^T(x,x^*_\varepsilon) \geq 0 \quad \text{for all } x \in K(\varepsilon) \quad \text{where} \qquad (4.2.1\text{-}10)$$

$$K(\varepsilon) = \{x | g(x,\varepsilon) \geq 0, h(x,\varepsilon) = 0\}. \qquad (4.2.1\text{-}11)$$

Theorem 4.8 *Implicit function theorem*: Let the conditions of Theorem 4.7 be satisfied for $VI(0)$ with $F(x^*), g(x^*), h(x^*), \lambda, \mu$ replaced by $F(x^*,0), g(x^*,0), h(x^*,0), \lambda^*, \mu^*$, respectively; with gradients $\nabla g_i(x^*,0), i$ such that $g_i(x^*,0) = 0$ and $\nabla h_i(x^*,0), i = 1,\ldots p$, linearly independent, and in addition, let the strict complementary slackness condition

$$\lambda_i^* > 0 \quad \text{where} \quad g_i(x^*,0) = 0 \qquad (4.2.1\text{-}12)$$

be satisfied. Then

a. λ^* and μ^* is unique;

b. In a neighborhood of $\varepsilon = 0$, there exists a unique once continuously differentiable function $[x(\varepsilon)^T, \lambda(\varepsilon)^T, \mu(\varepsilon)^T]^T$, where $x(\varepsilon)$ is a locally unique solution to $VI(\varepsilon)$ and $\lambda(\varepsilon)$, $\mu(\varepsilon)$ are unique associated multipliers satisfying the conditions of Theorem 4.7 for a locally unique solution for $VI(\varepsilon)$, and with $\left[x(0)^T, \lambda(0)^T, \mu(0)^T\right]^T = \left[x^{*T}, \lambda^{*T}, \mu^{*T}\right]^T$.

c. In a neighborhood of $\varepsilon = 0$, the set of binding inequality is unchanged , strict complementary slackness holds, and the binding constraints gradients are linearly independent at $x(\varepsilon)$. For $\varepsilon = 0$ and $\left[x^T, \lambda^T, \mu^T\right]^T = \left[x^{*T}, \lambda^{*T}, \mu^{*T}\right]^T$, by Theorem 4.5:

$$F(x,\varepsilon) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x,\varepsilon)^T - \sum_{i=1}^{p} \mu_i \nabla h_i(x,\varepsilon)^T = 0 \qquad (4.2.1\text{-}13)$$

$$\lambda_i g_i(x,\varepsilon) = 0, \qquad i = 1,\dots,m \qquad (4.2.1\text{-}14)$$

$$h_i(x,\varepsilon) = 0, \qquad i = 1,\dots,p. \qquad (4.2.1\text{-}15)$$

Let the Jacobian matrix of the system (4.2.1-13), (4.2.1-14), and (4.2.1-15) with respect to $y = (x, \lambda, \mu)$ be denoted by J_y^* and with respected to ε as J_ε^*.

Corollary 4.1 *Derivatives of the solution vector of* $VI(0)$ *with respect to* ε: Under the assumptions of Theorem 4.8, the inverse of J_y^* exists and the partial derivatives of (x^*, λ^*, μ^*) with respect to ε are given by

$$\nabla_\varepsilon y = \left[\nabla_\varepsilon x^* \quad \nabla_\varepsilon \pi^* \quad \nabla_\varepsilon \mu^*\right]^T = \left[J_y^*\right]^{-1}\left[-J_\varepsilon^*\right]. \qquad (4.2.1\text{-}16)$$

Corollary 4.2 *First-order approximation of solution to* $VI(\varepsilon)$ *for* ε *Near Zero*: Under assumptions of Theorem 4.8, a first-order approximation of $\left[x(\varepsilon)^T, \lambda(\varepsilon)^T, \mu(\varepsilon)^T\right]^T$ in a neighborhood of $\varepsilon = 0$ is given by

$$\begin{bmatrix} x(\varepsilon) \\ \lambda(\varepsilon) \\ \mu(\varepsilon) \end{bmatrix} = \begin{bmatrix} x^* \\ \lambda^* \\ \mu^* \end{bmatrix} + \left[J_y^*\right]^{-1}\left[J_\varepsilon^*\right]\varepsilon, \qquad (4.2.1\text{-}17)$$

where

$$\left[x^*, \lambda^*, \mu^*\right] = \left[x(0), \lambda(0), \mu(0)\right], \qquad (4.2.1\text{-}18)$$

$$J_y^* = J_y(0), \qquad (4.2.1\text{-}19)$$

$$J_\varepsilon^* = J_\varepsilon(0). \qquad (4.2.1\text{-}20)$$

Equilibrium network flow problems could be represented in variational inequality form

(Dafermos, [80]): find $\mathbf{f}^* \in \Omega$ such that

$$c(\mathbf{f}^*)^T(\mathbf{f}-\mathbf{f}^*) \geq 0. \tag{4.2.1-21}$$

for all $\mathbf{f} \in \Omega$, where

$$\Omega = \{\mathbf{f} \mid \Delta\mathbf{h} = \mathbf{f}, \Lambda\mathbf{h} = \mathbf{T}, \mathbf{h} \geq 0\}. \tag{4.2.1-22}$$

If c(**f**) is strictly monotone, the equilibrium flow vector $\mathbf{f}^*$ is unique. In general, the perturbed equilibrium network flow problem can be written as follows: find $\mathbf{f}_\varepsilon{}^* \in \Omega(\boldsymbol{\varepsilon})$ such that

$$c(\mathbf{f}_\varepsilon{}^*, \boldsymbol{\varepsilon})^T(\mathbf{f}-\mathbf{f}_\varepsilon{}^*) \geq 0 \tag{4.2.1-23}$$

for $\mathbf{f} \in \Omega(\boldsymbol{\varepsilon})$, where

$$\Omega(\boldsymbol{\varepsilon}) = \{\mathbf{f} \mid \Delta\mathbf{h} = \mathbf{f}, \Lambda\mathbf{h} = \mathbf{T}(\boldsymbol{\varepsilon}), \mathbf{h} \geq 0\}, \tag{4.2.1-24}$$

and $\boldsymbol{\varepsilon}$ is a vector of perturbation parameters. Assume that $c(\mathbf{x}, \boldsymbol{\varepsilon})$ is continuously differentiable in $(\mathbf{x}, \boldsymbol{\varepsilon})$, and $\mathbf{T}(\boldsymbol{\varepsilon})$ is once continuously differentiable in $\boldsymbol{\varepsilon}$. The equilibrium path flows are generally not unique and are contained in the convex polytope

$$\Gamma(\boldsymbol{\varepsilon}) = \{\mathbf{f} \mid \Delta\mathbf{h} = \mathbf{f}^*, \Lambda\mathbf{h} = \mathbf{T}(\boldsymbol{\varepsilon}), \mathbf{h} \geq 0\}, \tag{4.2.1-25}$$

where $\mathbf{f}^*$ solves (4.2.1-23) and (4.2.1-24). Because for any vector $\boldsymbol{\varepsilon}$ the set of path flow solutions in $\Gamma(\boldsymbol{\varepsilon})$ is a convex set, derivates of a solution $\mathbf{h}^*$ with respect to the perturbation parameters do not exist. If the perturbed variational inequality is written entirely in terms of **h**, then the perturbed variational inequality has the form: Find $\mathbf{h}^* \in \Omega(\varepsilon)$ such that

$$\hat{c}(\mathbf{h}^*, \boldsymbol{\varepsilon})(\mathbf{h}-\mathbf{h}^*) \geq 0 \tag{4.2.1-26}$$

for all $\mathbf{h} \in \Omega'(\boldsymbol{\varepsilon})$, where

$$\boldsymbol{\Omega}'(\boldsymbol{\varepsilon}) = \{\mathbf{h} \mid \boldsymbol{\Lambda}\mathbf{h} = \mathbf{T}(\boldsymbol{\varepsilon}), \mathbf{h} \geq \mathbf{0}\}. \tag{4.2.1-27}$$

Tobin and Friesz [65] tried to select one particular path flow solution, in particular an extreme point of $\Gamma(0)$, that is, $\mathbf{h}^*$ in which the number of paths with positive flows is equal to the rank of $[\mathbf{\Delta}|\mathbf{\Lambda}]^T$, the same assumption as Dafermos and Nagurney [8]. Therefore, we can reduce the network under consideration to that which contains only arcs which have positive flow in the solution and consider only the paths on these arcs. Since $\mathbf{h}^*$ is a solution to the perturbed variational inequality (4.2-26) and (4.2-27) at $\varepsilon = 0$, by Theorem 4.5 there exists a solution to the system

$$\hat{\mathbf{c}}(\mathbf{h}^*,\mathbf{0}) - \boldsymbol{\lambda} - \mathbf{\Lambda}^{\mathbf{T}}\boldsymbol{\mu} = \mathbf{0} \qquad (4.2.1\text{-}28)$$

$$\lambda^T \mathbf{h}^* = \mathbf{0} \qquad (4.2.1\text{-}29)$$

$$\mathbf{\Lambda h}^* - \mathbf{T(0)} = \mathbf{0} \qquad (4.2.1\text{-}30)$$

$$\lambda \geq 0 \qquad (4.2.1\text{-}31)$$

Since all path flow variables are positive in this restricted system and will remain so for perturbations in a neighborhood of 0, the nonnegativity constraints on **h** are not bind and may be eliminated without changing the solution in a neighborhood of 0. The system then reduce to

$$\hat{\mathbf{c}}^{\mathbf{o}}(\mathbf{h}^*,\mathbf{0}) - \mathbf{\Lambda}^{\mathbf{OT}}\boldsymbol{\mu} = \mathbf{0} \qquad (4.2.1\text{-}32)$$

$$\mathbf{\Lambda}^{\mathbf{O}}\mathbf{h}^{\mathbf{o}*} - \mathbf{T(0)} = \mathbf{0}\,. \qquad (4.2.1\text{-}33)$$

The Jocobian matrix of the system (4.2.1-32) and (4.2.1-33) with respect to $(\mathbf{h}^{\mathbf{o}}, \boldsymbol{\mu})$ and evaluated at $\boldsymbol{\varepsilon} = \mathbf{0}$ is

$$\mathbf{J}_{\mathbf{h}^{\mathbf{O}},\boldsymbol{\mu}} = \begin{bmatrix} \nabla\hat{\mathbf{c}}^{\mathbf{o}}(\mathbf{h}^*,\mathbf{0}) & -\mathbf{\Lambda}^{\mathbf{OT}} \\ \mathbf{\Lambda}^{\mathbf{O}} & \mathbf{0} \end{bmatrix}. \qquad (4.2.1\text{-}34)$$

The Jocobian matrix of the system (4.2.1-32) and (4.2.1-33) with respect to ε and evaluated at zero is

$$\mathbf{J}_{\boldsymbol{\varepsilon}} = \begin{bmatrix} \nabla_{\boldsymbol{\varepsilon}}\hat{\mathbf{c}}^{\mathbf{O}}(\mathbf{h}^*,\mathbf{0}) \\ -\nabla_{\boldsymbol{\varepsilon}}\mathbf{T(0)} \end{bmatrix}. \qquad (4.2.1\text{-}35)$$

Then

$$\begin{bmatrix} \nabla_{\varepsilon}\mathbf{h}^{\mathbf{O}} \\ \nabla_{\varepsilon}\boldsymbol{\mu} \end{bmatrix} = \left[\mathbf{J}_{\mathbf{h}^{\mathbf{O}},\boldsymbol{\mu}}\right]^{-1}\mathbf{J}_{\varepsilon}. \tag{4.2.1-36}$$

Yang [63] applied Tobin and Frieszs' work, adding elastic demand to his sensitivity analysis to equilibrium network problem. The OD demand is assumed not to be fixed but rather a function of equilibrium OD travel cost between all OD pairs, i.e., $T_w = D_w(\mathbf{U}), w \in W$, where **U** is a vector of the shortest path costs between all OD pairs. Let –D be strictly monotone, then demand function is invertible, $u_w = D_w^{-1}(\mathbf{T})$. The original problem was formulated as follows: Find $(\mathbf{f}^*, \mathbf{T}^*)$ such that

$$c(\mathbf{f}^*)^T(\mathbf{f} - \mathbf{f}^*) - \mathbf{D}^{-1}(\mathbf{T}^*)^T(\mathbf{T} - \mathbf{T}^*) \geq 0 \tag{4.2.1-37}$$

for all $\mathbf{f}, \mathbf{T} \in \Omega$, where

$$\Omega = \{(\mathbf{f}, \mathbf{T}) | \mathbf{f} = \boldsymbol{\Delta}\mathbf{h}, \mathbf{T} = \boldsymbol{\Lambda}\mathbf{h}, \mathbf{h} \geq \mathbf{0}\}. \tag{4.2.1-38}$$

Considering separable cost and demand functions, i.e., the travel cost on each link is independent of the flow on other links and the demand between an OD pair in the network depends on the travel cost between the OD pair only. The elastic-demand network equilibrium problem could be solved by the Frank-Wolfe convex combination method. (Sheffi, [1]) In the same way, the general perturbed variational inequality for equilibrium network problem for elastic demand could be written in the following: Find $(\mathbf{f}^*, \mathbf{T}^*) \in \Omega(\boldsymbol{\varepsilon})$ such that

$$\mathbf{c}(\mathbf{f}_{\varepsilon}^*, \boldsymbol{\varepsilon})^{\mathrm{T}}(\mathbf{f} - \mathbf{f}_{\varepsilon}^*) - \mathbf{D}^{-1}(\mathbf{T}_{\varepsilon}^*, \boldsymbol{\varepsilon})^{\mathrm{T}}(\mathbf{T} - \mathbf{T}_{\varepsilon}^*) \geq 0 \tag{4.2.1-39}$$

for all $(\mathbf{f}, \mathbf{T}) \in \Omega(\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon}$ is a vector of perturbation parameters. Assume that $\mathbf{c}(\mathbf{f}, \boldsymbol{\varepsilon})$ and $\mathbf{D}^{-1}(\mathbf{T}, \boldsymbol{\varepsilon})$ are once continuously differentiable in $\boldsymbol{\varepsilon}$, Yang(1997) adopted the restricted network equilibrium approach. It is assumed that a solution $\mathbf{T}^*$, $\mathbf{f}^*$, and $\mathbf{h}^*$ existing to the above perturbed problem (4.2.1-39) for $\boldsymbol{\varepsilon} = 0$ and $\mathbf{D}(\mathbf{U}, \boldsymbol{\varepsilon})$ and $\mathbf{c}(\mathbf{f}, \boldsymbol{\varepsilon})$ are strongly monotone in **U** and **f** respectively. In addition, the demand between every OD pair is strictly positive at equilibrium $T_w^* = D_w(U_w^*) > 0$, $w \in W$. Finally, every link should carry positive flow. Let $\mathbf{h}^* > 0$ be a no degenerate extreme point in the region $\Omega^* = \{(\mathbf{h}, \mathbf{T}) | \mathbf{f}^* = \mathbf{\Delta h}, \mathbf{T}^* = \mathbf{\Lambda h}, \mathbf{h} \geq \mathbf{0}\}$ of equilibrium path flows. The necessary conditions for the perturbed network equilibrium problem in (4.2.1-39) at $\boldsymbol{\varepsilon} = 0$ is that there exists a solution to the following system:

$$\hat{\mathbf{c}}(\mathbf{h}^*, \mathbf{0}) - \boldsymbol{\lambda} - \mathbf{\Lambda}^{\mathbf{T}}\mathbf{U} = \mathbf{0} \qquad (4.2.1\text{-}40)$$

$$\boldsymbol{\lambda}^{\mathbf{T}}\mathbf{h}^* = 0 \qquad (4.2.1\text{-}41)$$

$$\mathbf{\Lambda h}^* - \mathbf{D}(\mathbf{U}, \mathbf{0}) = \mathbf{0} \qquad (4.2.1\text{-}42)$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{h}^* \geq \mathbf{0} \qquad (4.2.1\text{-}43)$$

Under the situation of considering the nondegenerate extreme point of positive path flows solutions, the system then reduces to:

$$\hat{\mathbf{c}}^{\mathbf{o}}(\mathbf{h}^*, \mathbf{0}) - \mathbf{\Lambda}^{\mathbf{OT}}\mathbf{U} = \mathbf{0} \qquad (4.2.1\text{-}44)$$

$$\mathbf{\Lambda}^{\mathbf{O}}\mathbf{h}^{\mathbf{o}*} - \mathbf{D}(\mathbf{U}, \mathbf{0}) = \mathbf{0}. \qquad (4.2.1\text{-}45)$$

Differentiating both sides of the system of (4.2.1-44) and (4.2.1-45) with respect to the perturbation parameter $\boldsymbol{\varepsilon}$, we obtain

$$\begin{bmatrix} \nabla_{\boldsymbol{\varepsilon}}\mathbf{h}^{\mathbf{O}} \\ \nabla_{\boldsymbol{\varepsilon}}\mathbf{U} \end{bmatrix} = \left[\mathbf{J}_{\mathbf{h}^{\mathbf{O}}, U}\right]^{-1} \mathbf{J}_{\boldsymbol{\varepsilon}} = \begin{bmatrix} \nabla_{\mathbf{h}}\mathbf{c}^{\mathbf{o}}(\mathbf{h}^*, \mathbf{0}) & -\mathbf{\Lambda}^{\mathbf{OT}} \\ \mathbf{\Lambda}^{\mathbf{O}} & -\nabla_{\mathbf{U}}\mathbf{D}(\mathbf{U}, \mathbf{0}) \end{bmatrix}^{-1} \begin{bmatrix} -\nabla_{\boldsymbol{\varepsilon}}\mathbf{c}^{\mathbf{O}}(\mathbf{h}^*, \mathbf{0}) \\ \nabla_{\boldsymbol{\varepsilon}}\mathbf{D}(\mathbf{U}, \mathbf{0}) \end{bmatrix}.$$

(4.2.1-46)

Other derivatives such as that of OD demand in perturbation parameters could be obtained as

$$\nabla_{\boldsymbol{\varepsilon}}\mathbf{T} = \nabla_{\boldsymbol{\varepsilon}}\mathbf{D}(\mathbf{U}, \mathbf{0}) + \nabla_{\mathbf{U}}\mathbf{D}(\mathbf{U}, \mathbf{0})\nabla_{\boldsymbol{\varepsilon}}\mathbf{U}. \qquad (4.2.1\text{-}47)$$

Finally it should be motioned that the above results of sensitivity analysis are independent of the choice of the nondengerate path flow solution, consistent in Tobin and Friesz [65].

4.2.2 Relaxation on Feasible Solution Space of Sensitivity Analysis to Equilibrium Network Problem

Although in section 4.2.1, we could have the derivatives of decision variables (link flows) and constraint multiplies (OD travel cost) with respect to a variety of perturbation parameters in demand function and link cost functions of the network equilibrium problem, there is still limited to the assumption: $\mathbf{h}^*$ in which the number of paths with positive flows is equal to the rank of $[\boldsymbol{\Delta}|\boldsymbol{\Lambda}]^T$.

Thus Cho[81] proposed generalized inverse approach to transform feasible path flow space to link flow space in traffic equilibrium models, avoiding the non-uniqueness problem. Assume the network includes paths of positive flow, and number of paths is larger than that of arcs. He cut link-path incidence matrix into two submatrices: $\Delta = [\Delta^O | \Lambda^r]$, letting $[\Delta^O | \Lambda]$ be full of row rank and then derived derivatives of flow variables with respect to the perturbation parameters. From (4.2.1-25), we could have a solution

$$\mathbf{h}(\boldsymbol{\varepsilon}) = \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}} \\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^{\mathbf{O}} \\ \mathbf{T}(\boldsymbol{\varepsilon}) \end{bmatrix} + \left(\mathbf{I} - \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}} \\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix} \right) \mathbf{k} \quad (4.2.2\text{-}1)$$

where $\mathbf{k}$ is an arbitrary column vector which let $\mathbf{h}(\boldsymbol{\varepsilon})$ is positive. Therefore, the feasible arc flows set could be written as follows:

$$\Omega(\boldsymbol{\varepsilon}) = \{ (\mathbf{f} = \begin{bmatrix} \mathbf{f}^{\mathbf{O}} \\ \mathbf{f}^{\mathbf{r}} \end{bmatrix} \| \boldsymbol{\Delta}^{\mathbf{r}}\mathbf{h}(\boldsymbol{\varepsilon}) = \mathbf{f}^{\mathbf{r}}, \mathbf{f} \geq \mathbf{0} \} . \quad (4.2.2\text{-}2)$$

Let

$$\begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}} \\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad (4.2.2\text{-}3)$$

then

$$\mathbf{h}(\boldsymbol{\varepsilon}) = \mathbf{C}' + \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{\mathbf{o}} \\ \mathbf{T}(\boldsymbol{\varepsilon}) \end{bmatrix} \quad (4.2.2\text{-}4)$$

where

$$\mathbf{C}' = \mathbf{k} - \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}} \\ \boldsymbol{\Lambda} \end{bmatrix} \mathbf{k}\,. \tag{4.2.2-5}$$

Therefore

$$\Omega(\boldsymbol{\varepsilon}) = \{(\mathbf{f} = \begin{bmatrix} \mathbf{f}^{\mathbf{O}} \\ \mathbf{f}^{\mathbf{r}} \end{bmatrix} \Bigg| \boldsymbol{\Delta}^{\mathbf{r}} \left(\mathbf{C}' + \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{\mathbf{o}} \\ \mathbf{T}(\boldsymbol{\varepsilon}) \end{bmatrix} \right) = \mathbf{f}^{\mathbf{r}}, \mathbf{f} \geq \mathbf{0}\}; \tag{4.2.2-6}$$

let

$$\mathbf{B}_1 = \boldsymbol{\Delta}^{\mathbf{r}} \boldsymbol{\Delta}^{\mathbf{OT}} \mathbf{A}_{11} + \boldsymbol{\Delta}^{\mathbf{r}} \boldsymbol{\Lambda}^{\mathbf{T}} \mathbf{A}_{21}\,, \tag{4.2.2-7}$$

$$\mathbf{B}_2 = \boldsymbol{\Delta}^{\mathbf{r}} \boldsymbol{\Delta}^{\mathbf{OT}} \mathbf{A}_{12} + \boldsymbol{\Delta}^{\mathbf{r}} \boldsymbol{\Lambda}^{\mathbf{T}} \mathbf{A}_{22}\,, \tag{4.2.2-8}$$

$$\mathbf{C} = \boldsymbol{\Delta}^{\mathbf{r}} \mathbf{C}'\,, \tag{4.2.2-9}$$

then $\boldsymbol{\Omega}(\boldsymbol{\varepsilon}) = \{(\mathbf{f} = \begin{bmatrix} \mathbf{f}^{\mathbf{O}} \\ \mathbf{f}^{\mathbf{r}} \end{bmatrix} \Big| \mathbf{C} + \mathbf{B}_1 \mathbf{f}^{\mathbf{O}} + \mathbf{B}_2 [\mathbf{T}(\boldsymbol{\varepsilon})] = \mathbf{f}^{\mathbf{r}}, \mathbf{f} \geq \mathbf{0}\}$. According to theorem 4.5, for the system (4.2.1-24) there exists a solution as follows:

$$\mathbf{c}(\mathbf{f}^*, 0) - \lambda - [\mathbf{B}_1 \quad -\mathbf{I}]^T \mu = 0 \tag{4.2.2-10}$$

$$\lambda^T \mathbf{f}^* = 0 \tag{4.2.2-11}$$

$$\mathbf{C} + \mathbf{B}_1 \mathbf{f}^{\mathbf{O}} + \mathbf{B}_2 [\mathbf{T}(\boldsymbol{\varepsilon})] = \mathbf{f}^{\mathbf{r}} \tag{4.2.2-12}$$

$$\lambda \geq 0, \mathbf{f}^* \geq 0; \tag{4.2.2-13}$$

deleting the nonnegative constraints, the system is rewritten in the following:

$$c(\mathbf{f}^*, \boldsymbol{\varepsilon}) - [\mathbf{B}_1 \quad -\mathbf{I}]^T \mu = 0 \tag{4.2.2-14}$$

$$\mathbf{C} + \mathbf{B}_1 \mathbf{f}^{\mathbf{O}} + \mathbf{B}_2 [\mathbf{T}(\boldsymbol{\varepsilon})] - \mathbf{f}^{\mathbf{r}} = 0\,. \tag{4.2.2-15}$$

The Jocobian matrix of the system (4.2.2-14) and (4.2.2-15) with respect to $(\mathbf{f}, \boldsymbol{\mu})$ and evaluated at $\boldsymbol{\varepsilon} = 0$ is

$$\mathbf{J}_{\mathbf{f},\boldsymbol{\mu}} = \begin{bmatrix} \nabla \mathbf{c}(\mathbf{f}^*, \mathbf{0}) & -[\mathbf{B}_2 \quad -\mathbf{I}]^{\mathbf{T}} \\ [\mathbf{B}_1 \quad -\mathbf{I}] & \mathbf{0} \end{bmatrix}. \tag{4.2.2-16}$$

The Jocobian matrix of the system (4.2.2-14) and (4.2.2-15) with respect to ε and evaluated at zero is

$$\mathbf{J}_{\varepsilon} = \begin{bmatrix} \nabla_{\varepsilon}\mathbf{c}(\mathbf{f}^{*},\mathbf{0}) \\ \mathbf{B}_2\nabla_{\varepsilon}\mathbf{T}(\mathbf{0}) \end{bmatrix}. \tag{4.2.2-17}$$

Then

$$\begin{bmatrix} \nabla_{\varepsilon}\mathbf{f} \\ \nabla_{\varepsilon}\boldsymbol{\mu} \end{bmatrix} = \left[\mathbf{J}_{\mathbf{f},\boldsymbol{\mu}}\right]^{-1}\mathbf{J}_{\varepsilon}. \tag{4.2.2-18}$$

Although generalized inverse matrix approach could be used to relax limitation in network topology, it could not guarantee positive path flows when solving equilibrium network flow problems. Thus Cho and Lo [82] developed an algorithm to ensure positive path flows due to properties of generalized inverse matrix.

In summary, Yang's work could not apply to general network topology basing on specific assumption in restriction approach. Next, we start from Dafemos and Nagurney's model. In section 4.3 we will modify their model, and from the modified model, extending to sensitivity analysis to equilibrium network problem.

4.3 The Modified Model

In this section, we come back to *Dafermos and Nagurney's model* in Chapter 3. For convenience, we transform the model of Dafermos and Nagurney into a simpler form. $\left[\boldsymbol{\Delta}^{T}\mathbf{G}\boldsymbol{\Delta}\middle|\boldsymbol{\Delta}^{T}\right] = [\mathbf{M}]$, $\left[\boldsymbol{\Lambda}\middle|\mathbf{0}\right] = [\mathbf{N}]$, $\left[\mathbf{h}'\middle|-\mathbf{U}'\right]^{T} = \mathbf{y}$, then

$$\begin{bmatrix} \mathbf{M} \\ \hline \mathbf{N} \end{bmatrix}\mathbf{y} = \begin{bmatrix} -\boldsymbol{\rho} \\ \hline -\mathbf{e} \end{bmatrix}\mathbf{h}_{n}. \tag{4.3-1}$$

Therefore, the inverse of $\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}$ does not exist. Then partition $\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix}$ as

$$\begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix}=\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\\ \hline \mathbf{M}^{r}\\ \mathbf{N}^{r}\end{bmatrix}, \tag{4.3-2}$$

here we let $\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}$ be full row rank by Gaussian-Jordan elimination. Therefore we rewrite (4.3-1),

$$\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\\ \hline \mathbf{M}^{r}\\ \mathbf{N}^{r}\end{bmatrix}\mathbf{y}=\begin{bmatrix}\begin{bmatrix}-\boldsymbol{\rho}\\ \hline -\mathbf{e}\end{bmatrix}^{o}\\ \hline \begin{bmatrix}-\boldsymbol{\rho}\\ \hline -\mathbf{e}\end{bmatrix}^{r}\end{bmatrix}\mathbf{h}_{n}. \tag{4.3-3}$$

According to Theorem 4.4, for an arbitrary column vector **k**,column vector ***y*** is obtained as follows:

$$\mathbf{y}=\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}^{-}\begin{bmatrix}-\boldsymbol{\rho}\\ \hline -\mathbf{e}\end{bmatrix}^{o}\mathbf{h}_{n}+\left\{\mathbf{I}-\left(\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}^{-}\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}\right)\right\}\mathbf{k} \tag{4.3-4}$$

where

$$\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}^{-}=\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}^{T}\left(\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}^{T}\right)^{-1}, \tag{4.3-5}$$

k denotes an arbitrary column vector.

Obviously, (4.3-4) should satisfy the rest of (4.3-3). Let $\begin{bmatrix}\mathbf{M}^{r}\\ \mathbf{N}^{r}\end{bmatrix}=\begin{bmatrix}\lambda_1\ \lambda_2\end{bmatrix}\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}$, the fact can be fulfilled owing to linear relation between $\begin{bmatrix}\mathbf{M}^{o}\\ \mathbf{N}^{o}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{M}^{r}\\ \mathbf{N}^{r}\end{bmatrix}$:

$$\begin{bmatrix}\mathbf{M}^r\\ \mathbf{N}^r\end{bmatrix}\mathbf{y}=[\lambda_1\,\lambda_2]\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}^T\left(\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}^T\right)^{-1}\begin{bmatrix}-\boldsymbol{\rho}\\ -\mathbf{e}\end{bmatrix}^o\mathbf{h}_n$$

$$+\left\{[\lambda_1\,\lambda_2]\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}-[\lambda_1\,\lambda_2]\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}^T\left(\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}\begin{bmatrix}\mathbf{M}^o\\ \mathbf{N}^o\end{bmatrix}^T\right)^{-1}\right\}\mathbf{k} \quad (4.3\text{-}6)$$

$$=[\lambda_1\,\lambda_2]\begin{bmatrix}-\boldsymbol{\rho}\\ -\mathbf{e}\end{bmatrix}^o\mathbf{h}_n=\begin{bmatrix}-\boldsymbol{\rho}\\ -\mathbf{e}\end{bmatrix}^r\mathbf{h}_n.$$

Back to (4.3-4), the answer could be expanded as follows:

$$\mathbf{y}=\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^T\left(\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^T\right)^{-1}\begin{bmatrix}-\boldsymbol{\rho}\\ -\mathbf{e}\end{bmatrix}^{\mathbf{o}}\mathbf{h}_n+\left\{\mathbf{I}-\left(\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^T\left(\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^T\right)^{-1}\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}\right)\right\}\mathbf{k}$$

$$=\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^T\left(\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^{\mathrm{T}}\right)^{-1}\left(\begin{bmatrix}-\boldsymbol{\rho}\\ -\mathbf{e}\end{bmatrix}^{\mathbf{o}}\mathbf{h}_n+\begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\end{bmatrix}\right)-\begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\end{bmatrix}$$

(4.3-7)

Let $\left(\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}\begin{bmatrix}\mathbf{M}^0\\ \mathbf{N}^0\end{bmatrix}^{\mathrm{T}}\right)^{-1}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}$,

where $\mathbf{A}_{11}=[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}+[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}\mathbf{M}^0\mathbf{N}^{0\mathrm{T}}[\mathbf{N}^0\mathbf{N}^{\mathrm{T}}-\mathbf{N}^0\mathbf{M}^{0\mathrm{T}}[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}\mathbf{M}^0\mathbf{N}^{0\mathrm{T}}]^{-1}\mathbf{N}^0\mathbf{M}^{0\mathrm{T}}[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}$;

$\mathbf{A}_{12}=-[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}\mathbf{M}^0\mathbf{N}^{0\mathrm{T}}[\mathbf{N}^0\mathbf{N}^{\mathrm{T}}-\mathbf{N}^0\mathbf{M}^{0\mathrm{T}}[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}\mathbf{M}^0\mathbf{N}^{0\mathrm{T}}]^{-1}$;

$\mathbf{A}_{21}=-[\mathbf{N}^0\mathbf{N}^{0\mathrm{T}}-\mathbf{N}^0\mathbf{M}^{0\mathrm{T}}[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}\mathbf{M}^0\mathbf{N}^{0\mathrm{T}}]^{-1}\mathbf{N}^0\mathbf{M}^{0\mathrm{T}}[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}$;

$\mathbf{A}_{22}=[\mathbf{N}^0\mathbf{N}^{0\mathrm{T}}-\mathbf{N}^0\mathbf{M}^{0\mathrm{T}}[\mathbf{M}^0\mathbf{M}^{0\mathrm{T}}]^{-1}\mathbf{M}^0\mathbf{N}^{0\mathrm{T}}]^{-1}$.

Also let

$\mathbf{B}_1=-\boldsymbol{\rho}^0\mathbf{h}_\mathbf{n}+\mathbf{k}_1$;

$\mathbf{B}_2=-\mathbf{e}^0\mathbf{h}_\mathbf{n}+\mathbf{k}_2$.

Then

$$\mathbf{y} = \begin{bmatrix} \mathbf{M}^0 \\ \mathbf{N}^0 \end{bmatrix}^T \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{k}_1 \\ \hline \mathbf{k}_2 \end{bmatrix}. \tag{4.3-8}$$

Let

$\mathbf{D}_1 = \mathbf{M}^{0\mathrm{T}}\mathbf{A}_{11} + \mathbf{N}^{0\mathrm{T}}\mathbf{A}_{21}$;

$\mathbf{D}_2 = \mathbf{M}^{0\mathrm{T}}\mathbf{A}_{12} + \mathbf{N}^{0\mathrm{T}}\mathbf{A}_{22}$.

Then

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{k}_1 \\ \hline \mathbf{k}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{D}_1\mathbf{B}_1 + \mathbf{D}_1\mathbf{B}_2 - \mathbf{k}_1 \\ \mathbf{D}_2\mathbf{B}_1 + \mathbf{D}_2\mathbf{B}_2 - \mathbf{k}_2 \end{bmatrix}. \end{aligned} \tag{4.3-9}$$

Therefore, if all elements in [$\mathbf{D}_2\mathbf{B}_1 + \mathbf{D}_2\mathbf{B}_2 - \mathbf{k}_2$] are larger than zero, it means that adding a new path connecting a particular OD will not result in Braess' paradox.

However, the arbitrary vector **k** may not guarantee positive flows. Although in theorems, vector **k** is arbitrary. But this is not in transportation network. Negative path flows are meaningless. Therefore vector **k** which we need is that will ensure positive path flows. Numerous methods of solving the positive flow problem exist: for example, the algorithm presented in Cho and Lo [82] is applied to obtain vector k. The algorithm can guarantee this vector k ensure positive path flows.

First we express equilibrium network flows as

$$\begin{bmatrix} \Delta \\ \Lambda \end{bmatrix} \mathbf{h} = \begin{bmatrix} \mathbf{f} \\ \mathbf{T} \end{bmatrix}. \tag{4.3-10}$$

Exploit Theorem .4.4, we have

$$\mathbf{h} = \left(\begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\Delta}^0\boldsymbol{\Delta}^{0\mathrm{T}} & \boldsymbol{\Delta}^0\boldsymbol{\Lambda}^{0\mathrm{T}} \\ \boldsymbol{\Lambda}^0\boldsymbol{\Delta}^{0\mathrm{T}} & \boldsymbol{\Lambda}^0\boldsymbol{\Lambda}^{0\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{T}^0 \end{bmatrix} \right) + \left(\mathbf{I} - \begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\Delta}^0\boldsymbol{\Delta}^{0\mathrm{T}} & \boldsymbol{\Delta}^0\boldsymbol{\Lambda}^{0\mathrm{T}} \\ \boldsymbol{\Lambda}^0\boldsymbol{\Delta}^{0\mathrm{T}} & \boldsymbol{\Lambda}^0\boldsymbol{\Lambda}^{0\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix} \right) \mathbf{k} \tag{4.3-11}$$

Transform (4.3-11) as (4.3-12) for convenience as

$$\mathbf{h} = \begin{bmatrix} \mathbf{a_1} + (\mathbf{1} - \mathbf{b_{11}}) \times \mathbf{k_1} - \mathbf{b_{12}} \times \mathbf{k_2} - \ldots - \mathbf{b_{1n}} \times \mathbf{k_n} \\ \mathbf{a_2} + \mathbf{b_{21}} \times \mathbf{k_1} + (\mathbf{1} - \mathbf{b_{22}}) \times \mathbf{k_2} - \ldots - \mathbf{b_{2n}} \times \mathbf{k_n} \\ \vdots \\ \vdots \\ \mathbf{a_n} - \mathbf{b_{n1}} \times \mathbf{k_1} - \mathbf{b_{n2}} \times \mathbf{k_2} - \ldots + (\mathbf{1} - \mathbf{b_{nn}}) \times \mathbf{k_n} \end{bmatrix}, \tag{4.3-12}$$

where

$$\begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \boldsymbol{\Delta}^0 \boldsymbol{\Delta}^{0\mathbf{T}} & \boldsymbol{\Delta}^0 \boldsymbol{\Lambda}^{0\mathbf{T}} \\ \boldsymbol{\Lambda}^0 \boldsymbol{\Delta}^{0\mathbf{T}} & \boldsymbol{\Lambda}^0 \boldsymbol{\Lambda}^{0\mathbf{T}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{T}^0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}^{\mathbf{T}}, \tag{4.3-13}$$

$$\begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \boldsymbol{\Delta}^0 \boldsymbol{\Delta}^{0\mathbf{T}} & \boldsymbol{\Delta}^0 \boldsymbol{\Lambda}^{0\mathbf{T}} \\ \boldsymbol{\Lambda}^0 \boldsymbol{\Delta}^{0\mathbf{T}} & \boldsymbol{\Lambda}^0 \boldsymbol{\Lambda}^{0\mathbf{T}} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{b2} & \ldots & b_{nn} \end{bmatrix}. \tag{4.3-14}$$

Then a simplified version of the algorithm is presented as follows:

Step 1:Assume all elements of $\boldsymbol{k}$ are the same. Rewrite (4.3-12) as (4.3-13),

$$\mathbf{h} = \begin{bmatrix} a_1 + c_1 \times k \\ a_2 + c_2 \times k \\ \vdots \\ a_n + c_n \times k \end{bmatrix} \tag{4.3-13}$$

where

$$\begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T = \begin{bmatrix} 1 - b_{11} - b_{12} - \ldots - b_{1n} \\ 1 - b_{21} - b_{22} - \ldots - b_{2n} \\ \vdots \\ 1 - b_{n1} - b_{n2} - \ldots - b_{nn} \end{bmatrix}. \tag{4.3-14}$$

Step 2: If there exists any $c_i = 0$, check whether a_i is positive. After processing the problem of $c_i = 0$, let

$$\mathbf{a}_{\min} = Min[\mathbf{a}_i],\ \forall i \in [1, n] \tag{4.3-15}$$

$$\mathbf{c}_{\min} = Min[\mathbf{c}_i],\ \forall i \in \{[1, n] \setminus i',\ where\ i' \ni \mathbf{c}_{i'} = 0\}. \tag{4.3-16}$$

Step 3:Check $\mathbf{c}_{\min}$ and if $\mathbf{c}_{\min} \neq 0$, there are two situations,

i. $\mathbf{c}_{\min} > 0$, select $\mathbf{k}'$ such that $a_{\min} + c_{\min}\mathbf{k}' = 0$. If $\mathbf{k}' \geq 0$, then

$\forall \mathbf{k} \in \left[\frac{-\mathbf{a}_{\min}}{\mathbf{c}_{\min}}, \infty\right)$ satisfies the positive flow criterion. Otherwise, select any

$\mathbf{k} \geq 0$.

ii. $\mathbf{c}_{\min} < 0$, select $\mathbf{k}'$ such that $a_{\min} + c_{\min}\mathbf{k}' = 0$. If $\mathbf{k}' \geq 0$, then

$\forall \mathbf{k} \in \left(-\infty, \frac{-\mathbf{a}_{\min}}{\mathbf{c}_{\min}}\right]$. Otherwise, check two situations: $\mathbf{c}_i > 0$ and $\mathbf{c}_i < 0$.

From the intersection of the two situations, select $\mathbf{k} \ni \mathbf{a}_i + \mathbf{c}_i\mathbf{k} \geq 0$.

Since Braess' paradox is concerned with a tiny change in the network capacity, this study uses the network before the new path addition to obtain vector **k**. In next section, an illustrative example will be showed. The solution is an analysis solution, although vector **k** may lead to multiple solutions. However, the results from sensitivity analysis to equilibrium network problems could be applied to show that vector **k** would not affect the solution. In Cho and Lin [78], they exploited minimum distance method to prove the independency of the chosen path flow and the independency of the link/path incidence matrix in sensitivity analysis for equilibrium network flow problems. When choosing different path flow or different link/path incidence matrix, the network sensitivity analysis could get the same result. Tobin and Friesz [65] also proved similar theorem that different chosen path would not affect results for network sensitivity analysis. Combining the above results, vector **k** is independent of results for network sensitivity. The new added link can be considered as already exist in the network, but travel cost is too high to no traveler use it. In this way, addition of new link could be transformed into perturbations in link performance function and network sensitivity model could be formulated.

In Chapter 2, we understand that Braess' paradox depends on network topology, demand and link performance function. In the following, we tried to modify Yang's work, in the same way, by using generalized inverse approach to relax feasible link flow space. The perturbation network equilibrium is written as follows:

Find $(\mathbf{f}^*, \mathbf{T}^*)$ such that

$$c(\mathbf{f}^*)^T(\mathbf{f}-\mathbf{f}^*)-\mathbf{D}^{-1}(\mathbf{T}^*)^T(\mathbf{T}-\mathbf{T}^*)\geq 0 \tag{4.3-17}$$

for all $\mathbf{f},\mathbf{T}\in\Omega$, where

$$\Omega=\{(\mathbf{f},\mathbf{T})|\mathbf{f}=\boldsymbol{\Delta}\mathbf{h},\mathbf{T}=\boldsymbol{\Lambda}\mathbf{h},\mathbf{h}\geq\mathbf{0}\}. \tag{4.3-18}$$

Note here the demand between every OD pair is strictly positive at equilibrium

$T_w^*=D_w(U_w^*)>0$, $w\in W$ .Let

$$\boldsymbol{\Delta}^{\mathbf{o}}\mathbf{h}(\boldsymbol{\varepsilon})=\mathbf{f}^{\mathbf{o}} \tag{4.3-19}$$

$$\boldsymbol{\Lambda}\mathbf{h}(\boldsymbol{\varepsilon})=\mathbf{T}=\mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon}), \tag{4.3-20}$$

where $\boldsymbol{\Delta}^o$ is the matrix letting $\left[\boldsymbol{\Delta}^{\mathbf{o}}\boldsymbol{\Lambda}\right]^T$ be full of row rank. $\mathbf{h}(\boldsymbol{\varepsilon})$ is path flows after perturbation. From theorem 4.4, $\mathbf{h}(\boldsymbol{\varepsilon})$ is obtained as (4.2-48), except for demand is function of OD cost and perturbation parameters:

$$\mathbf{h}(\boldsymbol{\varepsilon})=\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{o}}\\ \boldsymbol{\Lambda}\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}}\\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{f}^{\mathbf{O}}\\ \mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon})\end{bmatrix}+\left(\mathbf{I}-\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{o}}\\ \boldsymbol{\Lambda}\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}}\\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{o}}\\ \boldsymbol{\Lambda}\end{bmatrix}\right)\mathbf{k}. \tag{4.3-21}$$

If existing vector **k** which let $\mathbf{h}(\boldsymbol{\varepsilon})$ be positive, then feasible link flow space could be written as follows:

$$\begin{aligned}\Omega(\boldsymbol{\varepsilon})=\{\mathbf{f}=\begin{bmatrix}\mathbf{f}^{\mathbf{O}}\\ \mathbf{f}^{\mathbf{r}}\end{bmatrix}|\boldsymbol{\Delta}^{\mathbf{r}}(&\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{o}}\\ \boldsymbol{\Lambda}\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}}\\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{f}^{\mathbf{O}}\\ \mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon})\end{bmatrix}+\\ &\left(\mathbf{I}-\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{o}}\\ \boldsymbol{\Lambda}\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}}\\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{\Delta}^{\mathbf{o}}\\ \boldsymbol{\Lambda}\end{bmatrix}\right)\mathbf{k})=\mathbf{f}^{\mathbf{r}},\mathbf{f}\geq\mathbf{0}\}.\end{aligned} \tag{4.3-22}$$

Let

$$\begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Delta}^{\mathbf{O}}\boldsymbol{\Lambda}^{\mathbf{T}} \\ \boldsymbol{\Lambda}^{\mathbf{O}}\boldsymbol{\Delta}^{\mathbf{OT}} & \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathbf{T}} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix}, \tag{4.3-23}$$

then

$$\mathbf{h}_{\boldsymbol{\varepsilon}} = \mathbf{F} + \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{\mathbf{o}} \\ \mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon}) \end{bmatrix} \tag{4.3-24}$$

where

$$\mathbf{F}' = \mathbf{k} - \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{O}} \\ \boldsymbol{\Lambda} \end{bmatrix} \mathbf{k}\,. \tag{4.3-25}$$

Therefore

$$\Omega(\boldsymbol{\varepsilon}) = \{\mathbf{f} = \begin{bmatrix} \mathbf{f}^{\mathbf{O}} \\ \mathbf{f}^{\mathbf{r}} \end{bmatrix} \Bigg| \boldsymbol{\Delta}^{\mathbf{r}} \left(\mathbf{F}' + \begin{bmatrix} \boldsymbol{\Delta}^{\mathbf{o}} \\ \boldsymbol{\Lambda} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{\mathbf{o}} \\ \mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon}) \end{bmatrix} \right) = \mathbf{f}^{\mathbf{r}}, \mathbf{f} \geq \mathbf{0}\}; \tag{4.3-26}$$

let

$$\mathbf{G}_1 = \boldsymbol{\Delta}^{\mathbf{r}}\boldsymbol{\Delta}^{\mathbf{OT}}\mathbf{E}_{11} + \boldsymbol{\Delta}^{\mathbf{r}}\boldsymbol{\Lambda}^{\mathbf{T}}\mathbf{E}_{21}, \tag{4.3-27}$$

$$\mathbf{G}_2 = \boldsymbol{\Delta}^{\mathbf{r}}\boldsymbol{\Delta}^{\mathbf{OT}}\mathbf{E}_{12} + \boldsymbol{\Delta}^{\mathbf{r}}\boldsymbol{\Lambda}^{\mathbf{T}}\mathbf{E}_{22}, \tag{4.3-28}$$

$$\mathbf{F} = \boldsymbol{\Delta}^{\mathbf{r}}\mathbf{F}', \tag{4.3-29}$$

then $\boldsymbol{\Omega}(\boldsymbol{\varepsilon}) = \{\mathbf{f} = \begin{bmatrix} \mathbf{f}^{\mathbf{O}} \\ \mathbf{f}^{\mathbf{r}} \end{bmatrix} \Big| \mathbf{F}' + \mathbf{G}_1\mathbf{f}^{\mathbf{O}} + \mathbf{G}_2[\mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon})] = \mathbf{f}^{\mathbf{r}}, \mathbf{f} \geq \mathbf{0}\}$. According to theorem 4.5, for the system (4.3-16) there exists a solution as follows:

$$\mathbf{c}(\mathbf{f}^*,0) - \lambda - [\mathbf{G}_1 \quad -\mathbf{I}]^T \mathbf{U} = 0 \tag{4.3-30}$$

$$\lambda^T \mathbf{f}^* = 0 \tag{4.3-31}$$

$$\mathbf{F} + \mathbf{G}_1\mathbf{f}^{\mathbf{O}} + \mathbf{G}_2[\mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon})] = \mathbf{f}^{\mathbf{r}} \tag{4.3-32}$$

$$\lambda \geq 0, \mathbf{f}^* \geq 0; \tag{4.3-33}$$

The Jocobian matrix of the system from (4.3-30) to (4.3-33) with respect to $(\mathbf{f},\mathbf{U})$ and evaluated at $\boldsymbol{\varepsilon} = 0$ is

$$\mathbf{J}_{\mathbf{f},\mathbf{U}} = \begin{bmatrix} \nabla \mathbf{c}(\mathbf{f}^*,\mathbf{0}) & -[\mathbf{G}_2 \quad -\mathbf{I}]^{\mathrm{T}} \\ [\mathbf{G}_1 \quad -\mathbf{I}] & -\nabla_{\mathbf{U}} \mathbf{D}(\mathbf{U},\mathbf{0}) \end{bmatrix}. \tag{4.3-34}$$

The Jocobian matrix of the system from (4.3-30) to (4.3-33) with respect to ε and evaluated at zero is

$$\mathbf{J}_{\boldsymbol{\varepsilon}} = \begin{bmatrix} \nabla_{\boldsymbol{\varepsilon}} \mathbf{c}(\mathbf{f}^*,\mathbf{0}) \\ \mathbf{G}_2 \nabla_{\boldsymbol{\varepsilon}} \mathbf{D}(\mathbf{U},\boldsymbol{\varepsilon}) \end{bmatrix}. \tag{4.3-35}$$

Then

$$\begin{bmatrix} \nabla_{\boldsymbol{\varepsilon}} \mathbf{f} \\ \nabla_{\boldsymbol{\varepsilon}} \mathbf{U} \end{bmatrix} = [\mathbf{J}_{\mathbf{f},\mathbf{U}}]^{-1} \mathbf{J}_{\boldsymbol{\varepsilon}}. \tag{4.3-36}$$

Through sensitivity analysis, the derivatives of link flows, OD demands, OD costs and other solution variables can obtained and expressed explicitly in terms of the equilibrium flow solutions. Therefore, we could analyze the expanded network assuming the new link is already in the network, finding out when demand or other variables varies, how the trend of change to travel cost moves.

4.4 Numerical Example

In this section a simple numerical example is presented. This network, which is illustrated in Fig. 4.4-1 involves a set of 7 nodes, N={1,...,7} together with a set of 11 links, A={a_1,…,a_{11}},yielding a set of 17 possible paths, P_1 ={p_1,…,p_{17}} between the single OD pair (1,7).

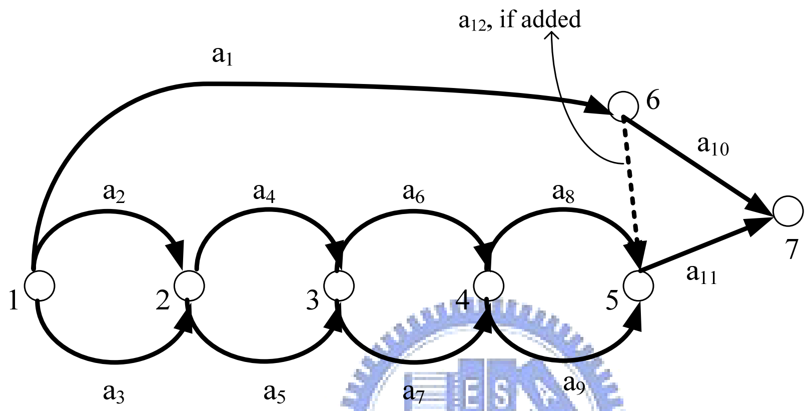


Figure 4.4-1 Illustrated Network Comprising Seven Nodes and Eleven Links.

Table 4.4-1 Link Performance Function

c_1=5f_1	c_2=10+2f_2	c_3=10+2f_3	c_4=2f_4+50	c_5=2f_5+50	c_6=2f_6
c_7=2f_7	c_8=10+f_8	c_9=10+f_9	c_{10}=70+3.5f_{10}	c_{11}=5f_{11}	c_{12}=2+f_{12} *(if adding a new path)*

Based on link performance function as listed in Table 4.4-1, given number of trips between OD pair (1,7) is $T=[24]^T$, and all possible paths are defined as follows.

Path 1: Link 1-Link 10,

Path 2: Link 2-Link 4-Link 6 –Link 8- Link 10;

Path 3: Link 2-Link 4-Link 6 –Link 9- Link 11;

Path 4: Link 2-Link 4-Link 7 –Link 8- Link 11;

Path 5: Link 2-Link 4-Link 7 –Link 9- Link 11;

Path 6: Link 2-Link 5-Link 6 –Link 8- Link 11;

Path 7: Link 2-Link 5-Link 6 –Link 9- Link 11;

Path 8: Link 2-Link 5-Link 7 –Link 8- Link 11;

Path 9: Link 2-Link 5-Link 7 –Link 9- Link 11;

Path 10: Link 3-Link 4-Link 6 –Link 8- Link 11;

Path 11: Link 3-Link 4-Link 6 –Link 9- Link 11;

Path 12: Link 3-Link 4-Link 7 –Link 8- Link 11;

Path 13: Link 3-Link 4-Link 7 –Link 9- Link 11;

Path 14: Link 3-Link 5-Link 6 –Link 8- Link 11;

Path 15: Link 3-Link 5-Link 6 –Link 9- Link 11;

Path 16: Link 3-Link 5-Link 7 –Link 8- Link 11;

Path 17: Link 3-Link 5-Link 7 –Link 9- Link 11.

The user equilibrium link flows before adding a path is $\mathbf{f} = \begin{bmatrix}12 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 12 & 12\end{bmatrix}^T$, the modified model in section 4.3 is applied to clarify whether a paradox may occur.

First, the algorithm developed by Cho and Lo is applied to obtain a column vector **k**. List all corresponding incidence matrices and express the equilibrium network form in Figure 4.4-1:

$$\mathbf{\Delta} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (4.4\text{-}1)$$

$$\mathbf{\Lambda}=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \tag{4.4-2}$$

$$\begin{bmatrix}\mathbf{\Delta}\\ \mathbf{\Lambda}\end{bmatrix}\mathbf{h}=\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}\mathbf{h}$$

$$=\begin{bmatrix}12 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 12 & 12 & 24\end{bmatrix}^{T} \tag{4.4-3}$$

$$\begin{bmatrix}\mathbf{\Delta^0}\\ \mathbf{\Lambda^0}\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}\mathbf{\Delta^0\Delta^{0T}} & \mathbf{\Delta^0\Lambda^{0T}}\\ \mathbf{\Lambda^0\Delta^{0T}} & \mathbf{\Lambda^0\Lambda^{0T}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{\Delta^0}\\ \mathbf{\Lambda^0}\end{bmatrix}=\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ b_{n1} & b_{b2} & \cdots & b_{nn}\end{bmatrix}$$

$$=\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{5}{16}&\frac{3}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&-\frac{1}{16}&-\frac{3}{16}\\
0&\frac{3}{16}&\frac{5}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{3}{16}&-\frac{1}{16}\\
0&\frac{3}{16}&\frac{1}{16}&\frac{5}{16}&\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&-\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}\\
0&\frac{1}{16}&\frac{3}{16}&\frac{3}{16}&\frac{5}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{3}{16}&-\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}\\
0&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{5}{16}&\frac{3}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&-\frac{1}{16}&-\frac{3}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}\\
0&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&\frac{5}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}\\
0&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{5}{16}&\frac{3}{16}&-\frac{1}{16}&-\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}\\
0&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&\frac{3}{16}&\frac{5}{16}&-\frac{3}{16}&-\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}\\
0&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&-\frac{1}{16}&-\frac{3}{16}&\frac{5}{16}&\frac{3}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}\\
0&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{3}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{5}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}\\
0&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&-\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{5}{16}&\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}\\
0&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{3}{16}&-\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&\frac{3}{16}&\frac{5}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}\\
0&\frac{1}{16}&-\frac{1}{16}&-\frac{1}{16}&-\frac{3}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{5}{16}&\frac{3}{16}&\frac{3}{16}&\frac{1}{16}\\
0&-\frac{1}{16}&\frac{1}{16}&-\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&\frac{5}{16}&\frac{1}{16}&\frac{3}{16}\\
0&-\frac{1}{16}&-\frac{3}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{5}{16}&\frac{3}{16}\\
0&-\frac{3}{16}&-\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&-\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{3}{16}&\frac{1}{16}&\frac{3}{16}&\frac{3}{16}&\frac{5}{16}
\end{bmatrix} \tag{4.4-4}$$

$$
\begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T = \begin{bmatrix} 1 - b_{11} - b_{12} - \dots - b_{1n} \\ 1 - b_{21} - b_{22} - \dots - b_{2n} \\ \vdots \\ 1 - b_{n1} - b_{n2} - \dots - b_{nn} \end{bmatrix} = [\mathbf{0}]_{17\times 17} \tag{4.4-5}
$$

$$
\begin{bmatrix} \boldsymbol{\Delta}^0 \\ \boldsymbol{\Lambda}^0 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \boldsymbol{\Delta}^0\boldsymbol{\Delta}^{0\mathbf{T}} & \boldsymbol{\Delta}^0\boldsymbol{\Lambda}^{0\mathbf{T}} \\ \boldsymbol{\Lambda}^0\boldsymbol{\Delta}^{0\mathbf{T}} & \boldsymbol{\Lambda}^0\boldsymbol{\Lambda}^{0\mathbf{T}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{T}^0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}^{\mathbf{T}}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
3/16 & 1/8 & 1/8 & 1/8 & 1/8 & -3/16 \\
1/16 & 1/8 & 1/8 & 1/8 & -1/8 & -1/16 \\
1/16 & 1/8 & 1/8 & -1/8 & 1/8 & -1/16 \\
-1/16 & 1/8 & 1/8 & -1/8 & -1/8 & 1/16 \\
1/16 & 1/8 & -1/8 & 1/8 & 1/8 & -1/16 \\
-1/16 & 1/8 & -1/8 & 1/8 & -1/8 & 1/16 \\
-1/16 & 1/8 & -1/8 & -1/8 & 1/8 & 1/16 \\
-3/16 & 1/8 & -1/8 & -1/8 & -1/8 & 3/16 \\
1/16 & -1/8 & 1/8 & 1/8 & 1/8 & -1/16 \\
-1/16 & -1/8 & 1/8 & 1/8 & -1/8 & 1/16 \\
-1/16 & -1/8 & 1/8 & -1/8 & 1/8 & 1/16 \\
-3/16 & -1/8 & 1/8 & -1/8 & -1/8 & 3/16 \\
-1/16 & -1/8 & -1/8 & 1/8 & 1/8 & 1/16 \\
-3/16 & -1/8 & -1/8 & 1/8 & -1/8 & 3/16 \\
-3/16 & -1/8 & -1/8 & -1/8 & 1/8 & 3/16 \\
-5/16 & -1/8 & -1/8 & -1/8 & -1/8 & 5/16
\end{bmatrix} \cdot \begin{bmatrix} 12 \\ 6 \\ 6 \\ 6 \\ 6 \\ 24 \end{bmatrix} = \begin{bmatrix} 12 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix} \tag{4.4-6}
$$

After some manipulation, the outcome shows that vector **k** would not affect the final result. This finding suggests that when obtaining **y**, the existence of vector **k** could be neglected. This outcome would not affect the general case. Therefore, only the values of $\left(\left[\mathbf{M}^0\mathbf{N}^0\right]^T\right)^- \left[-\boldsymbol{\rho}^0 - \mathbf{e}^0\right]^T$ is required; a situation in which a positive change in OD cost occurs implies that the new path may lead to Braess' paradox.

Second, all corresponding incidence matrices are listed as follows:

$$\mathbf{\Delta}_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{4.4-7}$$

$$\mathbf{G} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \tag{4.4-8}$$

So (4.4-1), (4.4-2),(4.4-7) and (4.4-8) are inserted into (4.3-1) as (4.4-9):

$$\begin{bmatrix} 8.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 12 & 11 & 10 & 9 & 10 & 9 & 8 & 7 & 10 & 9 & 8 & 7 & 8 & 7 & 6 & 5 & 1 \\ 0 & 11 & 12 & 9 & 10 & 9 & 10 & 7 & 8 & 9 & 10 & 7 & 8 & 7 & 8 & 5 & 6 & 1 \\ 0 & 10 & 9 & 12 & 11 & 8 & 7 & 10 & 9 & 8 & 7 & 10 & 9 & 6 & 5 & 8 & 7 & 1 \\ 0 & 9 & 10 & 11 & 12 & 7 & 8 & 9 & 10 & 7 & 8 & 9 & 10 & 5 & 6 & 7 & 8 & 1 \\ 0 & 10 & 9 & 8 & 7 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 10 & 9 & 8 & 7 & 1 \\ 0 & 9 & 10 & 7 & 8 & 11 & 12 & 9 & 10 & 7 & 8 & 5 & 6 & 9 & 10 & 7 & 8 & 1 \\ 0 & 8 & 7 & 10 & 9 & 10 & 9 & 12 & 11 & 6 & 5 & 8 & 7 & 8 & 7 & 10 & 9 & 1 \\ 0 & 7 & 8 & 9 & 10 & 9 & 10 & 11 & 12 & 5 & 6 & 7 & 8 & 7 & 8 & 9 & 10 & 1 \\ 0 & 10 & 9 & 8 & 7 & 8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 & 10 & 9 & 8 & 7 & 1 \\ 0 & 9 & 10 & 7 & 8 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 & 9 & 10 & 7 & 8 & 1 \\ 0 & 8 & 7 & 10 & 9 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 8 & 7 & 10 & 9 & 1 \\ 0 & 7 & 8 & 9 & 10 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 7 & 8 & 9 & 10 & 1 \\ 0 & 8 & 7 & 6 & 5 & 10 & 9 & 8 & 7 & 10 & 9 & 8 & 7 & 12 & 11 & 10 & 9 & 1 \\ 0 & 7 & 8 & 5 & 6 & 9 & 10 & 7 & 8 & 9 & 10 & 7 & 8 & 11 & 12 & 9 & 10 & 1 \\ 0 & 6 & 5 & 8 & 7 & 8 & 7 & 10 & 9 & 8 & 7 & 10 & 9 & 10 & 9 & 12 & 11 & 1 \\ 0 & 5 & 6 & 7 & 8 & 7 & 8 & 9 & 10 & 7 & 8 & 9 & 10 & 9 & 10 & 11 & 12 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -1 \end{bmatrix} h_r \tag{4.4-9}$$

The left matrix in (4.4-9) is partitioned to obtain the full row rank matrix as (4.4-10) and its corresponding right matrix as (4.4-11).

$$
\begin{bmatrix} \mathbf{M}^o \\ \mathbf{N}^o \end{bmatrix} =
\begin{bmatrix}
8.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 12 & 11 & 10 & 9 & 10 & 9 & 8 & 7 & 10 & 9 & 8 & 7 & 8 & 7 & 6 & 5 & 1 \\
0 & 10 & 9 & 12 & 11 & 8 & 7 & 10 & 9 & 8 & 7 & 10 & 9 & 6 & 5 & 8 & 7 & 1 \\
0 & 9 & 10 & 11 & 12 & 7 & 8 & 9 & 10 & 7 & 8 & 9 & 10 & 5 & 6 & 7 & 8 & 1 \\
0 & 9 & 10 & 7 & 8 & 11 & 12 & 9 & 10 & 7 & 8 & 5 & 6 & 9 & 10 & 7 & 8 & 1 \\
0 & 8 & 7 & 10 & 9 & 10 & 9 & 12 & 11 & 6 & 5 & 8 & 7 & 8 & 7 & 10 & 9 & 1 \\
0 & 10 & 9 & 8 & 7 & 8 & 7 & 6 & 6 & 12 & 11 & 10 & 9 & 10 & 9 & 8 & 7 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\tag{4.4-10}
$$

$$
\begin{bmatrix} -\boldsymbol{\rho} \\ \hline -\mathbf{e} \end{bmatrix}^o \mathbf{h}_r = [-5-5-5-5-5-5-5-1]^T \mathbf{h}_n \tag{4.4-11}
$$

The final step involves inserting (4.4-10) and (4.4-11) into (4.3-4). Owing to ignorance of the existence of **k**, the terms behind the plus sign do not need to be calculated.

$$
\begin{aligned}
\mathbf{y} &= \begin{bmatrix} \mathbf{M}^o \\ \mathbf{N}^o \end{bmatrix}^- \begin{bmatrix} -\boldsymbol{\rho} \\ \hline -\mathbf{e} \end{bmatrix}^o \mathbf{h}_n \\
&= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1]^T \mathbf{h}_n
\end{aligned}
\tag{4.4-12}
$$

Following some manipulation $\mathbf{y}=[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ {-1}]^T\mathbf{h}_n$ is obtained. Because we assume positive path flows before and after path addition, $\mathbf{h}_n$ is always larger than zero, that is, the last element in vector **y** is less than 0. In this numerical example, it means that adding a new link a_{12} may lead to increase in travelers' cost between OD pair (1, 7). Therefore adding a new link in this example network may result in Braess' paradox. According to network sensitivity analysis, vector **k** would not affect final solutions, thus the algorithm for finding a column vector **k** could be eliminated. Only the form as (4.4-12) is needed to calculate when predicting occurrence of Braess' paradox for one OD pair.

5. Discussions and Conclusions

This study adopted Dafermos and Nagurney's model and modified the specific assumption on the link/path incidence matrix. The rank assumption was relaxed by the generalized inverse matrix approach, thus overcoming restriction on small network. Moreover, the reasonability and usability of the modified model was demonstrated through a numerical example. We also applied generalized inverse approach to Yang (1997)'s work, which is about network sensitivity, modifying feasible link flow space. Applying some mathematical characteristics, we relaxed the rank assumption and cited previous results from sensitivity analysis method to complement shortcomings of the modified model.

Dafermos and Nagurney's model was based on equilibrium network problem, and their model was the first model which could determine occurrence of Braess' paradox. However, the rank assumption reduces application in real world; the model can only be implemented simple networks. Generalized inverse matrix method is used to relax the rank assumption, so it is operable in every network. Otherwise, the arbitrary vector **k** may cause multiple solutions due to properties of generalized inverse matrix method. Here, we applied an algorithm for finding a set of vector **k** which could guarantee positive path flows. Note we do not develop a direct mechanism guaranteeing vector k is independent of results of the model.

In Dafermos and Nagurney 's model, the original link cost was not changed after path addition. This does not suit to real situation. Logically thinking, road investment should affect original network situation. On the other hand, model was formulated from path information, not link. Even if the restriction is removed, there is no idea about how many paths would be produced due to addition of the new link. But their model is suitable for planning process. It provided transportation planners concept about network design: joining an OD pair of a network by a new path which contains none of the original links of the network will result in a decrease in travel cost for users of the OD pair.

Future research may focus on several topics: one is to develop formulas for multiple OD pairs, which is inevitably complicated. How to design a simple and fast algorithm to conquer complexity of the problem is need to be explored. One is to construct sensitivity analysis based on **k** vector. Here we only cite others' argument but not carry out practically. What if k vector calculated from the other positive path flows algorithm is not independent of the original algorithm? This will affect solutions. Therefore we should investigate sensitivity of vector **k**.

Braess' paradox told us sometimes we need to see things in different ways. Every seeming-good decision is not always good; from the other way, it may bring adverse effects. That is why we should do sensitivity analysis or simulation before any investments.

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