國立交通大學

資訊科學與工程研究所

碩 士 論 文

更多關於 Magnus-Derek Game 的研究

More on the Magnus-Derek Game

研 究 生：林進之

指導教授：蔡錫鈞 教授

中 華 民 國 九 十 七 年 六 月

More on the Magnus-Derek Game

Jinn-Jy Lin

March 15, 2010

Abstract

We consider the so called Magnus-Derek game, which is a two-person game played on a round table with n positions. The two players are called Magnus and Derek. Initially there is a token placed at position 0. In each round Magnus chooses a positive integer $m \leq n/2$ as the distance of the targeted position from his current position for the token to move, and Derek decides a direction, clockwise or counterclockwise, to move the token. The goal of Magnus is to maximize the total number of positions visited, while Derek's is to minimize this number. If both players play optimally, we prove that Magnus, the maximizer, can achieve his goal in $O(n)$ rounds, which improves a previous result with $O(n \log n)$ rounds. Then we consider a modified version of Magnus-Derek game, where one of the players reveals his moves in advance and the other player plays optimally. In this case we prove that Derek has an $O(n^3)$ algorithm to achieve his goal if Magnus reveals his moves in advance. On the other hand, Magnus has an $O(n)$ algorithm. We also consider the circumstance that both players play randomly, and we show that the expected time to visit all positions is $O(n \log n)$.

Acknowledgements

I want to give my thanks to my advisor, Dr. Shi-Chun Tsai, for his guidance and encouragement. I thank my family for their spiritual support.

Also, I appreciate the support from all members in CCIS lab. They really help me a lot during my graduate study.

Specially, I am thankful to the accompany of Tom Wang, Pei-Chi Tsai, Mon-Jang Hsieh, Prof. Ting-Lu Huang, Fr. Jaime Valenciano, Fr. Rolly Sia, and all members in Catholic Student Association. They are all my best friends.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Magnus-Derek game was first introduced by Nedev and Muthukrishnan [5]. The game is played on a round table with n positions and a token is placed at position 0 initially. For convenience, we label the positions with elements in $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, clockwise consecutively. Suppose the current position is i. In a round, Magnus chooses a positive integer m, where $m \leq \frac{n}{2}$ for the token to move, and Derek choose a direction, either $+1$ (clockwise) or -1 (counterclockwise) for the token to move. Then the token is moved to position $(i + m) \mod n$ or $(i - m) \mod n$ according to Derek's decision. In the game, Magnus wants to visit as many positions as possible, while Derek wants to minimize the number of positions visited. This game can be used to model a mobile agent for distributed computing and network maintenance task. We refer to [5] for more related references.

Nedev [4], and Nedev and Muthukrishnan [5] showed that Magnus could visit all positions in $n-1$ rounds if $n = 2^k$ for some nonnegative integer k, and for other cases, Magnus could visit $f^*(n) = (p-1)n/p$ positions, where p is the smallest odd prime factor for n. The number of rounds needed for these cases are listed as follows:

- If $n = 2^k p$, where p is a prime and k is a nonnegative integer, then Magnus needs $O(p^2 + n)$ rounds.
- If n is a prime, then Magnus needs $O(n^2)$ rounds.
- Otherwise, Magnus needs $O(\frac{n^2}{p})$ rounds, where p is the smallest odd prime factor

for n.

Later, Hurkens et. al. [1] reduced the bound down to $O(n \log n)$ rounds and showed that Derek could always limit the number of visited positions to $f^*(n) = (p-1)n/p$. In this paper, we improve the bound on the rounds further to $O(n)$.

Consider the situation in a ring network, Derek plays the role of an adversary and tries to reduce the visited positions in order to perform some malicious acts in the network, and Magnus plays the role of an agent in the network and tries to visit as many positions as possible to prevent malicious acts. We can modify the game in two ways: (1) Magnus predetermines a sequence of magnitudes, and Derek tries to design appropriate responses to minimize the number of positions that Magnus can visit. This is an open problem asked by Nedev and Muthukrishnan [5]. (2) Derek predetermines a sequence of directions, and Magnus tries to design appropriate response to visit as many positions as possible. In the first case, we provide an $O(n^3)$ time algorithm for Derek to minimize the number of positions that Magnus can visit and answer the above mentioned open question. For the second case, we provide an $O(n)$ time algorithm for Magnus to visit all of the positions. Furthermore, we consider the case that both players play randomly, that is, they choose their moves in every round uniformly at random. In this case, both players have no effective strategy and just adopt the random strategy. This is somewhat like performing a random walk on the n positions. We show that the expected number of rounds to visit all of the n positions is $O(n \log n)$, which is similar to the Coupon collection problem[3].

Throughout this paper, we assume that both players know the factors of n and all of the arithmetic operations are under $\mathbb{Z}_n$ unless stated otherwise. We organize the rest of the paper as follows. In chapter 2, we prove that Magnus can visit the maximum number of possible positions in $O(n)$ rounds. In chapter 3 and 4, we prove how a player can achieve the best possible result when he knows his rival's moves beforehand. In chapter 5, we consider both players play randomly.

Chapter 2

Visit $f^*(n)$ positions in $O(n)$ rounds

In this section we give a new strategy for Magnus to visit $f^*(n)$ positions in $O(n)$ rounds. Previous results show that when n is prime this problem can be the hardest. For this case, Nedev and Muthukrishnan [5] showed that Magnus could visit $f^*(n)$ positions in $O(n^2)$ rounds. Hurkens et. al. [1] reduced it to $O(n \log n)$ rounds. We show that Magnus only needs $O(n)$ rounds to visit $f^*(n)$ positions. We adopt the idea of Hurkens et. al. with some modification to obtain a better bound. We first focus on the case when n is an odd prime and then extend it for general n.

Let A and B be two subsets of $\mathbb{Z}_n$, and define $A + B = \{a + b \mid a \in A, b \in B\}$.

Definition 1. *Let $n \geq 3$ be an odd integer. For any two elements $a, b \in \mathbb{Z}_n$, the midpoint of a and b, denoted as $Mid(a, b)$, is $(a + b)/2$ if $a + b$ is even; else $(a + b + n)/2$. If S is a subset of $\mathbb{Z}_n$, define MID(S)=$\{Mid(a, b) \mid a, b \in S\}$, SUM(S)=$\{a + b \mid a, b \in S\}$ and $SUM^k(S) = \{a + b \mid a, b \in SUM^{k-1}(S)\}$*

By the definition we have the following immediate fact.

Fact 1. *If S is a proper subset of $\mathbb{Z}_n$ and SUM(S)=$\mathbb{Z}_n$, then any $x \in \mathbb{Z}_n$ is the midpoint of some elements $a, b \in S$, i.e., $x = Mid(a, b)$.*

The following theorem is a very useful tool in our proofs.

Theorem 1. *(Cauchy-Davenport [2]) If p is a prime, and A, B are two non-empty subsets of $\mathbb{Z}_p$, then*

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Now we are ready to prove our result.

Lemma 1. *Assume S_0 is a subset of $\mathbb{Z}_n$ and $\lceil \frac{n}{2^{k-1}} \rceil \geq |S_0| > \lceil \frac{n}{2^k} \rceil$ for some k, where n is a prime and $1 \leq k \leq \log n$. Let S_i=SUM(S_{i-1}) for $i \geq 1$. Then $S_k = SUM^k(S_0) = \mathbb{Z}_n$.*

Proof. We prove the lemma by induction on k, where k satisfies $\lceil \frac{n}{2^{k-1}} \rceil \geq |S_0| > \lceil \frac{n}{2^k} \rceil$.
Basis: When $k = 1$, we have $|S_0| > \lceil \frac{n}{2} \rceil$. By Theorem 1, we have $|S_0 + S_0| \geq \min\{n, |S_0| + |S_0| - 1\} \geq n$, so S_1=$SUM(S_0)$=$S_0 + S_0 = \mathbb{Z}_n$.
Inductive Step: Assume the lemma is true for $k = m - 1$, that is, if $S_0 > \lceil \frac{n}{2^k} \rceil = \lceil \frac{n}{2^{m-1}} \rceil$, then $S_{m-1} = \mathbb{Z}_n$. Now we consider the case for $k = m$. We have $|S_0| > \lceil \frac{n}{2^m} \rceil$, which implies $|S_0| \geq \lceil \frac{n}{2^m} \rceil + 1$. Then $|S_1|$= $|SUM(S_0)|$. If $S_1 = \mathbb{Z}_n$, then we are done. Suppose not. By Theorem 1, we have $|S_1| \geq 2|S_0| - 1 \geq 2(\lceil \frac{n}{2^m} \rceil + 1) - 1 = 2\lceil \frac{n}{2^m} \rceil + 1 > \lceil \frac{n}{2^{m-1}} \rceil$. By the induction hypothesis, we have $S_m = SUM^{m-1}(S_1) = \mathbb{Z}_n$. Thus, it holds for the case $k = m$. □

Theorem 2. *If n is a prime, then Magnus can visit $f^*(n) = n - 1$ positions in $2n$ rounds.*

Proof. Let C_0 be the set of unvisited positions, which is $\mathbb{Z}_n$ initially. By Lemma 1, we know $SUM(C_0)$=$\mathbb{Z}_n$ if $|C_0| > n/2$. By Fact 1, it implies that any position can be the middle point of 2 unvisited positions in C_0. Thus, as long as $|C_0| > n/2$, Magnus can visit a new position in each round.

In general for $\lceil \frac{n}{2^{k-1}} \rceil \geq |C_0| > \lceil \frac{n}{2^k} \rceil$, $1 \leq k \leq \log n$, we claim that Magnus can visit a new position in C_0 in every k rounds. The theorem follows by the claim, since

$$\sum_{k=1}^{\log n} k \lfloor \frac{n}{2^k} \rfloor \leq 2n.$$

We have shown the basis case (k=1) of the claim. Now assume the claim holds up to $k - 1$. Now consider the case when $\lceil \frac{n}{2^{k-1}} \rceil \geq |C_0| > \lceil \frac{n}{2^k} \rceil$. Let C_1 =$MID(C_0)$. Note that $|C_1| = |C_0 + C_0| > \lceil \frac{n}{2^{k-1}} \rceil$. It is clear that $SUM^{k-1}(C_1)$=$\mathbb{Z}_n$, by Lemma 1. By induction hypothesis, we know Magnus can visit a new position in C_1 in every $k - 1$ rounds. Then from a position in C_1, Magnus can visit a new position in C_0 in another round, since every element in C_1 is the middle point of two elements $a, b \in C_0$, where if $a = b$, then $a, b \in C_1$, which implies Magnus may visit a new position in C_0 in at most k rounds. This completes the proof of the claim. The remaining one unvisited position is not reachable for Magnus when Derek plays optimally. Thus the theorem holds. □

As in [5], we use $C(l,d,s) = \{s + i \cdot d \mid 0 \le i < l\}$ to denote a set of l positions starting from s and the distance between each pair of adjacent positions in the set is d.

Suppose that $n = mp$ is an odd positive integer and p is the smallest prime factor of n. Let $C_j = C(m,p,j) \subset \mathbb{Z}_n$, $j \in \mathbb{Z}_p$. We have the following general property.

Lemma 2. *Let $S_0 = C_i \cup R$ for some $i \in \mathbb{Z}_p$, where $R \subset \mathbb{Z}_n$ and $R \cap C_i = \emptyset$, and $S_i = SUM(S_{i-1})$ for $i \ge 1$. If $\lceil \frac{p}{2^{k-1}} \rceil \ge l > \lceil \frac{p}{2^k} \rceil$ for some k, where $1 \le k \le \log p$ and l is the number of C_j, $j \ne i$, intersecting with R, then $S_{k+1} = \mathbb{Z}_n$.*

Proof. For convenience, let $\mathcal{C}$ be the collection $\{C_j | j \ne i, C_j \cap R \ne \emptyset\}$ and $|\mathcal{C}| = l$. Let $S' = \{j \mid C_j \in \mathcal{C}\}$. By Lemma 1, we have $SUM^k(S') = \mathbb{Z}_p$. Note that $\{a\} + C_i = C_{(a+i) \bmod p} \subseteq SUM(\{a\} \cup C_i)$. $SUM^k(S') = \mathbb{Z}_p$ implies that $\mathbb{Z}_n \subseteq SUM^{k+1}(S_0)$. Thus $S_{k+1}=SUM^{k+1}(S_0)=\mathbb{Z}_n$. □

Let u be an odd integer. Hurkens et. al. [1] (Lemma 3.2) proved that: *if Magnus has a strategy to visit $f^*(u)$ positions in $g(u)$ rounds, then, for any integer n with u as its largest odd factor, Magnus has a strategy to visit $f^*(n)$ positions in $g(u) + n - u$ rounds.* Thus to prove a linear upper bound on the round number, it suffices to focus on odd integers.

Theorem 3. *Let $n = mp$ be an odd integer, where p is the smallest prime factor of n. Then there is a strategy for Magnus to visit $f^*(n) = (p-1)n/p$ positions in at most $3n$ rounds.*

Proof. Let $C_i = C(n/p, p, i), i \in \mathbb{Z}_p$, and S_0 be the unvisited positions, which is $\mathbb{Z}_n$ initially. Note that when Derek plays optimally, he can always keep one of C_i's, say C_0, from Magnus' visiting[5]. By Lemma 2, we know $SUM^{k+1}(S_0) = \mathbb{Z}_n$ as long as S_0 intersects with t C_i's other than C_0 and $\lceil \frac{p}{2^{k-1}} \rceil \ge t > \lceil \frac{p}{2^k} \rceil$. As in the proof of Theorem 2, it implies Magnus can visit a new position in S_0 within $k+1$ rounds. The smaller the t is, the more rounds Magnus needs to visit a new position. The best strategy for Derek is to force Magnus to visit C_i one after another in order to make t smaller.

Therefore, it takes at most

$$\sum_{k=1}^{\log p} (k+1) \lfloor \frac{p}{2^k} \rfloor (n/p) \le \sum_k (k+1)(\frac{n}{2^k}) \le 3n \text{ rounds.}$$

□

Chapter 3

When Derek knows the moves of Magnus

In this section we consider a variant of the game, where Magnus reveals all of his moves $m_1, m_2, \ldots, m_r$ to Derek. The goal of Derek is to design a sequence of directions $d_1, d_2, \ldots, d_r$ such that the number of positions Magnus can visit is minimal. To derive the algorithm, we define the predicate $T[i,j,k,d]$ as follows.

Definition 2. *$T[i,j,k,d]$ is a predicate indicating whether the following conditions can be satisfied simultaneously: (1) At the end of round i, the token can be moved to position j; (2) During the i rounds, k distinct positions are visited; (3) The choice of direction in round i is d, which is either +1 (clockwise) or -1 (counterclockwise).*

For each $T[i,j,k,d]$, we need an n-bit 0-1 vector to record the positions visited.

Definition 3. *$B[i,j,k,d]$ is an n-bit vector $(b_0, b_1, \ldots, b_{n-1})$, which records the k visited positions in the first i rounds when $T[i,j,k,d]$ is true. If position p has been visited, then $b_p = 1$. If $T[i,j,k,d]$ is false, then $b_p = 0$, for all $p \in \mathbb{Z}_n$.*

Note that b_k must be 1 when $T[i,j,k,d]$ is true. For convenience, sometimes we use $B[i,j,k,d]_l$ to indicate b_l. Since we want to minimize the number of visited positions in r rounds, the optimal solution must be the minimal k such that $T[r,j,k,d]$ is true for some j, d.

But how to compute $T[i,j,k,d]$? Since we start the game at position 0, we have $T[0,0,1,1] = T[0,0,1,-1] =$ true, and $B[0,0,1,1]_0 = B[0,0,1,-1]_0 = 1$. Moreover, $T[1,m_1,2,1]$ and $T[1,n-m_1,2,-1]$ are also true, because after round 1, the token is either moved to position m_1 or $n-m_1$ and there are two positions visited. It is also clear that if $k > i+1$, then $T[i,j,k,d]$ must be false, since at most $i+1$ different positions can be visited in i rounds. Initially, set $T[i,j,k,d] = \perp$ and $B[i,j,k,d] = \vec{0}$ for all i,j,k,d, except $T[0,0,1,1] = T[0,0,1,-1] = true$; $B[0,0,1,1]_0 = B[0,0,1,-1]_0 = 1$. Observe that if $T[i,j,k,1]$ is true, then at least one of $T[i-1,j-m_i,k,1]$, $T[i-1,j-m_i,k,-1]$, $T[i-1,j-m_i,k-1,1]$ and $T[i-1,j-m_i,k-1,-1]$ must be true. In general, we have the following recursive formula:

$$\begin{aligned}
T[i,j,k,1] &= T[i-1,j-m_i,k,1] \vee T[i-1,j-m_i,k,-1] \\
&\quad \vee T[i-1,j-m_i,k-1,1] \vee T[i-1,j-m_i,k-1,-1]. \\
T[i,j,k,-1] &= T[i-1,j+m_i,k,1] \vee T[i-1,j+m_i,k,-1] \\
&\quad \vee T[i-1,j+m_i,k-1,1] \vee T[i-1,j+m_i,k-1,-1].
\end{aligned}$$

More concisely, we have

$$\begin{aligned}
T[i,j,k,d] &= T[i-1,j-dm_i,k,1] \vee T[i-1,j-dm_i,k,-1] \\
&\quad \vee T[i-1,j-dm_i,k-1,1] \vee T[i-1,j-dm_i,k-1,-1].
\end{aligned}$$

Let max be the maximum number of different positions that Magnus can visit by revealing r moves in advance, then max is the smallest k such that $T[r,j,k,d]$ is true for some j and d, i.e.,

$$max = \min \ \{k \mid \exists j \exists d, T[r,j,k,d] = true\}.$$

By the recursive formula, we give an algorithm for Derek to check $T[r,j,k,d]$. We can use it to find max and the directions that lead to the minimum number of positions. Given r and $m_1,\ldots,m_r$, we can query $T[r,j,k,d]$ starting from $k=2$. For each k, we try all possible j, d and check if $T[r,j,k,d]$ is true or not. Return the k, which is the max, when $T[r,j,k,d]$ is true; otherwise repeat the above by increasing k by 1 until $k=r$. Therefore, checking $T[r,j,k,d]$ is crucial and it will be carried out by Algorithm Derek, as shown in Figure 3.1.

Algorithm $\text{Derek}(M, N, I, J, K, D)$

Input: M : array of size I storing Magnus' moves;

N : the number of positions;

I : the number of rounds;

J : the final position of the token after I rounds;

K : the number of visited positions so far, thus $K \le I$;

D : the direction from the previous position to current position;

Output: return whether Magnus stops at position J and visit K positions in I rounds;

for $i \leftarrow 1$ **to** I **do**

  for $j \leftarrow 0$ **to** $N-1$ **do**

    for $k \leftarrow 1$ **to** K **do**

      $T[i,j,k,1] \leftarrow T[i,j,k,-1] \leftarrow \perp$; $B[i,j,k,1] \leftarrow B[i,j,k,-1] \leftarrow \vec{0}$;

for $j \leftarrow 1$ **to** $N-1$ **do**

  for $k \leftarrow 2$ **to** I **do** $T[0,j,k,1] \leftarrow T[0,j,k,-1] \leftarrow false$;

$T[0,0,1,1] \leftarrow T[0,0,1,-1] \leftarrow true$; $B[0,0,1,1]_0 \leftarrow B[0,0,1,-1]_0 \leftarrow 1$;

return $\text{LookUp}(I, J, K, D)$;

Algorithm $\text{LookUp}(i, j, k, d)$

if $(k = 0)$ **then return** $false$;

if $(T[i,j,k,d] \ne \perp)$ **then return** $T[i,j,k,d]$;

if $((\text{LookUp}(i-1, j-dM[i], k, 1) = true)$ **and** $(B[i-1, j-dM[i], k, 1]_k = 1))$

  then $\{B[i,j,k,d] \leftarrow B[i-1, j-dM[i], k, 1]$; **return** $T[i,j,k,d] \leftarrow true;\}$

else if $((\text{LookUp}(i-1, j-dM[i], k, -1) = true)$ **and** $(B[i-1, j-dM[i], k, -1]_k = 1))$

  then $\{B[i,j,k,d] \leftarrow B[i-1, j-dM[i], k, -1]$; **return** $T[i,j,k,d] \leftarrow true;\}$

else if $((\text{LookUp}(i-1, j-dM[i], k-1, 1) = true)$ **and** $(B[i-1, j-dM[i], k, 1]_k = 0))$

  then $\{B[i,j,k,d] \leftarrow B[i-1, j-dM[i], k-1, 1]; b_k \leftarrow 1$; **return** $T[i,j,k,d] \leftarrow true;\}$

else if $((\text{LookUp}(i-1, j-dM[i], k-1, -1) = true)$ **and** $(B[i-1, j-dM[i], k, -1]_k = 0))$

  then $\{B[i,j,k,d] \leftarrow B[i-1, j-dM[i], k-1, -1]; b_k \leftarrow 1$; **return** $T[i,j,k,d] \leftarrow true;\}$

return $T[i,j,k,d] \leftarrow false$;

Figure 3.1: Recursive algorithm for checking $T[i,j,k,d]$.

Algorithm $\textsc{PrintDirection}(i, j, k, d)$

if $((T[i, j, k, d] = \perp)$ **or** $(T[i, j, k, d] = false))$ **then return**;

if $(i = 0)$ **return**;

if $((T[i-1, j-dM[i], k, 1] = true)$ **and** $(B[i-1, j-dM[i], k, 1]_k = 1))$

  $\textsc{PrintDirection}(i-1, j-dM[i], k, 1)$;

else if $((T[i-1, j-dM[i], k, -1] = true)$ **and** $(B[i-1, j-dM[i], k, -1]_k = 1))$

  $\textsc{PrintDirection}(i-1, j-dM[i], k, -1)$;

else if $((T[i-1, j-dM[i], k-1, 1] = true)$ **and** $(B[i-1, j-dM[i], k-1, 1]_k = 0))$

  $\textsc{PrintDirection}(i-1, j-dM[i], k-1, 1)$;

else if $((T[i-1, j-dM[i], k-1, -1] = true)$ **and** $(B[i-1, j-dM[i], k-1, -1]_k = 0))$

  $\textsc{PrintDirection}(i-1, j-dM[i], k-1, -1)$;

Print $(D[i] = d)$;

Figure 3.2: Recursive algorithm for printing the directions.

Theorem 4. *Algorithm* Derek *is correct and has time complexity* $O(n^3)$.

Proof. To compute the value of $T[i, j, k, d]$, we simply apply the corresponding recursive formula. The recursive call terminates when i reaches 0. Thus, the algorithm always terminates and the correctness follows by the recursive formula.

The table for $T[i, j, k, d]$ has $2n^3$ entries. For each entry, it checks at most four possible sub-cases. Hence, the time complexity is $O(n^3)$. □

Chapter 4

When Magnus knows Derek's moves

Here Derek gives all his moves first, and Magnus will try to find a set of magnitudes such that he can visit as many positions as possible. Assume there are n positions on the round table. Let $d_1, d_2, \ldots, d_k$ be the sequence given by Derek. For all $1 \leq i \leq k$, d_i will be either +1 (clockwise) or -1 (counterclockwise). The sequence of magnitudes from Magnus is denoted as $m_1, m_2, \ldots, m_k$. Here we show that Magnus actually has an advantage over Derek. Let $k = n - 1$ and we obtain the following result.

Theorem 5.

(a) If n is even, Magnus can always visit all n positions regardless of $d_1, d_2, \ldots, d_{n-1}$.

(b) If n is odd and Magnus can choose any magnitude in the set $\{1, \ldots, \lceil \frac{n}{2} \rceil\}$, then Magnus can visit all n positions regardless of $d_1, d_2, \ldots, d_{n-1}$.

(c) If n is odd, by examining $d_{\frac{n-5}{2}}, d_{\frac{n-3}{2}}, \ldots, d_{\frac{n+3}{2}}, d_{\frac{n+5}{2}}$ in advance, Magnus can design appropriate response to visit all n positions.

We give two different strategies for even n and odd n, respectively. We show the strategy for the case when n is even first and then the odd case. Actually for odd n, we can prove the case for magnitudes from the set $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. However, the proof is lengthy and we omit it.

4.1 Strategy for even n

We can determine the moves $m_1, m_2, \ldots, m_{n-1}$ by observing the pattern of visited positions.

Definition 4. *We call the visited positions on the round table* k**-balanced**, *if the visited positions consist of two disjoint sets of consecutive positions, i.e.,* $S_0 = \{j, \ldots, j+k-1\}$ *and* $S_1 = \{j+n/2, \ldots, j+n/2+k-1\}$ *for some* k *and* $j \in \mathbb{Z}_n$, *and the token is sitting at one of the four end positions:* $j, j+k-1, j+n/2$ *and* $j+n/2+k-1$.

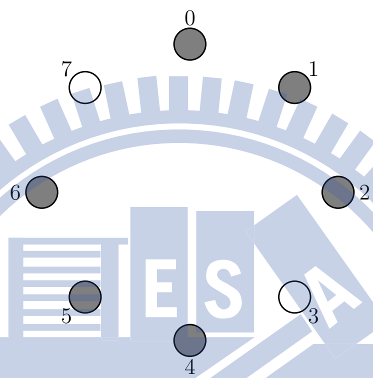

Figure 4.1: An example for 3**-balanced** with $n = 8$, where the gray nodes are visited.

Without loss of generality, we assume position 0 is in S_0. The following is a simple algorithm that determines the magnitudes $m_1, m_2, \ldots, m_{n-1}$. We prove that the algorithm correctly generates the magnitudes with which Magnus can visit all positions.

Let $m_1, m_2, \ldots, m_{n-1}$ be the moves generated by Algorithm Magnus. At round i, if i is odd, then $m_i = n/2$. Otherwise, if the position at ($current\ position + d_i$) is not visited then $m_i = 1$, else $m_i = i/2$. During the even rounds, the set of visited positions holding the token will be extended with a newly visited position. While during the odd rounds, the set of visited positions without the token will be extended with a newly visited position. We prove the correctness of the algorithm with the following lemma.

Lemma 3. *For* $i = 2k-1$, $k \in \mathbb{Z}_n$ *and* $1 \le k \le \frac{n}{2}$, *right after round* i, *the visited positions are in* k**-balanced** *form.*

Algorithm MAGNUS(D, N, M)

Input: D : array of size $N-1$ storing Derek's moves, $D[i] \in \{1, -1\}$;

N : the number of positions;

Output: M : array of size $N-1$ storing Magnus' moves;

$CurrentPos \leftarrow 0$; $Visited[0] \leftarrow true$; /* Initialize the current position */

for $i \leftarrow 1$ **to** $N-1$ **do** $Visited[i] \leftarrow false$; /* Initialize the other positions unvisited */

for $i \leftarrow 1$ **to** $N-1$ **do**

  if (i is odd) **then** $M[i] \leftarrow n/2$;

  else if $((CurrentPos + D[i])$ is not visited) **then** $M[i] \leftarrow 1$;

  else $M[i] \leftarrow i/2$;

  $CurrentPos \leftarrow CurrentPos + D[i] * M[i]$;

  $Visited[CurrentPos] \leftarrow true$;

Figure 4.2: Generate the moves for Magnus with Derek's revealed in advance.

Proof. We prove it by induction on k.

Basis: When $k = 1$, $m_1 = \frac{n}{2}$. After round 1, clearly, the visited positions are in **1-balanced** form and the token is at $n/2$.

Inductive hypothesis: Assume the statement is true for $k = \ell$ for some $\ell < \frac{n}{2}$, and the visited positions are in ℓ**-balanced** form, i.e. the visited positions are in $S_0 = \{j, \ldots, j + \ell - 1\}$ and $S_1 = \{j + \frac{n}{2}, \ldots, j + \frac{n}{2} + \ell - 1\}$ for some j. Note that $|S_0| = |S_1| = \ell$. Assume the token's current position is at $p \in S_i$, where i is either 0 or 1 and p is one of the end positions in S_i.

Inductive step: Note that positions $j-1, j+\ell, j+\frac{n}{2}-1$ and $j+\frac{n}{2}+\ell$ are unvisited. If $p + d_{2\ell}$ is unvisited, then it implies that along the direction of $d_{2\ell}$ there is a vacant position next to p and we can set $m_{2\ell} = 1$. The token will be relocated to $p' = p + d_{2\ell}$ in round 2ℓ and S_i becomes $S'_i = S_i \cup \{p + d_{2\ell}\}$. While, if $p + d_{2\ell}$ is already visited, then it implies that S_i can be extended at the other end, i.e., we can move the token from p to $p' = p + d_{2\ell}\ell$ and S_i becomes $S'_i = S_i \cup \{p + d_{2\ell}\ell\}$. After round 2ℓ, $|S'_i| = |S_{1-i}| + 1 = \ell + 1$.

At round $2\ell+1$, the algorithm sets $m_{2\ell+1} = \frac{n}{2}$ and moves the token to position $p'' = p' + \frac{n}{2}$, which is independent of $d_{2\ell+1}$. Now S_{1-i} becomes $S'_{1-i} = S_{1-i} \cup \{p' + \frac{n}{2}\}$ and $|S'_i| = |S'_{1-i}| = \ell+1$. We summarize the movement of the token in Table 4.1.

Table 4.1: The positions of the token during rounds 2ℓ and $2\ell+1$, where p, p' and p'' are the positions of the token at round $2\ell-1, 2\ell$ and $2\ell+1$, respectively.

	$d_{2\ell} = 1$	$d_{2\ell} = -1$
$p = j$	$p' = j + \ell;\ p'' = j + \ell + n/2$	$p' = j - 1;\ p'' = j - 1 + n/2$
$p = j + \ell - 1$	$p' = j + \ell;\ p'' = j + \ell + n/2$	$p' = j - 1;\ p'' = j - 1 + n/2$
$p = j + n/2$	$p' = j + \ell + n/2;\ p'' = j + \ell$	$p' = j - 1 + n/2;\ p'' = j - 1$
$p = j + n/2 + \ell - 1$	$p' = j + \ell + n/2;\ p'' = j + \ell$	$p' = j - 1 + n/2;\ p'' = j - 1$

Note that, for $d_{2\ell} = 1$, we have $S'_0 = S_0 \cup \{j + \ell\}$ and $S'_1 = S_1 \cup \{j + n/2 + \ell\}$. For $d_{2\ell} = -1$, we have $S'_0 = S_0 \cup \{j - 1\}$ and $S'_1 = S_1 \cup \{j + n/2 - 1\}$. Both are clearly in $(\ell+1)$**-balanced** form. □

From above, we prove the correctness of algorithm MAGNUS and part (a) of Theorem 5.

4.2 Strategy for the case of odd n with some relaxation

Since $\frac{n}{2}$ is not an integer, the strategy for even n does not work directly. Here Magnus is allowed to choose magnitude from $\{1, \ldots, \lceil \frac{n}{2} \rceil\}$. We can also determine the moves $m_1, m_2, \ldots, m_{n-1}$ by observing the pattern of visited positions. The pattern of visited positions is slightly different from the case of even n.

Definition 5. *We call the visited positions on the round table k**-skew-balanced**, if the visited positions consist of two disjoint sets of consecutive positions, i.e., $S_0 = \{j, \ldots, j + k - 1\}$ and $S_1 = \{j + \lfloor n/2 \rfloor, \ldots, j + \lfloor n/2 \rfloor + k\}$ for some k and $j \in \mathbb{Z}_n$, and the token is sitting at one of the four end positions: $j, j + k - 1, j + \lfloor n/2 \rfloor$ and $j + \lfloor n/2 \rfloor + k$*

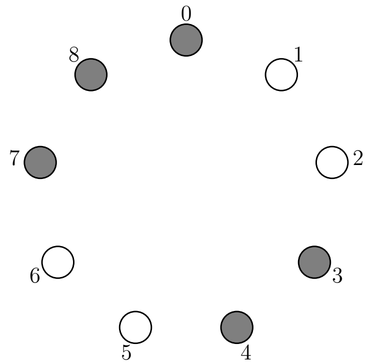

Figure 4.3: An example for **2-skew-balanced** with $n = 9$, where the gray nodes are visited.

Let $m_1, m_2, \ldots, m_{n-1}$ be the response of Magnus. At round i, if $d_i = +1$, then $m_i = \lfloor n/2 \rfloor$; otherwise $m_i = \lceil n/2 \rceil$. In every two rounds, each set of visited positions will be extended with a newly visited position. We prove the correctness of the strategy with the following lemma.

Lemma 4. *For $i = 2k$, $k \in \mathbb{Z}_n$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$, right after round i, the visited positions are in k-**skew-balanced** form for $j = \lfloor \frac{n}{2} \rfloor - k + 1$ and the token is sitting at position $n - k$.*

Proof. We prove it by induction on k.

Basis When $k = 1$, after round 1 and 2, the token will be moved to position $\lfloor \frac{n}{2} \rfloor$ and $n-1$, respectively, since $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$. Clearly, the visited positions are in 1-**skew-balanced** form for $j = \lfloor \frac{n}{2} \rfloor$ and the token is at $n - 1 = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.

Inductive hypothesis: Assume the statement is true for $k = \ell$ for some $\ell < \lfloor \frac{n}{2} \rfloor$, and the visited positions are in ℓ-**skew-balanced** form for $j = \lfloor \frac{n}{2} \rfloor - \ell + 1$, i.e. the visited positions are in $S_0 = \{\lfloor \frac{n}{2} \rfloor - \ell + 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and $S_1 = \{n - \ell, n - \ell + 1, \ldots, 0\}$. Moreover, the token is sitting at position $n - \ell$.

Inductive step: Note that the two positions $n - \ell - 1$ and $\lfloor \frac{n}{2} \rfloor - \ell$ are unvisited. In round $2\ell + 1$, if $d_{2\ell+1} = +1$, then we can set $m_{2\ell+1} = \lfloor \frac{n}{2} \rfloor$ and the token will be moved to position $n - \ell + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - \ell$. Otherwise, we set $m_{2\ell+1} = \lceil \frac{n}{2} \rceil$ and the token will be moved to position $n - \ell - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor - \ell$. Similarly, at the end of round $2\ell + 2$, the token will be moved to position $n - \ell - 1$. The visited positions become $S_0' = S_0 \cup \{\lfloor \frac{n}{2} \rfloor - \ell\}$ and $S_1' = S_1 \cup \{n - \ell - 1\}$, and they are in $(\ell+1)$**-skew-balanced** form for $j = \lfloor \frac{n}{2} \rfloor - (\ell+1) + 1$. $\square$

From above, we complete the proof of part (b) of Theorem 5.

4.3 Strategy for odd n without any relaxation

In this case, we will use a different approach. The following lemma shows the strategy we adapt in about first half of the steps.

Lemma 5. *We can design $m_1, m_2 \ldots, m_i$ to visit $i+1$ consecutive positions for $i \le \frac{n}{2}$ regardless of $d_1, d_2, \ldots, d_i$, the position we visited in round i is either positive biased or negative biased.*

Proof. The proof is based on the mathematical induction, when $i = 1$, the token is at position 0 and we can let $m_1 = 1$ and the token will move to position 1 or position $n-1$ and we can visit 2 consecutive positions. If we visit position 1 in round 1, position 1 is *positive biased*; otherwise, we visit position $n-1$ and it is *negative biased*. Now assume that the statement is still true when $i = k$. Now consider the situation $i = k+1$, by induction hypothesis , assume we have visited consecutive positions $j, j+1, \ldots, j+k$ in previous steps. Obviously, the token will be located on position j or $j+k$. Since $k < \frac{n}{2}$ and n is even, we have $k+1 \le \frac{n}{2}$. Hence, we can always choose a magnitude not larger than $\frac{n}{2}$ to visit one of position $j-1$ or $j+k+1$, we can still visit consecutive positions and the position we visit in round $k+1$ is either *positive biased* or *negative biased*.

The following lemma can be proved by similar method, the proof of the lemma reflects the strategy

Lemma 6. *Assume we have visited $l+1$ consecutive positions in first l rounds, where $l \ge \frac{n}{2}$. If the token is on the positive biased position and $d_{l+1} = 1$ or it is on the negative biased position and $d_{l+1} = -1$, then we can visit the rest $n-l-1$ positions in $n-l-1$ steps.*

Proof. Assume the locations we have visited is the set $\{j, j+1, \ldots, j+l\}$. Consider the case that the token is on position $j+l$, and $d_{l+1} = 1$. Obviously, the position $j+l$ is positive biased.

Now we need to check the sequence $d_{l+1}, d_{l+2}, \ldots, d_{n-1}$, if there is consecutive appearance of i 1's in this sequence start from d_{l+1}, i.e., $d_{l+1} = d_{l+2} = \ldots = d_{l+i} = 1$ and $d_{l+i+1} = -1$, then in round $l+1$ to $l+i-1$, we choose $m_{l+1} = ml+2 = m_{l+i-1} = 1$ to visit the positions $j+l+1, j+l+2, \ldots, j+l+i-1$, and we pick $m_{l+1} = n-l-i$, which is smaller than $\frac{n}{2}$ to visit position $j-1$. After round $l+i$ finished, the set of visited positions becomes $\{j-1, j, \ldots, j+l+i-1\}$, and the position $j-1$, on which the token is located, becomes negative biased. We can repeat the same action to visit the rest positions. The argument for the other case is symmetrical.□

From the two lemmas above, we are able to prove that we can visit all n positions in $n-1$ rounds easily if the sequence $d_1, d_2, \ldots, d_{n-1}$ satisfies the conditions mentioned in the following corollary.

Corollary 1. *If the sequence $d_1, d_2, \ldots, d_{n-1}$ satisfies $d_{\frac{n-1}{2}} = d_{\frac{n+1}{2}}$, then we can visit all positions in $n-1$ rounds.*

Proof. Consider the case that $d_{\frac{n-1}{2}} = d_{\frac{n+1}{2}} = 1$. From lemma 7, we can design $m_1, m_2, \ldots, m_{\frac{n-1}{2}}$ to visit $\frac{n+1}{2} \geq \frac{n}{2}$ consecutive positions. Furthermore, at the end of round $\frac{n-1}{2}$, the token is on a positive biased location. Since $d_{\frac{n+1}{2}} = 1$, by using the strategy mentioned in lemma 6, we are able to visit the rest $\frac{n-1}{2}$ positions in $\frac{n-1}{2}$ rounds.

The argument for the case $d_{\frac{n-1}{2}} = d_{\frac{n+1}{2}} = -1$ is symmetrical.□

Now we need to prove that, for the rest possible cases, we are still able to design the magnitudes $m_1, m_2, \ldots, m_{n-1}$ to visit all possible positions. The additional cost is that we need to examine the sequence $d_1, d_2, \ldots, d_{n-1}$ carefully. The following lemma exhibits the corresponding strategy and completes the proof of part (c) of Theorem 5.

Lemma 7. *If the sequence $d_1, d_2, \ldots, d_{n-1}$ does not satisfy $d_{\frac{n-1}{2}} = d_{\frac{n+1}{2}}$, by examining $d_{\frac{n-5}{2}}, d_{\frac{n-3}{2}}, \ldots, d_{\frac{n+3}{2}}, d_{\frac{n+5}{2}}$ in advance, Magnus is able to design magnitudes $m_1, m_2, \ldots, m_k$ to visit all n positions.*

Proof. First, we examine the value of $d_{\frac{n-3}{2}}, d_{\frac{n-1}{2}}, d_{\frac{n+1}{2}}, d_{\frac{n+3}{2}}$, since $d_{\frac{n-1}{2}} \neq d_{\frac{n+1}{2}}$, there are eight possible cases, in here we only prove four of them, the rest cases are symmetrical to the cases we proved.

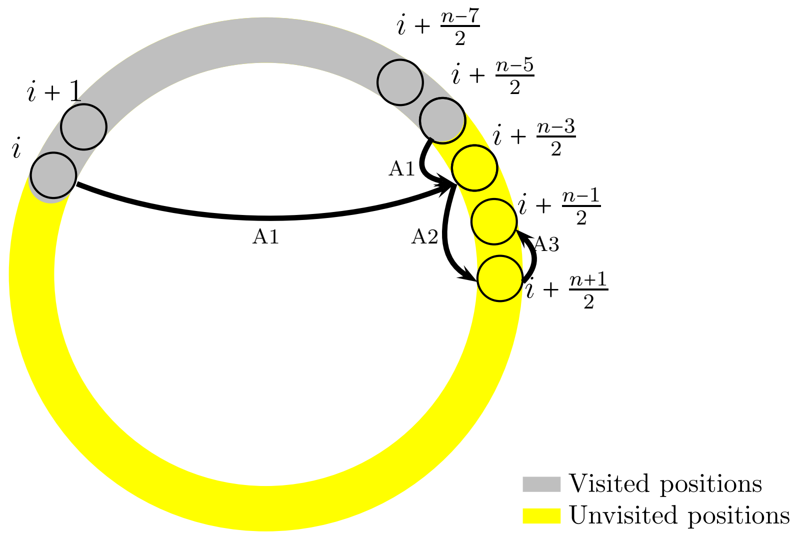

Figure 4.4: Case 1.

Case 1. $d_{\frac{n-3}{2}} = d_{\frac{n-1}{2}} = d_{\frac{n+3}{2}} = 1$ and $d_{\frac{n+1}{2}} = -1$. First, since $\frac{n-5}{2} \leq \frac{n}{2}$, from lemma 7, we can visit $\frac{n-3}{2}$ consecutive positions in first $\frac{n-5}{2}$ rounds. Assume the consecutive positions we have visited are $\{i, i+1, \ldots, i+\frac{n-5}{2}\}$, from lemma 7, the location of the token is either on position i or $i + \frac{n-5}{2}$. For this case, we adapt the following actions:

A1. In round $\frac{n-3}{2}$, move to position $i + \frac{n-3}{2}$, since $d_{\frac{n-3}{2}} = 1$ and the distance between position $\frac{n-3}{2}$ and current position of the token is at most $\frac{n-3}{2}$ in clockwise direction, we are able to perform the action.

A2. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = 2$ and move the token to position $i + \frac{n+1}{2}$.

A3. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 1$ and move the token to position $i+\frac{n-1}{2}$. At the end of this round, we have visited $\frac{n+1}{2}$ consecutive positions $\{i, i+1, \ldots, i+\frac{n-1}{2}\}$ and position $i + \frac{n+1}{2}$. Since position $i + \frac{n-1}{2}$ is positive biased related to the consecutive positions $\{i, i+1, \ldots, i+\frac{n-1}{2}\}$, and $d_{\frac{n+3}{2}} = 1$, from lemma 6, we are able to visit the rest $\frac{n-3}{2}$ positions in $\frac{n-3}{2}$ rounds.

The argument for proving the case $d_{\frac{n-3}{2}} = d_{\frac{n-1}{2}} = d_{\frac{n+3}{2}} = -1$ and $d_{\frac{n+1}{2}} = 1$ is symmetrical.

Case 2. $d_{\frac{n-3}{2}} = d_{\frac{n-1}{2}} = 1$ and $d_{\frac{n+1}{2}} = d_{\frac{n+3}{2}} = -1$. From lemma 7, we can visit $\frac{n-3}{2}$ consecutive positions in first $\frac{n-5}{2}$ rounds. Assume the consecutive positions we have visited

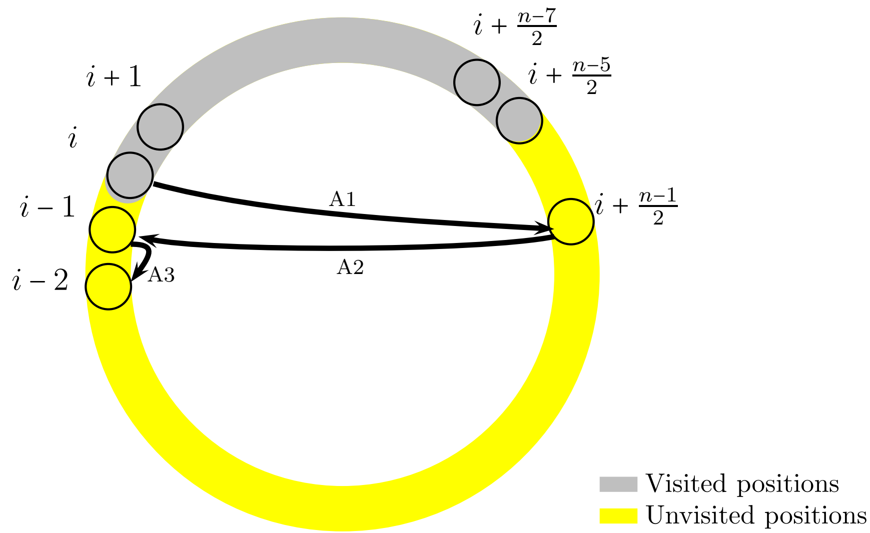

Figure 4.5: Case 2a.

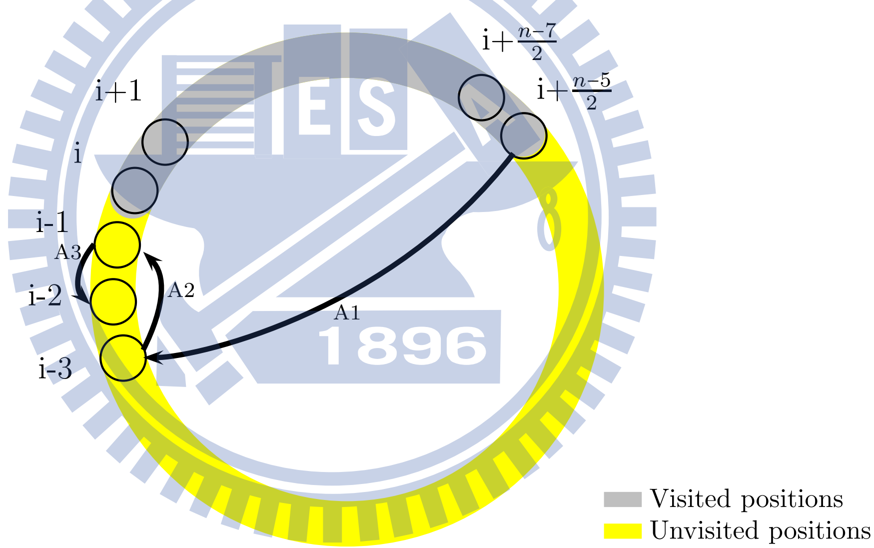

Figure 4.6: Case 2b.

are $\{i, i+1, \ldots, i+\frac{n-5}{2}\}$, from lemma 7, the location of the token is on position i or $i+\frac{n-5}{2}$. We consider the two subcases separately:

(a) If the token is on position i, then we adapt the following actions:

A1. In round $\frac{n-3}{2}$, set $m_{\frac{n-3}{2}} = \frac{n-1}{2}$ and move the token to position $i + \frac{n-1}{2}$.

A2. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = \frac{n-1}{2}$ and move the token to position $(i+\frac{n-1}{2}+\frac{n-1}{2}) \equiv i-1$.

A3. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 1$ and move the token to position $i-2$. At the end of the round, since we also visited position $i-1$ and $i-2$, the consecutive position we have visited becomes $\{i-2, i-1, i, \ldots, i+\frac{n-5}{2}\}$, which contains $\frac{n+1}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i-2$ which is negative biased and we have $d_{\frac{n+3}{2}} = -1$. From lemma 6, we are able to visit the rest $\frac{n-3}{2}$ positions in $\frac{n-3}{2}$ rounds.

(b) Otherwise, the token is on position $i+\frac{n-5}{2}$, we adapt the following actions:

A1. In round $\frac{n-3}{2}$, set $m_{\frac{n-3}{2}} = \frac{n-1}{2}$ and move the token to position $(i+\frac{n-5}{2}+\frac{n-1}{2})\%n = i-3$.

A2. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = 2$ and move the token to position $i-1$.

A3. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 1$ and move the token to position $i-2$. At the end of the round, since we also visited position $i-1$ and $i-2$, the consecutive position we have visited becomes $\{i-2, i-1, i, \ldots, i+\frac{n-5}{2}\}$, which contains $\frac{n+1}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i-2$ which is negative biased and we have $d_{\frac{n+3}{2}} = -1$. From lemma 6, we are able to visit the rest $\frac{n-3}{2}$ positions in $\frac{n-3}{2}$ rounds.

The argument for proving the case $d_{\frac{n-3}{2}} = d_{\frac{n-1}{2}} = -1$ and $d_{\frac{n+1}{2}} = d_{\frac{n+3}{2}} = 1$ is symmetrical.

Case 3. $d_{\frac{n-3}{2}} = d_{\frac{n+1}{2}} = d_{\frac{n+3}{2}} = -1$ and $d_{\frac{n-1}{2}} = 1$. From lemma 7, we can visit $\frac{n-3}{2}$ consecutive positions in first $\frac{n-5}{2}$ rounds. Assume the consecutive positions we have visited are $\{i, i+1, \ldots, i+\frac{+}{n-5}2\}$, we adapt the following actions:

A1. In round $\frac{n-3}{2}$, move the token to position $i-2$. Since the token is on position i or $i+\frac{n-5}{2}$ and $d_{\frac{n-3}{2}} = -1$. We can set $m_{\frac{n-3}{2}}$ to 2 or $\frac{n-1}{2}$ to accomplish this action.

A2. In round $\frac{n-1}{2}$, we set $m_{\frac{n-1}{2}} = 1$ and move the token to position $i-1$.

A3. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 2$ and move the token to position $i-3$. At the end of the round, the consecutive position we have visited becomes $\{i-3, i-2, i-1, i, \ldots, i+\frac{n-5}{2}\}$, which contains $\frac{n+3}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i-3$ which is negative biased and we have $d_{\frac{n+3}{2}} = -1$. From lemma 6, we are able to visit the rest $\frac{n-3}{2}$ positions in $\frac{n-3}{2}$ rounds.

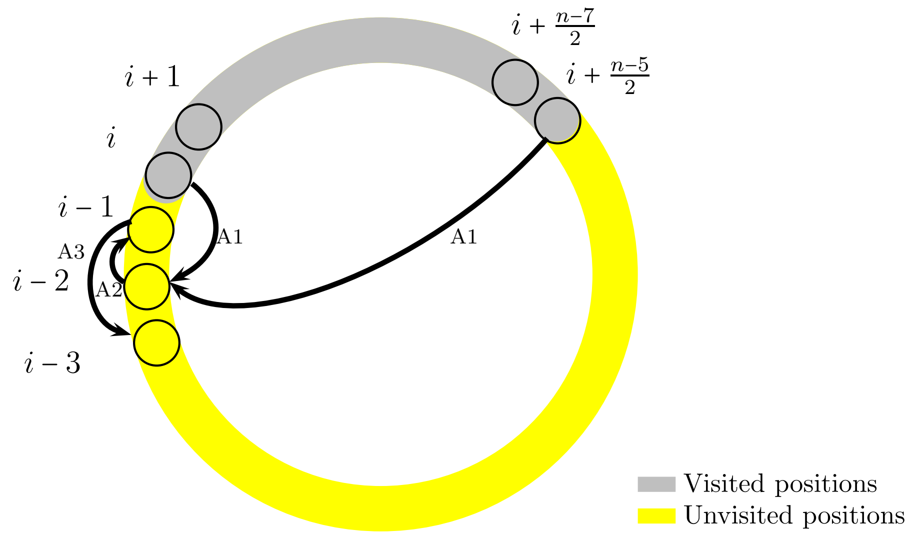

Figure 4.7: Case 3.

The argument for proving the case $d_{\frac{n-3}{2}} = d_{\frac{n+1}{2}} = d_{\frac{n+3}{2}} = 1$ and $d_{\frac{n-1}{2}} = -1$ is symmetrical.

Case 4. $d_{\frac{n-3}{2}} = d_{\frac{n+1}{2}} = -1$ and $d_{\frac{n-1}{2}} = d_{\frac{n+3}{2}} = 1$. This case is more complex then other cases, and we also consider the value of $d_{\frac{n-5}{2}}$ and $d_{\frac{n+5}{2}}$. From lemma 7, we can visit $\frac{n-5}{2}$ consecutive positions in first $\frac{n-7}{2}$ rounds. Assume the consecutive positions we have visited are $\{i, i+1, \ldots, i+\frac{n-7}{2}\}$, there are four possible cases:

(a) $d_{\frac{n-5}{2}} = d_{\frac{n+5}{2}} = -1$, we adapt the following actions:

A1. In round $\frac{n-5}{2}$, move the token to position $i-1$. Since the token is on position i or $i+\frac{n-7}{2}$ and $d_{\frac{n-5}{2}} = -1$. We can set $m_{\frac{n-5}{2}}$ to 1 or $\frac{n-5}{2}$ to accomplish this action.

A2. In round $\frac{n-3}{2}$, set $m_{\frac{n-3}{2}} = 2$ and move the token to position $i-3$.

A3. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = 1$ and move the token to position $i-2$.

A4. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 3$ and move the token to position $i-5$.

A5. In round $\frac{n+3}{2}$, set $m_{\frac{n-1}{2}} = 1$ and move the token to position $i-4$. At the end of the round, since we visited position $i-1, i-2, i-3, i-4$ We have a set of consecutive positions $\{i-4, i-3, i-2, i-1, i, \ldots, i+\frac{n-7}{2}\}$, which contains $\frac{n+3}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i-2$ which is negative biased related to these positions and we have $d_{\frac{n+5}{2}} = -1$. From lemma 6, we are able to visit the rest $\frac{n-5}{2}$ positions in $\frac{n-5}{2}$ rounds.

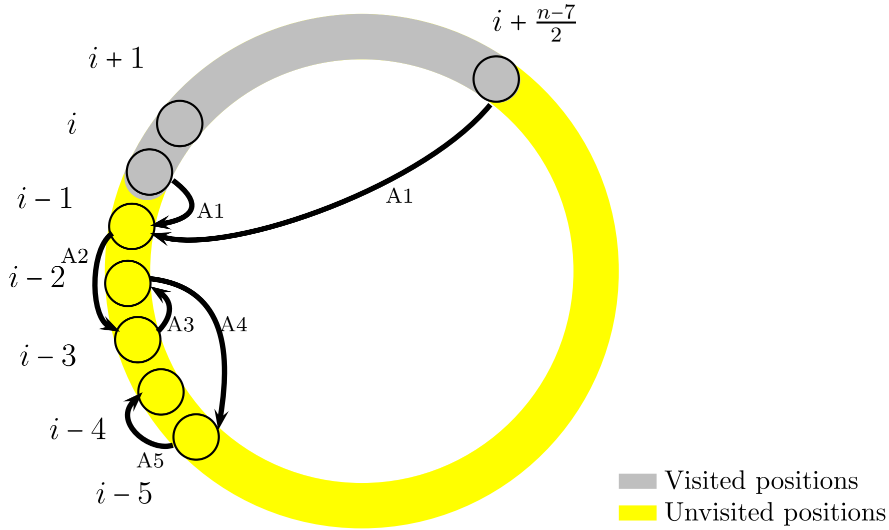

Figure 4.8: Case 4a.

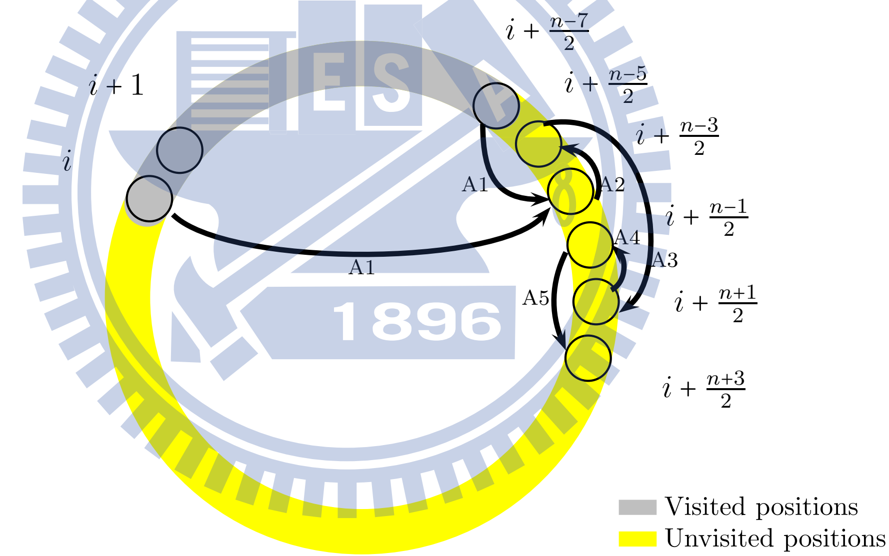

Figure 4.9: Case 4b.

(b) $d_{\frac{n-5}{2}} = d_{\frac{n+5}{2}} = 1$, we adapt the following actions:

A1. In round $\frac{n-5}{2}$, move the token to position $i+\frac{n-3}{2}$. Since the token is on position i or $i+\frac{n-7}{2}$ and $d_{\frac{n-5}{2}} = 1$. We can set $m_{\frac{n-5}{2}}$ to 2 or $\frac{n-3}{2}$ to accomplish this action.

A2. In round $\frac{n-3}{2}$, set $m_{\frac{n-3}{2}} = 1$ and move the token to position $i+\frac{n-5}{2}$.

A3. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = 3$ and move the token to position $i + \frac{n+1}{2}$.

A4. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 1$ and move the token to position $i + \frac{n-1}{2}$.

A5. In round $\frac{n+3}{2}$, set $m_{\frac{n-1}{2}} = 2$ and move the token to position $i + \frac{n+3}{2}$. At the end of the round, since we visited the five positions $i + \frac{n-5}{2}, i + \frac{n-3}{2}, i + \frac{n-1}{2}, i + \frac{n+1}{2}, i + \frac{n+3}{2}$. We have a set of consecutive positions $\{i, i+1, \ldots, i + \frac{n+3}{2}\}$, which contains $\frac{n+5}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i + \frac{n+3}{2}$ which is positive biased related to these positions and we have $d_{\frac{n+5}{2}} = 1$. From lemma 6, we are able to visit the rest $\frac{n-5}{2}$ positions in $\frac{n-5}{2}$ rounds.

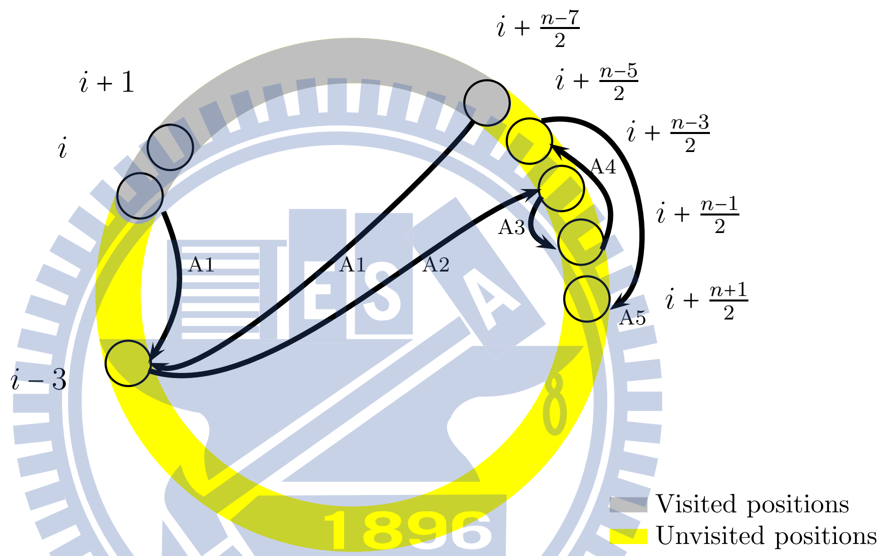

Figure 4.10: Case 4c.

(c) $d_{\frac{n-5}{2}} = -1$ and $d_{\frac{n+5}{2}} = 1$, we adapt the following actions:

A1. In round $\frac{n-5}{2}$, move the token to position $i-3$. Since the token is on position i or $i + \frac{n-7}{2}$ and $d_{\frac{n-5}{2}} = 1$. We can set $m_{\frac{n-5}{2}}$ to 3 or $\frac{n-1}{2}$ to accomplish this action.

A2. In round $\frac{n-3}{2}$, set $m_{\frac{n-3}{2}} = \frac{n-3}{2}$ and move the token to position $i + \frac{n-3}{2}$.

A3. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = 1$ and move the token to position $i + \frac{n-1}{2}$.

A4. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 2$ and move the token to position $i + \frac{n-5}{2}$.

A5. In round $\frac{n+3}{2}$, set $m_{\frac{n-1}{2}} = 3$ and move the token to position $i + \frac{n+1}{2}$. At the end of the round, since we visited the four positions $i + \frac{n-5}{2}, i + \frac{n-3}{2}, i + \frac{n-1}{2}, i + \frac{n+1}{2}$. We have a set of consecutive positions $\{i, i+1, \ldots, i+\frac{n+1}{2}\}$, which contains $\frac{n+3}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i+\frac{n+1}{2}$ which is positive biased related to these positions and we have $d_{\frac{n+5}{2}} = 1$. From lemma 6, we are able to visit the rest $\frac{n-5}{2}$ positions in $\frac{n-5}{2}$ rounds.

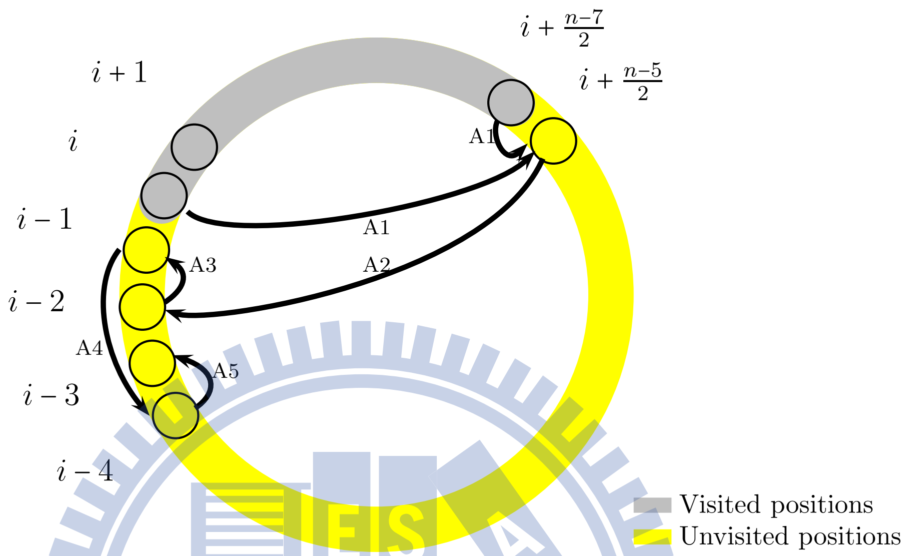

Figure 4.11: Case 4d.

(d) $d_{\frac{n-5}{2}} = 1$ and $d_{\frac{n+5}{2}} = -1$, we adapt the following actions:

A1. In round $\frac{n-5}{2}$, move the token to position $i+\frac{n-5}{2}$. Since the token is on position i or $i+\frac{n-7}{2}$ and $d_{\frac{n-5}{2}} = 1$. We can set $m_{\frac{n-5}{2}}$ to $\frac{n-5}{2}$ or 1 to accomplish this action.

A2. In round $\frac{n-3}{2}$, set $m_{\frac{n-3}{2}} = \frac{n-1}{2}$ and move the token to position $i-2$.

A3. In round $\frac{n-1}{2}$, set $m_{\frac{n-1}{2}} = 1$ and move the token to position $i-1$.

A4. In round $\frac{n+1}{2}$, set $m_{\frac{n+1}{2}} = 3$ and move the token to position $i-4$.

A5. In round $\frac{n+3}{2}$, set $m_{\frac{n-1}{2}} = 1$ and move the token to position $i-3$. At the end of the round, we have a set of consecutive positions $\{i-3, i-2, i-1, i, \ldots, i+\frac{n-5}{2}\}$, which contains $\frac{n+3}{2} \geq \frac{n}{2}$ positions. Since the token is now on position $i-3$ which is negative biased related to these positions and we have $d_{\frac{n+5}{2}} = -1$. From lemma 6, we are able to visit the rest $\frac{n-5}{2}$ positions in $\frac{n-5}{2}$ rounds.

The argument for proving the case $d_{\frac{n-3}{2}} = d_{\frac{n+1}{2}} = 1$ and $d_{\frac{n-1}{2}} = d_{\frac{n+3}{2}} = -1$ is symmetrical.$\square$

Chapter 5

When Derek and Magnus play randomly

Here, we consider the case when both players play randomly. The token will visit the positions on the circle randomly. Assume that the token is at position i, Magnus chooses m uniformly from $\{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and Derek chooses the direction d uniformly from $\{1, -1\}$. Let $p_{i,j}$ be the probability that the token is moved from position i to position j. For any $i, j \in \mathbb{Z}_n$ and $i \neq j$, let $\ell \leq \lfloor n/2 \rfloor$ be the distance between i and j. Then $\Pr[m = \ell] = 1/\lfloor \frac{n}{2} \rfloor$ and $\Pr[d = \text{ the direction from } i \text{ to } j] = 1/2$. If n is odd, then $p_{i,j} = 1/\lfloor \frac{n}{2} \rfloor \times 1/2 = 1/\frac{n-1}{2} \times 1/2 = 1/(n-1)$. Thus, for odd n, we have:

$$p_{i,j} := \begin{cases} 0 & \text{if } i = j, \\ \frac{1}{n-1} & \text{otherwise.} \end{cases}$$

Similarly, for even n, we have:

$$p_{i,j} := \begin{cases} 0 & \text{if } i = j, \\ \frac{2}{n} & \text{if } j \equiv i + \frac{n}{2} \bmod n, \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

We are interested in the cover time, which is the number of rounds needed to visit all positions. We show that the number of rounds needed is $\Theta(n \log n)$. Define $c_{(i,i+1)}, i \in \mathbb{Z}_n$, to be the number of rounds needed to change from a state with i positions visited to a state with $i+1$ positions visited. Since the token is at position 0 initially, we denote the cover time C_n as

$$C_n = \sum_{i=1}^{n-1} c_{(i,i+1)},$$

and the expected cover time is

$$E[C_n] = \sum_{i=1}^{n-1} E[c_{(i,i+1)}].$$

Lemma 8. *(a) When n is odd, $E[c_{(i,i+1)}] = \frac{n-1}{n-i}$; (b) When n is even, $\frac{n}{n-i+1} \leq E[c_{(i,i+1)}] \leq \frac{n}{n-i}$.*

Proof. (a) Suppose that there are $n-i$ unvisited positions. The probability to visit one of the unvisited positions is $p_i = \frac{n-i}{n-1}$. Note that $c_{(i,i+1)}$ is a geometric random variable with parameter p_i, and thus

$$E[c_{(i,i+1)}] = \frac{1}{p_i} = \frac{n-1}{n-i}.$$

(b) Assume the token is at position x. For even n, position $x+\frac{n}{2}$ mod n has a greater chance to be visited. If $x+\frac{n}{2}$ mod n has been visited, then the probability to visit a new position is

$$p_i = \frac{n-i}{n}.$$

If $x+\frac{n}{2}$ mod n hasn't been visited, then the probability to visit a position is

$$p_i = \frac{1}{n} \times (n-i-1) + \frac{2}{n} = \frac{n-i+1}{n}.$$

To bound the value of $E[c_{(i,i+1)}]$, we know that the above cases can happen, and we let p_i^* be the probability to visit a new position, where $\frac{n-i}{n} \leq p_i^* \leq \frac{n-i+1}{n}$. Note that p_i* depends on the current position and is well bounded. Let c' and c'' be two geometric random variables with parameter $\frac{n-i}{n}$ and $\frac{n-i+1}{n}$, respectively. Then we have

$$\frac{n}{n-i+1} = E[c''] \leq E[c_{(i,i+1)}] \leq E[c'] = \frac{n}{n-i}.$$

□

Since we know the range of $E[c_{i,i+1}]$ for all $i \in \{1,\ldots,n-1\}$, we can bound the expected cover time. We show that $E[C_n] = \Theta(n \log n)$ with the following theorem.

Theorem 6. *(a) When n is odd, $E[C_n] = (n-1)H_{n-1}$, where $H_n = \sum_{i=1}^{n} \frac{1}{i}$; (b) When n is even, $nH_n - n \le E[C_n] \le nH_n - 1$.*

Proof. (a) From part (a) of Lemma 8, $E[c_{(i,i+1)}] = \frac{n-1}{n-i}$. Hence,

$$E[C_n] = \sum_{i=1}^{n-1} E[c_{(i,i+1)}] = \sum_{i=1}^{n-1} \frac{n-1}{n-i} = (n-1)\sum_{i=1}^{n-1}\frac{1}{i} = (n-1)H_{n-1}.$$

(b) From part (b) of Lemma 8, $\frac{n}{n-i+1} \le E[c_{(i,i+1)}] \le \frac{n}{n-i}$ and $c_{1,2} = 1$. Hence,

$$\sum_{i=1}^{n-1} \frac{n}{n-i+1} \le E[C_n] \le \sum_{i=1}^{n-1} \frac{n}{n-i}.$$

We know that

$$\sum_{i=1}^{n-1} \frac{n}{n-i+1} = n\sum_{i=2}^{n}\frac{1}{i} = n(\sum_{i=1}^{n}\frac{1}{i} - 1) = nH_n - n,$$

and

$$\sum_{i=1}^{n-1} \frac{n}{n-i} = \sum_{i=1}^{n-1}\frac{n}{i} = n\sum_{i=1}^{n}\frac{1}{i} - \frac{n}{n} = nH_n - 1.$$

Since $H_n = \Theta(\log n)$, we know that the expected cover time is $\Theta(n \log n)$. ☐

Chapter 6

Conclusion

In here, we just state our conclusion in this paper.

In this paper we have answered two open questions in [5, 1], i.e. we prove that (1) Magnus can visit the maximum number of positions in $O(n)$ rounds; (2) Derek can find an optimal strategy in $O(n^3)$ with Magnus' moves revealed in advance.

	Number of rounds to visit maximum number of positions
Nedev and Muthukrishnan[5]	$O(n^2)$
Hurkens et. al. [1]	$O(n \log n)$
This paper	$O(n)$

Table 6.1: Table of results.

Moreover, we prove that Magnus always has full advantage with Derek's moves revealed in advance. We also proved that all positions will be visited in $O(\log n)$ rounds when both players play randomly. Several other questions raised in [5] remain open.

Bibliography

[1] R. Pendavingh C. Hurkens and G. Woeginger. Magnus-derek game revisited. *Information Processing Letters*, 109:38–40, 2008.

[2] S. Jukna. Extremal combinatorics with applications in computer science. *Springer-Verlag Berlin Heidelberg*, 2001.

[3] M. Mitzenmacher and E. Upfal. Probability and computing: Randomized algorithms and probabilistic analysis. *Cambridge University Press*, 2005.

[4] Z. Nedev. Universal set and the vector game. *INTEGERS: Electronic Journal of Combinatorial number theory*, 8, 2008.

[5] Z. Nedev and S. Muthukrishnan. The magnus–derek game. *Theoretical Computer Science*, 393:124–132, 2008.