附錄　最可能頻率偏移估計的推導

A.1 分割矩陣(Partitioned Matrix)

A 是一個 $N\,x\,M$ 矩陣，此矩陣可以用子矩陣(Submatrices)表示，此矩陣稱為(Partitioned Matrix)，使用分割矩陣的優點是可以把每個子矩陣當成是矩陣元素(Element)來處理。舉例說明如下:

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

其中，

A_{11}為$N_1\,x\,M_1$子矩陣
A_{21}為$(N-N_1)\,x\,M_1$子矩陣
A_{12}為$N_1\,x\,(M-M_1)$子矩陣
A_{22}為$(N-N_1)\,x\,(M-M_1)$子矩陣

如果 A_{11} 及 A_{21} 的反矩陣(Inverse Matrix) A_{11}^{-1} 及 A_{22}^{-1} 存在，A 的反矩陣為 A^{-1} 可表示如下:

$$A^{-1}=\begin{pmatrix} (A_{11}-A_{12}\,A_{22}^{-1}A_{21})^{-1} & -(A_{11}-A_{12}\,A_{22}^{-1}A_{21})^{-1}A_{12}A_{21}^{-1} \\ -(A_{22}-A_{21}{}_1A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22}-A_{21}\,A_{11}^{-1}A_{12})^{-1} \end{pmatrix} \quad \text{(A.1)}$$

當 A_{11} 是一個 $(N-1)\,x\,(N-1)$ 矩陣，A_{12} 是 $(N-1)\,x\,1$ column vector，A_{21} 是 $1x(N-1)$ row vector，A_{22} 是一個 scalar，則式(A.1)可表示如下[17]:

$$A^{-1}=\begin{pmatrix} A_{11}^{-1}+\beta A_{11}^{-1}A_{12}\,A_{21}A_{11}^{-1} & -\beta A_{11}^{-1}A_{12} \\ -\beta A_{21}A_{11}^{-1} & \beta \end{pmatrix} \quad \text{(A.2)}$$

其中, $\beta=(A_{22}-A_{21}A_{11}^{-1}\,A_{12})^{-1}$，式(A.2)對接下來的推導是相當有幫助的一個式子。

A.2 最可能頻率偏移估計

M 個相互複數高斯隨機變數(Jointly Complex Gaussian Random Variables) $\{z_i\}, i=0,1,2\cdots M-1,\ z_i=x_i+jy_i$ 。$\boldsymbol{Z}$ 是一個 $M\,x\,1$ column vector 使得 $\boldsymbol{Z}=[z_i]$，mean為 $\{\langle z_i\rangle\}$　R 是一個 $M\,x\,M$ 共變異矩陣(Covariance Matrix),

$$R=\frac{1}{2}<[\boldsymbol{Z}-<\boldsymbol{Z}>][\boldsymbol{Z}-<\boldsymbol{Z}>]^{H}> \quad \text{(A.3)}$$

其中，$\boldsymbol{Z}^{T}$ 是$\boldsymbol{Z}$的共軛轉置矩陣(Conjugate Transpose of Matrix)，M對實數高斯變數(x_i,y_i)的 joint p.d.f 可以表示如下[13]:

$$p(\{x_i,y_i\})=\frac{1}{(2\pi)^{M}\det \boldsymbol{R}}\exp[-1/2(\boldsymbol{Z}-<\boldsymbol{Z}>)^{H}\boldsymbol{R}^{-1}(\boldsymbol{Z}-<\boldsymbol{Z}>)] \quad \text{(A.4)}$$

$z(n)=e^{-j2\pi n\varepsilon/N}s(n)+w(n)$, $w(n)$為AWGN雜訊

$$E[z_0z_0^{*}]=E[z_1z_1^{*}]=E[z_2z_2^{*}]=....=E[z_{M-1}z_{M-1}^{*}]=\sigma_S^2+\sigma_n^2$$

$$E[z_kz_{k+n}^{*}]=\sigma_s^2e^{j2\pi\varepsilon n/N}$$

$$E[|s(n)|^2]=\sigma_s^2 \text{ , } E[|w(n)|^2]=\sigma_n^2$$

$<\boldsymbol{Z}>$ 為Null Vector

Case1: 向量$\boldsymbol{Z}$只含有 1 個 element

$$R_1=1/2<[\boldsymbol{Z}-<\boldsymbol{Z}>][\boldsymbol{Z}-<\boldsymbol{Z}>]^{H}>$$

$$\boldsymbol{Z}=[z_0]$$

$$R_1=\frac{1}{2}E[z_0^{*}z_0]$$

$$=\frac{1}{2}(\sigma_S^2+\sigma_n^2)$$

$$R^{-1}=\frac{2}{(\sigma_S^2+\sigma_n^2)}$$

$$\Rightarrow -1/2(\boldsymbol{Z}-<\boldsymbol{Z}>)^{H}R_1^{-1}(\boldsymbol{Z}-<\boldsymbol{Z}>)$$

$$=\frac{-|z_0|^2}{(\sigma_S^2+\sigma_n^2)}$$

Case2: 向量$\boldsymbol{Z}$含有 2 個 elements

$$\boldsymbol{Z}=[z_0 \quad z_1]^{T}$$

$$\boldsymbol{R}_2=1/2<[\boldsymbol{Z}-<\boldsymbol{Z}>][\boldsymbol{Z}-<\boldsymbol{Z}>]^{H}>$$

$$=\frac{1}{2}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & \sigma_s^2e^{j2\pi\varepsilon/N}\\ \sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$\boldsymbol{R}_2^{-1} = \frac{1}{2}\frac{1}{\det \boldsymbol{R}_2}\begin{bmatrix} \sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 \end{bmatrix}$$

$$= \frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix} \sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 \end{bmatrix}$$

$$\Rightarrow -1/2(\boldsymbol{Z}-<\boldsymbol{Z}>)^H \boldsymbol{R}_2^{-1}(\boldsymbol{Z}-<\boldsymbol{Z}>)$$

$$= \frac{-1}{4\det \boldsymbol{R}_2}\begin{bmatrix} z_0^* & z_1^* \end{bmatrix}\begin{bmatrix} \sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$$

$$= \frac{-1}{4\det R_2}[(\sigma_S^2+\sigma_n^2)(|z_0|^2+|z_1|^2)-2\sigma_S^2\operatorname{Re}(e^{j2\pi\varepsilon/N}z_0^* z_1)]$$

Case3: 向量 $\boldsymbol{Z}$ 含有 3 個 elements

$$R_3 = 1/2 < [\boldsymbol{Z}-<\boldsymbol{Z}>][\boldsymbol{Z}-<\boldsymbol{Z}>\}^H >$$

$$\boldsymbol{Z} = [z_0\ z_1\ z_2]^T$$

$$\boldsymbol{R}_3 = \frac{1}{2}\begin{bmatrix} \sigma_S^2+\sigma_n^2 & \sigma_s^2 e^{j2\pi\varepsilon/N} & \sigma_s^2 e^{j2\pi 2\varepsilon/N} \\ \sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 & \sigma_s^2 e^{j2\pi\varepsilon/N} \\ \sigma_s^2 e^{-j2\pi 2\varepsilon/N} & \sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 \end{bmatrix}$$

此時可利用式(A.1)與式(A.2)求出 $\boldsymbol{R}_3^{-1}$

$$A_{21}A_{11}^{-1}A_{12}$$

$$= \frac{1}{2}\begin{bmatrix} \sigma_s^2 e^{-j2\pi 2\varepsilon/N} & \sigma_s^2 e^{-j2\pi\varepsilon/N} \end{bmatrix}\frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix} \sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sigma_s^2 e^{j2\pi 2\varepsilon/N} \\ \frac{1}{2}\sigma_s^2 e^{j2\pi\varepsilon/N} \end{bmatrix}$$

$$= \frac{\sigma_S^4}{2\sigma_S^2+\sigma_n^2}$$

$$\beta = \left[\frac{1}{2}(\sigma_S^2+\sigma_n^2)-\frac{\sigma_S^4}{2\sigma_S^2+\sigma_n^2}\right]^{-1}$$

$$= \frac{2(2\sigma_S^2+\sigma_n^2)}{3\sigma_S^2\sigma_n^2+\sigma_n^4}$$

$$-\beta A_{11}^{-1}A_{12}=\frac{-2(2\sigma_S^2+\sigma_n^2)}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\sigma_s^2e^{j2\pi2\varepsilon/N}\\ \frac{1}{2}\sigma_s^2e^{j2\pi\varepsilon/N}\end{bmatrix}$$

$$=\frac{-2\sigma_S^2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}e^{j2\pi2\varepsilon/N}\\ e^{j2\pi\varepsilon/N}\end{bmatrix}$$

$$-\beta A_{21}A_{11}^{-1}=\frac{-2(2\sigma_S^2+\sigma_n^2)}{3\sigma_S^2\sigma_n^2+\sigma_n^4}.\frac{1}{2}\begin{bmatrix}\sigma_s^2e^{-j2\pi2\varepsilon/N} & \sigma_s^2e^{-j2\pi\varepsilon/N}\end{bmatrix}.$$

$$\frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$=\frac{-2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}\sigma_s^2e^{-j2\pi2\varepsilon/N} & \sigma_s^2e^{-j2\pi\varepsilon/N}\end{bmatrix}$$

$$\beta A_{11}^{-1}A_{12}A_{21}A_{11}^{-1}$$

$$=\frac{2(2\sigma_S^2+\sigma_n^2)}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\sigma_s^2e^{j2\pi2\varepsilon/N}\\ \frac{1}{2}\sigma_s^2e^{j2\pi\varepsilon/N}\end{bmatrix}.$$

$$\frac{1}{2}\begin{bmatrix}\sigma_s^2e^{-j2\pi2\varepsilon/N} & \sigma_s^2e^{-j2\pi\varepsilon/N}\end{bmatrix}\frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$=\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{1}{2\sigma_S^2+\sigma_n^2}\cdot\sigma_S^4\cdot\begin{bmatrix}1 & e^{j2\pi\varepsilon/N}\\ e^{-j2\pi\varepsilon/N} & 1\end{bmatrix}$$

$$A_{11}^{-1}+\beta A_{11}^{-1}A_{12}A_{21}A_{11}^{-1}$$

$$=\frac{2}{2\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2\end{bmatrix}+$$

$$\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{1}{2\sigma_S^2+\sigma_n^2}\cdot\sigma_S^4\cdot\begin{bmatrix}1 & e^{j2\pi\varepsilon/N}\\ e^{-j2\pi\varepsilon/N} & 1\end{bmatrix}$$

$$=\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}2\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$\Rightarrow \boldsymbol{R}_3^{-1}=\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}2\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N} & -\sigma_s^2e^{j2\pi2\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi2\varepsilon/N} & -\sigma_s^2e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2\end{bmatrix}$$

Case4: 向量 **Z** 含有 4 個 elements

$$R_4 = 1/2 < [\boldsymbol{Z} - < \boldsymbol{Z} >][\boldsymbol{Z} - < \boldsymbol{Z} >]^H >$$

$$\boldsymbol{Z} = [z_0 \ \ z_1 \ z_2 \ \ z_3]^T$$

$$\boldsymbol{R}_4 = \frac{1}{2}\begin{bmatrix} \sigma_S^2 + \sigma_n^2 & \sigma_s^2 e^{j2\pi\varepsilon/N} & \sigma_s^2 e^{j2\pi 2\varepsilon/N} & \sigma_s^2 e^{j2\pi 3\varepsilon/N} \\ \sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2 + \sigma_n^2 & \sigma_s^2 e^{j2\pi\varepsilon/N} & \sigma_s^2 e^{j2\pi 2\varepsilon/N} \\ \sigma_s^2 e^{-j2\pi 2\varepsilon/N} & \sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2 + \sigma_n^2 & \sigma_s^2 e^{j2\pi\varepsilon/N} \\ \sigma_s^2 e^{-j2\pi 3\varepsilon/N} & \sigma_s^2 e^{-j2\pi 2\varepsilon/N} & \sigma_s^2 e^{-j2\pi\varepsilon/N} & \sigma_S^2 + \sigma_n^2 \end{bmatrix}$$

$$A_{21} A_{11}^{-1} A_{12}$$

$$= \frac{1}{2}\sigma_S^2 \begin{bmatrix} e^{-j2\pi 3\varepsilon/N} & e^{-j2\pi 2\varepsilon/N} & e^{-j2\pi\varepsilon/N} \end{bmatrix} \cdot$$

$$\frac{2}{3\sigma_S^2\sigma_n^2 + \sigma_n^4}\begin{bmatrix} 2\sigma_S^2 + \sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} & -\sigma_s^2 e^{j2\pi 2\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2 + \sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi 2\varepsilon/N} & -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2 + \sigma_n^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2}\sigma_s^2 e^{j2\pi 3\varepsilon/N} \\ \frac{1}{2}\sigma_s^2 e^{j2\pi 2\varepsilon/N} \\ \frac{1}{2}\sigma_s^2 e^{j2\pi\varepsilon/N} \end{bmatrix}$$

$$= \frac{1}{2}\frac{3\sigma_S^4}{(3\sigma_S^2 + \sigma_n^2)}$$

$$\beta = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

$$= 2\left[(\sigma_S^2 + \sigma_n^2) - \frac{3\sigma_S^4}{3\sigma_S^2 + \sigma_n^2}\right]^{-1}$$

$$= \frac{2(3\sigma_S^2 + \sigma_n^2)}{4\sigma_S^2\sigma_n^2 + \sigma_n^4}$$

$$-\beta A_{21} A_{11}^{-1}$$

$$= \frac{-2(3\sigma_S^2 + \sigma_n^2)}{4\sigma_S^2\sigma_n^2 + \sigma_n^4} \cdot \frac{1}{2}\sigma_S^2 \begin{bmatrix} e^{-j2\pi 3\varepsilon/N} & e^{-j2\pi 2\varepsilon/N} & e^{-j2\pi\varepsilon/N} \end{bmatrix}.$$

$$\frac{2}{3\sigma_S^2\sigma_n^2 + \sigma_n^4}\begin{bmatrix} 2\sigma_S^2 + \sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} & -\sigma_s^2 e^{j2\pi 2\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2 + \sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi 2\varepsilon/N} & -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2 + \sigma_n^2 \end{bmatrix}$$

$$= \frac{-2\sigma_S^2}{4\sigma_S^2\sigma_n^2 + \sigma_n^4}\begin{bmatrix} e^{-j2\pi 3\varepsilon/N} & e^{-j2\pi 2\varepsilon/N} & e^{-j2\pi\varepsilon/N} \end{bmatrix}$$

.

$-\beta A_{11}^{-1}A_{12}$

$$= -\beta \cdot \frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot \begin{bmatrix} 2\sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} & -\sigma_s^2 e^{j2\pi2\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N}\\ -\sigma_s^2 e^{-j2\pi2\varepsilon/N} & -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2\end{bmatrix}\cdot \begin{bmatrix}\frac{1}{2}\sigma_s^2 e^{j2\pi3\varepsilon/N}\\ \frac{1}{2} \sigma_s^2 e^{j2\pi2\varepsilon/N}\\ \frac{1}{2}\sigma_s^2 e^{j2\pi\varepsilon/N}\end{bmatrix}$$

$$= -\frac{2(3\sigma_S^2+\sigma_n^2)}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot(\frac{1}{2}\sigma_s^2\sigma_n^2)\begin{bmatrix}e^{j2\pi3\varepsilon/N}\\ e^{j2\pi2\varepsilon/N}\\ e^{j2\pi\varepsilon/N}\end{bmatrix}$$

$$= \frac{-2\sigma_S^2}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\begin{bmatrix}e^{j2\pi3\varepsilon/N}\\ e^{j2\pi2\varepsilon/N}\\ e^{j2\pi\varepsilon/N}\end{bmatrix}$$

$\beta A_{11}^{-1}A_{12}\,A_{21}A_{11}^{-1}$

$$= \frac{2\sigma_S^2}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\begin{bmatrix}e^{j2\pi3\varepsilon/N}\\ e^{j2\pi2\varepsilon/N}\\ e^{j2\pi\varepsilon/N}\end{bmatrix}\cdot\frac{\sigma_S^2}{2}\begin{bmatrix}e^{-j2\pi3\varepsilon/N} & e^{-j2\pi2\varepsilon/N} & e^{-j2\pi\varepsilon/N}\end{bmatrix}\cdot$$

$$\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\begin{bmatrix} 2\sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} & -\sigma_s^2 e^{j2\pi2\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N}\\ -\sigma_s^2 e^{-j2\pi2\varepsilon/N} & -\sigma_s^2 e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$= \frac{2\sigma_S^4}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{1}{3\sigma_S^2+\sigma_n^2}\begin{bmatrix}1 & e^{j2\pi\varepsilon/N} & e^{j2\pi2\varepsilon/N}\\ e^{-j2\pi\varepsilon/N} & 1 & e^{j2\pi\varepsilon/N}\\ e^{-j2\pi2\varepsilon/N} & e^{-j2\pi\varepsilon/N} & 1\end{bmatrix}$$

$$A_{11}^{-1}+\beta A_{11}^{-1}A_{12}A_{21}A_{11}^{-1}$$

$$=\frac{2}{3\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}2\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N} & -\sigma_s^2e^{j2\pi2\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi2\varepsilon/N} & -\sigma_s^2e^{-j2\pi\varepsilon/N} & 2\sigma_S^2+\sigma_n^2\end{bmatrix}+$$

$$\frac{2\sigma_S^4}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\cdot\frac{1}{3\sigma_S^2+\sigma_n^2}\begin{bmatrix}1 & e^{j2\pi\varepsilon/N} & e^{j2\pi2\varepsilon/N}\\ e^{-j2\pi\varepsilon/N} & 1 & e^{j2\pi\varepsilon/N}\\ e^{-j2\pi2\varepsilon/N} & e^{-j2\pi\varepsilon/N} & 1\end{bmatrix}$$

$$=\frac{2}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}3\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N} & -\sigma_s^2e^{j2\pi2\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & 3\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi2\varepsilon/N} & -\sigma_s^2e^{-j2\pi\varepsilon/N} & 3\sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$\Rightarrow \boldsymbol{R}_4^{-1}=\frac{2}{4\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}3\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N} & -\sigma_s^2e^{j2\pi2\varepsilon/N} & -\sigma_s^2e^{j2\pi3\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & 3\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N} & -\sigma_s^2e^{j2\pi2\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi2\varepsilon/N} & -\sigma_s^2e^{-j2\pi\varepsilon/N} & 3\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi3\varepsilon/N} & -\sigma_s^2e^{-j2\pi2\varepsilon/N} & -\sigma_s^2e^{-j2\pi\varepsilon/N} & 3\sigma_S^2+\sigma_n^2\end{bmatrix}$$

由$\boldsymbol{R}_1^{-1},\boldsymbol{R}_2^{-1},\boldsymbol{R}_3^{-1}$及$\boldsymbol{R}_4^{-1}$可歸納出任意$R_k^{-1}$的規則性，

$$\boldsymbol{R}_k^{-1}=\frac{2}{k\sigma_S^2\sigma_n^2+\sigma_n^4}\begin{bmatrix}(k-1)\sigma_S^2+\sigma_n^2 & -\sigma_s^2e^{j2\pi\varepsilon/N} & \cdots & -\sigma_s^2e^{j2\pi(k-1)\varepsilon/N}\\ -\sigma_s^2e^{-j2\pi\varepsilon/N} & (k-1)\sigma_S^2+\sigma_n^2 & \cdots & -\sigma_s^2e^{j2\pi(k-2)\varepsilon/N}\\ \vdots & \vdots & \ddots & \vdots\\ -\sigma_s^2e^{-j2\pi(k-1)\varepsilon/N} & -\sigma_s^2e^{-j2\pi(k-2)\varepsilon/N} & \cdots & (k-1)\sigma_S^2+\sigma_n^2\end{bmatrix}$$

$$R_k=\frac{1}{2}\begin{bmatrix}\sigma_S^2+\sigma_n^2 & \sigma_s^2e^{j2\pi\varepsilon/N} & \cdots & \sigma_s^2e^{j2\pi(k-1)\varepsilon/N}\\ \sigma_s^2e^{-j2\pi\varepsilon/N} & \sigma_S^2+\sigma_n^2 & \cdots & -\sigma_s^2e^{j2\pi(k-2)\varepsilon/N}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_s^2e^{-j2\pi(k-1)\varepsilon/N} & \sigma_s^2e^{-j2\pi(k-2)\varepsilon/N} & \cdots & \sigma_S^2+\sigma_n^2\end{bmatrix}$$

$\boldsymbol{R}_k.\boldsymbol{R}_k^{-1}=\boldsymbol{I}_k$ ，所以$\boldsymbol{R}_k$與$\boldsymbol{R}_k^{-1}$的推導是正確的。

$$A = \frac{-1}{2}\boldsymbol{Z}^H \boldsymbol{R}_k^{-1}\boldsymbol{Z}$$

$$= \frac{-1}{2}[z_0\ z_1\ z_2 \cdots z_{k-1}]^* \cdot$$

$$\frac{2}{k\sigma_s^2\sigma_n^2+\sigma_n^4}\begin{bmatrix} (k-1)\sigma_s^2+\sigma_n^2 & -\sigma_s^2 e^{j2\pi\varepsilon/N} & \cdots & -\sigma_s^2 e^{j2\pi(k-1)\varepsilon/N} \\ -\sigma_s^2 e^{-j2\pi\varepsilon/N} & (k-1)\sigma_s^2+\sigma_n^2 & \cdots & -\sigma_s^2 e^{j2\pi(k-2)\varepsilon/N} \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_s^2 e^{-j2\pi(k-1)\varepsilon/N} & -\sigma_s^2 e^{-j2\pi(k-2)\varepsilon/N} & \cdots & (k-1)\sigma_s^2+\sigma_n^2 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{k-1}\end{bmatrix}$$

$$= \frac{-1}{k\sigma_s^2\sigma_n^2+\sigma_n^4}\Bigg[\sum_{i=0}^{k-1}|z_i|^2.\left((k-1)\sigma_s^2+\sigma_n^2\right)-2\sigma_s^2\,\mathrm{Re}\left\{e^{j2\pi\varepsilon/N}\sum_{i=0}^{k-2}z_i^* z_{i+1}\right\}$$
$$-2\sigma_s^2\,\mathrm{Re}\left\{e^{j2\pi 2\varepsilon/N}\sum_{i=0}^{k-3}z_i^* z_{i+2}\right\}-2\sigma_s^2\,\mathrm{Re}\left\{e^{j2\pi 3\varepsilon/N}\sum_{i=0}^{k-4}z_i^* z_{i+3}\right\}\cdots\Bigg]$$

$$= \frac{-1}{k\sigma_s^2\sigma_n^2+\sigma_n^4}\left[\sum_{i=0}^{k-1}|z_i|^2.\left((k-1)\sigma_s^2+\sigma_n^2\right)-2\sigma_s^2\sum_{m=1}^{k-1}\mathrm{Re}\left\{e^{j2\pi mD\varepsilon/N}\sum_{i=0}^{k-m-1}z_i^* z_{i+m}\right\}\right]$$

$$= \frac{1}{k\sigma_s^2\sigma_n^2+\sigma_n^4}2\sigma_s^2\sum_{m=1}^{k-1}\mathrm{Re}\left\{e^{j2\pi mD\varepsilon/N}\sum_{i=0}^{k-m-1}z_i^* z_{i+m}\right\}_i - \underbrace{\frac{1}{k\sigma_s^2\sigma_n^2+\sigma_n^4}\sum_{i=0}^{k-1}|z_i|^2.\left((k-1)\sigma_s^2+\sigma_n^2\right)}_{independent\ of\ \varepsilon}$$

$$\Rightarrow \hat{\varepsilon} = \max_{\varepsilon}\ \arg\left\{\sum_{m=1}^{k-1}\mathrm{Re}\left\{e^{j2\pi mD\varepsilon/N}\sum_{i=0}^{k-m-1}z_i^* z_{i+m}\right\}\right\}$$