國 立 交 通 大 學

電控工程研究所

碩士論文

整合 SDRE 和 ISMC 設計

之非線性系統可靠度控制理論與應用

Reliable Nonlinear Control

via Combining SDRE and ISMC Approaches

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摘要

本論文探討非線性系統使用state-dependent Riccati equation (SDRE)理論的可靠度控制設計，並應用於衛星之姿態控制。由於引進了integral sliding mode control (ISMC)理論合併使用，研究發現可以大大增進系統的穩健性和可靠度。然而，傳統的SDRE設計必須先拆解漂流項成為**f**(**x**)=A(**x**)**x**的形式，然後再利用A(**x**)及線性理論來判斷系統在該狀態的可穩定性和可觀測性以確保對應之SDRE存在正定解。但當系統動態足夠複雜時，這些判斷條件不容易被檢驗，此外，目前文獻也沒有提供不同拆解方式的分析與比較。因此，本論文提出另一種SDRE的拆解方式，並探討能保證對應的Riccati方程式存在正定唯一解的充分且必要條件，此充要條件只需要系統動態在該狀態之資訊。本論文中也發現如果採用固定拆解方式的傳統SDRE設計方法能夠正常工作，那麼本論文探討的SDRE拆解方式一樣可以正常工作。透過例子，我們說明了本論文提出拆解方法的好處。

Reliable Nonlinear Control

via Combining SDRE and ISMC Approaches

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ABSTRACT

In this thesis, we investigate the nonlinear reliable control issues via the state-dependent Riccati equation (SDRE) scheme with application to the attitude control of a satellite. Owing to incorporating with the integral sliding mode control (ISMC) design, both the robustness and the reliability performances are greatly improved. However, it is known that the conventional SDRE scheme has to symbolically factorize the drift term in the form of $\mathbf{f}(\mathbf{x})=A(\mathbf{x})\mathbf{x}$, and then using this $A(\mathbf{x})$ to check system's stabilizability and observability symbolically at every nonzero state for ensuring the solvability of an associated SDRE. These checking conditions are in general not easy to implement when the system dynamics is complicated, and there is no guideline provided for performing the factorization. As a result, this study also presents an alternative approach of factorization, which only requires the information of the system dynamics at every state and guarantees the existence of a unique positive definite solution of the associated Riccati equation when a mild condition is satisfied. It is shown that the alternative approach always works if the conventional SDRE approach adopting any specific factorization for $\mathbf{f}(\mathbf{x})$ is successfully operated. An illustrative example is also given to demonstrate the benefits of the alternative approach of factorization.

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CHAPTER ONE INTRODUCTION

1.1 Motivation

Numerous design methodologies are known to exist for the control design of highly nonlinear systems [12]. These include a huge number of linear design techniques used in conjunction with gain scheduling [38]; nonlinear design methodologies such as dynamic inversion [17] and sliding mode control [33]; and adaptive techniques which encompass both linear adaptive and nonlinear adaptive control. Lesser known but promising nonlinear design procedures are those that involve state-dependent Riccati equations (SDRE) [13]-[14].

Recently, the study of SDRE approach among the variety of control schemes for nonlinear systems has attracted considerable attention (see e.g., [10]-[13], and [36]) due to its remarkable benefits. These include: 1) concept of SDRE approach is intuitive which directly adopts the LQR design at every nonzero state; 2) SDRE approach can directly address system performance through the specification of the performance index by adjusting the state and the control weightings with predictable results, for instance, the engineer may tune up the weightings on system state to speed up the response at the expense of more control effort; 3) SDRE approach possesses an extra design degree of freedom arose from the non-uniqueness of the SDC representation of the nonlinear drift term, which can be utilized to enhance controller performance; 4) SDRE approach preserves the essential system nonlinearities, since it does not truncate any system's nonlinear term. Many practical and meaningful applications which are successfully performed by the SDRE design include advanced guidance law development, autopilot design, integrated guidance and control design, satellite and spacecraft control and estimation, process control, magnetic levitation, control of systems with parasitic effects, control of artificial human pancreas, robotics, simultaneous state and parameter estimation, fan control, and various benchmark problems (see [10], [15], [36] and the references therein).

SDRE scheme to the stabilization of nonlinear control systems is known to need to symbolically factorize the drift term in the form of $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$, and then using this $A(\mathbf{x}, t)$ to check system's stabilizability and observability symbolically at every state for ensuring the solvability of an associated state-dependent Riccati equations. In doing so, the SDRE algorithm fully captures the nonlinearities of the system, bringing the nonlinear system to a (non-unique) linear structure having state-dependent coefficient (SDC) matrices, and minimizing a nonlinear performance index having a quadratic-like structure. Moreover, the nonuniqueness of the factorization creates extra degrees of freedom, which can be used to enhance controller performance, such as robustness. But, there is no guideline provided for the factorization $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$, specifically to improve robustness. However, with the help of Integral-type Sliding Mode Control (ISMC), we can still improve robustness using SDC factorization. The ISMC approach does not have reaching phase and possesses the advantages of robustness and ease of implementation. When the uncertainty and disturbance are matched regarding the nominal healthy subsystem, the state trajectories of the nominal healthy subsystem and the uncertain system are identical. Thus, in this study, we adopt the SDRE strategy for the nominal system, and the ISMC strategy to completely nullify the matched uncertainty and disturbance. In addition to robustness, we are also interested in the reliability issue related to SDRE only and SDRE-ISMC combined designs.

However, we encounter some difficulties during the SDRE design. If the system dynamics is sufficiently complicated, the checking conditions of stabilizability and observability are generally not easy to implement, and there is no guideline provided for performing the factorization fulfilling some predetermined control objectives. Moreover, if SDRE fails some checking conditions at a system state, then the system may just stuck in the state since SDRE can not guarantee a feasible control related to the unique positive definite solution of the associated Riccati equation. As a result, this study also presents an alternative approach for the factorization, which only requires the information of the system dynamics at every state and guarantees the existence of a unique positive definite solution of the associated Riccati equation when a mild condition is satisfied. To be more detailed, we give a necessary and sufficient condition for that solution as well as the implementing algorithm on how to factorize $\mathbf{f}(\mathbf{x}, t)$. Moreover, it is shown that the alternative approach always works if the conventional SDRE approach adopting any specific factorization for $\mathbf{f}(\mathbf{x}$,t) is successfully operated. An illustrative example is also given to demonstrate that we adopt conventional approach at almost all system states, but at some states (which fails to operate under conventional approach), instead we resort to the alternative approach for a different factorization of A which works.

1.2 Outline

The rest of this thesis is organized as follows. Chapter 2 sketches the SDRE and ISMC designs. Then we investigate the robustness and reliability issues related to both the SDRE and SDRE+ISMC combined designs with analytical simulation results. In Chapter 5, we formulate an alternative SDRE problem and describes our solution. Finally, we provide a short conclusion and give some suggestions of future research related afterwards.

CHAPTER TWO
PRELIMINARIES

2.1 State Dependent Riccati Equation (SDRE)

Consider the following class of time-variant nonlinear control systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u} \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$ denote the system states and control inputs, respectively, $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^n$, $B(\mathbf{x}, t) \in \mathbb{R}^{n\times m}$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. In addition, we consider the following performance index

$$J = \int_0^\infty \left[\mathbf{x}^T Q(\mathbf{x})\mathbf{x} + \mathbf{u}^T R(\mathbf{x})\mathbf{u}\right] dt \tag{2.2}$$

where $Q^T(\mathbf{x}) = Q(\mathbf{x}) \geq 0$, $R^T(\mathbf{x}) = R(\mathbf{x}) > 0$ and $(\cdot)^T$ denotes the transpose of a vector or a matrix. In this study, we assume that $B(\mathbf{x}, t) \neq 0$ and $Q(\mathbf{x}) \neq 0$ for any nonzero state $\mathbf{x}$.

SDRE techniques are increasingly being used in nonlinear control applications [15] and entails factorization of the nonlinear dynamics into the state vector and the product of a matrix-valued function that depends on the state itself [10]. In doing so, the SDRE algorithm fully captures the nonlinearities of the system, bringing the nonlinear system to a (non-unique) linear structure having state-dependent coefficient (SDC) matrices, and minimizing a nonlinear performance index having a quadratic-like structure.

To solve the SDRE problem, almost all the existing studies adopted the following procedure:

- Symbolically factorize $\mathbf{f}(\mathbf{x}, t)$ into the form of $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$, where $A(\mathbf{x}, t) \in \mathbb{R}^{n\times n}$.

- Check the stabilizability of $[A(\mathbf{x},t), B(\mathbf{x},t)]$ and the observability of $[A(\mathbf{x},t), C(\mathbf{x})]$ symbolically, where $C(\mathbf{x}) \in \mathbb{R}^{p\times n}$ has full rank and satisfies $Q(\mathbf{x}) = C^T(\mathbf{x})C(\mathbf{x})$, to ensure the solvability of the following SDRE [24]:

$$A^T(\mathbf{x},t)P(\mathbf{x}) + P(\mathbf{x})A(\mathbf{x},t) - P(\mathbf{x})B(\mathbf{x},t)R^{-1}(\mathbf{x})B^T(\mathbf{x},t)P(\mathbf{x}) + Q(\mathbf{x}) = 0. \quad (2.3)$$

- Solve the SDRE for $P(\mathbf{x})$ to produce the SDRE controller $\mathbf{u} = -R^{-1}(\mathbf{x})B^T(\mathbf{x},t)P(\mathbf{x})\mathbf{x}$.

2.2 Integral Sliding Mode Control (ISMC)

The design concept of Integral Sliding Mode Control (ISMC) is quite similar to Sliding Mode Control (SMC, see e,g, [18], [26], [44], and [45]), and the main difference is that ISMC adopts the integral-type sliding surface and results no reaching phase, i.e., the system trajectories will start on the sliding manifold from the first time instant. Moreover, when the system is on the sliding manifold, the system trajectories is determined by the control law applied to the related nominal subsystem, and this control law can be any control laws fulfilling design objectives. In the following, we describe the design of ISMC([6], [7], and [28]).

Consider the following class of time-variant nonlinear control systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u} + \mathbf{d} \quad (2.4)$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$ denote the system states and control inputs, respectively. $\mathbf{f}(\mathbf{x},t) \in \mathbb{R}^n$ and $B(\mathbf{x},t) \in \mathbb{R}^{n\times m}$ are both smooth functions. $\mathbf{d}$ denotes possible system uncertainties and disturbances. Here we assume that $\mathbf{d}$ has only matched part with regard to B, thus we write (2.4) as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + B(\mathbf{x},t)(\mathbf{u} + \mathbf{d}_m) \quad (2.5)$$

where $\mathbf{d}_m = B^+(\mathbf{x},t)\cdot\mathbf{d}$, $B^+(\mathbf{x},t)$ is the pseudo-inverse matrix of $B(\mathbf{x},t)$, and $||\mathbf{d}_m|| \leq \rho_m(\mathbf{x},t)$, $\rho_m(\mathbf{x},t) > 0$. Then we design the control law composed of two parts:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \quad (2.6)$$

where $\mathbf{u}_0$ is the control input applied to the nominal subsystem, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + B(\mathbf{x},t)\cdot\mathbf{u}$. And $\mathbf{u}_1$ is a discontinuous control input designed to compensate disturbances such that the system trajectories can remain on the sliding manifold, as follows:

$$\mathbf{u}_1 = \begin{cases} 0 & \text{if } \mathbf{s} = \mathbf{0} \\ -\rho(\mathbf{x}, t) \cdot \frac{[DB(\mathbf{x},t)]^T \mathbf{s}}{||[DB(\mathbf{x},t)]^T \mathbf{s}||} & \text{if } \mathbf{s} \neq \mathbf{0} \end{cases} \tag{2.7}$$

where $\rho(\mathbf{x}, t) > \rho_m(\mathbf{x}, t)$, and the sliding surface is designed to be

$$\begin{aligned} \mathbf{s}(\mathbf{x}, t) &= D \cdot \left\{ \mathbf{x}(t) - \mathbf{x}(t_0) - \int_{t_0}^{t} [\mathbf{f}(\mathbf{x}(\tau), \tau) + B(\mathbf{x}(\tau), \tau) \cdot \mathbf{u}_0(\tau)] \cdot d\tau \right\} \\ &= \mathbf{0} \end{aligned} \tag{2.8}$$

with $D \in \mathbb{R}^{m \times n}$ and $DB(\mathbf{x}, t)$ having full rank. From (2.8), we observe that $\mathbf{s}(\mathbf{x}, t_0) = \mathbf{0}$, which implies the system trajectories start on the manifold from the first time instant (t_0). On the other hand, when system is on the sliding manifold, i.e., $\mathbf{s} = \dot{\mathbf{s}} = \mathbf{0}$, from (2.4) and (2.8), we obtain

$$\begin{aligned} \dot{\mathbf{s}} &= D \cdot \{\dot{\mathbf{x}} - [\mathbf{f} + B(\mathbf{x}, t)\mathbf{u}_0]\} \\ &= D \cdot \{[\mathbf{f} + B(\mathbf{x}, t)\mathbf{u} + B(\mathbf{x}, t)\mathbf{d}_m] - [\mathbf{f} + B(\mathbf{x}, t)\mathbf{u}_0]\} \\ &= DB(\mathbf{x}, t) \cdot (\mathbf{u} + \mathbf{d}_m - \mathbf{u}_0) \end{aligned}$$

thus $\mathbf{u} = \mathbf{u}_0 - \mathbf{d}_m$, substitute into (2.4) and obtain

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t) \cdot \mathbf{u}_0$$

which explains that the system trajectories staying on the sliding manifold is identical to that of the nominal system.

On the other hand, to see that $\mathbf{u}_1$ keeps the system stay on the sliding manifold. When $\mathbf{s} \neq \mathbf{0}$, we choose the Lyapunov function $V = \frac{1}{2}\mathbf{s}^T\mathbf{s}$, differentiate V and from (2.4)-(2.8), we know

$$\begin{aligned} \dot{V} &= \mathbf{s}^T\dot{\mathbf{s}} = \mathbf{s}^T DB(\mathbf{x}, t) \cdot (\mathbf{u} + \mathbf{d}_m - \mathbf{u}_0) \\ &= \mathbf{s}^T DB(\mathbf{x}, t) \cdot \left\{ -\rho(\mathbf{x}, t) \cdot \frac{[DB(\mathbf{x}, t)]^T \mathbf{s}}{||[DB(\mathbf{x}, t)]^T \mathbf{s}||} + \mathbf{d}_m \right\} \\ &\leq -\rho(\mathbf{x}, t) \cdot ||[DB(\mathbf{x}, t)]^T \mathbf{s}|| + ||\mathbf{d}_m|| \cdot ||[DB(\mathbf{x}, t)]^T \mathbf{s}|| \\ &\leq [-\rho(\mathbf{x}, t) + \rho_m(\mathbf{x}, t)] \cdot ||[DB(\mathbf{x}, t)]^T \mathbf{s}|| \\ &< 0. \end{aligned}$$

Since DB($\mathbf{x}$,t) is assumed full rank and $\mathbf{s}(\mathbf{x}, t_0) = \mathbf{0}$, the control law (2.6) and (2.7) guarantees the system remain on the sliding manifold, i.e., $\mathbf{s} = \mathbf{0}$, $\forall \mathbf{t} \in [\mathbf{t_0}, \infty)$.

CHAPTER THREE
STUDY OF ROBUSTNESS PERFORMANCE OF SDRE+ISMC SCHEME

SDRE can be used to enhance the performance of robustness through the extra design degree of freedom arose from the non-uniqueness of the SDC representation of the nonlinear drift term (see e.g., [10] and [12]). But, there is no guideline provided for the factorization $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$ to improve robustness. However, with the help of ISMC, we can still improve robustness using SDC factorization. The ISMC approach does not have reaching phase and possesses the advantages of robustness and ease of implementation. When the uncertainty and disturbance are matched regarding the nominal healthy subsystem, the state trajectories of the nominal healthy subsystem and the uncertain system are identical. Thus, in this chapter, we adopt the SDRE strategy for the nominal system, and the ISMC strategy to completely nullify the matched uncertainty and disturbance.

In Section 3.1, we define the system type, cost function, and control objective. Then we detailed the design of control law of SDRE and SDRE+ISMC in Section 3.2. Finally, we apply the control law to the satellite attitude control and analyze the simulating results.

3.1 Problem Statement

Consider a set of n 2nd-order time-variant nonlinear control systems as described by

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x}, t) + \tilde{B}(\mathbf{x}, t)\mathbf{u} + \tilde{\mathbf{d}}. \end{cases} \tag{3.1}$$

Here, $\mathbf{x}_1 = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $\mathbf{x}_2 = (x_{n+1}, \cdots, x_{2n})^T \in \mathbb{R}^n$ and $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T)^T$ are the system states, $\mathbf{u} = (u_1, \cdots, u_m)^T \in \mathbb{R}^m$ are the control inputs and $m \geq n$, $\tilde{\mathbf{d}} = (d_1, \cdots, d_n)^T \in \mathbb{R}^n$ denote possible model uncertainties and/or external disturbances and

$(\cdot)^T$ denotes the transpose of a vector or a matrix. Note that System(3.1) is equivalent to the following system dynamic:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u} + \mathbf{d} \tag{3.2}$$

where $\mathbf{f}(\mathbf{x}, t) = [\mathbf{x}_2^T \vdots \tilde{\mathbf{f}}^T(\mathbf{x}, t)]^T$, $B(\mathbf{x}, t) = [0_{n\times m}^T \vdots \tilde{B}^T(\mathbf{x}, t)]^T$, and $\mathbf{d} = (\mathbf{0}_{n\times 1}^T \vdots \tilde{\mathbf{d}}^T)^T$.

Assumption 3.1 : $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^n$ and $B(\mathbf{x}, t) \in \mathbb{R}^{n\times m}$ are smooth functions with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Assumption 3.2 : For all states, $B(\mathbf{x}, t)$ is full rank.

Moreover, we define the quadratic performance index

$$J = \int_0^\infty \left[\mathbf{x}^T Q(\mathbf{x})\mathbf{x} + \mathbf{u}^T R(\mathbf{x})\mathbf{u}\right] dt \tag{3.3}$$

where $Q(\mathbf{x}) = Q(\mathbf{x})^T \geq 0$ and $R(\mathbf{x}) = R(\mathbf{x})^T > 0$.

The control objective is to compare the performances of the two control strategies (SDRE and SDRE+ISMC) when there are possible model uncertainties and/or external disturbances. To be more precisely, we study whether the system can be stabilized and use the cost function (and others mentioned later) defined in (3.3) as an index to compare the performance.

3.2 Design of Control Law

3.2.1 SDRE

Under Assumption 3.1, we can factorize the drift term in the form of $\mathbf{f}(\mathbf{x},t) = A(\mathbf{x},t)\mathbf{x}$ and let every element of system (3.1) state appearing in $\mathbf{f}(\mathbf{x},t)$ contributes as an element in $A(\mathbf{x},t)$, i.e. capture their state dependency in the proper entry of SDC matrix. To achieve this goal, we adopt some factorizing techniques given by [10]. The following are some examples to illustrate:

$$
\begin{aligned}
& x_6\cos(x_3)\cos(x_2) \\
&\quad = x_6\frac{\cos(x_2)-1}{x_2}x_2 + x_6\frac{\cos(x_3)-1}{x_3}x_3 + [1+(\cos(x_3)-1)(\cos(x_2)-1)]x_6 \\
&\quad = \begin{bmatrix} 0 & x_6\frac{\cos(x_2)-1}{x_2} & x_6\frac{\cos(x_3)-1}{x_3} & 0 & 0 & [1+(\cos(x_3)-1)(\cos(x_2)-1)] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.
\end{aligned} \tag{3.4}
$$

The drift term, $x_6\cos(x_3)\cos(x_2)$, has three state components, x_2, x_3, and x_6, thus they contribute in the $(1,2)$, $(1,3)$, and $(1,6)$ entries of the corresponding SDC matrix, respectively.

$$
\begin{aligned}
& \tfrac{1}{2}\cos^2(x_3)\sin(2x_1) \\
&\quad = \frac{1}{4}\frac{\sin(2x_1)}{x_1}x_1 + \frac{1}{4}\cos^2(x_3)\frac{\sin(2x_1)}{x_1}x_1 + \frac{1}{4}\frac{\cos^2(x_3)-1}{x_3}\sin(2x_1)x_3 \\
&\quad = \begin{bmatrix} \frac{1}{4}\frac{\sin(2x_1)}{x_1} + \frac{1}{4}\cos^2(x_3)\frac{\sin(2x_1)}{x_1} & 0 & \frac{1}{4}\frac{\cos^2(x_3)-1}{x_3}\sin(2x_1) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.
\end{aligned} \tag{3.5}
$$

The drift term, $\frac{1}{2}\cos^2(x_3)\sin(2x_1)$, has three state components, x_1, and x_3, thus they contribute in the $(1,1)$, and $(1,3)$ entries of the corresponding SDC matrix, respectively.

$x_5 \sin(x_3) \sin(x_2)$

$$= \frac{1}{3} x_5 \sin(x_3) \frac{\sin(x_2)}{x_2} x_2 + \frac{1}{3} x_5 \frac{\sin(x_3)}{x_3} \sin(x_2) x_3 + \frac{1}{3} \sin(x_3) \sin(x_2) x_5$$

$$= \left[\begin{array}{cccccc} 0 & \frac{1}{3} x_5 \sin(x_3) \frac{\sin(x_2)}{x_2} & \frac{1}{3} x_5 \frac{\sin(x_3)}{x_3} \sin(x_2) & 0 & \frac{1}{3} \sin(x_3) \sin(x_2) & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \right]. \tag{3.6}$$

The drift term, $x_5 \sin(x_3) \sin(x_2)$, has three state components, x_2, x_3, and x_5, thus they contribute in the $(1, 2)$, $(1, 3)$, and $(1, 5)$ entries of the corresponding SDC matrix, respectively.

After symbolically factorize $\mathbf{f}(\mathbf{x}, t)$ into the form of $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$, where $A(\mathbf{x}, t) \in \mathbb{R}^{n \times n}$, we adopt the following procedures to solve the SDRE problem:

- Check the stabilizability of $[A(\mathbf{x}, t), B(\mathbf{x}, t)]$ and the observability of $[A(\mathbf{x}, t), C(\mathbf{x})]$ symbolically, where $C(\mathbf{x}) \in \mathbb{R}^{p \times n}$ has full rank and satisfies $Q(\mathbf{x}) = C^T(\mathbf{x})C(\mathbf{x})$, to ensure the solvability of the following SDRE [24]:

$$A^T(\mathbf{x}, t)P(\mathbf{x}) + P(\mathbf{x})A(\mathbf{x}, t) - P(\mathbf{x})B(\mathbf{x}, t)R^{-1}(\mathbf{x})B^T(\mathbf{x}, t)P(\mathbf{x}) + Q(\mathbf{x}) = 0. \tag{3.7}$$

- Solve the SDRE for $P(\mathbf{x})$ to produce the SDRE controller $\mathbf{u} = -R^{-1}(\mathbf{x})B^T(\mathbf{x}, t)P(\mathbf{x})\mathbf{x}$.

3.2.2 ISMC

Consider System (3.1), first we need following assumptions.

Assumption 3.3 : There exist $\rho_m(\mathbf{x}, t) > 0$ such that

$$||\tilde{\mathbf{d}}_m|| \leq \rho_m(\mathbf{x}, t) \tag{3.8}$$

where $\tilde{\mathbf{d}}_m = \tilde{B}^+(\mathbf{x}, t) \cdot \tilde{\mathbf{d}}$, and $\tilde{B}^+(\mathbf{x}, t)$ is the pseudo-inverse matrix of $\tilde{B}(\mathbf{x}, t)$.

Assumption 3.4 : The origin of the nominal subsystem $\dot{\mathbf{x}}_1 = \mathbf{x}_2$ and $\dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x}, t) + \tilde{B}(\mathbf{x}, t)\mathbf{u}$ is uniformly asymptotically stabilizable, that is, there exists a control $\mathbf{u}_0$ and a continuously differentiable function $V(\mathbf{x}, t)$ such that

$$\gamma_1(||\mathbf{x}||) \leq V(\mathbf{x}, t) \leq \gamma_2(||\mathbf{x}||) \tag{3.9}$$

$$\text{and} \quad \frac{\partial V(\mathbf{x}, t)}{\partial t} + \left(\frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}}\right)^T \cdot \left[\tilde{\mathbf{f}}(\mathbf{x}, t) + \tilde{B}(\mathbf{x}, t)\mathbf{u}_0\right] \leq -\gamma_3(||\mathbf{x}||) \tag{3.10}$$

where $\gamma_1, \gamma_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are class $\mathcal{K}_\infty$ functions and γ_3 is a class $\mathcal{K}$ function.

Under Assumptions 3.3 and 3.4, the control law is designed into two parts:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \tag{3.11}$$

where $\mathbf{u}_0$ can be any control law which satisfies Assumption 3.4 and creates a desired system trajectory for the state of the uncertain system to follow. In this chapter, $\mathbf{u}_0$ adopts the SDRE strategy. On the other hand, $\mathbf{u}_1$ is designed to compensate for the disturbances such that the system state can remain on the sliding manifold.

Along the ISMC design procedure, the sliding manifold is introduced as (3.12) below:

$$\begin{aligned} \bar{\mathbf{s}} &= \bar{\mathbf{s}}(\mathbf{x}, t) \\ &:= \overline{D} \cdot \left\{\mathbf{x}(t) - \mathbf{x}(t_0) - \int_{t_0}^{t} \left[\mathbf{f}(\mathbf{x}(\tau), \tau) + B(\mathbf{x}(\tau), \tau) \cdot \mathbf{u}_0(\tau)\right] \cdot d\tau\right\} \end{aligned} \tag{3.12}$$

where $\overline{D} = (D_1, D)$ and $D_1 \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$, $D \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$. Note that $\overline{D} \cdot B(\mathbf{x}, t) = D \cdot \tilde{B}(\mathbf{x}, t)$.

Differentiate (3.12), the sliding manifold (3.12) is simplified to be

$$\begin{aligned} \mathbf{s}(\mathbf{x}, t) &= D \cdot \left\{\mathbf{x}_2(t) - \mathbf{x}_2(t_0) - \int_{t_0}^{t} \left[\tilde{\mathbf{f}}(\mathbf{x}(\tau), \tau) + \tilde{B}(\mathbf{x}(\tau), \tau) \cdot \mathbf{u}_0(\tau)\right] \cdot d\tau\right\} \\ &= \mathbf{0}. \end{aligned} \tag{3.13}$$

Note that $\forall \mathbf{x}$, $D\tilde{B}(\mathbf{x},t)$ is full rank.

When the system is on the sliding manifold, $\mathbf{x} = 0$ and $\dot{\mathbf{s}} = 0$. From (3.1) and (3.12), it is obtained that

$$\begin{aligned}
\dot{\mathbf{s}} &= D \cdot \left\{ \dot{\mathbf{x}}_2 - [\tilde{\mathbf{f}} + \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \right\} \\
&= D \cdot \left\{ \tilde{\mathbf{f}} + \tilde{B}(\mathbf{x},t)\mathbf{u} + \tilde{\mathbf{d}} - [\tilde{\mathbf{f}} + \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \right\} \\
&= D \cdot [\tilde{B}(\mathbf{x},t)\mathbf{u} + \tilde{\mathbf{d}} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \\
&= \mathbf{0}.
\end{aligned}$$

Hence $\mathbf{u} = -\tilde{B}^{+}(\mathbf{x},t) \cdot [\tilde{\mathbf{d}} + \tilde{B}(\mathbf{x},t)\mathbf{u}_0]$. By substituting this $\mathbf{u}$ into (3.1), the system resembles the nominal system.

The other part of control law, $\mathbf{u}_1$, the discussion separate into two cases: one is when $\mathbf{s} = \mathbf{0}$, $\mathbf{u}_1 = \mathbf{0}$; The other is when $\mathbf{s} \neq \mathbf{0}$, $\mathbf{u}_1$ is designed to keep $\mathbf{s} = \mathbf{0}$, let

$$\mathbf{u}_1 = -\rho(\mathbf{x},t)\frac{[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}}{||[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}||} \tag{3.14}$$

where $\rho(\mathbf{x},t) > \rho_m(\mathbf{x},t)$. By choosing the Lyapunov function as $V = \frac{1}{2}\mathbf{s}^T\mathbf{s}$, then differentiate V and substitute into (3.11) and (3.12),

$$\begin{aligned}
\dot{V} &= \mathbf{s}^T\dot{\mathbf{s}} \\
&= \mathbf{s}^T D \cdot [\tilde{B}(\mathbf{x},t)\mathbf{u} + \mathbf{d} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \\
&= \mathbf{s}^T D\tilde{B}(\mathbf{x},t) \cdot [\mathbf{u}_0 + \mathbf{u}_1 + \tilde{B}^{+}(\mathbf{x},t)\mathbf{d} - \mathbf{u}_0] \\
&\leq -\rho(\mathbf{x},t) \cdot ||[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}|| + ||\tilde{B}^{+}(\mathbf{x},t)\mathbf{d}|| \cdot ||[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}|| \\
&\leq [-\rho(\mathbf{x},t) + \rho_m(\mathbf{x},t)] \cdot ||[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}|| \\
&< 0.
\end{aligned}$$

To conclude, the following theorem is presented.

Theorem 3.1 For the nonlinear 2nd-order system (3.1) under Assumptions 3.2-3.4, if adopting the following control law:

$$\mathbf{u} = \begin{cases} \mathbf{u}_0 & \text{if } \mathbf{s} = \mathbf{0} \\ \mathbf{u}_0 - \rho(\mathbf{x},t) \cdot \frac{[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}}{||[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}||} & \text{if } \mathbf{s} \neq \mathbf{0} \end{cases} \tag{3.15}$$

then the origin of this system is globally asymptotically stable (GAS).

3.3 Application to Satellite Attitude Control

3.3.1 Satellite Dynamics

An attitude model for a spacecraft along a circular orbit can be described in the same form as (3.1) with $n = 3$ [32]. The three Euler's angles (ϕ, θ, ψ) and their derivatives are adopted as the six state variables. For simplicity, we assume in this study that the thruster is the only applied control force. Let $\mathbf{x} = (\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})^T$ and $\tilde{\mathbf{f}}(\mathbf{x}, t) = (\tilde{f}_1(\mathbf{x}, t), \tilde{f}_2(\mathbf{x}, t), \tilde{f}_3(\mathbf{x}, t))^T$. The overall system dynamics has parameters described as below:

$$\begin{aligned}
\tilde{f}_1(\mathbf{x}, t) &= \omega_0 x_6 c x_3 c x_2 - \omega_0 x_5 s x_3 s x_2 + \frac{I_y - I_z}{I_x}\Big[x_5 x_6 + \omega_0 x_5 c x_1 s x_3 s x_2 + \omega_0 x_5 c x_3 s x_1 \\
&\quad + \omega_0 x_6 c x_3 c x_1 + \frac{1}{2}\omega_0^2 s(2x_3) c^2 x_1 s x_2 + \frac{1}{2} w_0^2 c^2 x_3 s(2x_1) - \omega_0 x_6 s x_3 s x_2 s x_1 \\
&\quad - \frac{1}{2}\omega_0^2 s^2 x_2 s^2 x_3 s(2x_1) - \frac{1}{2}\omega_0^2 s(2x_3) s x_2 s^2 x_1 - \frac{3}{2}\omega_0^2 c^2 x_2 s(2x_1) \Big], \qquad (3.16) \\
\tilde{f}_2(\mathbf{x}, t) &= \omega_0 x_6 s x_3 c x_1 + \omega_0 x_4 c x_3 s x_1 + \omega_0 x_6 c x_3 s x_2 s x_1 + \omega_0 x_5 s x_3 c x_2 s x_1 + \omega_0 x_4 s x_3 s x_2 c x_1 \\
&\quad + \frac{I_z - I_x}{I_y}\Big[x_4 x_6 + \omega_0 x_4 c x_1 s x_3 s x_2 + \omega_0 x_4 c x_3 s x_1 - \omega_0 x_6 s x_3 c x_2 \\
&\quad - \frac{1}{2}\omega_0^2 s(2x_2) s^2 x_3 c x_1 - \frac{1}{2} w_0^2 c x_2 s x_1 s(2x_3) + \frac{3}{2}\omega_0^2 s(2x_2) c x_1 \Big], \qquad (3.17) \\
\tilde{f}_3(\mathbf{x}, t) &= \omega_0 x_4 s x_1 s x_3 s x_2 - \omega_0 x_6 c x_1 c x_3 s x_2 - \omega_0 x_5 c x_1 s x_3 c x_2 + \omega_0 x_6 s x_3 s x_1 - \omega_0 x_4 c x_3 c x_1 \\
&\quad + \frac{I_x - I_y}{I_z}\Big[x_4 x_5 + \omega_0 x_4 c x_3 c x_1 - \omega_0 x_4 s x_3 s x_2 s x_1 - \omega_0 x_5 s x_3 c x_2 \\
&\quad - \frac{1}{2}\omega_0^2 s(2x_3) c x_2 c x_1 + \frac{1}{2} w_0^2 s^2 x_3 s x_1 s(2x_2) - \frac{3}{2}\omega_0^2 s(2x_2) s x_1 \Big], \qquad (3.18) \\
\tilde{B}(\mathbf{x}, t) &= \tilde{B} = \begin{pmatrix} 0.67 & 0.67 & 0.67 & 0.67 \\ 0.69 & -0.69 & -0.69 & 0.69 \\ 0.28 & 0.28 & -0.28 & -0.28 \end{pmatrix}. \qquad (3.19)
\end{aligned}$$

Here, I_x, I_y, and I_z are the inertia with respect to the three body coordinate axes, ω_0 denotes the constant orbital rate, and c and s denote the cos and sin functions, respectively. Note that, Assumptions 3.1 and 3.2 are obviously satisfied, since $B(\mathbf{x}, t)$ is a constant matrix and any three columns taking from B is invertible. Therefore, the system is found to be controllable for any control inputs and Assumption 3.4 is also satisfied.

The control objective is to compare the performances of the two control strategies (SDRE and SDRE+ISMC) when there are possible model uncertainties and/or external disturbances. To be more precisely, we study whether the system can be stabilized and use the cost function (and others mentioned later) defined in (3.3) as an index to compare the performance.

3.3.2 Simulation Results

In this section, we use MATLAB software to simulate the satellite attitude control under SDRE and ISMC approach. For both control approaches, we check whether the system with disturbances can be stabilized and compare their performances (e.g. quadratic performance index and convergence time).

The Table 3.1 shows the simulating parameters in this chapter: (Note that for SDRE approach, the procedure of factorizing $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$ is described in Appendix)

Table 3.1. Simulation parameters.

I_x	$2000\ N \cdot m \cdot s^2$
I_y	$400\ N \cdot m \cdot s^2$
I_z	$2000\ N \cdot m \cdot s^2$
ω_0	$1.0312 \times 10^{-3}\ rad/s$
$\tilde{\mathbf{d}}$	$(0.05\sin(t), 0.05\cos(2t), 0.05\sin(3t))^T$
$A(\mathbf{x}, t)$	see Appendix 3A
D	I_3
Q	I_6
R	I_4
$\mathbf{u}_0$	*SDRE approach*
$\rho(\mathbf{x}, t), \rho_m(\mathbf{x}, t)$	$\lVert B^+(\mathbf{x}, t)\mathbf{d}\rVert_\infty + 1$
$\mathbf{x}_0$	$(-0.7, -0.07, 1.5, 0.3, 1.3, -0.2)^T$

Furthermore, to alleviate chattering, we modify the control law (3.15) into:

$$\mathbf{u} = \begin{cases} \mathbf{u}_0 - \rho(\mathbf{x}, t) \cdot \frac{[D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}}{\lVert [D\tilde{B}(\mathbf{x},t)]^T\mathbf{s}\rVert} & \text{if } \lVert [D\tilde{B}(\mathbf{x}, t)]^T\mathbf{s}\rVert \geq \epsilon \\ \mathbf{u}_0 - \rho(\mathbf{x}, t) \cdot \frac{[D\tilde{B}(\mathbf{x},t)]^T\epsilon}{\epsilon} & \text{if } \lVert [D\tilde{B}(\mathbf{x}, t)]^T\mathbf{s}\rVert < \epsilon \end{cases} \tag{3.20}$$

where we choose $\epsilon = 0.02$.

The simulation results are shown in Figs. 3.1-3.3, and the summary of comparison of performance are shown in Table 3.2.

We denote the results:

- SDRE : the system without disturbance (nominal system) under SDRE approach only
- SDREd : the disturbed system under SDRE approach only

- SDRE+ISMCd : the disturbed system using SDRE-ISMC combined approaches

In addition, in Table 3.2, we also compare the performances under the Sliding Mode Control (SMC, see Section 3.2.3 in [42]), LQR (see Section 3.2.2 in [42]), and LQR-ISMC combined approach (see Section 3.2.1 in [42]), respectively.

- SMC : the disturbed system under nonlinear SMC approach only
- LQR : the system without disturbance (nominal system) under nonlinear LQR approach only
- LQRd : the disturbed system under nonlinear LQR approach only
- LQR+ISMCd : the disturbed system using LQR-ISMC combined approaches

From Fig. 3.1, we observe that SDRE approach stabilizes the nominal system but fails to stabilize when there exists disturbances. However, resorting to ISMC, the system with disturbances can still be stabilized. In addition, it is interesting to find that the trajectory of SDRE+ISMCd and SDRE for nominal design are almost identical (this is why we seem to see only two trajectories in this figure), this agrees with the theoretical conclusion. Moreover, the persistent oscillation of the state trajectory of SDREd comes from the effect of the disturbance $\tilde{\mathbf{d}}$, which also contributes to the oscillating control inputs of SDREd and SDRE+ISMCd in Fig. 3.2. From Fig. 3.2, we see that the control inputs of SDRE+ISMCd experiences larger oscillating amplitude than SDREd, this is because the additional part of control inputs in SDRE+ISMCd than SDREd, $\mathbf{u}_1$ in (3.11), which contributes to compensate disturbances while SDRE control scheme has no such mechanism. Finally, in Fig. 3.3, it is obvious that sliding variables of SDRE+ISMCd start on the sliding manifold and remain on it afterwards, which again agrees with the theoretical results that ISMC has no reaching phase.

Table 3.2 shows the comparison of performance, including energy consumption $\int \mathbf{u}^T\mathbf{u}$, quadratic performance index $\int(\mathbf{x}^T\mathbf{x}+\mathbf{u}^T\mathbf{u})$, required maximum control magnitude $||\mathbf{u}||_\infty$, and convergence time (when the magnitude of state is less than 0.01 at first time). For nominal system, LQR [42] approach seems to have better performance than SDRE in energy consumption $\int \mathbf{u}^T\mathbf{u}$, state regulation $\int \mathbf{x}^T\mathbf{x}$, quadratic performance index $\int(\mathbf{x}^T\mathbf{x}+\mathbf{u}^T\mathbf{u})$ and convergence time. But SDRE scheme has smaller maximum control magnitude $||\mathbf{u}||_\infty$ since LORd uses Taylor's series approximation up to 3rd-order for the real LQR solution associated to a Hamiltonian-Jacobian equation of the the nonlinear system [49]. For the system with disturbances, LQR+ISMCd approach also have better performance than SDRE+ISMCd in energy consumption $\int \mathbf{u}^T\mathbf{u}$, state regulation $\int \mathbf{x}^T\mathbf{x}$, quadratic performance index $\int(\mathbf{x}^T\mathbf{x}+\mathbf{u}^T\mathbf{u})$ and convergence time. Moreover, both SDRE+ISMCd and LQR+ISMCd consumes more control energy than the corresponding nominal control law SDRE and LQR [42], respectively. This is because the additional part, $\mathbf{u}_1$ in (3.11), is required in the ISMC design. Last but not least, we see that SMCd has the least convergence time among all approaches, and can be explained by the fact that Sliding Mode Control (SMC) inherently possesses robustness to model uncertainties and/or external disturbances [6]-[7], [16], [18], and [45].

To sum up, we conclude that SDRE (so as LQR) is not a robust control law.

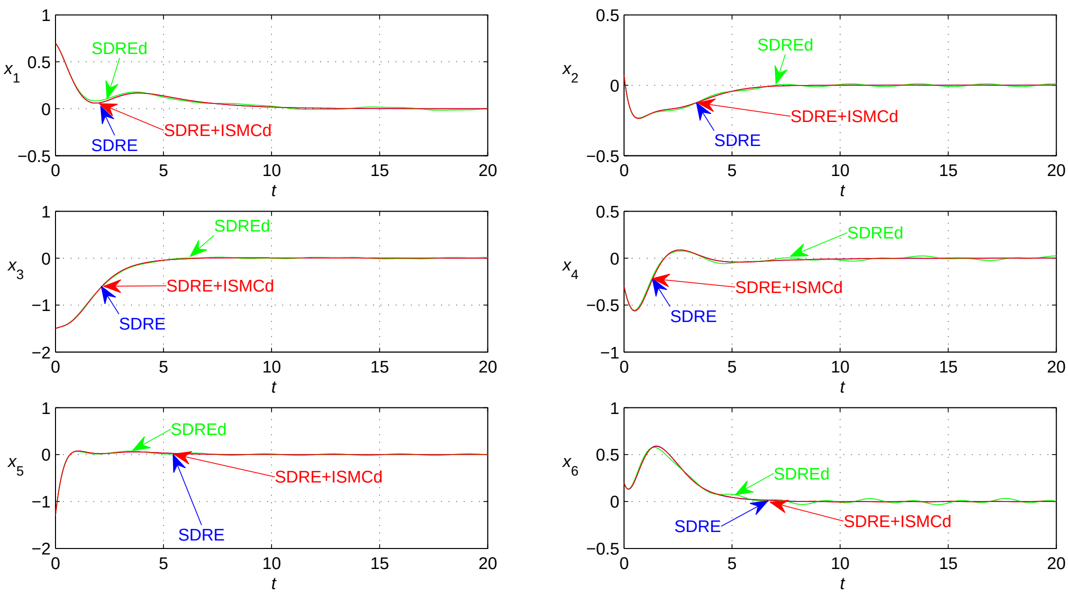

Fig. 3.1. Time history of the six state variables.

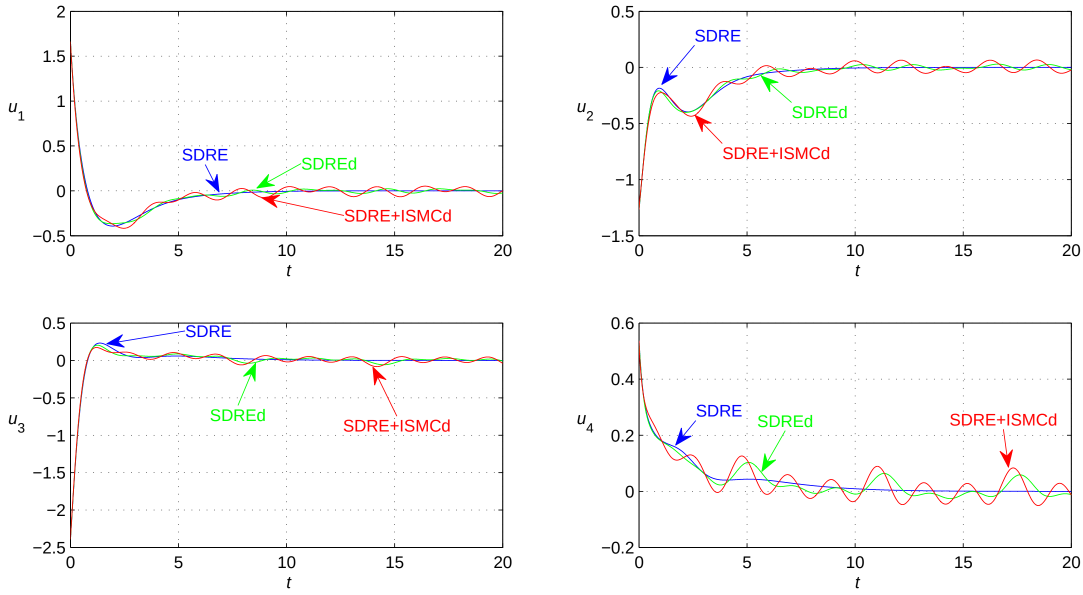

Fig. 3.2. Time history of the four control inputs.

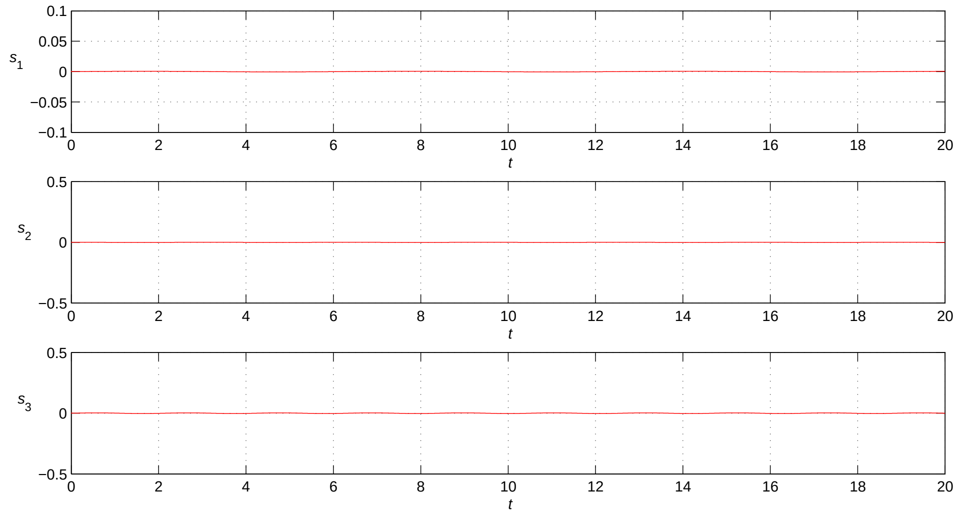

Fig. 3.3. Time history of the sliding variables (SDRE+ISMCd).

Table 3.2. Comparison of performance.

Controller	Performance Index					
	$\|\|\mathbf{x}(t)\|\|_{t\to\infty} < 10^{-3}$	$\int \mathbf{u}^T\mathbf{u}$	$\int \mathbf{x}^T\mathbf{x}$	$\int(\mathbf{x}^T\mathbf{x}+\mathbf{u}^T\mathbf{u})$	$\|\|\mathbf{u}\|\|_\infty$	Convergence time
LQR+ISMCd	Yes	2.1259	4.6294	6.7553	2.5099	5.173
SDRE+ISMCd		2.7979	4.7618	7.5597	2.3917	11.981
SMCd	Yes	2.4605	4.8981	7.3586	2.6305	4.633
LQRd	No	**X**	**X**	**X**	**X**	**X**
SDREd						
LQR	Yes	1.9517	4.6277	6.5794	2.5099	5.157
SDRE		2.7756	4.7638	7.5395	2.3917	11.598

Appendix 3A

We denote

- I_1, I_2, and I_3 denote $\frac{I_y-I_z}{I_x}$, $\frac{I_z-I_x}{I_y}$, and $\frac{I_x-I_y}{I_z}$, respectively.
- c and s denote the cos and sin functions, respectively.

As Section 3.2.1 illustrates, first we reformulate the drift terms (3.16)-(3.18) into:

Reformed Drift Terms :

$\tilde{f}_1$ is reformed as:

$$
\begin{aligned}
&\omega_0 x_6 \frac{cx_2-1}{x_2} x_2 + \omega_0 x_6 \frac{cx_3-1}{x_3} x_3 + \omega_0[1+(cx_3-1)(cx_2-1)]x_6 - \frac{1}{3}\omega_0 x_5 sx_3 \frac{sx_2}{x_2} x_2 \\
&-\frac{1}{3}\omega_0 x_5 \frac{sx_3}{x_3} sx_2 x_3 - \frac{1}{3}\omega_0 sx_3 sx_2 x_5 + \frac{1}{2} I_1 x_6 x_5 + \frac{1}{2} I_1 x_5 x_6 + \frac{1}{4} I_1 \omega_0 x_5 \frac{cx_1-1}{x_1} sx_3 sx_2 x_1 \\
&+\frac{1}{4} I_1 \omega_0 x_5 cx_1 sx_3 \frac{sx_2}{x_2} x_2 + \frac{1}{4} I_1 \omega_0 x_5 cx_1 \frac{sx_3}{x_3} sx_2 x_3 + \frac{1}{4} I_1 \omega_0 (cx_1 sx_3 sx_2 + sx_3 sx_2) x_5 \\
&+\frac{1}{3} I_1 \omega_0 x_5 cx_3 \frac{sx_1}{x_1} x_1 + I_1 \omega_0 \frac{1}{3} x_5 \frac{cx_3-1}{x_3} sx_1 x_3 + \frac{1}{3} I_1 \omega_0 (cx_3 sx_1 + sx_1) x_5 + I_1 \omega_0 x_6 \frac{cx_1-1}{x_1} x_1 \\
&+I_1 \omega_0 x_6 \frac{cx_3-1}{x_3} x_3 + I_1 \omega_0 [(cx_3-1)(cx_1-1)+1] x_6 + \frac{1}{6} I_1 \omega_0^2 s(2x_3) \frac{sx_2}{x_2} x_2 \\
&+\frac{1}{6} I_1 \omega_0^2 s(2x_3) \frac{c^2 x_1 - 1}{x_1} sx_2 x_1 + \frac{1}{6} I_1 \omega_0^2 sx_3 \frac{sx_2}{x_2} x_2 + \frac{1}{6} I_1 \omega_0^2 \frac{s(2x_3)}{x_3} c^2 x_1 sx_2 x_3 + \frac{1}{4} I_1 \omega_0^2 \frac{sx_1}{x_1} x_1 \\
&+\frac{1}{4} I_1 \omega_0^2 c^2 x_3 \frac{s(2x_1)}{x_1} x_1 + \frac{1}{4} I_1 \omega_0^2 \frac{c^2 x_3 - 1}{x_3} s(2x_1) x_3 - \frac{1}{4} I_1 \omega_0 x_6 sx_3 sx_2 \frac{sx_1}{x_1} x_1 \\
&-\frac{1}{4} I_1 \omega_0 x_6 sx_3 \frac{sx_2}{x_2} sx_1 x_2 - \frac{1}{4} I_1 \omega_0 x_6 \frac{sx_3}{x_3} sx_2 sx_1 x_3 - \frac{1}{4} I_1 \omega_0 sx_3 sx_2 sx_1 x_6 \\
&+\frac{1}{6} I_1 \omega_0^2 s^2 x_2 s^2 x_3 \frac{s(2x_1)}{x_1} x_1 - \frac{1}{6} I_1 \omega_0^2 \frac{s^2 x_2}{x_2} s^2 x_3 s(2x_1) x_2 - \frac{1}{6} I_1 \omega_0^2 s^2 x_2 \frac{s^2 x_3}{x_3} s(2x_1) x_3 \\
&-\frac{1}{6} I_1 \omega_0^2 s(2x_3) sx_2 \frac{s^2 x_1}{x_1} x_1 - \frac{1}{6} I_1 \omega_0^2 s(2x_3) \frac{sx_2}{x_2} s^2 x_1 x_2 - \frac{1}{6} I_1 \omega_0^2 \frac{s(2x_3)}{x_3} sx_2 s^2 x_1 x_3 \\
&-\frac{3}{4} I_1 \omega_0^2 c^2 x_2 \frac{s(2x_1)}{x_1} x_1 - \frac{3}{4} I_1 \omega_0^2 \frac{s(2x_1)}{x_1} x_1 - \frac{3}{4} I_1 \omega_0^2 \frac{c^2 x_2 - 1}{x_2} s(2x_1) x_2.
\end{aligned}
\tag{3A.1}
$$

$\tilde{f}_2$ is reformed as:

$$
\begin{aligned}
&\frac{1}{3}\omega_0 x_6 sx_3 \frac{cx_1-1}{x_1} x_1 + \frac{1}{3}\omega_0 x_6 \frac{sx_3}{x_3} cx_1 x_3 + \frac{1}{3}\omega_0 (sx_3 cx_1 + sx_3) x_6 + \frac{1}{3}\omega_0 x_4 cx_3 \frac{sx_1}{x_1} x_1 \\
&+\frac{1}{3}\omega_0 x_4 \frac{cx_3-1}{x_3} sx_1 x_3 + \frac{1}{3}\omega_0 (cx_3 sx_1 + sx_1) x_4 + \frac{1}{4}\omega_0 x_6 cx_3 sx_2 \frac{sx_1}{x_1} x_1 + \frac{1}{4}\omega_0 x_6 cx_3 \frac{sx_2}{x_2} sx_1 x_2 \\
&+\frac{1}{4}\omega_0 x_5 \frac{cx_3-1}{x_3} sx_2 sx_1 x_3 + \frac{1}{4}\omega_0 (cx_3 sx_2 sx_1 + sx_2 sx_1) x_6 + \frac{1}{4}\omega_0 x_5 sx_3 cx_2 \frac{sx_1}{x_1} x_1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\omega_0 x_5 sx_3 \frac{cx_2-1}{x_2} sx_1 x_2 + \frac{1}{4}\omega_0 x_5 \frac{sx_3}{x_3} cx_2 sx_1 x_3 + \frac{1}{4}\omega_0 (sx_3 cx_2 sx_1 + sx_3 sx_1) x_5 \\
&+\frac{1}{4}\omega_0 x_4 sx_3 sx_2 \frac{cx_1-1}{x_1} x_1 + \frac{1}{4}\omega_0 x_4 sx_3 \frac{sx_2}{x_2} cx_1 x_2 + \frac{1}{4}\omega_0 x_4 \frac{sx_3}{x_3} sx_2 cx_1 x_3 \\
&+\frac{1}{4}\omega_0 (sx_3 sx_2 cx_1 + sx_3 sx_2) x_4 + \frac{1}{2} I_2 x_6 x_4 + \frac{1}{2} I_2 x_4 x_6 + \frac{1}{4} I_2 \omega_0 x_4 \frac{cx_1-1}{x_1} sx_3 sx_2 x_1 \\
&+\frac{1}{4} I_2 \omega_0 x_4 cx_1 sx_3 \frac{sx_2}{x_2} x_2 + \frac{1}{4} I_2 \omega_0 x_4 cx_1 \frac{sx_3}{x_3} sx_2 x_3 + \frac{1}{4} I_2 \omega_0 (cx_1 sx_3 sx_2 + sx_3 sx_2) x_4 \\
&+\frac{1}{3} I_2 \omega_0 x_4 cx_3 \frac{sx_1}{x_1} x_1 + \frac{1}{3} I_2 \omega_0 x_4 \frac{cx_3-1}{x_3} sx_1 x_3 + \frac{1}{3} I_2 \omega_0 (cx_3 sx_1 + sx_1) x_4 \\
&-\frac{1}{3} I_2 \omega_0 x_6 sx_3 \frac{cx_2-1}{x_2} x_2 - \frac{1}{3} I_2 \omega_0 x_6 \frac{sx_3}{x_3} cx_2 x_3 - \frac{1}{4} I_2 \omega_0 (sx_3 cx_2 + sx_3) x_6 \\
&-\frac{1}{6} I_2 \omega_0^2 sx_2 s^2 x_3 \frac{cx_1-1}{x_1} x_1 - \frac{1}{6} I_2 \omega_0^2 \frac{s(2x_2)}{x_2} (s_3^x cx_1 + s_3^x) x_2 - \frac{1}{6} I_2 \omega_0^2 s(2x_2) \frac{s^2 x_3}{x_3} cx_1 x_3 \\
&-\frac{1}{6} I_2 \omega_0^2 cx_2 \frac{sx_1}{x_1} s(2x_3) x_1 - \frac{1}{6} I_2 \omega_0^2 \frac{cx_2-1}{x_2} sx_1 s(2x_3) x_2 - \frac{1}{6} I_2 \omega_0^2 \frac{sx_1}{x_1} s(2x_3) x_1 \\
&-\frac{1}{6} I_2 \omega_0^2 cx_2 sx_1 \frac{s(2x_3)}{x_3} x_3 + \frac{3}{4} I_2 \omega_0^2 s(2x_2) \frac{cx_1-1}{x_1} x_1 + \frac{3}{4} I_2 \omega_0^2 \frac{s(2x_2)}{x_2} (cx_1+1) x_2. \qquad \text{(3A.2)}
\end{aligned}
$$

$\tilde{f}_3$ is reformed as:

$$
\begin{aligned}
&\frac{1}{4}\omega_0 x_4 \frac{sx_1}{x_1} sx_2 sx_3 x_1 + \frac{1}{4}\omega_0 x_4 sx_1 \frac{sx_2}{x_2} sx_3 x_2 + \frac{1}{4}\omega_0 sx_1 sx_2 \frac{sx_3}{x_3} x_3 + \frac{1}{4}\omega_0 sx_1 sx_2 sx_3 x_4 \\
&-\omega_0 x_6 \frac{cx_1-1}{x_1} (cx_3-1) sx_2 x_1 - \omega_0 x_6 (cx_1-1) \frac{sx_2}{x_2} x_2 - \omega_0 x_6 \frac{cx_3-1}{x_3} sx_2 x_3 - \omega_0 sx_2 x_6 \\
&-\omega_0 x_5 sx_3 \frac{cx_1}{x_1} x_1 - \omega_0 x_5 sx_3 (cx_1-1) \frac{cx_2-1}{x_2} x_2 - \omega_0 x_5 \frac{sx_3}{x_3} (cx_2-1) x_3 - \omega_0 sx_3 x_5 \\
&+\frac{1}{3}\omega_0 x_6 sx_3 \frac{sx_1}{x_1} x_1 + \frac{1}{3}\omega_0 x_6 \frac{sx_3}{x_3} sx_1 x_3 + \frac{1}{3}\omega_0 sx_3 sx_1 x_6 - \omega_0 x_4 \frac{cx_1-1}{x_1} x_1 - \omega_0 x_4 \frac{cx_3-1}{x_3} x_3 \\
&-\omega_0 [(cx_3-1)(cx_1-1)+1] x_4 + \frac{1}{2} I_3 x_5 x_4 + \frac{1}{2} I_3 x_4 x_5 + I_3 \omega_0 x_4 \frac{cx_1-1}{x_1} x_1 + I_3 \omega_0 x_4 \frac{cx_3-1}{x_3} x_3 \\
&+I_3 \omega_0 [(cx_3-1)(cx_1-1)+1] x_4 - \frac{1}{4} I_3 \omega_0 x_4 sx_3 sx_2 \frac{sx_1}{x_1} x_1 - \frac{1}{4} I_3 \omega_0 x_4 sx_3 \frac{sx_2}{x_2} sx_1 x_2 \\
&-\frac{1}{4} I_3 \omega_0 x_3 \frac{sx_3}{x_3} sx_2 sx_1 x_3 - \frac{1}{4} I_3 \omega_0 sx_3 sx_2 sx_1 x_4 - \frac{1}{3} I_3 \omega_0 x_5 sx_3 \frac{cx_2-1}{x_2} x_2 \\
&-\frac{1}{3} I_3 \omega_0 \frac{sx_3}{x_3} cx_2 x_3 \frac{1}{3} I_3 \omega_0 (sx_3 cx_2 + sx_3) x_5 - \frac{1}{2} I_3 \omega_0^2 s(2x_3) \frac{cx_1-1}{x_1} x_1 - \frac{1}{2} I_3 \omega_0^2 s(2x_3) \frac{cx_2-1}{x_2} x_2 \\
&-\frac{1}{2} I_3 \omega_0^2 \frac{s(2x_3)}{x_3} [(cx_2-1)(cx_1-1)+1] x_3 + \frac{1}{6} I_3 \omega_0^2 s^2 x_3 \frac{sx_1}{x_1} s(2x_2) x_1 + \frac{1}{6} I_3 \omega_0^2 s^2 x_3 sx_1 \frac{sx_2}{x_2} x_2 \\
&+\frac{1}{6} I_3 \omega_0^2 \frac{s^2 x_3}{x_3} sx_1 s(2x_2) x_3 - \frac{3}{4} I_3 \omega_0^2 s(2x_2) \frac{sx_1}{x_1} x_1 - \frac{3}{4} I_3 \omega_0^2 \frac{s(2x_2)}{x_2} sx_1 x_2. \qquad \text{(3A.3)}
\end{aligned}
$$

Then we can factorize the drift term of System (3.1) into $\mathbf{f} = A(\mathbf{x}, t) \cdot \mathbf{x}$ and the elements of $A(\mathbf{x}, t)$ is described as below:

Factorization of the Drift Terms :

$A(\mathbf{x},t) = [a_{ij}(\mathbf{x},t)]$

$$
\begin{aligned}
a_{1j} &= 0, j = 1,2,3,5,6;\ \text{and } \mathrm{a}_{14} = 1. \\
a_{2j} &= 0, j = 1,2,3,4,6;\ \text{and } \mathrm{a}_{15} = 1. \\
a_{3j} &= 0, j = 1,2,3,4,5;\ \text{and } \mathrm{a}_{16} = 1. \\
a_{41} &= \frac{1}{4}I_1\omega_0 x_5\frac{cx_1-1}{x_1}sx_3sx_2 + \frac{1}{3}I_1\omega_0x_5cx_3\frac{sx_1}{x_1} + I_1\omega_0x_6\frac{cx_1-1}{x_1} \\
&+\frac{1}{6}I_1\omega_0^2 s(2x_3)\frac{c^2x_1-1}{x_1}sx_2 + \frac{1}{4}I_1\omega_0^2\frac{sx_1}{x_1} + \frac{1}{4}I_1\omega_0^2c^2x_3\frac{s(2x_1)}{x_1} \\
&-\frac{1}{4}I_1\omega_0x_6sx_3sx_2\frac{sx_1}{x_1} + \frac{1}{6}I_1\omega_0^2s^2x_2s^2x_3\frac{s(2x_1)}{x_1} - \frac{1}{6}I_1\omega_0^2s(2x_3)sx_2\frac{s^2x_1}{x_1} \\
&-\frac{3}{4}I_1\omega_0^2c^2x_2\frac{s(2x_1)}{x_1} - \frac{3}{4}I_1\omega_0^2\frac{s(2x_1)}{x_1}. \\
a_{42} &= \omega_0x_6\frac{cx_2-1}{x_2} - \frac{1}{3}\omega_0x_5sx_3\frac{sx_2}{x_2} + \frac{1}{4}I_1\omega_0x_5cx_1sx_3\frac{sx_2}{x_2} \\
&+\frac{1}{6}I_1\omega_0^2s(2x_3)\frac{sx_2}{x_2} + \frac{1}{6}I_1\omega_0^2sx_3\frac{sx_2}{x_2} - \frac{1}{4}I_1\omega_0x_6sx_3\frac{sx_2}{x_2}sx_1 \\
&-\frac{1}{6}I_1\omega_0^2\frac{s^2x_2}{x_2}s^2x_3s(2x_1) - \frac{1}{6}I_1\omega_0^2s(2x_3)\frac{sx_2}{x_2}s^2x_1 - \frac{3}{4}I_1\omega_0^2\frac{c^2x_2-1}{x_2}s(2x_1). \\
a_{43} &= \omega_0x_6\frac{cx_3-1}{x_3} - \frac{1}{3}\omega_0x_5\frac{sx_3}{x_3}sx_2 + \frac{1}{4}I_1\omega_0x_5cx_1\frac{sx_3}{x_3}sx_2 \\
&+I_1\omega_0\frac{1}{3}x_5\frac{cx_3-1}{x_3}sx_1 + I_1\omega_0x_6\frac{cx_3-1}{x_3} + \frac{1}{6}I_1\omega_0^2\frac{s(2x_3)}{x_3}c^2x_1sx_2 \\
&+\frac{1}{4}I_1\omega_0^2\frac{c^2x_3-1}{x_3}s(2x_1) - \frac{1}{4}I_1\omega_0x_6\frac{sx_3}{x_3}sx_2sx_1 - \frac{1}{6}I_1\omega_0^2s^2x_2\frac{s^2x_3}{x_3}s(2x_1) \\
&-\frac{1}{6}I_1\omega_0^2\frac{s(2x_3)}{x_3}sx_2s^2x_1. \\
a_{44} &= 0. \\
a_{45} &= -\frac{1}{3}\omega_0sx_3sx_2 + \frac{1}{2}I_1x_6 + \frac{1}{4}I_1\omega_0(cx_1sx_3sx_2 + sx_3sx_2) + \frac{1}{3}I_1\omega_0(cx_3sx_1 + sx_1). \\
a_{46} &= \omega_0[1 + (cx_3-1)(cx_2-1)] + \frac{1}{2}I_1x_5 + I_1\omega_0[(cx_3-1)(cx_1-1)+1] \\
&-\frac{1}{4}I_1\omega_0sx_3sx_2sx_1. \\
a_{51} &= \frac{1}{3}\omega_0x_6sx_3\frac{cx_1-1}{x_1} + \frac{1}{3}\omega_0x_4cx_3\frac{sx_1}{x_1} + \frac{1}{4}\omega_0x_6cx_3sx_2\frac{sx_1}{x_1} \\
&+\frac{1}{4}\omega_0x_5sx_3cx_2\frac{sx_1}{x_1} + \frac{1}{4}\omega_0x_4sx_3sx_2\frac{cx_1-1}{x_1} + \frac{1}{4}I_2\omega_0x_4\frac{cx_1-1}{x_1}sx_3sx_2 \\
&+\frac{1}{3}I_2\omega_0x_4cx_3\frac{sx_1}{x_1} - \frac{1}{6}I_2\omega_0^2sx_2s^2x_3\frac{cx_1-1}{x_1} - \frac{1}{6}I_2\omega_0^2cx_2\frac{sx_1}{x_1}s(2x_3) \\
&-\frac{1}{6}I_2\omega_0^2\frac{sx_1}{x_1}s(2x_3) + \frac{3}{4}I_2\omega_0^2s(2x_2)\frac{cx_1-1}{x_1}.
\end{aligned}
$$

$$
\begin{aligned}
a_{52} &= \frac{1}{4}\omega_0 x_6 cx_3 \frac{sx_2}{x_2} sx_1 + \frac{1}{4}\omega_0 x_5 sx_3 \frac{cx_2 - 1}{x_2} sx_1 + \frac{1}{4}\omega_0 x_4 sx_3 \frac{sx_2}{x_2} cx_1 \\
&\quad + \frac{1}{4} I_2 \omega_0 x_4 cx_1 sx_3 \frac{sx_2}{x_2} - \frac{1}{3} I_2 \omega_0 x_6 sx_3 \frac{cx_2 - 1}{x_2} - \frac{1}{6} I_2 \omega_0^2 \frac{s(2x_2)}{x_2}(s_3^x cx_1 + s_3^x) \\
&\quad - \frac{1}{6} I_2 \omega_0^2 \frac{cx_2 - 1}{x_2} sx_1 s(2x_3) + \frac{3}{4} I_2 \omega_0^2 \frac{s(2x_2)}{x_2}(cx_1 + 1). \\
a_{53} &= \frac{1}{3}\omega_0 x_6 \frac{sx_3}{x_3} cx_1 + \frac{1}{3}\omega_0 x_4 \frac{cx_3 - 1}{x_3} sx_1 + \frac{1}{4}\omega_0 x_5 \frac{cx_3 - 1}{x_3} sx_2 sx_1 \\
&\quad + \frac{1}{4}\omega_0 x_5 \frac{sx_3}{x_3} cx_2 sx_1 + \frac{1}{4}\omega_0 x_4 \frac{sx_3}{x_3} sx_2 cx_1 + \frac{1}{4} I_2 \omega_0 x_4 cx_1 \frac{sx_3}{x_3} sx_2 \\
&\quad + \frac{1}{3} I_2 \omega_0 x_4 \frac{cx_3 - 1}{x_3} sx_1 - \frac{1}{3} I_2 \omega_0 x_6 \frac{sx_3}{x_3} cx_2 - \frac{1}{6} I_2 \omega_0^2 s(2x_2) \frac{s^2 x_3}{x_3} cx_1 \\
&\quad - \frac{1}{6} I_2 \omega_0^2 cx_2 sx_1 \frac{s(2x_3)}{x_3}. \\
a_{54} &= \frac{1}{3}\omega_0 (cx_3 sx_1 + sx_1) + \frac{1}{4}\omega_0 (sx_3 sx_2 cx_1 + sx_3 sx_2) + \frac{1}{4} I_2 \omega_0 (cx_1 sx_3 sx_2 + sx_3 sx_2) \\
&\quad + \frac{1}{3} I_2 \omega_0 (cx_3 sx_1 + sx_1). \\
a_{55} &= \frac{1}{4}\omega_0 (sx_3 cx_2 sx_1 + sx_3 sx_1). \\
a_{56} &= \frac{1}{3}\omega_0 (sx_3 cx_1 + sx_3) + \frac{1}{4}\omega_0 (cx_3 sx_2 sx_1 + sx_2 sx_1) + \frac{1}{2} I_2 x_6 x_4 + \frac{1}{2} I_2 x_4 \\
&\quad - \frac{1}{4} I_2 \omega_0 (sx_3 cx_2 + sx_3). \\
a_{61} &= \frac{1}{4}\omega_0 x_4 \frac{sx_1}{x_1} sx_2 sx_3 - \omega_0 x_6 \frac{cx_1 - 1}{x_1}(cx_3 - 1) sx_2 - \omega_0 x_5 sx_3 \frac{cx_1}{x_1} \\
&\quad + \frac{1}{3}\omega_0 x_6 sx_3 \frac{sx_1}{x_1} - \omega_0 x_4 \frac{cx_1 - 1}{x_1} + I_3 \omega_0 x_4 \frac{cx_1 - 1}{x_1} \\
&\quad - \frac{1}{4} I_3 \omega_0 x_4 sx_3 sx_2 \frac{sx_1}{x_1} - \frac{1}{2} I_3 \omega_0^2 s(2x_3) \frac{cx_1 - 1}{x_1} + \frac{1}{6} I_3 \omega_0^2 s^2 x_3 \frac{sx_1}{x_1} s(2x_2) \\
&\quad - \frac{3}{4} I_3 \omega_0^2 s(2x_2) \frac{sx_1}{x_1}. \\
a_{62} &= \frac{1}{4}\omega_0 x_4 sx_1 \frac{sx_2}{x_2} sx_3 - \omega_0 x_6 (cx_1 - 1) \frac{sx_2}{x_2} - \omega_0 x_5 sx_3 (cx_1 - 1) \frac{cx_2 - 1}{x_2} \\
&\quad - \frac{1}{4} I_3 \omega_0 x_4 sx_3 \frac{sx_2}{x_2} sx_1 - \frac{1}{3} I_3 \omega_0 x_5 sx_3 \frac{cx_2 - 1}{x_2} - \frac{1}{2} I_3 \omega_0^2 s(2x_3) \frac{cx_2 - 1}{x_2} \\
&\quad + \frac{1}{6} I_3 \omega_0^2 s^2 x_3 sx_1 \frac{sx_2}{x_2} - \frac{3}{4} I_3 \omega_0^2 \frac{s(2x_2)}{x_2} sx_1. \\
a_{63} &= \frac{1}{4}\omega_0 sx_1 sx_2 \frac{sx_3}{x_3} - \omega_0 x_6 \frac{cx_3 - 1}{x_3} sx_2 - \omega_0 x_5 \frac{sx_3}{x_3}(cx_2 - 1) \\
&\quad + \frac{1}{3}\omega_0 x_6 \frac{sx_3}{x_3} sx_1 - \omega_0 x_4 \frac{cx_3 - 1}{x_3} + I_3 \omega_0 x_4 \frac{cx_3 - 1}{x_3} \\
&\quad - \frac{1}{4} I_3 \omega_0 x_3 \frac{sx_3}{x_3} sx_2 sx_1 - \frac{1}{2} I_3 \omega_0^2 \frac{s(2x_3)}{x_3}[(cx_2 - 1)(cx_1 - 1) + 1] \\
&\quad + \frac{1}{6} I_3 \omega_0^2 \frac{s^2 x_3}{x_3} sx_1 s(2x_2). \\
a_{64} &= \frac{1}{4}\omega_0 sx_1 sx_2 sx_3 - \omega_0[(cx_3 - 1)(cx_1 - 1) + 1] + \frac{1}{2} I_3 x_5 + I_3 \omega_0[(cx_3 - 1)(cx_1 - 1) + 1]
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{4}I_3\omega_0 sx_3 sx_2 sx_1. \\
a_{65} &= -\omega_0 sx_3 + \frac{1}{2}I_3 x_4 - \frac{1}{3}I_3\omega_0\frac{sx_3}{x_3}cx_2 x_3 \frac{1}{3}I_3\omega_0(sx_3 cx_2 + sx_3). \\
a_{66} &= -\omega_0 sx_2 + \frac{1}{3}\omega_0 sx_3 sx_1. \qquad (3A.4)
\end{aligned}
$$

CHAPTER FOUR
STUDY OF RELIABILITY PERFORMANCE OF SDRE+ISMC SCHEME

In Chapter 3, we found that SDRE is not a robust scheme via numerical simulation, but when combined with Integral-type Sliding Mode Control (ISMC), the closed-loop system is less sensitive to disturbances. In this chapter, we investigate the reliability issue of SDRE. Since there is no guideline provided for the factorization $\mathbf{f}(\mathbf{x}, t) = A(\mathbf{x}, t)\mathbf{x}$, specifically, to improve reliability, we resort to ISMC approach to improve robustness performance for a specific factorization of the nonlinear drift term.

From the approach viewpoint, reliable control can be classified as active [3]-[5], [19], [31], [35], [39], [50], [51] or passive [23], [27], [29], [46]-[49]. In a passive reliable design, we need to separate the healthy actuators from those actuators that might malfunction before it applies on the system. Nevertheless, it is difficult to retrieve such information in advance. On the contrary, in the active reliable control design, faults are detected and identified by a fault detection and diagnosis (FDD) mechanism, and then the controllers are reconfigured in real time in accordance with the online detection results. Therefore, we only consider the active reliable design in this chapter.

In Section 4.1, we define the system type, cost function, and control objective. Then we detailed the design of FDD and control law of SDRE and ISMC in Section 4.2. Finally, we apply the control law to the satellite attitude control and analyze the simulating results.

4.1 Problem Statement

In this study, we assume that the actuators' fault has been successfully detected and diagnosed by a Fault Detection and Diagnosis (FDD) mechanism. The fault may be time varying and include degradation, amplification and outage [30], [41]. Before the occurrence of faults, the engineers may take any kind of control strategy to fulfill their desired system performance. When the fault is detected and diagnosed, the control law is guided to switch to an active reliable law for ensuring system performance. Thus, after the fault is detected, we may divide the actuators into two groups $\mathcal{H}$ and $\mathcal{F}$, within which we assume that all of the actuators in $\mathcal{H}$ are healthy, while those in $\mathcal{F}$ experience faults. Therefore, System (3.1) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x}, t) + \tilde{B}_{\mathcal{H}}(\mathbf{x}, t)\mathbf{u}_{\mathcal{H}} + \tilde{B}_{\mathcal{F}}(\mathbf{x}, t)\mathbf{u}_{\mathcal{F}} + \tilde{\mathbf{d}} \end{cases} \tag{4.1}$$

where $\mathbf{x}_1 = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $\mathbf{x}_2 = (x_{n+1}, \cdots, x_{2n})^T \in \mathbb{R}^n$ and $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T)^T$ are the system states, $\mathbf{u}_{\mathcal{H}} \in \mathbb{R}^k$ and $\mathbf{u}_{\mathcal{F}} \in \mathbb{R}^{m-k}$ are the control inputs. $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^n$ and $\tilde{B}(\mathbf{x}, t) = [\tilde{B}_{\mathcal{H}}(\mathbf{x}, t) \in \mathbb{R}^{n\times k} \vdots \tilde{\mathrm{B}}_{\mathcal{F}}(\mathbf{x}, \mathrm{t}) \in \mathbb{R}^{n\times(m-k)}] \in \mathbb{R}^{n\times m}$, where $m \geq k \geq n$. $\tilde{\mathbf{d}} = (d_1, \cdots, d_n)^T \in \mathbb{R}^n$ denote possible model uncertainties and/or external disturbances. Note that System(4.1) is equivalent to the following system dynamic:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u} + \mathbf{d} \tag{4.2}$$

where $\mathbf{f}(\mathbf{x}, t) = [\mathbf{x}_2^T \vdots \tilde{\mathbf{f}}^T(\mathbf{x}, t)]^T$, $B(\mathbf{x}, t) = [0_{n\times m}^T \vdots \tilde{B}^T(\mathbf{x}, t)]^T$, $\mathbf{u} = (\mathbf{u}_{\mathcal{H}}^T \vdots \mathbf{u}_{\mathcal{F}}^T)^T$, and $\mathbf{d} = (\mathbf{0}_{n\times 1}^T \vdots \tilde{\mathbf{d}}^T)^T$.

Assumption 4.1 : $\mathbf{f}(\mathbf{x}, t)$ and $B(\mathbf{x}, t)$ are smooth functions with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Assumption 4.2 : For all states, $B(\mathbf{x}, t)$ is full rank.

Moreover, we define the quadratic performance index

$$J = \int_0^\infty \left[\mathbf{x}^T Q(\mathbf{x})\mathbf{x} + \mathbf{u}^T R(\mathbf{x})\mathbf{u}\right] dt \tag{4.3}$$

where $Q(\mathbf{x}) = Q(\mathbf{x})^T \geq 0$ and $R(\mathbf{x}) = R(\mathbf{x})^T > 0$.

The control objective is to compare the performances of the two control strategies (SDRE and SDRE+ISMC) when there are possible model uncertainties and/or external disturbances, especially when some actuators malfunction. To be more precisely, We study whether the system can be stabilized and use the cost function (and others mentioned later) defined in (4.3) as an index to compare the performance.

4.2 Design of Active Reliable Control Law

We assume that the output values of the faulty actuators are successfully diagnosed by an FDD mechanism as

$$\mathbf{u}_{\mathcal{F}} = \hat{\mathbf{u}}_{\mathcal{F}} + \Delta\mathbf{u}_{\mathcal{F}} \tag{4.4}$$

where $\hat{\mathbf{u}}_{\mathcal{F}}$ and $\Delta\mathbf{u}_{\mathcal{F}}$ denote the estimated value and the estimated error, respectively. Then System (4.1) can be written as

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2 \quad \text{and} \quad \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{B}_{\mathcal{H}}(\mathbf{x},t)\mathbf{u}_{\mathcal{H}} + \tilde{B}_{\mathcal{F}}(\mathbf{x},t)(\hat{\mathbf{u}}_{\mathcal{F}} + \Delta\mathbf{u}_{\mathcal{F}}) + \tilde{\mathbf{d}} \tag{4.5}$$

where $\mathbf{u}_{\mathcal{H}} \in \mathbb{R}^k$ and $\hat{\mathbf{u}}_{\mathcal{F}}$, $\Delta\mathbf{u}_{\mathcal{F}} \in \mathbb{R}^{m-k}$.

4.2.1 SDRE

Similar to Section 3.2.1 but with slight modification, we symbolically factorize $\mathbf{f}(\mathbf{x},t)$ into the form of $\mathbf{f}(\mathbf{x},t) = A(\mathbf{x},t)\mathbf{x}$ (see Appendix 3A in Chapter 3), where $A(\mathbf{x},t) \in \mathbb{R}^{n\times n}$, and then adopt the following procedures to solve the SDRE problem:

- Check the stabilizability of $[A(\mathbf{x},t), B_{\mathcal{H}}(\mathbf{x},t)]$ and the observability of $[A(\mathbf{x},t), C(\mathbf{x})]$ symbolically, where $B_{\mathcal{H}}(\mathbf{x},t) = [0^T_{n\times(m-k)} \vdots \tilde{B}^T_{\mathcal{H}}(\mathbf{x},t)]^T$, $C(\mathbf{x}) \in \mathbb{R}^{p\times n}$ has full rank and satisfies $Q(\mathbf{x}) = C^T(\mathbf{x})C(\mathbf{x})$, to ensure the solvability of the corresponding SDRE [24].

- Solve the SDRE for $P(\mathbf{x})$ to produce the SDRE controller $\mathbf{u} = -R^{-1}(\mathbf{x})B^T_{\mathcal{H}}(\mathbf{x},t)P(\mathbf{x})\mathbf{x}$.

4.2.2 ISMC

Under Assumption 4.2, System (4.5) is rewritten into:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2 \quad \text{and} \quad \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{B}_{\mathcal{H}}(\mathbf{x},t)\cdot(\mathbf{u}_{\mathcal{H}} + \Delta\mathbf{d}_m) + \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\cdot\hat{\mathbf{u}}_{\mathcal{F}} \tag{4.6}$$

where $\Delta\mathbf{d}_m = \tilde{B}^+_{\mathcal{H}}(\mathbf{x},t)\cdot[\tilde{B}_{\mathcal{F}}(\mathbf{x},t)\Delta\mathbf{u}_{\mathcal{F}} + \tilde{\mathbf{d}}]$ and $\tilde{B}^+_{\mathcal{H}}(\mathbf{x},t)$ is the pseudo-inverse matrix of $\tilde{B}_{\mathcal{H}}(\mathbf{x}$,t).

Assumption 4.3 : There exist $\rho_m(\mathbf{x},t) > 0$ such that

$$||\Delta\mathbf{d}_m|| \le \rho_m(\mathbf{x},t) \tag{4.7}$$

Assumption 4.4 : The origin of the nominal subsystem

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{B}(\mathbf{x},t)\mathbf{u} \end{cases} \tag{4.8}$$

is uniformly asymptotically stabilizable, that is, there exists a control $\mathbf{u}_0$ and a continuously differentiable function $V(\mathbf{x},t)$ such that

$$\gamma_1(||\mathbf{x}||) \leq V(\mathbf{x},t) \leq \gamma_2(||\mathbf{x}||) \tag{4.9}$$

$$\text{and} \quad \frac{\partial V(\mathbf{x},t)}{\partial t} + \left(\frac{\partial V(\mathbf{x},t)}{\partial \mathbf{x}}\right)^T \cdot \left[\tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{B}(\mathbf{x},t)\mathbf{u}_0\right] \leq -\gamma_3(||\mathbf{x}||) \tag{4.10}$$

where $\gamma_1, \gamma_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are class $\mathcal{K}_\infty$ functions and γ_3 is a class $\mathcal{K}$ function.

Along the ISMC design procedure, the sliding manifold is introduced as (4.11) below:

$$\begin{aligned} \mathbf{s}(\mathbf{x},t) &= D \cdot \left\{\mathbf{x}_2(t) - \mathbf{x}_2(t_0) - \int_{t_0}^{t} \left[\tilde{\mathbf{f}}(\mathbf{x}(\tau),\tau) + \tilde{B}(\mathbf{x}(\tau),\tau) \cdot \mathbf{u}_0(\tau)\right] \cdot d\tau\right\} \\ &= \mathbf{0} \end{aligned} \tag{4.11}$$

where $D \in \mathbb{R}^{n\times n}$ and $D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)$ is full rank $\forall \mathbf{x}$.

When the system's trajectory is on the sliding manifold, $\mathbf{s} = \mathbf{0}$, $\dot{\mathbf{s}} = \mathbf{0}$, from (4.6) and (4.11), it is obtained that

$$\begin{aligned} \dot{\mathbf{s}} &= D \cdot [\mathbf{x}_2 - \tilde{\mathbf{f}} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \\ &= D \cdot [\tilde{B}_{\mathcal{H}}(\mathbf{x},t) \cdot (\mathbf{u}_{\mathcal{H}} + \Delta\mathbf{u}_m) + \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] = \mathbf{0} \\ \Rightarrow \ & \mathbf{u}_{\mathcal{H}} = -\tilde{B}_{\mathcal{H}}^{+}(\mathbf{x},t) \cdot [\tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] - \Delta\mathbf{d}_m. \end{aligned}$$

Substitute $\mathbf{u}_{\mathcal{H}}$ into (4.6), the equivalent system dynamics is obtained

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{B}(\mathbf{x},t)\mathbf{u}_0 \end{cases} \tag{4.12}$$

which agrees with the nominal system defined in Assumption (4.4).

In order to keep the system state on the sliding manifold, it is chosen that

$$\mathbf{u}_{\mathcal{H}} = \begin{cases} \tilde{B}_{\mathcal{H}}^{+}(\mathbf{x},t) \cdot [\tilde{B}(\mathbf{x},t)\mathbf{u}_0 - \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}}] & \text{if } \mathbf{s} = \mathbf{0}; \\ \tilde{B}_{\mathcal{H}}^{+}(\mathbf{x},t) \cdot [\tilde{B}(\mathbf{x},t)\mathbf{u}_0 - \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}}] + \mathbf{u}_1 & \text{if } \mathbf{s} \neq \mathbf{0} \end{cases} \tag{4.13}$$

and

$$\mathbf{u}_1 = -\rho(\mathbf{x},t)\frac{[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}}{||[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}||} \tag{4.14}$$

where $\rho(\mathbf{x},t) > \rho_m(\mathbf{x},t)$. Note that $\mathbf{u}_1$ is designed to keep the state on the sliding manifold. By choosing the Lyapunov function as $V = \frac{1}{2}\mathbf{s}^T\mathbf{s}$, then differentiate V and substitute into (4.11), (4.13) and (4.14),

$$\begin{aligned}
\dot{V} &= \mathbf{s}^T\dot{\mathbf{s}} \\
&= \mathbf{s}^T D \cdot [\tilde{B}_{\mathcal{H}}(\mathbf{x},t)\cdot(\mathbf{u}_{\mathcal{H}} + \Delta\mathbf{d}_m) + \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \\
&= \mathbf{s}^T D \cdot [\tilde{B}(\mathbf{x},t)\mathbf{u}_0 - \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}} + \tilde{B}_{\mathcal{H}}(\mathbf{x},t)\mathbf{u}_1 + \tilde{B}_{\mathcal{H}}(\mathbf{x},t)\Delta\mathbf{d}_m + \tilde{B}_{\mathcal{F}}(\mathbf{x},t)\hat{\mathbf{u}}_{\mathcal{F}} - \tilde{B}(\mathbf{x},t)\mathbf{u}_0] \\
&= \mathbf{s}^T D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)\cdot\left\{-\rho(\mathbf{x},t)\frac{[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}}{||[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}||} + \Delta\mathbf{d}_m\right\} \\
&\leq -||[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}||\cdot\rho(\mathbf{x},t) + ||[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}||\cdot||\Delta\mathbf{d}_m|| \\
&\leq ||[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}||\cdot[-\rho(\mathbf{x},t) + \rho_m(\mathbf{x},t)] \\
&< 0.
\end{aligned}$$

To sum up with an important theorem,

Theorem 4.1 : Suppose that System (4.1) experiences actuator faults at the control channels in $\mathcal{F}$ with estimated value $\hat{\mathbf{u}}_{\mathcal{F}}$ and error $\Delta\mathbf{u}_{\mathcal{F}}$ given by FDD mechanism (4.5). Then the origin of System (4.1) under Assumptions 4.1 - 4.4 and the control law given by (4.13)-(4.14) is globally asymptotically stable (GAS).

4.3 Application to Satellite Attitude Control

In this section, we use the same satellite attitude control model as in Section 3.3.1. In the following, we first detail the design of fault detection and diagnosis (FDD) and compare the simulating results using different control methods.

4.3.1 Design of Fault Diagnosis and Detection (FDD)

In this section, we investigate the deign of FDD observer mentioned in Section 3.3.1 for the satellite attitude control. The main idea of this design is to decouple the control input so that the fault associated with each channel can be diagnosed and distinguished from the healthy ones. And the following system dynamics, same as (3.1), is considered.

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = \tilde{\mathbf{f}}(\mathbf{x},t) + \tilde{B}(\mathbf{x},t)\mathbf{u} + \tilde{\mathbf{d}}. \end{cases} \tag{4.15}$$

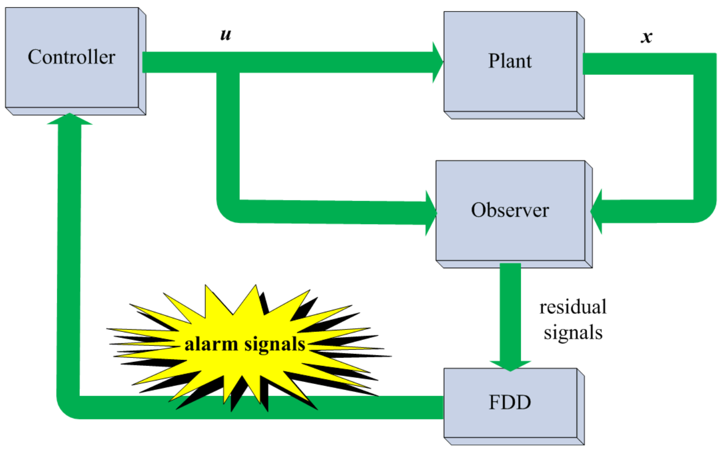

Fig. 4.1. FDD diagram

Fig. 4.1 shows the relation between FDD and system. Since the three Euler rates can be expressed in terms of angular velocity vector, which is available through accelerometer and gyroscope [34], in this section, we assume that all of the state variables are available for measurement and that $\tilde{B}(\mathbf{x}, t)$ in (4.15) is a constant matrix. We adopt the observer and residual signals r_i from [32] as (4.16) and (4.17) below:

$$\xi_i = f_i^{new}(\mathbf{z}) + u_i + l_i u_4 + k_i \cdot (z_{i+3} - \xi_i), \quad i = 1, 2, 3 \tag{4.16}$$

and

$$r_i = z_{i+3} - \xi_i, \quad i = 1, 2, 3 \tag{4.17}$$

where $k_i > 0$. It was shown in [32] that any single actuator fault can be detected and diagnosed at an exponential rate depending on k_i.

When the residual signals are larger than a selected threshold, the alarm will be set to be on.

4.3.2 Simulation Results

In this section, we still use MATLAB software to simulate the satellite attitude control under SDRE and ISMC approach. For both control approaches, we check whether the system with disturbances can be stabilized and compare their performances (e.g. quadratic performance index and convergence time).

The following Table 4.1 shows the simulating parameters in this chapter: (Note that for SDRE approach, the procedure of factorizing $\mathbf{f} = A(\mathbf{x}, t)\mathbf{x}$ is described in Appendix)

Table 4.1. Simulation parameters.

I_x	$2000\ N \cdot m \cdot s^2$
I_y	$400\ N \cdot m \cdot s^2$
I_z	$2000\ N \cdot m \cdot s^2$
ω_0	$1.0312 \times 10^{-3}\ rad/s$
$\tilde{\mathbf{d}}$	$(0.01\sin(t), 0.01\cos(2t), 0.01\sin(3t))^T$
$A(\mathbf{x}, t)$	see Appendix 3A in Chapter 3
D	I_3
Q	I_6
R	I_4
$\mathbf{u}_0$	*SDRE approach*
$\rho(\mathbf{x}, t), \rho_m(\mathbf{x}, t)$	$\|\|\Delta\mathbf{d}_m\|\|_\infty + 0.5$
$\mathbf{x}_0$	$(0.7, 0.07, -1.5, -0.3, -1.3, 0.2)^T$
k_1	10
k_2	10
k_3	10

Furthermore, to alleviate chattering, we modify the control law (4.13) into:

$$\mathbf{u} = \begin{cases} \tilde{B}_{\mathcal{H}}^{+}(\mathbf{x}, t)[\tilde{B}(\mathbf{x}, t)k_0 - \tilde{B}_{\mathcal{F}}(\mathbf{x}, t)\hat{\mathbf{u}}_{\mathcal{F}}] - \rho(\mathbf{x}, t)\frac{[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}}{\|[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}\|} & \text{if } \|[D\tilde{B}_{\mathcal{H}}(\mathbf{x}, t)]^T\mathbf{s}\| \geq \epsilon \\ \tilde{B}_{\mathcal{H}}^{+}(\mathbf{x}, t)[\tilde{B}(\mathbf{x}, t)k_0 - \tilde{B}_{\mathcal{F}}(\mathbf{x}, t)\hat{\mathbf{u}}_{\mathcal{F}}] - \rho(\mathbf{x}, t)\frac{[D\tilde{B}_{\mathcal{H}}(\mathbf{x},t)]^T\mathbf{s}}{\epsilon} & \text{if } \|[D\tilde{B}_{\mathcal{H}}(\mathbf{x}, t)]^T\mathbf{s}\| < \epsilon \end{cases} \quad (4.18)$$

where we choose $\epsilon = 0.02$. We simulate the faulty situation by that u_2 fails at time 1 and alarm signals as soon as $|r_i| \geq 0.01$.

The simulation results are shown in Figs. 4.2-4.6, and the summary of comparison of performance are shown in Table 4.2.

We denote the results:

- SDRE : the system without disturbance (nominal system) under SDRE approach only
- SDREr : the disturbed and actuator-failed system under SDRE approach only
- SDRE+ISMCr : the disturbed and actuator-failed system using SDRE-ISMC combined approaches

In addition, in Table 4.2, we also compare the performances under the Sliding Mode Control (SMC, see Section 4.2.2 in [42]), LQR (see Section 4.2.3 in [42]), and LQR-ISMC combined approach (see Section 4.2.1 in [42]), respectively.

- SMCr : the disturbed and actuator-failed system under nonlinear SMC approach only
- LQR : the system without disturbance (nominal system) under nonlinear LQR approach only
- LQRr : the disturbed system and actuator-failed under nonlinear LQR approach only
- LQR+ISMCr : the disturbed and actuator-failed system using LQR-ISMC combined approaches

It is observed from Fig. 4.2 that the stabilization performance is, as expected, achieved for the SDRE and the SDRE+ISMC designs. Besides, the state trajectories of the ISMC and those for nominal design (SDRE) are found almost identical, which agrees with the theoretical conclusion. From Fig. 4.6, the sliding variables of the SDRE+ISMC design are seen to keep at zero all the time. It implies that the system states remain on the sliding manifold for all t, which also agrees with the main results. In Fig. 4.4, the actuator fault is successfully detected by both designs, since the magnitude of the second residual signal exceeds the threshold near $t_{\text{SDRE+ISMC}} \approx 1.04$ and $t_{\text{SDRE}} \approx 1.067$, respectively. This can also be seen from the alarm signals given in Fig. 4.5 where alarm_2 denotes the fault of the second actuator. After the fault is successfully detected, the associated active reliable controllers are activated and the magnitude of the residual signals soon decreases, as shown in Fig. 4.4. The persistent oscillation of the residual signal comes from the effect of the disturbance $\tilde{\mathbf{d}}$, which also contributes to the oscillating control inputs $(\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_4)$ of SDREr and SDRE+ISMCr in Fig. 4.3. It is also noted from Figs. 4.3 that SDRE+ISMC design is observed to require larger control efforts than SDRE design due to the additional control $\mathbf{u}_1$ in (4.13) and (4.14). Finally, since the SDRE+ISMC design of this example adopts the SDRE scheme for the nominal healthy subsystem, its performances are close to those of SDRE except for the requirement of extra control component to compensate for the uncertainties.

Table 4.2 shows the comparison of performance, including energy consumption $\int \mathbf{u}^T\mathbf{u}$, quadratic performance index $\int(\mathbf{x}^T\mathbf{x}+\mathbf{u}^T\mathbf{u})$, required maximum control magnitude $||\mathbf{u}||_\infty$, and convergence time (when the magnitude of state is less than 0.01 at first time). For nominal system, LQR [42] approach has better performance than SDRE for all considered performance indexes. For the system with disturbances, LQR+ISMCr approach also has better performance than SDRE+ISMCr. Moreover, both SDRE+ISMCr and LQR+ISMCr consumes more control energy than the corresponding nominal control law SDRE and LQR [42], respectively. This is because the additional part, $\mathbf{u}_1$ in (4.13) and (4.14), is required in the ISMC design. Last but not least, we see that SMCr [42] and SDREr succeeds to stabilize, but LQRr [42] fails. To sum up, we conclude that in this study SDRE control law possesses certain robustness but not reliable.

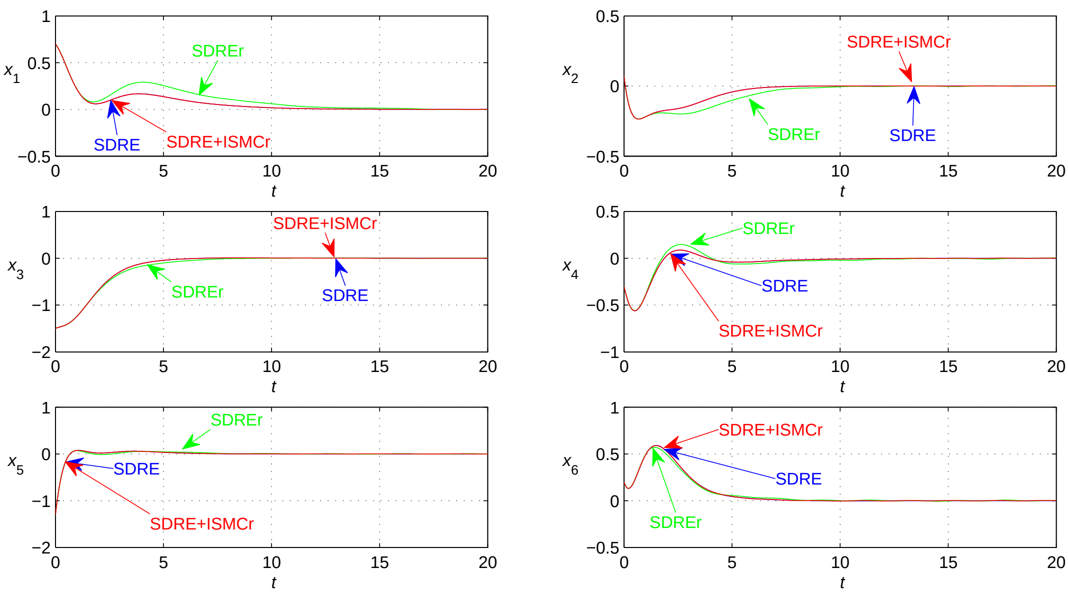

Fig. 4.2. Time history of the six state variables.

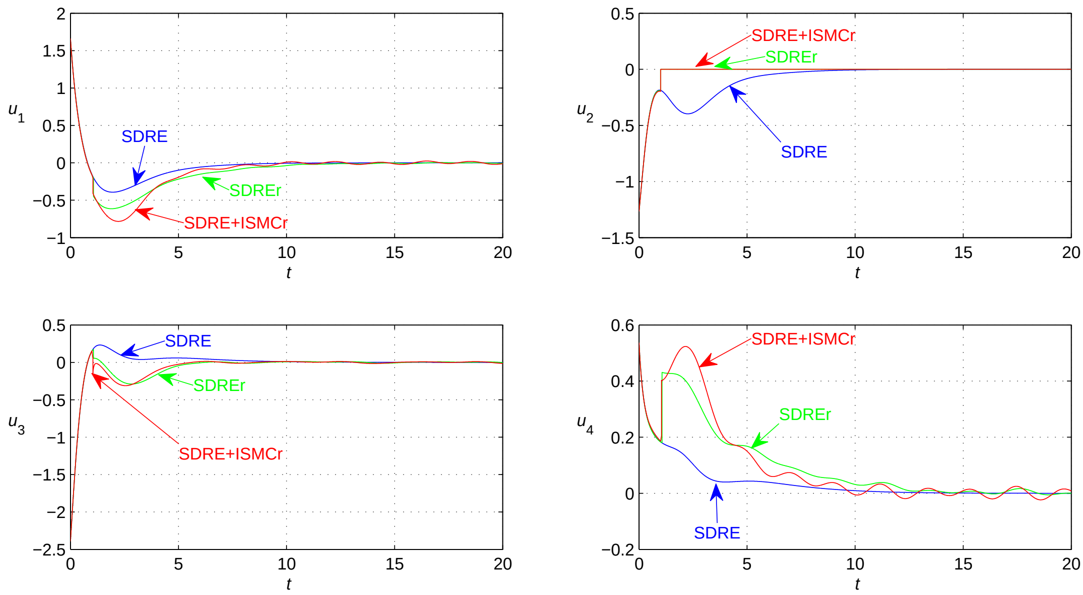

Fig. 4.3. Time history of the four control inputs.

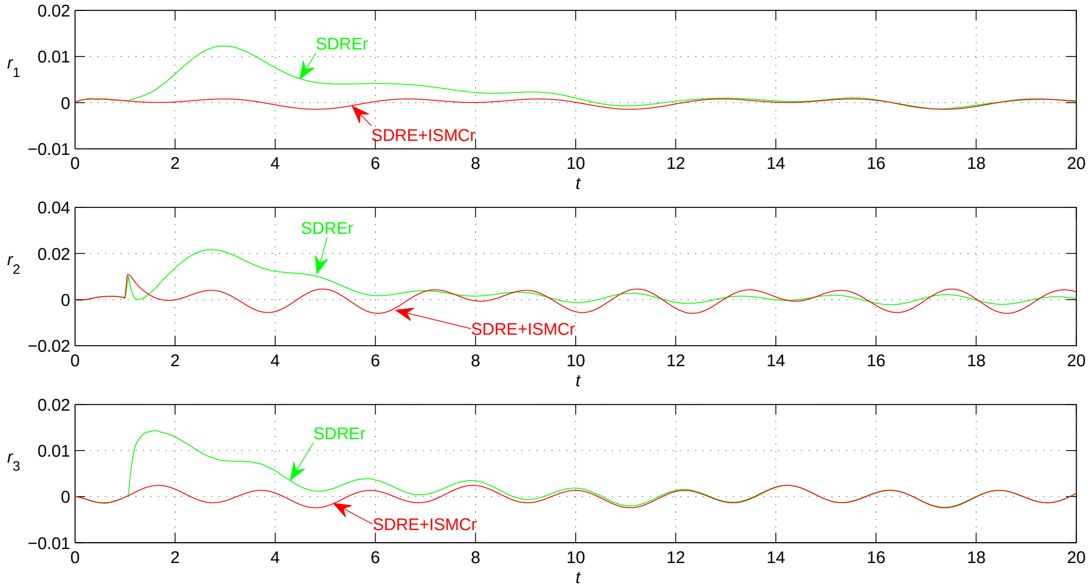

Fig. 4.4. Time history of the three residual signals.

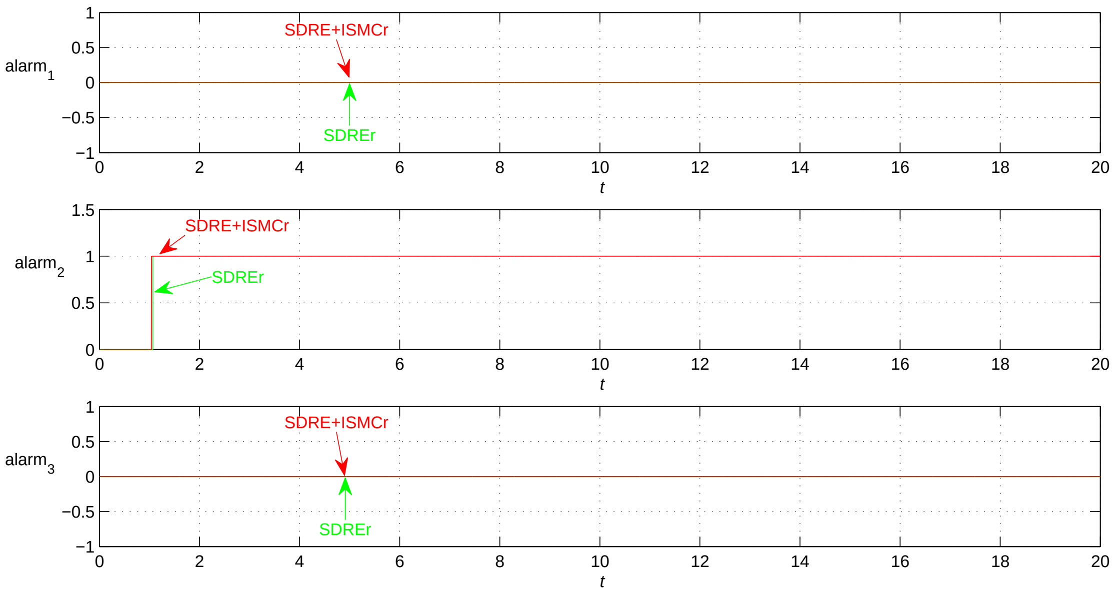

Fig. 4.5. Time history of the three alarm signals.

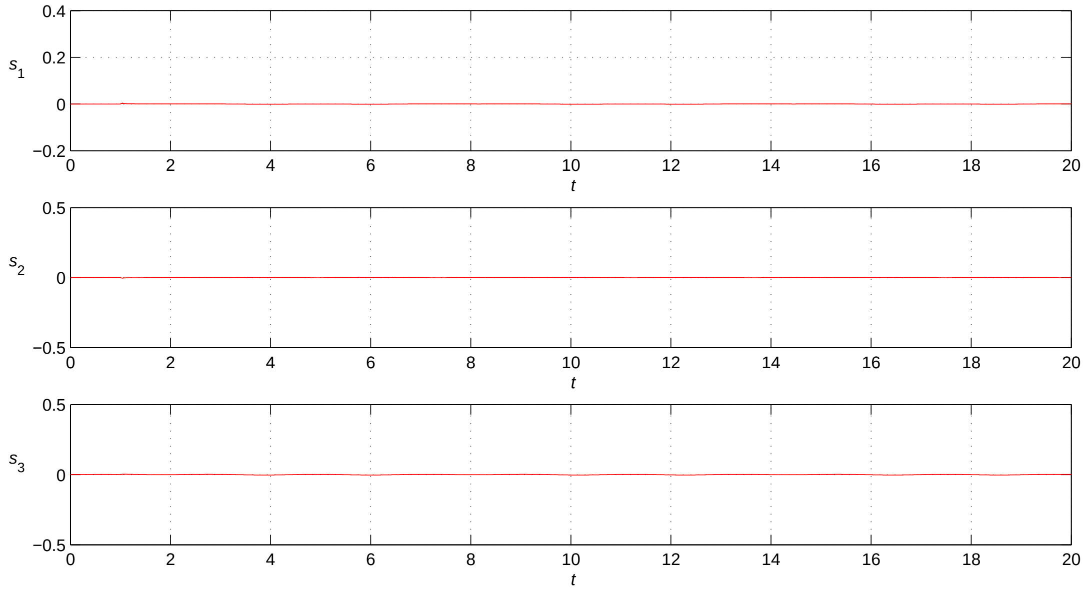

Fig. 4.6. Time history of the three sliding variables (SDRE+ISMCr).

Table 4.2. Comparison of performance.

Controller	Performance Index					
	$\|\|\mathbf{x}(t)\|\|_{t\to\infty} < 10^{-3}$	$\int \mathbf{u}^T\mathbf{u}$	$\int \mathbf{x}^T\mathbf{x}$	$\int(\mathbf{x}^T\mathbf{x}+\mathbf{u}^T\mathbf{u})$	$\|\|\mathbf{u}\|\|_\infty$	Convergence time
LQR+ISMCr	Yes	1.6149	4.4142	6.0291	2.1232	8.844
SDRE+ISMCr		4.0208	4.7619	8.7827	2.3917	11.61
SMCr	Yes	1.8763	6.0829	7.9592	2.2829	7.094
LQRr	No	**X**	**X**	**X**	**X**	**X**
SDREr	Yes	3.5434	5.066	8.6094	2.3917	16.982
LQR	Yes	1.5576	4.4156	5.9732	2.1232	8.799
SDRE		2.7756	4.7638	7.5395	2.3917	11.598

CHAPTER FIVE
ON FACTORIZATION OF THE DRIFT TERM IN SDRE SCHEME

5.1 Problem Statement

Although the SDRE algorithm fully captures the nonlinearities of the system, bringing the nonlinear system to a (non-unique) linear structure having state-dependent coefficient (SDC) matrices, and minimizing a nonlinear performance index having a quadratic-like structure, it has some drawbacks. First, it is known that the conditions"$[A(\mathbf{x},t), B(\mathbf{x},t)]$ is stabilizable" and"$[A(\mathbf{x},t), C(\mathbf{x})]$ is observable" are required for the existence of a unique positive definite solution $P(\mathbf{x})$ in Eq. (2.3) [24]; however, these symbolic checking conditions are in general not easy to implement, especially when the system dynamics is complicated. Next, there is no guideline provided for the factorization $\mathbf{f}(\mathbf{x},t) = A(\mathbf{x},t)\mathbf{x}$. To avoid these difficulty, in this study, we consider the following approach instead.

Problem A: At any nonzero state $\mathbf{x}$ and time t, $\mathbf{f} := \mathbf{f}(\mathbf{x},t)$ is a constant vector, while $B := B(\mathbf{x},t)$ and $C := C(\mathbf{x})$ are constant matrices. Find a matrix $A := A(\mathbf{x},t) \in \mathbb{R}^{n\times n}$ pointwise such that $A\mathbf{x} = \mathbf{f}$, (A, B) is stabilizable and (A, C) is observable.

To demonstrate the benefits of the alternative approach, we give an example below which shows the traditional SDRE scheme does not work when a specific factorization of $\mathbf{f} = A\mathbf{x}$ is adopted, but the alternative approach do work.

Example: Let $\mathbf{f} = (x_1 + x_1^2x_2^3, x_1^2x_2^2)^T$, $B = (0, 1)^T$ and $C = I_2$. Suppose that a specific factorization for $\mathbf{f} = A\mathbf{x}$ is given as $A := \begin{pmatrix} 1 & x_1^2x_2^2 \\ 0 & x_1^2x_2 \end{pmatrix}$. Clearly, (A, C) is observable, but (A, B) is not stabilizable when $x_1 = 0$ or $x_2 = 0$. Thus, the SDRE, given by (2.3), might fail to have a positive definite solution $P(\mathbf{x})$ when $x_1 = 0$ or $x_2 = 0$, which will result in the SDRE scheme failing to operate. However, since $Q(\mathbf{x}) = C^T(\mathbf{x})C(\mathbf{x}) = I_2$, Problem

A is solvable for this case (see Corollary 5.2). ■

It is also worth noting that Problem A is always solvable if the SDRE problem for some specific factorization can be continuously operated. We first consider the Problem A at a specific nonzero state, as described in Problem B below:

Problem B: Given two constant vectors $\mathbf{x}, \mathbf{f} \in \mathbb{R}^n$, and two constant matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ with $\mathbf{x} \neq \mathbf{0}$, rank$(B) \geq 1$ and rank$(C) \geq 1$, when does there exist a matrix $A \in \mathbb{R}^{n\times n}$ pointwise such that $A\mathbf{x} = \mathbf{f}$, (A, B) is stabilizable and (A, C) is observable?

Note that, Problem A (and B) are always solvable for the case of $n = 1$. Therefore, in the following we only consider the case of $n > 1$. To answer Problem B, we denote $(\mathbb{R}^n)^* = \{\mathbf{x}^{\mathrm{T}} | \mathbf{x} \in \mathbb{R}^n\}$, which is known to be the dual space of $\mathbb{R}^n$ [22]. Suppose that $\mathbf{p}_1, \cdots, \mathbf{p}_k \in \mathbb{R}^n$ and $\mathbf{q}_1^T, \cdots, \mathbf{q}_l^T \in (\mathbb{R}^n)^*$. We denote $\{\mathbf{p}_1, \cdots, \mathbf{p}_k\}^\perp = \{\mathbf{q}^T \in (\mathbb{R}^n)^* \mid \mathbf{q}^{\mathrm{T}}\mathbf{p}_{\mathrm{i}} = 0 \text{ for } 1 \leq i \leq k\}$ and $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_l^T\}^\perp = \{\mathbf{p} \in \mathbb{R}^n | \mathbf{q}_{\mathrm{i}}^{\mathrm{T}}\mathbf{p} = 0 \text{ for } 1 \leq i \leq l\}$. In addition, we denote $B^\perp := \{\mathbf{q}^T \in (\mathbb{R}^n)^* | \mathbf{q}^{\mathrm{T}}\mathrm{B} = 0\}$ and $C^\perp := \{\mathbf{p} \in \mathbb{R}^n | \mathrm{C}\mathbf{p} = 0\}$.

5.2 Solvability Condition

We assume that the matrix A is diagonalizable in the form of

$$A = MDM^{-1} \tag{5.1}$$

where $D = \mathrm{diag}[\lambda_1, \cdots, \lambda_n] \in \mathbb{R}^{n\times n}$, $M = [\mathbf{p}_1, \cdots, \mathbf{p}_n] \in \mathbb{R}^{n\times n}$, $M^{-1} = [\mathbf{q}_1, \cdots, \mathbf{q}_n]^T \in \mathbb{R}^{n\times n}$, and $\lambda_1, \cdots, \lambda_n$ are distinct. Clearly, $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, $\mathbf{p}_i$ and $\mathbf{q}_i^T$ are the right and the left eigenvectors of A associated with eigenvalues λ_i, respectively. We have the following lemma:

Lemma 5.1 Let A be factorized in the form of (5.1). Then

(i) $A\mathbf{x} = \mathbf{f} \iff \lambda_i \mathbf{q}_i^T \mathbf{x} = \mathbf{q}_i^T \mathbf{f}$ for all $i \iff \mathbf{q}_i^T(\lambda_i \mathbf{x} - \mathbf{f}) = 0$ for all i.

(ii) (A, C) is observable if and only if $\mathbf{p}_i \notin C^\perp$ for all $i = 1, \cdots, n$.

(iii) (A, B) is controllable if and only if $\mathbf{q}_i^T \notin B^\perp$ for all $i = 1, \cdots, n$.

(iv) (A, B) is stabilizable if and only if $\mathbf{q}_i^T \notin B^\perp$ whenever $\lambda_i \geq 0$.

Proof: (i) The result follows from writing $A\mathbf{x} = \mathbf{f}$ in the form of $DM^{-1}\mathbf{x} = M^{-1}\mathbf{f}$ and then comparing both sides componentwise.

(ii) It is known from the PBH test [8] that the pair (A, C) is observable if and only if $\text{rank}\left(\begin{pmatrix} C \\ \lambda_i I - A \end{pmatrix}\right) = n$ for all $i = 1, \cdots, n$, i.e., $\begin{pmatrix} C \\ \lambda_i I - A \end{pmatrix} \mathbf{p} \neq \mathbf{0}$ for any $\mathbf{p} \neq \mathbf{0}$. It is clear that $(\lambda_i I - A)\mathbf{p} = \mathbf{0}$ if and only if $(\lambda_i, \mathbf{p})$ is an eigenpair of A or $\mathbf{p} = \mathbf{0}$. It follows that (A, C) is observable if and only if $C\mathbf{p}_i \neq \mathbf{0}$ for all i, that is, $\mathbf{p}_i \notin C^\perp$ for all $i = 1, \cdots, n$.

(iii) It is known that (A, B) is controllable if and only if (A^T, B^T) is observable [8]. Since $(\lambda_i, \mathbf{q}_i)$, $i = 1, \cdots, n$, are eigenpairs of A^T, we have from the proof of (ii) that (A^T, B^T) is observable if and only if $B^T\mathbf{q}_i \neq 0$, i.e., $\mathbf{q}_i^T B \neq \mathbf{0}$, for all i. Thus, (A, B) is controllable if and only if $\mathbf{q}_i^T \notin B^\perp$ for all i.

(iv) (A, B) is stabilizable if and only if $\text{rank}([\lambda_i I - A \vdots B]) = n$ for those i in which $\lambda_i \geq 0$ [8]. This is equivalent to $\mathbf{q}_i^T B \neq 0$ whenever $\lambda_i \geq 0$, that is, $\mathbf{q}_i^T \notin B^\perp$ whenever $\lambda_i \geq 0$. ■

We also need the following three results:

Lemma 5.2 Let $\mathcal{V}$ be a k dimensional vector subspace of $(\mathbb{R}^n)^*$, $k < n$, and $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$ are linearly independent (LI) vectors with $\mathbf{q}_1^T \notin \mathcal{V}$. Then there exists $\mathbf{q}_{k+1}^T \in \mathcal{V}$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{k+1}^T\}$ are LI.

Proof: Suppose that such $\mathbf{q}_{k+1}^T$ does not exist. Then $\mathcal{V} \subset \text{span}\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$. Since both $\mathcal{V}$ and $\text{span}\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$ have dimension k, we have $\mathcal{V} = \text{span}\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$, and thus $\mathbf{q}_1^T \in \mathcal{V}$, a contradiction. This completes the proof. ■

Lemma 5.3 Let $\mathcal{V}$ be a $k-1$ dimensional vector subspace of $(\mathbb{R}^k)^*$ and $\{\mathbf{v}_1^T, \cdots, \mathbf{v}_k^T\}$ be a basis of $(\mathbb{R}^k)^*$ with $\mathbf{v}_i^T \notin \mathcal{V}$ for all i. Besides, let $\mathcal{W}_i := \text{span}\{\mathbf{v}_1^T, \cdots, \mathbf{v}_{i-1}^T, \mathbf{v}_{i+1}^T, \cdots, \mathbf{v}_k^T\}$. Then $\mathcal{V} \not\subset \cup_{i=1}^k \mathcal{W}_i$. As a result, there exists a nonzero $\mathbf{v}^T \in \mathcal{V}$ such that $\mathbf{v}^T = \sum_{i=1}^k \alpha_i \mathbf{v}_i^T$ and $\alpha_i \neq 0$ for all $i = 1, \cdots, k$.

Proof: Note that, for all $i = 1, \cdots, k$, $\mathcal{W}_i$ is a vector space of dimension $k-1$ and $\mathcal{V} \neq \mathcal{W}_i$; Otherwise, $\mathbf{v}_j^T \in \mathcal{V}$ for all $j \neq i$, which contradicts to the assumption $\mathbf{v}_j^T \notin \mathcal{V}$ for all j. Since $\cup_{i=1}^k \mathcal{W}_i$ is not a vector space, we thus have $\mathcal{V} \not\subset \cup_{i=1}^k \mathcal{W}_i$. This fact together with $\{\mathbf{v}_1^T, \cdots, \mathbf{v}_k^T\}$ being a basis implies there exists a nonzero $\mathbf{v}^T \in \mathcal{V}$ such that $\mathbf{v}^T = \sum_{i=1}^k \alpha_i \mathbf{v}_i^T$ with $\alpha_i \neq 0$ for all i; Otherwise, each $\mathbf{v} \in \mathcal{V}$ will belong $\mathcal{W}_i$ for some i, which contradicts $\mathcal{V} \not\subset \cup_{i=1}^k \mathcal{W}_i$. ■

Lemma 5.4 Let $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}^T\}$ are LI and $\mathbf{q}_n^T := \alpha_c \mathbf{c}^T + \sum_{j=1}^{n-1} \alpha_j \mathbf{q}_j^T$, $\alpha_c \neq 0$ and $\alpha_j \neq 0$ for all $j = 1, \cdots, n-1$. Then

(i) $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_n^T\}$ are LI.

(ii) For any $i \in \{1, \cdots, n\}$, the n vectors $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}^T\}$ are LI.

(iii) For any $i \in \{1, \cdots, n\}$, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp \not\subset (\mathbf{c}^T)^\perp$.

Proof: (i) Suppose that $\sum_{i=1}^n k_i \mathbf{q}_i^T = \mathbf{0}^\mathbf{T}$. Inserting the expression of $\mathbf{q}_n^T$ into the equation yields $\sum_{i=1}^{n-1}(k_i + k_n\alpha_i)\mathbf{q}_i^T + k_n\alpha_c\mathbf{c}^T = \mathbf{0}^\mathbf{T}$. Since $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}^T\}$ are LI, we have $k_n\alpha_c = 0$ and $k_i + k_n\alpha_i = 0$ for all $i = 1, \cdots, n-1$. Since $\alpha_c \neq 0$, we have $k_n = 0$ and $k_i = 0$ for $i = 1, \cdots, n-1$. This proves the linear independency of $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_n^T\}$.

(ii) Suppose that $\sum_{j\neq i} k_j \mathbf{q}_j^T + k_c\mathbf{c}^T = \mathbf{0}^\mathbf{T}$. Inserting $\mathbf{q}_n^T$ into the equation, we have $\sum_{j\neq i}^{n-1}(k_j + k_n\alpha_j)\mathbf{q}_j^T + k_n\alpha_i\mathbf{q}_i^T + (k_n\alpha_c + k_c)\mathbf{c}^T = \mathbf{0}^\mathbf{T}$. Since $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}^T\}$ are LI and $\alpha_i \neq 0$, we have from the coefficient of $\mathbf{q}_i^T$ that $k_n = 0$, and thus $k_c = 0$ and $k_j = 0$ for all $j \neq i$ and $j \leq n-1$. Thus, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}^T\}$ are LI.

(iii) Suppose, on the contrary, that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp \subset (\mathbf{c}^T)^\perp$. Then any nonzero vector $\mathbf{p} \in \{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp$ has the property $\mathbf{c}^T\mathbf{p} = 0$ and $\mathbf{q}_j^T\mathbf{p} = 0$ for all $j \neq i$. Since $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}^T\}$ is a basis for $(\mathbb{R}^n)^*$, it follows that $\mathbf{p}$ must be a zero vector, which contradicts the fact that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp$ is a vector space of dimension 1. This proves that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp \not\subset (\mathbf{c}^T)^\perp$. ■

In the following, we denote $\mathbb{R}^-$ the set of negative real numbers. A necessary and sufficient condition for Problem B is now stated as Theorem 5.1 below:

Theorem 5.1 Problem B is unsolvable if and only if $\{\mathbf{x}, \mathbf{f}\}$ are linearly dependent (LD) and $C\mathbf{x} = 0$.

Proof: We divide the proof into the following four cases:

Case 1: ($\{\mathbf{x}, \mathbf{f}\}$ are LI and $C[\mathbf{x}, \mathbf{f}] \neq \mathbf{0}$)

Note that, $C[\mathbf{x}, \mathbf{f}] \neq \mathbf{0}$ implies that there exists a nonzero row vector $\mathbf{c}^T$ of C with $\mathbf{c}^T \notin \{\mathbf{x}, \mathbf{f}\}^\perp$. Choose $\lambda_1, \cdots, \lambda_n \in \mathbb{R}^-$ such that the n vectors $\{\lambda_i\mathbf{x} - \mathbf{f} \mid i = 1, \cdots, n\}$ are distinct and $\mathbf{c}^T(\lambda_i\mathbf{x} - \mathbf{f}) \neq 0$ for all $i = 1, \cdots, n$. If $n > 2$, since $\dim((\lambda_i\mathbf{x} - \mathbf{f})^\perp) = n-1$ for all i, we may easily choose $\mathbf{q}_i^T \in (\lambda_i\mathbf{x} - \mathbf{f})^\perp$, $1 \leq i \leq n-2$, satisfying $\mathbf{q}_1^T \notin (\lambda_{n-1}\mathbf{x} - \mathbf{f})^\perp$,

$\mathbf{q}_1^T(\lambda_n\mathbf{x}-\mathbf{f}) > 0$, $\mathbf{q}_i^T(\lambda_n\mathbf{x}-\mathbf{f}) \geq 0$ for $i = 2, \cdots, n-2$ and $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-2}^T, \mathbf{c}^T\}$ are LI. Since $\mathbf{q}_1^T \notin (\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp$ and $\dim((\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp) = n-1$, it follows from Lemma 5.2 that there exists a $\mathbf{q}_{n-1}^T \in (\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}^T\}$ are LI. We also select $\mathbf{q}_{n-1}^T$ satisfying $\mathbf{q}_{n-1}^T(\lambda_n\mathbf{x}-\mathbf{f}) \geq 0$. Define $\mathbf{q}_n^T = \alpha\mathbf{c}^T + \sum_{i=1}^{n-1}\mathbf{q}_i^T$, $\alpha = -[\sum_{i=1}^{n-1}\mathbf{q}_i^T(\lambda_n\mathbf{x}-\mathbf{f})]/[\mathbf{c}^T(\lambda_n\mathbf{x}-\mathbf{f})]$. Clearly, $\alpha \neq 0$ since $\mathbf{c}^T(\lambda_n\mathbf{x}-\mathbf{f}) \neq 0$, $\mathbf{q}_1^T(\lambda_n\mathbf{x}-\mathbf{f}) > 0$ and $\mathbf{q}_i^T(\lambda_n\mathbf{x}-\mathbf{f}) \geq 0$ for $2 \leq i \leq n-1$. Moreover, it is easy to check that $\mathbf{q}_n^T(\lambda_n\mathbf{x}-\mathbf{f}) = 0$. Thus, from (i) of Lemma 5.1, we have $A\mathbf{x} = \mathbf{f}$. From the structure of $\mathbf{q}_n^T$, the fact $\mathbf{p}_i \in \{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp$ and Lemma 5.4, we have $\mathbf{p}_i \notin (\mathbf{c}^T)^\perp$ for all i. This together with $C^\perp \subset (\mathbf{c}^T)^\perp$ and (ii) of Lemma 5.1 implies that (A, C) is observable. Finally, (A, B) is stabilizable since $\lambda_i \in \mathbb{R}^-$ for all i. Thus, Problem B is solvable. If $n = 2$, the proof of this case can also be easily derived if we choose $\mathbf{q}_1^T \in (\lambda_1\mathbf{x}-\mathbf{f})^\perp$ and $\mathbf{q}_1^T(\lambda_2\mathbf{x}-\mathbf{f}) > 0$.

Case 2: ($\{\mathbf{x}, \mathbf{f}\}$ are LI and $C[\mathbf{x}, \mathbf{f}] = \mathbf{0}$)

This case implies that each nonzero row vector $\mathbf{c}^T$ of C satisfies $\mathbf{c}^T \in \{\mathbf{x}, \mathbf{f}\}^\perp$. Similar to that of Case 1, we choose $\lambda_1, \cdots, \lambda_n \in \mathbb{R}^-$ such that the n vectors $\{\lambda_i\mathbf{x}-\mathbf{f} \mid i = 1, \cdots, n\}$ are distinct. Suppose that $n > 2$. Since $(\lambda_i\mathbf{x}-\mathbf{f})^\perp \cap \mathbf{c}^\perp$ is a vector space of dimension $n-2$ for all i, we may choose $\mathbf{q}_i^T \in \{(\lambda_i\mathbf{x}-\mathbf{f})^\perp \cap \mathbf{c}^\perp\} \backslash (\lambda_n\mathbf{x}-\mathbf{f})^\perp$ for $1 \leq i \leq n-2$ and $\mathbf{q}_1^T \notin (\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-2}^T\}$ are LI. Since $\mathcal{W} := (\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp \cap \mathbf{c}^\perp$ has dimension $n-2$, $\mathcal{W} \subset \mathcal{V} := \mathbf{c}^\perp$ and $\mathbf{q}_1^T \notin \mathcal{W}$, we have from Lemma 5.2 that there exists a vector $\mathbf{q}_{n-1}^T \in \mathcal{W} \backslash (\lambda_n\mathbf{x}-\mathbf{f})^\perp$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T\}$ are LI. Under these settings, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}^T\}$ are also LI since $\mathbf{q}_i^T\mathbf{c} = 0$ for all $1 \leq i \leq n-1$. Now, from Lemma 5.3, there exists a $\mathbf{v}^T \in (\lambda_n\mathbf{x}-\mathbf{f})^\perp$ such that $\mathbf{v}^T = \sum_{i=1}^{n-1}\alpha_i\mathbf{q}_i^T$ and $\alpha_i \neq 0$ for all i. Since both $\mathbf{c}^T$ and $\mathbf{v}^T$ belong to $(\lambda_n\mathbf{x}-\mathbf{f})^\perp$, we have $\mathbf{q}_n^T := \mathbf{c}^T + \mathbf{v}^T \in (\lambda_n\mathbf{x}-\mathbf{f})^\perp$. Thus, from (i) of Lemma 5.1, we have $A\mathbf{x} = \mathbf{f}$. Besides, from the structure of $\mathbf{q}_n^T$, the fact $\mathbf{p}_i \in \{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp$ and Lemma 5.4, we have $\mathbf{p}_i \notin (\mathbf{c}^T)^\perp$ for all i. This together with $C^\perp \subset (\mathbf{c}^T)^\perp$ and (ii) of Lemma 5.1 implies that (A, C) is observable. Finally, (A, B) is stabilizable since $\lambda_i \in \mathbb{R}^-$ for all i. Thus, Problem B is solvable. The case for $n = 2$ can be similarly proved if we choose $\mathbf{q}_1^T \in (\lambda_1\mathbf{x}-\mathbf{f})^\perp \backslash (\lambda_2\mathbf{x}-\mathbf{f})^\perp$.

Case 3: ($\{\mathbf{x}, \mathbf{f}\}$ are LD and $C\mathbf{x} \neq 0$)

Let $\mathbf{c}^T$ be a nonzero row vector of C such that $\mathbf{c}^T\mathbf{x} \neq 0$, and $\mathbf{b}$ be a nonzero column vector of B. We choose $n-1$ distinct real numbers $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}^-$, and $n-1$ LI row vectors $\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T \in \mathbf{x}^\perp$. This implies that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}^T\}$ are LI since $\mathbf{c}^T \notin \mathbf{x}^\perp$. If $\{\mathbf{x}, \mathbf{b}\}$ are LD (i.e., $\mathbf{x}^\perp = \mathbf{b}^\perp$), we choose $\mathbf{q}_n^T := \alpha \mathbf{c}^T + \sum_{i=1}^{n-1} \mathbf{q}_i^T$, $\alpha \neq 0$. It follows that $\mathbf{q}_n^T \notin \mathbf{b}^\perp$, and thus $\mathbf{q}_n^T \notin B^\perp$ since $B^\perp \subset \mathbf{b}^\perp$. On the other hand, if $\{\mathbf{x}, \mathbf{b}\}$ are LI, the above-mentioned $\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T$ may be chosen from $\mathbf{x}^\perp$ satisfying $\mathbf{q}_1^T\mathbf{b} > 0$ and $\mathbf{q}_i^T\mathbf{b} \geq 0$ for all $i = 2, \cdots, n-1$. It follows that $\sum_{i=1}^{n-1} \mathbf{q}_i^T\mathbf{b} > 0$, and therefore there exists a nonzero constant α such that $(\alpha\mathbf{c}^T + \sum_{i=1}^{n-1} \mathbf{q}_i^T)\mathbf{b} = \alpha\mathbf{c}^T\mathbf{b} + \sum_{i=1}^{n-1} \mathbf{q}_i^T\mathbf{b} \neq 0$ no matter $\mathbf{c}^T\mathbf{b}$ is zero or not. Here, we also choose $\mathbf{q}_n^T := \alpha\mathbf{c}^T + \sum_{i=1}^{n-1} \mathbf{q}_i^T$ as before. Clearly, $\mathbf{q}_n^T \notin B^\perp$ since $\mathbf{q}_n^T \notin \mathbf{b}^\perp$ and $B^\perp \subset \mathbf{b}^\perp$. Finally, we choose λ_n such that $\mathbf{q}_n^T(\lambda_n\mathbf{x} - \mathbf{f}) = 0$. From these discussions, we have $\mathbf{q}_i^T(\lambda_i\mathbf{x} - \mathbf{f}) = 0$ for all $i = 1, \cdots, n$, which implies from (i) of Lemma 5.1 that $A\mathbf{x} = \mathbf{f}$. Besides, due to the special structure of $\mathbf{q}_n^T$ and (ii) of Lemma 5.4, we have $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}^T\}$ are LI for any $i = 1, \cdots, n$. This fact together with (iii) of Lemma 5.4 and $\mathbf{p}_i \in \{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp$ leads to $\mathbf{c}\mathbf{p}_i \neq 0$ for all i, and thus $C\mathbf{p}_i \neq 0$ for all i. That is, by (ii) of Lemma 5.1, (A, C) is observable. Since $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}^-$ and $\mathbf{q}_n^T \notin B^\perp$, (A, B) is stabilizable by (iv) of Lemma 5.1. Thus, Problem B is solvable.

Case 4: ($\{\mathbf{x}, \mathbf{f}\}$ are LD and $C\mathbf{x} = 0$)

Since $\{\mathbf{x}, \mathbf{f}\}$ are LD, we have $\mathbf{f} = \lambda\mathbf{x}$ for some constant λ. Suppose that there exists A such that $A\mathbf{x} = \mathbf{f}$. Then $A\mathbf{x} = \mathbf{f} = \lambda\mathbf{x}$, and thus $(\lambda, \mathbf{x})$ is an eigenpair of A. This fact together with the condition $C\mathbf{x} = 0$ results in $\begin{pmatrix} C \\ \lambda I - A \end{pmatrix} \mathbf{x} = 0$, which implies that (A, C) is unobservable and Problem B is unsolvable.

Summarizing the above three cases gives the result. ■

From Theorem 5.1, we have the next two trivial results:

Corollary 5.1 Problem B is solvable if and only if $C\mathbf{x} \neq \mathbf{0}$ or $\{\mathbf{x}, \mathbf{f}\}$ are LI.

Corollary 5.2 Problem A is always solvable if any one of the following two conditions holds:

(i) $Q(\mathbf{x})$ is a nonsingular matrix for all $\mathbf{x} \neq \mathbf{0}$.

(ii) $Q(\mathbf{x}) = Q$ is a constant matrix and $\text{rank}(Q) = n$.

5.3 Implementation

We look forward to implement by usage of orthogonal matrices because the condition number of them equals 1 [40]. In particular, we choose the Householder matrix [20], denoted as $H \in \mathbb{R}^{n\times n}$, which is orthogonal and possesses following properties:

1. $H = I - 2\mathbf{v}\mathbf{v}^T$, where I is the $(n \times n)$ identity matrix, $\mathbf{v} \in \mathbb{R}^n$ is an unit vector.

2. -1 is one eigenvalue of H with corresponding eigenvector, $\mathbf{v}$. Moreover, 1 of multiplicity (n-1) are the other eigenvalues with corresponding eigen-space being perpendicular to $\mathbf{v}$.

3. $H^T = H$, $\det(H) = -1$, and $\mathrm{tr}(H) = n - 2$.

4. Let $H' = H_2H_1$, where $H_i = I - 2\mathbf{v}_i\mathbf{v}_i^T$, $||\mathbf{v}_i|| = 1$, $i = 1, 2$, then H' is an orthogonal matrix. Moreover, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are LI, then the subspace perpendicular to $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ is the eigen-space of H' corresponding to eigenvalue 1 of multiplicity $(n - 2)$.

To implement, we need the following lemmas:

Lemma 5.5 Assume that $A \in \mathbb{R}^{n\times n}$ is orthogonal, diagonalizable, and has four distinct eigenvectors: $\mathbf{p}_1 = \mathbf{p}_R + i\mathbf{p}_I$, $\mathbf{p}_2 = \mathbf{p}_R - i\mathbf{p}_I$, $\mathbf{p}_3 = \mathbf{q}_R + i\mathbf{q}_I$, and $\mathbf{p}_4 = \mathbf{q}_R - i\mathbf{q}_I$, with corresponding distinct eigenvalues $\lambda_1 = \alpha_p + i\beta_p$, $\lambda_2 = \alpha_p - i\beta_p$, $\lambda_3 = \alpha_q + i\beta_q$, and $\lambda_4 = \alpha_q + i\beta_q$, where $\beta_p \neq 0$ and $\beta_q \neq 0$. Then

(i) A can be represented as $A = \Psi \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \Psi^{-1}$, where $S = \begin{pmatrix} \alpha_p & \beta_p \\ -\beta_p & \alpha_p \end{pmatrix} \in \mathbb{R}^{2\times 2}$, $T = \mathrm{diag}[\lambda_3, \lambda_4, \cdots, \lambda_n] \in \mathbb{R}^{(n-2)\times(n-2)}$, and $\Psi = \left[\mathbf{p}_R \vdots \mathbf{p}_I \vdots \mathbf{p}_3 \vdots \mathbf{p}_4 \vdots \cdots \vdots \mathbf{p}_n\right]$.

(ii) $\mathbf{p}_R \perp \mathbf{p}_I$ and $||\mathbf{p}_R|| = ||\mathbf{p}_I|| = \frac{1}{\sqrt{2}}$.

(iii) $(\mathbf{p}_R, \mathbf{p}_I, \mathbf{q}_R, \mathbf{q}_I)$ is an orthogonal set.

Proof: (i) Since A is diagonalizable, let $A = \sum_{i=1}^{n} \lambda_i \mathbf{p}_i \mathbf{q}_i^T = \sum_{i=1}^{2} \lambda_i \mathbf{p}_i \mathbf{q}_i^T + \sum_{i=3}^{n} \lambda_i \mathbf{p}_i \mathbf{q}_i^T$, where $\mathbf{p}_i$ and $\mathbf{q}_i$ are right and left eigenvectors of A associated with A's eigenvalues λ_i, respectively, for $i = 1, 2, \cdots, n$. Consider the first summation only,

$\sum_{i=1}^{2}\lambda_i\mathbf{p}_i\mathbf{q}_i^T = \left[\mathbf{p}_R+i\mathbf{p}_I \vdots \mathbf{p}_R-i\mathbf{p}_I\right]\begin{pmatrix}\alpha_p+i\beta_p & 0\\ 0 & \alpha_p-i\beta_p\end{pmatrix}\left[\mathbf{q}_1^T \vdots \mathbf{q}_2^T\right]^T = \left[\mathbf{w}_1 \vdots \mathbf{w}_2\right]\left[\mathbf{q}_1^T \vdots \mathbf{q}_2^T\right]^T$, where $\mathbf{w}_1 = \mathbf{p}_R\alpha_p - \mathbf{p}_I\beta_p + i(\mathbf{p}_R\beta_p + \mathbf{p}_I\alpha_p)$ and $\mathbf{w}_2 = \mathbf{p}_R\alpha_p - \mathbf{p}_I\beta_p - i(\mathbf{p}_R\beta_p + \mathbf{p}_I\alpha_p)$. Here $\mathbf{w}_1$ and $\mathbf{w}_2$ are complex LI vectors. In order to yield two real-valued LI vectors from $\mathbf{w}_1$ and $\mathbf{w}_2$, let $\mathbf{r}_1 = \frac{1}{2}(\mathbf{w}_1+\mathbf{w}_2) = \mathbf{p}_R\alpha_p - \mathbf{p}_I\beta_p$ and $\mathbf{r}_2 = \frac{-i}{2}(\mathbf{w}_1-\mathbf{w}_2) = \mathbf{p}_R\beta_p + \mathbf{p}_I\alpha_p$. Hence $\left[\mathbf{r}_1 \vdots \mathbf{r}_2\right] = \left[\mathbf{p}_R\alpha_p - \mathbf{p}_I\beta_p \vdots \mathbf{p}_R\beta_p + \mathbf{p}_I\alpha_p\right] = \left[\mathbf{p}_R \vdots \mathbf{p}_I\right]\begin{pmatrix}\alpha_p & \beta_p\\ -\beta_p & \alpha_p\end{pmatrix}$.

(ii) Since A is orthogonal, we have $\mathbf{p}_1^H\mathbf{p}_1 = 1 \Leftrightarrow (\mathbf{p}_R+i\mathbf{p}_I)^H(\mathbf{p}_R+i\mathbf{p}_I) = 1 \Leftrightarrow (\mathbf{p}_R^T\mathbf{p}_R+\mathbf{p}_I^T\mathbf{p}_I)+i(\mathbf{p}_I^T\mathbf{p}_R-\mathbf{p}_R^T\mathbf{p}_I) = 1$. Moreover, we have $\mathbf{p}_1^H\mathbf{p}_2 = 0 \Leftrightarrow (\mathbf{p}_R+i\mathbf{p}_I)^H(\mathbf{p}_R-i\mathbf{p}_I) = 0 \Leftrightarrow (\mathbf{p}_R^T\mathbf{p}_R-\mathbf{p}_I^T\mathbf{p}_I)-i(\mathbf{p}_I^T\mathbf{p}_R+\mathbf{p}_R^T\mathbf{p}_I) = 0$. Combining these two equations yields $||\mathbf{p}_R|| = ||\mathbf{p}_I|| = \frac{1}{\sqrt{2}}$ and $\mathbf{p}_R\perp\mathbf{p}_I$.

(iii) Since A is orthogonal, we have $\mathbf{p}_1^H\mathbf{p}_3 = 0 \Leftrightarrow (\mathbf{p}_R+i\mathbf{p}_I)^H(\mathbf{q}_R+i\mathbf{q}_I) = 0 \Leftrightarrow (\mathbf{p}_R^T\mathbf{q}_R+\mathbf{p}_I^T\mathbf{q}_I)-i(\mathbf{p}_I^T\mathbf{q}_R-\mathbf{p}_R^T\mathbf{q}_I) = 0$. Moreover, we have $\mathbf{p}_1^H\mathbf{p}_4 = 0 \Leftrightarrow (\mathbf{p}_R+i\mathbf{p}_I)^H(\mathbf{q}_R-i\mathbf{q}_I) = 0 \Leftrightarrow (\mathbf{p}_R^T\mathbf{q}_R-\mathbf{p}_I^T\mathbf{q}_I)-i(\mathbf{p}_I^T\mathbf{q}_R+\mathbf{p}_R^T\mathbf{q}_I) = 0$. Combining these two equations yields $2\mathbf{p}_R^T\mathbf{q}_R - 2i\mathbf{p}_I^T\mathbf{q}_R = 0$ and $2\mathbf{p}_I^T\mathbf{q}_I + 2i\mathbf{p}_R^T\mathbf{q}_I = 0$, i.e., $\mathbf{p}_R\perp\mathbf{q}_R$, $\mathbf{p}_R\perp\mathbf{q}_I$, $\mathbf{p}_I\perp\mathbf{q}_R$ and $\mathbf{p}_I\perp\mathbf{q}_I$.
■

Lemma 5.6 Consider $H = H_2H_1$, where $H_i = I - 2\mathbf{u}_i\mathbf{u}_i^T$ with LI unit vectors $\mathbf{u}_i \in \mathbb{R}^n$, for $i = 1, 2$, are both Householder matrices. Let $\alpha \pm i\beta$ denote two eigenvalues of H (else being 1) with corresponding eigenvectors, $\mathbf{p} = \mathbf{p}_R + i\mathbf{p}_I$ and $\bar{\mathbf{p}}$. Moreover, we choose LI unit vectors $\mathbf{u}_3, \mathbf{u}_4 \in \{\mathbf{u}_1, \mathbf{u}_2\}^\perp$, then

(i) span$(\mathbf{u}_1, \mathbf{u}_2)$=span$(\mathbf{p}_R, \mathbf{p}_I)$.

(ii) $\alpha = -1 + 2\cos^2\theta$ and $\beta = \sqrt{1-\alpha^2}$, where $\cos\theta = \mathbf{u}_1^T\mathbf{u}_2$.

(iii) $\mathbf{p}$ and $\bar{\mathbf{p}}$ will still be eigenvectors of H_3H and H_4H_3H, where $H_3 = I - 2\mathbf{u}_3\mathbf{u}_3^T$ and $H_4 = I - 2\mathbf{u}_4\mathbf{u}_4^T$, with unchanged corresponding eigenvalues $\alpha \pm i\beta$.

Proof: (i)Because $H(\mathbf{p}_R+i\mathbf{p}_I) = (\alpha+i\beta)(\mathbf{p}_R+i\mathbf{p}_I)$, we have $H\left[\mathbf{p}_R \vdots \mathbf{p}_I\right] = \left[\mathbf{p}_R \vdots \mathbf{p}_I\right]\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$. Let $H = \Gamma\begin{pmatrix}K & 0\\ 0 & I_{(n-2)\times(n-2)}\end{pmatrix}\Gamma^{-1}$, where $\Gamma = \left[\mathbf{p}_R+i\mathbf{p}_I \vdots \mathbf{p}_R-i\mathbf{p}_I \vdots \mathbf{p}_3 \vdots \mathbf{p}_4 \vdots \cdots \vdots \mathbf{p}_n\right]$, and $K = \begin{pmatrix}\alpha+i\beta & 0\\ 0 & \alpha-i\beta\end{pmatrix} \in \mathbb{R}^{2\times 2}$. By Lemma 5.5, we can rewrite H as $\Gamma'\begin{pmatrix}K' & 0\\ 0 & I_{(n-2)\times(n-2)}\end{pmatrix}\Gamma'^{-1}$, where $\Gamma' = \left[\mathbf{p}_R \vdots \mathbf{p}_I \vdots \mathbf{p}_3 \vdots \mathbf{p}_4 \vdots \cdots \vdots \mathbf{p}_n\right]$, and $K' = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix} \in \mathbb{R}^{2\times 2}$. Note that span$(\mathbf{p}_R, \mathbf{p}_I) = (\mathbf{p}_3, \mathbf{p}_4, \cdots, \mathbf{p}_n)^\perp$ and span$(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{p}_3, \mathbf{p}_4, \cdots, \mathbf{p}_n)^\perp$, therefore span$(\mathbf{p}_R, \mathbf{p}_I)$=span$(\mathbf{u}_1, \mathbf{u}_2)$.

(ii)From the definition of trace of a matrix, we have $tr(H_2H_1) = n-2+2\alpha$. In addition, $tr(H_2H_1) = tr\left[(I - 2\mathbf{u}_2\mathbf{u}_2^T)(I - 2\mathbf{u}_1\mathbf{u}_1^T)\right] = tr\left[I - 2\mathbf{u}_1\mathbf{u}_1^T - 2\mathbf{u}_2\mathbf{u}_2^T + 4(\mathbf{u}_2^T\mathbf{u}_1)(\mathbf{u}_2\mathbf{u}_1^T)\right] = n-4+4\cos^2\theta$, where $\cos\theta = \mathbf{u}_1^T\mathbf{u}_2 = tr(\mathbf{u}_1^T\mathbf{u}_2) = tr(\mathbf{u}_2^T\mathbf{u}_1)$. Combining these two results yields $\alpha = -1 + 2\cos^2\theta$. Moreover, since $det(H_2H_1) = 1$, we have $\alpha^2 + \beta^2 = 1 \Rightarrow \beta = \sqrt{1-\alpha^2}$.

(iii)By (i), span$(\mathbf{u}_1, \mathbf{u}_2)$ =span$(\mathbf{p}_R, \mathbf{p}_I)$. Given that $\mathbf{u}_3 \in \{\mathbf{u}_1, \mathbf{u}_2\}^\perp$, then $\mathbf{u}_3 \in \{\mathbf{p}_R, \mathbf{p}_I\}^\perp \Rightarrow \mathbf{p}^H\mathbf{u}_3 = \bar{\mathbf{p}}^H\mathbf{u}_3 = 0$. Hence $H_3H\mathbf{p} = (I - 2\mathbf{u}_3\mathbf{u}_3^T)H\mathbf{p} = (I - 2\mathbf{u}_3\mathbf{u}_3^T)(\alpha + i\beta)\mathbf{p} = (\alpha + i\beta)\mathbf{p}$. Similarly, it is true for $\bar{\mathbf{p}}$. By similar procedure, it is true for H_4H_3H.

■

From Lemma 5.6, we have the following two results:

Corollary 5.3 Recall that C is defined in Problem A in Section 5.1.

1. span$(\mathbf{u}_1, \mathbf{u}_2) \not\subset C^\perp \Leftrightarrow$ span$(\mathbf{p}_R, \mathbf{p}_I) \not\subset C^\perp \Leftrightarrow C^T(\mathbf{p}) = C^T(\mathbf{p}_R + i\mathbf{p}_I) \neq 0 \Leftrightarrow C^T(\bar{\mathbf{p}}) = C^T(\mathbf{p}_R - i\mathbf{p}_I) \neq 0$.

2. We categorize θ into the the following:

   (a) $\theta \in (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$: $\alpha > 0$.

   (b) $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$: $\alpha < 0$.

   (c) $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$: $(\alpha, \beta) = (0, 1)$.

   (d) $\theta = \frac{\pi}{2}$: $(\alpha, \beta) = (-1, 0)$.

By using above lemmas, we obtain the next important result described by Theorem 5.2 as below:

Theorem 5.2 We consider that rank(C) =rank$(B) = 1$(other cases can be discussed similarly) and denote $\mathbf{c}^T$ and $\mathbf{b}$ as one row vector of C and one column vector of B, respectively. Under the following two cases of assumptions:

(i) If n is odd, i.e., $n = 3, 5, 7, \cdots$, we assume that $\{\mathbf{x}, \mathbf{f}\}$ are LI, and $\{\mathbf{c}^T, \mathbf{f}\}$ are LI.

(ii) If n is even, i.e., $n = 4, 6, 8, \cdots$, we assume that $\{\mathbf{x}, \mathbf{f}, \mathbf{c}^T\}$ are LI. Moreover, if $\mathbf{f}^T\mathbf{x} \leq 0$, we also assume that $\{\mathbf{x}, \mathbf{f}, \mathbf{b}\}$ are LI.

Then we can factorize A, which solves Problem A, into products of Householder matrice, therefore the condition number of A equals 1 [40].

Proof: (i)Let $K = \frac{||\mathbf{f}||}{||\mathbf{x}||}$. If $\mathbf{f}^T\mathbf{x} \leq 0$, then let $\mathbf{u}_1 = \frac{K\mathbf{x}+\mathbf{f}}{||K\mathbf{x}+\mathbf{f}||} \in \mathbb{R}^n$; else, let $\mathbf{u}_1 = \frac{K\mathbf{x}-\mathbf{f}}{||K\mathbf{x}-\mathbf{f}||} \in \mathbb{R}^n$. Hence $H_1\mathbf{x} = \mathbf{f}$, where $H_1 = I - 2\mathbf{u}_1{\mathbf{u}_1}^T$. Then choose an unit vector $\mathbf{u}_2 \in \mathbb{R}^n$ such that $\mathbf{u}_2 \in \mathbf{f}^\perp$, and $\mathbf{u}_2 \notin C^\perp$. Since $\mathbf{u}_2 \in \mathbf{f}^\perp$, we have $H_2H_1\mathbf{x} = \mathbf{f}$, where $H_2 = I - 2\mathbf{u}_2{\mathbf{u}_2}^T$. If $\mathbf{f}^T\mathbf{x} \leq 0$, then let $\mathbf{u}_1 = \frac{K\mathbf{x}+\mathbf{f}}{||K\mathbf{x}+\mathbf{f}||} \in \mathbb{R}^n$. By Lemma 5.6, we know that the two eigenvalues ($\neq 1$) of H_2H_1 will have positive real parts; else, we require that $|\mathbf{u}_2^T\mathbf{u}_1| = \xi_{(2,1)} < \frac{1}{\sqrt{2}}$, hence the eigenvalues($\neq 1$) of H_2H_1 will have negative real parts, else if $n = 3$, we also require that $\mathbf{b} \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2)$.

When $n = 3$, we require that $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2)$. Then we can construct A. If $\mathbf{f}^T\mathbf{x} \leq 0$, then let $A = -H_2H_1$, hence all eigenvalues of A will have negative real parts, which implies that A is stabilizable. On the other hand, since $\mathbf{u}_2 \notin C^\perp$, by Lemma 5.6 and Corollary 5.3, we have the two right eigenvectors($\neq -1$) of A not perpendicular to $\mathbf{c}_T$. Moreover, since $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2)$, the right eigenvector corresponding to eigenvalue -1 will not be perpendicular to $\mathbf{c}^T$, which means A is observable; else, we let $A = H_2H_1$. Therefore A is stabilizable since the left eigenvector corresponding to the only eigenvalue of real part (1) is not perpendicular to $\mathbf{b}$. On the other hand, A is observable by same reason as the other case of $\mathbf{f}^T\mathbf{x} \leq 0$.

When $n \neq 3$, we need an iteration. Iterate this step over $i = 3, 5, 7, \cdots, n-2$. Choose unit vectors $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$, which form a sub-basis of $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{i-1}, \mathbf{f})^\perp$ and $\mathbf{c}^T\mathbf{u}_i \neq 0$. Therefore we have $H_{i+1} \cdots H_2H_1\mathbf{x} = \mathbf{f}$, and by Lemma 5.6, that the eigenvectors of $H_{i-1} \cdots H_2H_1$ corresponding to $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{i-1})$ will still be eigenvectors of $H_{i+1} \cdots H_2H_1$. If $i = n-2$, we also require that $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-1})$. Moreover, if $\mathbf{f}^T\mathbf{x} \leq 0$, then we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} > \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \neq \xi_{(i,i-1)} \neq \cdots \neq \xi_{(2,1)}$; else, we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} < \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \neq \xi_{(i,i-1)} \neq \cdots \neq \xi_{(2,1)}$, else if $i = n-2$, we also require that $\mathbf{b} \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-1})$. As all iterations are done, we still need the final step to fully construct A. If $\mathbf{f}^T\mathbf{x} \leq 0$, then $A = -H_{n-1} \cdots H_2H_1$, where $H_j = I - 2\mathbf{u}_j\mathbf{u}_j^T, \forall j = 1, 2, \cdots, n-1$. By Lemma 5.6, we have all eigenvalues of A having negative real parts, which implies that A is stabilizable. On the other hand, since $\mathbf{u}_i \notin \mathbf{c}^\perp$, by Lemma 5.6 and Corollary 5.3, we have the n-1 eigenvectors of A corresponding to eigenvalues($\neq -1$) not perpendicular to $\mathbf{c}^T$. Moreover, since $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-1})$, the right eigenvector of A corresponding to eigenvalue -1 will not be perpendicular to $\mathbf{c}^T$, which means A is observable; else, we let $A = H_{n-1} \cdots H_2 H_1$. Therefore A is stabilizable since the left eigenvector corresponding to the only eigenvalue of positive real part, which equals 1 and this left eigenvector is perpendicular to $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-1})$, is not perpendicular to $\mathbf{b}$. On the other hand, A is observable by same reason as the other case of $\mathbf{f}^T\mathbf{x} \leq 0$.

(ii)Let k and $\mathbf{u}_1$ as given in (i). Then choose an unit vector $\mathbf{u}_2 \in \mathbb{R}^n$ such that $\mathbf{u}_2 \in \mathbf{f}^\perp$ and $\mathbf{u}_2 \notin c^\perp$. Since $\mathbf{u}_2 \in \mathbf{f}^\perp$, we have $H_2 H_1 \mathbf{x} = \mathbf{f}$, where $H_2 = I - 2\mathbf{u}_2{\mathbf{u}_2}^T$. If $\mathbf{f}^T\mathbf{x} \leq 0$, we require that $|\mathbf{u}_2^T \mathbf{u}_1| = \xi_{(2,1)} > \frac{1}{\sqrt{2}}$, and by Lemma 5.6, the two eigenvalues ($\neq 1$) of $H_2 H_1$ will have positive real parts; else, we require that $|\mathbf{u}_2^T \mathbf{u}_1| = \xi_{(2,1)} < \frac{1}{\sqrt{2}}$, hence the eigenvalues($\neq 1$) of $H_2 H_1$ have negative real parts.

When $n = 4$, we choose an unit vector $\mathbf{u}_3 \in \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{f}\}^\perp$ and require that $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{f})$, which implies $\mathbf{c}^T\mathbf{u}_3 \neq 0$. Finally, we can construct A. If $\mathbf{f}^T\mathbf{x} \leq 0$, then $A = -H_2 H_1$, hence all eigenvalues of A have negative real parts, which implies that A is stabilizable. On the other hand, since $\mathbf{u}_2 \notin c^\perp$ and $\mathbf{c}^T\mathbf{u}_3 \neq 0$, by Lemma 5.6 and Corollary 5.3, we know that the three right eigenvectors of A corresponding to eigenvalues ($\neq -1$) is not perpendicular to $\mathbf{c}^T$. Moreover, since the right eigenvector of A corresponding to eigenvalue -1 is perpendicular to $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{x}, \mathbf{f}, \mathbf{c}\}$ are LI, we obtain that $\{\mathbf{u}_1, \mathbf{c}\}$ are LI and therefore this right eigenvector is not perpendicular to $\mathbf{c}$, which means A is observable; else, we let $A = H_2 H_1$. By similar derivation of $\mathbf{c}$, we conclude that the right eigenvector corresponding to eigenvalue 1 is not perpendicular to $\mathbf{b}$. Therefore A is stabilizable since the left eigenvector corresponding to the only eigenvalue of real parts (equals 1) is not perpendicular to $\mathbf{b}$. On the other hand, A is observable by same reason as the other case of $\mathbf{f}^T\mathbf{x} \leq 0$.

When $n \neq 4$, we need an iteration. Iterate this step over $i = 3, 5, \cdots, n-3$. Choose unit vectors $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$, which form a sub-basis of $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{i-1}, \mathbf{f})^\perp$ and $\mathbf{c}^T\mathbf{u}_i \neq 0$. Therefore we have $H_{i+1} \cdots H_2 H_1 \mathbf{x} = \mathbf{f}$, and by Lemma 5.6, that the eigenvectors of $H_{i-1} \cdots H_2 H_1$ corresponding to $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{i-1})$ are still eigenvectors of

$H_{i+1}\cdots H_2H_1$. If $i=n-3$, we also require that $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{n-2},\mathbf{f})$. Moreover, if $\mathbf{f}^T\mathbf{x}\le 0$, then we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} > \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \ne \xi_{(i,i-1)} \ne \cdots \ne \xi_{(2,1)}$, ; else, we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} < \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \ne \xi_{(i,i-1)} \ne \cdots \ne \xi_{(2,1)}$. As all iterations are done, we choose unit vector $\mathbf{u}_{n-1} \in \{\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{n-2},\mathbf{f}\}^\perp$. Finally, we can construct A. If $\mathbf{f}^T\mathbf{x}\le 0$, then $A=-H_{n-1}\cdots H_2H_1$, where $H_j = I-2\mathbf{u}_j\mathbf{u}_j^T, \forall j = 1,2,\cdots,n-1$. By Lemma 5.6, we have all eigenvalues of A having negative real parts, which implies that A is stabilizable. On the other hand, since $\mathbf{u}_i \notin \mathbf{c}^\perp$ and $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{n-2},\mathbf{f})$, by Lemma 5.6 and Corollary 5.3, we have the n-1 eigenvectors of A corresponding to eigenvalues ($\ne -1$) not perpendicular to $\mathbf{c}^T$. Moreover, since this right eigenvector of A corresponding to eigenvalue -1 is perpendicular to $\text{span}(\mathbf{u}_1,\cdots,\mathbf{u}_{n-1})$ and $\{\mathbf{x},\mathbf{f},\mathbf{c}\}$ are LI, we have $\{\mathbf{u}_1,\mathbf{c}\}$ are LI and thus this eigenvector is not perpendicular to $\mathbf{c}$, which means A is observable; else, we let $A=H_{n-1}\cdots H_2H_1$. By similar derivation of $\mathbf{c}$, we conclude that the right eigenvector corresponding to eigenvalue 1 is not perpendicular to $\mathbf{b}$. Therefore A is stabilizable since the left eigenvector corresponding to the only eigenvalue of positive real part (equals 1) is not perpendicular to $\mathbf{b}$. On the other hand, A is observable by same reason as the other case of $\mathbf{f}^T\mathbf{x}\le 0$. ■

5.4 Algorithm

Prerequisite: $\{\mathbf{x},\mathbf{f}\}$ are LI or $C\mathbf{x}\ne 0$.

Case 1: $\{\mathbf{x},\mathbf{f}\}$ are LI, and $\{\mathbf{c}^T,\mathbf{f}\}$ are LI. ($n=3,5,7,\cdots$)

Step.1. Let $K=\frac{||\mathbf{f}||}{||\mathbf{x}||}$. If $\mathbf{f}^T\mathbf{x}\le 0$, then let $\mathbf{u}_1 = \frac{K\mathbf{x}+\mathbf{f}}{||K\mathbf{x}+\mathbf{f}||} \in \mathbb{R}^n$; else, let $\mathbf{u}_1 = \frac{K\mathbf{x}-\mathbf{f}}{||K\mathbf{x}-\mathbf{f}||} \in \mathbb{R}^n$.

Step.2. Choose an unit vector $\mathbf{u}_2\in\mathbb{R}^n$ such that $\mathbf{u}_2\in\mathbf{f}^\perp$, and $\mathbf{u}_2\notin C^\perp$. If $\mathbf{f}^T\mathbf{x}\le 0$, then we require that $|\mathbf{u}_2^T\mathbf{u}_1| = \xi_{(2,1)} > \frac{1}{\sqrt{2}}$; else, we require that $|\mathbf{u}_2^T\mathbf{u}_1| = \xi_{(2,1)} < \frac{1}{\sqrt{2}}$, else if $n=3$, we also require that $\mathbf{b}\notin\text{span}(\mathbf{u}_1,\mathbf{u}_2)$. Note that if $n=3$, we require that $\mathbf{c}^T\notin\text{span}(\mathbf{u}_1,\mathbf{u}_2)$ and then go to Step.4.

Step.3. Iterate this step over $i=3,5,7,\cdots,n-2$. Choose unit vectors $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$, which form a sub-basis of $(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{i-1},\mathbf{f})^\perp$ and $\mathbf{c}^T\mathbf{u}_i\ne 0$. If $i=n-2$, we also require that $\mathbf{c}^T\notin\text{span}(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_{n-1})$. Moreover, if $\mathbf{f}^T\mathbf{x}\le 0$, then we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} > \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \neq \xi_{(i,i-1)} \neq \cdots \neq \xi_{(2,1)}$; else, we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} < \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \neq \xi_{(i,i-1)} \neq \cdots \neq \xi_{(2,1)}$, else if $i = n-2$, we also require that $\mathbf{b} \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-1})$.

Step.4. If $\mathbf{f}^T\mathbf{x} \leq 0$, then $A = -H_{n-1}\cdots H_2H_1$; else $A = H_{n-1}\cdots H_2H_1$, where $H_i = I - 2\mathbf{u}_i\mathbf{u}_i^T, \forall i = 1, 2, \cdots, n-1$.

Case 1: $(n = 4, 6, 8, \cdots)$

Prerequisite: $\{\mathbf{x}, \mathbf{f}, \mathbf{c}^T\}$ are LI. If $\mathbf{f}^T\mathbf{x} \leq 0$, we also require that $\{\mathbf{x}, \mathbf{f}, \mathbf{b}\}$ are LI.

Step.1. Let $K = \frac{||\mathbf{f}||}{||\mathbf{x}||}$. If $\mathbf{f}^T\mathbf{x} \leq 0$, then let $\mathbf{u}_1 = \frac{K\mathbf{x}+\mathbf{f}}{||K\mathbf{x}+\mathbf{f}||} \in \mathbb{R}^n$; else, let $\mathbf{u}_1 = \frac{K\mathbf{x}-\mathbf{f}}{||K\mathbf{x}-\mathbf{f}||} \in \mathbb{R}^n$.

Step.2. Choose an unit vector $\mathbf{u}_2 \in \mathbb{R}^n$ such that $\mathbf{u}_2 \in \mathbf{f}^\perp$, and $\mathbf{u}_2 \notin C^\perp$. If $\mathbf{f}^T\mathbf{x} \leq 0$, then we require that $|\mathbf{u}_2^T\mathbf{u}_1| = \xi_{(2,1)} > \frac{1}{\sqrt{2}}$; else, we require that $|\mathbf{u}_2^T\mathbf{u}_1| = \xi_{(2,1)} < \frac{1}{\sqrt{2}}$. Note that if $n = 4$, we require that $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{f})$ and then go to Step.4.

Step.3. Iterate this step over $i = 3, 5, 7, \cdots, n-3$. Choose unit vectors $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$, which form a sub-basis of $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{i-1}, \mathbf{f})^\perp$ and $\mathbf{c}^T\mathbf{u}_i \neq 0$. If $i = n-3$, we also require that $\mathbf{c}^T \notin \text{span}(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-2}, \mathbf{f})$. Moreover, if $\mathbf{f}^T\mathbf{x} \leq 0$, then we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} > \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \neq \xi_{(i,i-1)} \neq \cdots \neq \xi_{(2,1)}$; else, we require that $|\mathbf{u}_{i+1}^T\mathbf{u}_i| = \xi_{(i+1,i)} < \frac{1}{\sqrt{2}}$ and $\xi_{(i+1,i)} \neq \xi_{(i,i-1)} \neq \cdots \neq \xi_{(2,1)}$.

Step.4. Choose an unit vector $\mathbf{u}_{n-1} \in \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-2}, \mathbf{f}\}^\perp$.

Step.5. If $\mathbf{f}^T\mathbf{x} \leq 0$, then $A = -H_{n-1}\cdots H_2H_1$; else $A = H_{n-1}\cdots H_2H_1$, where $H_i = I - 2\mathbf{u}_i\mathbf{u}_i^T, \forall i = 1, 2, \cdots, n-1$.

Case 2: $\{\mathbf{x}, \mathbf{f}\}$ are LI, and $\{\mathbf{c}^T, \mathbf{f}\}$ are LD. $(n > 2)$

(As case 1 of Theorem.5.1 describes)

Step.1. Choose $\lambda_i \in \mathbb{R}^-$ such that $\{\lambda_i\mathbf{x} - \mathbf{f}\}$ are all distinct and $\mathbf{c}^T(\lambda_i\mathbf{x} - \mathbf{f}) \neq 0$ $\forall i = 1, 2, \cdots, n$.

Step.2. Choose $\mathbf{q}_i^T \in (\lambda_i\mathbf{x}-\mathbf{f})^\perp$, $1 \le i \le n-2$, satisfying $\mathbf{q}_1^T \notin (\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp$, $\mathbf{q}_1^T(\lambda_n\mathbf{x}-\mathbf{f}) > 0$, $\mathbf{q}_i^T(\lambda_n\mathbf{x}-\mathbf{f}) \ge 0$ $\forall i = 2,3,\cdots,n-2$ and $\{\mathbf{q}_1,\cdots,\mathbf{q}_{n-2},\mathbf{c}^T\}$ are LI.

Step.3. Find $\mathbf{q}_{n-1}^T \in (\lambda_{n-1}\mathbf{x}-\mathbf{f})^\perp$ such that $\{\mathbf{q}_1^T,\cdots,\mathbf{q}_{n-1}^T,\mathbf{c}^T\}$ are LI and $\mathbf{q}_{n-1}^T(\lambda_n\mathbf{x}-\mathbf{f}) \ge 0$.

Step.4. Define $\mathbf{q}_n^T = \alpha\mathbf{c}^T + \sum_{i=1}^{n-1}\mathbf{q}_i^T$, where $\alpha = \frac{-[\sum_{i=1}^{n-1}\mathbf{q}_i^T(\lambda_n\mathbf{x}-\mathbf{f})]}{[\mathbf{c}^T(\lambda_n\mathbf{x}-\mathbf{f})]}$.

Case 2: $(n = 2)$

We choose $\mathbf{q}_1^T \in (\lambda_1\mathbf{x}-\mathbf{f})^\perp$ such that $\mathbf{q}_1^T(\lambda_2\mathbf{x}-\mathbf{f}) > 0$. Furthermore, λ_1, λ_2 and $\mathbf{q}_2^T$ can be chosen by the above algorithm(case $n > 2$).

Case 3: $\{\mathbf{x},\mathbf{f}\}$ are LD, and $\mathbf{c}^T\mathbf{f} \ne 0$.

(As case 3 of Theorem.5.1 describes)

Step.1. Choose distinct $\lambda_i \in \mathbb{R}^-$ and LI row vectors $\mathbf{q}_i^T \in \mathbf{x}^\perp$, $\forall i = 1,2,\cdots,n-1$.

Step.2. If $\{\mathbf{x},\mathbf{b}\}$ are LD, choose any $\alpha \in \mathbb{R} \ne 0$; else, we additionally require that $\mathbf{q}_1^T\mathbf{b} > 0$ and $\mathbf{q}_j^T\mathbf{b} \ge 0, \forall j = 2,3,\cdots,n-1$. Moreover, we choose α such that $(\alpha\mathbf{c}^T + \sum_{i=1}^{n-1}\mathbf{q}_i^T)\mathbf{b} \ne 0$.

Step.3. $\mathbf{q}_n^T := \alpha\mathbf{c}^T + \sum_{i=1}^{n-1}\mathbf{q}_i^T$.

Step.4. Choose $\lambda_n \in \mathbb{R}$ such that $\mathbf{q}_n^T(\lambda_n\mathbf{x}-\mathbf{f}) = 0$.

5.5 Illustrative Example - 2-Dim Single-Input Affine System

Consider the single-input affine system [2]

$$\dot{x}_1 = u \tag{5.2}$$

$$\dot{x}_2 = x_2 - x_1^3. \tag{5.3}$$

Let state vector $\mathbf{x} = (x_1, x_2)^T$. For demonstrating, we choose the performance index to be: $J = \int_0^\infty \left[\mathbf{x}^TQ\mathbf{x} + u^2\right] dt$, where $Q = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ge 0$, and the initial condition is

$\mathbf{x} = (0.5, 1)^T$. We adopt two approaches to simulate and then compare their performance. One is a continuous stabilizer concerning center manifold (CM) as given in [1],[2], $u_{CM} = -x_2 + x_1 + \frac{4}{3}x_1^{\frac{1}{3}} - x_2^3$; the other adopts SDRE to obtain the feedback controller, u_{SDRE}. For both approaches, it can seen from Fig. 5.1 that the system state can be stabilized.

On the other hand, we attempt to use the SDRE approach and see what might happen. At first, we try the fixed factorization of $A(\mathbf{x}, t) = \begin{pmatrix} 1 & -x_2^2 \\ 0 & 0 \end{pmatrix}$, feed this factorization into the associated Riccati equation and obtain the control law. However, we find that the associated Riccati equation fails to give a positive semi-definite solution since (A, C) is not detectable for $x_2 = 1$ [24]. As a result, we resort to a different factorization of A near (0.5,1) as described in previous sections. Note that $\mathbf{f}(1, 1) = \mathbf{0}$, thus $\{\mathbf{x}, \mathbf{f}\}$ are LI. By Theorem 5.1, we know that Problem A is solvable. After adopting this different factorization of A near (0.5,1) once, we still use the original fixed factorization to obtain the control law. From Fig. 5.1, it can be seen that the system state will finally also be stabilized.

From Table. 5.1, the SDRE approach is found to results better performance than the given stabilizer in quadratic performance $\int \mathbf{x}^T Q(\mathbf{x})\mathbf{x} + u^2$, energy consumption $\int u^2$ and the convergence time.

Table 5.1. Comparison of performance

Controller	Performance Index				
	$\lvert\lvert\mathbf{x}(t)\rvert\rvert_{t\to\infty} < 10^{-3}$	$\int u^2$	$\int(\mathbf{x}^T\mathbf{x} + u^2)$	$\lvert u\rvert_\infty$	Convergence time
CM	Yes	2.567	6.182	0.735	140
SDRE		0.149	2.176	1.086	55

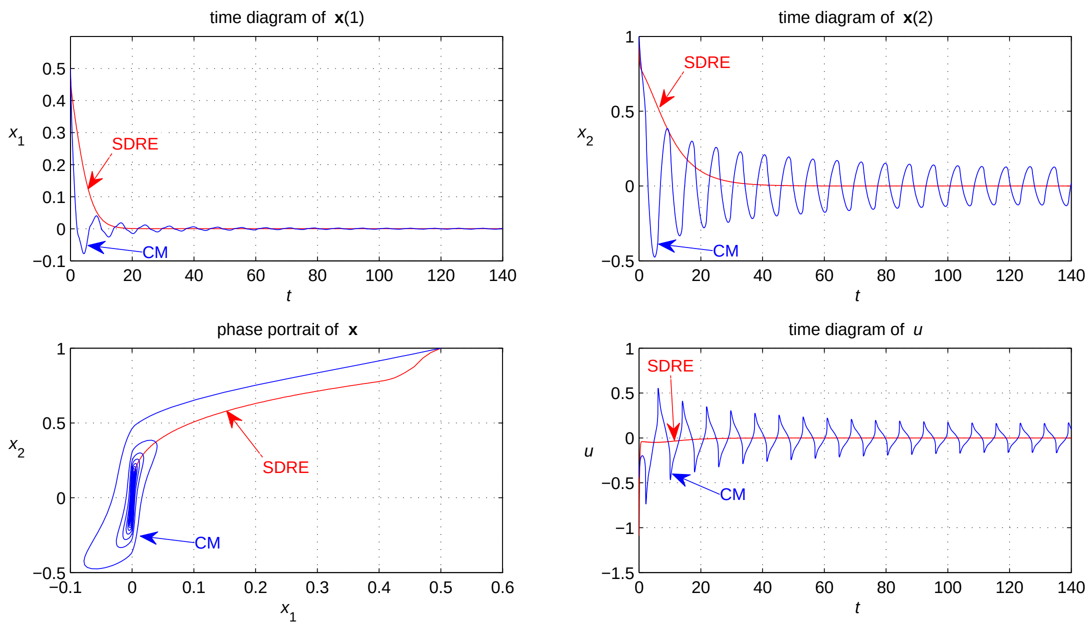

Fig. 5.1. State variables (x) and control input (u).

CHAPTER SIX
CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 Conclusions

In this thesis, we have investigated several interesting issues. In Chapter 3, we show from simulation that SDRE is not a robust scheme, but when incorporating with ISMC, the robustness performance can be greatly improved. In Chapter 4, we also consider the reliability issue of SDRE. By using the same dynamical system as in Chapter 3, we show that SDRE is not a reliable design, either. Again, we resort to ISMC and organize a reliable controller that can tolerate some actuators' outage. After encountering some difficulties during the design using SDRE, e.g.,

- hard to symbolicly check the conditions for the existence of the unique positive definite solution of the related Riccati equation
- no guidelines existed to factorize the drift term to satisfy some design criterion

we present an alternative approach to ease the implementation of traditional SDRE design. In Chapter 5, we formulate an alternative approach to factorize the drift term in SDRE scheme and give a necessary and sufficient condition (Theorem 5.1) that is much easier to check whether Problem A is solvable than the fixed factorization of the traditional SDRE. If Problem A is solvable, we give an alternative approach to construct the SDC matrix instead of using the fixed factorization of the traditional SDRE. By an illustrative example (Section 5.5), we demonstrate that while the traditional SDRE fails at some state, we still can resort to this alternative approach for a small deviation from this state and then adopts the original fixed factorization.

6.2 Suggestions for Further Research

There are still many interesting topics related to this thesis that are worth further studying, as listed below:

1. Try to parameterize all the SDC matrices satisfying the necessary and sufficient condition (Theorem 5.1).

2. Among all the parameterizations for a specific state, try to formulate an algorithm to find the (local) optimal solution at every state in the sense of minimizing the corresponding performance index.

3. Try to extend local optimum to global optimum (Dynamic programming [25] may be a possible direction to solve this problem).

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