國 立 交 通 大 學

電機與控制工程學系

碩 士 論 文

對於多射頻訊號尋找最小取樣頻率之有效率遞迴演算法

An Efficient Iterative Algorithm for Finding the Minimum Sampling Frequency of Multiple Bandpass Signals

研 究 生：劉 懿 德

指導教授：林 源 倍　博士

中 華 民 國 九 十 八 年 八 月

對於多射頻訊號尋找最小取樣頻率之有效率遞迴演算法

An Efficient Iterative Algorithm for Finding the Minimum Sampling Frequency of Multiple Bandpass Signals

研　究　生：劉懿德　　Student：Yi-De Liu

指導教授：林源倍　博士　　Advisor：Yuan-Pei Lin

國 立 交 通 大 學

電 機 與 控 制 工 程 學 系

碩 士 論 文

A Thesis

Submitted to Department of Electrical and Control Engineering

College of Electrical Engineering and Computer Engineering

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Master

in

Computer and Information Science
August 2009
Hsinchu, Taiwan, Republic of China

中華民國九十八年八月

對於多射頻訊號尋找最小取樣頻率之有效率遞迴演算法

學生:劉懿德

指導教授:林源倍 博士

國立交通大學電機與控制工學系（研究所）碩士班

摘要

我們提出一個有效率的遞迴演算法來針對多射頻訊號尋找其最小取樣頻率，這在軟體無線電(Software Radio)上有重要的應用，我們可以同時對多射頻訊號做不重疊失真取樣來降低成本。首先我們同時對兩個射頻訊號取樣，提出新的條件來達到不失真取樣，並且這些條件僅需要少量的運算。藉由遞迴增加取樣頻率來滿足所有的不失真條件，我們可以找到最小的不失真取樣頻率。我們可以將演算法擴充到同時取樣多個射頻訊號，並且在取樣後於不同的射頻訊號間加上保護頻帶(Guard Band)。模擬的結果顯示，我們提出的演算法能較之前的方法有效地降低運算量。

誌謝

感謝指導教授 林源倍教授在這碩士班的兩年裡，有耐心地給予我專業領域中的教導，在研究遇到困難時，老師也都能適時地給予幫助，讓我能順利完成碩士論文，並且獲益良多。另外也感謝林清安教授和蔡尚澕助理教授能撥冗來參加我的碩士論文口試，並在口試過程中給予我諸多的建議，使我的論文能夠更加完善。

感謝實驗室的成員建樟、鈞麟、宗堯、素卿、芳儀、翔澤、士軒、人予、虹君，不僅在研究和課業上給予我不少幫助，也讓實驗室氣氛變得很活潑融洽。也感謝室友政興、忠傑在生活上的陪伴。最後感謝我的家人和琳瑋，因為有你們在背後的支持，讓我能順利地完成研究所的學業。

An Efficient Iterative Algorithm for Finding the Minimum Sampling Frequency of Multiple Bandpass Signals

Yi-De Liu

Advisor: Dr. Yuan-Pei Lin
Department of Electrical and Control Engineering
National Chiao Tung University

August 18, 2009

Abstract

In this thesis, we propose an efficient iterative algorithm for finding the minimum sampling frequency for a signal that consists of multiple bandpass signals. This finds important application in software radio where it is desirable to downconvert multiple bandpass signals simultaneously. We will derive a new set of conditions for alias-free sampling for signals that contain two bandpass signals. The conditions can be easily examined with few computations. The minimum sampling frequency can be found by iteratively increasing the sampling frequency to meet the alias-free conditions. We will show how the algorithm can be extended to find the minimum sampling frequency for signals that consist of more than two bandpass signals. Furthermore we will generalize the result to the case when a guard band is required between different bandpass signals after sampling. The simulations demonstrate that the proposed method has a much lower complexity than existing algorithms.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Bandpass sampling has important applications in downcoverting radio frequency (RF) signals. In the application of software defined radio systems, it is desirable to downconvert multiple RF signals simultaneously to save cost [1]-[6]. The signal to be sampled may consist of more than one bandpass signal. Sampling theorem for a bandpass signal (two passbands) is well-known [7, 8]. The minimum frequency for alias-free sampling can be found in a closed form [9]. The minimum sampling frequency is usually significantly lower than the carrier frequency of the bandpass signal.

For signals with more than two passbands, the minimum sampling frequency can not be found in a closed from due to the nonlinear nature of spectrum folding in the process of sampling. Sampling for multi-band signals is extended in [6]. An example of a spectrum that consists of two bandpass signals is shown in Fig. 1.1. Conditions for alias-free sampling of multi-band signals are derived [6]. A systematic algorithm for finding valid sampling frequencies is developed in [10]. In [11][12][13], the complexity for finding valid sampling frequency is considerably reduced by imposing constraints on the ordering of the bands in the folded spectrum. These results may not yield the minimum frequency for alias-free sampling due to the ordering constraints. An efficient algorithm for finding valid sampling frequency range is proposed in [14]. By exhausting all possible orderings of the bands in the folded spectrum and categorizing all possible cases, the computational complexity can be reduced. An algorithm for finding the

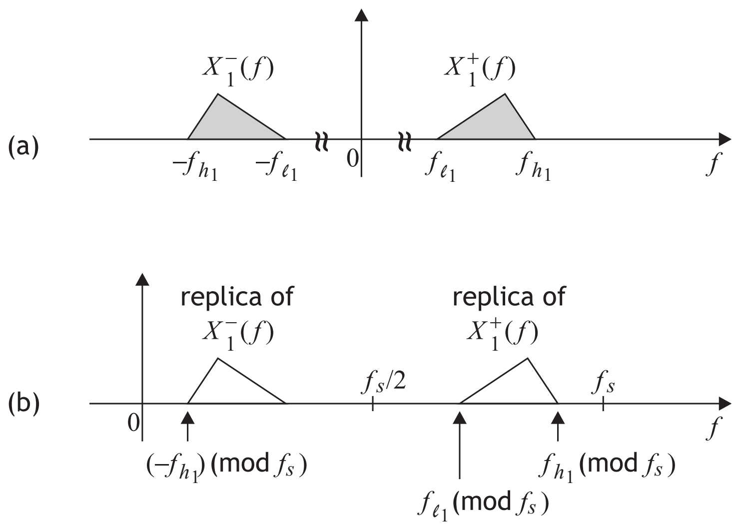

Figure 1.1: An example of spectrum that consists of two bandpass signals.

minimum sampling frequency is developed in [15] by finding the intersection of valid sampling frequencies for every two signal bands.

In this thesis, we propose an efficient algorithm for finding the minimum sampling frequency for a signal consisting of two or more bandpass signals. We will first derive a set of conditions for alias-free sampling of signals that consist of two bandpass signals (four bands). These conditions can be checked with very few computations. When one of these conditions is not satisfied, the sampling frequency can be adjusted with minimum increment so that the condition becomes satisfied. By iteratively increasing the sampling frequency to meet the conditions for alias-free sampling, an algorithm for finding the minimum sampling frequency can be developed. There is no need to consider the ordering of the sinal band in the folded spectrum. The algorithm can be extended to find the minimum sampling frequency for multiple bandpass signals. We can also generalize the algorithm to the case when a guard band is required between different bandpass signals after sampling. We will see that the algorithm based on the conditions derived in this thesis requires fewer computations when compared to previously reported methods.

1.1 Outline

- Chapter 2: The problem of bandpass sampling is formulated.

- Chapter 3: Section 3.1, we review a low-cost algorithm proposed by S. Bose, V. Khaitan, and A. Chaturvedi [13]. The algorithm finds the minimum sampling frequency when an ordering constraint is placed on the passbands. Section 3.2 introduces an efficient algorithm for finding valid sampling frequency ranges proposed by C. H. Tseng and S. C. Chou [14]. Section 3.3 introduces a searching algorithm for minimum sampling frequency by finding the intersection of valid sampling frequencies for every two signal passbands. This is proposed by J. Bae and J. Park [15].

- Chapter 4: Section 4.1 describes a set of conditions for alias-free sampling of two bandpass signals. An efficient algorithm for finding the minimum sampling frequency of two bandpass signals is shown in section 4.2. A complexity analysis is given in section 4.3.

- Chapter 5: Section 5.1 extends the alias-free conditions when there is an user-specified minimum guard band. A method for finding a valid sampling frequency range is shown in 5.2. Section 5.3 extends the case of two bandpass signals into the case of multi-band signals.

- Chapter 6: Simulations and comparisons of the previously reported methods and the proposed method are given.

1.2 Notations

- The notation $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

- The notation $\lceil x \rceil$ denotes the smallest integer larger than or equal to x.

Chapter 2

Problem Formulation

A receiver front end design of software defined radio is shown in Fig. 2.1. A wide-band RF signal is received from the antenna and amplified with a low-noise amplifier (LNA). Then the signal is filtered with N parallel bandpass filters. Thus the input signal to the analog-to-digital converter (ADC) is a multi-band RF signal as shown in Fig. 2.2. The sampling frequency of the ADC should be properly chosen so that there is no aliasing. The minimum sampling frquency provides an attractive alternative to sampling at twice the carrier frequency (Nyquist rate) [6]. Our goal here is to find the minimum sampling frequency efficiently given a multiband signal like the one in Fig. 2.2.

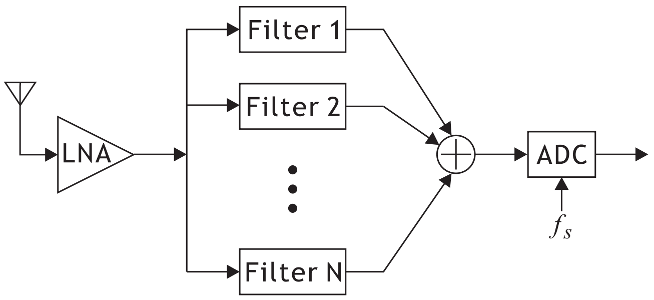

Figure 2.1: The software defined radio receiver front end.

If an analog signal $x(t)$ (with $X(f)$ denoting its Fourier transform) is sampled with a sampling frequency f_s, the spectrum will be folded back and there will be

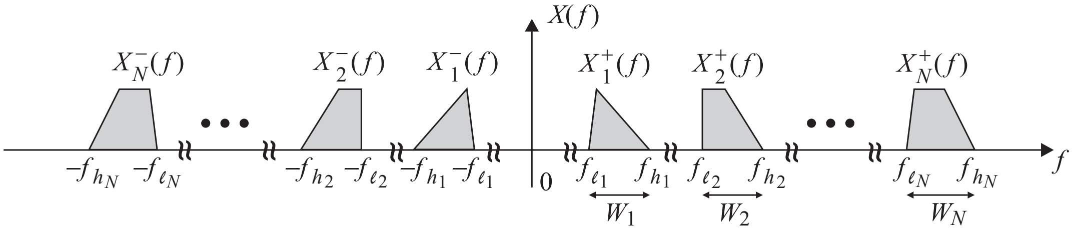

Figure 2.2: A spectrum that consists of N bandpass signals.

a copy of $X(f)$ every f_s,

$$\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Consider a bandpass signal (two passbands) $X(f)$ as shown in Fig. 2.3(a). Assume that $X(f) \neq 0$ for $f_\ell < |f| < f_h$, where f_ℓ and f_h are band edges, and $W = f_h - f_\ell$ is the one-sided bandwidth as indicated in the figure. If we are to sample $X(f)$ without causing aliasing, the replicas of $X^-(f)$ should not overlap with $X^+(f)$. Suppose we shift $X^-(f)$ shifts by mf_s and the copy $X^-(f - mf_s)$ is located at the right side of $X^+(f)$ as shown in Fig. 2.3(b). The smallest m for this is

$$m = \lfloor 2f_h/f_s \rfloor \tag{2.1}$$

To avoid aliasing, we can have

$$-f_\ell + (m-1)f_s \leq f_\ell,$$

and

$$-f_h + mf_s \geq f_h.$$

Combining the above two conditions, we have a valid sampling frequency range [7, 8]

$$\frac{2f_h}{m} \leq f_s \leq \frac{2f_\ell}{m-1} \tag{2.2}$$

Since the lowest possible sampling frequency for no aliasing is $2W$, the maximum of m is $m_{max} = \lfloor f_h/W \rfloor$. Thus we can have a closed form of the minimum sampling frequency as

$$f_{s,min} = \frac{2f_h}{m_{max}} = \frac{2f_h}{\lfloor f_h/W \rfloor} \tag{2.3}$$

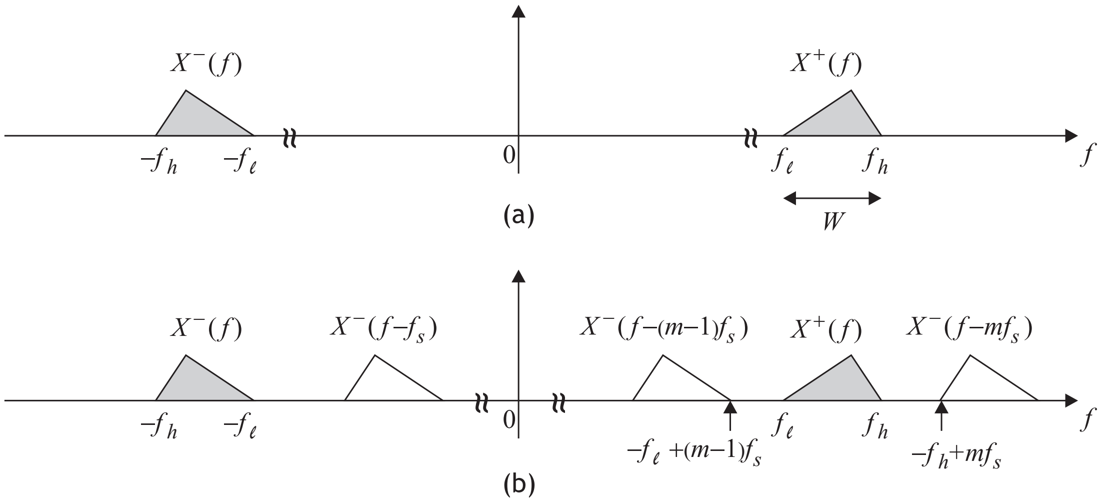

Figure 2.3: (a) A spectrum of single bandpass signal. (b) The spectrum sampled with f_s.

For signals with more than two passbands (Fig. 2.2), the minimum sampling frequency can not be found in a closed from due to the nonlinear nature of spectrum folding in the process of sampling. Upon sampling with frequency f_s, replicas of each passband appear each f_s, resulting in a periodic spectrum; we can simply consider the period $[0, f_s)$. Fig. 2.4 gives an example of a signal spectrum sampled with an alias-free sampling frequency f_s. For alias-free sampling, there are two types of constraints: one is referred to as boundary constraint and another is referred to as neighbor constraint.

Boundary constraint. Boundary constraint means that the replicas of $X_i^+(f)$ and $X_i^-(f)$ should not overlap at the edge in $[0, f_s/2)$ (replica of $X_N^+(f)$, $X_N^-(f)$ and replica of $X_1^+(f)$, $X_1^-(f)$ in Fig. 2.4) need to be completely positioned within $[0, f_s/2)$. If 0 or $f_s/2$ is contained inside the band of these replicas, there will be aliasing.

Neighbor constraint. Neighbor constraint means that the replicas of $X_i(f)$ and $X_j(f)$ should not overlap each other in $[0, f_s/2)$. For example in Fig. 2.4, $X_i^-(f)$ should not overlap $X_{i+1}^-(f)$ for $i = 1, 2, \cdots, N-1$.

Existing algorithm for finding the minimum sampling frequency are reviewed

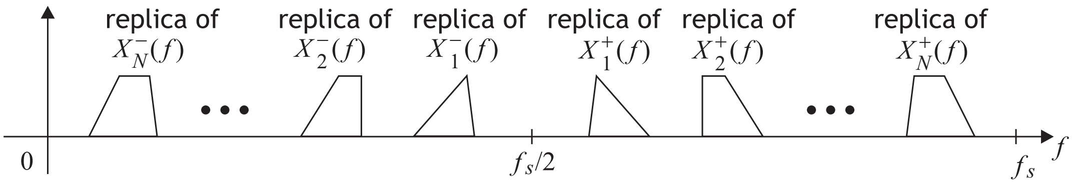

Figure 2.4: An example of a signal spectrum sampled with an alias-free sampling frequency f_s.

in the next chapter.

Chapter 3

Previously Reported Methods

In this chapter we briefly review the previously methods for finding the valid bandpass sampling frequency. Section 3.1, we review a low-cost algorithm proposed by S. Bose, V. Khaitan, and A. Chaturvedi [13]. The algorithm finds the minimum sampling frequency when an ordering constraint is placed on the passbands. Section 3.2 introduces an efficient algorithm for finding valid sampling frequency ranges proposed by C. H. Tseng and S. C. Chou [14]. Section 3.3 introduces a searching algorithm for minimum sampling frequency by finding the intersection of valid sampling frequencies for every two signal passbands. This is proposed by J. Bae and J. Park [15].

3.1 Efficient Method with An Ordering Constraint [13]

In this section, we review a low-cost algorithm proposed by S. Bose, V. Khaitan, and A. Chaturvedi. The algorithm finds the minimum sampling frequency when an ordering constraint is placed on the passbands. Section 3.1.1 introduces the assumption of ordering constraint and analyzes the constraint for alias-free sampling under this assumption. Section 3.1.2 provides a low-cost algorithm finding the minimum sampling frequency with the ordering constraint.

3.1.1 Alias-free Conditions with Ordering Constraints

Consider a signal that consists of N bandpass signals as shown in Fig. 3.1(a). Assume $X_i(f) \neq 0$ for $f_{\ell_i} < |f| < f_{h_i}$, $i = 1, 2, \cdots, N$, where f_{ℓ_i} and f_{h_i} are band edges, and $W_i = f_{h_i} - f_{\ell_i}$ are one-sided bandwidths as indicated in the figure. Let $f_i = (f_{\ell_i} + f_{h_i})/2$ denote the center frequency of $X_i^+(f)$. Upon sampling with frequency f_s, replicas of each passband appear each f_s, resulting in a periodic spectrum as shown in Fig. 3.1(b), where r_i (referred to as frequency shifting parameter) is given by

$$r_i = \left\lfloor \frac{f_{\ell_i} - GB}{f_s} \right\rfloor.$$

Assume that after alias-free bandpass sampling, the ordering of the bandpass signals in the interval $[0, f_s/2)$ does not change. Let $f_{IF_i} = f_i \pmod{f_s}$ denote the center frequency of the replica of $X_i^+(f)$ in $[0, f_s/2)$ after sampling. The ordering constraint is such that

$$f_{IF_1} < f_{IF_2} < \cdots < f_{IF_N}$$

as indicating in the figure. Between every two replicas after bandpass sampling, an user-specified minimum guard band GB is required for practical considerations.

To avoid aliasing after sampling, two basic constraints must be satisfied: a boundary constraint in the sampled bandwidth and a neighbor constraint between adjacent passbands.

Boundary constraint. The boundary constraint is that $X_1^+(f)$ should be positioned within $[r_1 f_s, (r_1+0.5)f_s]$ and $X_N^+(f)$ should be positioned within $[r_N f_s, (r_N+0.5)f_s]$ respectively so that aliasing by the negative frequency part of each signal should not occur at both boundaries. Two boundary constraints can be obtained as follows

$$r_1 f_s \leq f_{\ell_1} - GB, \tag{3.1}$$

and

$$(r_N + 0.5)f_s \geq f_{h_N} + GB. \tag{3.2}$$

Neighbor constraint. The neighbor constraint is that adjacent passbands should not overlap each other, it can be expressed as

$$f_{h_i} - r_i f_s \leq f_{\ell_{i+1}} - GB - r_{i+1} f_s \quad \text{for } i = 1, 2, \cdots, N-1 \tag{3.3}$$

Combining equation (3.1), (3.2), (3.3), the alias-free sampling frequency range can be expressed as

$$\frac{f_{h_N} + GB}{r_N + 0.5} \leq f_s \leq \min\{f_{UB_{0,1}}, f_{UB_{1,2}}, \cdots, f_{UB_{N-1,N}}\} \tag{3.4}$$

where

$$f_{UB_{i,i+1}} = \frac{f_{\ell_{i+1}} - f_{h_i} - GB}{r_{i+1} - r_i} \quad \text{for } i = 0, 1, \cdots, N-1$$

$f_{h_0} = 0$ and $r_0 = 0$. Since the lowest possible sampling frequency for alias-free sampling is $f_s = 2\{W_1 + W_2 + \cdots + W_N + (N+1)GB\}$, r_i can be bounded as

$$1 \leq r_i \leq \left\lfloor \frac{f_{\ell_i} - GB}{2\{W_1 + W_2 + \cdots + W_N + (N+1)GB\}} \right\rfloor, \quad \text{for } i = 1, 2, \cdots, N \tag{3.5}$$

Furthermore, observe that a valid sampling frequency f_s for N bandpass signals will also be a valid sampling frequency for $k < N$ bandpass signals. Thus, equation (3.4) can be extended as

$$\frac{f_{h_k} + GB}{r_k + 0.5} \leq f_s \leq \min\{f_{UB_{0,1}}, f_{UB_{1,2}}, \cdots, f_{UB_{k-1,k}}\} \tag{3.6}$$

for $k = 1, 2, \cdots, N$.

To ensure the existence of f_s, the LHS of equation (3.6) should be less or equal to the RHS of equation (3.6), and it can be expressed as follows

$$\frac{f_{h_k} + GB}{r_k + 0.5} \leq f_{UB_{k-1,k}} = \frac{f_{\ell_k} - f_{h_{k-1}} - GB}{r_k - r_{k-1}} \quad , \text{for } k = 1, 2, \cdots, N \tag{3.7}$$

and

$$\frac{f_{h_k} + GB}{r_k + 0.5} \leq \min\{f_{UB_{0,1}}, \cdots, f_{UB_{k-2,k-1}}\} \quad , \text{for } k = 2, 3, \cdots, N \tag{3.8}$$

From equation (3.7), it follows that

$$r_k \leq \left\lfloor \frac{r_{k-1}(f_{h_k} + GB) + 0.5(f_{\ell_k} - f_{h_{k-1}} - GB)}{W_k + f_{h_{k-1}} + 2GB} \right\rfloor \quad , \text{for } k = 1, 2, \cdots, N \tag{3.9}$$

which gives the upper bound of r_k, and equation (3.8) can be expressed as

$$r_k \geq \left\lceil \frac{f_{h_k} + GB}{\min\{f_{UB_{0,1}}, \cdots, f_{UB_{k-2,k-1}}\}} - 0.5 \right\rceil \quad , \text{ for } k = 2, 3, \cdots, N \tag{3.10}$$

which gives the lower bound of r_k.

These two equations (3.9) and (3.10) define the valid range for r_k. Note that equation (3.10) does not provide the lower bound of r_1. Since r_1 is an positive integer, the lower bound can be set as 0. Besides, the upper bound of r_1 from (3.9) is given by

$$r_1 \leq \left\lfloor \frac{f_{\ell_1} - GB}{2(W_1 + 2GB)} \right\rfloor,$$

which does not take into account all passbands. Thus it can be modified with a tighter bound by taking into account all passbands

$$r_1 \leq \left\lfloor \frac{f_{\ell_1} - GB}{2\{W_1 + W_2 + \cdots + W_N + (N+1)GB\}} \right\rfloor \tag{3.11}$$

3.1.2 The Algorithm for Finding the Minimum Sampling Frequency with Ordering Constraint

From the above section, the constraints of each r_k include the alias-free constraint, the purpose is to find the N-tuple of valid integers $\{r_1, r_2, \cdots, r_N\}$ which satisfy (3.9) and (3.10). Let the lower bound and upper bound of r_k be denoted as $r_{k_{min}}$ and $r_{k_{max}}$. For a given $r_1, r_2, \cdots, r_{k-1}$, if $r_{k_{min}} > r_{k_{max}}$, then given $k-1$-tuple is not a valid one. Therefore, by iterating each r_k, all valid tuple $\{r_1, r_2, \cdots, r_N\}$ can be obtained. Furthermore, equation (3.6) shows that to choose the minimum of f_s, r_k needs to choose maximum as possible and r_k depends on the induction of $r_1, r_2, \cdots, r_{k-1}$. The searching algorithm is given as follows:

1. Set the minimum guard band GB.

2. Initialize $f_{s,min}$ to the Nyquist rate $2(f_{h_N} + GB)$.

3. Set $r_{1_{min}}$ to 0 and evaluate $r_{1_{max}}$ from equation (3.9). Then set r_1 to $r_{1_{max}}$.

4. Compute $r_{2_{min}}$ and $r_{2_{max}}$ from equation (3.9)(3.10).

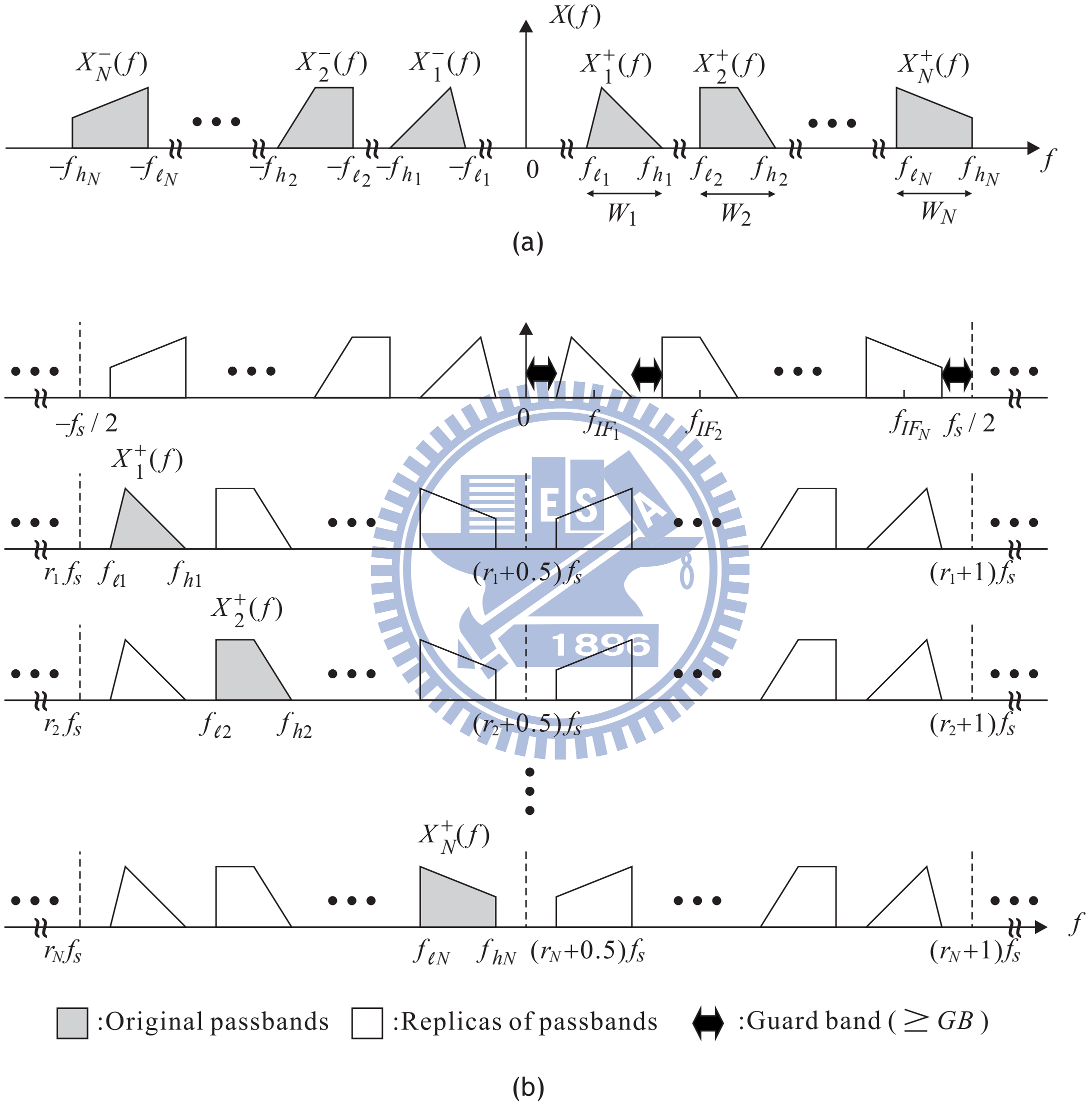

Figure 3.1: (a) A signal that consists of N bandpass signals. (b) The signals spectrum after alias-free sampling with ordering constraint.

5. If $r_{2_{min}} > r_{2_{max}}$, decrease r_1 by 1.

6. If $r_1 = r_{1_{min}}$, go to step 14; else go to step 4.

7. Set r_2 to $r_{2_{max}}$.

8. Compute $r_{3_{min}}$ and $r_{3_{max}}$ from equation (3.9)(3.10).

9. If $r_{3_{min}} > r_{3_{max}}$, decrease r_2 by 1.

10. If $r_2 < r_{2_{min}}$, decrease r_1 by 1 and go to step 4; else go to step 8.

11. Continue this procedure until obtaining a valid set of $r_1, r_2, \cdots, r_{N-1}$.

12. Compute $r_{N_{min}}$ and $r_{N_{max}}$ from equation (3.9)(3.10).

13. If $r_{N_{min}} > r_{N_{max}}$, decrease r_{N-1} by 1 and go to step 11; else compute $f_{s,min}$ as

$$f_{s,min} = \frac{f_{h_N} + GB}{r_{N_{max}} + 0.5} \tag{3.12}$$

14. Output $f_{s,min}$.

3.2 Method in [14]

In this section, we review an efficient algorithm for finding valid sampling frequency range proposed by C. H. Tseng and S. C. Chou. By exhausting all possible orderings of the bands in the folded spectrum and categorizing all possible cases, the computational complexity can be reduced. Section 3.2.1 analyzes the all possible replica orders of the signal spectrum after bandpass sampling and derives the constraints for alias-free sampling. Section 3.2.2 presents a searching algorithm for the ranges of alias-free sampling frequency by iterating each index of the segment, and the minimum sampling frequency can be obtained from the valid ranges.

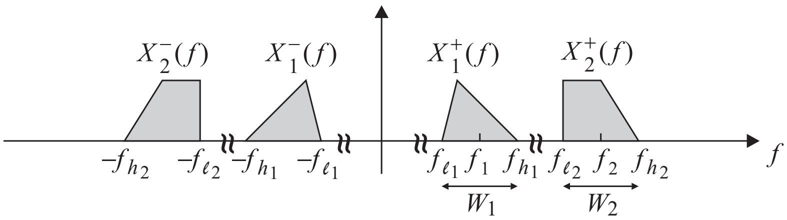

Figure 3.2: A signal that consists of two bandpass signals.

3.2.1 Constraints of Valid Sampling Frequency Ranges for Multiple Bandpass Signals

To sample without aliasing, the sampling frequency f_s needs to be chosen without causing spectral overlapping after bandpass sampling. First consider the problem of sampling a signal that consists of two bandpass signals (four passbands) whose spectrum is shown in Fig. 3.2, there will be 8 possible replica orders after bandpass sampling without causing aliasing as shown in Fig. 3.3. The signal spectrum after sampling is separated into many segments and n_1 and n_2 are the index of the segment where the original spectrum $X_1^+(f)$ and $X_2^+(f)$ are located and can be obtained as $n_1 = \lfloor f_{\ell_1}/f_s \rfloor$, $n_2 = \lfloor f_{\ell_2}/f_s \rfloor$ respectively, where f_1 and f_2 are the center frequency of $X_1^+(f)$ and $X_2^+(f)$. The four passbands are symmetric to the center of each segment.

For a given replica order, there are two types of constraints: one is referred to as the neighbor constraint and the other is referred to as the boundary constraint. Consider the case 1 in Fig. 3.3 as an example.

Boundary constraint. The boundary constraint for case 1 is that the passband '1' and '2' should be completely inside the half of each segment, which lead to two boundary constraints as $f_{l1} \geq n_1 f_s$ and $f_{h2} \leq (n_2 + 1/2) f_s$, or equivalently

$$f_s \leq \frac{f_{\ell_1}}{n_1} \tag{3.13}$$

$$f_s \geq \frac{f_{h_2}}{n_2 + 1/2} \tag{3.14}$$

Neighbor constraint. The neighbor constraint is that the passband '1' does not

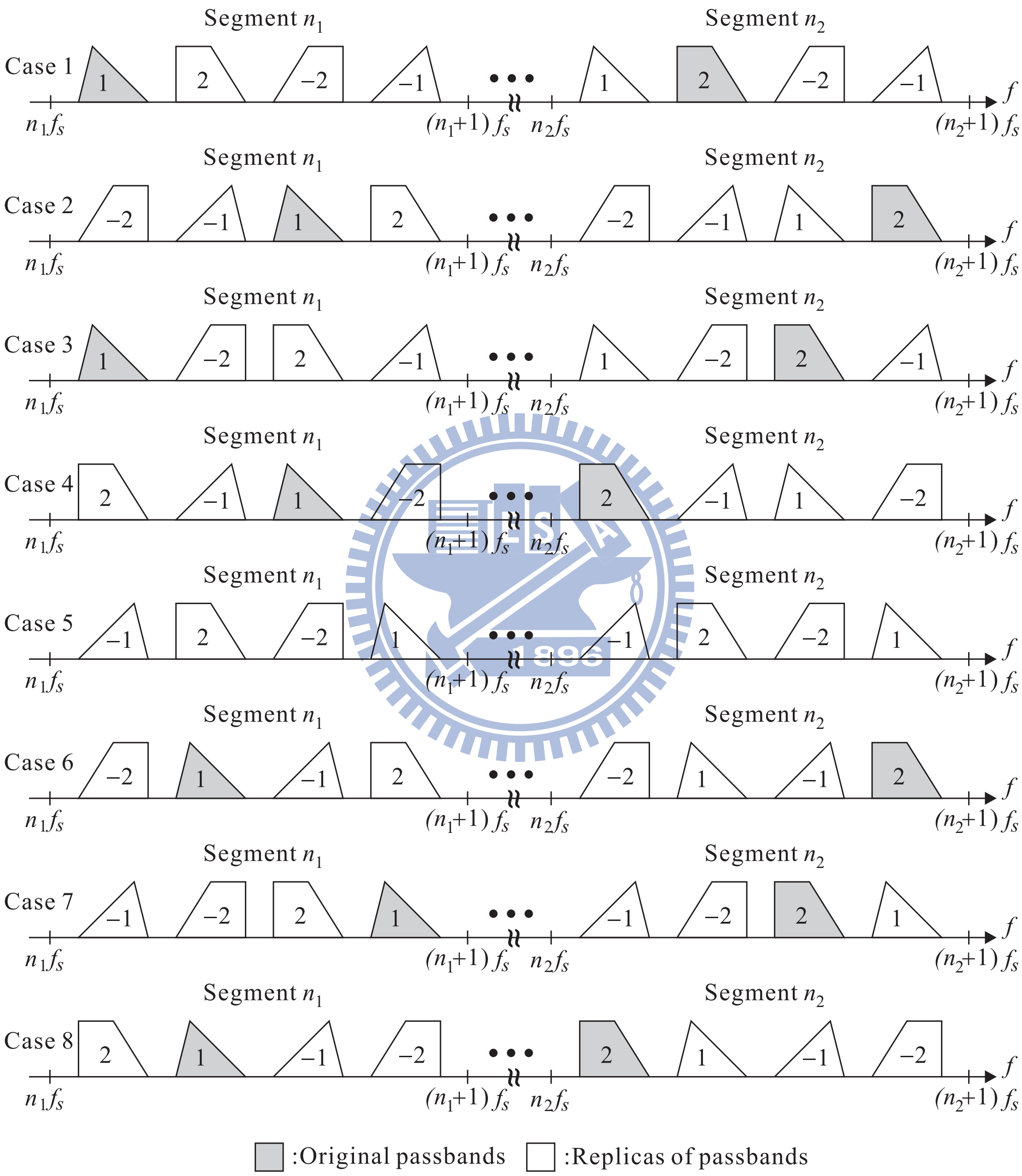

Figure 3.3: The 8 possible replica orders after bandpass sampling.

overlap the passband '2'. This means $f_{h1} - n_1 f_s \le f_{l2} - n_2 f_s$, or equivalently

$$f_s \le \frac{f_{\ell_2} - f_{h_1}}{n_2 - n_1} \tag{3.15}$$

Combining (3.13)-(3.15), a range of alias-free sampling frequency for case 1 can be found as

$$\frac{f_{h_2}}{n_2 + 1/2} \le f_s \le \min\{\frac{f_{\ell_1}}{n_1}, \frac{f_{\ell_2} - f_{h_1}}{n_2 - n_1}\}$$

By examining all the other replica orders, the ranges of alias-free sampling frequency are summarized in table 3.1.

Case	Range of Valid f_s
1	$\frac{f_{h_2}}{n_2+1/2} \le f_s \le \min\{\frac{f_{\ell_1}}{n_1}, \frac{f_{\ell_2}-f_{h_1}}{n_2-n_1}\}$
2	$\frac{f_{h_2}}{n_2+1} \le f_s \le \min\{\frac{f_{\ell_1}}{n_1+1/2}, \frac{f_{\ell_2}-f_{h_1}}{n_2-n_1}\}$
3	$\frac{f_{h_1}+f_{h_2}}{n_1+n_2+1} \le f_s \le \min\{\frac{f_{\ell_1}}{n_1}, \frac{f_{\ell_2}}{n_2+1/2}\}$
4	$\frac{f_{h_1}+f_{h_2}}{n_1+n_2+1} \le f_s \le \min\{\frac{f_{\ell_1}}{n_1+1/2}, \frac{f_{\ell_2}}{n_2}\}$
5	$\max\{\frac{f_{h_1}}{n_1+1}, \frac{f_{h_2}}{n_2+1/2}\} \le f_s \le \frac{f_{\ell_1}+f_{\ell_2}}{n_1+n_2+1}$
6	$\max\{\frac{f_{h_1}}{n_1+1/2}, \frac{f_{h_2}}{n_2+1}\} \le f_s \le \frac{f_{\ell_1}+f_{\ell_2}}{n_1+n_2+1}$
7	$\max\{\frac{f_{h_1}}{n_1+1}, \frac{f_{h_2}-f_{\ell_1}}{n_2-n_1}\} \le f_s \le \frac{f_{\ell_2}}{n_2+1/2}$
8	$\max\{\frac{f_{h_1}}{n_1+1/2}, \frac{f_{h_2}-f_{\ell_1}}{n_2-n_1}\} \le f_s \le \frac{f_{\ell_2}}{n_2}$

Table 3.1: The ranges of alias-free sampling frequency for two bandpass signals.

Consider a signal that consists of N bandpass signals ($2N$ passbands) as shown in Fig. 2.2. The signal spectrum after bandpass sampling are the combinations of all replicas of the $2N$ passbands. As the case of two bandpass signals, the spectrum after sampling can be separated into many segments, and then consider all possible replica orders in a segment. Note that there are two ways a passband is located in a segment: one is in the first half of the segment and the other is in the second half. Since there are N passbands in a segment after sampling, there are 2^N possibilities. In the half of the segment, there are $N!$ ways of ordering the allocated replicas. Therefore, the total number of all possible replica orders is $2^N \times N!$. For each possible replica order, there are 2 boundary constraints and $N-1$ neighbor constraints. Taking one of the possible replica order as shown in

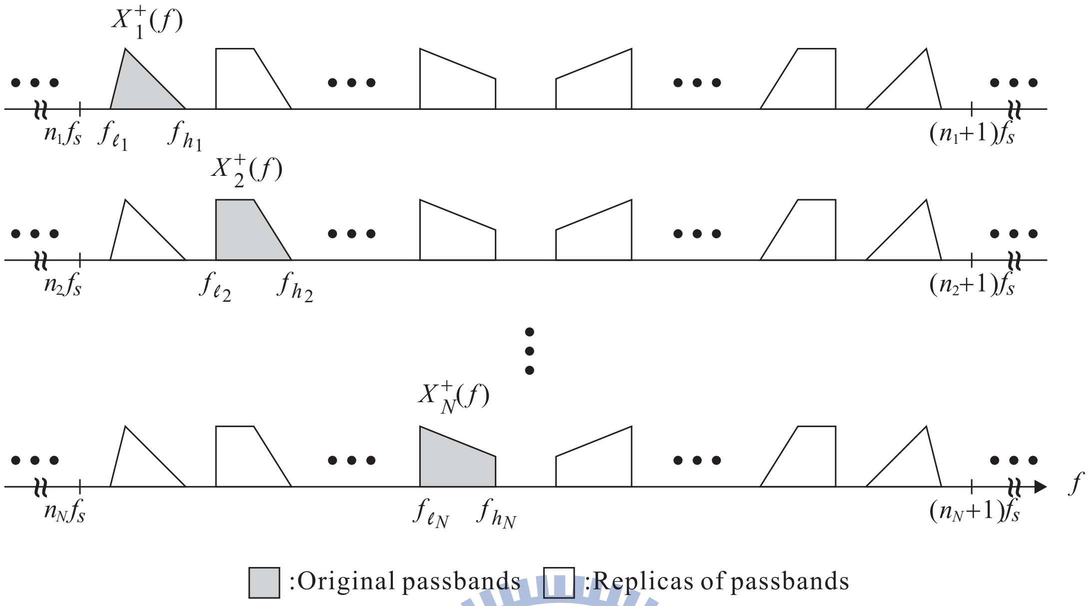

Figure 3.4: An example of one of the possible replica order of N bandpass signals.

Fig. 3.4 as an example, there are 2 boundary conditions as

$$f_s \leq \frac{f_{\ell_1}}{n_1},$$

and

$$f_s \geq \frac{f_{h_N}}{n_N + 1/2}.$$

and $N-1$ neighbor constraints as

$$f_s \leq \frac{f_{h_{i+1}} - f_{\ell_i}}{n_{i+1} - n_i} \quad , \text{ for } i = 1, 2, \cdots, N-1.$$

3.2.2 Algorithm for Searching the Ranges of Alias-free Sampling Frequency

Section 3.2.1 shows that given a particular segment index $(n_1, n_2, \cdots, n_N)$, the range of alias-free sampling frequency can be obtained. n_i is the index of segment where the original spectrum $X_i^+(f)$ is located, and can be obtained as

$$n_i = \left\lfloor \frac{f_{\ell_i}}{f_s} \right\rfloor \leq \left\lfloor \frac{f_{\ell_i}}{2(W_1 + W_2 + \cdots + W_N)} \right\rfloor, i = 1, 2, \cdots, N \tag{3.16}$$

where the inequality is obtained by the fact that f_s must be larger than the lowest possible sampling frequency. Since f_i is bounded in each corresponding segment

$$n_i f_s < f_i < (n_i + 1) f_s, i = 2, 3, \cdots, N \tag{3.17}$$

For a given n_1, a tighter bound of n_i for $i = 2, 3, \cdots, N$ can be obtained by multiplying (3.17) by R_i/f_s and taking floor operation to each side, where $R_i = f_{i+1}/f_i$ for $i = 1, 2, \cdots, N-1$

$$\lfloor R_i n_i \rfloor < n_{i+1} < \lfloor R_i(n_i + 1) \rfloor, i = 1, 2, \cdots, N-1 \tag{3.18}$$

The possible values of n_{i+1} can be obtained for a given n_i, $i = 1, 2, \cdots, N-1$ and $1 \leq n_1 \leq \lfloor \frac{f_1}{2(W_1+W_2+\cdots+W_N)} \rfloor$. Knowing all the possible values of n_i, there are two approaches to obtain the ranges of alias-free sampling frequency. The first approach is that for a given $(n_1, n_2 \cdots, n_N)$, the all ranges of alias-free sampling can be obtained by combining the neighbor and boundary constraints for each of the $2^N \times N!$ replica orders. The second approach is that choose two bandpass signals from the N bandpass signals, and evaluate the ranges of alias-free sampling frequency of the two bandpass signals, then there are total C_2^N tables which is like table 3.1. Thus the ranges for N bandpass signals can be obtained from the combination of the C_2^N tables. It is shown that the second approach is more computationally efficient than the first approach.

3.3 Method in [15]

In this section, we review a searching algorithm for minimum sampling frequency proposed by J. Bae and J. Park, which is achieved by finding the intersection of valid sampling frequencies for every two signal passbands. Section 3.3.1 provides the all valid sampling frequency ranges for N bandpass signals by the intersection of the ranges from any two passbands. Section 3.3.2 shows the valid sampling frequency ranges with user-specified minimum guard band and provides a procedure for searching the minimum sampling frequency.

3.3.1 Valid Sampling Frequency Ranges for Multiple Bandpass Signals

Consider the signal that consists of N bandpass signals ($2N$ passbands) as shown in Fig. 3.5(a), where f_i denotes the center frequency of each corresponding passband. First consider the valid sampling frequency rages of any two passbands $X_m(f)$ and $X_n(f)$, where $m, n \in \{\pm1, \pm2, \cdots, \pm N\}$ as shown in Fig. 3.5(b). Assume that sampling $X_m(f)$ and $X_n(f)$ with an alias-free sampling frequency denoting $f_{s_{m,n}}$ as shown in Fig. 3.5(c). To avoid aliasing, f_s needs to satisfy the following two constrains

$$f_n - \frac{W_n}{2} - r_{m,n} f_{s_{m,n}} \geq f_m + \frac{W_m}{2}$$

and

$$f_n + \frac{W_n}{2} - (r_{m,n}+1) f_{s_{m,n}} \leq f_m - \frac{W_m}{2}$$

which lead to the valid sampling frequency range

$$\frac{f_{n-m} + W_{m+n}/2}{r_{m,n}+1} \leq f_{s_{m,n}} \leq \frac{f_{n-m} - W_{m+n}/2}{r_{m,n}} \tag{3.19}$$

where $f_{n-m} = f_n - f_m$, $W_{m+n} = W_m + W_n$, and $r_{m,n}$ is an integer given by

$$0 \leq r_{m,n} \leq \left\lfloor \frac{f_{n-m} - W_{m+n}/2}{W_{m+n}} \right\rfloor \tag{3.20}$$

The all valid sampling frequency ranges can be obtained from the intersection $f_{s_{m,n}}$ of any two passbands $X_m(f)$ and $X_n(f)$, where $m, n \in \{\pm1, \pm2, \cdots, \pm N\}$. The number of $f_{s_{m,n}}$ is C_2^{2N} and the valid ranges for N bandpass signals can be expressed as

$$f_{s,all} = f_{s\_N-} \cap f_{s\_(N-1)-} \cap \cdots \cap f_{s\_1-} \cap f_{s\_1+} \cap \cdots \cap f_{s\_(N-1)+} \tag{3.21}$$

where

$$f_{s\_N-} = [\bigcap_{k=(N-1)-}^{1-} f_{s_{N-,k}}] \cap [\bigcap_{k=1+}^{N+} f_{s_{N-,k}}]$$

$$f_{s\_(N-1)-} = [\bigcap_{k=(N-2)-}^{1-} f_{s_{(N-1)-,k}}] \cap [\bigcap_{k=1+}^{N+} f_{s_{(N-1)-,k}}]$$

$$f_{s\_1-} = \bigcap_{k=1+}^{N+} f_{s_{1-,k}}$$

$$f_{s\_1+} = \bigcap_{k=2+}^{N+} f_{s_{1+,k}}$$

and

$$f_{s\_(N-1)+} = f_{s_{(N-1)+,N+}}$$

Note that $f_{s_{m,n}}$ and $f_{s_{-m,-n}}$ ('-' denotes the counterpart of the signal) have the same range for $m \neq n$ by symmetry. $f_{s_{m,-n}}$ and $f_{s_{-m,n}}$ have the same range for $m \neq n$ similarly. Therefore, the number of $f_{s_{m,n}}$ is reduced to $N+(C_2^{2N}-N)/2 = N^2$ and (3.21) can be modified as

$$f_{s,all} = f_{s\_N-} \cap f_{s\_(N-1)-} \cap f_{s\_(N-2)-} \cap \cdots \cap f_{s\_2-} \cap f_{s\_1-} \tag{3.22}$$

where

$$f_{s\_N-} = [\bigcap_{k=(N-1)-}^{1-} f_{s_{N-,k}}] \cap [\bigcap_{k=1+}^{N+} f_{s_{N-,k}}]$$

$$f_{s\_(N-1)-} = [\bigcap_{k=(N-2)-}^{1-} f_{s_{(N-1)-,k}}] \cap [\bigcap_{k=1+}^{(N-1)+} f_{s_{(N-1)-,k}}]$$

$$f_{s\_(N-2)-} = [\bigcap_{k=(N-3)-}^{1-} f_{s_{(N-2)-,k}}] \cap [\bigcap_{k=1+}^{(N-2)+} f_{s_{(N-2)-,k}}]$$

$$f_{s\_2-} = f_{s_{2-,1-}} \cap \bigcap_{k=1+}^{2+} f_{s_{2-,k}}$$

and

$$f_{s\_1-} = f_{s_{1-,1+}}$$

Note that the upper bound of $r_{m,n}$ in (3.20) is obtained from only considering the two passbands $X_m(f)$ and $X_n(f)$. To consider the all N bandpass signals, the bound for $r_{m,n}$ can be modified

$$0 \leq r_{m,n} \leq \left\lfloor \frac{f_{n-m} - W_{m+n}/2}{f_{bound}} \right\rfloor \tag{3.23}$$

where $f_{bound} = 2(W_1 + W_2 + \cdots + W_N)$, which is the lowest possible sampling frequency for no aliasing.

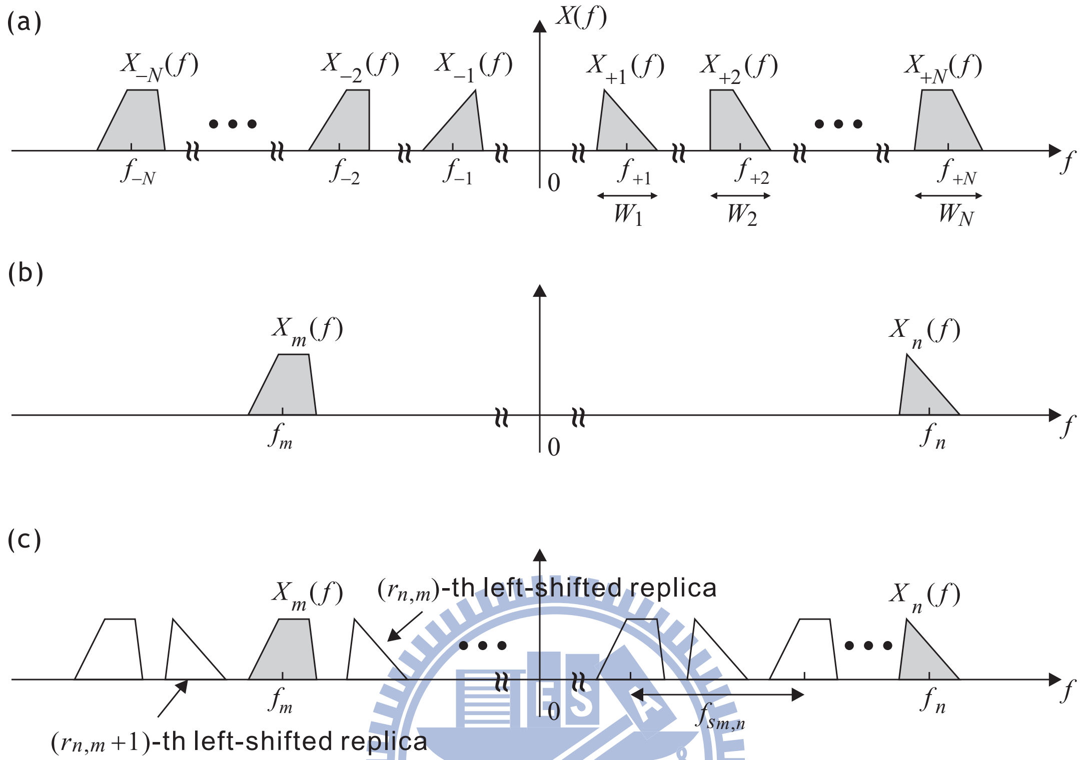

Figure 3.5: (a) Signal spectrum of N RF Signals. (b) Any two passbands of the $2N$ passbands. (c) The signal spectrum in (b) after bandpass sampling.

3.3.2 Algorithm for Searching The Minimum Sampling Frequency with User-Specified Minimum Guard-Band

To insert user-specified minimum guard band GB, the half of the guard band is added on both sides of each passbands band edges as shown in Fig. 3.6. The new valid sampling frequency range can be obtained by substituting W_{m+n} in (3.19) with $W_{m+n+2GB} = W_m + W_n + 2GB$, and substituting f_{bound} in (3.23) with $f_{GB_{bound}} = 2(W_1 + W_2 + \cdots + W_N + NGB)$. The two equations become

$$\frac{f_{n-m} + W_{m+n+2GB}/2}{r_{m,n} + 1} \le f_{s_{m,n}} \le \frac{f_{n-m} - W_{m+n+2GB}/2}{r_{m,n}} \tag{3.24}$$

$$0 \le r_{GB_{m,n}} \le \left\lfloor \frac{f_{n-m} - W_{m+n+2GB}/2}{f_{GB_{bound}}} \right\rfloor \tag{3.25}$$

Furthermore, it is shown that $2f_{GB_{bound}}$ is enough large to be an upper bound for minimum sampling frequency from numerical experiments. Thus, the bound of

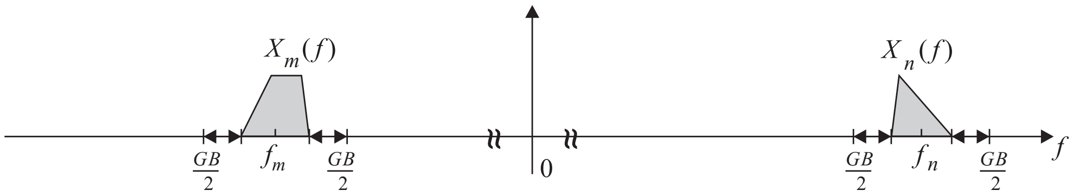

Figure 3.6: Signal spectrum after introducing a user-specified minimum guard band.

$r_{GB_{m,n}}$ (3.25) can be modified as

$$\left\lfloor \frac{f_{n-m} - W_{m+n+2GB}/2}{2f_{GB_{bound}}} \right\rfloor \leq r_{GB_{m,n}} \leq \left\lfloor \frac{f_{n-m} - W_{m+n+2GB}/2}{f_{GB_{bound}}} \right\rfloor \quad (3.26)$$

Based on the above discussion, the minimum sampling frequency can be obtained as follows:

1. Specify the value of the minimum guard band GB.

2. Evaluate the ranges of $r_{GB_{m,n}}$ for each $f_{s_{m,n}}$ using (3.26)

3. Evaluate the ranges of $f_{s_{m,n}}$ corresponding to each $r_{GB_{m,n}}$ using (3.24)

4. The minimum sapling frequency can be obtained as

$$f_{s,GB} = \min\{f_{s\_N-} \cap f_{s\_(N-1)-} \cap \cdots \cap f_{s\_1-}\} \quad (3.27)$$

Chapter 4

The Proposed Algorithm

In this chapter, we propose an efficient algorithm for finding the minimum sampling frequency for a signal consists of two bandpass signals. First we start up the analysis for a signal consists of two bandpass signals, which leads to four constraints of f_s for causing no aliasing. This will be discussed in section 4.1. In section 4.2 we introduce an efficient algorithm finding the minimum sampling frequency. Section 4.3 demonstrates the complexity analysis.

4.1 Conditions for Alias-free Sampling of Two Bandpass Signals

Conditions for alias-free sampling can be stated in various ways in terms of the band edges and bandwidths of the member bandpass signals. The conditions that are employed affect the complexity of ensuing algorithms. In this section, we derive a new set of conditions for alias-free sampling that will lead to an efficient algorithm in the next section.

First we consider the case of two bandpass signals for simplicity. Suppose we are to sample a signal $X(f)$ that consists of two bandpass signals $X_1(f)$ and $X_2(f)$ as shown in Fig. 4.1. Assume $X_i(f) \neq 0$ for $f_{\ell_i} < |f| < f_{h_i}$, $i = 1, 2$, where f_{ℓ_i} and f_{h_i} are band edges, and $W_i = f_{h_i} - f_{\ell_i}$ are one-sided bandwidths as indicated in the figure. Let $X_i^+(f)$, and $X_i^-(f)$ denote respectively the positive frequency part and negative frequency part of $X_i(f)$. There are four signal bands,

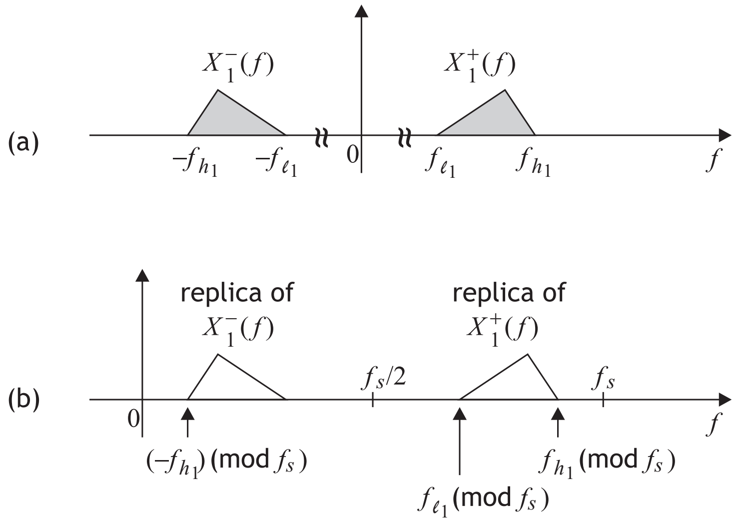

Figure 4.1: An example of spectrum that consists of two bandpass signals.

including $X_1^+(f)$, $X_1^-(f)$, $X_2^+(f)$, and $X_2^-(f)$. Since the replicas of any two bands may overlap and result in aliasing after sampling, there are a total of $C_2^4 = 6$ cases. Note that $X_1^+(f)$ and $X_1^-(f)$ are symmetric with respect to 0, and so are $X_2^+(f)$ and $X_2^-(f)$. If $X_1^+(f)$ and $X_2^+(f)$ are not aliasing after sampling, then $X_1^-(f)$ and $X_2^-(f)$ will not be aliasing by symmetry. Similarly, if $X_1^-(f)$ and $X_2^+(f)$ are not aliasing after sampling, then $X_1^+(f)$ and $X_2^-(f)$ will not be aliasing. Thus, we need to consider only 4 cases:

$$\begin{array}{ll} \text{(a)} & \{X_1^+(f), X_1^-(f)\} \\ \text{(b)} & \{X_2^+(f), X_2^-(f)\} \\ \text{(c)} & \{X_1^+(f), X_2^+(f)\} \\ \text{(d)} & \{X_1^-(f), X_2^+(f)\}. \end{array} \tag{4.1}$$

Case (a). If we consider only the pair $\{X_1^+(f), X_1^-(f)\}$ as shown in Fig. 4.2(a), this is the same as the case of one bandpass signal. For convenience, we will derive a condition in terms of the band edge f_{h_1} and one-sided bandwidth W_1. Upon sampling with frequency f_s, replicas of $X_1^+(f)$ and $X_1^-(f)$ appear every f_s, resulting in a periodic spectrum; we can simply consider the period $[0, f_s)$. Since $X_1^+(f)$ and $X_1^-(f)$ are symmetric with respect zero, the replicas of $X_1^+(f)$ and $X_1^-(f)$ are symmetric with respect to $\frac{f_s}{2}$ in the interval $[0, f_s)$ (Fig. 4.2(b)).

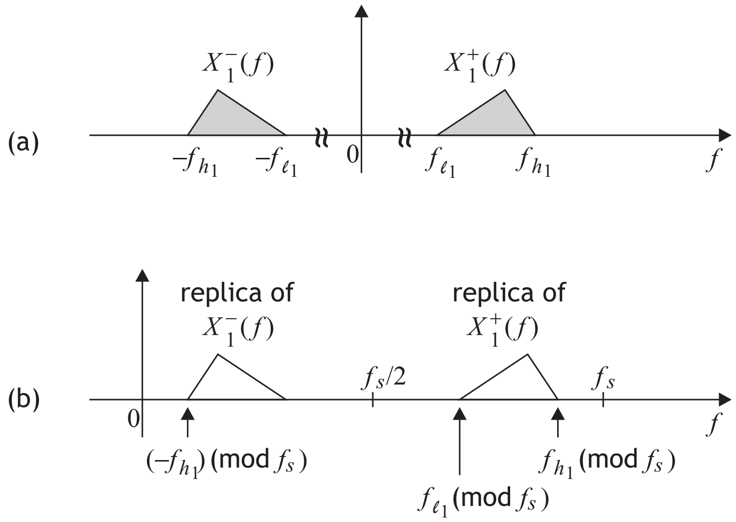

Figure 4.2: (a) The spectrum of $X_1^+(f)$ and $X_1^-(f)$. (b) An example of the folded spectrum for the interval $[0, f_s)$.

Observe that if 0 or f_s is not contained inside the band of replicas of $X_1^+(f)$ and $X_1^-(f)$, there will not be aliasing. One necessary and sufficient condition for alias-free sampling is thus $f_{h_1} \pmod{\frac{f_s}{2}} = 0$, or $f_{h_1} \pmod{\frac{f_s}{2}} \geq W_1$. Equivalently, we have

$$\begin{aligned} & 2f_{h_1} \pmod{f_s} = 0 \\ \text{or} \quad & 2f_{h_1} \pmod{f_s} \geq 2W_1 \end{aligned} \tag{4.2}$$

Case (b). Similar to case (a), if we consider the pair $\{X_2^+(f), X_2^-(f)\}$ as shown in Fig. 4.3(a), since $X_2^+(f)$ and $X_2^-(f)$ are symmetric with respect zero, the replicas of $X_2^+(f)$ and $X_2^-(f)$ are symmetric with respect to $\frac{f_s}{2}$ in the interval $[0, f_s)$ (Fig. 4.3(b)). Observe that if 0 or f_s is not contained inside the band of replicas of $X_2^+(f)$ and $X_2^-(f)$, there will not be aliasing. One necessary and sufficient condition for alias-free sampling is thus $f_{h_2} \pmod{\frac{f_s}{2}} = 0$, or $f_{h_2} \pmod{\frac{f_s}{2}} \geq W_2$. Equivalently, we have there will be no aliasing if and only if

$$\begin{aligned} & 2f_{h_2} \pmod{f_s} = 0, \\ \text{or} \quad & 2f_{h_2} \pmod{f_s} \geq 2W_2 \end{aligned} \tag{4.3}$$

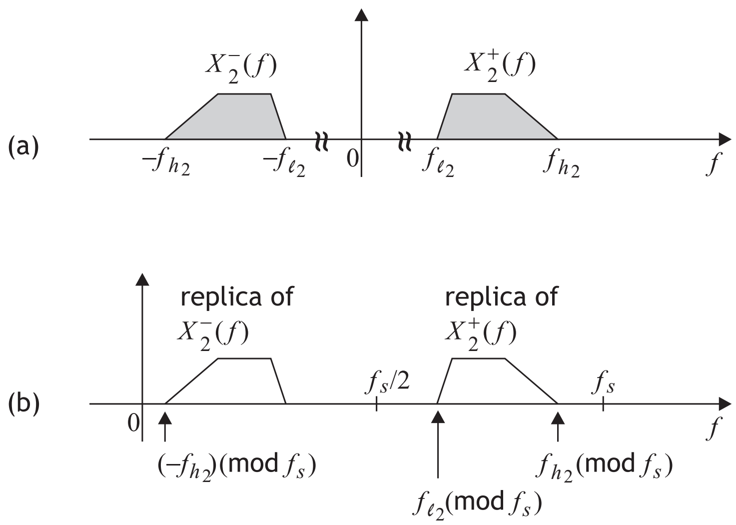

Figure 4.3: (a) The spectrum of $X_2^+(f)$ and $X_2^-(f)$. (b) An example of the folded spectrum for the interval $[0, f_s)$.

Case (c). Consider Fig. 4.4(a) where we have shown only the pair $\{X_1^+(f), X_2^+(f)\}$. First observe that there is no aliasing due to this pair if and only if there is no aliasing when we sample a shifted version of the pair $\{X_1^+(f + f_0), X_2^+(f + f_0)\}$ where f_0 is the shift. For convenience we will consider the condition for alias-free sampling of the pair with a shift. Suppose we choose f_0 as the midpoint of f_{ℓ_1} and f_{h_2}, i.e.,

$$f_0 = (f_{\ell_1} + f_{h_2})/2.$$

Then the shifted pair is as shown in Fig. 4.4(b), where

$$a = \frac{f_{h_2} - f_{\ell_1}}{2},$$

$$b = f_{\ell_2} - (f_{\ell_1} + f_{h_2})/2,$$

$$c = f_{h_1} - (f_{\ell_1} + f_{h_2})/2.$$

If we consider the folded spectrum in the $[0, f_s)$ interval, the band edges a (mod f_s) and $(-a)$ (mod f_s) are equal-distanced from $f_s/2$. We now discuss two possible scenarios (i) a (mod f_s) $\geq$ $(-a)$ (mod f_s) and (ii) a (mod f_s) $<$ $(-a)$ (mod f_s). Examples of these two possible cases are shown respectively in Fig. 4.4(c) and (d).

(i) When $a \pmod{f_s} \geq (-a) \pmod{f_s}$ there will be no aliasing if and only if $(-a) \pmod{f_s} = a \pmod{f_s}$ or if the interval $((-a) \pmod{f_s}, a \pmod{f_s})$ is large enough to accommodate the two replicas. That is,

$$\begin{aligned} & a \pmod{f_s} - ((-a) \pmod{f_s}) = 0, \\ \text{or} \quad & a \pmod{f_s} - ((-a) \pmod{f_s}) \geq W_1 + W_2. \end{aligned}$$

The equivalent conditions are

$$\begin{aligned} & 2a \pmod{f_s} = 0, \\ \text{or} \quad & 2a \pmod{f_s} \geq W_1 + W_2 \end{aligned} \tag{4.4}$$

(ii) when $a \pmod{f_s} < (-a) \pmod{f_s}$ as shown in Fig. 4.4(d), there is some space between the two replicas and the space is of length $((-a) \pmod{f_s} - a \pmod{f_s})$. There will be no aliasing if and only if the remaining part of the $[0, f_s)$ interval is large enough to take in the two replicas. That is,

$$f_s - ((-a) \pmod{f_s}) - a \pmod{f_s}) \geq W_1 + W_2.$$

Or equivalently

$$2a \pmod{f_s} \geq W_1 + W_2$$

This is the same as the second condition in (4.4).

Substituting $a = (f_{h_2} - f_{\ell_1})/2$ to (4.4), we obtain one necessary and sufficient condition for alias-free sampling

$$\begin{aligned} & (f_{h_2} - f_{\ell_1}) \pmod{f_s} = 0, \\ \text{or} \quad & (f_{h_2} - f_{\ell_1}) \pmod{f_s} \geq W_1 + W_2 \end{aligned} \tag{4.5}$$

Case (d). Similarly, for the pair $\{X_1^-(f), X_2^+(f)\}$ as shown in Fig. 4.5(a), we can use the technique in case (c) to consider the condition for alias-free sampling of the pair with a shift where we choose f_0 as the midpoint of $-f_{h_1}$ and f_{h_2}, i.e.,

$$f_0 = (f_{h_2} - f_{h_1})/2.$$

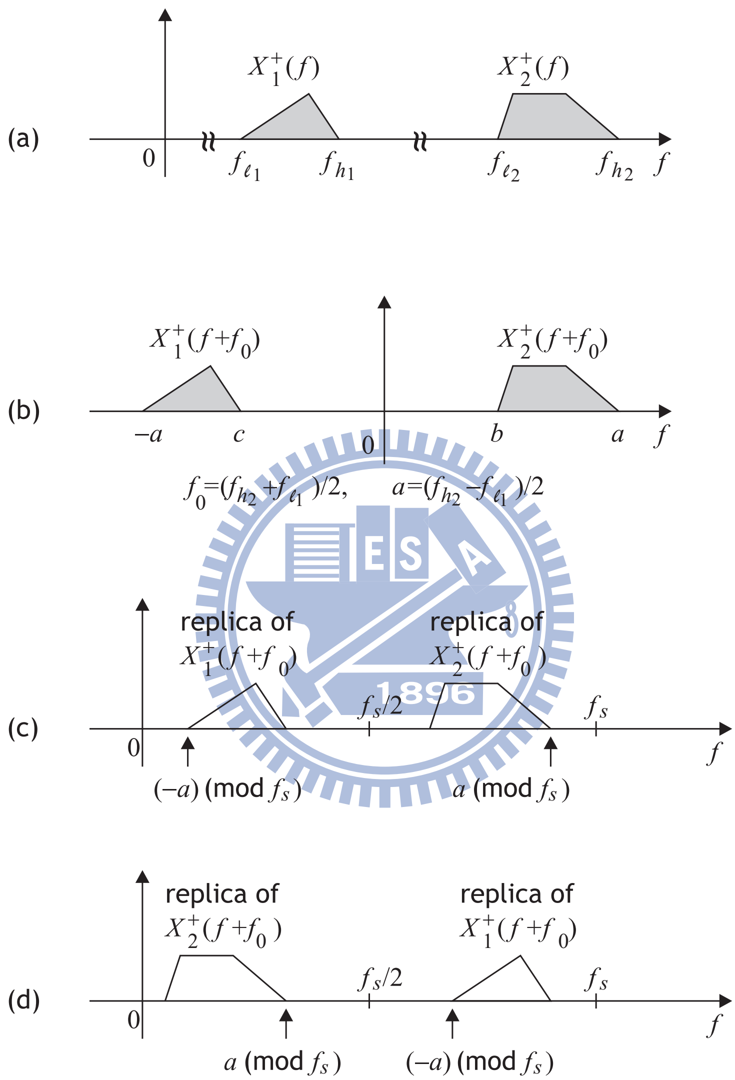

Figure 4.4: (a) The spectrum of $X_1^+(f)$ and $X_2^+(f)$. (b) The shifted spectrum $X_1^+(f+f_0)$ and $X_2^+(f+f_0)$, where $f_0 = (f_{h_2}+f_{\ell_1})/2$ and $a = (f_{h_2}-f_{\ell_1})/2$. (c) An example of the folded spectrum for the interval $[0, f_s)$ when $a \pmod{f_s} \geq (-a) \pmod{f_s}$. (d) An example of the folded spectrum for the interval $[0, f_s)$ when $a \pmod{f_s} < (-a) \pmod{f_s}$.

Then the shifted pair is as shown in Fig. 4.5(b), where

$$a = \frac{f_{h_1} + f_{h_2}}{2},$$

$$b = f_{\ell_2} - (f_{h_2} - f_{h_1})/2,$$

$$c = -f_{\ell_1} - (f_{h_2} - f_{h_1})/2.$$

If we consider the folded spectrum in the $[0, f_s)$ interval, the band edges $a \pmod{f_s}$ and $(-a) \pmod{f_s}$ are equal-distanced from $f_s/2$. We can discuss two possible scenarios (i) $a \pmod{f_s} \geq (-a) \pmod{f_s}$ and (ii) $a \pmod{f_s} < (-a) \pmod{f_s}$ as case (c) similarly and examples of these two possible cases are shown respectively in Fig. 4.5(c) and (d). Substituting $a = (f_{h_1} + f_{h_2})/2$ to (4.4), we obtain one necessary and sufficient condition for alias-free sampling

$$\begin{aligned} & (f_{h_1} + f_{h_2}) \pmod{f_s} = 0, \\ \text{or} \quad & (f_{h_1} + f_{h_2}) \pmod{f_s} \geq W_1 + W_2 \end{aligned} \tag{4.6}$$

Summarizing, for a given sampling frequency f_s, there will not be aliasing if the following four conditions are satisfied.

1. $2f_{h_1} \pmod{f_s} = 0$ or $2f_{h_1} \pmod{f_s} \geq 2W_1$
2. $2f_{h_2} \pmod{f_s} = 0$ or $2f_{h_2} \pmod{f_s} \geq 2W_2$
3. $(f_{h_2} - f_{\ell_1}) \pmod{f_s} = 0$ or $(f_{h_2} - f_{\ell_1}) \pmod{f_s} \geq W_1 + W_2$
4. $(f_{h_1} + f_{h_2}) \pmod{f_s} = 0$ or $(f_{h_1} + f_{h_2}) \pmod{f_s} \geq W_1 + W_2$

4.2 Proposed Algorithm for finding the Minimum Sampling Frequency of Two Bandpass Signals

In this section we propose an efficient algorithm for finding the minimum sampling frequency. For simplicity, first consider the case of two bandpass signals, which we have derive four alias-free conditions in section 4.1. For each of the four

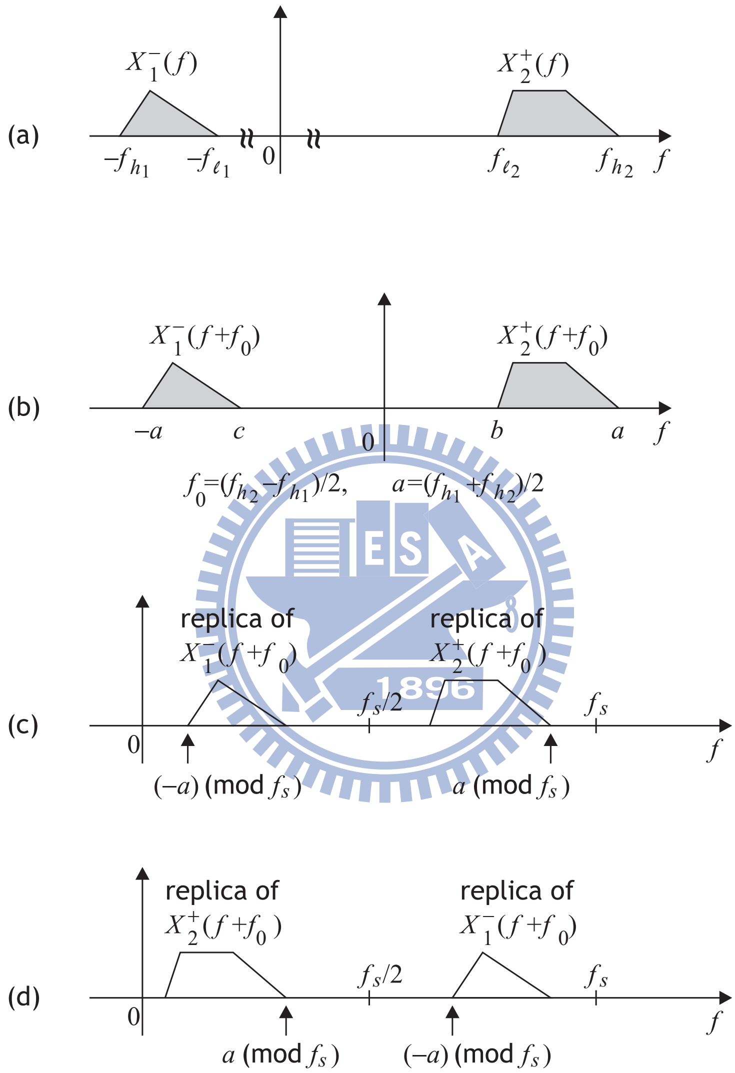

Figure 4.5: (a) The spectrum of $X_1^-(f)$ and $X_2^+(f)$. (b) The shifted spectrum $X_1^-(f+f_0)$ and $X_2^+(f+f_0)$, where $f_0 = (f_{h_2} - f_{h_1})/2$ and $a = (f_{h_1} + f_{h_2})/2$. (c) An example of the folded spectrum for the interval $[0, f_s)$ when $a \pmod{f_s} \geq (-a) \pmod{f_s}$. (d) An example of the folded spectrum for the interval $[0, f_s)$ when $a \pmod{f_s} < (-a) \pmod{f_s}$.

cases, we derive the minimum increment in sampling frequency such that the corresponding condition for alias-free sampling can be satisfied.

Case (a). Suppose the condition in (4.2) is not satisfied for a given sampling frequency f_s. Consider the folded spectrum for the interval $[0, f_s)$. We discuss the two cases (i) $0 < f_{h_1} \pmod{f_s} < f_s/2$ and (ii) $f_s/2 < f_{h_1} \pmod{f_s} < f_s$ separately.

(i) $0 < f_{h_1} \pmod{f_s} < f_s/2$: When we gradually increase the sampling frequency the band edge $f_{h_1} \pmod{f_s}$ of replica $X_1^+(f)$ moves towards 0 while the band edge $-f_{h_1} \pmod{f_s}$ of replica $X_1^-(f)$ moves towards f_s. When the sampling frequency is increased such that $f_{h_1} \pmod{f_s}$ decreases to 0, then the condition in (4.2) becomes satisfied.

(ii) $f_s/2 < f_{h_1} \pmod{f_s} < f_s$: Similarly the condition in (4.2) becomes satisfied when $f_{h_1} \pmod{f_s}$ decreases to $f_s/2$.

Therefore we can conclude that the alias-free condition (4.2) can be satisfied by increasing the sampling frequency such that f_{h_1} becomes an integer multiple of $f_s/2$. The smallest new sampling $f_{s,new}$ for this to happen can be computed as follows. Let

$$f_{h_1} = n_{h_1} f_s/2 + r_{h_1},$$

where $r_{h_1} = f_{h_1} \pmod{f_s/2}$ and $n_{h_1} = \lfloor f_{h_1}/(f_s/2) \rfloor$. Then we have $f_{h_1} = n_{h_1} f_{s,new}/2$, or equivalently

$$f_{s,new} = \frac{2f_{h_1}}{n_{h_1}} = \frac{2f_{h_1}}{\lfloor f_{h_1}/(f_s/2) \rfloor} = \frac{2f_{h_1}}{\lfloor 2f_{h_1}/f_s \rfloor}, \tag{4.7}$$

where we have used the fact that n_{h_1} can also be computed using $n_{h_1} = \lfloor 2f_{h_1}/f_s \rfloor$.

Case (b). Similar to case (a), if the condition in (4.3) is not satisfied, we can increase sampling frequency to

$$f_{s,new} = \frac{2f_{h_2}}{\lfloor 2f_{h_2}/f_s \rfloor}, \tag{4.8}$$

then (4.3) will become satisfied.

Case (c). Suppose the condition in (4.5) is not satisfied. Consider again the shifted spectrum in Fig. 4.4(b). Using the steps in case (a), we can verify that there will be not aliasing if we increase the sampling frequency so that $a \pmod{f_s}$ to be equal to 0 or $\frac{f_s}{2}$. Moreover the new sampling frequency can be obtained by

$$f_{s,new} = \frac{2a}{\lfloor a/\frac{f_s}{2} \rfloor} = \frac{f_{h_2} - f_{\ell_1}}{\lfloor (f_{h_2} - f_{\ell_1})/f_s \rfloor} \tag{4.9}$$

Case (d). Like case (c), if the condition in (4.6) is not satisfied, we can increase the sampling frequency to

$$f_{s,new} = \frac{f_{h_1} + f_{h_2}}{\lfloor (f_{h_1} + f_{h_2})/f_s \rfloor} \tag{4.10}$$

then (4.6) will be satisfied.

Proposed iterative algorithm

Using the conditions for alias-free sampling in section 4.1 and the methods for computing new sampling frequency for each case, we have the following iterative algorithm for finding the minimum sampling frequency. To start off, let $f_s = 2(W_1 + W_2)$, which is the lowest possible sampling frequency for no aliasing.

1. Examine if the condition for case (a) in (4.2) is satisfied. If it is, go to the next step. If it is not satisfied, compute the new sampling frequency using (4.7) and go to the next step.

2. If the condition (4.3) for case (b) is satisfied, go to the next step. If it is not satisfied, compute the new sampling frequency using (4.8) and go to step 1.

3. If the condition (4.5) for case (c) is satisfied, go to the next step. If it is not, compute the new sampling frequency using (4.9) and go to step 1.

4. If the condition (4.6) for case (d) is not satisfied, compute the new sampling frequency using (4.10) and go to step 1. If it is satisfied then we have found the minimum sampling frequency.

Usually not all four steps are performed in one iteration.

4.3 Complexity

In this section we will analyze the computation of the algorithm. In the algorithm for two bandpass signals, the main computations are in the inspection of conditions in (4.2), (4.3), (4.5) and (4.6), and the computation of new sampling frequency in (4.7)-(4.10). Few computations are required for these equations as we can borrow results from earlier evaluations. For example in step 1 we compute $2f_{h_1} \pmod{f_s}$ in (4.2). In the process we can also obtain the integer n_{h_1} which is used in computing the new sampling frequency (4.7). Similar conclusions can be drawn for steps 2. In step 3, we need to evaluate $f_{h_2} - f_{\ell_1} \pmod{f_s}$ which can be written as

$$\begin{aligned} & (f_{h_2} - f_{\ell_1}) \pmod{f_s} \\ = & \underbrace{(f_{h_2} \pmod{f_s} - f_{\ell_1} \pmod{f_s})}_{\text{call this } x} \pmod{f_s} \\ = & \begin{cases} x &, \quad x \geq 0, \\ x + f_s &, \quad \text{otherwise.} \end{cases} \end{aligned} \tag{4.11}$$

When we are in step 3, the conditions in step 1 are already satisfied, we can obtain $f_{\ell_1} \pmod{f_s}$ using

$$f_{\ell_1} \pmod{f_s} = f_{h_1} \pmod{f_s} - W_1.$$

if $f_{h_1} \pmod{f_s} \neq 0$. When $f_{h_1} \pmod{f_s} = 0$, $f_{\ell_1} \pmod{f_s} = f_s - W_1$. f_{h_1} and $2f_{h_1}$ can be expressed as follows.

$$f_{h_1} = mf_s + f_{h_1} \pmod{f_s}, \text{ where } m = \left\lfloor \frac{f_{h_1}}{f_s} \right\rfloor$$

$$2f_{h_1} = nf_s + (2f_{h_1}) \pmod{f_s}, \text{ where } n = \left\lfloor \frac{2f_{h_1}}{f_s} \right\rfloor$$

Comparing the above two equations, $f_{h_1} \pmod{f_s} = (2f_{h_1}) \pmod{f_s}/2$ if n is even and $f_{h_1} \pmod{f_s} = ((2f_{h_1}) \pmod{f_s} + f_s)/2$ if n is odd. Thus both $f_{h_1} \pmod{f_s}$ and $f_{h_2} \pmod{f_s}$ can be obtained from steps 1 and 2. The evaluation for step 3 requires at most 5 additions. Similarly in step 4, we need to evaluate $f_{h_1} + f_{h_2} \pmod{f_s}$ which can be written as

$$\begin{aligned} & (f_{h_1} + f_{h_2}) \pmod{f_s} \\ = & \underbrace{(f_{h_1} \pmod{f_s} + f_{h_2} \pmod{f_s})}_{\text{call this } y} \pmod{f_s} \\ = & \begin{cases} y & , \ y < f_s, \\ y + f_s & , \ \text{otherwise.} \end{cases} \end{aligned} \tag{4.12}$$

$f_{h_1} \pmod{f_s}$ and $f_{h_2} \pmod{f_s}$ are already obtained from step 3. The evaluation requires at most 2 additions.

Chapter 5

Generalization and Extensions

In this chapter, we extends the proposed algorithm to general case. In section 5.1, we extend the results in Sec. 4.1 and Ch. 3 to the case when there is a user-specified minimum guard band. In Sec. 5.2, we propose a method for finding a valid sampling frequency range. In section 5.3, we extend the case of two bandpass signals to the case of multi-band signals.

5.1 Guard Bands

In practice it is desirable to have guard bands between different bandpass signals after sampling. Suppose the minimum guard interval is GB. Then every 2 passbands should be spaced apart by at least GB after sampling. We can consider the spacing of every two replica as in Sec. 4.1, and there are a total of C_2^4 cases. Again due to the fact that $X_i^+(f)$ and $X_i^-(f)$ are symmetric with respect to 0, if there is a guard band of at least GB between the replica of $X_1^+(f)$ and $X_2^+(f)$, then replicas of $X_1^-(f)$ and $X_2^-(f)$ will be spaced apart by at least GB. Similar conclusion can be drown for the pair $\{X_1^-(f), X_2^+(f)\}$ and $\{X_1^+(f), X_2^-(f)\}$. Therefore, we only need to consider the spacing of pairs (c) and (d) in (4.1).

Case(c). *The pair* $\{X_1^+(f), X_2^+(f)\}$. Let us make the following adjustment of band edges for $X_1^+(f)$ and $X_2^+(f)$

$$\begin{array}{rcl} f'_{h_i} & = & f_{h_i} + GB/2 \\ f'_{\ell_i} & = & f_{\ell_i} - GB/2 \end{array}, \text{for } i = 1, 2. \tag{5.1}$$

Then $X_1^+(f)$ and $X_2^+(f)$ have expanded bandwidths

$$W_1' = W_1 + GB \quad \text{and} \quad W_2' = W_2 + GB \tag{5.2}$$

respectively, as illustrated in Fig. 5.1(a). We can verify expanding the passband like this, we are effectively placing a guard band of $GB/2$ on each side of $X_1^+(f)$ and $X_2^+(f)$, there will be a spacing of at least GB if the newly defined band edges and bandwidths satisfy the alias-free sampling condition in (4.5), that is,

$$\begin{aligned} & (f'_{h_2} - f'_{\ell_1}) \pmod{f_s} = 0, \\ \text{or} \quad & (f'_{h_2} - f'_{\ell_1}) \pmod{f_s} \geq W_1' + W_2' \end{aligned} \tag{5.3}$$

Fig. 5.1(b) shows an example of the replicas in the frequency range $[0, f_s)$ when the above condition is satisfied. The number a_0 and f_0 that are useful in the analysis of locations of replicas in case (c) of Sec. 4.1 are now respectively

$$a' = \frac{1}{2}(f'_{h_2} - f'_{\ell_1}), \quad f'_0 = \frac{1}{2}(f'_{h_2} + f'_{\ell_1}).$$

If the condition in (5.3) is not satisfied, we can increase the sampling frequency to

$$f_{s,new} = \frac{f'_{h_2} - f'_{\ell_1}}{\lfloor (f'_{h_2} - f'_{\ell_1})/f_s \rfloor} \tag{5.4}$$

Then (5.3) will become satisfied and there will be a spacing of GB.

Case (d). *The pair* $\{X_1^-(f), X_2^+(f)\}$. Similar to case (c) above, we can make the adjustment of band edges and bandwidths as in (5.1) and (5.2), which is shown in Fig. 5.2(a). There will be a guard band of GB if the new band edges satisfy the alias-free condition in (4.6), that is

$$\begin{aligned} & (f'_{h_1} + f'_{h_2}) \pmod{f_s} = 0, \\ \text{or} \quad & (f'_{h_1} + f'_{h_2}) \pmod{f_s} \geq W_1' + W_2' \end{aligned} \tag{5.5}$$

If the condition in (5.5) is not satisfied, we can increase the sampling frequency to

$$f_{s,new} = \frac{f'_{h_1} + f'_{h_2}}{\lfloor (f'_{h_1} + f'_{h_2})/f_s \rfloor} \tag{5.6}$$

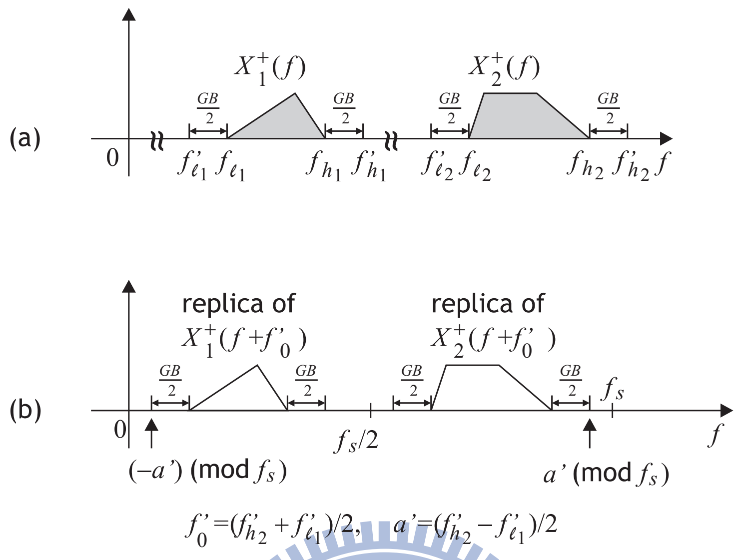

Figure 5.1: (a)The spectrum of $X_1^+(f)$ and $X_2^+(f)$ with expanded passbands. (b) An example of the folded spectrum for the interval $[0, f_s)$ with guard band.

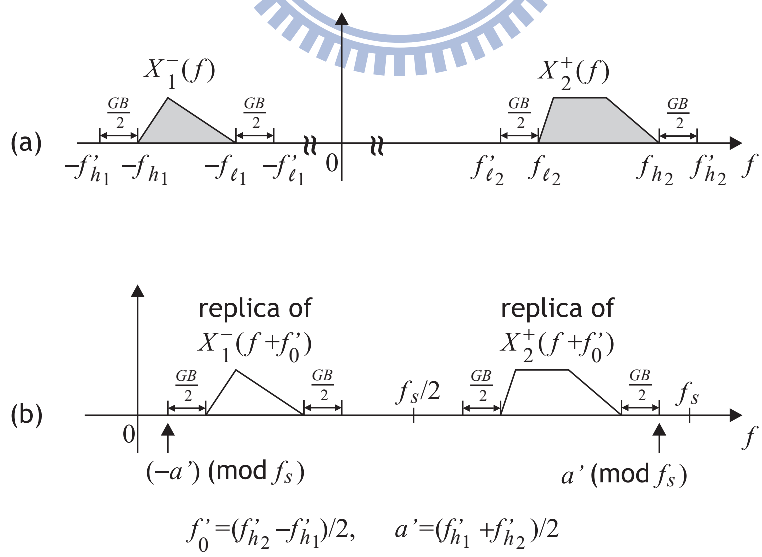

Figure 5.2: (a)The spectrum of $X_1^-(f)$ and $X_2^+(f)$ with expanded passbands. (b) An example of the folded spectrum for the interval $[0, f_s)$ with guard band.

Fig. 5.2(b) shows an example with the above condition satisfied.

The iterative algorithm in Sec. 4.2 can be modified for the case with guard band. The first 2 steps can be carried out as before. For the 3rd and 4th steps, we will use the new band edges and bandwidths in (5.1) and (5.2) to check whether there is enough spacing between guard bands ((5.3) and (5.5)) and to increase the sampling frequency ((5.4) and (5.6)) when the conditions are not satisfied.

5.2 Finding a Range of Valid Sampling Frequency

In practice it is desirable to have a sampling frequency range for alias-free sampling. From the proposed algorithm we obtain a minimum sampling frequency $f_{s_{min}}$ that meets the four alias-free conditions (4.1). For each case, we can gradually increase the sampling frequency and have a boundary when the alias-free condition becomes unsatisfied if we keep increasing the sampling frequency. We will derive the boundary for each case.

Case (a). *The pair* $\{X_1^+(f), X_1^-(f)\}$. Suppose the condition in (4.2) is satisfied for a given sampling frequency f_s. Consider the folded spectrum for the interval $[0, f_s)$. We discuss the two cases (i) $0 < f_{\ell_1} \pmod{f_s} < f_s/2$ and (ii) $f_s/2 < f_{\ell_1}$ $\pmod{f_s} < f_s$ separately as shown in Fig. 5.3(a) and (b).

(i) $0 < f_{\ell_1} \pmod{f_s} < f_s/2$: When we gradually increase the sampling frequency the band edge $f_{\ell_1} \pmod{f_s}$ of replica $X_1^+(f)$ moves towards 0 while the band edge $(-f_{\ell_1}) \pmod{f_s}$ of replica $X_1^-(f)$ moves towards f_s. When the sampling frequency is increased such that $f_{\ell_1} \pmod{f_s}$ decreases to 0, the condition in (4.2) becomes unsatisfied if we keep increasing the sampling frequency.

(ii) $f_s/2 < f_{\ell_1} \pmod{f_s} < f_s$: Similarly the condition in (4.2) becomes unsatisfied if we keep increasing the sampling frequency when $f_{\ell_1} \pmod{f_s}$ decreases to $f_s/2$.

Therefore we can conclude that the alias-free condition (4.2) becomes unsatisfied if we keep increasing the sampling frequency when f_{ℓ_1} becomes an integer

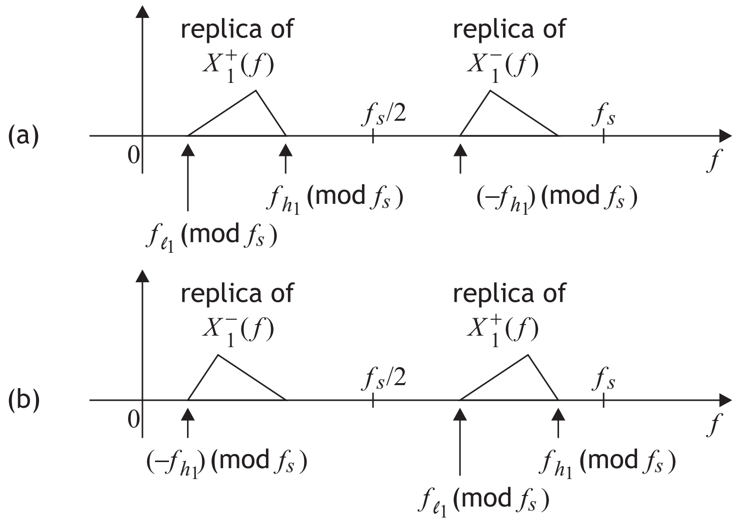

Figure 5.3: (a) An example of the folded spectrum for the interval $[0, f_s)$ when $0 < f_{\ell_1} \pmod{f_s} < f_s/2$. (b) An example of the folded spectrum for the interval $[0, f_s)$ when $f_s/2 < f_{\ell_1} \pmod{f_s} < f_s$.

multiple of $f_s/2$. The boundary for this case f_{u_a} can be computed as follows. Let

$$f_{\ell_1} = n_{\ell_1} f_s/2 + r_{\ell_1},$$

where $r_{\ell_1} = f_{\ell_1} \pmod{f_s/2}$ and $n_{\ell_1} = \lfloor f_{\ell_1}/(f_s/2) \rfloor$. Then we have $f_{\ell_1} = n_{\ell_1} f_{u_a}/2$, or equivalently

$$f_{u_a} = \frac{2f_{\ell_1}}{n_{\ell_1}} = \frac{2f_{\ell_1}}{\lfloor f_{\ell_1}/(f_s/2) \rfloor} = \frac{2f_{\ell_1}}{\lfloor 2f_{\ell_1}/f_s \rfloor}, \tag{5.7}$$

where we have used the fact that n_{ℓ_1} can also be computed using $n_{\ell_1} = \lfloor 2f_{\ell_1}/f_s \rfloor$.

Case (b). *The pair* $\{X_2^+(f), X_2^-(f)\}$. Similar to case (a), if the condition in (4.3) is satisfied, we can increase sampling frequency to

$$f_{u_b} = \frac{2f_{\ell_2}}{n_{\ell_2}} = \frac{2f_{\ell_2}}{\lfloor f_{\ell_2}/(f_s/2) \rfloor} = \frac{2f_{\ell_2}}{\lfloor 2f_{\ell_2}/f_s \rfloor}, \tag{5.8}$$

then (4.3) will become unsatisfied if we keep increasing the sampling frequency.

Case (c). *The pair* $\{X_1^+(f), X_2^+(f)\}$ as shown in Fig. 5.4(a). For convenience we consider the condition of the pair with a shift. Suppose we choose f_0 as the midpoint of f_{h_1} and f_{ℓ_2}, i.e.,

$$f_0 = (f_{h_1} + f_{\ell_2})/2.$$

Then the shifted pair is as shown in Fig. 5.4(b), where

$$x = \frac{f_{\ell_2} - f_{h_1}}{2}.$$

If we consider the folded spectrum in the $[0, f_s)$ interval, the band edges $x \pmod{f_s}$ and $(-x) \pmod{f_s}$ are equal-distanced from $f_s/2$. In Fig. 5.4(c) and (d) we show the two possible scenarios of $x \pmod{f_s} \geq (-x) \pmod{f_s}$ and $x \pmod{f_s} < (-x) \pmod{f_s}$ respectively. Using the steps in case (a), we can verify that if the condition in (4.5) is satisfied, we can increase the sampling frequency such that $x \pmod{f_s}$ to be equal to 0 or $f_s/2$. Moreover we can increase sampling frequency to

$$f_{u_c} = \frac{2x}{\lfloor x/(f_s/2) \rfloor} = \frac{f_{\ell_2} - f_{h_1}}{\lfloor (f_{\ell_2} - f_{h_1})/2 \rfloor}, \tag{5.9}$$

then (4.5) will become unsatisfied if we keep increasing the sampling frequency.

Case (d). *The pair* $\{X_1^-(f), X_2^+(f)\}$. Like case (c), if the condition in (4.6) is satisfied, we can increase sampling frequency to

$$f_{u_d} = \frac{f_{\ell_1} + f_{\ell_2}}{\lfloor (f_{\ell_1} + f_{\ell_2})/2 \rfloor}, \tag{5.10}$$

then (4.6) will become unsatisfied if we keep increasing the sampling frequency.

For a given minimum frequency $f_{s,min}$ for alias-free sampling, we have derived four boundaries for each case using (5.7), (5.8), (5.9), and (5.10). To ensure the four alias-free conditions to be satisfied, the upper bound for the valid range can be obtained by choosing the minimum of the four boundaries. Then we have a valid sampling frequency range

$$f_{s,min} \leq f_s \leq \min\{f_{u_a}, f_{u_b}, f_{u_c}, f_{u_d}\}. \tag{5.11}$$

5.3 Multiple-Bandpass Signals

We can extend the proposed algorithm to find the minimum sampling frequency for multiple bandpass signals. Suppose we are to sample a signal consist N bandpass signals ($2N$ bands). Since every two of the passbands may cause aliasing, we need to consider C_2^{2N} cases. In the analysis of two bandpass signals, we note

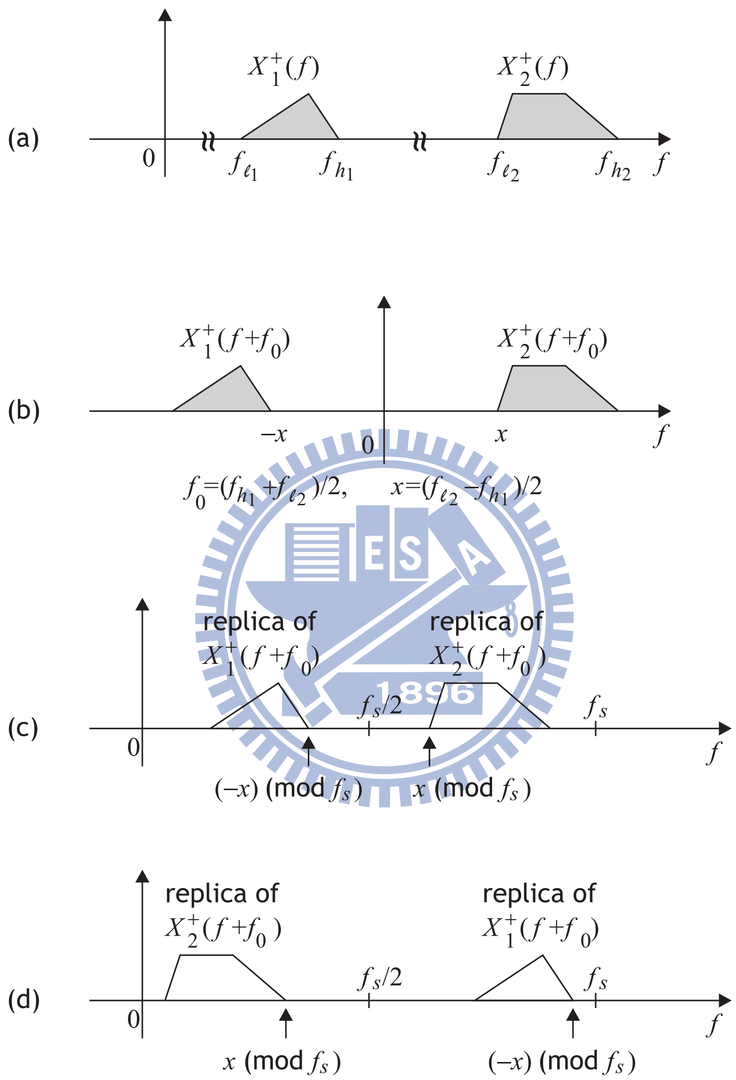

Figure 5.4: (a) The spectrum of $X_1^+(f)$ and $X_2^+(f)$. (b) The shifted spectrum $X_1^+(f+f_0)$ and $X_2^+(f+f_0)$, where $f_0 = (f_{h_1}+f_{\ell_2})/2$ and $a = (f_{\ell_2}-f_{h_1})/2$. (c) An example of the folded spectrum for the interval $[0, f_s)$ when $x \pmod{f_s} \geq (-x) \pmod{f_s}$. (d) An example of the folded spectrum for the interval $[0, f_s)$ when $x \pmod{f_s} < (-x) \pmod{f_s}$.

that $X_i^+(f)$ and $X_i^-(f)$ are symmetric to 0, which is shown in Fig. 5.5(a), we can have N alias-free conditions

$$\begin{aligned} & 2f_{h_i} \pmod{f_s} = 0 \\ \text{or} \quad & 2f_{h_i} \pmod{f_s} \geq 2W_i, \text{ for } i = 1, 2, \cdots, N. \end{aligned} \tag{5.12}$$

Consider the other $C_2^{2N} - N$ cases, we note that if $X_i^+(f)$ and $X_j^+(f)$ are not aliasing after sampling, then $X_i^-(f)$ and $X_j^-(f)$ will not be aliasing due to symmetry, which is shown in Fig. 5.5(b). The corresponding condition is

$$\begin{aligned} & (f_{h_j} - f_{\ell_i}) \pmod{f_s} = 0, \\ \text{or} \quad & (f_{h_j} - f_{\ell_i}) \pmod{f_s} \geq W_i + W_j, \text{ for } 1 \leq i < j \leq N. \end{aligned} \tag{5.13}$$

Similarly, if $X_i^+(f)$ and $X_j^-(f)$ are not aliasing after sampling, then $X_i^-(f)$ and $X_j^+(f)$ will not be aliasing, which is shown in Fig. 5.5(c). This requires

$$\begin{aligned} & (f_{h_i} + f_{h_j}) \pmod{f_s} = 0, \\ \text{or} \quad & (f_{h_i} + f_{h_j}) \pmod{f_s} \geq W_i + W_j, \text{ for } 1 \leq i < j \leq N. \end{aligned} \tag{5.14}$$

There are N conditions in (5.12), $N(N-1)/2$ conditions in (5.13) and $N(N-1)/2$ conditions in (5.14). Combining (5.12), (5.13) and (5.14), we have a total of N^2 sufficient and necessary conditions for alias-free sampling. We can examine each of the N^2 conditions. If one condition is not satisfied, we can always increase the sampling frequency so that the condition becomes satisfied. By iteratively examining the conditions and increasing the frequency, we can find the minimum sampling frequency for alias-free sampling.

Remark. For multiple bandpass signals, we can also leave guard bands between different bandpass signals after sampling. Similar to the two bandpass signals case, we can make the adjustment of band edges and bandwidths, and use the conditions in (5.13) and (5.14). The new band edges and bandwidths are respectively

$$\begin{aligned} f'_{\ell_i} &= f_{\ell_i} - GB/2 \\ f'_{h_i} &= f_{h_i} + GB/2 \text{ , for } i = 1, 2, \cdots, N. \\ W'_i &= W_i + GB \end{aligned} \tag{5.15}$$

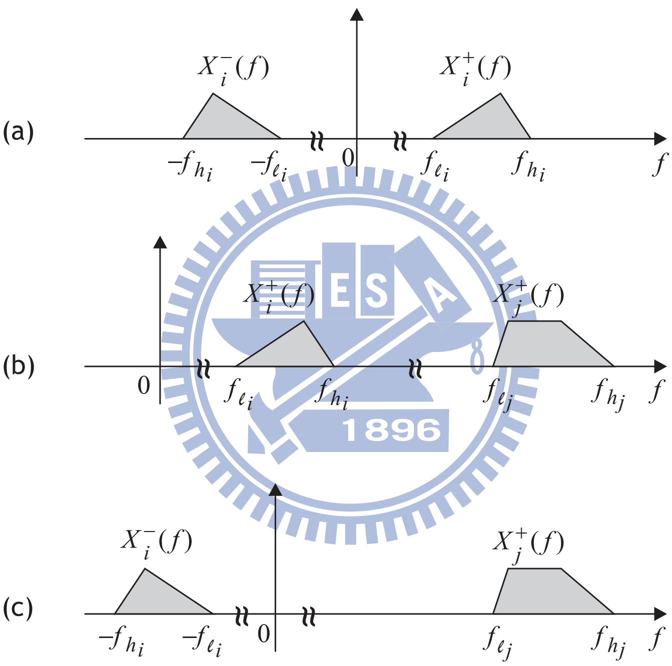

Figure 5.5: (a)The spectrum of $X_i^-(f)$ and $X_i^+(f)$. (b) The spectrum of $X_i^+(f)$ and $X_j^+(f)$. (c)The spectrum of $X_i^-(f)$ and $X_j^+(f)$.

Chapter 6

Simulations and Comparisons

In this chapter, we apply the proposed algorithm to wireless applications. The bandpass signals considered in the simulations are GSM 900 (935-960 MHz, one-sided bandwidth 25 MHz), GSM 1800 (1805-1880 MHz, one-sided bandwidth 75 MHz) [17], DAB Eureka-147 L-Band (1472.286-1473.822 MHz, one-sided bandwidth 1536 KHz) [18], IEEE 802.11g (2412-2432 MHz, one-sided bandwidth 20 MHz) [19], and WCDMA (2119-2124 MHz, one-sided bandwidth 5 MHz).

6.1 Complexity Comparisons of The Proposed Algorithm and Previously Reported Methods

In this section, we compare the number of addition and multiplication for finding the minimum sampling frequency using method in [14], method in [15], and our proposed method. Table 6.1 lists the complexity in finding the minimum sampling frequency for different combinations of wireless systems. The simulation result demonstrates that the proposed method can reduce the number of additions and multiplications significantly. The required numbers of additions and multiplications are reduced respectively by around 35-41% and 36-53% in two bandpass signals case, 28-58% and 61-76% in three bandpass signals case. As another example, we consider GSM 900 application with spectrum divided to 125 users and the bandwidth of each user is 200 kHz. Table 6.2 lists the complexity for

Case	Method in [14]		Method in [15]		Proposed Method	
	ADD	MUL	ADD	MUL	ADD	MUL
GSM900, GSM1800	29	50	38	50	17	28
DAB, 802.11g	65	122	161	296	42	78
GSM900, WCDMA	35	62	101	176	21	29
DAB, WCDMA	119	230	452	878	77	141
GSM900, GSM1800, 802.11g	105	186	87	109	60	42
DAB, GSM1800, 802.11g	75	126	99	133	41	36
GSM900, DAB, WCDMA	183	342	198	331	77	84

Table 6.1: Complexity for finding the minimum sampling frequency of multiple bandpass signals in terms of additions (ADD) and multiplications (MUL).

User Index	Method in [14]		Method in [15]		Proposed Method	
	ADD	MUL	ADD	MUL	ADD	MUL
‘57’, ‘101’	329	650	3585	7144	11	13
‘37’, ‘49’	1109	2210	3554	7082	15	22
‘10’, ‘45’, ‘88’	2553	5082	4772	9479	1014	962
‘30’, ‘50’, ‘95’	4359	8694	4780	9495	2325	2332
‘8’, ‘46’, ‘74’, ‘102’	3270	6492	6001	11884	812	575
‘25’, ‘39’, ‘77’, ‘125’	8952	17856	6013	11908	4525	3089
‘25’, ‘50’, ‘75’, ‘100’, ‘125’	2336	4592	7253	14317	1734	1174
‘11’, ‘39’, ‘78’, ‘110’, ‘119’	12062	24044	7251	14313	4164	2172
‘20’, ‘40’, ‘60’, ‘80’, ‘100’, ‘120’	3765	7410	8488	16702	2476	1021
‘8’, ‘15’, ‘36’, ‘73’, ‘99’, ‘111’	19593	39066	8467	16660	2986	1206

Table 6.2: Complexity for finding the minimum sampling frequency for GSM 900 with multiple users. For the ‘i’-th user, $f_{\ell_i} = 935 + 0.2(i-1)$ Mhz, $W_i = 200$ kHz, $i = 1 - 125$.

finding the minimum sampling frequency. The required numbers of additions and multiplications are reduced to around 25-98% and 73-99% compared to the other two methods. We can see that the proposed algorithm is much more efficient for finding the minimum bandpass sampling frequency.

Case	GB (MHz)	$f_{s,min}$ (MHz)	ADD	MUL
GSM900, GSM1800	0	240	17	28
	12.5	240	15	13
DAB, WCDMA	0	13.9737	77	141
	0.768	15.3357	50	88
GSM900, GSM1800, 802.11g	0	320	60	42
	10	320	37	20
GSM900, DAB, WCDMA	0	77.2364	77	84
	0.768	80.1509	70	71

Table 6.3: Complexity for finding the minimum sampling frequency with and without guard band.

6.2 Complexity Comparisons for Finding the Minimum Sampling Frequency with and without Guard Band

Table 6.3 lists the complexity with and without guard band. The simulation results shows that introducing a larger guard band as larger, the minimum sampling frequency is in general larger. Furthermore, having guard band between different bandpass signals may increase or decrease the complexity, depending on the length of guard band. Consider the case of the signals that consists of three bandpass signals, GSM900, GSM1800, and IEEE 802.11g. We apply the proposed algorithm to find the minimum sampling frequency $f_{s,min}$. It is equal to 320 MHz. Fig. 6.1 shows the folded spectrum. When $GB = 10$ MHz, the minimum sampling frequency is also equal to 320 MHz. In this particular example, including a guard band of 10 MHz does not increase the minimum sampling frequency. In fact we can see in Fig. 6.1 that, the replicas of different bandpass signals are spaced apart by at least 13 MHz. Applying the proposed algorithm with $GB \leq 13$ MHz will yield the same $f_{s,min}$.

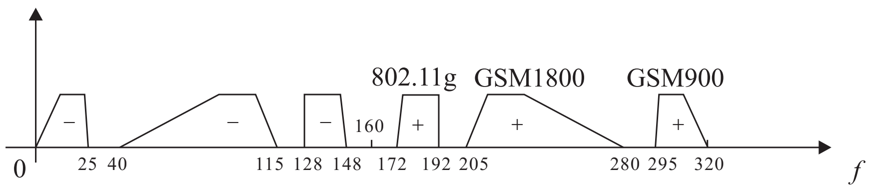

Figure 6.1: An example of the sampled signal in $[0, f_s)$ with $f_s = 320$ MHz.

Case	$f_{s,min}$ (MHz) with ordering constraint [13]	$f_{s,min}$ (MHz) without ordering constraint
GSM900, GSM1800	417.7778	240
DAB, WCDMA	14.0198	13.9737
GSM900, GSM1800, 802.11g	4864	320
GSM900, DAB, WCDMA, 802.11g	4864	137.1429

Table 6.4: Minimum sampling frequency comparisons with and without an ordering constraint.

6.3 Minimum Sampling Frequency Comparisons with and without Ordering Constraint

Table 6.4 lists the minimum sampling frequency when there is a ordering among the replicas [13]. The constraint is such that in the $[0, f_s)$ frequency range the replica of $X_i^+(f)$ is at the left of $X_{i+1}^+(f)$. We have also listed the minimum sampling frequency obtained using the proposed iterative algorithm without an ordering constraint. We can see that the minimum sampling frequency without an ordering constraint can be much smaller that with an ordering constraint. Consider a signal that consists of two bandpass signals, GSM 900 and GSM 1800. The minimum sampling frequency is 240 MHz without an ordering constraint as shown in Fig. 6.2(a), and 417.778 MHz with an ordering constraint as shown in Fig. 6.2(b). Since the proposed algorithm does not impose the ordering constraint, it can obtain the true minimum sampling frequency.

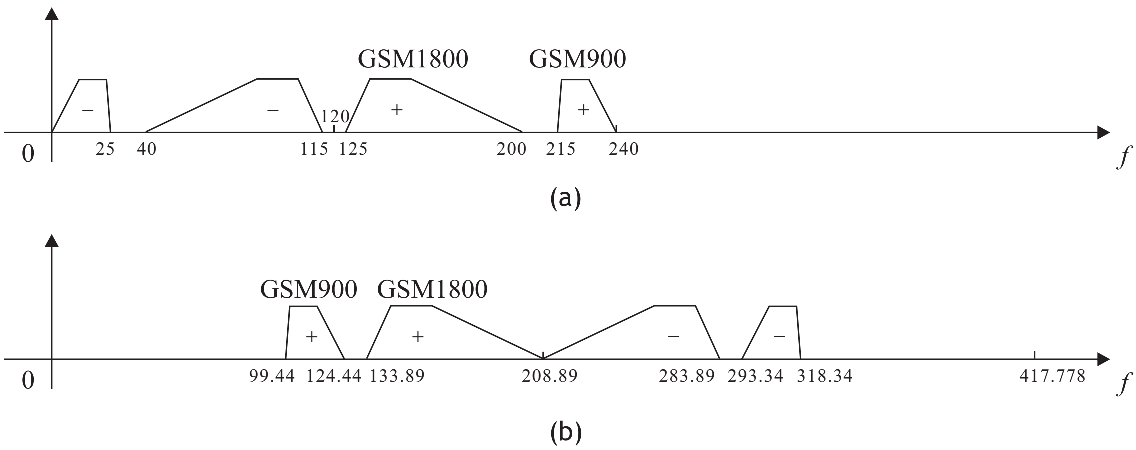

Figure 6.2: An example of the sampled signal in $[0, f_s)$ (a) $f_{s,min}$ = 240 MHz without an ordering constraint. (b) $f_{s,min}$ = 417.778 MHz with an ordering constraint.

Case	Valid sampling frequency range (MHz)
GSM900, GSM1800	$240.0000 \leq f_s \leq 240.6667$
DAB, 802.11g	$46.7880 \leq f_s \leq 46.7986$
GSM900, WCDMA	$64.3636 \leq f_s \leq 64.3889$
DAB, WCDMA	$13.9737 \leq f_s \leq 13.9739$
GSM900, GSM1800, 802.11g	$320.0000 \leq f_s \leq 321.6000$
DAB, GSM1800, 802.11g	$209.6139 \leq f_s \leq 209.7391$
GSM900, DAB, WCDMA	$77.2364 \leq f_s \leq 77.2667$

Table 6.5: Valid ranges of the sampling frequency.

6.4 Range of Valid Sampling Frequency

Table 6.5 lists a valid sampling frequency range for different combinations of wireless systems. Consider a signal that consists of GSM 900, GSM 1800, and IEEE 802.11g. The minimum sampling frequency is 320 MHz as shown in Fig. 6.3(a). When we gradually increase the sampling frequency, the positive part of the replica of 802.11g moves towards $f_s/2$. When we increase the sampling frequency to $321.6MHz$, the positive part and negative part of replica of 802.11g will be aliasing if we keep increasing the sampling frequency as shown in Fig. 6.3(b). The frequencies in the range of 320-321.6 MHz are all alias-free sampling frequency.

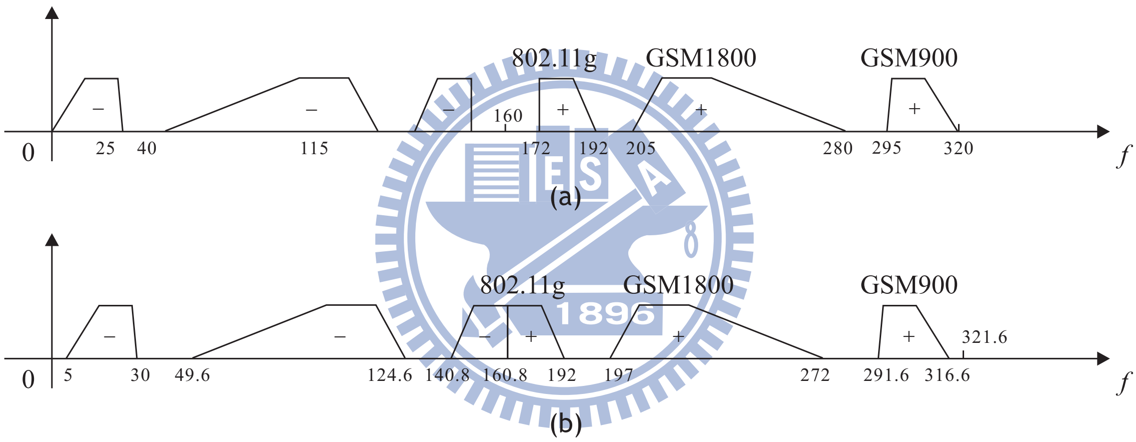

Figure 6.3: An example of the sampled signal in $[0, f_s)$ (a) $f_{s,min} = 320$ MHz. (b) $f_s = 321.6$ MHz.

Chapter 7

Conclusions

In this thesis, we have proposed an efficient algorithm for finding the minimum sampling frequency for signals that contain multi-passband signals. We have derived a set of necessary and sufficient conditions for alias-free sampling that can be checked with few computations. These conditions lead to an iterative algorithm for finding the minimum sampling frequency. This is done by iteratively increasing the sampling frequency to meet the alias-free conditions. The complexity for finding the minimum sampling frequency is much lower than existing methods. There is no need to consider ordering of the signal bands in the folded spectrum. The method can be easily extended the case when a guard band is required.

Bibliography

[1] J. Mitola, “The software radio architecture” *IEEE Commun. Mag.*, vol. 33, no. 5, pp. 26-38, May 1995.

[2] E. Buracchini, “The software radio concept,” *IEEE Commun. Mag.*, vol. 38, no. 9, pp. 138-143, Sept. 2000.

[3] T. Hentschel, M. Henker, and G. Fettweis, “The digital front-end of softsoftware radio terminals,” *IEEE Pers. Commun.*, vol. 6, no. 4, pp. 40V46, Aug. 1999.

[4] Jeffery A. Wepman,“Analog-to-digital converters and their applications in radio receivers,” *IEEE Commun. Mag.*, vol. 33, no. 5, pp. 39-45, May 1995.

[5] K. C. Zangi, and R. D. Koilpillai, “Software radio issues in cellular base stations,” *IEEE Journal on Selected Areas in Commun.*, vol. 33, no. 5, pp. 39-45, May 1995.

[6] D. M. Akos, M. Stockmaster, and J. B. Y. Tsui, “Direct bandpass sampling of multiple distinct RF signals,” *IEEE Trans. Commun.*, vol. 47, no. 7, pp. 983-988, July 1999.

[7] J. D. Gaskell, “Linear Systems, Fourier Transforms, and Optics,” New York: Wiley, 1978.

[8] R. G. Vaughan, N. L. Scott, and D. R. White, “The theory of bandpass sampling,” *IEEE Trans. Signal Process.*, vol. 39, no. 9, pp. 1973-1983, Sept. 1991.

[9] R. Qi, F. P. Coakley, and B. G. Evans, “Practical consideration for bandpass sampling,” *Electronics Letters*, vol. 321, no. 20, pp. 1861-1862, Sept. 1996.

[10] N.Wong and T. S. Ng, "An efficient algorithm for down-converting multimultiple bandpass signals using bandpass sampling," in *Proc. IEEE ICC 2001*, vol. 3, pp. 910-914, June 2001.

[11] M. Choe and K. Kim, "Bandpass sampling algorithm with normal and inverse placements for multiple RF signals," *IEICE Trans. Commun.*, vol. E88, no. 2, pp. 754-757, Feb. 2005.

[12] J. Bae and J. Park, "An efficient algorithm for bandpass sampling of multiple RF signals," *IEEE Signal Process. Lett.*, vol. 13, no. 4, pp. 193-196, Apr. 2006.

[13] S. Bose, V. Khaitan, and A. Chaturvedi, "A low-cost algorithm to find the minimum sampling frequency for multiple bandpass sampling," *IEEE Signal Process. Lett.*, vol. 15, pp. 877-880, Apr. 2008.

[14] C. H. Tseng and S. C. Chou, "Direct down-conversion of multiband RF signals using bandpass sampling," *IEEE Trans. Wireless Commun.*, vol. 5, no. 1, pp. 72-76, Jan. 2006.

[15] J. Bae and J. Park, "A searching algorithm for minimum bandpass sampling frequency in simultaneous down-conversion of multiple RF signals," *Journal of Communications and Networks*, vol. 10, no. 1, pp. 55-62, Mar. 2008.

[16] A. V. Oppenheim, R. W. Schafer, *Discrete-time signal processing*, Prentice Hall, 1999.

[17] *Digital Cellular Telecommunications System (Phase 2+); Radio Transmission and Reception (GSM 05.05 Version 85.1 Release 1999)*, ETSI EN 300 910 Ver, 8.5.1 (2000-11)

[18] ETSI (European Telecommunications Standards Institute), "Digital Audio Broadcasting (DAB) to mobile, portable and fixed receivers," ETSI EN 300 401 v1.3.3, May 2001.

[19] *Part 11: Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications Amendment 4:Further Higher Data Rate Extension in the 2.4 GHz Band*, IEEE Std. 802.11g-2003.