

國立交通大學

機械工程學系

碩士論文

以有限元素法分析旋轉傾斜尤拉梁的穩態變形與
自由振動

Steady state and free vibration analysis of a rotating inclined
Euler beam by finite element method

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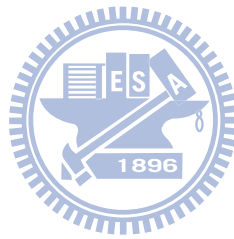
本研究主要利用共旋轉有限元素法推導旋轉傾斜尤拉梁的運動方程式，探討設定角為 0° 具不同傾斜角之等速旋轉傾斜尤拉梁的穩態變形及以穩態變形為平衡點的自然振動頻率。

本文利用非線性梁理論的一致線性化、d'Alembert 原理和虛功原理在當前的旋轉元素座標上推導梁元素的節點變形力、節點慣性力、元素剛度矩陣、元素向心力剛度矩陣(centripetal stiffness matrix)，元素質量矩陣(mass matrix)，元素陀螺矩陣(gyroscopic matrix)。本研究中將變形參數對時間的微分視為擾動量，故僅取到一次項，元素節點變形內力取到變形參數的二次項，元素節點慣性內力僅取到變形參數的一次項且忽略變形參數與變形參數對時間的微分的耦合項。將系統的非線性運動方程式中對時間的微分的項去掉即為系統的穩態平衡方程式，將系統運動方程式用泰勒級數在穩態

變形的位置展開，取到一次項，即為旋轉梁微小振動的運動方程式。

本文利用基於牛頓法的增量迭代法求出軸向位移及側向位移的穩態解。旋轉傾斜梁的振動方程式中存在陀螺矩陣，所以其自然振動頻率對應的振動模態為複變數，其頻率方程式(frequency equations)為一組代數齊次方程式，該組齊次方程式為一個二次特徵值問題，其係數形成之矩陣的行列式值為零時的根，即為自然振動頻率。本文以二分法來求行列式值為零時的根。

本研究以無因次化的數值例題探討傾斜角、無因次轉速、無因次轉軸半徑及細長比對旋轉傾斜梁無因次側向穩態變形和無因次自然頻率的影響，本研究還探討旋轉傾斜梁的兩個振動頻率接近時，對應之振動模態的耦合現象、特徵值曲線轉向(eigenvalue curve veering)及振態交換的現象。



Steady state and free vibration analysis of a rotating inclined Euler beam by
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Abstract

In this paper a co-rotational finite element formulation is proposed to derive the equations of motion for a rotating inclined Euler beam with constant angular velocity. The steady state deformation and natural frequency of the infinitesimal free vibration measured from the position of the corresponding steady state deformation are investigated for rotating inclined Euler beams with zero setting angle.

The element deformation nodal forces, inertia nodal forces, stiffness matrix, centripetal stiffness matrix, mass matrix and gyroscopic matrix are systematically derived by consistent linearization of the fully geometrically non-linear beam theory using the d'Alembert principle and the virtual work principle in the current rotating element coordinates. In this paper the terms up to the second order of deformation parameters and their spatial derivatives corresponding to the steady state element deformations are retained. However, only the terms up to the first order of deformation parameters, their spatial derivatives, and time derivatives corresponding to the free vibration of the beam element are retained. The coupling among deformation parameters and their

time derivatives are neglected. The steady state equilibrium equations may be obtained by dropping the terms of the time derivatives in the equation of motion. The governing equations for linear vibration may be obtained by the first order power series expansion of the equation of motion at the position of the corresponding steady state deformation.

An incremental-iterative method based on the Newton-Raphson method is employed to solve the steady state deformation. The frequency equations for free vibration of rotating inclined beam are a set of homogeneous equations. The natural frequencies may be determined by solving the homogeneous equations using the bisection method.

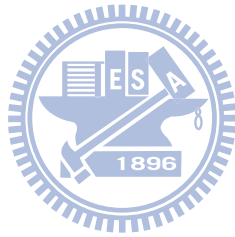
Dimensionless numerical examples are studied to investigate the dimensionless steady lateral deformation and the dimensionless natural frequency of rotating inclined beams with different inclined angle, dimensionless angular velocities, dimensionless radius of the hub, and slenderness ratios. The phenomenon of the coupling of mode shapes, the phenomenon of eigenvalue curve veering and mode shape changes are also investigated for rotating inclined beams that has two modes with closely spaced natural frequencies.

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第一章 導論

旋轉梁結構在實際上的應用是很重要的，像吊扇、渦輪的葉片、直升機的旋轉翼、風力發電機的葉片、衛星的支臂、飛機的螺旋槳和機械手臂。振動分析在旋轉梁結構的設計和分析上扮演很重要的角色，文獻上在這方面已有很多的研究。[1-16]

關於旋轉梁結構的振動分析可從文獻[1,2]有詳細的探討與回顧，Schilhansl [3]在考慮離心力，但忽略科氏力的情況下，導出了如圖一所示之等速旋轉梁振動的微分方程式。Lee 與 Kuo[4]探討了如圖一所示之旋轉 Euler 梁，對其旋轉軸的中心半徑、設定角及轉速對旋轉梁彎矩振動自然頻率的影響。Yokoyama[5]將旋轉慣量及剪變形、旋轉軸的中心半徑和設定角合併到有限元素的模式中，探討其對自然頻率的影響。Lee and Lin[6]用線性梁理論去推導旋轉 Timoshenko 梁之運動方程式，並探討了旋轉速度和質量慣性矩(mass moment of inertia)的耦合效應、設定角和旋轉速度對彎矩自然頻率的影響。Eick and Mignolet [7]探討旋轉梁在不同旋轉軸中心半徑與旋轉梁長度之比值下，其受軸向壓應力挫屈時之臨界轉速。

文獻[3-7]均用線性梁理論推導旋轉梁的運動方程式，且在作其振動分析時都不考慮科氏力，但均無討論其適當性或影響，在文獻[8]Simo and Vu-Quoc 提到在分析旋轉結構需要用幾何非線性梁理論(至少取到二次項)才能適當的計算離心力對彎矩剛度的影響，若用線性梁理論(只取到一次項)將會產生虛假的彎矩剛度流失，所以文獻[3-7]中推導的旋轉梁之運動方程式及所求得之振動的自然頻率應是不正確的。

文獻[9,10]利用非線性梁理論的一致線性化、虛功原理和 d'Alembert 原理在旋轉座標上推導旋轉 Timoshenko 梁正確的線性運動方程式，文獻[9,10]

在分析時考慮了軸向變形及科氏力。旋轉梁的自然振動是指以其穩態解為平衡點的微小振動，故須先求出其穩態解，除了設定角為 0° 或 90° 外，旋轉梁之穩態變形是三維的變形，且其自然振動是軸向、側向與扭轉耦合的三維振動，文獻[9,10]僅分析設定角為 0° 或 90° 之旋轉梁，並僅考慮軸向變形及一個側方向的位移與旋轉的二維振動，文獻[9]提出一套旋轉梁之自然頻率的級數解法及計算其自然頻率的數值計算程序，並探討科氏力對旋轉梁之自然頻率的影響。文獻[10]以文獻[9]提出的方法及細長比很大的 Timoshenko 梁模擬旋轉 Euler 梁，文獻[10]發現在低轉速時，科氏力對細長比很大的旋轉梁的自然頻率影響不大，但文獻[9,10]中並無高轉速的結果，因在高轉速時，文獻[9,10]的數值方法對細長比很大的旋轉梁無法收斂。

文獻[11]分析如圖一所示，設定角為 0° 或 90° 的旋轉 Euler 梁，利用虛功原理與 d'Alembert 原理，配合非線性梁理論的一致線性化，在旋轉座標上推導旋轉 Euler 梁正確的線性運動方程式，並僅考慮軸向變形及一個側方向的位移與旋轉的二維振動，文獻[11]將旋轉梁分成數段，每段稱為一個元素，每個元素用一個級數解來表示其自由振動，文獻[11]發現當細長比很大時，在高轉速下僅用一個元素無法求得正確的自然頻率，需將旋轉梁分成兩個以上的元素，才能求得精確的自然頻率，但文獻[11]並未探討其原因。文獻[12]考慮一具雙軸對稱之三維旋轉 Timoshenko 梁，利用共旋轉有限元素法(Co-rotational finite element formulation)和虛功原理配合非線性梁理論的一致線性化，推導梁元素節點慣性力與節點變形力。具雙軸對稱之三維旋轉梁的穩態解包含軸向和扭轉變形，文獻[12]保留軸向和扭轉變形的穩態解到二次項及扭轉率的三次項，文獻[12]將旋轉梁的運動方程式中的時間函數去掉求得系統穩態平衡式，再用牛頓法求得穩態解，文獻[12]用泰勒級數在穩態平衡點將運動方程式一致線性化，求得旋轉梁的振動方程式，文獻[12]

探討旋轉速度和設定角對三維旋轉梁之穩態變形及自然頻率的影響。

文獻[13-16]探討，如圖二所示，具有設定角與傾斜角的旋轉梁之自由振動。當傾斜角不為零時，若設定角不為 90° ，旋轉梁之軸向位移及側向位移的穩態解都不為零，但文獻[13]僅考慮軸向的穩態變形對自然頻率的影響，並未考慮側向穩態位移的影響，因文獻[13-15]忽略了側向位移的穩態解，故其旋轉梁之自然頻率可能不準確。當傾斜角不為零時，僅有設定角為 0° 或 90° 時，旋轉傾斜尤拉梁的穩態解及自然振動是二維運動，文獻[16]用虛功原理與 d'Alembert 原理及幾何非線性梁理論的一致線性化，推導設定角為 0° 或 90° 之旋轉 Euler 梁的二維運動方程式。當設定角為 90° 時，旋轉傾斜尤拉梁的側向穩態解為零，且因為科氏加速度的值為零，旋轉梁的軸向與側向振動不耦合，所以文獻[16]用與文獻[11]相同的方法，求得軸向穩態變形與自然振動頻率及振態，並探討傾斜角、轉速、轉軸半徑及細長比對等速旋轉傾斜尤拉梁之自然振動頻率的影響；當設定角為 0° 時，旋轉傾斜尤拉梁之側向穩態變形不為零，且因為科氏加速度的值不為零，其軸向與側向振動耦合。文獻[16]假設側向穩態變形為小變形，文獻[16]先求出其軸向穩態解析解，再用級數解法求得旋轉梁之穩態解。文獻[16]將穩態解代入運動方程式，用一致線性化求得振動的統御方程式，但文獻[16]並未求其自然振動頻率及振態。

文獻[16]僅保留到變形的二次項，所以側向穩態變形太大時可能不準確，且文獻[16]以旋轉梁變形前所受的離心力求其軸向及側向穩態變形，但旋轉梁所受的離心力為與結構變形位置相關的外力(configuration dependent load)，當旋轉梁的側向穩態變形不是很小時，必須考慮幾何非線性，才能得到可靠的側向穩態變形。當旋轉傾斜梁的細長比很大，在高轉速時，其側向位移的穩態解可能很大，其軸向和側向穩態變形會互相耦合，為高度

的幾何非線性問題，若將旋轉梁分成數段(元素)，用共旋轉法描述旋轉梁的變形，則可除去梁元素的剛體旋轉，故可以僅保留到變形的二次項，但仍維持解的精度，不過須以迭代的方式求得正確的穩態變形。如果採用文獻[16]的級數解，求穩態變形的過程將會很複雜，文獻[12]利用共旋轉有限元素法成功的求出旋轉梁之軸向及扭轉耦合的穩態變形，故本研究擬採用共旋轉有限元素法求軸向和側向耦合的穩態變形。本研究的主要目的為以共旋轉有限元素法探討設定角為 0° 之旋轉傾斜梁的穩態變形及自然振動頻率。本文在第二章中先以梁變形前的形心軸之長度為獨立變數推導梁元素的變形，本研究利用文獻[16]中旋轉梁的變形機制，由梁元素在當前元素座標之位置向量及轉速求得梁元素的加速度，再以虛功原理、有限元素法及非線性梁理論的一致線性化，在梁元素當前之元素座標上推導節點慣性力和節點變形力，將元素的節點力轉到總體座標後組合成系統的非線性運動方程式。本研究將旋轉梁的運動方程式的時間函數項去掉求得系統穩態平衡方程式，再用基於牛頓法的增量迭代法求出軸向位移及側向位移的穩態解，將運動方程式在穩態平衡位置用泰勒級數展開，取到一次項，求得旋轉傾斜梁的振動方程式。假設自然振動頻率存在，可獲得一組代數齊次方程式，該組齊次方程式係數形成之矩陣的行列式為零時，即可求得旋轉梁以穩態解為平衡點的自然振動的頻率及其對應的振態。本研究擬探討傾斜角、旋轉速度、轉軸半徑及細長比等對旋轉 Euler 梁之穩態變形、自然頻率及振態的影響。

第二章 理論推導

2.1 問題描述

如圖二所示，本文考慮一長度為 L_T 具均勻斷面且雙軸對稱之尤拉梁，其支承端以設定角(setting angle) β 與傾斜角(inclination angle) α 剛接在一半徑為 R 剛性圓柱上，該圓柱以等角速率 Ω 繞其軸心旋轉。本文中所有梁的位移、變形和振動指的是在一個以等角速率 Ω 繞圓柱中心軸旋轉的旋轉座標上描述的位移、變形和振動。本文中僅考慮梁的軸向位移，單一個側向位移及旋轉。以等角速率旋轉的傾斜梁存在著一個含軸向及側向的穩態變形。本文中所有的振動都是指以該穩態變形為平衡點的振動。本文中考慮的振動是線性振動，所以由振動造成的位移、速度和加速度都視為是一微小量(infinitesimal quantity)。



2.2 基本假設

本文對梁元素的推導，做如下的假設：

- (1) Euler-Bernoulli 假說成立。
- (2) 梁元素的形心軸之單位長度伸長量(unit extension)為均勻的伸長。
- (3) 梁元素的變形與應變皆為小變形與小應變。

2.3 座標系統描述

本研究是使用共旋轉有限元素法(co-rotational finite element formulation)，將梁分割成若干個兩個節點的梁元素，節點1及節點2為元素的兩個端點。為了描述旋轉梁系統的運動，本文中使用了以下二個座標系統：

- (1) 總體座標系統(Global coordinates) X_1 、 X_2 、 X_3

總體座標系統是以等角速率 Ω 繞圓柱中心軸旋轉，如圖三與圖四所示，總體座標系統的原點是取在旋轉梁斷面的形心軸與旋轉圓柱的交點(即 O 點)上，其 X_1 軸和梁變形前的斷面形心軸一致，其 X_2 和 X_3 軸是取旋轉梁變形前的斷面主軸方向，將圓柱的轉軸方向繞 X_1 軸逆時鐘方向轉 β 角即為和 X_3 軸的方向。本文中假設梁只有在 X_1 、 X_2 方向的變形，本文中旋轉梁的節點座標、節點位移、節點速度、節點加速度及整個系統的運動方程式均在此座標系統中定義。

(2)元素座標系統(Element coordinates) x_1 、 x_2 、 x_3

元素座標系統是建立在每個元素當前的位置上，且以一個等角速率 Ω 繞圓柱中心軸旋轉，如圖五所示，元素座標系統的原點是定義在節點1(即 o 點)上，令 o 點當前的總體座標為 $(X_o, Y_o, 0)$ ， x_1 軸的方向為梁元素兩節點連線的方向，令 x_1 軸與總體座標的 X_1 軸間的夾角為 θ_e ，因本文中假設只有 $x_1 x_2$ 平面的變形，所以 x_3 軸與總體座標 X_3 軸的方向一致， x_2 軸的方向由右手定則決定，本文中梁元素的位移、變形、速度、加速度及運動方程式，均在此座標系統定義。

元素座標系統與總體座標系統關係可表示成

$$\mathbf{X} = \mathbf{A}_{GE} \mathbf{x} \quad (2.3.1)$$

$$\mathbf{A}_{GE} = \begin{bmatrix} \cos \theta_e & -\sin \theta_e & 0 \\ \sin \theta_e & \cos \theta_e & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.3.2)$$

其中 $\mathbf{x} = \{x_1, x_2, x_3\}$ ， $\mathbf{X} = \{X_1, X_2, X_3\}$ 。本文中 $\{\}$ 代表行矩陣。

令 $\boldsymbol{\Omega}_X$ 為旋轉梁的角速度向量在總體座標上的表示式，其分量可表示如下：

$$\boldsymbol{\Omega}_X = \Omega \{0, \sin \beta, \cos \beta\} \quad (2.3.3)$$

由(2.3.1)、(2.3.3)式可得，令 $\boldsymbol{\Omega}$ 為旋轉梁的角速度向量在元素座標上的表示

式如下：

$$\boldsymbol{\Omega} = \{\Omega_1, \Omega_2, \Omega_3\} = \mathbf{A}_{GE}^t \boldsymbol{\Omega}_X = \boldsymbol{\Omega} \mathbf{n} \quad (2.3.4)$$

$$\mathbf{n} = \{n_1, n_2, n_3\} = \{\sin \beta \sin \theta_e, \sin \beta \cos \theta_e, \cos \beta\} \quad (2.3.5)$$

其中 \mathbf{n} 為旋轉軸的單位向量， $n_i (i=1,2,3)$ 為其在元素座標軸 x_i 的分量， β 為梁的設定角， θ_e 為梁元素變形後元素座標之 x_1 軸與總體座標之 X_1 軸的夾角。

因除了 $\beta=0^\circ$ 及 90° 外，傾斜旋轉梁的穩態解是三維的，故本文假設僅適用於 $\beta=0^\circ$ 及 90° ，為了推導上的方便，本文在推導過程中仍視為 β 變數。當 $\beta=90^\circ$ 時，穩態解僅有軸向位移，[16]用級數解詳盡的探討了其自然振動頻率。當 $\beta=0^\circ$ 時，穩態解有側向和軸向位移，故其所受離心力為變形位置的函數，文獻上仍沒有其側向位移的非線性穩態解及其自然振動頻率，故本研究主要探討如圖六所示設定角 $\beta=0^\circ$ 之旋轉傾斜梁的軸向、側向位移的穩態解及自然振動頻率，當 $\beta=0^\circ$ 時(2.3.4)式的角速度向量 $\boldsymbol{\Omega}$ 退化成

$$\boldsymbol{\Omega} = \{\Omega_1, \Omega_2, \Omega_3\} = \boldsymbol{\Omega} \mathbf{n} \quad (2.3.6)$$

$$\mathbf{n} = \{n_1, n_2, n_3\} = \{0, 0, 1\} \quad (2.3.7)$$

2.4 Euler 梁的變形描述

本文是在旋轉元素座標上描述梁元素的變形。由2.2節中的基本假設可知，梁元素的變形可由其形心軸在元素座標上的位移及其斷面的旋轉決定。本文採用梁變形前形心軸的長度為獨立變數。

2.4.1 梁元素之位移

圖七中的 P 點為梁中的任意點， Q 點為 P 點在形心軸上的對應點，即 P 點與 Q 點位於梁的同一斷面上。在元素座標上， Q 點在梁變形前後的位置向量

可分別表示為 $\{x, 0, 0\}$ 與 $\{x_p(x,t), v(x,t), 0\}$ 。其中 t 為時間， $x_p(x,t)$ 及 $v(x,t)$ 分別是 Q 點在 x_1 與 x_2 軸方向的座標。

P 點在梁變形前後的位置向量可分別表示如下

$$\mathbf{r}_0 = \{x, y, z\} \quad (2.4.1)$$

$$\mathbf{r} = \{r_1, r_2, r_3\} = \{x_p - y \sin \theta, y \cos \theta + v, z\} \quad (2.4.2)$$

其中 x 、 y 與 z 分別為梁變形前 P 點在 x_i ($i = 1, 2, 3$) 軸的座標， θ 為 x_1 軸和形心軸的切線向量的夾角。

(2.4.2)式之 $\sin \theta$ 及 $\cos \theta$ 可表示成如下

$$\sin \theta = \frac{\partial v}{\partial s} = \frac{\partial x}{\partial s} \frac{\partial v}{\partial x} = \frac{1}{1 + \varepsilon_0} \frac{\partial v}{\partial x} = \frac{v_{,x}}{1 + \varepsilon_0} \quad (2.4.3)$$

$$\cos \theta = \frac{\partial x_p}{\partial s} = \frac{\partial x_p}{\partial x} \frac{\partial x}{\partial s} = (1 - \sin^2 \theta)^{1/2} \quad (2.4.4)$$

$$\varepsilon_0 = \frac{\partial s - \partial x}{\partial x} = \frac{\partial s}{\partial x} - 1 \quad (2.4.5)$$

其中 s 為圖七中 o 點到 Q 點間形心軸在變形後的弧長， ε_0 為形心軸的單位伸長量。

由(2.4.3)-(2.4.5)式可得

$$x_p(x,t) = u_1 + \int_0^x [(1 + \varepsilon_0)^2 - v_{,x}^2]^{1/2} dx \quad (2.4.6)$$

其中 u_1 為節點1在 x_1 方向上的位移，由元素座標系統的定義，其值為零，但其變分及對時間的微分並不為零。

由小變形的假設，利用近似式 $[(1 + \varepsilon_0)^2 - v_{,x}^2]^{1/2} \approx (1 + \varepsilon_0 - \frac{1}{2} v_{,x}^2)$ ，(2.4.6)式

可表示成

$$x_p(x,t) = u_1 + \int_0^x (1 + \varepsilon_0 - \frac{1}{2}v_{,x}^2) dx \quad (2.4.7)$$

由(2.4.7)式及梁元素的形心軸之單位伸長量為均勻的假設，可以得到形心軸的單位伸長量 ε_0

$$\varepsilon_0 = \frac{l-L}{L} + \frac{1}{2L} \int_0^L v_{,x}^2 dx \quad (2.4.8)$$

$$l = L + u_2 - u_1 \quad (2.4.9)$$

其中 L 為梁元素變形前的長度， l 為梁元素之形心軸變形後的弦長， u_2 為節點2在 x_1 方向的位移。

本文中假設梁元素變形後的形心軸的側向位移 $v(x,t)$ 為 x 的三次 Hermitian 多項式。因此 $v(x,t)$ 可表示成

$$v(x,t) = \{N_1, N_2, N_3, N_4\}^t \{v_1, v_1', v_2, v_2'\} = \mathbf{N}_b^t \mathbf{u}_b \quad (2.4.10)$$

$$N_1 = \frac{1}{4}(1-\xi)^2(2+\xi), \quad N_2 = \frac{L}{8}(1-\xi^2)(1-\xi), \quad (2.4.11)$$

$$N_3 = \frac{1}{4}(1+\xi)^2(2-\xi), \quad N_4 = \frac{L}{8}(-1+\xi^2)(1+\xi)$$

$$\xi = -1 + \frac{2x}{L} \quad (2.4.12)$$

$$\mathbf{u}_b = \mathbf{u}_b(t) = \{v_1, v_1', v_2, v_2'\} \quad (2.4.13)$$

其中 $v_j (j=1,2)$ 是 v 在節點 j 的節點值， v_j' 則是 $v' = \frac{\partial v}{\partial x}$ 在節點 $j (j=1,2)$ 之節點值， $N_i (i=1-4)$ 代表形狀函數(shape function)。

在小變形的假設下，由(2.4.2)式及近似式 $\sin \theta \approx \theta$ ， $\cos \theta \approx 1 - \frac{1}{2}\theta^2$ ，可以將

位置向量 \mathbf{r} 重新寫成如下

$$\mathbf{r} = \{r_1, r_2, r_3\} = \{x_p - y\theta, y(1 - \frac{1}{2}\theta^2) + v, z\} \quad (2.4.14)$$

將(2.4.10)式代入(2.4.8)式整理可得

$$\varepsilon_0 = \frac{1}{L}(\mathbf{G}_a^t \mathbf{u}_a + \frac{1}{2} \int_0^L v_{,x}^2 dx) = \frac{1}{L}(\mathbf{G}_a^t \mathbf{u}_a + \frac{1}{2} \mathbf{G}_b^t \mathbf{u}_b) \quad (2.4.15)$$

其中

$$\mathbf{u}_a = \{u_1, u_2\} \quad (2.4.16)$$

$$\mathbf{G}_a = \{-1, 1\} \quad (2.4.17)$$

$$\mathbf{G}_b = \{G_{b1}, G_{b2}, G_{b3}, G_{b4}\} = \int \mathbf{N}'_b v_{,x} dx \quad (2.4.18)$$

將(2.4.15)式代入(2.4.7)式整理可得

$$x_p(x, t) = \mathbf{N}_a^t \mathbf{u}_a + x + \frac{x}{2L} \mathbf{G}_b^t \mathbf{u}_b - \frac{1}{2} \int_0^x v_{,x}^2 dx \quad (2.4.19)$$

$$\mathbf{N}_a = \left\{ \frac{1-\xi}{2}, \frac{1+\xi}{2} \right\} \quad (2.4.20)$$

2.4.2 梁之位置向量的變分

將(2.4.14)式的位置向量 \mathbf{r} 變分可表示成

$$\delta \mathbf{r} = \{\delta r_1, \delta r_2, 0\} = \{-y\delta\theta + \delta x_p, -y\theta\delta\theta + \delta v, 0\} \quad (2.4.21)$$

在小變形的假設下，利用近似式 $\sin \theta \approx \theta$ 、 $\frac{1}{(1 + \varepsilon_0)} \approx (1 - \varepsilon_0)$ ，將(2.4.3)式及

變分可得

$$\delta\theta = -\delta\varepsilon_0 v_{,x} + (1 - \varepsilon_0)\delta v_{,x} \quad (2.4.22)$$

由(2.4.15)式的變分可得

$$\delta\varepsilon_0 = \frac{1}{L}(\delta\mathbf{u}_a^t \mathbf{G}_a + \delta\mathbf{u}_b^t \mathbf{G}_b) \quad (2.4.23)$$

將(2.4.10)式對 x 的一次微分可表示成

$$v_{,x} = \mathbf{N}'_b^t \mathbf{u}_b \quad (2.4.24)$$

由(2.4.24)式的變分可得

$$\delta v_{,x} = \delta\mathbf{u}_b^t \mathbf{N}'_b \quad (2.4.25)$$

由(2.4.19)式的變分可得

$$\delta x_p = \delta\mathbf{u}_a^t \mathbf{N}_a + \frac{x}{L} \delta\mathbf{u}_b^t \mathbf{G}_b - \int_0^x v_{,x} \delta v_{,x} dx \quad (2.4.26)$$

將(2.4.23)式代入(2.4.22)式整理可得

$$\delta v_{,x} \approx (1 + \varepsilon_0)\delta\theta + v_{,x}\delta\varepsilon_0 \approx (1 + \varepsilon_0)\delta\theta + \frac{v_{,x}}{L}(\delta\mathbf{u}_a^t \mathbf{G}_a + \delta\mathbf{u}_b^t \mathbf{G}_b) \quad (2.4.27)$$

將(2.4.22)及(2.4.26)式代入(2.4.21)式，位置向量的變分 $\delta\mathbf{r}$ 可以寫成

$$\delta r_1 = \underline{y v_{,x} \delta\varepsilon_0} - y(1 - \varepsilon_0)\delta v_{,x} + \delta\mathbf{u}_a^t \mathbf{N}_a + \underline{\frac{x}{L} \delta\mathbf{u}_b^t \mathbf{G}_b} - \underline{\int_0^x v_{,x} \delta v_{,x} dx} \quad (2.4.28)$$

$$\delta r_2 = \underline{y(1 - \varepsilon_0)v_{,x}^2 \delta\varepsilon_0} - \underline{y(1 - \varepsilon_0)^2 v_{,x} \delta v_{,x}} + \delta v$$

因為梁元素為小變形，所以 $v_{,x}$ 在元素較多時都將趨近於零，故在計算慣性力時，上式中畫底線的項可以忽略。

2.5 梁的應變及其變分、速度、加速度

2.5.1 梁的應變及其變分

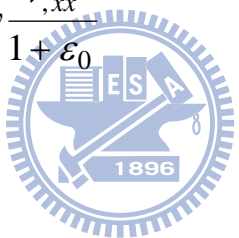
本文中的應變採用工程應變。為了推導上的方便，本文中先推導出 Green strain ε_{ij} ，再由 Green strain 求得與其對應之工程應變。Euler 梁的 Green strain 非為零的應變只有 ε_{11} ，可表示成

$$\varepsilon_{11} = \frac{1}{2}(\mathbf{g}_1^t \mathbf{g}_1 - 1) \quad (2.5.1)$$

其中

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}}{\partial x} \quad (2.5.2)$$

將(2.4.7)、(2.4.14)式代入(2.5.2)式，可得 \mathbf{g}_1 的分量 g_{11} 和 g_{12} 如下

$$\begin{aligned} g_{11} &= 1 + \varepsilon_0 - \frac{1}{2}v_{,x}^2 - y \frac{v_{,xx}}{1 + \varepsilon_0} \\ g_{12} &= v_{,x} - y \frac{v_{,x}v_{,xx}}{(1 + \varepsilon_0)^2} \end{aligned} \quad (2.5.3)$$


由小變形的假設，將(2.5.3)式及近似式 $\frac{1}{(1 + \varepsilon_0)} \approx (1 - \varepsilon_0)$ ，代入(2.5.1)式，且

保留變形參數及其微分到二次項，可得：

$$\varepsilon_{11} = \varepsilon_0 + \frac{1}{2}\varepsilon_0^2 - yv_{,xx} + \frac{1}{2}y^2v_{,xx}^2 \quad (2.5.4)$$

Green strain ε_{11} 與對應之工程應變 e_{11} 的關係可表示成[17]：

$$e_{11} = (1 + 2\varepsilon_{11})^{1/2} - 1 \quad (2.5.5)$$

將(2.5.4)式代入(2.5.5)式，且保留變形參數及其微分到二次項可得：

$$e_{11} = \varepsilon_0 - (1 - \varepsilon_0)yv_{,xx} \quad (2.5.6)$$

由(2.5.6)式的變分可表示成

$$\delta e_{11} = \delta \varepsilon_0 + y v_{,xx} \delta \varepsilon_0 - (1 - \varepsilon_0) y \delta v_{,xx} \quad (2.5.7)$$

將(2.4.10)式對 x 的二次微分可表示成

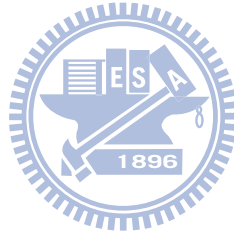
$$v_{,xx} = \mathbf{N}_b''^t \mathbf{u}_b$$

由上式的變分可得

$$\delta v_{,xx} = \delta \mathbf{u}_b^t \mathbf{N}_b'' \quad (2.5.8)$$

將(2.4.23)、(2.5.8)式，代入(2.5.7)式，可得

$$\delta e_{11} = \frac{(1 + y v_{,xx})}{L} \delta \mathbf{u}_a^t \mathbf{G}_a + \frac{(1 + y v_{,xx})}{L} \delta \mathbf{u}_b^t \mathbf{G}_b - (1 - \varepsilon_0) y \delta \mathbf{u}_b^t \mathbf{N}_b''$$



(2.5.9)

2.5.2 梁的速度及加速度

因梁的位置向量是在旋轉元素座標上描述，所以 P 點的絕對速度在當前元素座標的分量可表示成

$$\mathbf{v} = \{v_1, v_2, v_3\} = \mathbf{v}_o + \boldsymbol{\Omega} \times \mathbf{r} + \dot{\mathbf{r}} \quad (2.5.10)$$

$$\mathbf{v}_o = \boldsymbol{\Omega} \{v_{o1}, v_{o2}, v_{o3}\} = \boldsymbol{\Omega} \times \mathbf{r}_o \quad (2.5.11)$$

$$\mathbf{r}_o = \{r_{o1}, r_{o2}, r_{o3}\} = \mathbf{A}_{GE}^t \mathbf{r}_{oG} \quad (2.5.12)$$

$$\mathbf{r}_{oG} = \{R \cos \alpha + X_o, -R \sin \alpha \cos \beta + Y_o, R \sin \alpha \sin \beta\} \quad (2.5.13)$$

其中 \mathbf{v}_o 為 o 點的絕對速度， $\dot{\mathbf{r}}$ 為 P 點對元素座標原點 o 的速度， \mathbf{r}_{oG} 為元素座標原點 o 在總體座標的位置向量的表示式， \mathbf{r}_o 為原點 o 在當前元素座標

的表示式。

將(2.3.2)、(2.3.4)式代入(2.5.11)、(2.5.12)式可得：

$$v_{o1} = n_2 r_{o3} - n_3 r_{o2} \quad (2.5.14)$$

$$v_{o2} = n_3 r_{o1} - n_1 r_{o3}$$

$$v_{o3} = n_1 r_{o2} - n_2 r_{o1}$$

$$r_{o1} = \cos \theta_e (R \cos \alpha + X_o) - \sin \theta_e (R \sin \alpha \cos \beta - Y_o) \quad (2.5.15)$$

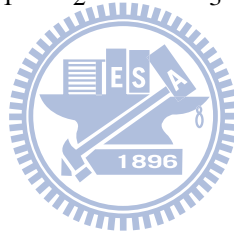
$$r_{o2} = -\sin \theta_e (R \cos \alpha + X_o) - \cos \theta_e (R \sin \alpha - Y_o)$$

$$r_{o3} = R \sin \alpha \sin \beta$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.5.14)、(2.5.15)

式會退化成

$$v_{o1} = -r_{o2} \quad , \quad v_{o2} = r_{o1} \quad , \quad v_{o3} = 0 \quad (2.5.16)$$



$$r_{o1} = \cos \theta_e (R \cos \alpha + X_o) - \sin \theta_e (R \sin \alpha - Y_o) \quad (2.5.17)$$

$$r_{o2} = -\sin \theta_e (R \cos \alpha + X_o) - \cos \theta_e (R \sin \alpha - Y_o)$$

$$r_{o3} = 0$$

將(2.4.14)式對時間 t 微分， P 點對元素座標原點 o 的速度可表示成

$$\dot{\mathbf{r}} = \{\dot{r}_1, \dot{r}_2, \dot{r}_3\} = \{\dot{x}_p - y\dot{\theta}, \dot{y} - y\dot{\theta}, 0\} \quad (2.5.18)$$

由小變形的假設下，利用近似式 $\sin \theta \approx \theta$ 、 $\frac{1}{(1 + \varepsilon_0)} \approx (1 - \varepsilon_0)$ ，將(2.4.3)、

(2.4.15)、(2.4.19)式對時間 t 微分可得

$$\dot{x}_p = \mathbf{N}_a^t \dot{\mathbf{u}}_a + \frac{x}{L} \mathbf{G}_b^t \dot{\mathbf{u}}_b - \int_0^x v_{,x} \dot{v}_{,x} dx \quad (2.5.19)$$

$$\dot{\theta} = (1 - \varepsilon_0)\dot{v}_{,x} - \underline{\dot{\varepsilon}_0}v_{,x} \quad (2.5.20)$$

$$\dot{\varepsilon}_0 = \frac{1}{L}(\mathbf{G}_a^t \dot{\mathbf{u}}_a + \int_0^L v_{,x} \dot{v}_{,x} dx) = \frac{1}{L}(\mathbf{G}_a^t \dot{\mathbf{u}}_a + \underline{\mathbf{G}_b^t \dot{\mathbf{u}}_b}) \quad (2.5.21)$$

因為梁元素為小變形，所以 $v_{,x}$ 在元素較多時都將趨近於零，故在計算慣性力時，畫底線的項可以忽略。

由(2.3.4)、(2.4.2)式，及近似式 $\sin \theta \approx \theta$ 、 $\cos \theta \approx 1$ ，(2.5.10)式之 $\boldsymbol{\Omega} \times \mathbf{r}$ 可表示成

$$\boldsymbol{\Omega} \times \mathbf{r} = \Omega \{v_{p1}, v_{p2}, v_{p3}\} \quad (2.5.22)$$

$$v_{p1} = n_2 z - n_3 (y + v)$$

$$v_{p2} = n_3 (x_p - y\theta) - n_1 z$$

$$v_{p3} = n_1 (y + v) - n_2 (x_p - y\theta)$$

將(2.5.14)、(2.5.15)、(2.4.18)及(2.5.22)式代入(2.5.10)式， P 點的絕對速度分量可表示成

$$v_1 = \Omega v_{o1} + \Omega [n_1 z - n_3 (y\theta + v)] + \dot{x}_p - y\dot{\theta} \quad (2.5.23)$$

$$v_2 = \Omega_{o2} + \Omega [n_3 (x_p - y\theta) - n_1 z] + \dot{v} - y\dot{\theta}\theta$$

$$v_3 = \Omega v_{o3} + [n_1 (y + v) - n_2 (x_p - y\theta)]$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.5.23)式會退化成

$$v_1 = \Omega v_{o1} - \Omega (y\theta + v) + \dot{x}_p - y\dot{\theta} \quad (2.5.24)$$

$$v_2 = \Omega_{o2} + \Omega (x_p - y\theta) + \dot{v} - y\dot{\theta}\theta$$

$$v_3 = \Omega v_{o3}$$

因梁的位置向量是在旋轉元素座標上描述，所以 P 點的絕對加速度在當前元

素座標的分量可表示成

$$\mathbf{a} = \{a_1, a_2, a_3\} = \mathbf{a}_o + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} + \ddot{\mathbf{r}} \quad (2.5.25)$$

$$\mathbf{a}_o = \Omega^2 \{a_{o1}, a_{o2}, a_{o3}\} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_o) \quad (2.5.26)$$

其中 \mathbf{a}_o 為 o 點的絕對加速度， $\ddot{\mathbf{r}}$ 為 P 點對元素座標原點 o 的加速度，因為本研究僅考慮等角速度的旋轉，所以 $\dot{\boldsymbol{\Omega}}$ 為零。

由(2.3.4)、(2.5.12)式代入(2.5.26)式之 $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_o)$ 可表示成

$$a_{o1} = n_1 n_2 r_{o2} - (n_2^2 + n_3^2) r_{o1} + n_1 n_3 r_{o3} \quad (2.5.27)$$

$$a_{o2} = n_2 n_3 r_{o3} - (n_1^2 + n_3^2) r_{o2} + n_1 n_2 r_{o1}$$

$$a_{o3} = n_1 n_3 r_{o1} - (n_1^2 + n_2^2) r_{o3} + n_2 n_3 r_{o2}$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.5.27)式會退化成

$$a_{o1} = -r_{o1}, \quad a_{o2} = -r_{o2}, \quad a_{o3} = 0 \quad (2.5.28)$$

將(2.3.4)、(2.3.5)及(2.4.2)式代入(2.5.25)式之 $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ 可得

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \Omega^2 \{a_{p1}, a_{p2}, a_{p3}\} \quad (2.5.29)$$

$$a_{p1} = n_1 n_2 r_2 - (n_2^2 + n_3^2) r_1 + n_1 n_3 r_3$$

$$a_{p2} = n_2 n_3 r_3 - (n_1^2 + n_3^2) r_2 + n_1 n_2 r_1$$

$$a_{p3} = n_1 n_3 r_1 - (n_1^2 + n_2^2) r_3 + n_2 n_3 r_2$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.5.29)式會退化成

$$a_{p1} = -(x_p - y \sin \theta), \quad a_{p2} = -(v + y \cos \theta), \quad a_{p3} = 0 \quad (2.5.30)$$

將(2.3.4)、(2.5.18)式代入(2.5.25)式之 $\boldsymbol{\Omega} \times \dot{\mathbf{r}}$ 可表示成

$$\boldsymbol{\Omega} \times \dot{\mathbf{r}} = \Omega \{n_2 \dot{r}_3 - n_3 \dot{r}_2, n_3 \dot{r}_1 - n_1 \dot{r}_3, n_1 \dot{r}_2 - n_2 \dot{r}_1\}$$

$$= \Omega \{-n_3(\dot{v} - y\dot{\theta}), n_3(\dot{x}_p - y\dot{\theta}), n_1(\dot{v} - y\dot{\theta}) - n_2(\dot{x}_p - y\dot{\theta})\} \quad (2.5.31)$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.5.31)式會退化成

$$\mathbf{\Omega} \times \dot{\mathbf{r}} = \Omega \{-(\dot{v} - y\dot{\theta}), \dot{x}_p - y\dot{\theta}, 0\} \quad (2.5.32)$$

將(2.5.18)式對時間 t 微分， P 點對元素座標原點 o 的加速度可表示成

$$\ddot{\mathbf{r}} = \{\ddot{r}_1, \ddot{r}_2, \ddot{r}_3\} = \{\ddot{x}_p - y\ddot{\theta}, \dot{v} - y\dot{\theta}^2 - y\ddot{\theta}, 0\} \quad (2.5.33)$$

將(2.5.19)、(2.5.20)、(2.5.21)式對時間 t 微分可得

$$\begin{aligned} \ddot{x}_p &= \mathbf{N}_a^t \ddot{\mathbf{u}}_a + \frac{x}{L} \int_0^L (\underline{v_{,x} \ddot{v}_{,x}} + \dot{v}_{,x}^2) dx - \int_0^x (\underline{v_{,x} \ddot{v}_{,x}} + \dot{v}_{,x}^2) dx \\ &= \mathbf{N}_a^t \ddot{\mathbf{u}}_a + \frac{x}{L} (\underline{\mathbf{G}_b^t \ddot{\mathbf{u}}_b} + \dot{\mathbf{G}}_b^t \dot{\mathbf{u}}_b) - \int_0^x (\underline{v_{,x} \ddot{v}_{,x}} + \dot{v}_{,x}^2) dx \end{aligned} \quad (2.5.34)$$

$$\ddot{\theta} = (1 - \varepsilon_0) \ddot{v}_{,x} - 2\varepsilon_0 \dot{v}_{,x} - \ddot{\varepsilon}_0 v_{,x} \quad (2.5.35)$$

因本文僅保留擾動量到一次項，且不考慮變形與速度、加速度的耦合項，所以畫底線的項可以忽略。

將(2.5.27)、(2.5.29)、(2.5.31)、(2.5.33)、(2.5.34)及(2.5.35)式代入(2.5.25)式，

忽略畫底線的項，且保留擾動量到一次項， P 點的絕對加速度分量可表示成

$$\begin{aligned} a_1 &= \Omega^2 a_{o1} + \Omega^2 [n_1 n_2 (y + v) - (n_2^2 + n_3^2) (\mathbf{N}_a^t \mathbf{u}_a + x) + n_1 n_3 z] - 2\Omega n_3 \dot{v} + \mathbf{N}_a^t \ddot{\mathbf{u}}_a \\ &\quad - y \ddot{v}_{,x} \end{aligned} \quad (2.5.36)$$

$$\begin{aligned} a_2 &= \Omega^2 a_{o2} + \Omega^2 [n_2 n_3 z - (n_1^2 + n_3^2) (y + v) + n_1 n_2 (\mathbf{N}_a^t \mathbf{u}_a + x)] \\ &\quad + 2\Omega n_3 (\mathbf{N}_a^t \dot{\mathbf{u}}_a - y \dot{v}_{,x}) + \ddot{v} \end{aligned}$$

$$a_3 = \Omega^2 a_{o3} + \Omega^2 [n_1 n_3 (\mathbf{N}_a^t \mathbf{u}_a + x) - (n_1^2 + n_2^2) z + n_2 n_3 (y + v)]$$

$$+ 2\Omega n_3 [n_1 \dot{v} - n_2 (\mathbf{N}_a^t \dot{\mathbf{u}}_a - y \dot{v}_{,x})]$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.5.36)式會退化成

$$a_1 = -\Omega^2 r_{o1} - \Omega^2 (\mathbf{N}_a^t \mathbf{u}_a + x) - 2\Omega \dot{v} + \mathbf{N}_a^t \ddot{\mathbf{u}}_a - y \ddot{v}_{,x} \quad (2.5.37)$$

$$a_2 = -\Omega^2 r_{o2} - \Omega^2 (y + v) + 2\Omega (\mathbf{N}_a^t \dot{\mathbf{u}}_a - y \dot{v}_{,x}) + \ddot{v}$$

$$a_3 = 0$$

2.6 元素節點內力之推導

本文利用虛功原理及 d'Alembert 原理在當前的元素座標上推導元素節點內力。若給元素節點 $j (j=1,2)$ 虛位移 δu_j 、 δv_j 及虛旋轉 $\delta \theta_j$ ，則由虛功原理可得

$$\delta W_{ext} = \delta W_{int} \quad (2.6.1)$$

$$\delta W_{ext} = \delta \mathbf{q}_\theta^t \mathbf{f}_\theta \quad (2.6.2)$$

$$\delta W_{int} = \int_V E e_{11} \delta e_{11} dV + \int_V \rho \ddot{\mathbf{a}}^t \delta \mathbf{r} dV = \delta \mathbf{q}^t \mathbf{f} \quad (2.6.3)$$

$$\delta \mathbf{q}_\theta = \{\delta u_1, \delta v_1, \delta \theta_1, \delta u_2, \delta v_2, \delta \theta_2\} \quad (2.6.4)$$

$$\delta \mathbf{q} = \{\delta u_1, \delta v_1, \delta v_1', \delta u_2, \delta v_2, \delta v_2'\} \quad (2.6.5)$$

$$\mathbf{f}_\theta = \mathbf{f}_\theta^D + \mathbf{f}_\theta^I \quad (2.6.6)$$

$$\mathbf{f}_\theta^D = \{f_{11}^\theta, f_{21}^\theta, m_1^\theta, f_{12}^\theta, f_{22}^\theta, m_2^\theta\} \quad (2.6.7)$$

$$\mathbf{f}_\theta^I = \{f_{11}^{I\theta}, f_{21}^{I\theta}, m_1^{I\theta}, f_{12}^{I\theta}, f_{22}^{I\theta}, m_2^{I\theta}\} \quad (2.6.8)$$

$$\mathbf{f} = \mathbf{f}^D + \mathbf{f}^I \quad (2.6.9)$$

$$\mathbf{f}^D = \{f_{11}, f_{21}, m_1, f_{12}, f_{22}, m_2\} \quad (2.6.10)$$



$$\mathbf{f}^I = \{f_{11}^I, f_{21}^I, m_1^I, f_{12}^I, f_{22}^I, m_2^I\} \quad (2.6.11)$$

其中內力所作的虛功包含虛應變 δe_{11} 所作的虛功及慣性力所作的虛功，因虛應變 δe_{11} 為虛位移向量 $\delta \mathbf{q}$ 的函數，所以內力所作的虛功可以表示成 $\delta \mathbf{q}^T \mathbf{f}$ 。本文採用的元素節點內力為對應於虛位移向量 $\delta \mathbf{q}_\theta$ 的 \mathbf{f}_θ ，(2.6.6)及(2.6.9)式中 \mathbf{f}^D 、 \mathbf{f}_θ^D 為元素變形節點內力， \mathbf{f}^I 、 \mathbf{f}_θ^I 為元素慣性節點內力，由(2.4.27)式可知 $\delta \theta_j$ 和 $\delta v'_j$ 不同，對應於 $\delta \theta_j$ 的廣義力 m_j^θ 為傳統的力矩，對應於 $\delta v'_j$ 的廣義力 m_j 為一廣義的力矩。所以由內力所作的虛功得到的 \mathbf{f} 和 \mathbf{f}_θ 不同，兩者的關係將在以下推導。

由 (2.4.17)、(2.4.18)及(2.4.27)式可得

$$\delta \mathbf{q} = \mathbf{T}_\theta \delta \mathbf{q}_\theta \quad (2.6.12)$$

$$\mathbf{T}_\theta = \mathbf{T}_\theta^1 + \mathbf{T}_\theta^2 \quad (2.6.13)$$

$$\mathbf{T}_\theta^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-v'_1}{L} & 0 & 1 + \varepsilon_0 & \frac{v'_1}{L} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-v'_2}{L} & 0 & 0 & \frac{v'_2}{L} & 0 & 1 + \varepsilon_0 \end{bmatrix} \quad (2.6.14)$$

$$\mathbf{T}_\theta^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v'_1 G_{b1}}{L} & \frac{v'_1 G_{b2}}{L} & 0 & \frac{v'_1 G_{b3}}{L} & \frac{v'_1 G_{b4}}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v'_2 G_{b1}}{L} & \frac{v'_2 G_{b2}}{L} & 0 & \frac{v'_2 G_{b3}}{L} & \frac{v'_2 G_{b4}}{L} \end{bmatrix} \quad (2.6.15)$$

其中 \mathbf{T}_θ^1 為變形的零次和一次項， \mathbf{T}_θ^2 為變形的二次項

將(2.6.2)、(2.6.3)及(2.6.12)式代入(2.6.1)式可得

$$\mathbf{f}_\theta = \mathbf{T}_\theta^t \mathbf{f} \quad (2.6.16)$$

因本文中的穩態變形含大位移及旋轉，為考慮軸向與側向變形的耦合，故(2.6.16)式中保留變形到二次項及部分三次項，但本文僅考慮微小的振動，所以(2.6.16)式中僅保留到速度及加速度的一次項。本文中假設慣性力中速度及加速度與變形耦合項可以忽略，所以由(2.6.8)、(2.6.11)式可得以下之近似式

$$\mathbf{f}_\theta^I \approx \mathbf{f}^I \quad (2.6.17)$$

為了推導上的方便，本文中將 $\delta \mathbf{q}$ 分成 $\delta \mathbf{u}_a$ 及 $\delta \mathbf{u}_b$ ， \mathbf{u}_a 及 \mathbf{u}_b 在(2.4.13)式及

(2.4.16)式已定義，與 $\delta \mathbf{q}$ 對應的 \mathbf{f} 分成 \mathbf{f}_a 及 \mathbf{f}_b ，並定義如下

$$\mathbf{f}_a = \mathbf{f}_a^D + \mathbf{f}_a^I \quad (2.6.18)$$

$$\mathbf{f}_b = \mathbf{f}_b^D + \mathbf{f}_b^I \quad (2.6.19)$$

$$\mathbf{f}_a^D = \{f_{11}, f_{12}\}$$

$$\mathbf{f}_a^I = \{f_{11}^I, f_{12}^I\}$$

$$\mathbf{f}_b^D = \{f_{21}, m_1, f_{22}, m_2\}$$

$$\mathbf{f}_b^I = \{f_{21}^I, m_1^I, f_{22}^I, m_2^I\}$$

將 $\delta \mathbf{q}_\theta$ 分成 $\delta \mathbf{u}_a^\theta = \{\delta u_1 \quad \delta u_2\}$ 及 $\delta \mathbf{u}_b^\theta = \{\delta v_1 \quad \delta \theta_1 \quad \delta v_2 \quad \delta \theta_2\}$ ，與 $\delta \mathbf{q}_\theta$ 對應的 \mathbf{f}^θ

分成 \mathbf{f}_a^θ 及 \mathbf{f}_b^θ ，並定義如下

$$\mathbf{f}_a^\theta = \mathbf{f}_a^{D\theta} + \mathbf{f}_a^{I\theta} \quad (2.6.20)$$

$$\begin{aligned}\mathbf{f}_a^{D\theta} &= \{f_{11}^\theta, f_{12}^\theta\} \\ \mathbf{f}_a^{I\theta} &= \{f_{11}^{I\theta}, f_{12}^{I\theta}\} \\ \mathbf{f}_b^\theta &= \mathbf{f}_b^{D\theta} + \mathbf{f}_b^{I\theta} \\ \mathbf{f}_b^{D\theta} &= \{f_{21}^\theta, m_1^\theta, f_{22}^\theta, m_2^\theta\} \\ \mathbf{f}_b^{I\theta} &= \{f_{21}^{I\theta}, m_1^{I\theta}, f_{22}^{I\theta}, m_2^{I\theta}\}\end{aligned}\tag{2.6.21}$$

由(2.6.14)、(2.6.16)、(2.6.18)-(2.6.21)式可得

$$\mathbf{f}_a^\theta = \mathbf{f}_a + \mathbf{T}_{ba}^t \mathbf{f}_b \tag{2.6.22}$$

$$\mathbf{T}_{ba} = \begin{bmatrix} 0 & 0 \\ \frac{-v'_1}{L} & \frac{v'_1}{L} \\ 0 & 0 \\ \frac{-v'_2}{L} & \frac{v'_2}{L} \end{bmatrix} \tag{2.6.23}$$



$$\mathbf{f}_b^\theta = \mathbf{T}_b^t \mathbf{f}_b \tag{2.6.24}$$

$$\mathbf{T}_b = \mathbf{T}_b^1 + \mathbf{T}_b^2 \tag{2.6.25}$$

$$\mathbf{T}_b^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + \varepsilon_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 + \varepsilon_0 \end{bmatrix} \tag{2.6.26}$$

$$\mathbf{T}_b^2 = [\mathbf{0}, v'_1 \mathbf{G}_b, \mathbf{0}, v'_2 \mathbf{G}_b]^t \tag{2.6.27}$$

將(2.4.28)、(2.5.9)、(2.5.36)式代入(2.6.3)式可得

$$\mathbf{f}_a^D = EA\varepsilon_0 \mathbf{G}_a - \frac{EI(1-\varepsilon_0)}{L} \int v_{,xx}^2 dx \mathbf{G}_a \tag{2.6.28}$$

$$\mathbf{f}_b^D = EA\varepsilon_0 \int \mathbf{N}'_b v_{,x} dx + EI(1-\varepsilon_0)^2 \int \mathbf{N}''_b v_{,xx} dx - EI(1-\varepsilon_0)/L \int v_{,xx}^2 dx \mathbf{G}_b \tag{2.6.29}$$

$$\begin{aligned} \mathbf{f}_a^I &= \rho A \int \mathbf{N}_a \mathbf{N}_a^t \ddot{\mathbf{u}}_a dx + \Omega^2 a_{o1} \rho A \int \mathbf{N}_a dx + \Omega^2 n_1 n_2 \rho A \int \mathbf{N}_a v dx \\ &\quad - \Omega^2 \rho A (n_2^2 + n_3^2) \int \mathbf{N}_a (\mathbf{N}_a^t \mathbf{u}_a + x) dx - 2\Omega n_3 \rho A \int \mathbf{N}_a \dot{v} dx \end{aligned} \quad (2.6.30)$$

$$\begin{aligned} \mathbf{f}_b^I &= \Omega^2 \rho A \int \mathbf{N}_b a_{o2} dx - \Omega^2 (n_1^2 + n_3^2) \rho A \int \mathbf{N}_b v dx + \Omega^2 n_1 n_2 \rho A \int \mathbf{N}_b (\mathbf{N}_a^t \mathbf{u}_a + x) dx \\ &\quad + \rho A \int \mathbf{N}_b \ddot{v} dx + 2\Omega n_3 \rho A \int \mathbf{N}_b \mathbf{N}_a^t dx \dot{\mathbf{u}}_a + \rho I \int \mathbf{N}_b' \ddot{v}_{,x} dx - \Omega^2 n_1 n_2 \rho I \int \mathbf{N}_b' dx \end{aligned} \quad (2.6.31)$$

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.6.30)、(2.6.31)式會退化

$$\begin{aligned} \mathbf{f}_a^I &= \rho A \int \mathbf{N}_a \mathbf{N}_a^t \ddot{\mathbf{u}}_a dx + \Omega^2 \rho A (a_{o1} \int \mathbf{N}_a dx - \int \mathbf{N}_a (\mathbf{N}_a^t \mathbf{u}_a + x) dx) \\ &\quad - 2\Omega \rho A \int \mathbf{N}_a \dot{v} dx \end{aligned} \quad (2.6.32)$$

$$\begin{aligned} \mathbf{f}_b^I &= \Omega^2 \rho A \int \mathbf{N}_b a_{o2} dx - \Omega^2 \rho A \int \mathbf{N}_b v dx + \rho A \int \mathbf{N}_b \ddot{v} dx \\ &\quad + 2\Omega \rho A \int \mathbf{N}_b \mathbf{N}_a^t dx \dot{\mathbf{u}}_a + \rho I \int \mathbf{N}_b' \ddot{v}_{,x} dx \end{aligned} \quad (2.6.33)$$

由(2.4.10)、(2.6.22)、(2.6.23)、(2.6.28)及(2.6.29)式，保留部份的三次項，可得 $\mathbf{f}_a^{D\theta}$

$$f_{12}^\theta = EA\varepsilon_0 - \frac{EI}{L} \varepsilon_0 \int v_{,xx}^2 dx \quad (2.6.34)$$

$$\mathbf{f}_a^{D\theta} = EA\varepsilon_0 \mathbf{G}_a - \frac{EI\varepsilon_0}{L} \int v_{,xx}^2 dx \mathbf{G}_a \quad (2.6.35)$$

(2.6.34)式在附錄 A 中有詳細推導。

由(2.6.24) - (2.6.27)及(2.6.29)式，保留部份的三次項，可得 $\mathbf{f}_b^{D\theta}$

$$\mathbf{f}_b^{D\theta} = \mathbf{T}_b^{lt} EI (1 - \varepsilon_0)^2 \int \mathbf{N}_b'' v_{,xx} dx + \mathbf{f}_{12}^\theta \int \mathbf{N}_b' v_{,x} dx \quad (2.6.36)$$

(2.6.36)式在附錄 A 中有詳細推導。

由(2.6.17)式可得

$$\begin{aligned}\mathbf{f}_a^{I\theta} &\approx \mathbf{f}_a^I \\ \mathbf{f}_b^{I\theta} &\approx \mathbf{f}_b^I\end{aligned}\quad (2.6.37)$$

2.7 元素剛度矩陣及慣性矩陣之推導

因為在數值計算時，對於系統運動方程式的平衡迭代過程中需要先求得梁元素切線剛度矩陣和質量矩陣。依元素切線剛度矩陣的定義可知，對應於 $\delta\mathbf{q}^\theta$ 元素剛度矩陣 \mathbf{k}^θ 可以表示成

$$\mathbf{k}^\theta = \frac{\partial \mathbf{f}^{D\theta}}{\partial \mathbf{q}_\theta} = \mathbf{T}_\theta^t \frac{\partial \mathbf{f}^D}{\partial \mathbf{q}} \mathbf{T}_\theta \quad (2.7.1)$$

$$\frac{\partial \mathbf{f}^D}{\partial \mathbf{q}} = \mathbf{k} \quad (2.7.2)$$

其中 \mathbf{k} 為對應於 $\delta\mathbf{q}$ 元素剛度矩陣可以由下列之子矩陣 \mathbf{k}_{ij} ($i = a, b, j = a, b$)組合而成

$$\mathbf{k}_{aa} = \frac{\partial \mathbf{f}_a^D}{\partial \mathbf{u}_a} = \frac{EA}{L} \mathbf{G}_a \mathbf{G}_a^t + \frac{EI}{L^2} \int v_{,xx}^2 dx \mathbf{G}_a \mathbf{G}_a^t \quad (2.7.3)$$

$$\mathbf{k}_{ab} = \mathbf{k}_{ba}^t = \frac{\partial \mathbf{f}_a^D}{\partial \mathbf{u}_b} = \frac{EA}{L} \mathbf{G}_a \mathbf{G}_b^t + \frac{EI}{L^2} \int v_{,xx}^2 dx \mathbf{G}_a \mathbf{G}_b^t - 2 \frac{EI}{L} (1 - \varepsilon_0) \mathbf{G}_a \int v_{,xx} N_b^{''t} dx \quad (2.7.4)$$

$$\begin{aligned}\mathbf{k}_{bb} = \frac{\partial \mathbf{f}_b^D}{\partial \mathbf{u}_b} &= \frac{EA}{L} \mathbf{G}_b \mathbf{G}_b^t + EA\varepsilon_0 \int \mathbf{N}_b' \mathbf{N}_b^{''t} dx - \frac{2EI(1 - \varepsilon_0)}{L} \int \mathbf{N}_b'' v_{,xx} dx \mathbf{G}_b^t \\ &\quad + EI(1 - \varepsilon_0)^2 \int \mathbf{N}_b'' \mathbf{N}_b^{''t} dx - \frac{2EI(1 - \varepsilon_0)}{L} \int N_b^{''t} v_{,xx} dx \mathbf{G}_b\end{aligned} \quad (2.7.5)$$

因為 $v_{,x}$ 在元素較多時都將趨近於零，所以畫底線項可以忽略。

由(2.6.35)、(2.6.36)及(2.7.1)式，可得對應於 $\delta\mathbf{q}^\theta$ 元素剛度矩陣 \mathbf{k}^θ 的顯式， \mathbf{k}^θ

可以由下列之子矩陣 \mathbf{k}_{ij}^θ ($i = a, b, j = a, b$) 組合而成

$$\mathbf{k}_{aa}^\theta = \frac{\partial \mathbf{f}_a^{D\theta}}{\partial \mathbf{u}_a^\theta} = \left(\frac{EA}{L} - \frac{EI}{L^2} \int v_{,xx}^2 dx \right) \mathbf{G}_a \mathbf{G}_a^t \quad (2.7.6)$$

$$\mathbf{k}_{ab}^\theta = \mathbf{k}_{ba}^\theta = \frac{\partial \mathbf{f}_a^{D\theta}}{\partial \mathbf{u}_b^\theta} = \frac{\partial \mathbf{f}_a^{D\theta}}{\partial \mathbf{u}_b} \frac{\partial \mathbf{u}_b}{\partial \mathbf{u}_b^\theta} = -\frac{2EI\varepsilon_0}{L} \mathbf{G}_a \int v_{,xx} \mathbf{N}_b^{''t} dx \mathbf{T}_{\theta b}^1 \quad (2.7.7)$$

$$\mathbf{k}_{bb}^\theta = \frac{\partial \mathbf{f}_b^{D\theta}}{\partial \mathbf{u}_b^\theta} = \mathbf{T}_{\theta b}^{1t} (EI(1-\varepsilon_0)^2 \int \mathbf{N}_b'' \mathbf{N}_b^{''t} dx) \mathbf{T}_{\theta b}^1 + f_{12}^\theta \int \mathbf{N}_b' \mathbf{N}_b^{''t} dx \quad (2.7.8)$$

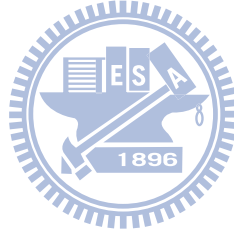
由(2.6.32)、(2.6.33)式可以知道元素慣性力 \mathbf{f}_θ^I 與元素節點位移、速度及加速度有關，對應於 $\delta \mathbf{q}_\theta$ 、 $\delta \dot{\mathbf{q}}_\theta$ 及 $\delta \ddot{\mathbf{q}}_\theta$ 之慣性力改變量 $\delta \mathbf{f}_\theta^I$ 可以表示成

$$\delta \mathbf{f}_\theta^I = \Omega^2 \mathbf{k}_\Omega \delta \mathbf{q}_\theta + \Omega \mathbf{c} \delta \dot{\mathbf{q}}_\theta + \mathbf{m} \delta \ddot{\mathbf{q}}_\theta \quad (2.7.9)$$

$$\mathbf{k}_\Omega = \frac{\partial \mathbf{f}^{\theta I}}{\Omega^2 \partial \mathbf{q}_\theta} \quad (2.7.10)$$

$$\mathbf{c} = \frac{\partial \mathbf{f}^{\theta I}}{\Omega \partial \dot{\mathbf{q}}_\theta} \quad (2.7.11)$$

$$\mathbf{m} = \frac{\partial \mathbf{f}^{\theta I}}{\partial \ddot{\mathbf{q}}_\theta} \quad (2.7.12)$$



其中 \mathbf{k}_Ω 為元素向心力剛度矩陣(centripetal stiffness matrix)， \mathbf{m} 為元素質量矩陣(mass matrix)， \mathbf{c} 為元素陀螺矩陣(gyroscopic matrix)。

本文中假設慣性力中速度及加速度與變形耦合項可以忽略，所以由(2.7.10)至(2.7.12)式可得以下之近似式

$$\mathbf{k}_\Omega = \frac{\partial \mathbf{f}^I}{\Omega^2 \partial \mathbf{q}} \quad (2.7.13)$$

$$\mathbf{c} = \frac{\partial \mathbf{f}^I}{\Omega \partial \dot{\mathbf{q}}} \quad (2.7.14)$$

$$\mathbf{m} = \frac{\partial \mathbf{f}^I}{\partial \ddot{\mathbf{q}}} \quad (2.7.15)$$

\mathbf{k}_Ω 、 \mathbf{c} 、 \mathbf{m} 的顯式可以由下列之子矩陣 $\mathbf{k}_{\Omega ij}$ 、 \mathbf{c}_{ij} 、 \mathbf{m}_{ij} ($i = a, b, j = a, b$)組合而成

$$\mathbf{k}_{\Omega aa} = \frac{\partial \mathbf{f}_a^I}{\Omega^2 \partial \mathbf{u}_a} = -\rho A (n_2^2 + n_3^2) \int \mathbf{N}_a \mathbf{N}_a^t dx \quad (2.7.16)$$

$$\mathbf{k}_{\Omega ab} = \mathbf{k}_{\Omega ba}^t = \frac{\partial \mathbf{f}_a^I}{\Omega^2 \partial \mathbf{u}_b} = n_1 n_2 \rho A \int \mathbf{N}_a \mathbf{N}_b^t dx$$

$$\mathbf{k}_{\Omega bb} = \frac{\partial \mathbf{f}_b^I}{\Omega^2 \partial \mathbf{u}_b} = -(n_1^2 + n_3^2) \rho A \int \mathbf{N}_b \mathbf{N}_b^t dx$$

$$\mathbf{c}_{aa} = \frac{\partial \mathbf{f}_a^I}{\partial \dot{\mathbf{u}}_a} = \mathbf{0} \quad (2.7.17)$$

$$\mathbf{c}_{ab} = -\mathbf{c}_{ba}^t = \frac{\partial \mathbf{f}_a^I}{\Omega \partial \dot{\mathbf{u}}_b} = -2n_3 \rho A \int \mathbf{N}_a \mathbf{N}_b^t dx$$

$$\mathbf{c}_{bb} = \frac{\partial \mathbf{f}_b^I}{\partial \dot{\mathbf{u}}_b} = \mathbf{0}$$

$$\mathbf{m}_{aa} = \frac{\partial \mathbf{f}_a^I}{\partial \ddot{\mathbf{u}}_a} = \rho A \int \mathbf{N}_a \mathbf{N}_a^t dx \quad (2.7.18)$$

$$\mathbf{m}_{ab} = \mathbf{m}_{ba}^t = \frac{\partial \mathbf{f}_a^I}{\partial \ddot{\mathbf{u}}_b} = \mathbf{0}$$

$$\mathbf{m}_{bb} = \frac{\partial \mathbf{f}_b^I}{\partial \ddot{\mathbf{u}}_b} = \rho A \int \mathbf{N}_b \mathbf{N}_b^t dx + \rho I \int \mathbf{N}_b' \mathbf{N}_b'^t dx$$

其中 \mathbf{f}_a^I 及 \mathbf{f}_b^I 已在(2.6.30)及(2.6.31)式中定義。

由(2.3.6)式可知，當 $\beta = 0^\circ$ 時， $n_1 = n_2 = 0$ ， $n_3 = 1$ ，故(2.7.16) - (2.7.18)式會退化

$$\mathbf{m}_{\Omega aa} = \frac{\partial \mathbf{f}_a^I}{\Omega^2 \partial \mathbf{u}_a} = -\rho A \int \mathbf{N}_a \mathbf{N}_a^t dx \quad (2.7.19)$$

$$\mathbf{m}_{\Omega ab} = \mathbf{m}_{\Omega ba}^t = \frac{\partial \mathbf{f}_a^I}{\partial \mathbf{u}_b} = \mathbf{0}$$

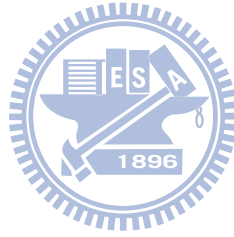
$$\mathbf{m}_{\Omega bb} = \frac{\partial \mathbf{f}_b^I}{\Omega^2 \partial \mathbf{u}_b} = -\rho A \int \mathbf{N}_b \mathbf{N}_b^t dx$$

$$\mathbf{c}_{aa} = \frac{\partial \mathbf{f}_a^I}{\partial \dot{\mathbf{u}}_a} = \mathbf{0} \quad (2.7.20)$$

$$\mathbf{c}_{ab} = -\mathbf{c}_{ba}^t = \frac{\partial \mathbf{f}_a^I}{\partial \dot{\mathbf{u}}_b} = -2\Omega \rho A \int \mathbf{N}_a \mathbf{N}_b^t dx$$

$$\mathbf{c}_{bb} = \frac{\partial \mathbf{f}_b^I}{\partial \dot{\mathbf{u}}_b} = \mathbf{0}$$

$$\mathbf{m}_{aa} = \frac{\partial \mathbf{f}_a^I}{\partial \ddot{\mathbf{u}}_a} = \rho A \int \mathbf{N}_a \mathbf{N}_a^t dx \quad (2.7.21)$$



$$\mathbf{m}_{ab} = \mathbf{m}_{ba}^t = \frac{\partial \mathbf{f}_a^I}{\partial \ddot{\mathbf{u}}_b} = \mathbf{0}$$

$$\mathbf{m}_{bb} = \frac{\partial \mathbf{f}_b^I}{\partial \ddot{\mathbf{u}}_b} = \rho A \int \mathbf{N}_b \mathbf{N}_b^t dx + \rho I \int \mathbf{N}'_b \mathbf{N}'_b{}^t dx$$

由(2.7.17)式可知(2.7.14)式之陀螺矩陣 \mathbf{c} 有以下的性質

$$\mathbf{c}^t = -\mathbf{c} \quad (2.7.22)$$

2.8 系統的運動方程式

旋轉梁系統的運動方程式可表示成

$$\boldsymbol{\varphi} = \mathbf{F}^D(\bar{\mathbf{Q}}) + \mathbf{F}^I(\Omega^2, \bar{\mathbf{Q}}, \dot{\bar{\mathbf{Q}}}, \ddot{\bar{\mathbf{Q}}}) = \mathbf{0} \quad (2.8.1)$$

其中 $\boldsymbol{\varphi}$ 為系統的不平衡力、 \mathbf{F}^D 和 \mathbf{F}^I 為系統的節點變形力及慣性力， Ω 為旋轉梁的轉速， $\bar{\mathbf{Q}}$ 、 $\dot{\bar{\mathbf{Q}}} = \frac{\partial \bar{\mathbf{Q}}}{\partial t}$ 和 $\ddot{\bar{\mathbf{Q}}} = \frac{d^2 \bar{\mathbf{Q}}}{dt^2}$ 為在時間為 t 時系統的節點位移、速度和加速度。 \mathbf{F}^D 和 \mathbf{F}^I 可以由(2.6.28)至(2.6.31)式之元素節點變形力及慣性力從當前的元素座標轉換到總體座標後組合而成。

令 \mathbf{Q}_s 表示旋轉梁在轉速為 Ω 時的穩態節點位移，因 $\dot{\mathbf{Q}}_s = \ddot{\mathbf{Q}}_s = \mathbf{0}$ ，所以由(2.8.1)式，旋轉梁系統的穩態平衡方程式可表示成

$$\boldsymbol{\varphi} = \mathbf{F}_s^D(\mathbf{Q}_s) + \Omega^2 \mathbf{F}_{sref}^I(\mathbf{Q}_s) = \mathbf{0} \quad (2.8.2)$$

其中 $\mathbf{F}_s^D(\mathbf{Q}_s)$ 為對應於穩態變形 \mathbf{Q}_s 的系統節點變形力，可以由(2.6.28)及(2.6.29)式之元素節點變形力從元素座標轉換到總體座標後組合而成， $\Omega^2 \mathbf{F}_{sref}^I(\mathbf{Q}_s)$ 為對應於穩態變形 \mathbf{Q}_s 的系統節點慣性力，可以由(2.6.30)及(2.6.31)式之元素節點慣性力中將速度及加速度項去掉，再從元素座標轉換到總體座標後組合而成。(2.8.2)式為一非線性方程式，本文以基於牛頓法的增量迭代法求出在不同轉速 Ω 下的穩態解，詳細的數值方法及程序將在第三章中說明。

令 $\mathbf{Q}_d = \mathbf{Q}_d(t)$ 表示以 \mathbf{Q}_s 為平衡點的微小振動，將 $\bar{\mathbf{Q}} = \mathbf{Q}_d + \mathbf{Q}_s$ 代入(2.8.2)式，用泰勒級數在 $\bar{\mathbf{Q}} = \mathbf{Q}_s$ 展開，取到一次項，則旋轉梁微小振動的運動方程式可表示成

$$\mathbf{M}\ddot{\mathbf{Q}}_d + \Omega \mathbf{C}\dot{\mathbf{Q}}_d + (\mathbf{K} + \Omega^2 \mathbf{K}_\Omega)\mathbf{Q}_d = \mathbf{0} \quad (2.8.3)$$

其中 \mathbf{M} 為系統的質量矩陣， \mathbf{C} 為系統的陀螺矩陣 (gyroscopic matrix)， \mathbf{K} 和 \mathbf{K}_Ω 為系統的剛度矩陣及向心力剛度矩陣 (centripetal stiffness matrix)。 \mathbf{M} 、 \mathbf{C} 、 \mathbf{K} 及 \mathbf{K}_Ω 可以由(2.7.1)式及(2.7.13)至(2.7.15)式中元素之 \mathbf{m} 、 \mathbf{c} 、 \mathbf{k} 及 \mathbf{k}_Ω 從元素座標轉換到總體座標後組合而成。

若振動方程式(2.8.3)式存在自然振動頻率，則其解的形式可以表示如下

$$\mathbf{Q}_d = (\mathbf{Q}_R + i\mathbf{Q}_I)e^{i\lambda t} \quad (2.8.4)$$

其中 i 為虛數 $\sqrt{-1}$ ， λ 為自然頻率， \mathbf{Q}_R 和 \mathbf{Q}_I 為對應於 λ 之振態的實部和虛部。

將(2.8.4)式代入(2.8.3)式，並由實部和虛部分別為零，可得一個 λ 的二次特徵值問題

$$\mathbf{H}\Theta = \mathbf{0} \quad (2.8.5)$$

$$\mathbf{H} = \mathbf{H}(\lambda, \Omega) = \begin{bmatrix} \mathbf{K} + \Omega^2\mathbf{K}_\Omega - \lambda^2\mathbf{M} & \lambda\Omega\mathbf{C}^t \\ \lambda\Omega\mathbf{C} & \mathbf{K} + \Omega^2\mathbf{K}_\Omega - \lambda^2\mathbf{M} \end{bmatrix} \quad (2.8.6)$$

$$\Theta = \begin{Bmatrix} \mathbf{Q}_R \\ \mathbf{Q}_I \end{Bmatrix} \quad (2.8.7)$$

其中 \mathbf{H} 為一對稱矩陣。

因(2.8.5)式為一齊次方程式(homogeneous equation)，所以僅有當 \mathbf{H} 的行列式值 $\det|\mathbf{H}|=0$ ，才有非零解，在固定轉速 $\Omega \neq 0$ 時，本文採用二分法解出滿足 $\det|\mathbf{H}|=0$ 的 λ ，當轉速 $\Omega = 0$ 時，(2.8.5)式將退化成一廣義特徵值問題，本文採用子空間法(subspace method)求其特徵值 λ 及特徵向量 Θ ，詳細的數值方法及程序將在第三章中說明。



第三章 數值方法及程序

本章將數值計算的方法及程序分作穩態解及振動頻率分析兩個部份，本章中僅考慮設定角 $\beta = 0^\circ$ 之矩形斷面旋轉梁，令 L_T 為旋轉梁的長度， b 、 t 分別為矩形斷面寬與高，則斷面的二次矩可表示成 $I = \frac{bt^3}{12}$ 。因旋轉梁一般在彈性範圍內使用，大部分的工程材料的降伏應變 $\varepsilon_y \ll 1$ ，所以本文將限制旋轉梁的最大轉速，使其最大應變儘量不超過 0.01。若不考慮側向變形對慣性力的影響，旋轉傾斜梁之穩態解造成的最大膜應變(membrane strain)及撓曲應變(flexural strain)都發生在旋轉梁的根部，並可表示成(詳細推導在附錄 B)

$$\varepsilon_{0\max} = k^2 \left(r \cos \alpha + \frac{1}{2} \right) \quad (3.1)$$

$$\varepsilon_{b\max} = \frac{\sqrt{3}k^2 \eta r \sin \alpha}{2} \quad (3.2)$$

$$k = \Omega L_T \sqrt{\rho / E} \quad (3.3)$$

$$\eta = L_T \sqrt{A / I} \quad (3.4)$$

$$r = R / L_T \quad (3.5)$$

其中 k 為無因次轉速， ρ 為密度， E 為楊氏係數， r 為無因次轉軸半徑， η 為細長比， A 為斷面面積。

旋轉傾斜梁之穩態解造成的最大應變為(3.1)和(3.2)式的和，所以其最大應變與 k^2 成正比，由(3.1)、(3.2)式可以估算旋轉梁運轉時的最大允許轉速，但因(3.1)、(3.2)式為線性解，故分析時仍需要檢查非線性解的最大應變。為了方便分析與討論，本章及下一章中將採用無因次變數 k 、 r 、 η 及無因次自然頻率 K

$$K = \lambda L_T \sqrt{\rho / E} \quad (3.6)$$

3.1 穩態解

為了簡潔，在不造成混淆的情況下，本節中將(2.8.2)式之下標 s 及上標 D 省略，再令 $\mathbf{P}(\mathbf{Q}) = \frac{\Omega^2}{k^2} \mathbf{F}_{sref}^I(\mathbf{Q})$ ，則旋轉梁系統的穩態平衡方程式可改寫成

$$\boldsymbol{\phi} = \mathbf{F}(\mathbf{Q}) + k^2 \mathbf{P}(\mathbf{Q}) = \mathbf{0} \quad (3.7)$$

其中 \mathbf{P} 稱為參考負荷， k 稱為負荷參數。

本文以基於牛頓法的增量迭代法解非線性代數方程式(3.1)式，求得在不同無因次轉速 k 下，旋轉梁的節點位移向量 \mathbf{Q} 。

3.1.1 增量迭代數值計算方法

本文中將選定之最大負荷參數，即最大無因次轉速 k_{\max} 等分成數個增量負荷參數 Δk ，若第 I 個增量的平衡位置為已知，即其位移向量為 \mathbf{Q}_I 、負荷參數為 k_I 為已知，則對應於第 $I+1$ 個增量之負荷參數 $k = k_I + \Delta k$ 的初始增量位移向量 $\Delta \mathbf{Q}$ ，可利用尤拉預測值(Euler predictor)[19]求得

$$\Delta \mathbf{Q} = -(2k_I + \Delta k) \Delta k \mathbf{K}_T^{-1} \mathbf{P} \quad (3.8)$$

$$\mathbf{K}_T = \left. \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{Q}} \right|_{\mathbf{Q}=\mathbf{Q}_I}, \quad \mathbf{P} = \mathbf{P}(\mathbf{Q}_I) \quad (3.9)$$

其中 $\mathbf{K}_T = \mathbf{K} + \Omega^2 \mathbf{K}_\Omega$ 為第 I 個增量的平衡位置之系統切線剛度矩陣， \mathbf{K} 和 \mathbf{K}_Ω 為系統的剛度矩陣及向心力剛度矩陣。

由 $\mathbf{Q} = \mathbf{Q}_I + \Delta \mathbf{Q}$ 可求得每個元素當前的元素座標及節點變形位移，將其代入(2.6.30)、(2.6.31)、(2.6.35)及(2.6.36)式，可算出元素的節點變形力及慣性力，將元素的節點力轉換到總體座標，可組合得到(3.7)式之不平衡力 $\boldsymbol{\phi}$ ，再依牛頓法，可得位移修正量如下：

$$\delta \mathbf{Q} = -\mathbf{K}_T^{-1} \boldsymbol{\phi} \quad (3.10)$$

其中 $\boldsymbol{\phi}$ 為不平衡力， \mathbf{K}_T 為當前的系統切線剛度矩陣。

將求得的位移修正量 $\delta\mathbf{Q}$ ，加入上次迭代之 \mathbf{Q} 中，可得新的節點位移向量，再進行下一次迭代，此過程一直重複至(3.7)式中的不平衡力滿足斂準則為止。本文以不平衡力 $\boldsymbol{\phi}$ 的 weighted Euclidean norm 做為平衡迭代時的誤差度量，所使用的收斂準則為

$$e = \frac{\|\boldsymbol{\phi}\|}{k^2 \sqrt{N} \|\mathbf{P}\|} = e_{tol} \quad (3.11)$$

其中 $\|\cdot\|$ 為 \cdot 的歐幾里德範數(Euclidean norm)， N 為方程式的數目， e_{tol} 為一設定的容許誤差，本文中取 $e_{tol} = 10^{-5}$ 。

3.1.2 數值程序

本文所使用的增量迭代法之數值之數值程序可以分成三個部分：

1. 輸入與計算開始分析所需要的資料

- (a) 輸入結構資料及給定外力負荷參數的最大值。
- (b) 給定增量數、最大迭代數及收斂時的容許誤差。
- (c) 計算增量負荷參數、負荷參數、
- (d) 用(3.9)式計算系統切線剛度 $\mathbf{K}_T = \mathbf{K} + \Omega^2 \mathbf{K}_\Omega$ 、參考負荷 \mathbf{P} 。

2. 使用迭代法求在已知負荷參數的收斂解

- (a) 利用(3.8)式求初始增量位移向量 $\Delta\mathbf{Q}$ 。
- (b) 將前一個平衡位置的節點位移向量 \mathbf{Q}_I 加上 $\Delta\mathbf{Q}$ 得到 \mathbf{Q} 。
- (c) 由 \mathbf{Q} 中萃取元素之節點位移，計算出當前的元素座標及元素的節點力，再計算(3.7)式之不平衡力 $\boldsymbol{\phi}$ 。
- (d) 檢查(3.11)式的收斂準則，若滿足則進行(e)；若不滿足，檢查迭代次數，若迭代次數小於給定之最大迭代次數，則利用(3.10)式求得位移修正向量 $\delta\mathbf{Q}$ ，將當前的節點位移向量 \mathbf{Q} 加上 $\delta\mathbf{Q}$ 得到一個新的 \mathbf{Q} ，再回 2(c) 進行迭代；若迭代次數大於最大迭代次數則停止迭代並印出迭代相關資料。
- (e) 檢查增量次數是否大於最大增量次數，若滿足，則完成增量迭代步驟；若

不滿足，則進行步驟3。

3. 計算下一次增量所需要的資料

- (a) 計算(3.9)式中的切線剛度及參考負荷。
- (b) 計算下一次增量的負荷參數。
- (c) 回到2 執行迭代工作。

3.2 振動分析

本節將說明求旋轉梁自然頻率及振動模態的計算程序。本文先用本文採用子空間法(subspace method)[20]，求出當無因次轉速 $k=0$ 時的無因次自然頻率 K 及特徵向量 \mathbf{Q} ，將由3.1節之增量迭代法求得對應於無因次轉速 k 的穩態解代入(2.8.6)式中，再以二分法(bisection method)[21]解(2.8.5)式。

(2.8.5)式之 $\mathbf{H}(K)$ 可分解成 $\mathbf{H}(K) = \mathbf{L}\mathbf{D}\mathbf{L}^t$ ，其中 \mathbf{L} 為下三角矩陣， \mathbf{D} 為對角線矩陣，其對角線元素的值皆為1。令

$$D(K) = \det|\mathbf{H}(K)| = 0 \quad (3.12)$$

其中 $D(K)$ 為 $\mathbf{H}(K)$ 的行列式值，其值為 \mathbf{D} 矩陣之對角線元素的乘積。

若 K_B 滿足 $D(K_B) = 0$ ，則 K_B 為旋轉梁之一無因次自然頻率。由於 \mathbf{H} 的維數隨著元素數目的增加而變大，為了避免其行列式的數值過大，所以本文中將 $D(K)$ 做以下的標準化(normalization)處理：

$$\bar{D}(K) = \frac{D(K)}{D(K_0)} \quad (3.13)$$

其中 K_0 為一參考值。

若 $K_L < K_B < K_R$ ，其中 K_B 為一無因次自然振動頻率，則 $N_L < N_R$ ，其中 N_L 、 N_R 分別為 $\mathbf{D}(K_L)$ 、 $\mathbf{D}(K_R)$ 中負的對角線元素的個數。如已知 K_R 及 K_L ，則可由二分法求得 K_B 。

本文解(3.12)式所採用的計算程序如下：

設定需要的自然頻率的數目，先用子空間法(subspace method)求出當無

因次轉速 $k=0$ 時的無因次自然頻率 K 及特徵向量 \mathbf{Q} 。若第 I 個增量的穩態平衡位置已求出，即其位移向量為 \mathbf{Q}_I 、負荷參數 k_I 為已知，先以前一個無因次轉速 k_{I-1} 得到之無因次自然頻率為參考值，設定無因次自然頻率的起始值 K_0 及增量 ΔK ，計算 $\mathbf{D}(K_0)$ 中負的對角線元素的數目 N_0 ，設定容許誤差 e_{tol} 及 e_{tol2} ，本文中取 $e_{tol} = 10^{-7}$ ， $e_{tol2} = 10^{-5}$ 。

(A)

(1) 令 $K_n = K_0 + (n-1)\Delta K$ ($n=1, 2, 3, \dots$)，由 K_n 、 k_I 、 K 及 \mathbf{Q}_I 計算 $\mathbf{D}(K_n)$ ，一直到相鄰兩個 $\mathbf{D}(K_n)$ 中，負的對角線元素的數目不一樣。

(2) 令 $K_L = K_{n-1}$ ， $K_R = K_n$ ，其中下標 L 及 R 分別代表根的左邊極限及右邊極限。

(3) 以二分法取 $K_B = \frac{K_L + K_R}{2}$ ，並重新計算 $\mathbf{D}(K_B)$ ，及其負的對角線元素的數目 N_B 。

(4) 若 $N_B = N_0$ ，取 $K_L = K_B$ ；若 $N_B > N_0$ ，取 $K_R = K_B$ 。

(5) 若 $\frac{K_R - K_L}{K_B} < e_{tol}$ ，且 $\bar{D}(K_B) < e_{tol2}$ ，則取 K_B 為(3.12)式之解，繼續進行

步驟(B)求振態，否則回到步驟(3)。

(B)

(1) 將 $\mathbf{H}(K_B)$ 分解成 $\mathbf{H}(K_B) = \mathbf{L}\mathbf{D}\mathbf{L}^t$ 。

(2) 找出對角線矩陣 \mathbf{D} 中元素之絕對值有最小值的行，令該行為第 k 行。

(3) 令振態之第 k 個分量的值為 1，再將(2.8.5)式改寫成

$$\begin{bmatrix} H_{1,1} & \cdots & H_{1,k-1} & 0 & H_{1,k+1} & \cdots & H_{1,2N} \\ \vdots & & & \vdots & & & \\ H_{k-1,1} & & & 0 & & & H_{k-1,2N} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ H_{k+1,1} & & & 0 & & & H_{k+1,2N} \\ \vdots & & & \vdots & & & \vdots \\ H_{2N,1} & & & 0 & & & H_{2N,2N} \end{bmatrix}_{2N \times 2N} \begin{Bmatrix} \Theta_1 \\ \vdots \\ \Theta_{k-1} \\ \Theta_k \\ \Theta_{k+1} \\ \vdots \\ \Theta_{2N} \end{Bmatrix}_{2N \times 1} = \begin{Bmatrix} -H_{1,k} \\ \vdots \\ -H_{k-1,k} \\ 1 \\ -H_{k+1,k} \\ \vdots \\ -H_{N,k} \end{Bmatrix}_{2N \times 1} \quad (3.14)$$

(4) 利用高斯消去法求解(3.14)式得到振態 Θ 。

(5) 將 Θ 除以其分量中絕對值最大的分量 Θ_{\max} ，即

$$\bar{\Theta} = \frac{\Theta}{\Theta_{\max}} \quad (3.15)$$

(6) 以步驟(A)求得之 K_B 及前一個無因次轉速 k_{I-1} 得到之無因次自然頻率為

參值，設定無因次自然頻率的起始值 K_0 及增量 ΔK ，回到步驟(A)繼續求

下一個自然頻率及振動模態。



第四章 數值例題

如同第三章，本章中僅考慮設定角 $\beta = 0^\circ$ 之矩形斷面旋轉傾斜 Euler 梁，本章將分析不同細長比 η ((3.4)式)、無因次轉軸半徑 r ((3.5)式)、傾斜角 α 的旋轉傾斜梁在不同的無因次轉速 k ((3.3)式)下之穩態解、無因次自然頻率 K ((3.6)式)及振態。當無因次轉速 $k = 0$ 時，無因次振動頻率 K 僅與梁的細長比 η 有關且其軸向和側向振態不互相耦合；當 $r \neq 0$ 、 $k \neq 0$ 、 $\alpha = 0$ 時，由(2.5.17)、(2.5.28)、(2.6.32)、(2.6.33)式可知梁未變形時慣性力僅有軸向分量，故旋轉梁的穩態解僅有軸向變形，但由文獻[16]或(2.7.17)式可知因科氏力不為零，故其軸向和側向振態互相耦合；當 $r \neq 0$ 、 $k \neq 0$ 、 $\alpha \neq 0$ 時，由(2.5.17)、(2.5.28)、(2.6.32)、(2.6.33)式可知梁未變形時慣性力有軸向和側向分量，故旋轉梁的穩態解有軸向和側向變形，因穩態解有軸向和側向變形，所以其振態含軸向和側向分量。本章中第 i 個振態指的是對應於無因次轉速 $k = 0$ 時的第 i 個無因次自然頻率 K_i 之振態，軸向振態指的是該振態在無因次轉速 $k = 0$ 時的振動是軸向振動，側向振態指的是該振態在無因次轉速 $k = 0$ 時的振動是側向振動。

本章中將考慮不同的無因次轉軸半徑 r ((3.5)式)、細長比 η ((3.4)式)、傾斜角 α 。為了方便計算，但不失一般性，本章中採用旋轉梁的長度 $L_T = 1m$ 、梁的斷面寬度 $b = 0.1m$ 、密度 $\rho = 7.8 \times 10^3 kg / m^3$ 、彈性模數 $E = 2.1 \times 10^{11} N / m^2$ ，若給定 r 、 η 、 k ，則可決定梁的斷面的厚度 t 、面積 A 、斷面的二次矩 I 、轉軸半徑 R 、旋轉梁的轉速 Ω 。

因本文在推導時假設穩態解的應變 $\varepsilon \ll 1$ ，所以本文中將穩態解的允許最大應變定為 10^{-2} 。本文將限制旋轉梁的最大無因次轉速轉速 k ，使其最大膜應變(membrane strain) ε_{0max} 和最大撓曲應變(flexural strain) ε_{bmax} 的和儘量不超過 10^{-2} 。由附錄 B 可知旋轉梁的最大膜應變和最大撓曲應變為 k 、 r 、 η 、

α 的函數，所以具不同 η 、 α 、 r 的旋轉梁有不同的最大無因次轉速。

4.1 收斂分析

本節在探討使用不同的元素數目 N 時，旋轉傾斜 Euler 梁之自然頻率的收斂情形。在無因次轉軸半徑 $r=1.5$ 時，本節中考慮了傾斜角 $\alpha=5^\circ$ 、 30° 、 75° 、 90° 四種情況。表一至表六為旋轉傾斜 Euler 梁在細長比 $\eta=10$ 、 20 、 50 、 100 、 500 、 1000 時之最大膜應變 ε_0 、最大撓曲應變 ε_b 、無因次端點側向位移 V_{tip}/L_T 及前七個無因次自然頻率 K_i ($i=1\sim 7$)。由表一至六可發現，隨著元素數目的增加，相同細長比及傾斜角所對應的無因次振動頻率 K_i 變化幅度愈小，當元素數目為 50 時，振動頻率變化幅度大約小於 10^{-4} ，已足夠精確，因此在本章的其他例題皆使用 50 個元素。

4.2 準確性分析

為探討本文方法的準確性，本節中分析了文獻[14]的例題並與其結果比較。本節中首先考慮了無因次轉軸半徑 $r=1$ 、細長比 $\eta=70$ 、無因次轉速 $k=5/70$ 的旋轉梁，表七為本文的結果及文獻[14]的結果。由表七可以發現除了傾斜角 $\alpha=0^\circ$ 外，最大膜應變 ε_0 、最大撓曲應變 ε_b 都遠大於本文的允許最大應變 10^{-2} ，所以表七的結果僅供比較用。由表七可以發現當傾斜角 $\alpha=0^\circ$ 時，側向位移的穩態解為零，本文的結果及文獻[14]的結果相當接近，但因文獻[14]沒有考慮梁的轉動慣量項，故仍有些微的不同；當傾斜角 α 逐漸增加時，穩態側向位移逐漸增加，本文及文獻[14]的無因次振動頻率 K_1 之差異逐漸增加，這可能是因文獻[14]沒有考慮梁的側向位移的穩態解造成的。

本節中還分析了無因次轉軸半徑 $r=1$ 、不同細長比 η 、不同無因次轉速 k 之旋轉梁的穩態變形，並與附錄 B 的線性解及文獻[16]的結果比較，由表八可以發現當 $\eta=100$ 時，線性解與本文的非線性解的差異不太大，但當 $\eta=500$ 、 1000 時，線性解比本文的非線性解大很多，這應是離心力引起的

軸向拉力會增加梁的側向剛度，且其影響隨細長比 η 增加而增加。當 $V_{tip}/L_T < 0.2$ 時，文獻[16]的級數解與本文的非線性解的相當接近，但隨著側向位移的穩態解增大兩者的差異逐漸增大，這應是因文獻[16]沒有考慮離心力引起的側向力是側向位移的函數，且文獻[16]在推導其統御方程式時，僅取到變形二次項，所以側向位移太大時，其結果應不準確。由表八可以發現對相同的傾斜角 α ，當 ηk 有相同的值時，本文的 V_{tip}/L_T 之值非常接近，文獻[16]的 V_{tip}/L_T 有相同的值，由文獻[16]可以發現當 $\eta^2 k^2 r \sin\alpha$ 有相同的值時，旋轉傾斜梁在不同的無因次轉速下之無因次穩態側向位移 V/L_T 有相同的無因次統馭方程式，故其 V_{tip}/L_T 有相同的值。

4.3 個案分析

本節中將探討具不同無因次轉軸半徑 r 、細長比 η 及傾斜角 α 之旋轉傾斜 Euler 梁在不同無因次轉速 k 的穩態解、無因次振動頻率 K 及振態。

當無因次轉速 $k=0$ 時，無因次振動頻率 K 僅與細長比 η 有關且其軸向和側向振態不互相耦合，表九為 $k=0$ 時不同細長比 η 之旋轉梁的前七個無因次振動頻率 $K_i (i=1\sim 7)$ ，表九中(a)表示該振動頻率對應的振態為軸向振態，由表九可以發現當 $\eta=38-40$ 時，第一個軸向振態對應的振動頻率和第三個側向振態對應的振動頻率相當接近，且第二個軸向振態對應的振動頻率和第五個側向振態對應的振動頻率相當接近， $\eta=75-80$ 時，第一個軸向振態對應的振動頻率和第四個側向振態對應的振動頻率相當接近。

當 $r=0$ 、 $k \neq 0$ 時，如前所述，因科氏力不為零，故其軸向和側向振態互相耦合，表十為 $r=0$ 、 $k \neq 0$ 時，不同細長比 η 之旋轉梁在不同無因次轉速 k 的前七個無因次振動頻率 $K_i (i=1\sim 7)$ ，表十中(a)表示該振動頻率對應的振態在 $k=0$ 時為軸向振態，當 $r \neq 0$ 、 $k \neq 0$ 、 $\alpha=0$ 時，由文獻[16]或(2.5.17)、(2.5.28)、(2.6.33)式可知旋轉梁的穩態解僅有軸向變形，但由文獻[16]或(2.7.17)式可知因科氏力不為零，故其軸向和側向振態互相耦合。

表十一到表三十七為無因次轉軸半徑 $r=0.5、1、1.5$ ，傾斜角 $\alpha=0^\circ、5^\circ、10^\circ、15^\circ、30^\circ、45^\circ、60^\circ、75^\circ、90^\circ$ ，細長比 $\eta=10、20、50、100、500、1000$ 之旋轉梁在不同無因次轉速 k 的最大膜應變 ε_0 、撓曲應變 ε_b 、無因次端點側向位移 V_{tip}/L_T 及前七個無因次振動頻率 $K_i (i=1\sim7)$ 。由表中可以發現當傾斜角 $\alpha=0^\circ、5^\circ$ ，細長比 $\eta=10、20$ 時，旋轉傾斜梁最大的無因次轉速 k 主要由最大膜應變 ε_0 決定，當傾斜角 $\alpha\neq 0^\circ$ 、細長比 $\eta>20$ 時，旋轉傾斜梁最大的無因次轉速 k 主要由最大撓曲應變 ε_b 決定；本文中考慮的最大無因次轉速隨著傾斜角 α 和細長比 η 的增加而減少，當傾斜角 $\alpha=0^\circ、5^\circ、10^\circ、15^\circ$ 時，本文中最大無因次轉速取 $k=0.06、0.03、0.02、0.01$ ，當傾斜角 $\alpha\geq 30^\circ$ 、細長比 $\eta\leq 100$ 時，本文中最大無因次轉速取 $k=0.01$ ，當傾斜角 $\alpha\geq 30^\circ$ 、細長比 $\eta\geq 100$ 時，本文中最大無因次轉速 k 隨著傾斜角 α 和細長比 η 的增加而減少。由表十一到表三十七中可以發現無因次轉速 k 對無因次振動頻率 $K_i (i=1\sim7)$ 的影響隨著 i 的增加而減少，但隨著細長比 η 及無因次轉軸半徑 r 的增加而增加；側向振動頻率 K_i 隨著無因次轉速 k 增加而增加，且 i 愈小、細長比 η 愈大、無因次轉軸半徑 r 愈大時，其增加率愈明顯。在細長比 $\eta\leq 100$ ，相同無因次轉速 k 下，傾斜角 α 對側向振動頻率 K_i 似乎影響不是很大。

圖八至圖十六為細長比 $\eta=80、100、1000$ 、傾斜角 $\alpha=5^\circ、30^\circ、90^\circ$ 時，旋轉傾斜梁在不同無因次轉速 k 的穩態變形、穩態軸向及側向位移，圖中的穩態變形在 $X、Y$ 方向是依相同的尺度繪製，故為旋轉梁真正的變形圖。由圖八及圖十一可以發現當傾斜角 $\alpha=5^\circ、\eta=80、100$ 時，側向位移很小，因離心力的效應，軸向位移是正的；其餘的情況，因側向位移較大，其效應大於離心力的效應，故軸向位移是負的。

本章中將旋轉梁的振態分成 X_1 和 X_2 的分量， $U_R + iU_I = |U|e^{i\phi_u}$ 及

$V_R + iV_I = |V|e^{i\phi_v}$ ，其中 $|U| = (U_R^2 + U_I^2)^{1/2}$ 、 $|V| = (V_R^2 + V_I^2)^{1/2}$ 為振態的大小，

$\tan \phi_u = \frac{U_I}{U_R}$ 、 $\tan \phi_v = \frac{V_I}{V_R}$ ， ϕ_u 、 ϕ_v 為相角(Phase angle)。本章中的振動模態

圖中取 $-\pi \leq \phi_u \leq \pi$ 、 $-\pi \leq \phi_v \leq \pi$ ， $U = |U| \text{sign}(\sin \phi_u)$ 、 $V = |V| \text{sign}(\sin \phi_v)$ ，

其中 $\text{sign}(x) = 1$ 若 $x \geq 0$ ， $\text{sign}(x) = -1$ 若 $x < 0$ 。

圖十七至圖二十八為無因次轉軸半徑 $r = 1$ 、細長比 $\eta = 50$ 、 100 、 1000 、傾斜角 $\alpha = 0^\circ$ 、 5° 、 30° 、 90° 時，在不同無因次轉速 k 下，對應於前六個振動頻率的振動模態及相角。因為在無因次轉速 $k=0$ 時， $\phi_u = \phi_v = 0$ ，且在傾斜角 $\alpha = 0^\circ$ 、無因次轉速 $k \neq 0$ 時， $\phi_u = 0$ 、 $\phi_v = \frac{\pi}{2}$ ，所以在其振動模態圖中沒有繪出相角。由圖十七至圖二十八可以發現當無因次轉速 $k = 0$ 時，其軸向和側向振態不互相耦合。

由圖十七到二十四可以發現當細長比 $\eta = 50$ 及 100 時，其第四個振態及第五振態分別為其第一個軸向振態，當 $k \neq 0$ 時，可以看出該振態有側向分量，其側向分量的大小隨著無因次轉速 k 、傾斜角 α 和細長比 η 增加而增加，但其軸向分量的形狀和大小與 $k = 0$ 時幾乎一樣；由圖十七到二十四亦可以發現其餘的振態皆為側向振態，當 $k \neq 0$ 時，可以看出這些振態有軸向分量，其軸向分量的大小隨著無因次轉速 k 、傾斜角 α 和細長比 η 增加而增加，但其側向分量的形狀和大小與 $k = 0$ 時幾乎一樣。

由圖二十五到二十八可以發現當細長比 $\eta = 1000$ 時，所有的振態皆為側向振態，當 $k \neq 0$ 時，可以看出這些振態有軸向分量，其軸向分量的大小隨著無因次轉速 k 、傾斜角 α 增加而增加，且其側向分量的形狀和大小與 $k = 0$ 時有相當的改變。

當傾斜角 $\alpha \neq 0^\circ$ 時，側向穩態位移隨傾斜角 α 、無因次轉速 k 及細長比 η 增加而變大，所以以穩態解為平衡點的振動都是軸向與側向耦合的振動，

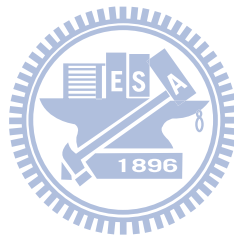
且其耦合的程度隨傾斜角 α 、無因次轉速 k 及細長比 η 增加而增加。當傾斜角 $\alpha = 0^\circ$ 時，旋轉梁的穩態解僅有軸向變形，但因科氏力不為零，故其軸向和側向振態互相耦合，由圖十七，細長比 $\eta = 50$ 的第四個振態及圖二十一，細長比 $\eta = 100$ 的第五振態可以發現振態耦合的現象，文獻[22]中提到當兩個振動頻率愈接近，其對應振動模態的耦合愈明顯，由表九可以發現在無因次轉速 $k = 0$ 時，細長比 $\eta = 50$ 的第三及第四個無因次振動頻率為 1.21530 及 1.57086，細長比 $\eta = 100$ 的第四及第五個無因次振動頻率為 1.20047 及 1.57086，這兩組振動頻率分別為 $\eta = 50$ 和 100 之前六個無因次振動頻率中最接近的無因次振動頻率，所以其耦合的現象最明顯。

4.4 旋轉梁之特徵值曲線轉向(Eigenvalue curve veering)分析

由上節的例題可以發現旋轉傾斜梁的無因次振動頻率會隨著無因次轉速 k 改變，且兩個振動頻率愈接近，其對應振動模態的耦合愈明顯，但上節的例題因振動頻率不夠接近，並未發現特徵值曲線轉向(Eigenvalue curve veering)及振態交換[23,24]的現象，本節中將考慮不同細長比 η 的旋轉梁，探討旋轉梁之特徵值曲線轉向及振態交換的現象。

表三十八到表四十五為無因次轉軸半徑 $r = 1$ ，細長比 $\eta = 38、38.5、39、40、77、78、79、80$ ，傾斜角 $\alpha = 0^\circ、5^\circ、10^\circ、15^\circ、30^\circ、45^\circ、60^\circ、75^\circ、90^\circ$ 之旋轉梁在不同無因次轉速 k 的前六個無因次振動頻率 $K_i (i = 1 - 6)$ ，圖二十九為無因次振動頻率-無因次轉速曲線，圖三十到四十五為細長比 $\eta = 38、39、40、77$ ，傾斜角 $\alpha = 0^\circ、5^\circ、30^\circ、90^\circ$ 時，在不同無因次轉速 k 下，對應於前六個振動頻率的振動模態。由上述的表及圖中可以發現當傾斜角 $\alpha = 0^\circ$ ，細長比 $\eta = 38.5、39、77、78、79、80$ 時，有無因次振動頻率曲線轉向及振態交換的現象，圖三十四中，無因次轉速 $k = 0、0.004$ ，振態 3 和 4 對應的無因次振動頻率分別為 K_3 和 K_4 ，但無因次轉速 $k = 0.005、0.006$ ，振態 3 和 4 對應的無因次振動頻率分別為 K_4 和 K_3 ；圖四十二中，

無因次轉速 $k=0、0.002$ ，振態 4 和 5 對應的無因次振動頻率分別為 K_4 和 K_5 ，但無因次轉速 $k=0.003、0.004$ ，振態 4 和 5 對應的無因次振動頻率分別為 K_5 和 K_4 。除了上述情況外，圖三十到四十五中，振態 $i(i=1-6)$ 對應的無因次自然頻率為 K_i 。由表九可以發現在無因次轉速 $k=0$ 時，細長比 $\eta=38、39、40$ 的旋轉梁除了第三及第四個無因次振動頻率外，第六及第七個無因次振動頻率亦相當接近，所以由圖三十到四十一中可以發現除了第三及第四個振態外，第六振態亦有明顯的振態耦合現象。



第五章 結論與展望

本研究採用共旋轉有限元素法探討設定角為 0° 之等速旋轉的傾斜尤拉梁的穩態變形及以該穩態變形為平衡點的微小振動之自然振動頻率。本文中僅考慮梁的軸向位移及單一個側向位移和旋轉二維運動，本研究採用d'Alembert原理、虛功原理、幾何非線性梁理論的一致線性化，在梁元素當前之元素座標上推導節點慣性力和節點變形力，將元素的節點力轉到總體座標後組合成系統的非線性運動方程式。當設定角為 0° 時，旋轉梁的軸向和側向穩態變形都不為零，本研究將旋轉梁的運動方程式的時間函數去掉求得系統穩態平衡方程式，再用基於牛頓法的增量迭代法求出軸向位移及側向位移的穩態解，將運動方程式在穩態平衡位置用泰勒級數展開，取到一次項，求得旋轉傾斜梁的振動方程式，假設自然振動頻率存在，可獲得一組代數齊次方程式，該組齊次方程式係數形成之矩陣的行列式為零時，即可求得旋轉梁以穩態解為平衡點的自然振動的頻率及其對應的振態。

由本研究的數值例題可以得到以下的結論：

1. 旋轉梁的側向穩態位移隨傾斜角 α 、無因次轉速 k 及細長比 η 增加而變大，且以穩態解為平衡點的振動都是軸向與側向耦合的振動，且其耦合的程度隨傾斜角 α 、無因次轉速 k 及細長比 η 增加而增加。
2. 旋轉梁的最大的無因次轉速 k 在傾斜角 $\alpha = 0^\circ$ 、 5° ，細長比 $\eta = 10$ 、 20 時，主要由穩態解之最大膜應變決定，當傾斜角 $\alpha \neq 0^\circ$ 、細長比 $\eta > 20$ 時，主要由穩態解最大撓曲應變決定。
3. 由離心力引起的軸向拉力會增加梁的側向剛度，且其影響隨細長比增加而增加。

4. 無因次轉速 k 對第 i 個無因次振動頻率 K_i 的影響隨著 i 的增加而減少，但隨著梁的細長比 η 及無因次轉軸半徑 r 的增加而增加；無因次側向振動頻率隨著無因次轉速 k 增加而增加，且 i 愈小、細長比 η 愈大、無因次轉軸半徑 r 愈大時，其增加率愈明顯。
5. 梁的無因次振動頻率會隨著無因次轉速 k 改變，且兩個振動頻率愈接近，其對應振動模態的耦合愈明顯。在傾斜角為 0° 且細長比 $\eta=38-40$ 、 $77-80$ 時，可發現當兩個振動頻率很接近時會發生特徵值曲線轉向(Eigenvalue curve veering)及振態交換的現象。

本研究僅探討設定角為 0° 時，旋轉傾斜尤拉梁穩態變形及振動，其運動是二維的運動，以後的研究應考慮設定角不為 0° 時的旋轉傾斜尤拉梁的穩態變形及振動，其運動是三維的運動。



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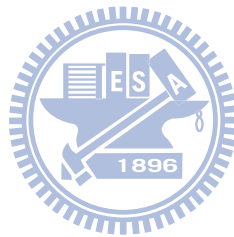
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表一 不同傾斜角與不同轉速的旋轉梁之振動頻率的收斂分析($\eta = 10$, $r = 1.5$)

α	元素 數目	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	V_{tip} / L_T (10^{-3})	K_1	$K_2 (a)$	K_3	K_4	$K_5 (a)$	K_6	K_7
5°	10	0.03	1.72672	1.00662	1.44842	.346300	1.57297	1.91534	4.64752	4.75786	7.82258	8.05791
	20		1.76145	1.00806	1.44846	.346288	1.57176	1.91526	4.64593	4.72569	7.81539	7.90662
	40		1.77854	1.00843	1.44847	.346286	1.57146	1.91524	4.64560	4.71775	7.81356	7.87006
	50		1.78193	1.00848	1.44847	.346285	1.57142	1.91524	4.64556	4.71681	7.81324	7.86581
	80		1.78701	1.00853	1.44847	.346285	1.57138	1.91524	4.64552	4.71578	7.81285	7.86126
	100		1.78870	1.00854	1.44847	.346285	1.57137	1.91524	4.64551	4.71554	7.81275	7.86022
30°	10	0.01	.173261	.647778	.936098	.343934	1.57247	1.91385	4.64995	4.75630	7.82680	8.05717
	20		.176628	.648601	.936101	.343933	1.57126	1.91380	4.64903	4.72357	7.82103	7.90479
	40		.178279	.648807	.936101	.343932	1.57096	1.91380	4.64894	4.71542	7.82046	7.86705
	50		.178607	.648832	.936101	.343932	1.57092	1.91380	4.64893	4.71445	7.82039	7.86255
	80		.179097	.648859	.936101	.343932	1.57088	1.91380	4.64893	4.71339	7.82032	7.85769
	100		.179260	.648865	.936101	.343932	1.57087	1.91380	4.64892	4.71314	7.82030	7.85657
75°	10	0.01	.0867650	1.25238	1.81023	.343750	1.57247	1.91378	4.65000	4.75630	7.82699	8.05717
	20		.0878387	1.25396	1.81024	.343749	1.57126	1.91374	4.64909	4.72357	7.82123	7.90479
	40		.0883433	1.25435	1.81024	.343749	1.57096	1.91373	4.64900	4.71542	7.82066	7.86705
	50		.0884416	1.25440	1.81024	.343749	1.57092	1.91373	4.64899	4.71445	7.82060	7.86255
	80		.0885874	1.25445	1.81024	.343749	1.57088	1.91373	4.64899	4.71339	7.82052	7.85769
	100		.0886356	1.25446	1.81024	.343749	1.57087	1.91373	4.64898	4.71314	7.82050	7.85657
90°	10	0.01	.0498854	1.29698	1.87490	.343672	1.57247	1.91375	4.65002	4.75630	7.82707	8.05717
	20		.0499869	1.29861	1.87490	.343671	1.57126	1.91370	4.64911	4.72357	7.82131	7.90479
	40		.0500053	1.29902	1.87490	.343671	1.57096	1.91370	4.64902	4.71542	7.82074	7.86705
	50		.0500064	1.29907	1.87490	.343671	1.57092	1.91370	4.64902	4.71445	7.82068	7.86255
	80		.0500064	1.29912	1.87490	.343671	1.57088	1.91370	4.64901	4.71339	7.82061	7.85769
	100		.0500059	1.29913	1.87490	.343671	1.57087	1.91370	4.64901	4.71314	7.82059	7.85657

表二 不同傾斜角與不同轉速的旋轉梁之振動頻率的收斂分析($\eta = 20$, $r = 1.5$)

α	元素 數目	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	V_{tip} / L_T (10^{-3})	K_1	K_2	$K_3 (a)$	K_4	$K_5 (a)$	K_6	K_7
5°	10	0.03	1.72680	1.93655	5.48934	.180882	1.06562	1.57335	2.82949	4.75639	5.19801	8.00220
	20		1.76150	1.94011	5.48989	.180860	1.06554	1.57214	2.82878	4.72359	5.19374	7.90467
	40		1.77856	1.94104	5.49003	.180855	1.06552	1.57184	2.82872	4.71541	5.19341	7.86675
	50		1.78195	1.94115	5.49004	.180854	1.06552	1.57180	2.82871	4.71442	5.19339	7.86221
	80		1.78702	1.94128	5.49006	.180853	1.06552	1.57176	2.82871	4.71336	5.19337	7.85729
	100		1.78871	1.94131	5.49007	.180853	1.06552	1.57175	2.82871	4.71312	5.19337	7.85615
30°	10	0.01	.173298	1.29046	3.72389	.175392	1.06018	1.57252	2.82541	4.75613	5.19573	8.00213
	20		.176647	1.29216	3.72393	.175390	1.06014	1.57131	2.82480	4.72333	5.19170	7.90455
	40		.178289	1.29258	3.72394	.175389	1.06014	1.57100	2.82476	4.71514	5.19143	7.86661
	50		.178615	1.29263	3.72394	.175389	1.06014	1.57097	2.82476	4.71416	5.19142	7.86207
	80		.179102	1.29269	3.72395	.175389	1.06014	1.57093	2.82476	4.71310	5.19141	7.85715
	100		.179264	1.29270	3.72395	.175389	1.06014	1.57092	2.82476	4.71285	5.19141	7.85601
75°	10	0.01	.0869040	2.50173	7.22633	.174999	1.05987	1.57253	2.82520	4.75612	5.19562	8.00207
	20		.0879121	2.50493	7.22636	.174998	1.05984	1.57132	2.82460	4.72332	5.19160	7.90451
	40		.0883807	2.50573	7.22637	.174997	1.05983	1.57102	2.82456	4.71513	5.19134	7.86659
	50		.0884715	2.50583	7.22637	.174997	1.05983	1.57098	2.82456	4.71415	5.19132	7.86205
	80		.0886060	2.50593	7.22638	.174997	1.05983	1.57094	2.82456	4.71309	5.19132	7.85713
	100		.0886504	2.50596	7.22638	.174997	1.05983	1.57093	2.82456	4.71284	5.19132	7.85599
90°	10	0.01	.0500346	2.59388	7.49567	.174830	1.05974	1.57253	2.82511	4.75612	5.19557	8.00206
	20		.0500657	2.59716	7.49569	.174829	1.05971	1.57132	2.82451	4.72331	5.19156	7.90451
	40		.0500454	2.59798	7.49570	.174829	1.05971	1.57102	2.82447	4.71513	5.19129	7.86659
	50		.0500385	2.59808	7.49570	.174829	1.05971	1.57098	2.82447	4.71415	5.19128	7.86205
	80		.0500263	2.59819	7.49570	.174829	1.05971	1.57094	2.82447	4.71309	5.19127	7.85712
	100		.0500217	2.59821	7.49570	.174829	1.05971	1.57093	2.82447	4.71284	5.19127	7.85599

表三 不同傾斜角與不同轉速的旋轉梁之振動頻率的收斂分析($\eta = 50$, $r = 1.5$)

α	元素 數目	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	V_{tip} / L_T	K_1	K_2	K_3	$K_4 (a)$	K_5	K_6	$K_7 (a)$
5°	10	0.03	1.72718	3.89539	.0250785	.0848761	.455824	1.23339	1.57392	2.37221	3.85294	4.75649
	20		1.76166	3.91291	.0250895	.0848304	.455686	1.23296	1.57270	2.36997	3.84395	4.72369
	40		1.77860	3.91756	.0250922	.0848189	.455654	1.23291	1.57239	2.36977	3.84326	4.71551
	50		1.78196	3.91814	.0250925	.0848175	.455650	1.23290	1.57236	2.36976	3.84323	4.71453
	80		1.78699	3.91878	.0250929	.0848161	.455646	1.23290	1.57232	2.36975	3.84320	4.71346
	100		1.78866	3.91893	.0250929	.0848157	.455645	1.23290	1.57231	2.36975	3.84320	4.71322
30°	10	0.01	.173543	3.14014	.0224082	.0718399	.439698	1.21709	1.57305	2.35587	3.83741	4.75622
	20		.176773	3.14519	.0224097	.0718343	.439671	1.21680	1.57183	2.35382	3.82874	4.72341
	40		.178350	3.14647	.0224101	.0718329	.439667	1.21678	1.57153	2.35368	3.82814	4.71523
	50		.178662	3.14662	.0224101	.0718327	.439667	1.21677	1.57149	2.35367	3.82812	4.71425
	80		.179129	3.14679	.0224101	.0718325	.439667	1.21677	1.57145	2.35367	3.82810	4.71318
	100		.179284	3.14683	.0224102	.0718325	.439666	1.21677	1.57145	2.35367	3.82810	4.71294
75°	10	0.01	.0878436	6.19826	.0444794	.0709001	.438758	1.21517	1.57464	2.35531	3.83649	4.75645
	20		.0883942	6.20706	.0444809	.0708973	.438738	1.21488	1.57342	2.35326	3.82783	4.72365
	40		.0886110	6.20927	.0444812	.0708966	.438736	1.21486	1.57311	2.35312	3.82724	4.71547
	50		.0886496	6.20953	.0444812	.0708966	.438735	1.21486	1.57307	2.35311	3.82722	4.71449
	80		.0887045	6.20982	.0444813	.0708965	.438735	1.21486	1.57303	2.35311	3.82720	4.71343
	100		.0887220	6.20989	.0444813	.0708964	.438735	1.21486	1.57303	2.35311	3.82720	4.71318
90°	10	0.01	.0510541	6.47860	.0466112	.0704770	.438377	1.21466	1.57485	2.35498	3.83613	4.75648
	20		.0505878	6.48726	.0466120	.0704755	.438359	1.21438	1.57363	2.35294	3.82748	4.72368
	40		.0502942	6.48943	.0466122	.0704751	.438357	1.21436	1.57332	2.35280	3.82689	4.71550
	50		.0502306	6.48969	.0466122	.0704751	.438357	1.21436	1.57328	2.35279	3.82687	4.71452
	80		.0501320	6.48997	.0466122	.0704750	.438357	1.21436	1.57325	2.35279	3.82685	4.71346
	100		.0500983	6.49004	.0466122	.0704750	.438357	1.21436	1.57324	2.35279	3.82685	4.71321

表四 不同傾斜角與不同轉速的旋轉梁之振動頻率的收斂分析($\eta=100$, $r=1.5$)

α	元素 數目	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	V_{tip} / L_T	K_1	K_2	K_3	K_4	$K_5 (a)$	K_6	K_7
5°	10	0.03	1.72775	5.09357	.0511913	.0591219	.254778	.652135	1.24001	1.57551	2.02262	2.99338
	20		1.76185	5.15588	.0512287	.0590608	.254541	.651746	1.23871	1.57422	2.01772	2.97847
	40		1.77853	5.17329	.0512379	.0590456	.254483	.651675	1.23857	1.57390	2.01730	2.97732
	50		1.78184	5.17548	.0512390	.0590437	.254476	.651668	1.23856	1.57386	2.01728	2.97726
	80		1.78678	5.17791	.0512402	.0590418	.254469	.651659	1.23855	1.57382	2.01726	2.97721
	100		1.78842	5.17847	.0512404	.0590413	.254467	.651657	1.23855	1.57381	2.01725	2.97721
30°	10	0.01	.174250	5.74549	.0788668	.0383279	.223645	.617615	1.20301	1.57860	1.98709	2.95692
	20		.177102	5.76051	.0788849	.0383174	.223613	.617446	1.20193	1.57731	1.98247	2.94234
	40		.178466	5.76440	.0788893	.0383147	.223606	.617431	1.20185	1.57700	1.98215	2.94131
	50		.178734	5.76488	.0788899	.0383144	.223606	.617430	1.20185	1.57696	1.98213	2.94126
	80		.179132	5.76540	.0788905	.0383141	.223605	.617428	1.20185	1.57692	1.98212	2.94124
	100		.179264	5.76552	.0788906	.0383140	.223605	.617428	1.20185	1.57691	1.98212	2.94123
75°	10	0.01	.0907471	11.9290	.166216	.0368604	.221453	.612094	1.19273	1.59872	1.99177	2.95541
	20		.0897082	11.9519	.166237	.0368544	.221434	.611941	1.19168	1.59719	1.98718	2.94085
	40		.0890411	11.9577	.166242	.0368529	.221430	.611929	1.19161	1.59684	1.98684	2.93981
	50		.0888954	11.9584	.166242	.0368527	.221430	.611928	1.19160	1.59680	1.98682	2.93976
	80		.0886691	11.9591	.166243	.0368525	.221430	.611928	1.19160	1.59676	1.98680	2.93974
	100		.0885916	11.9593	.166243	.0368524	.221429	.611928	1.19160	1.59675	1.98680	2.93973
90°	10	0.01	.0543132	12.7940	.179785	.0361111	.220583	.610488	1.19029	1.60306	1.99241	2.95467
	20		.0520411	12.8146	.179797	.0361073	.220569	.610339	1.18923	1.60148	1.98784	2.94012
	40		.0507494	12.8197	.179800	.0361063	.220567	.610328	1.18916	1.60113	1.98749	2.93907
	50		.0504783	12.8203	.179801	.0361062	.220567	.610327	1.18916	1.60109	1.98747	2.93903
	80		.0500633	12.8210	.179801	.0361061	.220566	.610327	1.18916	1.60105	1.98745	2.93900
	100		.0499228	12.8211	.179801	.0361060	.220566	.610327	1.18916	1.60104	1.98745	2.93899

表五 不同傾斜角與不同轉速的旋轉梁之振動頻率的收斂分析($\eta = 500$, $r = 1.5$)

α	元素 數目	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	V_{tip} / L_T	K_1	K_2	K_3	K_4	K_5	K_6	K_7
5°	10	0.03	1.73038	5.04954	.0803070	.0464280	.134390	.247496	.393200	.571561	.784736	1.03544
	20		1.76394	5.76426	.0804024	.0463541	.133696	.246343	.391776	.569491	.780885	1.02742
	40		1.77967	6.04540	.0804161	.0463376	.133519	.246063	.391472	.569153	.780442	1.02671
	50		1.78268	6.08649	.0804174	.0463356	.133498	.246030	.391438	.569118	.780406	1.02667
	80		1.78709	6.13455	.0804186	.0463336	.133475	.245995	.391401	.569083	.780372	1.02663
	100		1.78852	6.14641	.0804189	.0463331	.133470	.245986	.391392	.569075	.780365	1.02663
30°	10	0.008	.114268	8.46079	.352547	.0144174	.0553632	.133507	.250244	.406843	.604350	.843603
	20		.114306	8.60956	.352775	.0144002	.0553013	.133485	.250083	.405945	.601303	.835644
	40		.113982	8.65147	.352830	.0143958	.0552855	.133486	.250089	.405911	.601126	.835109
	50		.113885	8.65675	.352837	.0143953	.0552836	.133486	.250091	.405913	.601122	.835091
	80		.113717	8.66258	.352844	.0143947	.0552815	.133486	.250094	.405915	.601122	.835083
	100		.113655	8.66395	.352845	.0143946	.0552810	.133487	.250094	.405916	.601123	.835083
75°	10	0.003	.00848906	4.97759	.325332	.00799673	.0446289	.121364	.238962	.396654	.595333	.835863
	20		.00817969	4.99116	.325406	.00799381	.0446229	.121333	.238737	.395692	.592299	.828112
	40		.00799694	4.99460	.325425	.00799307	.0446215	.121331	.238720	.395623	.592080	.827535
	50		.00795804	4.99501	.325427	.00799298	.0446213	.121330	.238719	.395619	.592070	.827510
	80		.00789818	4.99546	.325429	.00799288	.0446211	.121330	.238719	.395617	.592064	.827495
	100		.00787782	4.99556	.325430	.00799286	.0446211	.121330	.238719	.395617	.592063	.827493
90°	10	0.003	.00526392	5.50358	.363581	.00774943	.0441721	.120239	.237680	.395230	.593820	.834230
	20		.00479652	5.51497	.363632	.00774726	.0441682	.120208	.237453	.394262	.590779	.826470
	40		.00453290	5.51777	.363645	.00774671	.0441673	.120206	.237435	.394191	.590556	.825889
	50		.00447769	5.51810	.363646	.00774664	.0441672	.120206	.237434	.394187	.590546	.825863
	80		.00439327	5.51846	.363648	.00774657	.0441671	.120206	.237433	.394185	.590539	.825847
	100		.00436471	5.51854	.363648	.00774655	.0441671	.120205	.237433	.394184	.590538	.825845

表六 不同傾斜角與不同轉速的旋轉梁之振動頻率的收斂分析($\eta=1000$, $r=1.5$)

α	元素 數目	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	V_{ip} / L_T	K_1	K_2	K_3	K_4	K_5	K_6	K_7
5°	10	0.03	1.73113	3.88613	.0835235	.0454694	.127442	.217642	.323061	.442957	.577154	.726538
	20		1.76526	5.17029	.0837847	.0453581	.126503	.215621	.320468	.439874	.573192	.720772
	40		1.78106	5.89061	.0838230	.0453354	.126247	.215078	.319816	.439172	.572419	.719852
	50		1.78397	6.01548	.0838255	.0453331	.126216	.215013	.319740	.439096	.572342	.719773
	80		1.78809	6.17213	.0838275	.0453309	.126183	.214944	.319660	.439015	.572263	.719697
	100		1.78938	6.21315	.0838279	.0453304	.126175	.214927	.319641	.438996	.572245	.719680
30°	10	0.008	.117176	8.73688	.429697	.0129066	.0405573	.0836364	.143457	.221474	.319570	.439088
	20		.116206	9.19910	.429937	.0128898	.0404478	.0836092	.143553	.221286	.318124	.434692
	40		.114745	9.34364	.429982	.0128852	.0404185	.0836045	.143617	.221403	.318213	.434607
	50		.114344	9.36265	.429987	.0128846	.0404148	.0836039	.143625	.221421	.318236	.434627
	80		.113663	9.38393	.429993	.0128840	.0404109	.0836032	.143634	.221442	.318264	.434657
	100		.113413	9.38899	.429994	.0128839	.0404101	.0836030	.143637	.221447	.318271	.434665
75°	10	0.003	.00938191	7.02594	.663462	.00571542	.0238973	.0587160	.115623	.193319	.292133	.412286
	20		.00834191	7.07182	.663647	.00570942	.0238954	.0587390	.115534	.192822	.290543	.408259
	40		.00771222	7.08294	.663692	.00570781	.0238942	.0587461	.115530	.192785	.290420	.407943
	50		.00757673	7.08424	.663697	.00570762	.0238940	.0587470	.115531	.192783	.290414	.407928
	80		.00736725	7.08564	.663703	.00570740	.0238939	.0587479	.115531	.192781	.290408	.407916
	100		.00729575	7.08595	.663705	.00570735	.0238938	.0587481	.115531	.192781	.290408	.407914
90°	10	0.003	.00632598	8.11019	.747141	.00561366	.0232167	.0566047	.113316	.190888	.289635	.409720
	20		.00489502	8.14495	.747230	.00560767	.0232188	.0566244	.113214	.190376	.288040	.405709
	40		.00405954	8.15225	.747251	.00560603	.0232182	.0566290	.113204	.190328	.287902	.405375
	50		.00388231	8.15303	.747254	.00560584	.0232181	.0566295	.113203	.190324	.287893	.405356
	80		.00360996	8.15384	.747256	.00560562	.0232180	.0566301	.113202	.190321	.287886	.405341
	100		.00351746	8.15402	.747257	.00560557	.0232180	.0566302	.113202	.190320	.287884	.405339

表七 不同傾斜角的旋轉梁之振動頻率的準確性分析($k = \frac{5}{70}$, $r = 1$, $\eta = 70$)

α	ε_0	ε_b	V_{tip} / L_T	K_1	K_1 [14]	K_2	K_2 [14]
0°	.00761579	0	0	.105427	.104966	.410792	.417489
10°	.00753381	.0220374	.119890	.105377	.104674	.410400	.416567
20°	.00728963	.0438841	.237606	.105225	.102756	.409219	.414031
30°	.00688882	.0653510	.351025	.104971	.100040	.407246	.410294
40°	.00634057	.0862526	.458121	.104612	.0960628	.404475	.404936
50°	.00565758	.106408	.557013	.104146	.0912700	.400900	.398364
60°	.00485594	.125641	.646008	.103568	.0850186	.396518	.389997
70°	.00395486	.143786	.723643	.102875	.0771729	.391331	.381219
80°	.00297641	.160683	.788722	.102058	.0683600	.385349	.371329
90°	.00194513	.176180	.840342	.101109	.0571429	.378595	.361034

表八 不同傾斜角與不同細長比的旋轉梁之端點位移的準確性分析($r = 1$)

α	η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_0^L (10^{-4})$	ε_b	ε_b^L	V_{tip} / L_T	V_{tip}^L / L_T	$V_{tip} / L_T [16]$
5°	500	0.02	5.94466	5.98478	.00335361	.0150958	.0734169	1.08945	.073612
	1000	0.01	1.48600	1.49619	.00167595	.00754791	.0733957	1.08945	.073612
	1000	0.03	13.3890	13.4658	.00519998	.0679312	.0825422	9.80502	.082773
10°	500	0.01	1.47373	1.48481	.00305801	.00751919	.119162	.542651	.12020
	1000	0.005	.368421	.371202	.00152880	.00375959	.119151	.542651	.12020
15°	500	0.004	.232945	.234548	.00133657	.00179315	.0825672	.129410	.082973
	1000	0.01	1.45421	1.46593	.00501583	.0224144	.218450	3.23524	.224477
30°	100	0.01	1.35791	1.36603	.00399154	.00433013	.0553226	.0625000	.055441
	500	0.008	.862862	.874256	.00691072	.0138564	.310627	1.00000	.331620
	1000	0.004	.215709	.218564	.00345508	.00692820	.310607	1.00000	.331620
45°	100	0.01	1.20113	1.20711	.00571738	.00612372	.0794779	.0883883	.079854
	1000	0.003	.106851	.108640	.00349175	.00551135	.367905	.795495	.406945
60°	100	0.008	.637717	.640000	.00464576	.00480000	.0656798	.0692820	.065896
	500	0.006	.349676	.360000	.00904678	.0135000	.473591	.974279	.568146
	1000	0.003	.0874165	.0900000	.00452322	.00675000	.473573	.974279	.568146
75°	100	0.008	.485134	.485644	.00525596	.00535370	.0745883	.0772741	.074912
	1000	0.003	.0645893	.0682937	.00544045	.00752865	.565891	1.08667	.756726
90°	100	0.008	.321229	.320000	.00552865	.00554256	.0788019	.0800000	.079189
	100	0.01	.502076	.500000	.00860877	.00866025	.121597	.125000	.123034
	1000	0.003	.0398360	.0450000	.00621305	.00779423	.642909	1.12500	.988891

表九 旋轉傾斜梁在不同細長比下的振動頻率($k=0$)

η	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	.343681	1.57086(a)	1.91364	4.64936	4.71413(a)	7.82132	7.86206(a)
20	.174787	1.05953	1.57086(a)	2.82431	4.71413(a)	5.19120	7.86206(a)
38	.0923781	.573449	1.57086(a)	1.58182	3.03502	4.71413(a)	4.88776
38.5	.0911822	.566163	1.56231	1.57086(a)	2.99909	4.71413(a)	4.83278
39	.0900168	.559057	1.54325	1.57086(a)	2.96396	4.71413(a)	4.77893
40	.0877730	.545362	1.50647	1.57086(a)	2.89600	4.67451	4.71413(a)
50	.0702550	.437859	1.21530	1.57086(a)	2.35176	3.82646	4.71413(a)
75	.0468608	.292950	.817035	1.57086(a)	1.59193	2.61236	3.86790
76	.0462448	.289117	.806427	1.57086(a)	1.57150	2.57930	3.81980
77	.0456446	.285383	.796090	1.55157	1.57086(a)	2.54705	3.77285
78	.0450599	.281744	.786013	1.53214	1.57086(a)	2.51558	3.72700
79	.0444900	.278196	.776186	1.51318	1.57086(a)	2.48487	3.68223
80	.0439342	.274736	.766600	1.49468	1.57086(a)	2.45488	3.63849
100	.0351520	.219989	.614602	1.20047	1.57086(a)	1.97619	2.93707
500	.00703197	.0440661	.123375	.241735	.399539	.596720	.833235
1000	.00351601	.0220341	.0616949	.120893	.199838	.298509	.416903

表十 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0$)

η	k	$\varepsilon_0 (10^{-4})$	ε_b	$\frac{V_{rip}}{L_T}$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.01	.499954	0	0	.343670	1.57092(a)	1.91370	4.64902	4.71445(a)	7.82068	7.86255(a)
	0.02	2.00007	0	0	.343639	1.57111(a)	1.91389	4.64802	4.71536(a)	7.81886	7.86394(a)
	0.04	8.00427	0	0	.343513	1.57185(a)	1.91465	4.64426	4.71880(a)	7.81245	7.86861(a)
	0.06	18.0246	0	0	.343306	1.57310(a)	1.91591	4.63867	4.72385(a)	7.80359	7.87457(a)
20	0.01	.499954	0	0	.174823	1.05972	1.57096(a)	2.82446	4.71417(a)	5.19127	7.86208(a)
	0.02	2.00007	0	0	.174930	1.06027	1.57127(a)	2.82489	4.71427(a)	5.19148	7.86213(a)
	0.04	8.00427	0	0	.175354	1.06248	1.57251(a)	2.82663	4.71467(a)	5.19234	7.86236(a)
	0.06	18.0246	0	0	.176054	1.06615	1.57458(a)	2.82952	4.71533(a)	5.19377	7.86273(a)
50	0.01	.499954	0	0	.0703844	.438455	1.21592	1.57096(a)	2.35238	3.82704	4.71417(a)
	0.02	2.00007	0	0	.0707689	.440240	1.21780	1.57125(a)	2.35425	3.82879	4.71426(a)
	0.04	8.00427	0	0	.0722524	.447313	1.22528	1.57240(a)	2.36171	3.83581	4.71466(a)
	0.06	18.0246	0	0	.0745530	.458872	1.23765	1.57433(a)	2.37409	3.84748	4.71531(a)
100	0.01	.499954	0	0	.0354205	.221216	.615938	1.20187	1.57096(a)	1.97760	2.93847
	0.02	2.00007	0	0	.0361954	.224860	.619929	1.20605	1.57124(a)	1.98183	2.94267
	0.04	8.00427	0	0	.0389181	.238890	.635620	1.22261	1.57239(a)	1.99865	2.95942
	0.06	18.0246	0	0	.0425305	.260607	.660876	1.24968	1.57431(a)	2.02636	2.98715
500	0.01	.499954	0	0	.00814757	.0498972	.130004	.248856	.406927	.604274	.840899
	0.02	2.00007	0	0	.0100978	.0642386	.147927	.268932	.428225	.626321	.863435
	0.04	8.00427	0	0	.0135687	.102725	.202962	.335905	.503267	.706764	.947569
	0.06	18.0246	0	0	.0164590	.145081	.268690	.421555	.605093	.821022	1.07120
1000	0.01	.499954	0	0	.00504927	.0321223	.0739765	.134505	.214203	.313344	.432052
	0.02	2.00007	0	0	.00677821	.0513635	.101497	.168006	.251762	.353634	.474228
	0.04	8.00427	0	0	.00942143	.0942683	.169116	.257543	.360229	.477630	.610650
	0.06	18.0246	0	0	.0116029	.138373	.240912	.355544	.483657	.624839	.779391

表十一 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=0^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	ε_b	$\frac{V_{tip}}{L_T}$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.01	.0994970	0	0	.343771	1.57092(a)	1.91374	4.64898	4.71445(a)	7.82057	7.86255(a)
	0.02	.398032	0	0	.344042	1.57111(a)	1.91404	4.64789	4.71536(a)	7.81843	7.86392(a)
	0.04	1.59285	0	0	.345120	1.57186(a)	1.91526	4.64377	4.71877(a)	7.81085	7.86841(a)
	0.06	3.58661	0	0	.346903	1.57311(a)	1.91730	4.63762	4.72373(a)	7.80021	7.87390(a)
20	0.01	.0994970	0	0	.175041	1.05988	1.57096(a)	2.82457	4.71417(a)	5.19132	7.86208(a)
	0.02	.398032	0	0	.175798	1.06092	1.57127(a)	2.82535	4.71427(a)	5.19171	7.86213(a)
	0.04	1.59285	0	0	.178785	1.06505	1.57252(a)	2.82847	4.71466(a)	5.19326	7.86236(a)
	0.06	3.58661	0	0	.183629	1.07192	1.57459(a)	2.83366	4.71533(a)	5.19584	7.86273(a)
50	0.01	.0994970	0	0	.0709373	.438931	1.21639	1.57096(a)	2.35284	3.82747	4.71417(a)
	0.02	.398032	0	0	.0729406	.442131	1.21966	1.57125(a)	2.35608	3.83052	4.71426(a)
	0.04	1.59285	0	0	.0803894	.454706	1.23266	1.57241(a)	2.36900	3.84268	4.71466(a)
	0.06	3.58661	0	0	.0912769	.474929	1.25401	1.57434(a)	2.39038	3.86289	4.71531(a)
100	0.01	.0994970	0	0	.0365097	.222183	.616926	1.20289	1.57096(a)	1.97864	2.93950
	0.02	.398032	0	0	.0402768	.228637	.623843	1.21012	1.57124(a)	1.98596	2.94678
	0.04	1.59285	0	0	.0524164	.252784	.650685	1.23857	1.57240(a)	2.01495	2.97574
	0.06	3.58661	0	0	.0674213	.288455	.692841	1.28440	1.57432(a)	2.06229	3.02335
500	0.01	.0994970	0	0	.0119675	.0540290	.134663	.253929	.412249	.609756	.846491
	0.02	.398032	0	0	.0200813	.0762383	.163340	.286920	.447870	.647045	.884889
	0.04	1.59285	0	0	.0370596	.130652	.243280	.387911	.564823	.775613	1.02188
	0.06	3.58661	0	0	.0541445	.188305	.332907	.507593	.711566	.945280	1.21033
1000	0.01	.0994970	0	0	.0100424	.0381228	.0816862	.143506	.224040	.323729	.442813
	0.02	.398032	0	0	.0185394	.0653273	.121659	.194022	.282569	.388114	.511474
	0.04	1.59285	0	0	.0356716	.123413	.212746	.316745	.434993	.567018	.713250
	0.06	3.58661	0	0	.0528045	.182500	.307134	.445787	.599005	.765364	.944405

表十二 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=5^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.005	.0248264	.00943349	.0136160	.343703	1.57088(a)	1.91366	4.64926	4.71421(a)	7.82113	7.86218(a)
	0.01	.0993087	.0377248	.0544395	.343771	1.57092(a)	1.91374	4.64898	4.71445(a)	7.82057	7.86255(a)
	0.02	.397280	.150753	.217366	.344040	1.57111(a)	1.91404	4.64789	4.71536(a)	7.81843	7.86392(a)
	0.03	.894049	.338649	.487608	.344488	1.57142(a)	1.91455	4.64613	4.71683(a)	7.81511	7.86595(a)
20	0.005	.0248264	.0188599	.0544314	.174851	1.05962	1.57089(a)	2.82438	4.71414(a)	5.19123	7.86206(a)
	0.01	.0993088	.0753371	.217236	.175040	1.05988	1.57096(a)	2.82457	4.71417(a)	5.19132	7.86208(a)
	0.02	.397280	.299724	.861205	.175794	1.06091	1.57127(a)	2.82535	4.71427(a)	5.19171	7.86213(a)
	0.03	.894050	.668432	1.90939	.177043	1.06264	1.57179(a)	2.82665	4.71443(a)	5.19235	7.86222(a)
50	0.005	.0248265	.0470270	.338773	.0704258	.438127	1.21557	1.57089(a)	2.35203	3.82671	4.71414(a)
	0.01	.0993089	.186410	1.33534	.0709353	.438929	1.21639	1.57096(a)	2.35284	3.82747	4.71417(a)
	0.02	.397281	.720242	5.04737	.0729336	.442122	1.21964	1.57127(a)	2.35608	3.83051	4.71427(a)
	0.03	.894048	1.53748	10.4047	.0761333	.447393	1.22500	1.57184(a)	2.36148	3.83557	4.71444(a)
100	0.005	.0248265	.0931927	1.33514	.0354959	.220538	.615182	1.20108	1.57089(a)	1.97680	2.93767
	0.01	.0993091	.359940	5.04454	.0365061	.222179	.616918	1.20288	1.57098(a)	1.97864	2.93949
	0.02	.397272	1.27852	16.5298	.0402677	.228619	.623782	1.20999	1.57150(a)	1.98602	2.94677
	0.03	.893967	2.48198	28.6429	.0457772	.238965	.635024	1.22164	1.57246(a)	1.99833	2.95888
500	0.005	.0248252	.371275	22.7407	.00857670	.0467568	.126287	.244824	.402727	.599970	.836523
	0.01	.0992814	1.03467	46.1594	.0119649	.0540176	.134605	.253821	.412091	.609550	.846218
	0.02	.397089	2.47033	64.4901	.0200785	.0762298	.163282	.286748	.447537	.646508	.884061
	0.03	.893645	3.93617	71.4377	.0285341	.102679	.201130	.333403	.500339	.703443	.943719
1000	0.005	.0248196	.517285	46.1559	.00598284	.0270110	.0673132	.126944	.206128	.304950	.423444
	0.01	.0992609	1.23470	64.4758	.0100409	.0381185	.0816572	.143422	.223882	.323491	.442492
	0.02	.397143	2.69658	75.0800	.0185380	.0653250	.121647	.193975	.282446	.387868	.511055
	0.03	.893990	4.13175	78.9386	.0270995	.0941368	.166441	.253819	.355873	.472835	.605502

表十三 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=10^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.005	.0246855	.0187952	.0271285	.343703	1.57088(a)	1.91366	4.64926	4.71421(a)	7.82113	7.86218(a)
	0.01	.0987451	.0751628	.108465	.343769	1.57092(a)	1.91374	4.64899	4.71445(a)	7.82057	7.86255(a)
	0.015	.222187	.169049	.243866	.343880	1.57100(a)	1.91386	4.64853	4.71483(a)	7.81967	7.86314(a)
	0.02	.395025	.300364	.433088	.344035	1.57111(a)	1.91404	4.64789	4.71536(a)	7.81843	7.86392(a)
20	0.005	.0246855	.0375765	.108449	.174850	1.05962	1.57089(a)	2.82438	4.71414(a)	5.19123	7.86206(a)
	0.01	.0987452	.150104	.432832	.175037	1.05988	1.57096(a)	2.82457	4.71417(a)	5.19132	7.86208(a)
	0.015	.222188	.336981	.970271	.175349	1.06031	1.57109(a)	2.82489	4.71421(a)	5.19148	7.86210(a)
	0.02	.395027	.597219	1.71605	.175785	1.06091	1.57128(a)	2.82535	4.71427(a)	5.19171	7.86213(a)
50	0.005	.0246856	.0936995	.674997	.0704242	438125	1.21557	1.57089(a)	2.35203	3.82671	4.71414(a)
	0.01	.0987458	.371451	2.66097	.0709292	438923	1.21638	1.57097(a)	2.35284	3.82746	4.71417(a)
	0.015	.222190	.823659	5.84641	.0717626	440249	1.21772	1.57111(a)	2.35418	3.82872	4.71421(a)
	0.02	.395032	1.43571	10.0625	.0729123	442098	1.21956	1.57135(a)	2.35607	3.83048	4.71428(a)
100	0.005	.0246857	.185701	2.66058	.0354928	.220535	.615179	1.20107	1.57089(a)	1.97680	2.93767
	0.01	.0987467	.717494	10.0569	.0364955	.222166	.616894	1.20284	1.57106(a)	1.97865	2.93948
	0.015	.222186	1.53142	20.7296	.0381045	.224855	.619711	1.20569	1.57149(a)	1.98178	2.94251
	0.02	.394995	2.55085	32.9871	.0402404	.228564	.623602	1.20957	1.57228(a)	1.98623	2.94675
500	0.005	.0246805	.741032	45.3966	.00857017	.0467402	.126231	.244745	.402627	.599852	.836378
	0.01	.0986356	2.06691	92.1628	.0119570	.0539832	.134432	.253499	.411625	.608941	.845417
	0.015	.221809	3.48592	115.715	.0159239	.0642325	.147134	.267622	.426439	.624048	.860520
	0.02	.394262	4.93625	128.676	.0200699	.0762036	.163106	.286233	.446545	.644923	.881650
1000	0.005	.0246582	1.03336	92.1559	.00597887	.0269938	.0672270	.126785	.205904	.304674	.423120
	0.01	.0985537	2.46722	128.648	.0100366	.0381054	.0815699	.143170	.223412	.322786	.441549
	0.015	.221780	3.92973	142.420	.0142671	.0513339	.100529	.166561	.249851	.351204	.471215
	0.02	.394474	5.38896	149.716	.0185334	.0653175	.121609	.193831	.282074	.387128	.509801

表十四 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=15^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.156492	.0179294	.0258795	.343695	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82120	7.86214(a)
	0.006	.352109	.0403387	.0582220	.343712	1.57088(a)	1.91367	4.64922	4.71425(a)	7.82105	7.86224(a)
	0.008	.625978	.0717070	.103489	.343736	1.57090(a)	1.91370	4.64912	4.71433(a)	7.82084	7.86237(a)
	0.01	.978104	.112030	.161667	.343768	1.57092(a)	1.91374	4.64899	4.71445(a)	7.82058	7.86255(a)
20	0.004	.156492	.0358505	.103479	.174827	1.05959	1.57088(a)	2.82435	4.71414(a)	5.19122	7.86206(a)
	0.006	.352109	.0806353	.232692	.174876	1.05966	1.57090(a)	2.82441	4.71415(a)	5.19124	7.86207(a)
	0.008	.625979	.143281	.413335	.174945	1.05975	1.57093(a)	2.82448	4.71415(a)	5.19128	7.86207(a)
	0.01	.978107	.223735	.645157	.175033	1.05987	1.57096(a)	2.82457	4.71417(a)	5.19132	7.86208(a)
50	0.004	.156492	.0894812	.645051	.0703617	.438028	1.21547	1.57088(a)	2.35193	3.82662	4.71414(a)
	0.006	.352111	.200857	1.44579	.0704948	.438239	1.21569	1.57090(a)	2.35214	3.82682	4.71415(a)
	0.008	.625985	.355903	2.55653	.0706808	.438535	1.21598	1.57093(a)	2.35244	3.82710	4.71415(a)
	0.01	.978121	.553765	3.96727	.0709192	.438914	1.21636	1.57097(a)	2.35283	3.82746	4.71417(a)
100	0.004	.156493	.177937	2.55629	.0353672	.220336	.614968	1.20085	1.57088(a)	1.97658	2.93745
	0.006	.352117	.396613	5.66435	.0356346	.220768	.615422	1.20132	1.57093(a)	1.97707	2.93793
	0.008	.625999	.696076	9.86035	.0360058	.221372	.616052	1.20196	1.57102(a)	1.97776	2.93860
	0.01	.978140	1.07029	15.0050	.0364778	.222145	.616853	1.20276	1.57118(a)	1.97866	2.93947
500	0.004	.156464	.761716	49.2689	.00804423	.0457777	.125164	.243606	.401446	.598645	.835150
	0.006	.351812	1.48039	85.4447	.00914593	.0478334	.127322	.245838	.403692	.600876	.837319
	0.008	.624910	2.27095	115.249	.0104759	.0505844	.130323	.248950	.406813	.603957	.840287
	0.01	.975640	3.09429	137.855	.0119437	.0539259	.134145	.252969	.410863	.607954	.844133
1000	0.004	.156224	1.13540	115.243	.00523820	.0252941	.0651716	.124510	.203500	.302189	.420578
	0.006	.350988	1.96781	154.772	.00674778	.0288771	.0693729	.128986	.208055	.306744	.425095
	0.008	.623450	2.82562	177.669	.00836615	.0332373	.0749036	.135153	.214484	.313247	.431579
	0.01	.973789	3.69537	192.213	.0100294	.0380837	.0814241	.142751	.222639	.321644	.440042

表十五 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=30^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.148580	.0346372	.0499958	.343693	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82120	7.86214(a)
	0.006	.334307	.0779295	.112479	.343708	1.57088(a)	1.91367	4.64922	4.71425(a)	7.82105	7.86224(a)
	0.008	.594330	.138531	.199932	.343730	1.57090(a)	1.91370	4.64912	4.71433(a)	7.82085	7.86237(a)
	0.01	.928654	.216433	.312334	.343757	1.57092(a)	1.91373	4.64899	4.71445(a)	7.82059	7.86255(a)
20	0.004	.148580	.0692600	.199914	.174823	1.05959	1.57088(a)	2.82435	4.71414(a)	5.19121	7.86206(a)
	0.006	.334308	.155786	.449566	.174868	1.05965	1.57090(a)	2.82440	4.71415(a)	5.19124	7.86207(a)
	0.008	.594334	.276832	.798628	.174931	1.05974	1.57093(a)	2.82447	4.71415(a)	5.19127	7.86207(a)
	0.01	.928663	.432305	1.24665	.175012	1.05986	1.57096(a)	2.82456	4.71417(a)	5.19132	7.86208(a)
50	0.004	.148581	.172899	1.24646	.0703529	.438020	1.21546	1.57088(a)	2.35192	3.82661	4.71414(a)
	0.006	.334316	.388196	2.79463	.0704751	.438222	1.21566	1.57090(a)	2.35213	3.82680	4.71415(a)
	0.008	.594357	.688087	4.94376	.0706461	.438503	1.21594	1.57095(a)	2.35242	3.82707	4.71416(a)
	0.01	.928716	1.07108	7.67594	.0708659	.438864	1.21629	1.57102(a)	2.35279	3.82741	4.71418(a)
100	0.004	.148586	.344016	4.94331	.0353499	.220320	.614949	1.20083	1.57090(a)	1.97657	2.93743
	0.006	.334337	.767509	10.9666	.0355969	.220732	.615371	1.20124	1.57102(a)	1.97705	2.93789
	0.008	.594407	1.34866	19.1194	.0359418	.221305	.615941	1.20178	1.57129(a)	1.97777	2.93854
	0.01	.928788	2.07668	29.1467	.0363833	.222037	.616645	1.20239	1.57180(a)	1.97874	2.93937
500	0.002	.0371501	.415273	29.1417	.00727916	.0444794	.123797	.242174	.399984	.597168	.833678
	0.004	.148469	1.49004	96.5785	.00799487	.0456820	.124887	.243241	.401001	.598129	.834529
	0.006	.333126	2.91366	168.308	.00908053	.0476379	.126599	.244787	.402350	.599279	.835362
	0.008	.590060	4.48524	227.218	.0104055	.0503120	.129142	.247046	.404274	.600853	.836444
1000	0.002	.0371172	.745007	96.5767	.00399752	.0228423	.0624517	.121651	.200581	.299243	.417623
	0.004	.147512	2.24248	227.205	.00520298	.0251579	.0645832	.123571	.202276	.300760	.418964
	0.006	.329805	3.89809	304.452	.00671105	.0287201	.0685214	.127308	.205649	.303769	.421631
	0.008	.584146	5.60339	348.562	.00832811	.0330971	.0740247	.133053	.211145	.308859	.426295

表十六 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=45^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.135993	.0489849	.0707057	.343690	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82120	7.86214(a)
	0.006	.305987	.110212	.159074	.343703	1.57088(a)	1.91367	4.64923	4.71425(a)	7.82106	7.86224(a)
	0.008	.543985	.195921	.282763	.343720	1.57090(a)	1.91369	4.64912	4.71433(a)	7.82086	7.86237(a)
	0.01	.849990	.306103	.441747	.343742	1.57092(a)	1.91373	4.64899	4.71445(a)	7.82061	7.86255(a)
20	0.004	.135994	.0979532	.282739	.174818	1.05958	1.57088(a)	2.82435	4.71414(a)	5.19121	7.86206(a)
	0.006	.305990	.220339	.635872	.174856	1.05964	1.57090(a)	2.82440	4.71415(a)	5.19124	7.86207(a)
	0.008	.543992	.391575	1.12972	.174909	1.05973	1.57093(a)	2.82446	4.71415(a)	5.19127	7.86207(a)
	0.01	.850008	.611558	1.76373	.174977	1.05983	1.57097(a)	2.82454	4.71417(a)	5.19131	7.86208(a)
50	0.004	.135997	.244592	1.76347	.0703388	.438008	1.21545	1.57088(a)	2.35191	3.82660	4.71414(a)
	0.006	.306005	.549378	3.95577	.0704437	.438194	1.21563	1.57091(a)	2.35210	3.82677	4.71415(a)
	0.008	.544039	.974308	7.00265	.0705909	.438454	1.21588	1.57098(a)	2.35238	3.82702	4.71416(a)
	0.01	.850114	1.51763	10.8821	.0707807	.438786	1.21617	1.57108(a)	2.35273	3.82734	4.71419(a)
100	0.004	.136007	.487119	7.00207	.0353222	.220295	.614919	1.20079	1.57093(a)	1.97655	2.93741
	0.006	.306049	1.08839	15.5633	.0355363	.220674	.615292	1.20113	1.57114(a)	1.97703	2.93784
	0.008	.544140	1.91629	27.2008	.0358385	.221199	.615774	1.20151	1.57166(a)	1.97777	2.93844
	0.01	.850256	2.95766	41.5870	.0362301	.221866	.616333	1.20185	1.57266(a)	1.97883	2.93923
500	0.003	.0764842	1.27694	86.8781	.00752478	.0449027	.124084	.242418	.400176	.597310	.833733
	0.004	.135750	2.15140	139.954	.00791215	.0455302	.124466	.242694	.400341	.597372	.833626
	0.005	.211495	3.15828	194.485	.00839937	.0463332	.124905	.242933	.400383	.597235	.833224
	0.006	.303365	4.25202	246.054	.00896911	.0473213	.125475	.243204	.400376	.596978	.832624
1000	0.003	.0758403	2.12594	246.045	.00448470	.0236623	.0627492	.121649	.200324	.298812	.417025
	0.004	.133598	3.29272	332.757	.00514289	.0249340	.0636582	.122163	.200505	.298748	.416744
	0.005	.206917	4.50987	397.106	.00587487	.0265538	.0651281	.123182	.201115	.299037	.416766
	0.006	.295795	5.75361	444.206	.00664893	.0284572	.0671547	.124804	.202295	.299848	.417276

表十七 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=60^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	1.19590	.0599949	.0865980	.343687	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82121	7.86214(a)
	0.006	2.69081	.134985	.194833	.343695	1.57088(a)	1.91367	4.64923	4.71425(a)	7.82107	7.86224(a)
	0.008	4.78373	.239966	.346338	.343706	1.57090(a)	1.91369	4.64913	4.71433(a)	7.82088	7.86237(a)
	0.01	7.47472	.374931	.541089	.343721	1.57092(a)	1.91372	4.64900	4.71445(a)	7.82063	7.86255(a)
20	0.004	1.19591	.119975	.346312	.174811	1.05958	1.57088(a)	2.82435	4.71414(a)	5.19121	7.86206(a)
	0.006	2.69084	.269898	.778925	.174840	1.05963	1.57090(a)	2.82439	4.71415(a)	5.19123	7.86207(a)
	0.008	4.78384	.479701	1.38407	.174880	1.05970	1.57093(a)	2.82444	4.71415(a)	5.19126	7.86207(a)
	0.01	7.47498	.749299	2.16122	.174932	1.05980	1.57097(a)	2.82452	4.71417(a)	5.19130	7.86207(a)
50	0.004	1.19596	.299686	2.16094	.0703204	.437992	1.21543	1.57088(a)	2.35190	3.82658	4.71414(a)
	0.006	2.69108	.673466	4.85053	.0704027	.438158	1.21559	1.57092(a)	2.35207	3.82674	4.71415(a)
	0.008	4.78454	1.19522	8.59432	.0705185	.438390	1.21580	1.57100(a)	2.35232	3.82696	4.71417(a)
	0.01	7.47658	1.86339	13.3707	.0706689	.438685	1.21603	1.57115(a)	2.35265	3.82724	4.71420(a)
100	0.004	1.19611	.597569	8.59368	.0352859	.220262	.614882	1.20075	1.57095(a)	1.97652	2.93737
	0.006	2.69173	1.33781	19.1488	.0354566	.220599	.615196	1.20100	1.57126(a)	1.97699	2.93776
	0.008	4.78606	2.36162	33.5787	.0357013	.221062	.615575	1.20121	1.57204(a)	1.97774	2.93831
	0.01	7.47867	3.65658	51.5416	.0360246	.221647	.615965	1.20124	1.57358(a)	1.97888	2.93903
500	0.001	.0747628	.186314	13.3687	.00707423	.0441520	.123463	.241827	.399632	.596813	.833328
	0.002	.299134	.731223	51.5340	.00720735	.0444011	.123669	.242024	.399815	.596982	.833467
	0.003	.672458	1.59370	108.905	.00744485	.0447970	.123862	.242147	.399860	.596957	.833321
	0.004	1.19176	2.71183	177.375	.00779555	.0453339	.123957	.242049	.399573	.596503	.832608
1000	0.001	.0747834	.365610	51.5338	.00360371	.0222017	.0618417	.121039	.199979	.298648	.417038
	0.002	.297937	1.35589	177.372	.00389785	.0226682	.0619880	.121061	.199888	.298486	.416803
	0.003	.661312	2.72290	316.194	.00440406	.0234508	.0620462	.120707	.199203	.297583	.415696
	0.004	1.15365	4.25712	429.053	.00505566	.0246273	.0624752	.120479	.198480	.296520	.414351

表十八 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=75^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	1.00489	.0669167	.0965896	.343683	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82121	7.86214(a)
	0.006	2.26102	.150562	.217318	.343686	1.57088(a)	1.91366	4.64923	4.71425(a)	7.82108	7.86224(a)
	0.008	4.01966	.267665	.386323	.343691	1.57090(a)	1.91368	4.64913	4.71433(a)	7.82089	7.86237(a)
	0.01	6.28085	.418223	.603587	.343696	1.57092(a)	1.91371	4.64901	4.71445(a)	7.82066	7.86255(a)
20	0.004	1.00490	.133825	.386298	.174802	1.05957	1.57088(a)	2.82434	4.71414(a)	5.19121	7.86206(a)
	0.006	2.26107	.301082	.868966	.174821	1.05962	1.57090(a)	2.82438	4.71415(a)	5.19123	7.86207(a)
	0.008	4.01980	.535195	1.54432	.174847	1.05968	1.57093(a)	2.82443	4.71415(a)	5.19125	7.86207(a)
	0.01	6.28119	.836120	2.41197	.174880	1.05976	1.57097(a)	2.82449	4.71416(a)	5.19128	7.86207(a)
50	0.004	1.00496	.334416	2.41170	.0702990	.437974	1.21541	1.57088(a)	2.35188	3.82657	4.71414(a)
	0.006	2.26136	.751960	5.41752	.0703547	.438117	1.21555	1.57093(a)	2.35203	3.82670	4.71415(a)
	0.008	4.02067	1.33562	9.60911	.0704337	.438315	1.21571	1.57102(a)	2.35225	3.82689	4.71417(a)
	0.01	6.28317	2.08447	14.9696	.0705373	.438568	1.21588	1.57120(a)	2.35254	3.82713	4.71420(a)
100	0.004	1.00515	.667774	9.60849	.0352433	.220224	.614840	1.20070	1.57097(a)	1.97648	2.93733
	0.006	2.26217	1.49846	21.4738	.0353624	.220512	.615093	1.20087	1.57136(a)	1.97692	2.93767
	0.008	4.02256	2.65356	37.8065	.0355376	.220906	.615368	1.20093	1.57234(a)	1.97767	2.93816
	0.01	6.28571	4.12454	58.3140	.0357766	.221398	.615595	1.20068	1.57431(a)	1.97884	2.93879
500	0.001	.0628291	.208424	14.9676	.00706102	.0441400	.123450	.241812	.399616	.596797	.833311
	0.002	.251417	.824818	58.3065	.00715769	.0443511	.123597	.241943	.399725	.596885	.833360
	0.0025	.392696	1.27764	89.1996	.00723839	.0445011	.123643	.241954	.399698	.596823	.833239
	0.003	.564857	1.81927	125.027	.00734494	.0446775	.123639	.241883	.399558	.596622	.832939
1000	0.001	.0628543	.412407	58.3062	.00357888	.0221767	.0618061	.120999	.199935	.298601	.416990
	0.002	.249711	1.56838	206.724	.00382250	.0225558	.0617268	.120742	.199523	.298094	.416383
	0.0025	.386233	2.35163	293.354	.00403219	.0228344	.0615122	.120295	.198890	.297343	.415521
	0.003	.548173	3.22129	376.281	.00429559	.0231971	.0612738	.119726	.198072	.296367	.414405

表十九 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=0.5$, $\alpha=90^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.799902	.0692784	.0999992	.343679	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82122	7.86214(a)
	0.006	1.79980	.155879	.224996	.343677	1.57088(a)	1.91366	4.64923	4.71425(a)	7.82109	7.86224(a)
	0.008	3.19971	.277126	.399988	.343674	1.57090(a)	1.91368	4.64914	4.71433(a)	7.82091	7.86237(a)
	0.01	4.99966	.433023	.624970	.343671	1.57092(a)	1.91370	4.64902	4.71445(a)	7.82068	7.86255(a)
20	0.004	.799911	.138557	.399966	.174793	1.05956	1.57088(a)	2.82434	4.71414(a)	5.19121	7.86206(a)
	0.006	1.79985	.311758	.899828	.174800	1.05960	1.57090(a)	2.82437	4.71415(a)	5.19122	7.86207(a)
	0.008	3.19986	.554251	1.59945	.174810	1.05965	1.57093(a)	2.82441	4.71415(a)	5.19124	7.86207(a)
	0.01	5.00002	.866044	2.49866	.174824	1.05972	1.57097(a)	2.82446	4.71416(a)	5.19127	7.86207(a)
50	0.004	.799975	.346391	2.49843	.0702760	.437954	1.21539	1.57088(a)	2.35186	3.82655	4.71414(a)
	0.006	1.80016	.779385	5.61696	.0703030	.438072	1.21550	1.57093(a)	2.35199	3.82666	4.71415(a)
	0.008	3.20079	1.38557	9.97430	.0703421	.438236	1.21563	1.57103(a)	2.35218	3.82682	4.71417(a)
	0.01	5.00216	2.16490	15.5613	.0703945	.438444	1.21575	1.57122(a)	2.35243	3.82702	4.71421(a)
100	0.004	.800182	.692755	9.97376	.0351974	.220183	.614798	1.20065	1.57098(a)	1.97644	2.93729
	0.006	1.80104	1.55847	22.3628	.0352597	.220421	.614994	1.20076	1.57139(a)	1.97684	2.93758
	0.008	3.20282	2.76949	39.5477	.0353571	.220743	.615180	1.20070	1.57248(a)	1.97753	2.93799
	0.01	5.00481	4.32362	61.3395	.0354991	.221139	.615273	1.20027	1.57467(a)	1.97867	2.93853
500	0.001	.0500194	.216470	15.5596	.00704669	.0441272	.123436	.241798	.399601	.596782	.833295
	0.002	.200183	.864647	61.3328	.00710214	.0442990	.123534	.241874	.399651	.596807	.833277
	0.0025	.312610	1.34817	94.6072	.00715405	.0444180	.123531	.241829	.399561	.596678	.833081
	0.003	.449395	1.93425	133.848	.00722824	.0445546	.123452	.241670	.399322	.596369	.832658
1000	0.001	.0500457	.432322	61.3326	.00355111	.0221506	.0617743	.120964	.199898	.298563	.416950
	0.002	.198042	1.69560	225.723	.00373046	.0224392	.0614996	.120477	.199228	.297782	.416054
	0.0025	.304202	2.58480	326.051	.00391126	.0226422	.0610578	.119757	.198288	.296707	.414852
	0.003	.426938	3.59170	423.658	.00415519	.0229167	.0605189	.118825	.197068	.295312	.413304

表二十 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1, \alpha=0^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	ε_b	$\frac{V_{tip}}{L_T}$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.01	.148998	0	0	.343872	1.57092(a)	1.91378	4.64895	4.71445(a)	7.82046	7.86255(a)
	0.02	.596058	0	0	.344444	1.57111(a)	1.91420	4.64776	4.71536(a)	7.81800	7.86390(a)
	0.04	2.38527	0	0	.346719	1.57186(a)	1.91588	4.64328	4.71873(a)	7.80924	7.86822(a)
	0.06	5.37076	0	0	.350462	1.57312(a)	1.91869	4.63657	4.72360(a)	7.79677	7.87329(a)
20	0.01	.148998	0	0	.175258	1.06004	1.57096(a)	2.82469	4.71417(a)	5.19138	7.86208(a)
	0.02	.596058	0	0	.176661	1.06156	1.57127(a)	2.82581	4.71427(a)	5.19194	7.86213(a)
	0.04	2.38527	0	0	.182150	1.06762	1.57252(a)	2.83030	4.71467(a)	5.19418	7.86236(a)
	0.06	5.37076	0	0	.190897	1.07766	1.57460(a)	2.83779	4.71533(a)	5.19792	7.86273(a)
50	0.01	.148998	0	0	.0714858	.439405	1.21686	1.57096(a)	2.35330	3.82790	4.71417(a)
	0.02	.596058	0	0	.0750486	.444012	1.22152	1.57125(a)	2.35791	3.83224	4.71426(a)
	0.04	2.38527	0	0	.0877636	.461975	1.23999	1.57241(a)	2.37627	3.84954	4.71466(a)
	0.06	5.37076	0	0	.105339	.490429	1.27012	1.57435(a)	2.40655	3.87823	4.71531(a)
100	0.01	.148998	0	0	.0375668	.223145	.617913	1.20392	1.57096(a)	1.97967	2.94053
	0.02	.596058	0	0	.0439754	.232350	.627728	1.21418	1.57124(a)	1.99008	2.95089
	0.04	2.38527	0	0	.0630513	.265905	.665347	1.25429	1.57240(a)	2.03110	2.99195
	0.06	5.37076	0	0	.0852550	.313660	.723130	1.31800	1.57433(a)	2.09747	3.05905
500	0.01	.148998	0	0	.0148226	.0578458	.139137	.258881	.417488	.615179	.852038
	0.02	.596058	0	0	.0265111	.0864577	.177158	.303589	.466489	.666975	.905720
	0.04	2.38527	0	0	.0505357	.153227	.276760	.432184	.618642	.837278	1.08978
	0.06	5.37076	0	0	.0746277	.222828	.385008	.578158	.800313	1.05084	1.33083
1000	0.01	.148998	0	0	.0132579	.0432330	.0885977	.151847	.233362	.333715	.453260
	0.02	.596058	0	0	.0252836	.0766152	.138402	.216170	.309504	.418994	.545501
	0.04	2.38527	0	0	.0494470	.146591	.247884	.364608	.495920	.640731	.799060
	0.06	5.37076	0	0	.0735850	.217470	.360269	.518160	.691561	.878499	1.07803

表二十一 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha=5^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.005	.0371543	.0188650	.0272283	.343728	1.57088(a)	1.91367	4.64926	4.71421(a)	7.82110	7.86218(a)
	0.01	.148622	.0754183	.108819	.343871	1.57092(a)	1.91378	4.64895	4.71445(a)	7.82046	7.86255(a)
	0.02	.594553	.301007	.433772	.344441	1.57111(a)	1.91420	4.64776	4.71536(a)	7.81800	7.86390(a)
	0.03	1.33799	.674794	.970406	.345388	1.57142(a)	1.91490	4.64584	4.71682(a)	7.81418	7.86588(a)
20	0.005	.0371543	.0377019	.108796	.174905	1.05966	1.57089(a)	2.82441	4.71414(a)	5.19124	7.86206(a)
	0.01	.148622	.150388	.433403	.175256	1.06004	1.57096(a)	2.82469	4.71417(a)	5.19138	7.86208(a)
	0.02	.594555	.594973	1.70572	.176655	1.06156	1.57128(a)	2.82581	4.71427(a)	5.19194	7.86213(a)
	0.03	1.33800	1.31504	3.73772	.178959	1.06408	1.57180(a)	2.82768	4.71443(a)	5.19287	7.86222(a)
50	0.005	.0371544	.0937658	.674872	.0705638	4.38245	1.21569	1.57089(a)	2.35214	3.82682	4.71414(a)
	0.01	.148623	.368372	2.62960	.0714819	4.39402	1.21685	1.57096(a)	2.35330	3.82790	4.71417(a)
	0.02	.594560	1.37849	9.53161	.0750365	4.43994	1.22144	1.57134(a)	2.35792	3.83222	4.71428(a)
	0.03	1.33800	2.81927	18.5488	.0805930	4.51541	1.22898	1.57208(a)	2.36562	3.83940	4.71448(a)
100	0.005	.0371547	.184151	2.62905	.0357700	.220780	.615428	1.20133	1.57089(a)	1.97706	2.93793
	0.01	.148624	.688752	9.52424	.0375607	.223136	.617890	1.20387	1.57105(a)	1.97969	2.94052
	0.02	.594541	2.24028	27.6708	.0439644	.232313	.627575	1.21380	1.57195(a)	1.99029	2.95088
	0.03	1.33786	4.02653	42.8179	.0528363	.246845	.643385	1.23009	1.57328(a)	2.00786	2.96808
500	0.005	.0371523	.623972	35.8503	.00964371	.0478904	.127486	.246092	.404032	.601297	.837856
	0.01	.148592	1.53057	59.7156	.0148209	.0578323	.139055	.258706	.417215	.614805	.851525
	0.02	.594466	3.35362	73.4169	.0265094	.0864514	.177104	.303407	.466095	.666282	.904574
	0.03	1.33806	5.16695	77.9682	.0384949	.119171	.224942	.363905	.536149	.743176	.986233
1000	0.005	.0371469	.765180	59.7101	.00741092	.0289187	.0695390	.129389	.208697	.307592	.426132
	0.01	.148600	1.67595	73.3957	.0132571	.0432299	.0885713	.151759	.233177	.333411	.452825
	0.02	.594715	3.47188	80.2091	.0252828	.0766140	.138394	.216134	.309404	.418777	.545103
	0.03	1.33890	5.19998	82.5422	.0373627	.111397	.192494	.289119	.400294	.525712	.665915

表二十二 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha=10^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.005	.0368725	.0375866	.0542495	.343728	1.57088(a)	1.91367	4.64926	4.71421(a)	7.82110	7.86218(a)
	0.01	.147495	.150264	.216812	.343869	1.57092(a)	1.91378	4.64895	4.71445(a)	7.82047	7.86255(a)
	0.015	.331879	.337786	.487132	.344103	1.57100(a)	1.91395	4.64845	4.71483(a)	7.81942	7.86313(a)
	0.02	.590045	.599746	.864286	.344432	1.57111(a)	1.91419	4.64776	4.71536(a)	7.81801	7.86390(a)
20	0.005	.0368725	.0751176	.216766	.174904	1.05966	1.57089(a)	2.82441	4.71414(a)	5.19124	7.86206(a)
	0.01	.147495	.299644	.863556	.175251	1.06004	1.57096(a)	2.82468	4.71417(a)	5.19138	7.86208(a)
	0.015	.331881	.671134	1.92998	.175830	1.06066	1.57109(a)	2.82515	4.71421(a)	5.19161	7.86210(a)
	0.02	.590052	1.18562	3.39920	.176636	1.06154	1.57128(a)	2.82580	4.71426(a)	5.19193	7.86213(a)
50	0.005	.0368727	.186831	1.34473	.0705607	4.38242	1.21568	1.57089(a)	2.35214	3.82681	4.71414(a)
	0.01	.147498	.734133	5.24094	.0714704	4.39390	1.21682	1.57099(a)	2.35329	3.82789	4.71417(a)
	0.015	.331891	1.60542	11.3083	.0729618	4.41293	1.21868	1.57121(a)	2.35522	3.82967	4.71423(a)
	0.02	.590072	2.74889	19.0110	.0750004	4.43941	1.22121	1.57163(a)	2.35793	3.83216	4.71432(a)
100	0.005	.0368734	.366998	5.23986	.0357642	.220774	.615419	1.20132	1.57091(a)	1.97706	2.93792
	0.01	.147501	1.37346	18.9963	.0375426	.223109	.617820	1.20373	1.57131(a)	1.97974	2.94050
	0.015	.331883	2.80853	36.9624	.0403322	.226941	.621727	1.20753	1.57238(a)	1.98436	2.94481
	0.02	.589994	4.47238	55.2464	.0439313	.232203	.627119	1.21266	1.57405(a)	1.99092	2.95085
500	0.005	.0368639	1.24607	71.5843	.00963699	.0478600	.127360	.245903	.403787	.601000	.837484
	0.01	.147373	3.05801	119.162	.0148156	.0577917	.138807	.258185	.416409	.613704	.850035
	0.015	.331594	4.87936	137.386	.0205807	.0712323	.156015	.277757	.437207	.635045	.871422
	0.02	.589691	6.70119	146.399	.0265043	.0864319	.176943	.302860	.464916	.664231	.901243
1000	0.005	.0368421	1.52880	119.151	.00740825	.0288984	.0694156	.129133	.208312	.307095	.425532
	0.01	.147405	3.34889	146.356	.0132546	.0432202	.0884912	.151491	.232621	.332510	.451545
	0.015	.331945	5.15678	155.338	.0192519	.0595841	.112452	.181853	.267806	.371091	.492432
	0.02	.590688	6.93832	159.879	.0252802	.0766099	.138369	.216026	.309101	.418120	.543898

表二十三 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha=15^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.232993	.0358565	.0517545	.343710	1.57087(a)	1.91366	4.64929	4.71418(a)	7.82118	7.86214(a)
	0.006	.524239	.0806657	.116422	.343747	1.57088(a)	1.91369	4.64921	4.71425(a)	7.82101	7.86224(a)
	0.008	.931991	.143377	.206907	.343799	1.57090(a)	1.91373	4.64910	4.71433(a)	7.82077	7.86237(a)
	0.01	1.45626	.223969	.323161	.343865	1.57092(a)	1.91377	4.64895	4.71445(a)	7.82047	7.86255(a)
20	0.004	.232994	.0716798	.206879	.174860	1.05961	1.57088(a)	2.82437	4.71414(a)	5.19122	7.86206(a)
	0.006	.524241	.161163	.464982	.174952	1.05971	1.57090(a)	2.82445	4.71415(a)	5.19126	7.86207(a)
	0.008	.931995	.286224	.825404	.175079	1.05985	1.57093(a)	2.82455	4.71415(a)	5.19131	7.86207(a)
	0.01	1.45627	.446646	1.28723	.175243	1.06003	1.57096(a)	2.82468	4.71417(a)	5.19138	7.86208(a)
50	0.004	.232995	.178621	1.28693	.0704475	438102	1.21554	1.57088(a)	2.35200	3.82668	4.71414(a)
	0.006	.524249	.399999	2.87568	.0706875	438404	1.21584	1.57090(a)	2.35231	3.82696	4.71415(a)
	0.008	.932020	.706460	5.06355	.0710224	438828	1.21626	1.57095(a)	2.35273	3.82736	4.71416(a)
	0.01	1.45632	1.09469	7.81586	.0714511	439370	1.21678	1.57102(a)	2.35328	3.82787	4.71418(a)
100	0.004	.233001	.353188	5.06288	.0355386	.220486	.615117	1.20100	1.57090(a)	1.97674	2.93761
	0.006	.524272	.780114	11.0871	.0360170	.221104	.615750	1.20164	1.57102(a)	1.97745	2.93829
	0.008	.932074	1.35273	18.9983	.0366779	.221965	.616617	1.20248	1.57128(a)	1.97848	2.93924
	0.01	1.45640	2.05010	28.3642	.0375124	.223064	.617706	1.20350	1.57175(a)	1.97983	2.94047
500	0.004	.232945	1.33657	82.5672	.00878114	.0464921	.125811	.244238	.402050	.599215	.835658
	0.006	.523761	2.40353	127.695	.0105614	.0493800	.128805	.247270	.405036	.602116	.838383
	0.008	.930567	3.49014	158.100	.0126123	.0531886	.133036	.251633	.409363	.606322	.842347
	0.01	1.45349	4.57922	178.071	.0148067	.0577239	.138395	.257330	.415102	.611946	.847700
1000	0.004	.232637	1.74492	158.089	.00630646	.0265964	.0665293	.125858	.204796	.303429	.421755
	0.006	.523165	2.83438	191.617	.00854092	.0314085	.0723732	.132190	.211269	.309887	.428128
	0.008	.930220	3.92558	208.459	.0108760	.0370658	.0798239	.140759	.220369	.319173	.437418
	0.01	1.45421	5.01583	218.450	.0132503	.0432039	.0883572	.151045	.231702	.331041	.449500

表二十四 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha=30^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.217169	.0692704	.0999838	.343707	1.57087(a)	1.91366	4.64929	4.71418(a)	7.82119	7.86214(a)
	0.006	.488635	.155839	.224918	.343740	1.57088(a)	1.91368	4.64921	4.71425(a)	7.82102	7.86224(a)
	0.008	.868696	.276998	.399741	.343786	1.57090(a)	1.91372	4.64910	4.71433(a)	7.82079	7.86237(a)
	0.01	1.35736	.432711	.624368	.343845	1.57092(a)	1.91377	4.64896	4.71445(a)	7.82049	7.86255(a)
20	0.004	.217170	.138483	.399691	.174853	1.05961	1.57088(a)	2.82437	4.71414(a)	5.19122	7.86206(a)
	0.006	.488640	.311386	.898436	.174936	1.05970	1.57090(a)	2.82444	4.71415(a)	5.19126	7.86207(a)
	0.008	.868711	.553076	1.59506	.175052	1.05983	1.57093(a)	2.82453	4.71415(a)	5.19130	7.86207(a)
	0.01	1.35740	.863181	2.48796	.175200	1.06000	1.57097(a)	2.82466	4.71416(a)	5.19137	7.86207(a)
50	0.004	.217177	.345204	2.48741	.0704300	.438086	1.21552	1.57088(a)	2.35199	3.82667	4.71414(a)
	0.006	.488671	.773414	5.56161	.0706487	.438369	1.21579	1.57093(a)	2.35228	3.82693	4.71415(a)
	0.008	.868804	1.36686	9.80109	.0709550	.438764	1.21615	1.57103(a)	2.35269	3.82730	4.71417(a)
	0.01	1.35761	2.11970	15.1438	.0713488	.439268	1.21657	1.57120(a)	2.35322	3.82777	4.71420(a)
100	0.004	.217197	.683351	9.79987	.0355048	.220453	.615073	1.20094	1.57097(a)	1.97673	2.93758
	0.006	.488757	1.51201	21.5075	.0359457	.221027	.615619	1.20142	1.57136(a)	1.97747	2.93822
	0.008	.869005	2.62751	36.9515	.0365619	.221821	.616314	1.20192	1.57227(a)	1.97863	2.93913
	0.01	1.35791	3.99154	55.3226	.0373502	.222830	.617116	1.20232	1.57395(a)	1.98028	2.94031
500	0.002	.0543143	.798138	55.3097	.00747272	.0446394	.123905	.242269	.400063	.597231	.833714
	0.004	.216965	2.62907	162.583	.00872059	.0462982	.125115	.243280	.400855	.597815	.833958
	0.006	.486672	4.75037	251.546	.0105026	.0490506	.127347	.245018	.402093	.598573	.834055
	0.008	.862862	6.91073	310.627	.0125602	.0528088	.131069	.248188	.404621	.600444	.835114
1000	0.002	.0542410	1.31451	162.579	.00436038	.0231505	.0625672	.121676	.200527	.299137	.417464
	0.004	.215709	3.45508	310.607	.00628041	.0264065	.0655495	.124158	.202506	.300707	.418660
	0.006	.484207	5.61912	374.732	.00851749	.0312451	.0712882	.129769	.207592	.305203	.422605
	0.008	.861283	7.78577	406.544	.0108529	.0369462	.0788909	.138171	.215887	.312999	.429813

表二十五 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha = 45^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.191997	.0979652	.0141402	.343702	1.57087(a)	1.91366	4.64929	4.71418(a)	7.82119	7.86214(a)
	0.006	.431998	.220400	.0318102	.343728	1.57088(a)	1.91368	4.64922	4.71425(a)	7.82103	7.86224(a)
	0.008	.768008	.391767	.0565383	.343765	1.57090(a)	1.91371	4.64911	4.71433(a)	7.82081	7.86237(a)
	0.01	1.20004	.612026	.0883145	.343813	1.57092(a)	1.91376	4.64897	4.71445(a)	7.82053	7.86255(a)
20	0.004	.191999	.195864	.0565319	.174842	1.05960	1.57088(a)	2.82436	4.71414(a)	5.19122	7.86206(a)
	0.006	.432007	.440463	.127094	.174911	1.05968	1.57090(a)	2.82442	4.71414(a)	5.19125	7.86206(a)
	0.008	.768038	.782471	.225689	.175008	1.05980	1.57093(a)	2.82451	4.71415(a)	5.19129	7.86207(a)
	0.01	1.20011	1.22146	.352126	.175132	1.05994	1.57097(a)	2.82462	4.71416(a)	5.19135	7.86207(a)
50	0.004	.192011	.488498	.352056	.0704020	.438062	1.21550	1.57089(a)	2.35197	3.82665	4.71414(a)
	0.006	.432069	1.09530	.787936	.0705867	.438313	1.21572	1.57096(a)	2.35223	3.82688	4.71416(a)
	0.008	.768224	1.93774	1.39041	.0708469	.438663	1.21598	1.57113(a)	2.35261	3.82720	4.71419(a)
	0.01	1.20053	3.00891	2.15186	.0711843	.439107	1.21625	1.57145(a)	2.35312	3.82761	4.71424(a)
100	0.004	.192052	.968774	1.39025	.0354506	.220402	.615005	1.20085	1.57107(a)	1.97670	2.93753
	0.006	.432241	2.14959	3.06198	.0358306	.220906	.615421	1.20110	1.57183(a)	1.97749	2.93811
	0.008	.768627	3.74874	5.28369	.0363729	.221596	.615862	1.20110	1.57367(a)	1.97883	2.93894
	0.01	1.20113	5.71738	7.94779	.0370830	.222461	.616235	1.20062	1.57710(a)	1.98091	2.94004
500	0.003	.108006	2.37654	15.7344	.00792274	.0451555	.123910	.242068	.399657	.596644	.832837
	0.004	.191502	3.83283	23.7537	.00861721	.0459857	.124039	.241840	.399096	.595793	.831570
	0.005	.298065	5.39008	30.8957	.00945862	.0470968	.124360	.241633	.398426	.594730	.830002
	0.006	.427412	6.98381	36.7920	.0104019	.0485093	.125076	.241706	.397955	.593799	.828543
1000	0.003	.106851	3.49175	36.7905	.00520115	.0242565	.0625541	.120920	.199178	.297385	.415351
	0.004	.188592	5.09603	45.2284	.00623616	.0260909	.0640141	.121694	.199390	.297176	.414802
	0.005	.293352	6.70054	50.5786	.00733889	.0283735	.0664014	.123470	.200564	.297868	.415117
	0.006	.421568	8.30520	54.1259	.00847821	.0309696	.0695305	.126204	.202707	.299497	.416342

表二十六 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha = 60^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	1.59191	.119986	.0173188	.343695	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82120	7.86214(a)
	0.006	3.58186	.269950	.0389627	.343713	1.57088(a)	1.91367	4.64922	4.71425(a)	7.82105	7.86224(a)
	0.008	6.36788	.479868	.0692552	.343739	1.57090(a)	1.91370	4.64912	4.71433(a)	7.82084	7.86237(a)
	0.01	9.95007	.749705	.108188	.343771	1.57092(a)	1.91374	4.64898	4.71445(a)	7.82057	7.86255(a)
20	0.004	1.59194	.239914	.0692486	.174828	1.05959	1.57088(a)	2.82436	4.71414(a)	5.19122	7.86206(a)
	0.006	3.58200	.539609	.155715	.174879	1.05966	1.57090(a)	2.82441	4.71414(a)	5.19124	7.86206(a)
	0.008	6.36832	.958813	.276593	.174950	1.05975	1.57093(a)	2.82448	4.71415(a)	5.19128	7.86207(a)
	0.01	9.95114	1.49716	.431705	.175043	1.05988	1.57097(a)	2.82458	4.71416(a)	5.19133	7.86207(a)
50	0.004	1.59213	.598775	.431633	.0703655	4.38030	1.21546	1.57090(a)	2.35194	3.82662	4.71414(a)
	0.006	3.58293	1.34391	.967280	.0705054	4.38241	1.21562	1.57100(a)	2.35217	3.82681	4.71416(a)
	0.008	6.37111	2.38084	1.70989	.0707047	4.38532	1.21579	1.57124(a)	2.35251	3.82708	4.71420(a)
	0.01	9.95750	3.70333	2.65209	.0709666	4.38899	1.21588	1.57171(a)	2.35298	3.82741	4.71428(a)
100	0.004	1.59275	1.19032	1.70972	.0353793	.220335	.614922	1.20074	1.57117(a)	1.97666	2.93746
	0.006	3.58552	2.65115	3.78356	.0356773	.220751	.615186	1.20073	1.57232(a)	1.97748	2.93796
	0.008	6.37717	4.64576	6.56798	.0361176	.221308	.615331	1.20019	1.57517(a)	1.97899	2.93869
	0.01	9.96627	7.12453	9.94533	.0367164	.221990	.615201	1.19873	1.58053(a)	1.98153	2.93967
500	0.001	.0995708	.370269	2.65157	.00710412	.0441743	.123473	.241833	.399633	.596811	.833319
	0.002	.398630	1.42467	9.94347	.00734584	.0444712	.123552	.241842	.399566	.596676	.833067
	0.003	.895261	3.00402	20.0010	.00779944	.0449239	.123279	.241278	.398722	.595591	.831610
	0.004	1.58240	4.90168	30.4970	.00846717	.0455709	.122699	.240117	.397048	.593498	.828946
1000	0.001	.0996573	.712333	9.94341	.00367296	.0222367	.0617842	.120950	.199862	.298514	.416886
	0.002	.395596	2.45079	30.4964	.00423367	.0227869	.0613626	.120111	.198677	.297113	.415275
	0.003	.874165	4.52322	47.3573	.00512761	.0238860	.0611240	.118999	.196928	.294949	.412763
	0.004	1.52872	6.63384	57.8448	.00617231	.0256517	.0620640	.118883	.196085	.293610	.411048

表二十七 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha = 75^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	1.20988	.133831	.0193175	.343687	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82121	7.86214(a)
	0.006	2.72229	.301112	.0434614	.343696	1.57088(a)	1.91367	4.64923	4.71425(a)	7.82107	7.86224(a)
	0.008	4.83976	.535292	.0772574	.343708	1.57090(a)	1.91369	4.64913	4.71433(a)	7.82087	7.86237(a)
	0.01	7.56239	.836356	.120700	.343723	1.57092(a)	1.91372	4.64900	4.71445(a)	7.82063	7.86255(a)
20	0.004	1.20992	.267629	.0772516	.174811	1.05958	1.57088(a)	2.82435	4.71414(a)	5.19121	7.86206(a)
	0.006	2.72246	.602055	.173753	.174841	1.05963	1.57090(a)	2.82439	4.71414(a)	5.19123	7.86206(a)
	0.008	4.84031	1.07005	.308735	.174883	1.05970	1.57093(a)	2.82445	4.71415(a)	5.19126	7.86206(a)
	0.01	7.56372	1.67140	.482078	.174938	1.05980	1.57097(a)	2.82452	4.71416(a)	5.19130	7.86206(a)
50	0.004	1.21015	.668484	.482015	.0703228	.437993	1.21542	1.57090(a)	2.35190	3.82658	4.71414(a)
	0.006	2.72363	1.50213	1.08182	.0704101	.438157	1.21553	1.57102(a)	2.35209	3.82674	4.71416(a)
	0.008	4.84380	2.66550	1.91635	.0705370	.438382	1.21559	1.57132(a)	2.35238	3.82694	4.71421(a)
	0.01	7.57166	4.15461	2.98011	.0707082	.438661	1.21552	1.57191(a)	2.35279	3.82719	4.71431(a)
100	0.004	1.21092	1.33266	1.91621	.0352951	.220258	.614832	1.20063	1.57124(a)	1.97660	2.93738
	0.006	2.72687	2.98161	4.26491	.0354939	.220575	.614945	1.20039	1.57270(a)	1.97740	2.93779
	0.008	4.85134	5.25596	7.45883	.0358065	.220983	.614804	1.19935	1.57639(a)	1.97903	2.93839
	0.01	7.58219	8.11670	11.3912	.0362605	.221459	.614181	1.19700	1.58348(a)	1.98195	2.93922
500	0.001	.0757132	.415405	2.97964	.00707818	.0441499	.123443	.241800	.399598	.596774	.833279
	0.002	.303269	1.62313	11.3894	.00725456	.0443646	.123361	.241618	.399312	.596396	.832748
	0.0025	.473408	2.48620	17.1154	.00741641	.0445017	.123093	.241199	.398746	.595706	.831861
	0.003	.679771	3.48824	23.4115	.00763826	.0446579	.122624	.240485	.397803	.594575	.830452
1000	0.001	.0758172	.811561	11.3894	.00362732	.0221834	.0616888	.120839	.199738	.298382	.416746
	0.002	.298516	2.89324	36.2410	.00413243	.0225406	.0606359	.119228	.197673	.296043	.414143
	0.0025	.457932	4.14845	47.5851	.00454687	.0228868	.0599396	.118028	.196109	.294259	.412171
	0.003	.645894	5.44045	56.5891	.00502734	.0234253	.0595231	.117027	.194724	.292633	.410357

表二十八 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1$, $\alpha=90^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.799911	.138557	.0199998	.343679	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82122	7.86214(a)
	0.006	1.79985	.311759	.0449992	.343677	1.57088(a)	1.91366	4.64923	4.71425(a)	7.82109	7.86224(a)
	0.008	3.19986	.554252	.0799975	.343674	1.57090(a)	1.91368	4.64914	4.71433(a)	7.82091	7.86237(a)
	0.01	5.00003	.866046	.124994	.343670	1.57092(a)	1.91370	4.64902	4.71445(a)	7.82068	7.86255(a)
20	0.004	.799949	.277114	.0799932	.174793	1.05956	1.57088(a)	2.82434	4.71414(a)	5.19121	7.86206(a)
	0.006	1.80004	.623516	.179965	.174801	1.05960	1.57090(a)	2.82437	4.71414(a)	5.19122	7.86206(a)
	0.008	3.20045	1.10850	.319889	.174811	1.05965	1.57093(a)	2.82441	4.71415(a)	5.19124	7.86206(a)
	0.01	5.00146	1.73208	.499725	.174826	1.05971	1.57097(a)	2.82446	4.71416(a)	5.19127	7.86206(a)
50	0.004	.800204	.692777	.499678	.0702768	.437953	1.21538	1.57090(a)	2.35186	3.82655	4.71414(a)
	0.006	1.80129	1.55871	1.12331	.0703070	.438068	1.21543	1.57103(a)	2.35201	3.82666	4.71417(a)
	0.008	3.20420	2.77081	1.99439	.0703545	.438223	1.21541	1.57135(a)	2.35223	3.82680	4.71422(a)
	0.01	5.01000	4.32854	3.11046	.0704248	.438412	1.21523	1.57200(a)	2.35257	3.82696	4.71432(a)
100	0.004	.801032	1.38534	1.99428	.0352036	.220177	.614745	1.20053	1.57127	1.97653	2.93730
	0.006	1.80479	3.11508	4.46722	.0352910	.220389	.614728	1.20013	1.57288	1.97726	2.93760
	0.008	3.21229	5.52865	7.88019	.0354544	.220646	.614359	1.19875	1.57704	1.97887	2.93806
	0.01	5.02076	8.60877	12.1597	.0357312	.220916	.613344	1.19577	1.58523	1.98193	2.93871
500	0.001	.0500975	.432814	3.11011	.00704972	.0441242	.123414	.241769	.399566	.596740	.833244
	0.002	.200817	1.72159	12.1583	.00714860	.0442553	.123200	.241437	.399113	.596183	.832514
	0.0025	.313142	2.66832	18.5342	.00726241	.0443213	.122774	.240833	.398338	.595264	.831370
	0.003	.448275	3.79060	25.7176	.00743838	.0443815	.122046	.239816	.397053	.593764	.829553
1000	0.001	.0502041	.860795	12.1582	.00357433	.0221288	.0616084	.120749	.199640	.298279	.416639
	0.002	.194123	3.21654	40.7207	.00400094	.0222722	.0599363	.118425	.196788	.295121	.413183
	0.0025	.290777	4.68882	54.0274	.00440751	.0224722	.0587409	.116668	.194632	.292733	.410601
	0.003	.398359	6.21305	64.2909	.00489484	.0228875	.0578803	.115176	.192739	.290593	.408272

表二十九 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=0^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	ε_b	$\frac{V_{tip}}{L_T}$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.01	.198500	0	0	.343973	1.57092(a)	1.91382	4.64892	4.71445(a)	7.82035	7.86255(a)
	0.02	.794084	0	0	.344846	1.57111(a)	1.91435	4.64763	4.71536(a)	7.81756	7.86388(a)
	0.04	3.17769	0	0	.348310	1.57186(a)	1.91650	4.64279	4.71870(a)	7.80763	7.86804(a)
	0.06	7.15492	0	0	.353985	1.57313(a)	1.92007	4.63552	4.72348(a)	7.79328	7.87272(a)
20	0.01	.198500	0	0	.175475	1.06020	1.57096(a)	2.82480	4.71417(a)	5.19144	7.86208(a)
	0.02	.794084	0	0	.177520	1.06221	1.57127(a)	2.82627	4.71427(a)	5.19217	7.86213(a)
	0.04	3.17769	0	0	.185453	1.07019	1.57252(a)	2.83214	4.71467(a)	5.19509	7.86236(a)
	0.06	7.15492	0	0	.197894	1.08337	1.57461(a)	2.84192	4.71534(a)	5.20000	7.86273(a)
50	0.01	.198500	0	0	.0720301	.439880	1.21732	1.57096(a)	2.35376	3.82833	4.71417(a)
	0.02	.794084	0	0	.0770980	.445886	1.22338	1.57125(a)	2.35974	3.83396	4.71426(a)
	0.04	3.17769	0	0	.0945541	.469124	1.24727	1.57241(a)	2.38351	3.85639	4.71466(a)
	0.06	7.15492	0	0	.117702	.505425	1.28599	1.57436(a)	2.42258	3.89351	4.71531(a)
100	0.01	.198500	0	0	.0385945	.224102	.618898	1.20494	1.57096(a)	1.98070	2.94156
	0.02	.794084	0	0	.0473812	.236000	.631585	1.21823	1.57124(a)	1.99418	2.95499
	0.04	3.17769	0	0	.0721063	.278366	.679633	1.26977	1.57240(a)	2.04709	3.00805
	0.06	7.15492	0	0	.0999008	.336839	.751962	1.35058	1.57434(a)	2.13195	3.09426
500	0.01	.198500	0	0	.0172021	.0614083	.143445	.263719	.422648	.620545	.857542
	0.02	.794084	0	0	.0316424	.0955023	.189772	.319172	.484211	.686184	.925973
	0.04	3.17769	0	0	.0610727	.172703	.305993	.471303	.666924	0.893468	1.15254
	0.06	7.15492	0	0	.0905376	.252467	.430045	.639373	.877838	1.14394	1.43827
1000	0.01	.198500	0	0	.0158242	.0477558	.0949072	.159645	.242235	.343339	.463416
	0.02	.794084	0	0	.0305566	.0863532	.153022	.235739	.333668	.447131	.576950
	0.04	3.17769	0	0	.0601094	.166457	.278177	.405893	.548602	.704779	.874116
	0.06	7.15492	0	0	.0896171	.247409	.406028	.580433	.771135	.975828	1.19320

表三十 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=5^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.005	.0494824	.0282946	.0408368	.343753	1.57088(a)	1.91368	4.64925	4.71421(a)	7.82108	7.86218(a)
	0.01	.197935	.113080	.163138	.343971	1.57092(a)	1.91382	4.64892	4.71445(a)	7.82035	7.86255(a)
	0.02	.791827	.450766	.649227	.344841	1.57111(a)	1.91435	4.64763	4.71536(a)	7.81757	7.86389(a)
	0.03	1.78193	1.00848	1.44847	.346285	1.57142(a)	1.91524	4.64556	4.71681(a)	7.81324	7.86581(a)
20	0.005	.0494824	.0565259	.163093	.174959	1.05970	1.57089(a)	2.82444	4.71414(a)	5.19126	7.86206(a)
	0.01	.197936	.225155	.648508	.175473	1.06020	1.57096(a)	2.82480	4.71417(a)	5.19144	7.86208(a)
	0.02	.791831	.885872	2.53403	.177511	1.06220	1.57128(a)	2.82627	4.71426(a)	5.19217	7.86213(a)
	0.03	1.78195	1.94115	5.49004	.180854	1.06552	1.57180(a)	2.82871	4.71442(a)	5.19339	7.86221(a)
50	0.005	.0494825	.140220	1.00833	.0707017	.438364	1.21580	1.57089(a)	2.35226	3.82692	4.71414(a)
	0.01	.197937	.546082	3.88463	.0720245	.439874	1.21731	1.57097(a)	2.35376	3.82833	4.71417(a)
	0.02	.791843	1.98428	13.5413	.0770825	.445857	1.22323	1.57144(a)	2.35976	3.83392	4.71429(a)
	0.03	1.78196	3.91814	25.0925	.0848175	.455650	1.23290	1.57236(a)	2.36976	3.84323	4.71453(a)
100	0.005	.0494829	.272974	3.88359	.0360421	.221021	.615673	1.20158	1.57090(a)	1.97732	2.93819
	0.01	.197939	.991225	13.5281	.0385867	.224088	.618855	1.20485	1.57114(a)	1.98075	2.94155
	0.02	.791821	3.01351	35.6750	.0473706	.235948	.631344	1.21758	1.57239(a)	1.99455	2.95498
	0.03	1.78184	5.17548	51.2390	.0590437	.254476	.651667	1.23856	1.57386(a)	2.01728	2.97726
500	0.005	.0494807	.817670	44.3501	.0106024	.0489976	.128676	.247355	.405335	.602621	.839187
	0.01	.197920	1.88557	66.1259	.0172010	.0613954	.143356	.263512	.422306	.620056	.856854
	0.02	.791934	3.99912	76.9430	.0316414	.0954978	.189728	.319005	.483819	.685449	.924684
	0.03	1.78268	6.08649	80.4174	.0463356	.133498	.246030	.391438	.569118	.780407	1.02667
1000	0.005	.0494786	.942627	66.1187	.00860106	.0307005	.0716908	.131795	.211248	.310231	.428825
	0.01	.197962	1.99825	76.9150	.0158237	.0477536	.0948853	.159564	.242052	.343020	.462936
	0.02	.792371	4.05799	82.0756	.0305562	.0863526	.153016	.235713	.333588	.446949	.576597
	0.03	1.78397	6.01549	83.8255	.0453331	.126216	.215013	.319740	.439096	.572342	.719773

表三十一 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=10^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.005	.0490596	.0563741	.0813632	.343753	1.57088(a)	1.91368	4.64925	4.71421(a)	7.82108	7.86218(a)
	0.01	.196245	.225304	.325040	.343968	1.57092(a)	1.91381	4.64892	4.71445(a)	7.82036	7.86255(a)
	0.015	.441571	.506213	.729800	.344327	1.57100(a)	1.91404	4.64838	4.71483(a)	7.81918	7.86312(a)
	0.02	.785067	.898152	1.29361	.344828	1.57111(a)	1.91435	4.64764	4.71536(a)	7.81758	7.86389(a)
20	0.005	.0490597	.112623	.324951	.174957	1.05970	1.57089(a)	2.82444	4.71414(a)	5.19125	7.86206(a)
	0.01	.196246	.448625	1.29219	.175465	1.06020	1.57096(a)	2.82480	4.71417(a)	5.19144	7.86208(a)
	0.015	.441576	1.00250	2.87930	.176309	1.06102	1.57110(a)	2.82540	4.71421(a)	5.19174	7.86210(a)
	0.02	.785081	1.76544	5.05037	.177484	1.06217	1.57128(a)	2.82626	4.71426(a)	5.19216	7.86212(a)
50	0.005	.0490602	.279401	2.00925	.0706971	.438359	1.21580	1.57089(a)	2.35226	3.82692	4.71414(a)
	0.01	.196251	1.08843	7.74345	.0720079	.439855	1.21726	1.57102(a)	2.35375	3.82831	4.71418(a)
	0.015	.441598	2.34914	16.4217	.0741434	.442333	1.21961	1.57136(a)	2.35626	3.83061	4.71425(a)
	0.02	.785127	3.95809	27.0172	.0770360	.445772	1.22277	1.57201(a)	2.35980	3.83382	4.71438(a)
100	0.005	.0490615	.544079	7.74139	.0360338	.221012	.615657	1.20155	1.57094	1.97732	2.93818
	0.01	.196260	1.97723	26.9912	.0385634	.224046	.618727	1.20457	1.57167	1.98087	2.94153
	0.015	.441595	3.90304	49.9992	.0424474	.229008	.623708	1.20927	1.57342	1.98700	2.94712
	0.02	.785042	6.01783	71.2332	.0473385	.235792	.630623	1.21568	1.57579	1.99567	2.95497
500	0.005	.0490526	1.63324	88.5459	.0105966	.0489579	.128494	.247069	.404953	.602151	.838591
	0.01	.196181	3.76742	131.904	.0171975	.0613566	.143089	.262894	.421296	.618625	.854865
	0.015	.441578	5.88407	146.430	.0243492	.0775561	.164368	.287590	.447903	.646167	.882631
	0.02	.785486	7.99105	153.391	.0316382	.0954838	.189594	.318503	.482645	.683263	.920925
1000	0.005	.0490439	1.88339	131.889	.00859928	.0306811	.0715578	.131491	.210766	.309586	.428027
	0.01	.196348	3.99293	153.335	.0158221	.0477466	.0948190	.159319	.241501	.342069	.461519
	0.015	.442307	6.07260	160.161	.0231753	.0667503	.123010	.195661	.284370	.389811	.512762
	0.02	.787232	8.10983	163.575	.0305546	.0863505	.152998	.235631	.333346	.446397	.575528

表三十二 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=15^\circ$)

η	k	$\varepsilon_0 (10^{-3})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-3})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.0309495	.0537813	.0776251	.343726	1.57087(a)	1.91367	4.64929	4.71418(a)	7.82116	7.86214(a)
	0.006	.0696370	.120981	.174598	.343782	1.57088(a)	1.91370	4.64920	4.71425(a)	7.82097	7.86224(a)
	0.008	.123800	.215010	.310253	.343861	1.57090(a)	1.91375	4.64908	4.71433(a)	7.82070	7.86237(a)
	0.01	.193441	.335818	.484481	.343962	1.57092(a)	1.91381	4.64892	4.71445(a)	7.82036	7.86255(a)
20	0.004	.0309496	.107488	.310199	.174894	1.05964	1.57088(a)	2.82439	4.71414(a)	5.19123	7.86206(a)
	0.006	.0696373	.241584	.696872	.175027	1.05977	1.57090(a)	2.82449	4.71415(a)	5.19128	7.86207(a)
	0.008	.123801	.428829	1.23621	.175214	1.05995	1.57093(a)	2.82462	4.71415(a)	5.19135	7.86207(a)
	0.01	.193443	.668737	1.92625	.175453	1.06019	1.57097(a)	2.82479	4.71417(a)	5.19143	7.86208(a)
50	0.004	.0309500	.267421	1.92566	.0705333	.438175	1.21561	1.57088(a)	2.35207	3.82675	4.71414(a)
	0.006	.0696391	.597455	4.28994	.0708799	.438569	1.21600	1.57092(a)	2.35247	3.82711	4.71415(a)
	0.008	.123807	1.05181	7.52242	.0713630	.439120	1.21652	1.57098(a)	2.35302	3.82762	4.71416(a)
	0.01	.193456	1.62331	11.5507	.0719802	.439825	1.21718	1.57110(a)	2.35374	3.82828	4.71419(a)
100	0.004	.0309512	.525825	7.52113	.0357094	.220635	.615265	1.20115	1.57093(a)	1.97691	2.93777
	0.006	.0696442	1.15128	16.2826	.0363965	.221438	.616069	1.20193	1.57116(a)	1.97785	2.93865
	0.008	.123819	1.97390	27.4868	.0373406	.222553	.617158	1.20294	1.57167(a)	1.97923	2.93989
	0.01	.193475	2.95281	40.3241	.0385245	.223976	.618517	1.20413	1.57254(a)	1.98108	2.94148
500	0.004	.0309468	1.79896	106.459	.00946359	.0472001	.126459	.244862	.402637	.599760	.836128
	0.006	.0695981	3.08931	152.668	.0118082	.0508952	.130344	.248780	.406467	.603445	.839546
	0.008	.123703	4.36918	180.195	.0144315	.0556872	.135817	.254481	.412137	.608946	.844717
	0.01	.193296	5.64182	196.988	.0171916	.0612918	.142645	.261880	.419662	.616350	.851783
1000	0.004	.0309252	2.18435	180.181	.00721610	.0278460	.0679212	.127287	.206199	.304787	.423059
	0.006	.0695996	3.45494	207.897	.0100128	.0337441	.0753004	.135456	.214650	.313264	.431442
	0.008	.123827	4.72050	221.085	.0128992	.0405083	.0844469	.146246	.226329	.325330	.443605
	0.01	.193667	5.98046	228.768	.0158194	.0477350	.0947077	.158910	.240586	.340509	.459239

表三十三 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=30^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.285759	.103900	.0149964	.343721	1.57087(a)	1.91366	4.64929	4.71418(a)	7.82117	7.86214(a)
	0.006	.642965	.233728	.0337319	.343771	1.57088(a)	1.91370	4.64920	4.71425(a)	7.82099	7.86224(a)
	0.008	1.14307	.415401	.0599427	.343842	1.57090(a)	1.91374	4.64908	4.71433(a)	7.82072	7.86237(a)
	0.01	1.78607	.648832	.0936101	.343932	1.57092(a)	1.91380	4.64893	4.71445(a)	7.82039	7.86255(a)
20	0.004	.285761	.207670	.0599330	.174884	1.05963	1.57088(a)	2.82438	4.71414(a)	5.19123	7.86206(a)
	0.006	.642976	.466801	.134661	.175004	1.05975	1.57090(a)	2.82447	4.71414(a)	5.19127	7.86206(a)
	0.008	1.14310	.828735	.238931	.175173	1.05992	1.57093(a)	2.82460	4.71415(a)	5.19134	7.86207(a)
	0.01	1.78615	1.29263	.372394	.175389	1.06014	1.57097(a)	2.82476	4.71416(a)	5.19142	7.86207(a)
50	0.004	.285776	.516920	.372289	.0705071	.438152	1.21558	1.57089(a)	2.35205	3.82673	4.71414(a)
	0.006	.643045	1.15569	.830125	.0708225	.438516	1.21591	1.57097(a)	2.35243	3.82706	4.71416(a)
	0.008	1.14331	2.03651	1.45737	.0712644	.439022	1.21632	1.57115(a)	2.35296	3.82753	4.71419(a)
	0.01	1.78662	3.14662	2.24101	.0718327	.439667	1.21677	1.57149(a)	2.35367	3.82812	4.71425(a)
100	0.004	.285821	1.01811	1.45714	.0356599	.220585	.615190	1.20103	1.57109	1.97691	2.93772
	0.006	.643236	2.23459	3.16418	.0362954	.221318	.615832	1.20151	1.57189	1.97795	2.93855
	0.008	1.14376	3.84233	5.35981	.0371829	.222328	.616599	1.20183	1.57374	1.97965	2.93973
	0.01	1.78734	5.76488	7.88899	.0383144	.223606	.617430	1.20185	1.57696	1.98213	2.94126
500	0.002	.0714904	1.15266	7.88670	.00766574	.0447961	.123986	.242326	.400093	.597237	.833677
	0.004	.285616	3.54846	20.9805	.00940468	.0469252	.125366	.243308	.400674	.597445	.833320
	0.006	.641279	6.11430	30.0040	.0117623	.0504960	.128400	.245619	.402257	.598330	.833313
	0.008	1.13885	8.65675	35.2837	.0143953	.0552836	.133486	.250091	.405913	.601122	.835091
1000	0.002	.0714036	1.77418	20.9799	.00470244	.0234641	.0626937	.121696	.200456	.299002	.417269
	0.004	.284703	4.32794	35.2810	.00719798	.0276442	.0667603	.125121	.203192	.301152	.418895
	0.006	.641188	6.85030	40.5251	.00999761	.0336000	.0742047	.132746	.210299	.307548	.424599
	0.008	1.14344	9.36264	42.9987	.0128846	.0404149	.0836039	.143625	.221421	.318236	.434627

表三十四 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=45^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.248001	.146941	.0212090	.343713	1.57087(a)	1.91366	4.64929	4.71418(a)	7.82118	7.86214(a)
	0.006	.558010	.330565	.0477086	.343754	1.57088(a)	1.91369	4.64921	4.71425(a)	7.82100	7.86224(a)
	0.008	.992037	.587540	.0847861	.343811	1.57090(a)	1.91373	4.64909	4.71433(a)	7.82076	7.86237(a)
	0.01	1.55010	.917769	.132420	.343884	1.57092(a)	1.91378	4.64895	4.71445(a)	7.82045	7.86255(a)
20	0.004	.248005	.293732	.0847741	.174867	1.05962	1.57088(a)	2.82438	4.71414(a)	5.19123	7.86206(a)
	0.006	.558031	.660372	.190520	.174967	1.05972	1.57090(a)	2.82445	4.71414(a)	5.19127	7.86206(a)
	0.008	.992104	1.17269	.338153	.175107	1.05987	1.57093(a)	2.82456	4.71415(a)	5.19132	7.86206(a)
	0.01	1.55026	1.82971	.527260	.175287	1.06006	1.57097(a)	2.82471	4.71416(a)	5.19139	7.86206(a)
50	0.004	.248034	.731720	.527129	.0704654	.438115	1.21554	1.57091(a)	2.35202	3.82669	4.71414(a)
	0.006	.558170	1.63777	1.17709	.0707307	.438431	1.21578	1.57104(a)	2.35236	3.82698	4.71417(a)
	0.008	.992520	2.89042	2.07049	.0711059	.438867	1.21602	1.57138(a)	2.35286	3.82738	4.71422(a)
	0.01	1.55120	4.47432	3.19124	.0715945	.439418	1.21618	1.57205(a)	2.35355	3.82787	4.71433(a)
100	0.004	.248125	1.44503	2.07020	.0355805	.220507	.615076	1.20086	1.57130(a)	1.97689	2.93765
	0.006	.558553	3.18433	4.51787	.0361312	.221131	.615479	1.20089	1.57292(a)	1.97807	2.93839
	0.008	.993424	5.50164	7.69684	.0369233	.221973	.615765	1.20024	1.57670(a)	1.98022	2.93946
	0.01	1.55261	8.29579	11.3931	.0379644	.223022	.615795	1.19858	1.58334(a)	1.98367	2.94090
500	0.003	.139610	3.33394	21.4531	.00832692	.0454066	.123645	.241572	.398947	.595752	.831679
	0.004	.247562	5.19859	30.7060	.00930348	.0464778	.123670	.240997	.397844	.594204	.829546
	0.005	.385611	7.10727	38.1086	.0104433	.0479552	.124184	.240736	.396936	.592763	.827492
	0.006	.553756	9.01112	43.7125	.0116840	.0498366	.125387	.241080	.396574	.591815	.825942
1000	0.003	.138435	4.50532	43.7104	.00584218	.0249203	.0627121	.120621	.198530	.296494	.414262
	0.004	.245339	6.39048	51.0738	.00716734	.0273072	.0649396	.122034	.199241	.296668	.414017
	0.005	.383287	8.26164	55.4241	.00855346	.0301769	.0682729	.124779	.201274	.298110	.415002
	0.006	.553007	10.1272	58.2072	.00997229	.0333560	.0724036	.128708	.204555	.300774	.417182

表三十五 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=60^\circ$)

η	k	$\varepsilon_0 (10^{-4})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.198793	.179973	.0259771	.343703	1.57087(a)	1.91366	4.64929	4.71418(a)	7.82119	7.86214(a)
	0.006	.447293	.404895	.0584381	.343731	1.57088(a)	1.91368	4.64922	4.71425(a)	7.82103	7.86224(a)
	0.008	.795210	.719706	.103864	.343771	1.57090(a)	1.91372	4.64911	4.71433(a)	7.82080	7.86237(a)
	0.01	1.24256	1.12432	.162236	.343822	1.57092(a)	1.91376	4.64897	4.71445(a)	7.82052	7.86255(a)
20	0.004	.198799	.359815	.103852	.174845	1.05960	1.57088(a)	2.82437	4.71414(a)	5.19122	7.86206(a)
	0.006	.447324	.809133	.233468	.174918	1.05969	1.57090(a)	2.82443	4.71414(a)	5.19125	7.86206(a)
	0.008	.795309	1.43733	.414557	.175021	1.05980	1.57093(a)	2.82452	4.71415(a)	5.19130	7.86206(a)
	0.01	1.24280	2.24357	.646744	.175154	1.05995	1.57098(a)	2.82464	4.71416(a)	5.19136	7.86205(a)
50	0.004	.198842	.897266	.646613	.0704108	.438067	1.21548	1.57092(a)	2.35198	3.82665	4.71415(a)
	0.006	.447534	2.01131	1.44665	.0706099	.438321	1.21562	1.57112(a)	2.35227	3.82688	4.71418(a)
	0.008	.795936	3.55683	2.55122	.0708963	.438668	1.21568	1.57162(a)	2.35272	3.82719	4.71426(a)
	0.01	1.24423	5.51964	3.94454	.0712769	.439099	1.21550	1.57262(a)	2.35337	3.82756	4.71442(a)
100	0.004	.198980	1.77823	2.55092	.0354753	.220405	.614938	1.20067	1.57153(a)	1.97685	2.93755
	0.006	.448113	3.93982	5.60471	.0359105	.220891	.615063	1.20019	1.57400(a)	1.97815	2.93818
	0.008	.797297	6.85215	9.62555	.0365677	.221521	.614785	1.19846	1.57993(a)	1.98079	2.93911
	0.01	1.24628	10.4061	14.3658	.0374757	.222275	.613860	1.19495	1.59043(a)	1.98539	2.94039
500	0.001	.0124417	.551847	3.94359	.00713528	.0441953	.123472	.241825	.399619	.596790	.833288
	0.002	.0498478	2.08080	14.3625	.00749782	.0445298	.123336	.241523	.399149	.596174	.832430
	0.003	.111919	4.25027	27.4813	.00818433	.0450453	.122528	.240164	.397282	.593888	.829537
	0.004	.197757	6.69837	39.5492	.00915509	.0458747	.121541	.238260	.394619	.590640	.825581
1000	0.001	.0124619	1.04039	14.3624	.00374895	.0222661	.0616767	.120794	.199664	.298290	.416636
	0.002	.0494384	3.34911	39.5483	.00457764	.0229389	.0607865	.119196	.197503	.295781	.413801
	0.003	.109668	5.85810	55.9504	.00578511	.0244655	.0608303	.118068	.195568	.293314	.410915
	0.004	.193409	8.33265	64.7602	.00712348	.0268352	.0626337	.118602	.195239	.292390	.409558

表三十六 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=75^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	1.41488	.200743	.0289755	.343692	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82120	7.86214(a)
	0.006	3.18358	.451651	.0651887	.343705	1.57088(a)	1.91367	4.64923	4.71425(a)	7.82106	7.86224(a)
	0.008	5.65995	.802882	.115875	.343724	1.57090(a)	1.91370	4.64912	4.71433(a)	7.82086	7.86237(a)
	0.01	8.84416	1.25440	.181024	.343749	1.57092(a)	1.91373	4.64899	4.71445(a)	7.82060	7.86255(a)
20	0.004	1.41496	.401411	.115865	.174820	1.05958	1.57088(a)	2.82435	4.71414(a)	5.19121	7.86206(a)
	0.006	3.18398	.902921	.260568	.174862	1.05964	1.57090(a)	2.82440	4.71414(a)	5.19124	7.86206(a)
	0.008	5.66119	1.60456	.462908	.174921	1.05973	1.57093(a)	2.82447	4.71415(a)	5.19128	7.86206(a)
	0.01	8.84715	2.50583	.722637	.174997	1.05983	1.57098(a)	2.82456	4.71415(a)	5.19132	7.86205(a)
50	0.004	1.41550	1.00220	.722528	.0703470	.438011	1.21542	1.57093(a)	2.35193	3.82660	4.71415(a)
	0.006	3.18659	2.25048	1.62014	.0704678	.438195	1.21546	1.57118(a)	2.35216	3.82677	4.71419(a)
	0.008	5.66902	3.98939	2.86597	.0706477	.438440	1.21534	1.57181(a)	2.35254	3.82698	4.71429(a)
	0.01	8.86496	6.20953	4.44812	.0708965	.438735	1.21486	1.57307(a)	2.35311	3.82722	4.71449(a)
100	0.004	1.41722	1.99454	2.86571	.0353506	.220289	.614793	1.20047	1.57170(a)	1.97677	2.93743
	0.006	3.19387	4.44822	6.34880	.0356431	.220619	.614646	1.19956	1.57488(a)	1.97813	2.93792
	0.008	5.68599	7.80121	11.0163	.0361261	.221012	.613820	1.19683	1.58269(a)	1.98115	2.93867
	0.01	8.88954	11.9584	16.6242	.0368527	.221430	.611928	1.19161	1.59680(a)	1.98682	2.93976
500	0.001	.0886451	.620857	4.44731	.00709709	.0441582	.123422	.241770	.399559	.596726	.833217
	0.002	.355552	2.39131	16.6211	.00737310	.0443605	.122979	.241098	.398660	.595634	.831812
	0.0025	.554825	3.61686	24.4522	.00763597	.0444769	.122296	.240118	.397400	.594146	.829984
	0.003	.795804	4.99501	32.5427	.00799298	.0446213	.121330	.238719	.395619	.592070	.827510
1000	0.001	.0888877	1.19565	16.6210	.00368659	.0221814	.0614983	.120583	.199426	.298036	.416365
	0.002	.348595	3.99716	47.2665	.00447598	.0225720	.0596109	.117788	.195925	.294117	.412061
	0.0025	.534911	5.54461	58.4738	.00506404	.0230877	.0588892	.116401	.194095	.292013	.409741
	0.003	.757671	7.08424	66.3697	.00570761	.0238941	.0587470	.115531	.192783	.290414	.407928

表三十七 旋轉傾斜梁在不同細長比與不同轉速下的振動頻率($r=1.5$, $\alpha=90^\circ$)

η	k	$\varepsilon_0 (10^{-5})$	$\varepsilon_b (10^{-3})$	$\frac{V_{tip}}{L_T} (10^{-2})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7
10	0.004	.799927	.207835	.0299998	.343679	1.57087(a)	1.91365	4.64930	4.71418(a)	7.82122	7.86214(a)
	0.006	1.79993	.467638	.0674988	.343677	1.57088(a)	1.91366	4.64923	4.71425(a)	7.82109	7.86224(a)
	0.008	3.20011	.831377	.119996	.343674	1.57090(a)	1.91368	4.64914	4.71433(a)	7.82091	7.86237(a)
	0.01	5.00064	1.29907	.187490	.343671	1.57092(a)	1.91370	4.64902	4.71445(a)	7.82068	7.86255(a)
20	0.004	.800011	.415670	.119990	.174793	1.05956	1.57088(a)	2.82434	4.71414(a)	5.19121	7.86206(a)
	0.006	1.80035	.935273	.269947	.174801	1.05960	1.57090(a)	2.82437	4.71414(a)	5.19122	7.86206(a)
	0.008	3.20144	1.66274	.479828	.174813	1.05965	1.57093(a)	2.82441	4.71415(a)	5.19125	7.86206(a)
	0.01	5.00385	2.59808	.749570	.174829	1.05971	1.57098(a)	2.82447	4.71415(a)	5.19128	7.86205(a)
50	0.004	.800586	1.03915	.749498	.0702781	.437952	1.21536	1.57094(a)	2.35187	3.82654	4.71415(a)
	0.006	1.80316	2.33792	1.68475	.0703135	.438061	1.21532	1.57120(a)	2.35204	3.82664	4.71419(a)
	0.008	3.20987	4.15539	2.99040	.0703752	.438200	1.21505	1.57188(a)	2.35233	3.82676	4.71430(a)
	0.01	5.02306	6.48969	4.66122	.0704751	.438357	1.21436	1.57328(a)	2.35279	3.82687	4.71452(a)
100	0.004	.802448	2.07760	2.99023	.0352140	.220167	.614656	1.20032	1.57177(a)	1.97667	2.93730
	0.006	1.81104	4.66797	6.68756	.0353429	.220337	.614290	1.19909	1.57532(a)	1.97797	2.93764
	0.008	3.22810	8.26754	11.7482	.0356146	.220490	.613028	1.19568	1.58431(a)	1.98111	2.93818
	0.01	5.04783	12.8203	17.9801	.0361062	.220567	.610327	1.18916	1.60109(a)	1.98747	2.93903
500	0.001	.0502277	.648908	4.66066	.00705477	.0441193	.123377	.241721	.399507	.596672	.833160
	0.002	.201893	2.56381	17.9777	.00722366	.0441871	.122676	.240754	.398276	.595220	.831351
	0.0025	.314261	3.93783	26.9195	.00743089	.0441828	.121667	.239394	.396589	.593272	.829019
	0.003	.447767	5.51810	36.3646	.00774664	.0441672	.120206	.237434	.394187	.590546	.825864
1000	0.001	.0504732	1.28190	17.9776	.00361187	.0220947	.0613474	.120412	.199237	.297838	.416158
	0.002	.191177	4.51428	53.6436	.00434099	.0221568	.0584298	.116472	.194500	.292648	.410551
	0.0025	.283431	6.34013	66.3355	.00494287	.0225120	.0571632	.114502	.192068	.289934	.407619
	0.003	.388229	8.15303	74.7254	.00560584	.0232181	.0566295	.113203	.190324	.287893	.405356

表三十八 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=38$)

α	k	K_1	K_2	$K_3 (a)$	K_4	K_5	$K_6 (a)$
0°	0	.0923781	.573449	1.57086	1.58182	3.03502	4.71413
	0.01	.0933104	.574596	1.57093	1.58297	3.03606	4.71416
	0.02	.0960486	.578022	1.57116	1.58637	3.03918	4.71424
	0.03	.100433	.583687	1.57158	1.59197	3.04436	4.71438
	0.04	.106241	.591528	1.57221	1.59975	3.05162	4.71457
	0.05	.113232	.601458	1.57305	1.60967	3.06092	4.71483
	0.06	.121175	.613377	1.57410	1.62169	3.07226	4.71514
5°	0.01	.0933074	.574593	1.57087	1.58302	3.03606	4.71416
	0.02	.0960381	.578009	1.57056	1.58697	3.03917	4.71423
	0.025	.0980332	.580557	1.57023	1.59003	3.04150	4.71427
	0.03	.100414	.583656	1.56983	1.59376	3.04435	4.71432
10°	0.005	.0926090	.573734	1.57086	1.58212	3.03528	4.71414
	0.01	.0932982	.574585	1.57071	1.58318	3.03605	4.71416
	0.015	.0944361	.575999	1.57013	1.58528	3.03734	4.71417
	0.02	.0960067	.577971	1.56900	1.58854	3.03914	4.71418
15°	0.004	.0925234	.573629	1.57086	1.58202	3.03518	4.71414
	0.006	.0927048	.573854	1.57082	1.58230	3.03539	4.71414
	0.008	.0929582	.574168	1.57070	1.58275	3.03567	4.71415
	0.01	.0932831	.574571	1.57046	1.58343	3.03604	4.71415
30°	0.004	0.0925101	.573618	1.57082	1.58205	3.03517	4.71414
	0.006	.0926750	.573828	1.57063	1.58246	3.03536	4.71414
	0.008	.0929059	.574121	1.57016	1.58326	3.03563	4.71414
	0.01	.0932029	.574496	1.56930	1.58454	3.03598	4.71413
45°	0.004	.0924058	.573487	1.57086	1.58187	3.03505	4.71413
	0.006	.0924888	.573600	1.57077	1.58208	3.03516	4.71414
	0.008	.0926275	.573787	1.57038	1.58268	3.03533	4.71413
	0.01	.0928224	.574047	1.56948	1.58389	3.03557	4.71412
60°	0.004	.0924610	.573576	1.57071	1.58211	3.03514	4.71413
	0.006	.0925653	.573733	1.57014	1.58287	3.03528	4.71413
	0.008	.0927130	.573951	1.56884	1.58445	3.03549	4.71411
	0.01	.0929054	.574227	1.56671	1.58695	3.03576	4.71405
75°	0.004	.0924285	.573549	1.57068	1.58213	3.03511	4.71413
	0.006	.0924927	.573671	1.56997	1.58299	3.03523	4.71413
	0.008	.0925846	.573840	1.56839	1.58481	3.03540	4.71410
	0.01	.0927063	.574053	1.56586	1.58767	3.03561	4.71403
90°	0.004	.0923937	.573519	1.57066	1.58212	3.03509	4.71413
	0.006	.0924144	.573605	1.56990	1.58300	3.03517	4.71412
	0.008	.0924457	.573722	1.56821	1.58489	3.03529	4.71409
	0.01	.0924900	.573868	1.56551	1.58787	3.03545	4.71402

表三十九 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=38.5$)

α	k	K_1	K_2	K_3	K_4	K_5	$K_6 (a)$
0°	0	.0911822	.566163	1.56231	1.57086(a)	2.99909	4.71413
	0.01	.0921270	.567326	1.56340	1.57100(a)	3.00015	4.71416
	0.02	.0949006	.570802	1.56655	1.57153(a)	3.00333	4.71423
	0.03	.0993372	.576549	1.57034(a)	1.57387	3.00863	4.71436
	0.04	.105208	.584500	1.57179(a)	1.58095	3.01603	4.71454
	0.05	.112264	.594566	1.57282(a)	1.59085	3.02552	4.71478
	0.06	.120272	.606642	1.57394(a)	1.60299	3.03709	4.71509
5°	0.01	.0921240	.567323	1.56330	1.57110	3.00015	4.71416
	0.02	.0948900	.570789	1.56500	1.57308	3.00332	4.71421
	0.025	.0969096	.573374	1.56537	1.57549	3.00570	4.71423
	0.03	.0993184	.576517	1.56544	1.57881	3.00861	4.71426
10°	0.005	.0914162	.566451	1.56255	1.57092	2.99935	4.71414
	0.01	.0921147	.567315	1.56303	1.57136	3.00014	4.71415
	0.015	.0932676	.568750	1.56304	1.57289	3.00146	4.71416
	0.02	.0948584	.570751	1.56224	1.57586	3.00330	4.71413
15°	0.004	.0913295	.566345	1.56246	1.57090	2.99925	4.71414
	0.006	.0915133	.566573	1.56259	1.57101	2.99946	4.71414
	0.008	.0917701	.566891	1.56267	1.57127	2.99975	4.71414
	0.01	.0920994	.567300	1.56262	1.57176	3.00013	4.71414
30°	0.004	.0913159	.566333	1.56239	1.57096	2.99924	4.71414
	0.006	.0914831	.566547	1.56230	1.57128	2.99944	4.71413
	0.008	.0917172	.566844	1.56186	1.57205	2.99971	4.71412
	0.01	.0920182	.567225	1.56095	1.57339	3.00007	4.71410
45°	0.004	.0912943	.566315	1.56230	1.57103	2.99923	4.71413
	0.006	.0914350	.566505	1.56193	1.57162	2.99940	4.71413
	0.008	.0916326	.566769	1.56091	1.57294	2.99965	4.71410
	0.01	.0918881	.567106	1.55919	1.57508	2.99997	4.71404
60°	0.004	.0912662	.566291	1.56221	1.57110	2.99921	4.71413
	0.006	.0913720	.566450	1.56156	1.57194	2.99936	4.71412
	0.008	.0915218	.566671	1.56006	1.57372	2.99957	4.71407
	0.01	.0917170	.566952	1.55769	1.57648	2.99984	4.71398
75°	0.004	.0912333	.566263	1.56213	1.57115	2.99918	4.71413
	0.006	.0912984	.566388	1.56129	1.57216	2.99930	4.71411
	0.008	.0913917	.566559	1.55945	1.57424	2.99947	4.71406
	0.01	.0915154	.566775	1.55665	1.57739	2.99969	4.71393
90°	0.004	.0911980	.566234	1.56209	1.57117	2.99916	4.71413
	0.006	.0912191	.566321	1.56115	1.57223	2.99924	4.71411
	0.008	.0912510	.566439	1.55917	1.57441	2.99937	4.71405
	0.01	.0912963	.566587	1.55617	1.57770	2.99953	4.71391

表四十 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1, \eta=39$)

α	k	K_1	K_2	K_3	K_4	K_5	$K_6 (a)$
0°	0	.0900168	.559057	1.54325	1.57086(a)	2.96396	4.71413
	0.01	.0909741	.560237	1.54439	1.57097(a)	2.96504	4.71415
	0.02	.0937830	.563763	1.54780	1.57130(a)	2.96829	4.71421
	0.03	.0982717	.569591	1.55343	1.57189(a)	2.97369	4.71431
	0.04	.104204	.577652	1.56112	1.57287(a)	2.98124	4.71447
	0.05	.111325	.587853	1.56961	1.57547(a)	2.99092	4.71469
	0.06	.119397	.600085	1.57330(a)	1.58523	3.00272	4.71498
5°	0.01	.0909710	.560234	1.54436	1.57100	2.96504	4.71415
	0.02	.0937723	.563750	1.54735	1.57176	2.96828	4.71416
	0.025	.0958163	.566372	1.54928	1.57264	2.97071	4.71415
	0.03	.0982529	.569559	1.55131	1.57405	2.97368	4.71411
10°	0.005	.0902539	.559349	1.54353	1.57090	2.96423	4.71414
	0.01	.0909617	.560225	1.54427	1.57109	2.96503	4.71414
	0.015	.0921296	.561681	1.54524	1.57168	2.96637	4.71411
	0.02	.0937404	.563711	1.54610	1.57302	2.96826	4.71401
15°	0.004	.0901660	.559242	1.54342	1.57088	2.96413	4.71414
	0.006	.0903523	.559473	1.54362	1.57093	2.96434	4.71414
	0.008	.0906125	.559796	1.54386	1.57104	2.96464	4.71413
	0.01	.0909462	.560211	1.54412	1.57123	2.96502	4.71412
30°	0.004	.0901523	.559230	1.54339	1.57090	2.96412	4.71413
	0.006	.0903217	.559446	1.54351	1.57103	2.96432	4.71412
	0.008	.0905589	.559748	1.54354	1.57132	2.96460	4.71409
	0.01	.0908640	.560134	1.54340	1.57191	2.96496	4.71403
45°	0.004	.0901304	.559211	1.54335	1.57093	2.96410	4.71413
	0.006	.0902730	.559404	1.54335	1.57115	2.96428	4.71411
	0.008	.0904734	.559672	1.54311	1.57171	2.96453	4.71404
	0.01	.0907324	.560013	1.54244	1.57279	2.96486	4.71391
60°	0.004	.0901019	.559187	1.54331	1.57095	2.96408	4.71413
	0.006	.0902092	.559349	1.54318	1.57127	2.96423	4.71409
	0.008	.0903611	.559573	1.54266	1.57208	2.96445	4.71399
	0.01	.0905592	.559857	1.54149	1.57364	2.96473	4.71378
75°	0.004	.0900686	.559159	1.54326	1.57097	2.96405	4.71412
	0.006	.0901347	.559285	1.54303	1.57136	2.96418	4.71408
	0.008	.0902294	.559459	1.54229	1.57235	2.96435	4.71395
	0.01	.0903551	.559677	1.54075	1.57424	2.96458	4.71368
90°	0.004	.0900328	.559129	1.54323	1.57098	2.96403	4.71412
	0.006	.0900543	.559217	1.54293	1.57140	2.96412	4.71407
	0.008	.0900868	.559338	1.54208	1.57245	2.96424	4.71394
	0.01	.0901331	.559487	1.54036	1.57447	2.96441	4.71364

表四十一 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=40$)

α	k	K_1	K_2	K_3	$K_4 (a)$	K_5	$K_6 (a)$
0°	0	.0877730	.545362	1.50647	1.57086	2.89600	4.67451
	0.01	.0887554	.546576	1.50765	1.57096	2.89713	4.67548
	0.02	.0916345	.550201	1.51120	1.57127	2.90050	4.67838
	0.03	.0962265	.556192	1.51709	1.57178	2.90612	4.68319
	0.04	.102280	.564471	1.52529	1.57251	2.91396	4.68982
	0.05	.109529	.574941	1.53572	1.57348	2.92402	4.69803
	0.06	.117725	.587482	1.54828	1.57477	2.93628	4.70664
5°	0.01	.0887522	.546573	1.50764	1.57098	2.89712	4.67547
	0.02	.0916237	.550188	1.51100	1.57147	2.90049	4.67826
	0.025	.0937162	.552883	1.51341	1.57197	2.90302	4.68026
	0.03	.0962077	.556158	1.51623	1.57269	2.90611	4.68262
10°	0.005	.0880164	.545663	1.50676	1.57089	2.89628	4.67475
	0.01	.0887426	.546564	1.50759	1.57102	2.89712	4.67544
	0.015	.0899404	.548061	1.50886	1.57136	2.89851	4.67652
	0.02	.0915913	.550147	1.51041	1.57208	2.90047	4.67789
15°	0.004	.0879261	.545552	1.50665	1.57088	2.89618	4.67466
	0.006	.0881173	.545790	1.50687	1.57091	2.89640	4.67484
	0.008	.0883844	.546122	1.50716	1.57098	2.89671	4.67509
	0.01	.0887268	.546549	1.50752	1.57109	2.89711	4.67539
30°	0.004	.0879121	.545540	1.50663	1.57089	2.89617	4.67464
	0.006	.0880860	.545762	1.50681	1.57096	2.89638	4.67480
	0.008	.0883295	.546073	1.50699	1.57112	2.89667	4.67498
	0.01	.0886427	.546470	1.50713	1.57143	2.89704	4.67515
45°	0.004	.0878897	.545521	1.50660	1.57090	2.89615	4.67462
	0.006	.0880360	.545719	1.50671	1.57102	2.89634	4.67473
	0.008	.0882418	.545994	1.50674	1.57131	2.89660	4.67481
	0.01	.0885079	.546345	1.50660	1.57189	2.89694	4.67481
60°	0.004	.0878604	.545496	1.50657	1.57091	2.89613	4.67460
	0.006	.0879707	.545662	1.50660	1.57108	2.89629	4.67465
	0.008	.0881268	.545892	1.50647	1.57151	2.89652	4.67463
	0.01	.0883305	.546184	1.50603	1.57236	2.89681	4.67444
75°	0.004	.0878263	.545467	1.50653	1.57092	2.89610	4.67457
	0.006	.0878942	.545596	1.50649	1.57113	2.89623	4.67458
	0.008	.0879918	.545775	1.50623	1.57165	2.89641	4.67446
	0.01	.0881214	.545999	1.50554	1.57271	2.89665	4.67411
90°	0.004	.0877895	.545436	1.50650	1.57093	2.89608	4.67454
	0.006	.0878117	.545526	1.50641	1.57115	2.89617	4.67451
	0.008	.0878456	.545650	1.50606	1.57171	2.89630	4.67434
	0.01	.0878938	.545804	1.50523	1.57284	2.89648	4.67389

表四十二 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=77$)

α	k	K_1	K_2	K_3	K_4	K_5	K_6
0°	0	.0456446	.285383	.796090	1.55157	1.57086(a)	2.54705
	0.01	.0475274	.287806	.798607	1.55415	1.57097(a)	2.54962
	0.02	.0527487	.294954	.806106	1.56182	1.57132(a)	2.55733
	0.03	.0603897	.306488	.818433	1.57127(a)	1.57514	2.57012
	0.04	.0695879	.321922	.835345	1.57226(a)	1.59254	2.58791
	0.05	.0797415	.340707	.856536	1.57316(a)	1.61497	2.61060
	0.06	.0904762	.362297	.881655	1.57424(a)	1.64193	2.63804
5°	0.01	.0475221	.287800	.798593	1.55379	1.57135	2.54962
	0.02	.0527362	.294925	.805992	1.55723	1.57616	2.55735
	0.025	.0563121	.300156	.811477	1.55835	1.58123	2.56313
	0.03	.0603752	.306428	.818132	1.55934	1.58777	2.57019
10°	0.005	.0461168	.285985	.796712	1.55212	1.57098	2.54769
	0.01	.0475062	.287780	.798548	1.55281	1.57240	2.54961
	0.015	.0497366	.290742	.801540	1.55205	1.57682	2.55284
	0.02	.0526986	.294837	.805649	1.55006	1.58410	2.55739
15°	0.004	.0459424	.285764	.796483	1.55189	1.57097	2.54745
	0.006	.0463123	.286238	.796967	1.55207	1.57134	2.54796
	0.008	.0468261	.286900	.797634	1.55197	1.57225	2.54868
	0.01	.0474797	.287748	.798475	1.55139	1.57392	2.54960
30°	0.004	.0459158	.285739	.796452	1.55165	1.57121	2.54743
	0.006	.0462548	.286181	.796881	1.55100	1.57246	2.54792
	0.008	.0467292	.286795	.797440	1.54916	1.57526	2.54861
	0.01	.0473383	.287577	.798100	1.54610	1.57975	2.54952
45°	0.004	.0458733	.285700	.796405	1.55132	1.57153	2.54740
	0.006	.0461624	.286091	.796750	1.54966	1.57385	2.54784
	0.008	.0465726	.286630	.797151	1.54607	1.57863	2.54850
	0.01	.0471080	.287309	.797542	1.54082	1.58577	2.54939
60°	0.004	.0458176	.285650	.796346	1.55098	1.57184	2.54735
	0.006	.0460404	.285976	.796592	1.54841	1.57513	2.54775
	0.008	.0463639	.286418	.796809	1.54338	1.58155	2.54835
	0.01	.0467976	.286966	.796890	1.53643	1.59090	2.54919
75°	0.004	.0457522	.285592	.796281	1.55072	1.57207	2.54729
	0.006	.0458960	.285842	.796426	1.54749	1.57602	2.54762
	0.008	.0461138	.286176	.796464	1.54148	1.58356	2.54815
	0.01	.0464205	.286578	.796249	1.53332	1.59449	2.54891
90°	0.004	.0456815	.285530	.796216	1.55058	1.57215	2.54723
	0.006	.0457381	.285702	.796271	1.54705	1.57636	2.54749
	0.008	.0458366	.285923	.796162	1.54056	1.58439	2.54791
	0.01	.0459954	.286176	.795716	1.53175	1.59611	2.54856

表四十三 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=78$)

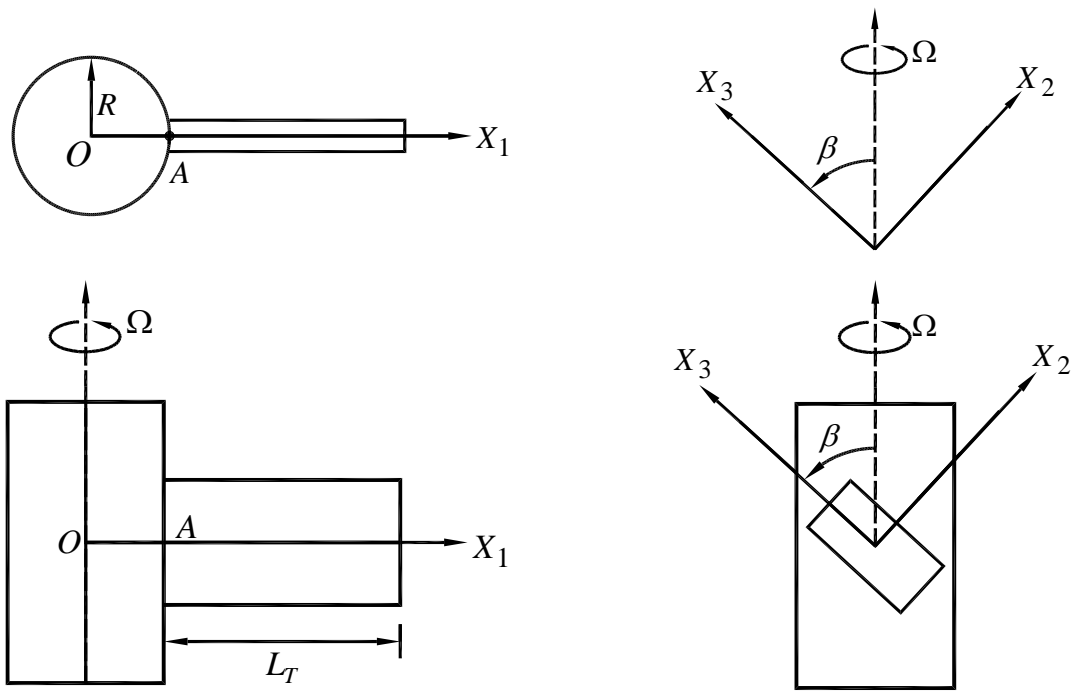
α	k	K_1	K_2	K_3	K_4	K_5	K_6
0°	0	.0450599	.281744	.786013	1.53214	1.57086(a)	2.51558
	0.01	.0469663	.284199	.788565	1.53476	1.57096(a)	2.51820
	0.02	.0522436	.291438	.796166	1.54258	1.57127(a)	2.52602
	0.03	.0599473	.303111	.808656	1.55549	1.57182(a)	2.53901
	0.04	.0692007	.318714	.825783	1.57125(a)	1.57470	2.55707
	0.05	.0793981	.337682	.847230	1.57309(a)	1.59649	2.58010
	0.06	.0901664	.359459	.872635	1.57421(a)	1.62377	2.60794
5°	0.01	.0469609	.284193	.788550	1.53458	1.57116	2.51819
	0.02	.0522311	.291409	.796050	1.54038	1.57374	2.52604
	0.025	.0558388	.296704	.801610	1.54380	1.57659	2.53192
	0.03	.0599330	.303050	.808353	1.54733	1.58068	2.53909
10°	0.005	.0455382	.282354	.786643	1.53274	1.57094	2.51623
	0.01	.0469449	.284173	.788504	1.53405	1.57176	2.51819
	0.015	.0492011	.287173	.791537	1.53504	1.57449	2.52147
	0.02	.0521936	.291319	.795702	1.53521	1.57969	2.52609
15°	0.004	.0453615	.282130	.786411	1.53251	1.57093	2.51599
	0.006	.0457362	.282610	.786902	1.53285	1.57114	2.51651
	0.008	.0462565	.283281	.787578	1.53314	1.57168	2.51724
	0.01	.0469182	.284140	.788430	1.53323	1.57270	2.51818
30°	0.004	.0453346	.282105	.786380	1.53237	1.57106	2.51597
	0.006	.0456780	.282553	.786814	1.53223	1.57181	2.51647
	0.008	.0461587	.283175	.787380	1.53140	1.57364	2.51717
	0.01	.0467756	.283966	.788046	1.52955	1.57694	2.51811
45°	0.004	.0452916	.282065	.786331	1.53217	1.57125	2.51593
	0.006	.0455846	.282462	.786680	1.53139	1.57270	2.51639
	0.008	.0460005	.283007	.787084	1.52918	1.57613	2.51707
	0.01	.0465433	.283694	.787476	1.52523	1.58203	2.51798
60°	0.004	.0452352	.282014	.786271	1.53196	1.57144	2.51589
	0.006	.0454612	.282344	.786519	1.53055	1.57358	2.51629
	0.008	.0457895	.282791	.786735	1.52706	1.57850	2.51691
	0.01	.0462300	.283345	.786810	1.52133	1.58670	2.51779
75°	0.004	.0451690	.281955	.786205	1.53178	1.57158	2.51583
	0.006	.0453150	.282209	.786350	1.52988	1.57422	2.51617
	0.008	.0455366	.282546	.786383	1.52546	1.58023	2.51672
	0.01	.0458490	.282951	.786155	1.51844	1.59012	2.51752
90°	0.004	.0450974	.281892	.786139	1.53167	1.57163	2.51577
	0.006	.0451551	.282067	.786193	1.52953	1.57448	2.51603
	0.008	.0452560	.282290	.786076	1.52463	1.58097	2.51648
	0.01	.0454190	.282544	.785612	1.51691	1.59171	2.51717

表四十四 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=79$)

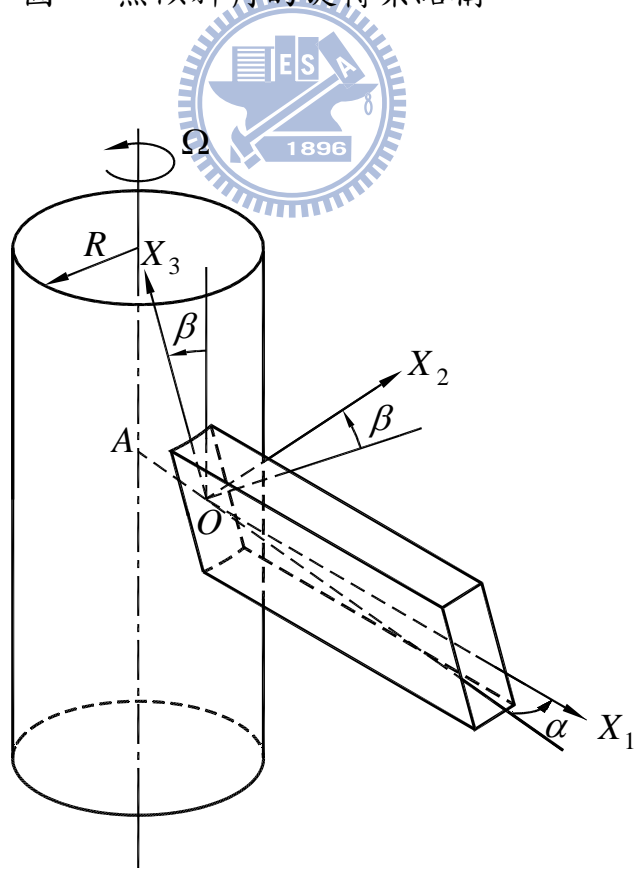
α	k	K_1	K_2	K_3	K_4	K_5	K_6
0°	0	.0444900	.278196	.776186	1.51318	1.57086(a)	2.48487
	0.01	.0464199	.280683	.778773	1.51584	1.57096(a)	2.48752
	0.02	.0517527	.288014	.786476	1.52379	1.57126(a)	2.49547
	0.03	.0595184	.299824	.799129	1.53692	1.57177(a)	2.50865
	0.04	.0688257	.315595	.816471	1.55502	1.57255(a)	2.52699
	0.05	.0790658	.334745	.838172	1.57256(a)	1.57894	2.55035
	0.06	.0898666	.356704	.863859	1.57415(a)	1.60610	2.57859
5°	0.01	.0464145	.280677	.778757	1.51571	1.57111	2.48752
	0.02	.0517403	.287984	.786358	1.52233	1.57298	2.49549
	0.025	.0553795	.293343	.791992	1.52673	1.57495	2.50146
	0.03	.0595043	.299763	.798825	1.53171	1.57768	2.50875
10°	0.005	.0449744	.278814	.776825	1.51381	1.57093	2.48553
	0.01	.0463983	.280656	.778711	1.51534	1.57155	2.48752
	0.015	.0486801	.283695	.781783	1.51706	1.57362	2.49085
	0.02	.0517030	.287893	.786005	1.51848	1.57763	2.49555
15°	0.004	.0447954	.278587	.776589	1.51357	1.57091	2.48529
	0.006	.0451749	.279074	.777087	1.51397	1.57108	2.48581
	0.008	.0457017	.279753	.777771	1.51440	1.57149	2.48655
	0.01	.0463715	.280623	.778635	1.51475	1.57227	2.48751
30°	0.004	.0447682	.278562	.776557	1.51346	1.57102	2.48526
	0.006	.0451161	.279015	.776997	1.51351	1.57159	2.48577
	0.008	.0456029	.279645	.777569	1.51311	1.57301	2.48649
	0.01	.0462277	.280447	.778243	1.51193	1.57567	2.48744
45°	0.004	.0447247	.278522	.776509	1.51331	1.57116	2.48523
	0.006	.0450216	.278923	.776861	1.51288	1.57228	2.48570
	0.008	.0454431	.279475	.777268	1.51138	1.57503	2.48639
	0.01	.0459935	.280170	.777660	1.50837	1.58004	2.48733
60°	0.004	.0446676	.278470	.776447	1.51314	1.57130	2.48518
	0.006	.0448967	.278804	.776697	1.51222	1.57298	2.48560
	0.008	.0452299	.279256	.776911	1.50965	1.57703	2.48624
	0.01	.0456773	.279817	.776980	1.50496	1.58426	2.48715
75°	0.004	.0446006	.278410	.776380	1.51300	1.57141	2.48512
	0.006	.0447488	.278667	.776525	1.51169	1.57350	2.48547
	0.008	.0449742	.279007	.776553	1.50828	1.57853	2.48604
	0.01	.0452925	.279416	.776311	1.50232	1.58745	2.48688
90°	0.004	.0445280	.278346	.776314	1.51290	1.57145	2.48506
	0.006	.0445869	.278522	.776365	1.51137	1.57371	2.48533
	0.008	.0446902	.278748	.776240	1.50753	1.57920	2.48580
	0.01	.0448576	.279003	.775757	1.50088	1.58898	2.48653

表四十五 旋轉梁在不同傾斜角與不同轉速下的振動頻率($r=1$, $\eta=80$)

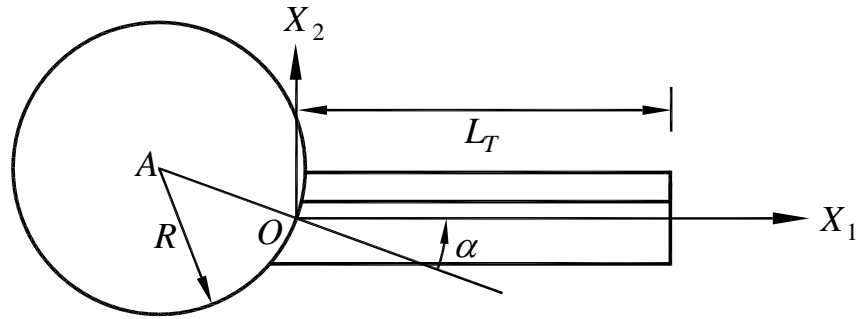
α	k	K_1	K_2	K_3	K_4	K_5	K_6
0°	0	.0439342	.274736	.766600	1.49468	1.57086(a)	2.45488
	0.01	.0458877	.277256	.769222	1.49738	1.57096(a)	2.45758
	0.02	.0512757	.284677	.777027	1.50544	1.57125(a)	2.46564
	0.03	.0591023	.296624	.789842	1.51876	1.57175(a)	2.47902
	0.04	.0684624	.312561	.807398	1.53718	1.57247(a)	2.49763
	0.05	.0787439	.331891	.829353	1.56032	1.57356(a)	2.52132
	0.06	.0895762	.354031	.855320	1.57395(a)	1.58903	2.54996
5°	0.01	.0458823	.277249	.769206	1.49728	1.57108	2.45758
	0.02	.0512633	.284647	.776907	1.50433	1.57264	2.46567
	0.025	.0549336	.290070	.782615	1.50923	1.57419	2.47173
	0.03	.0590884	.296564	.789537	1.51496	1.57627	2.47913
10°	0.005	.0444248	.275362	.767247	1.49532	1.57092	2.45555
	0.01	.0458660	.277228	.769158	1.49698	1.57146	2.45757
	0.015	.0481732	.280306	.772271	1.49909	1.57319	2.46096
	0.02	.0512261	.284555	.776549	1.50126	1.57654	2.46574
15°	0.004	.0442436	.275132	.767009	1.49508	1.57091	2.45531
	0.006	.0446278	.275626	.767513	1.49552	1.57105	2.45584
	0.008	.0451611	.276314	.768206	1.49602	1.57140	2.45659
	0.01	.0458389	.277194	.769081	1.49650	1.57206	2.45756
30°	0.004	.0442160	.275107	.766977	1.49499	1.57100	2.45529
	0.006	.0445684	.275566	.767421	1.49514	1.57148	2.45580
	0.008	.0450614	.276204	.768000	1.49496	1.57269	2.45653
	0.01	.0456941	.277015	.768681	1.49418	1.57499	2.45751
45°	0.004	.0441720	.275066	.766927	1.49486	1.57112	2.45525
	0.006	.0444728	.275472	.767283	1.49461	1.57207	2.45573
	0.008	.0449000	.276031	.767692	1.49351	1.57445	2.45643
	0.01	.0454579	.276734	.768085	1.49113	1.57889	2.45741
60°	0.004	.0441142	.275014	.766865	1.49472	1.57124	2.45520
	0.006	.0443466	.275351	.767116	1.49406	1.57267	2.45563
	0.008	.0446846	.275809	.767329	1.49203	1.57622	2.45629
	0.01	.0451389	.276376	.767391	1.48810	1.58277	2.45724
75°	0.004	.0440464	.274953	.766797	1.49459	1.57133	2.45514
	0.006	.0441969	.275213	.766941	1.49359	1.57312	2.45550
	0.008	.0444261	.275557	.766963	1.49082	1.57757	2.45609
	0.01	.0447503	.275970	.766707	1.48569	1.58578	2.45698
90°	0.004	.0439729	.274888	.766729	1.49450	1.57136	2.45508
	0.006	.0440330	.275066	.766778	1.49330	1.57331	2.45536
	0.008	.0441387	.275294	.766645	1.49013	1.57818	2.45585
	0.01	.0443105	.275551	.766141	1.48431	1.58725	2.45663



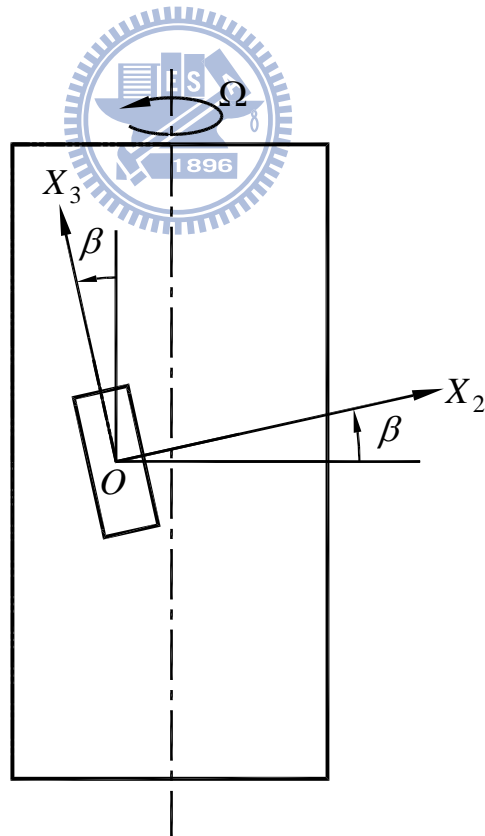
圖一 無傾斜角的旋轉梁結構



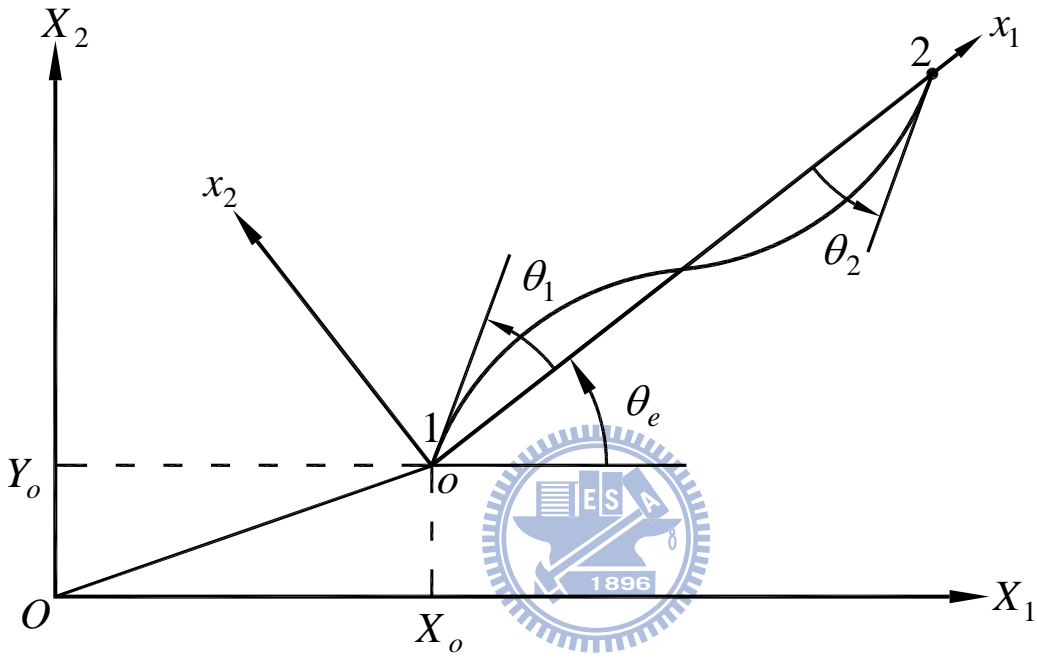
圖二 具傾斜角的旋轉梁結構



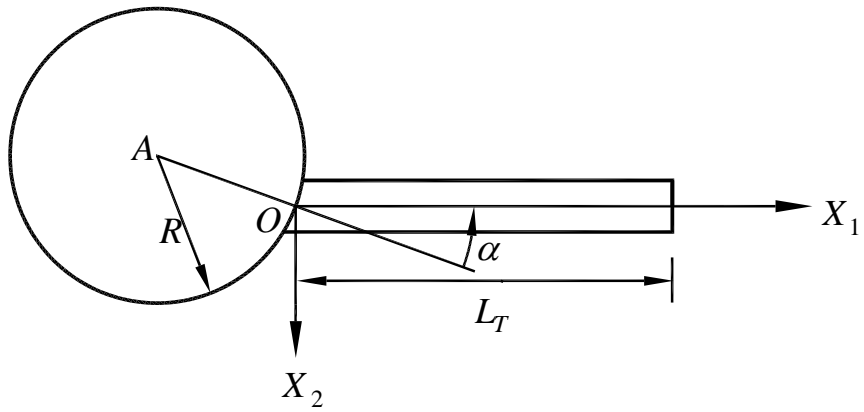
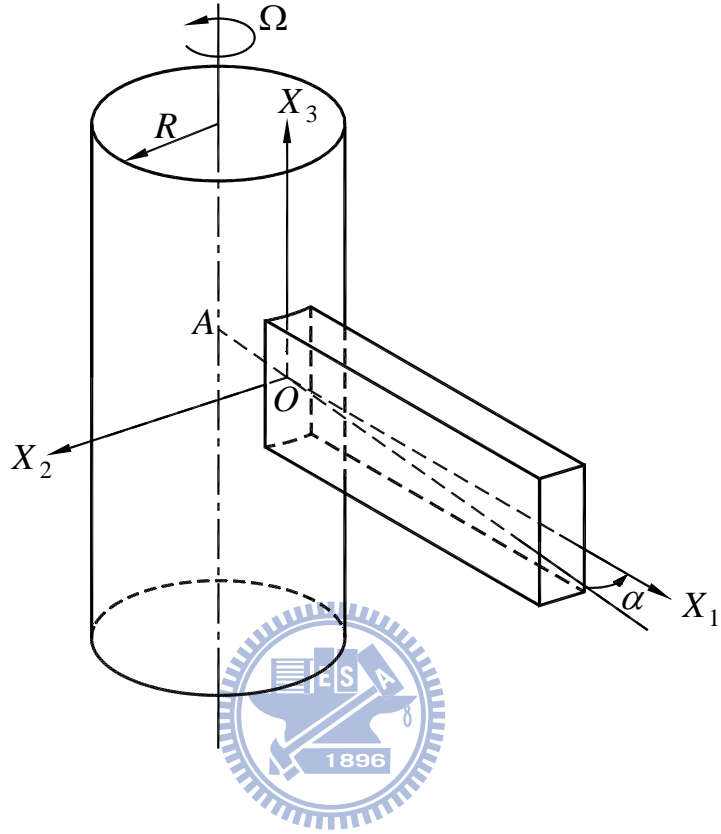
圖三 旋轉傾斜梁的上視圖



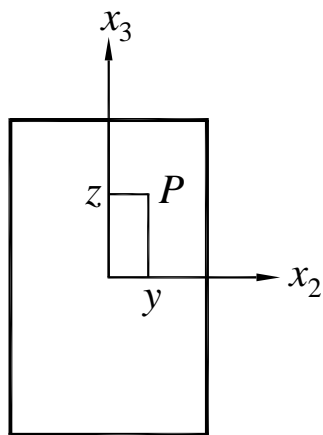
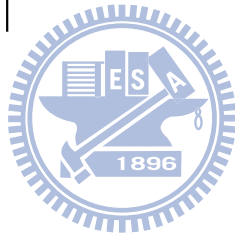
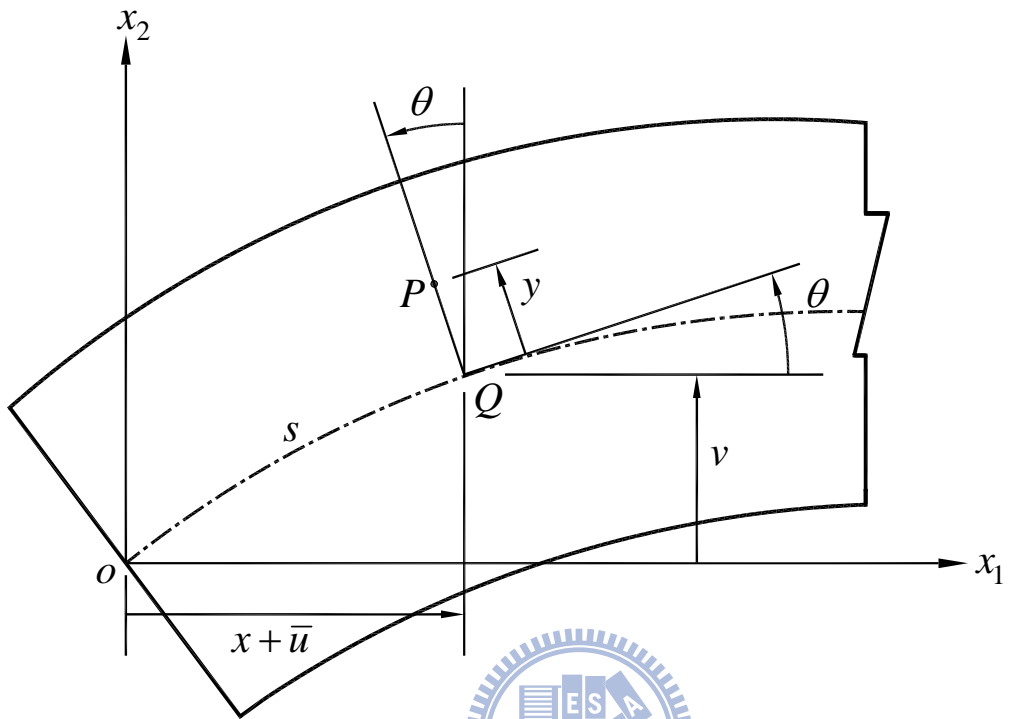
圖四 旋轉傾斜梁的側視圖



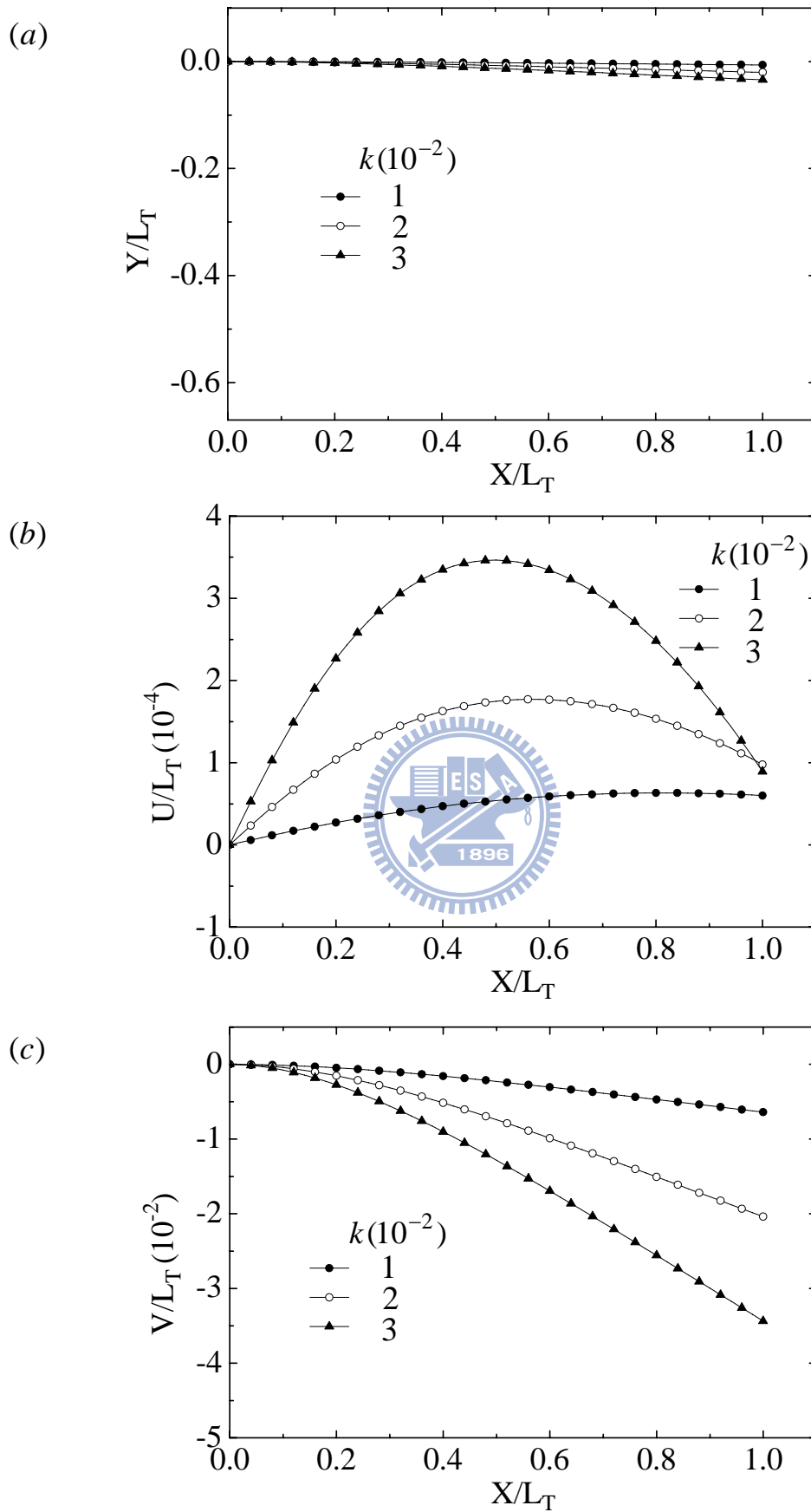
圖五 元素座標及總體座標關係圖



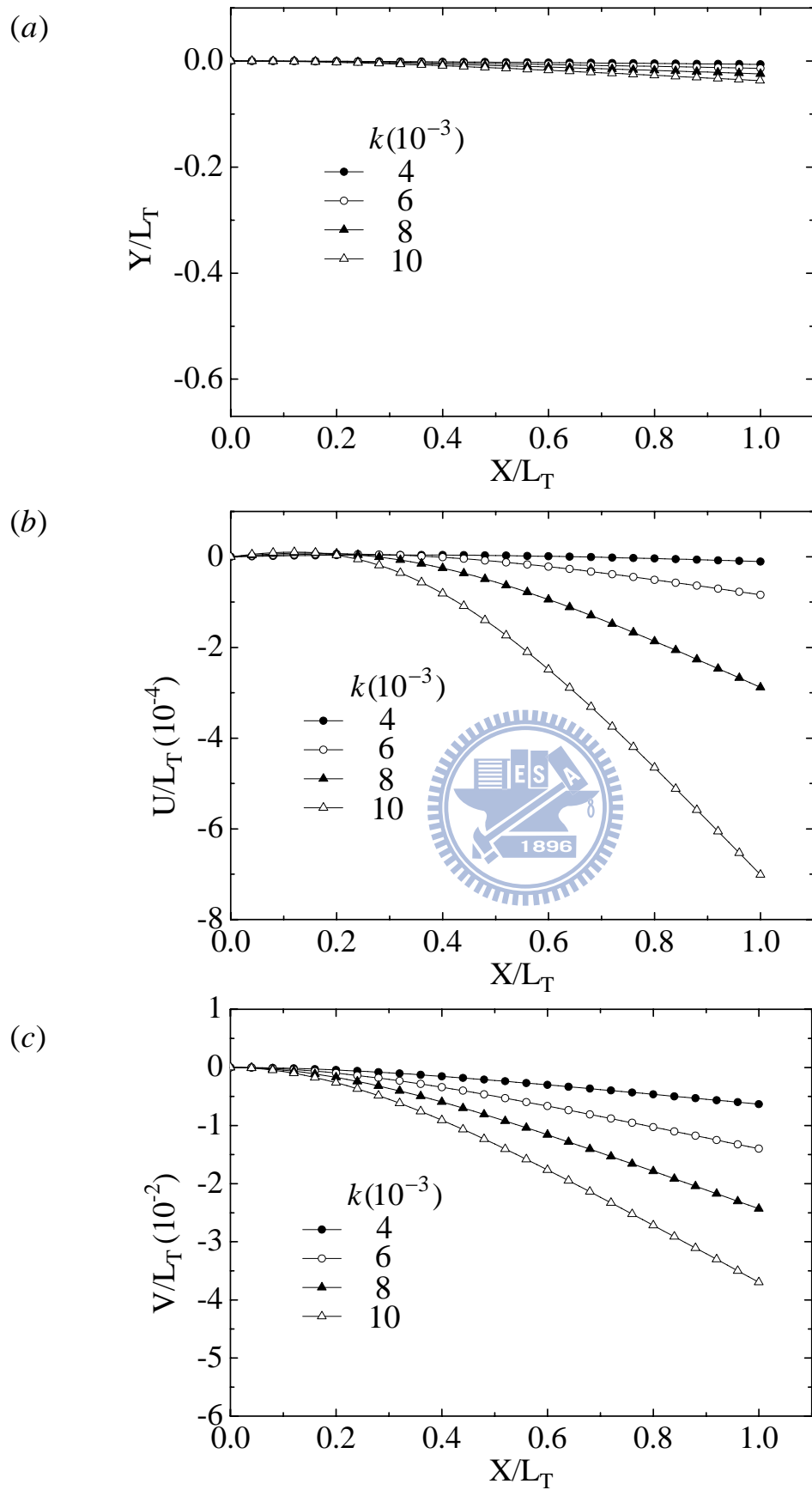
圖六 具傾斜角的旋轉梁結構($\beta = 0^\circ$)



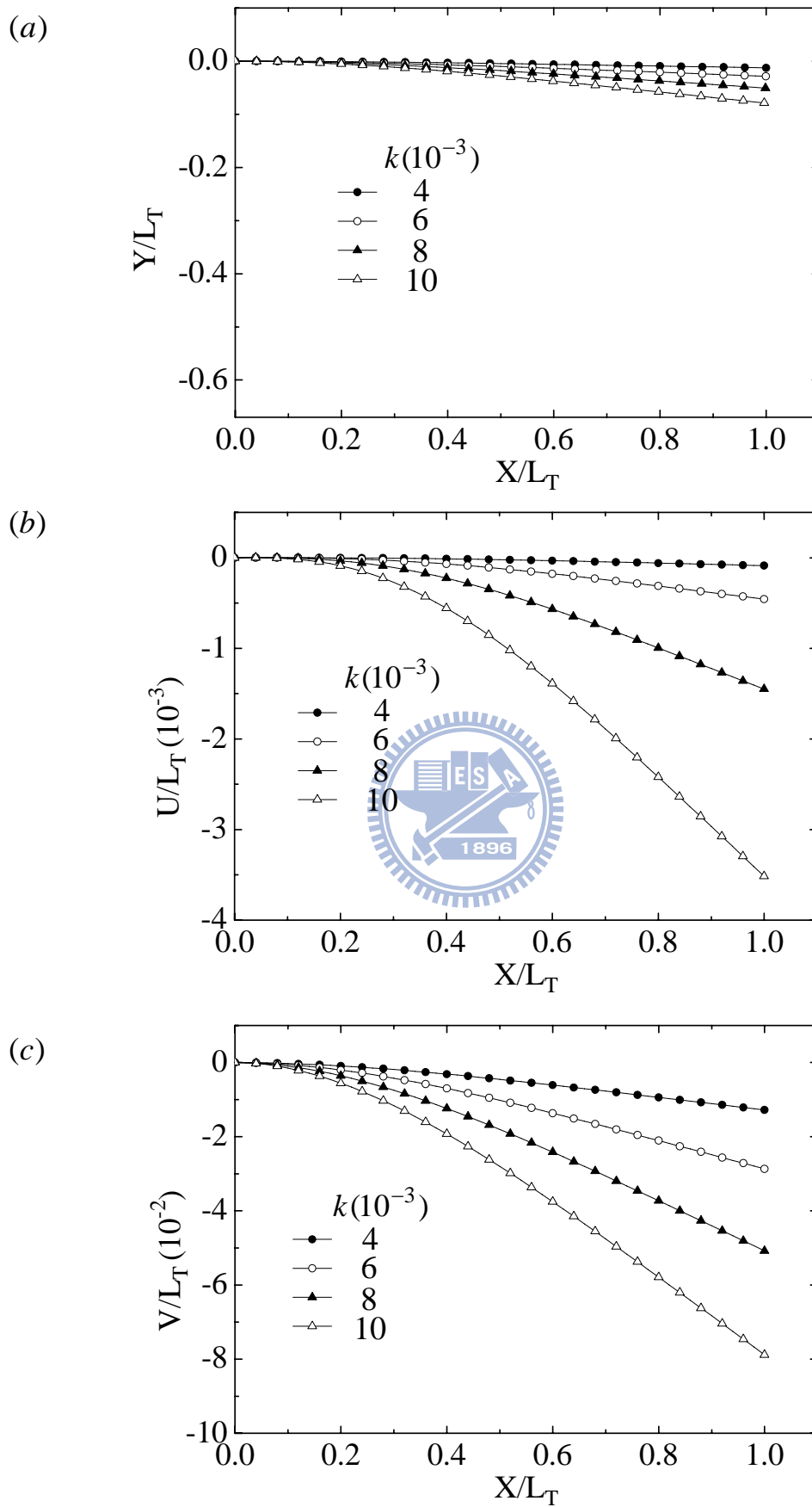
圖七 梁的變形圖



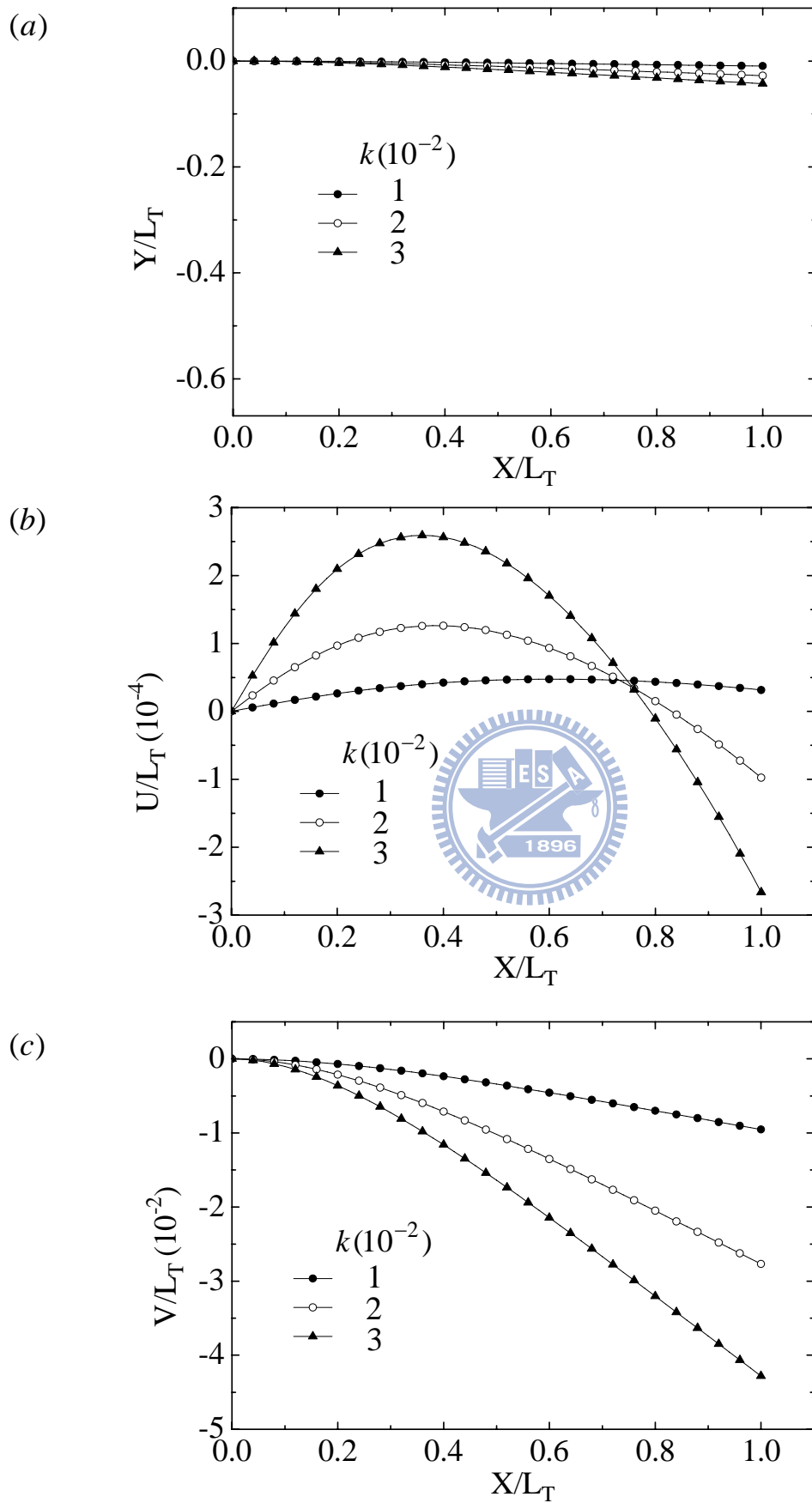
圖八 傾斜旋轉梁的 (a) 穩態變形 (b) 軸向位移 (c) 側向位移 ($\eta = 80$, $r = 1$, $\alpha = 5^\circ$)



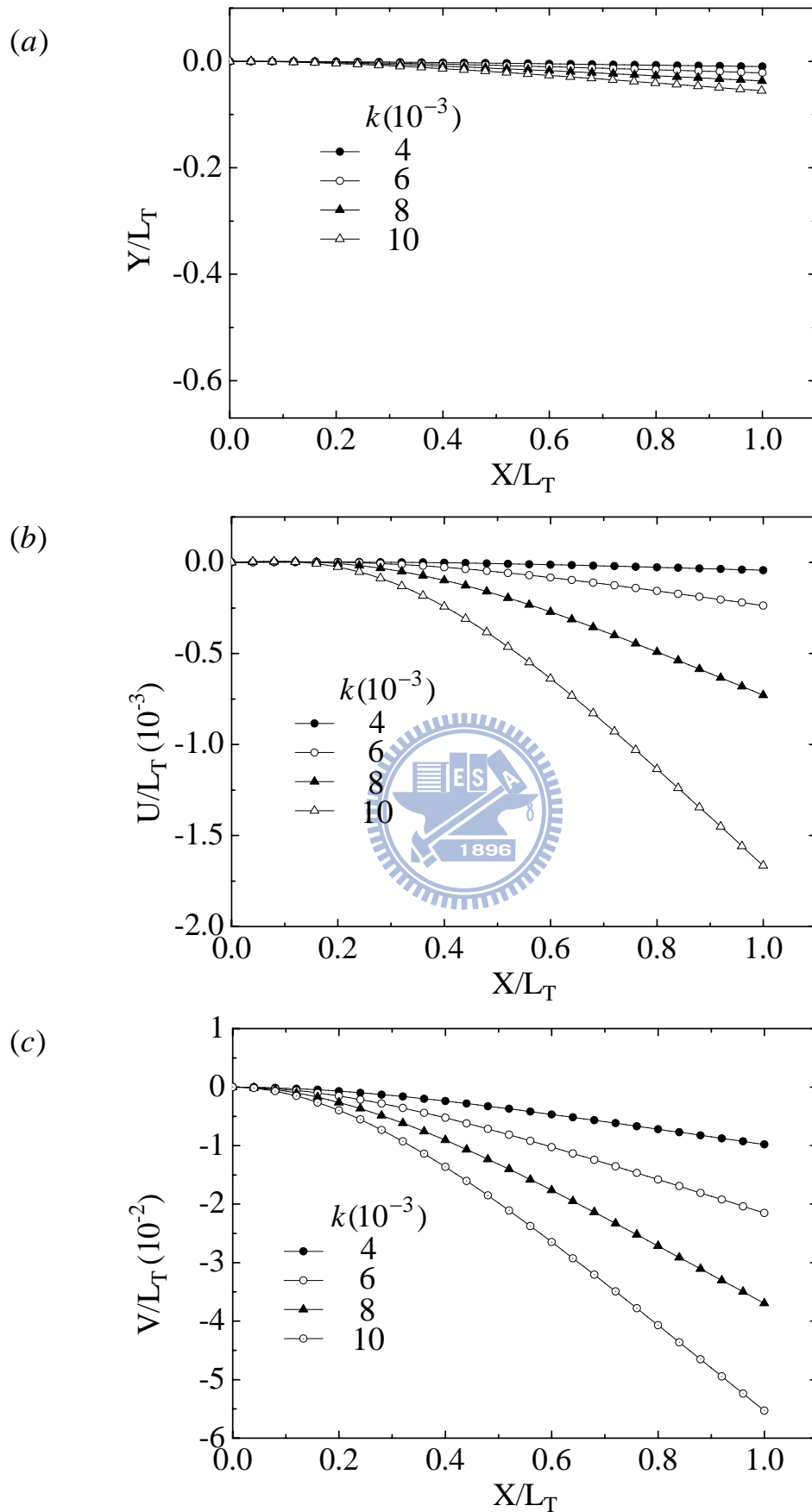
圖九 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=80$, $r=1$, $\alpha=30^\circ$)



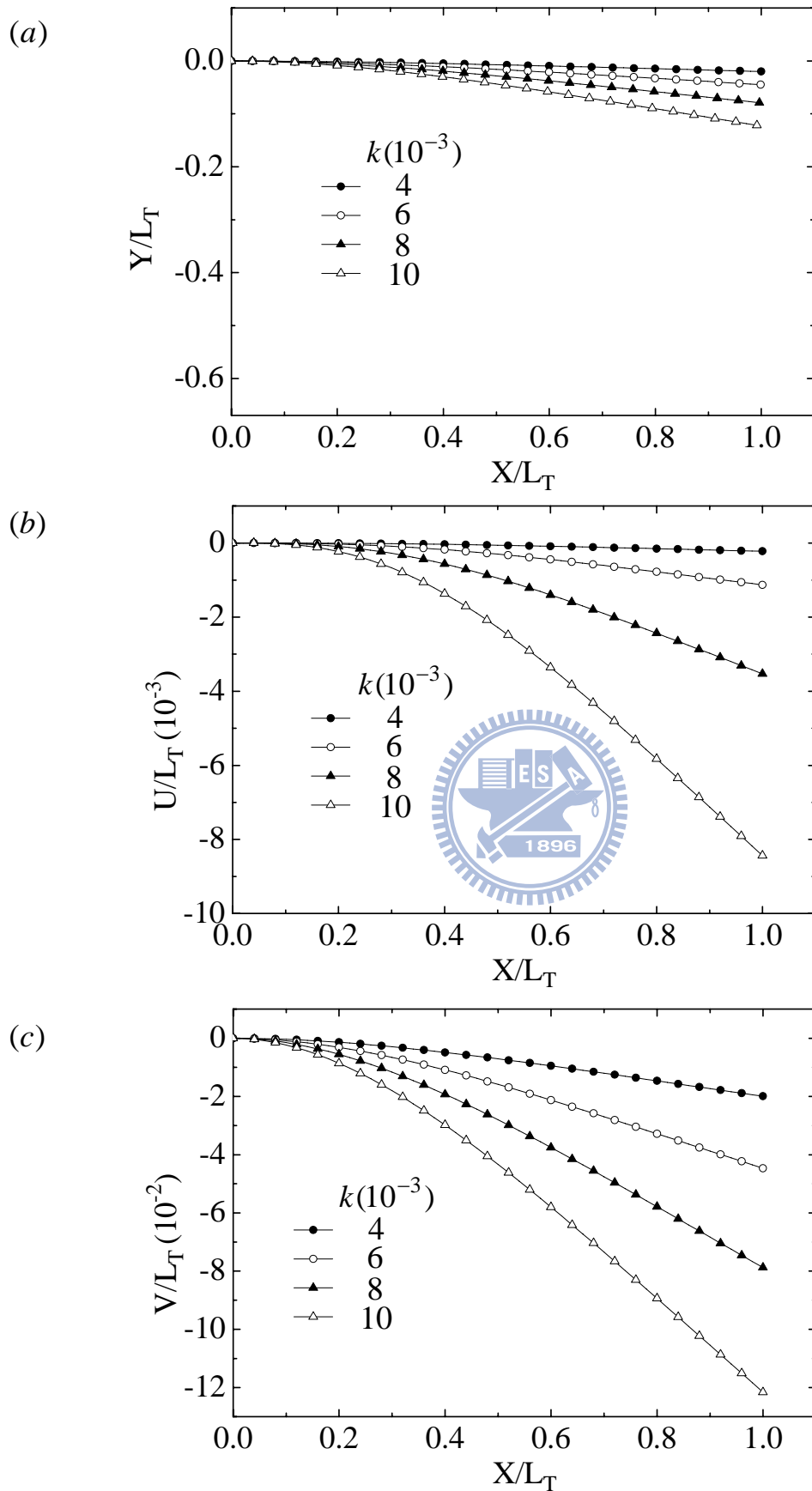
圖十 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta = 80$, $r = 1$, $\alpha = 90^\circ$)



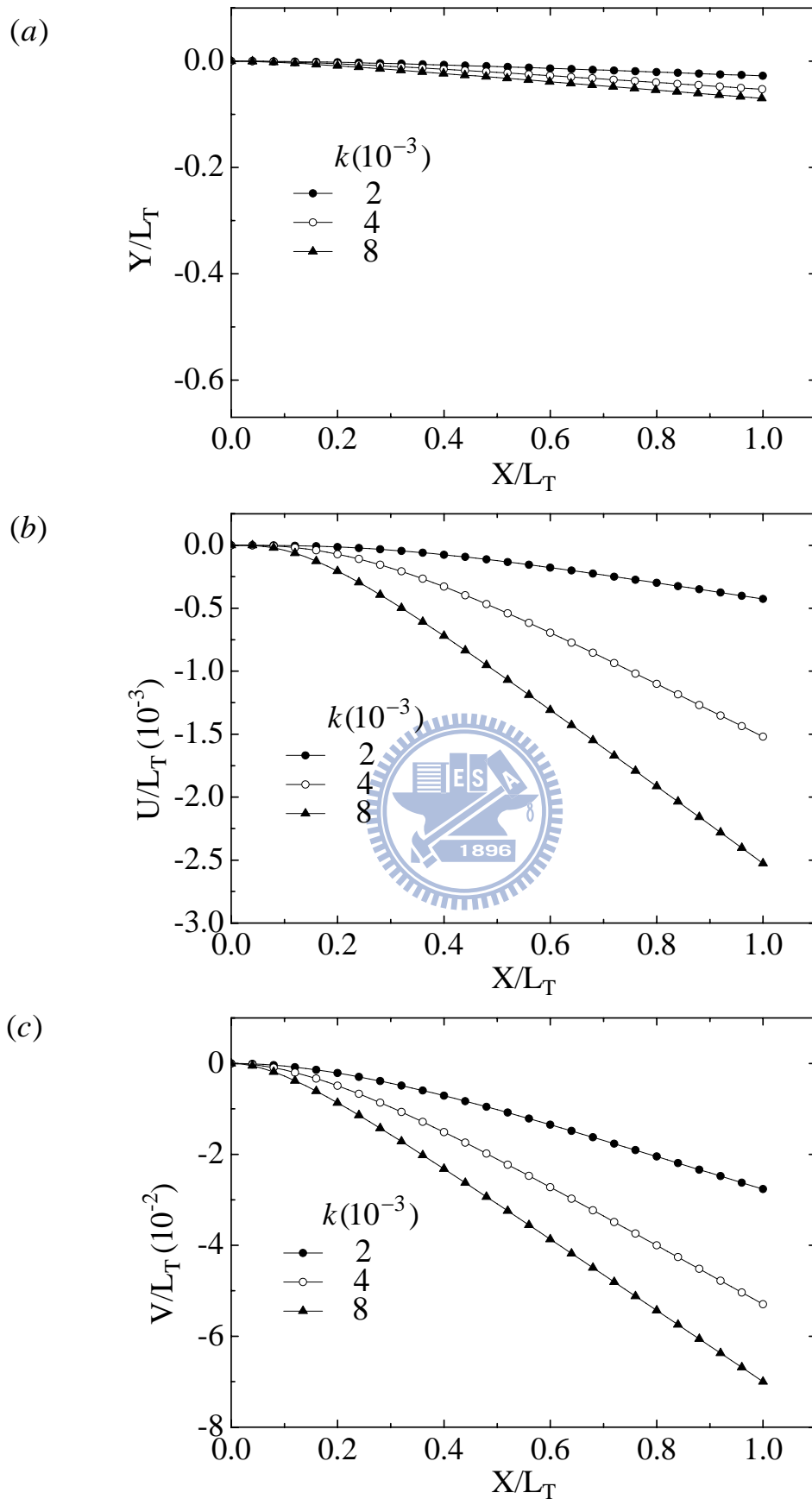
圖十一 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=100$, $r=1$, $\alpha=5^\circ$)



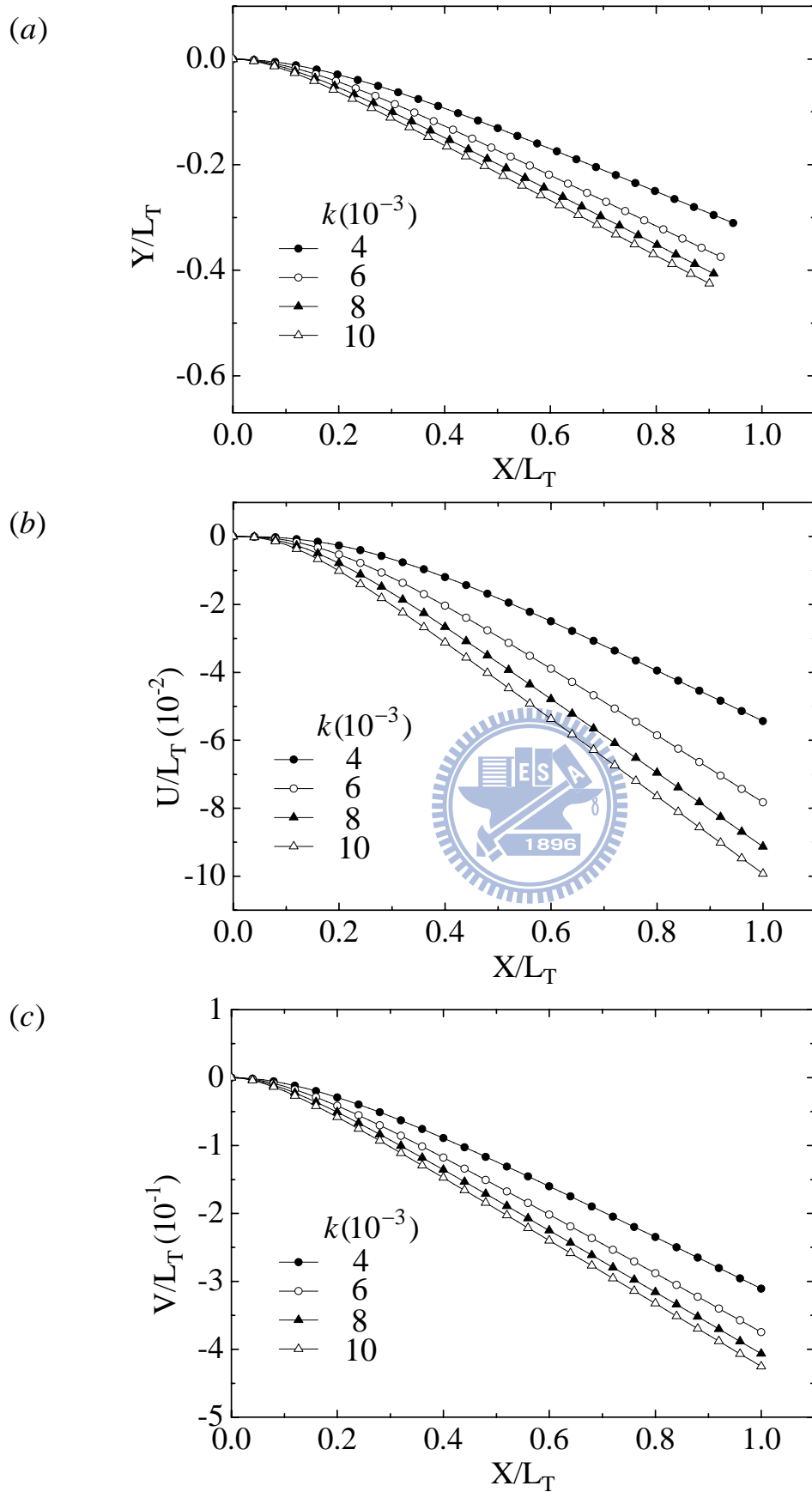
圖十二 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=100$, $r=1$, $\alpha=30^\circ$)



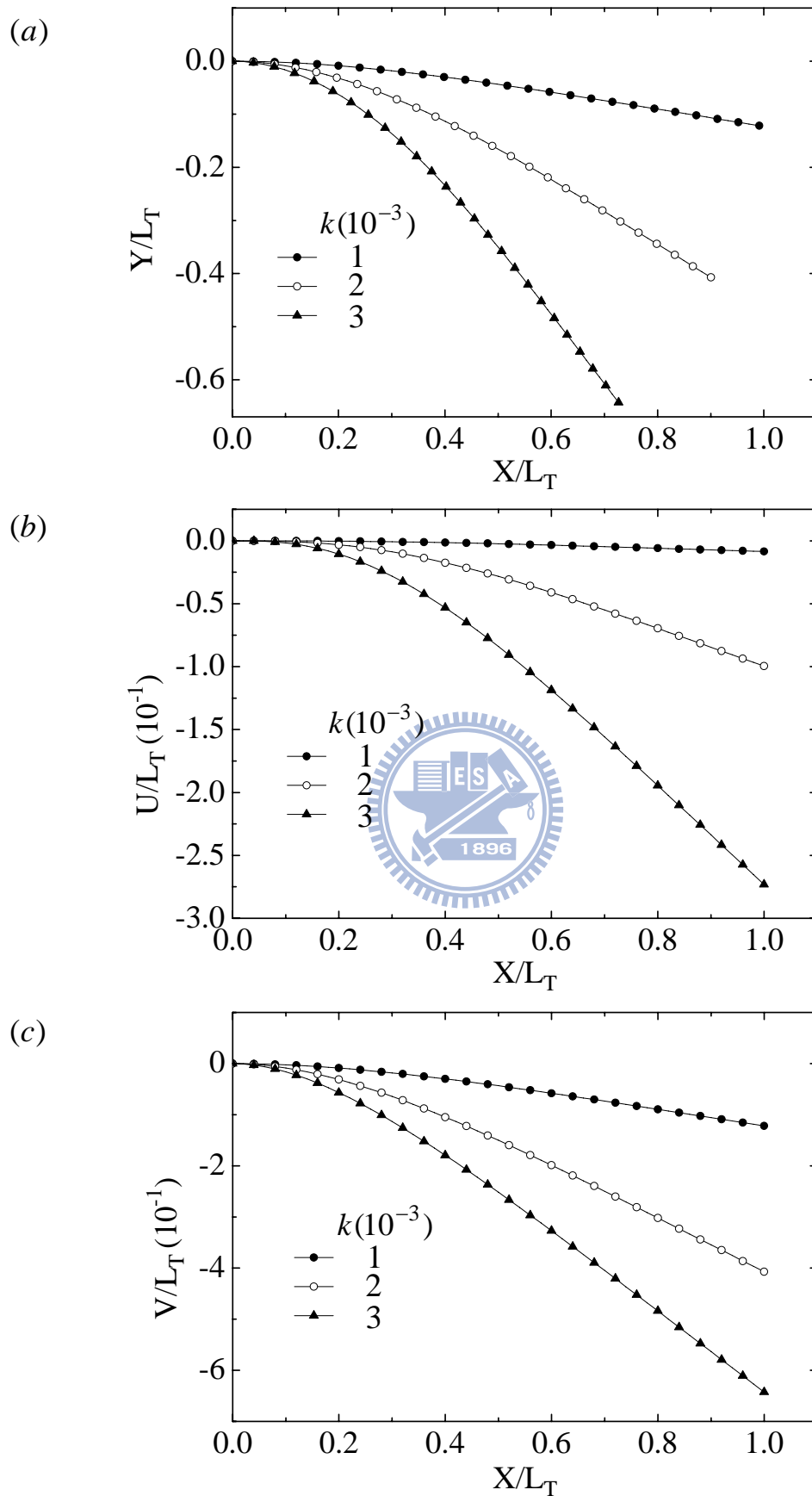
圖十三 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=100$, $r=1$, $\alpha=90^\circ$)



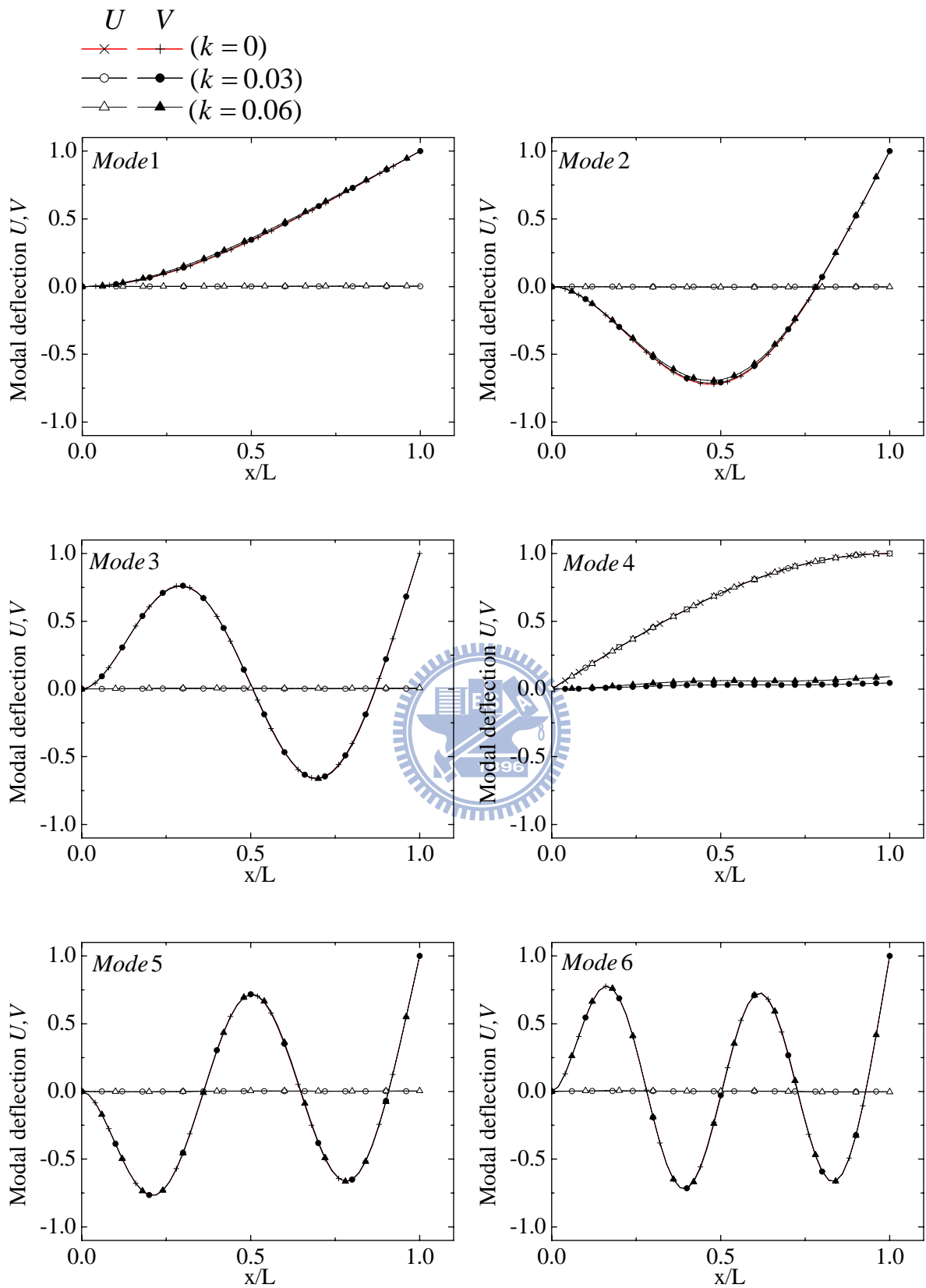
圖十四 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=1000$, $r=1$, $\alpha=5^\circ$)



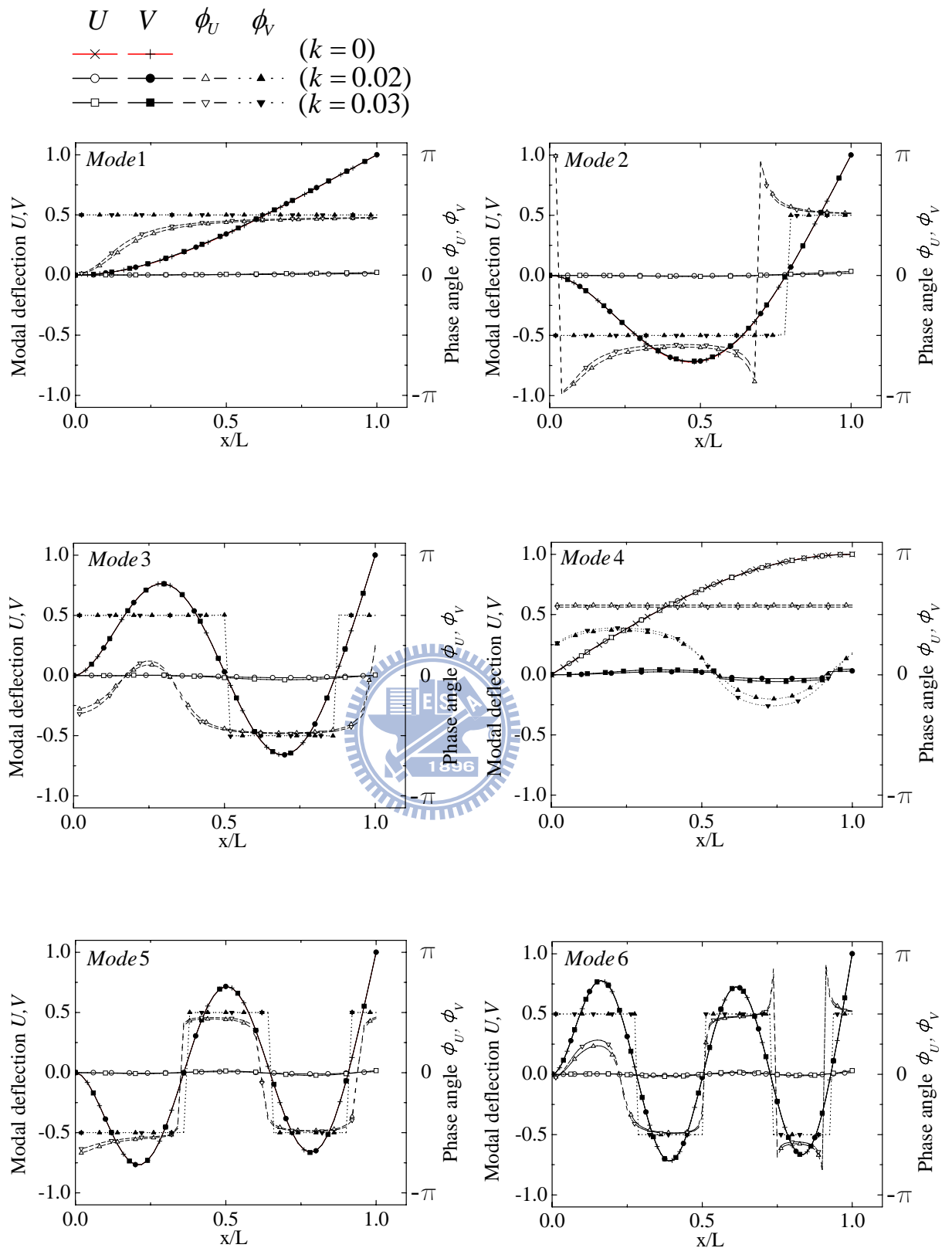
圖十五 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=1000$, $r=1$, $\alpha=30^\circ$)



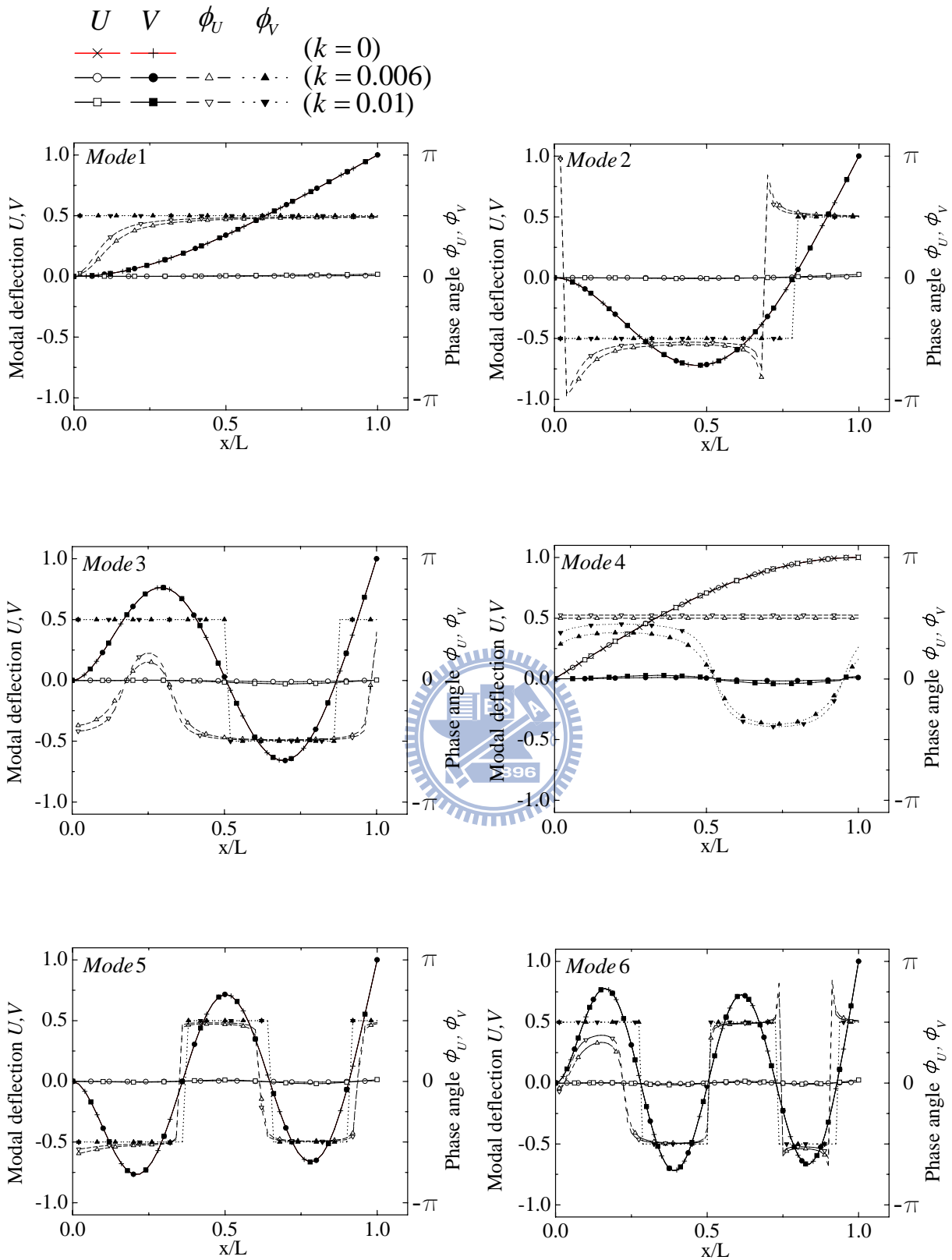
圖十六 傾斜旋轉梁的(a)穩態變形(b)軸向位移(c)側向位移($\eta=1000$, $r=1$, $\alpha=90^\circ$)



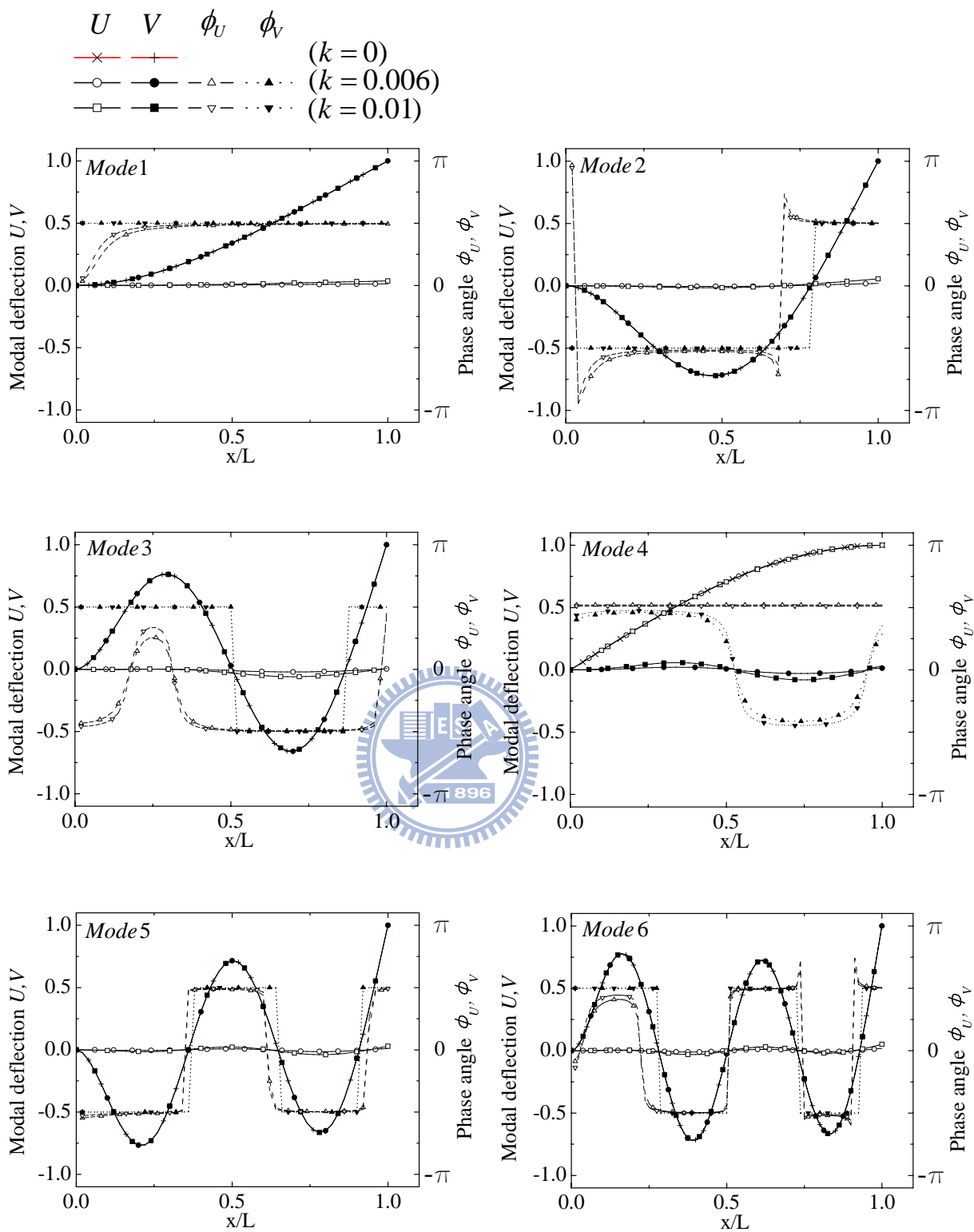
圖十七 不同轉速下的第一至第六振動模態($\alpha = 0^\circ$, $r = 1$, $\eta = 50$)



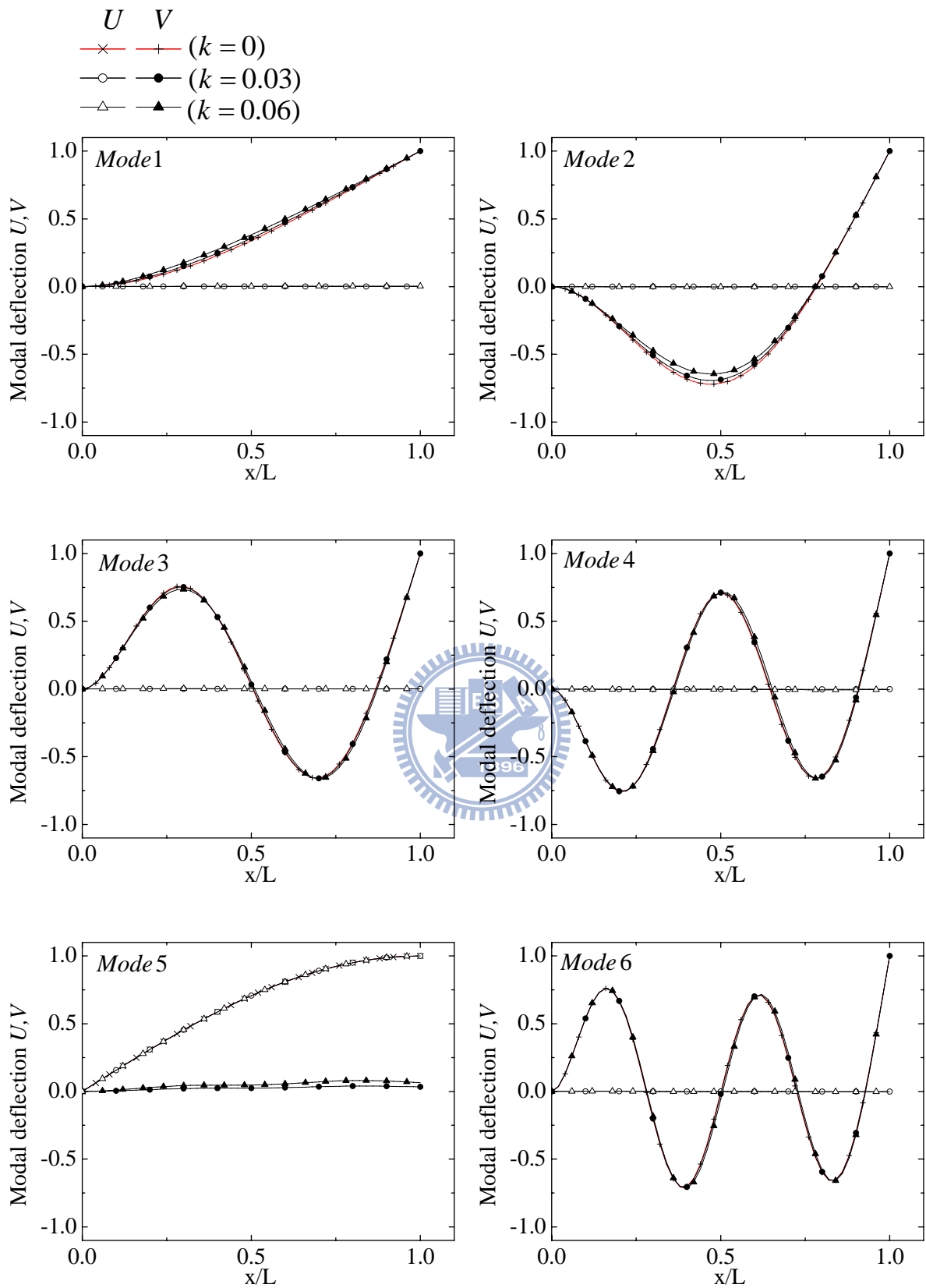
圖十八 不同轉速下的第一至第六振動模態($\alpha=5^\circ$, $r=1$, $\eta=50$)



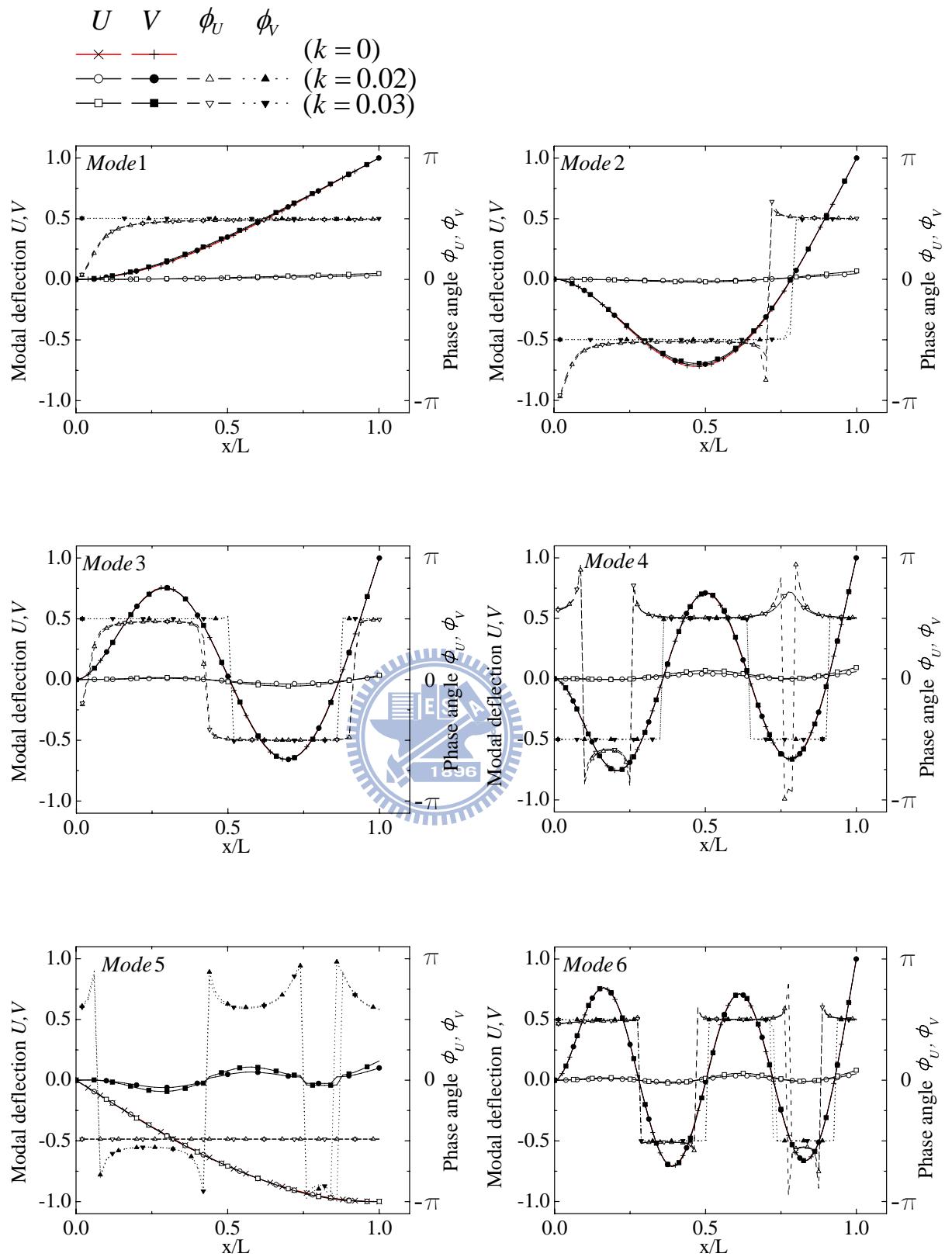
圖十九 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r=1$, $\eta=50$)



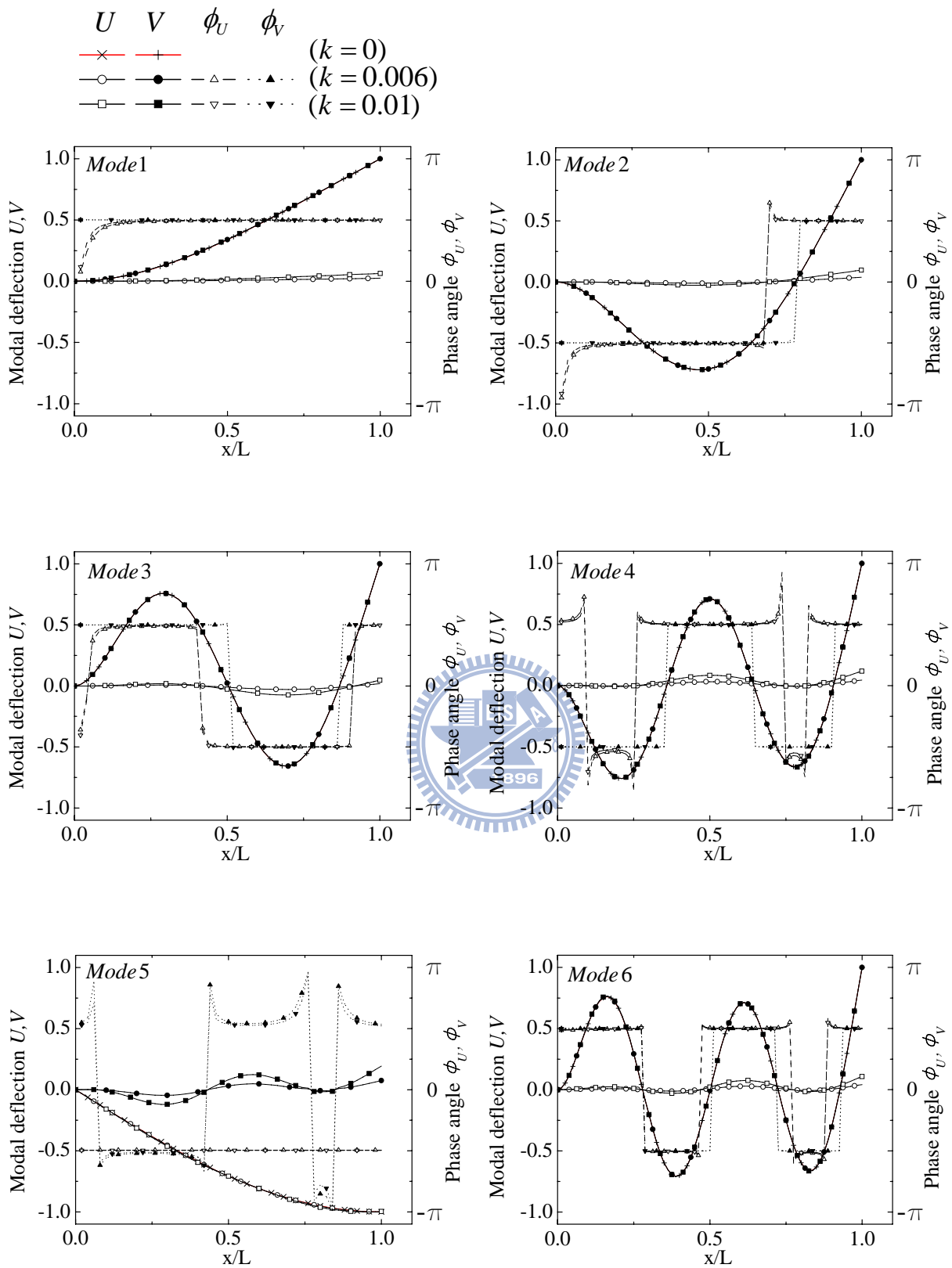
圖二十 不同轉速下的第一至第六振動模態($\alpha = 90^\circ$, $r = 1$, $\eta = 50$)



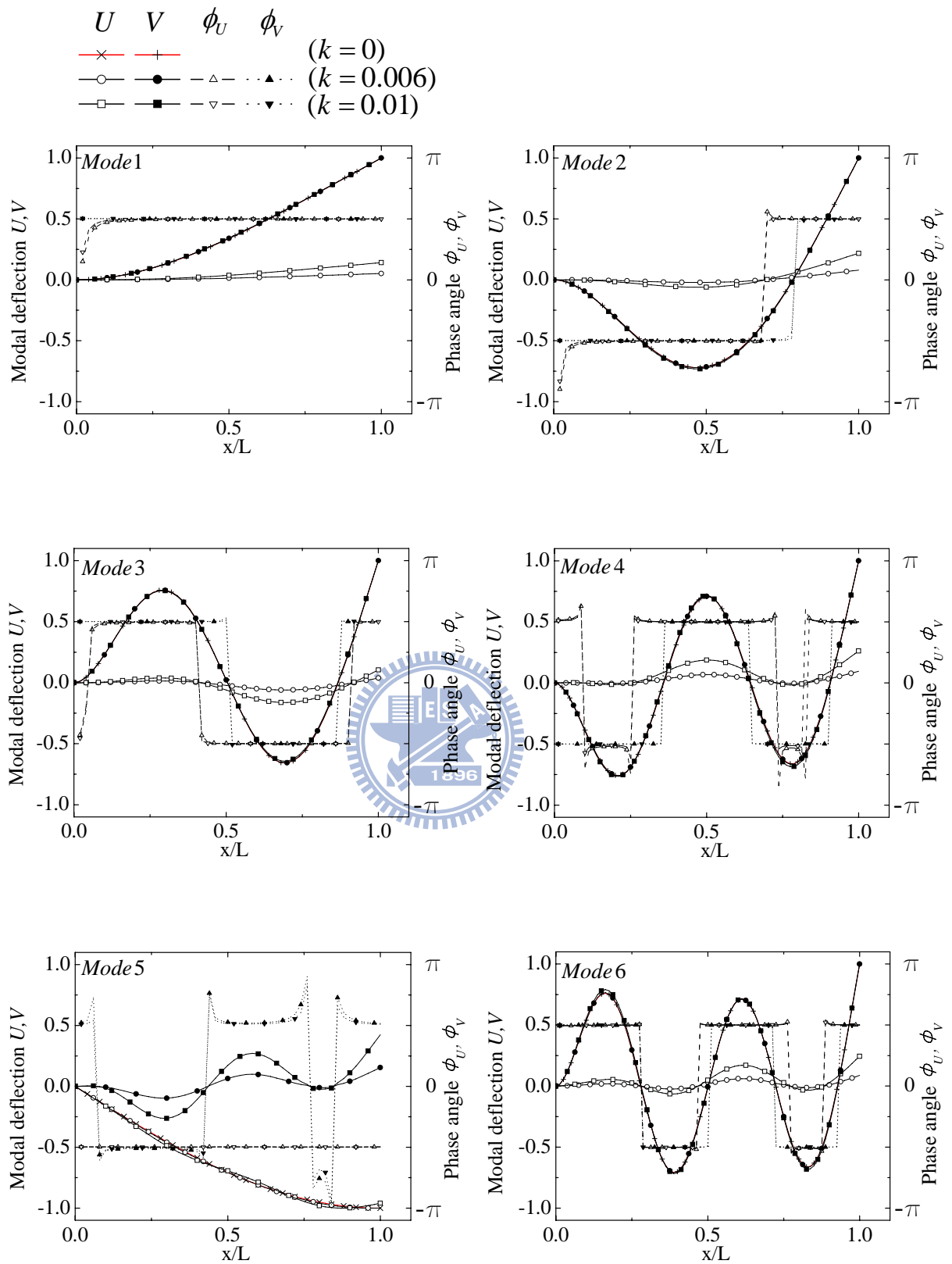
圖二十一 不同轉速下的第一至第六振動模態($\alpha = 0^\circ$, $r = 1$, $\eta = 100$)



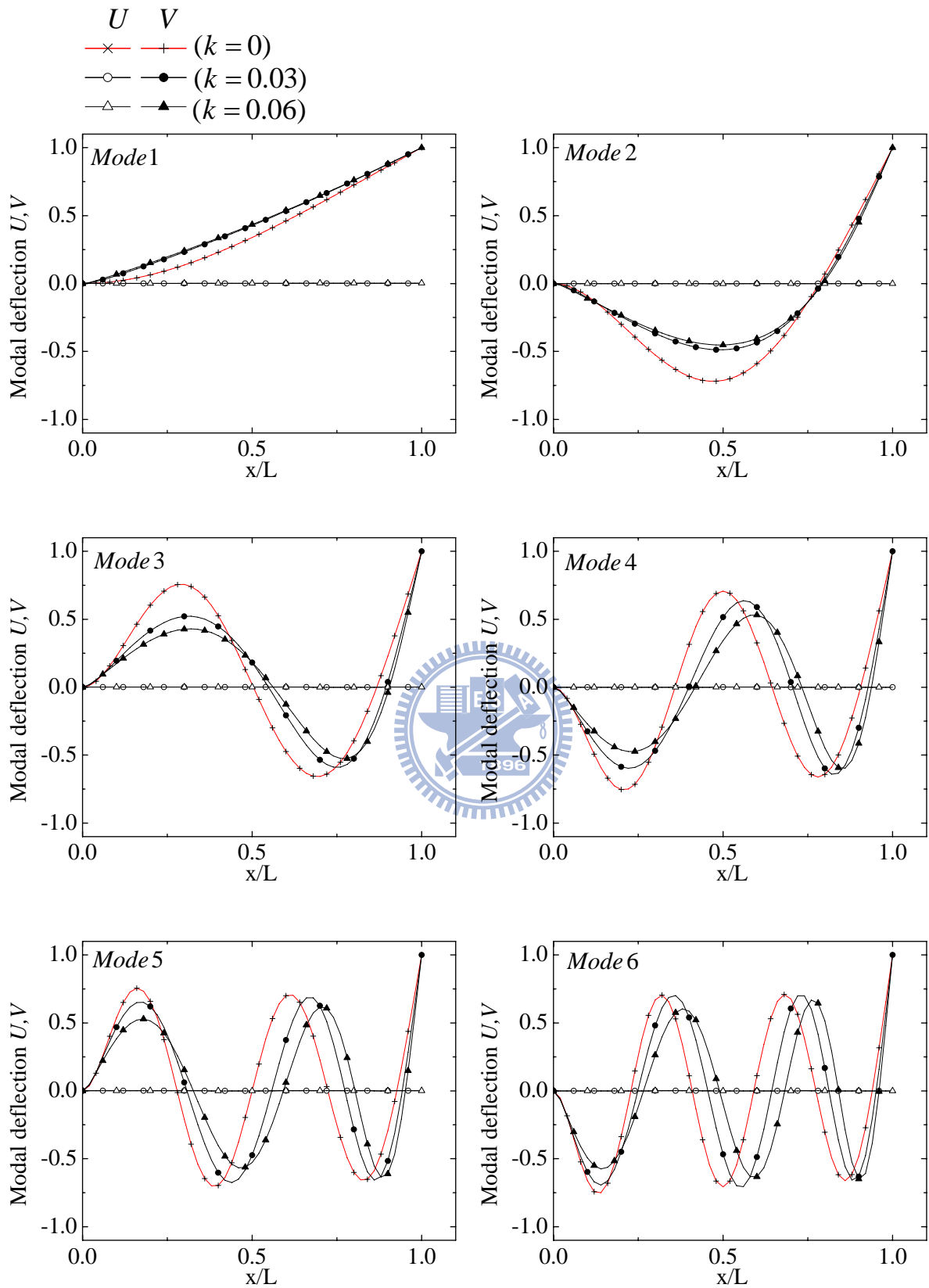
圖二十二 不同轉速下的第一至第六振動模態($\alpha = 5^\circ$, $r=1$, $\eta=100$)



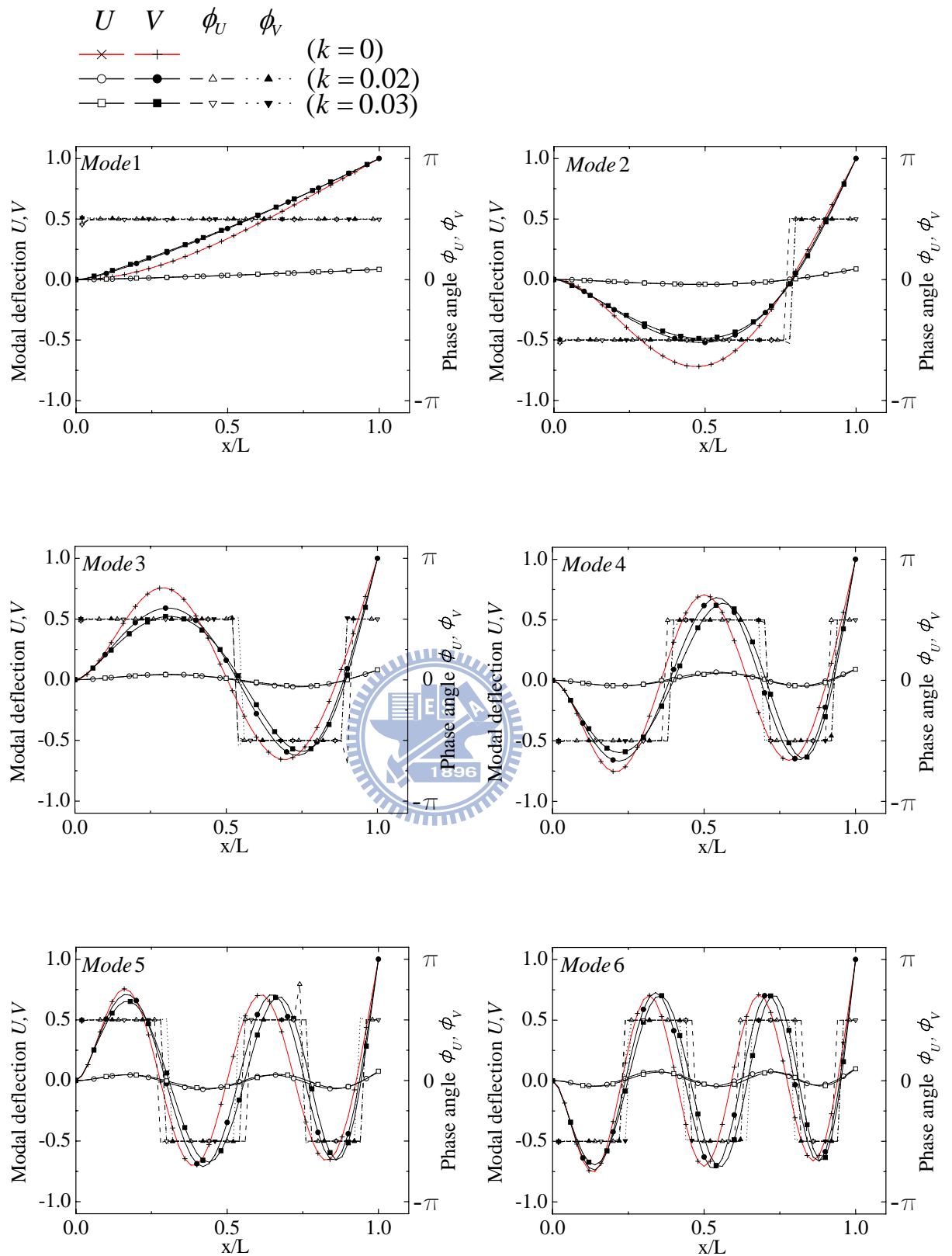
圖二十三 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r = 1$, $\eta = 100$)



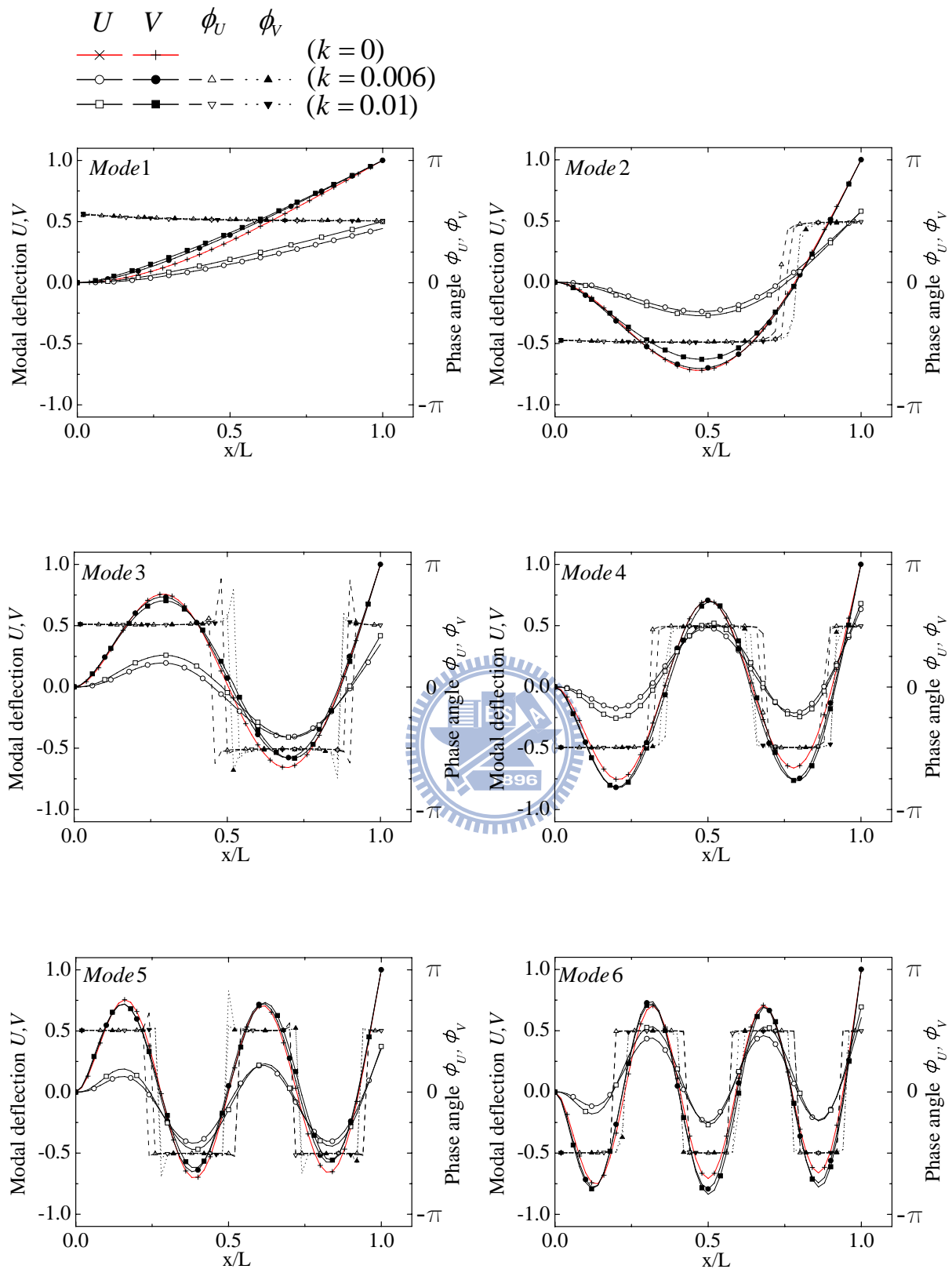
圖二十四 不同轉速下的第一至第六振動模態($\alpha=90^\circ$, $r=1$, $\eta=100$)



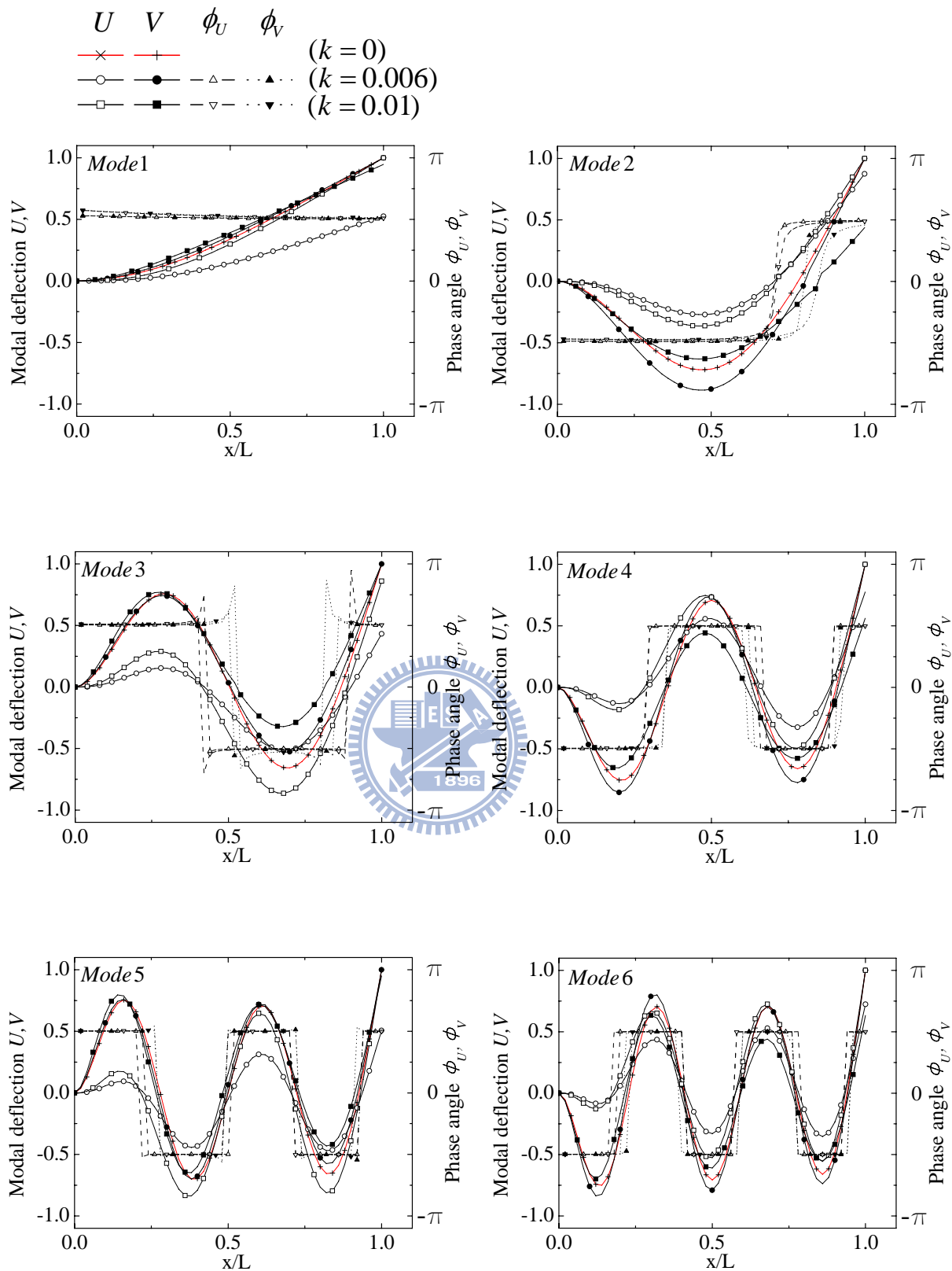
圖二十五 不同轉速下的第一至第六振動模態($\alpha = 0^\circ$, $r = 1$, $\eta = 1000$)



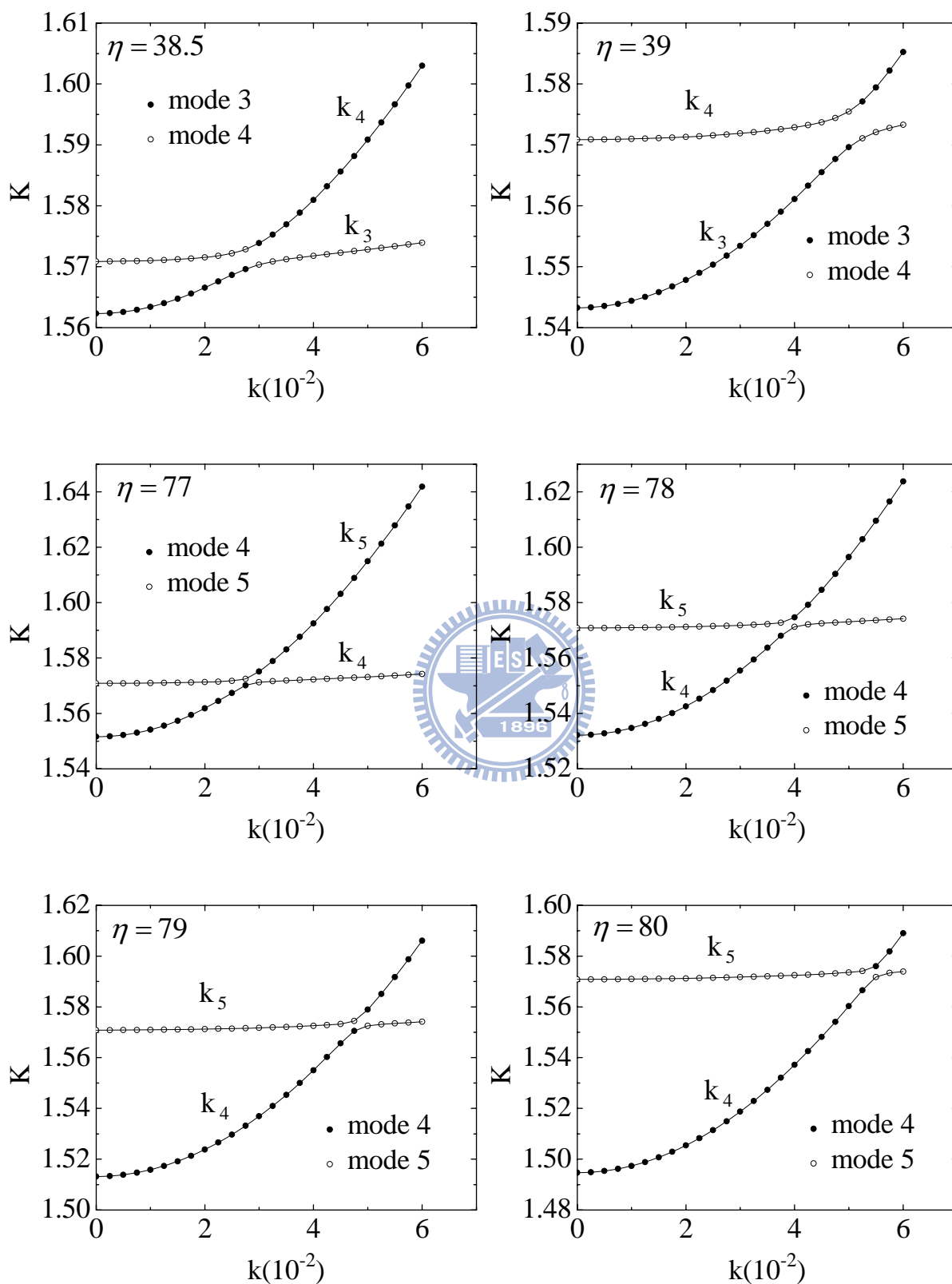
圖二十六 不同轉速下的第一至第六振動模態($\alpha=5^\circ$, $r=1$, $\eta=1000$)



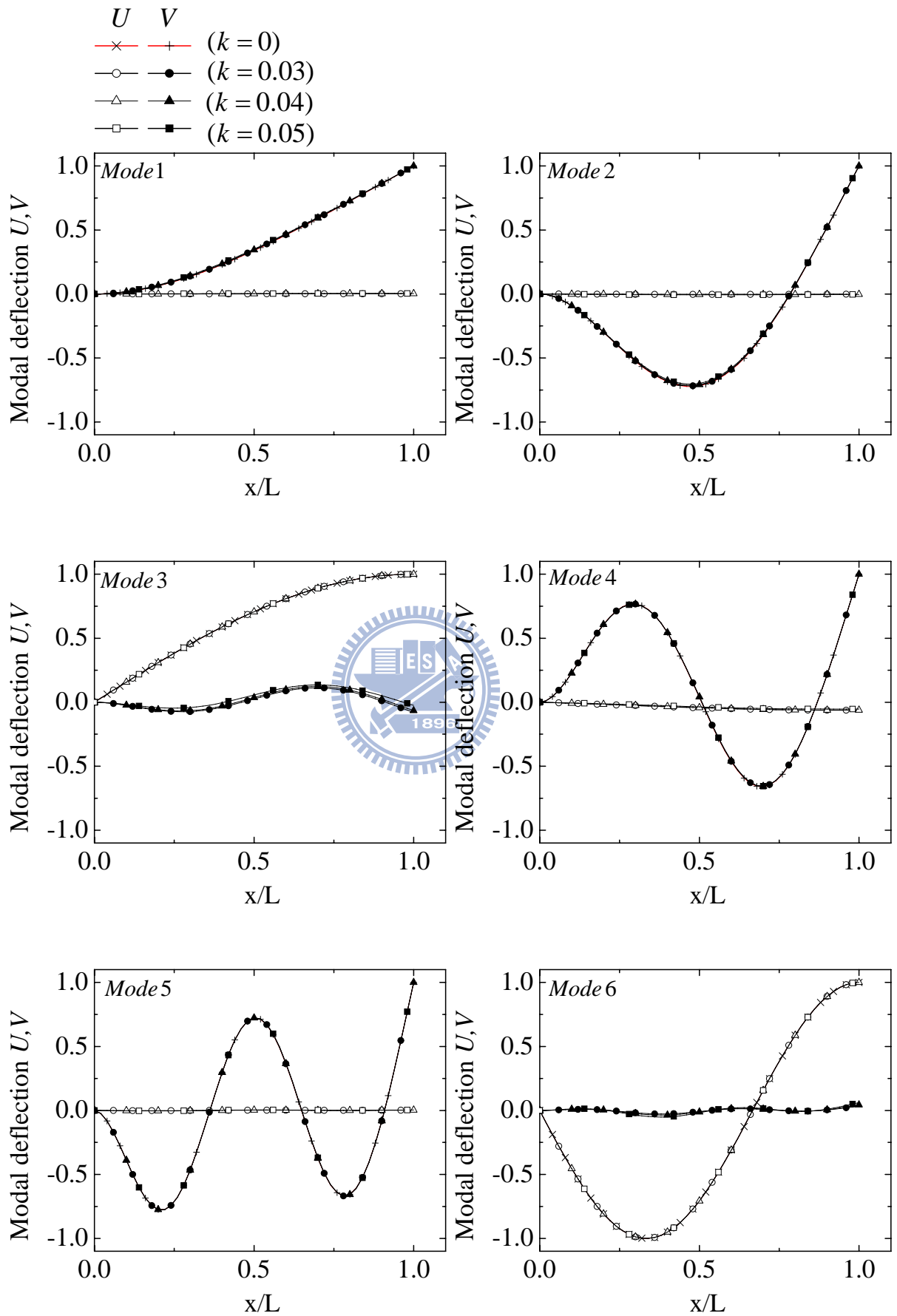
圖二十七 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r = 1$, $\eta = 1000$)



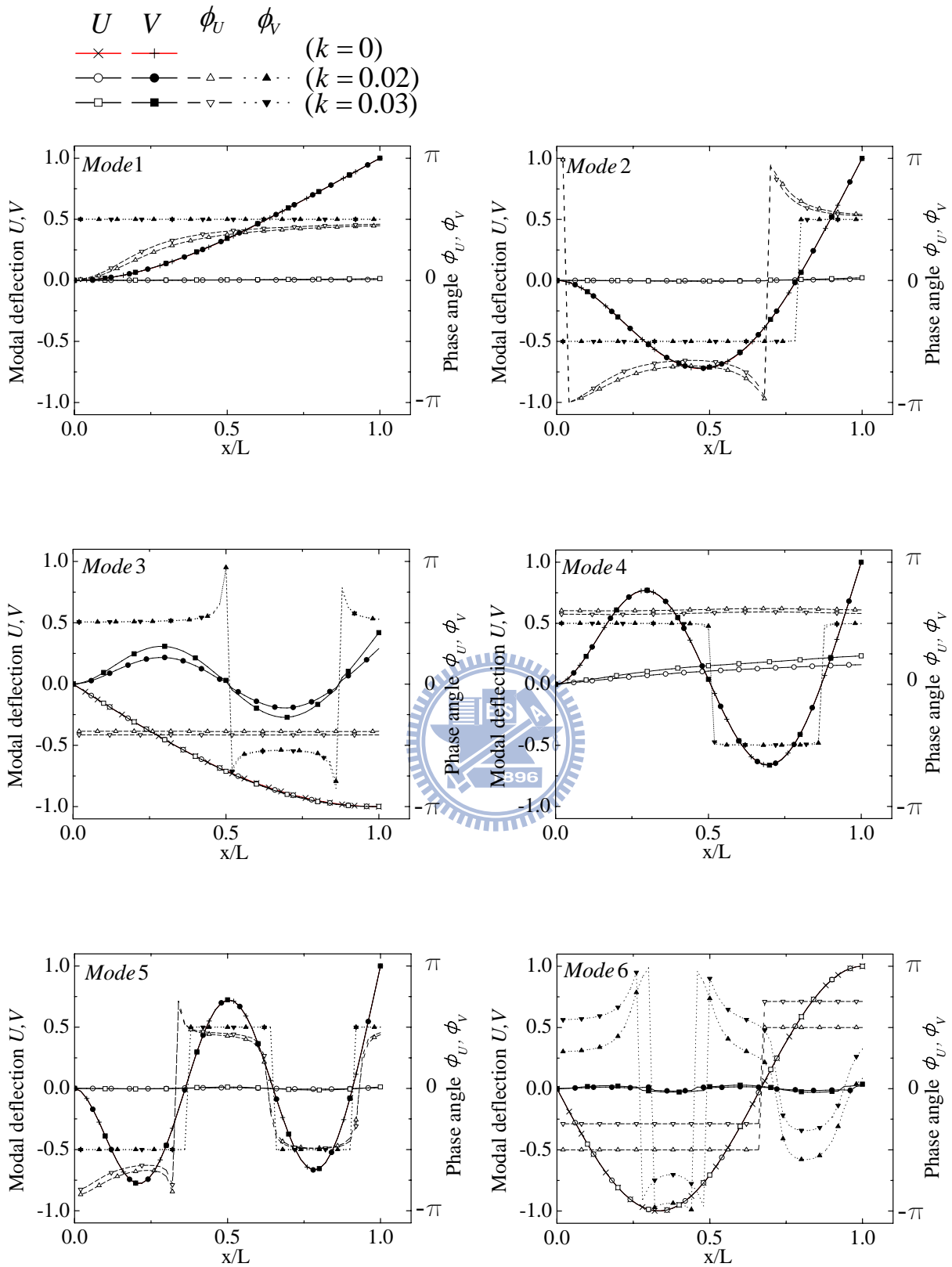
圖二十八 不同轉速下的第一至第六振動模態($\alpha=90^\circ$, $r=1$, $\eta=1000$)



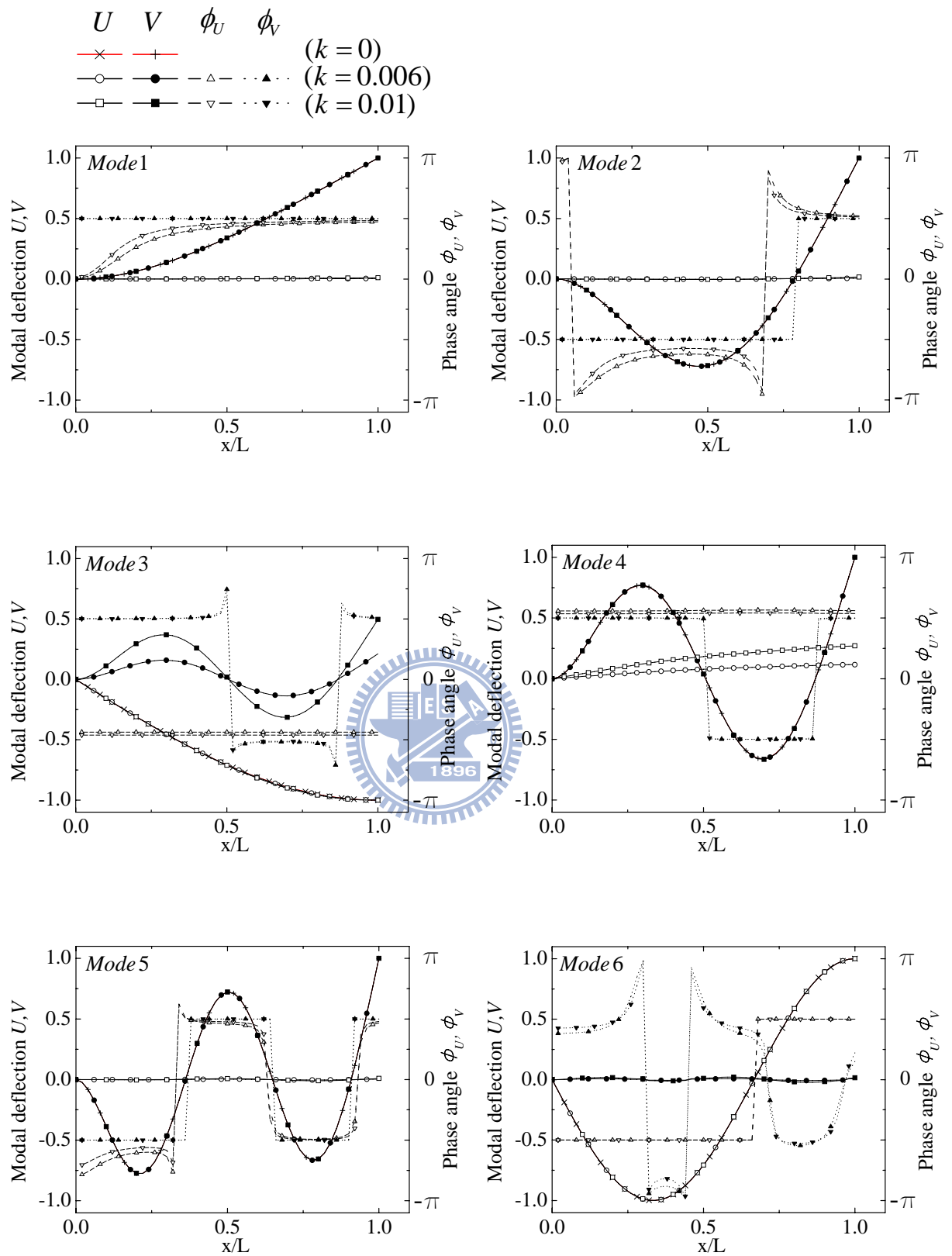
圖二十九 無因次振動頻率-無因次轉速曲線($r=1, \alpha=0^\circ$)



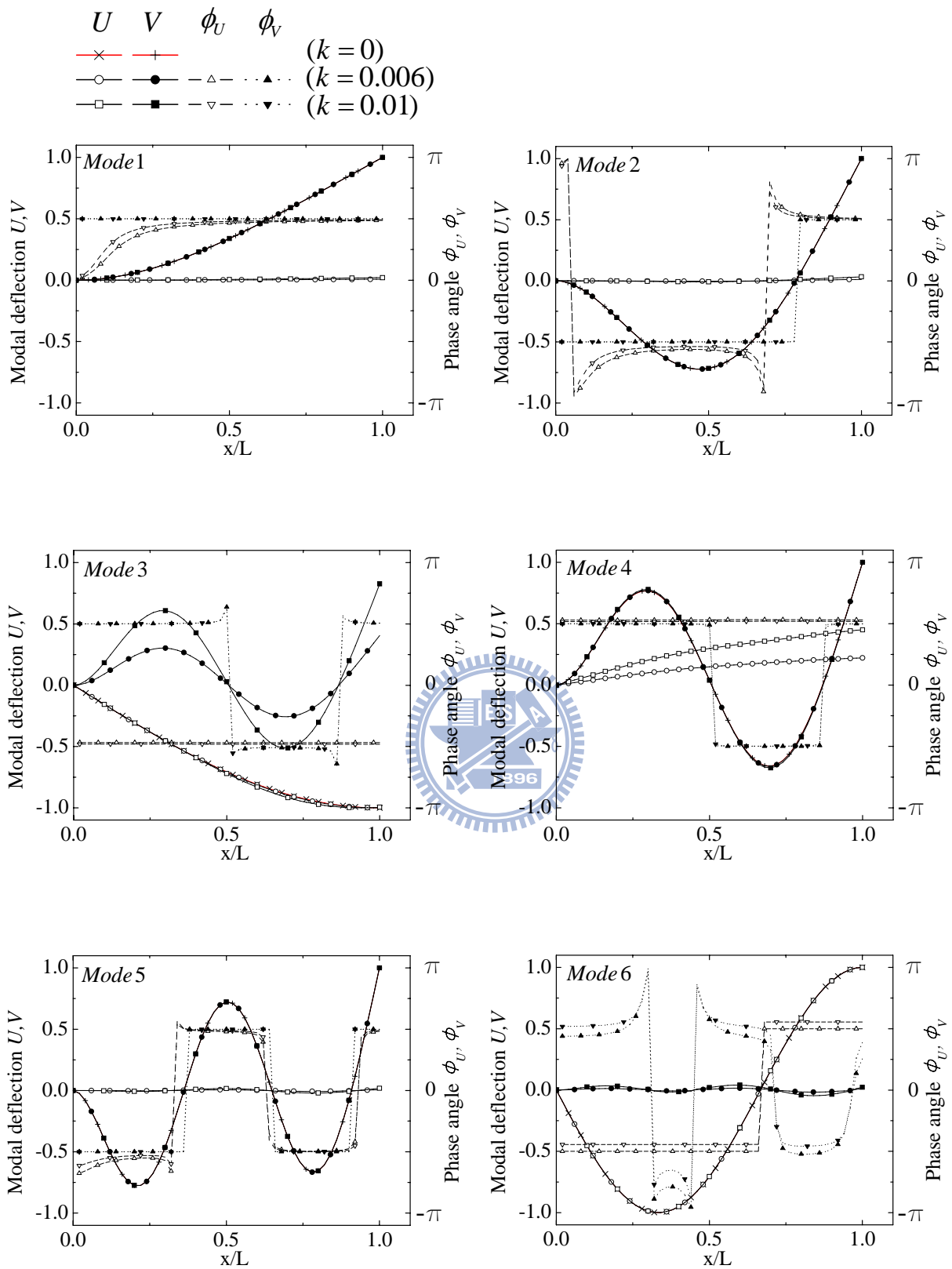
圖三十 不同轉速下的第一至第六振動模態($\alpha=0^\circ$, $r=1$, $\eta=38$)



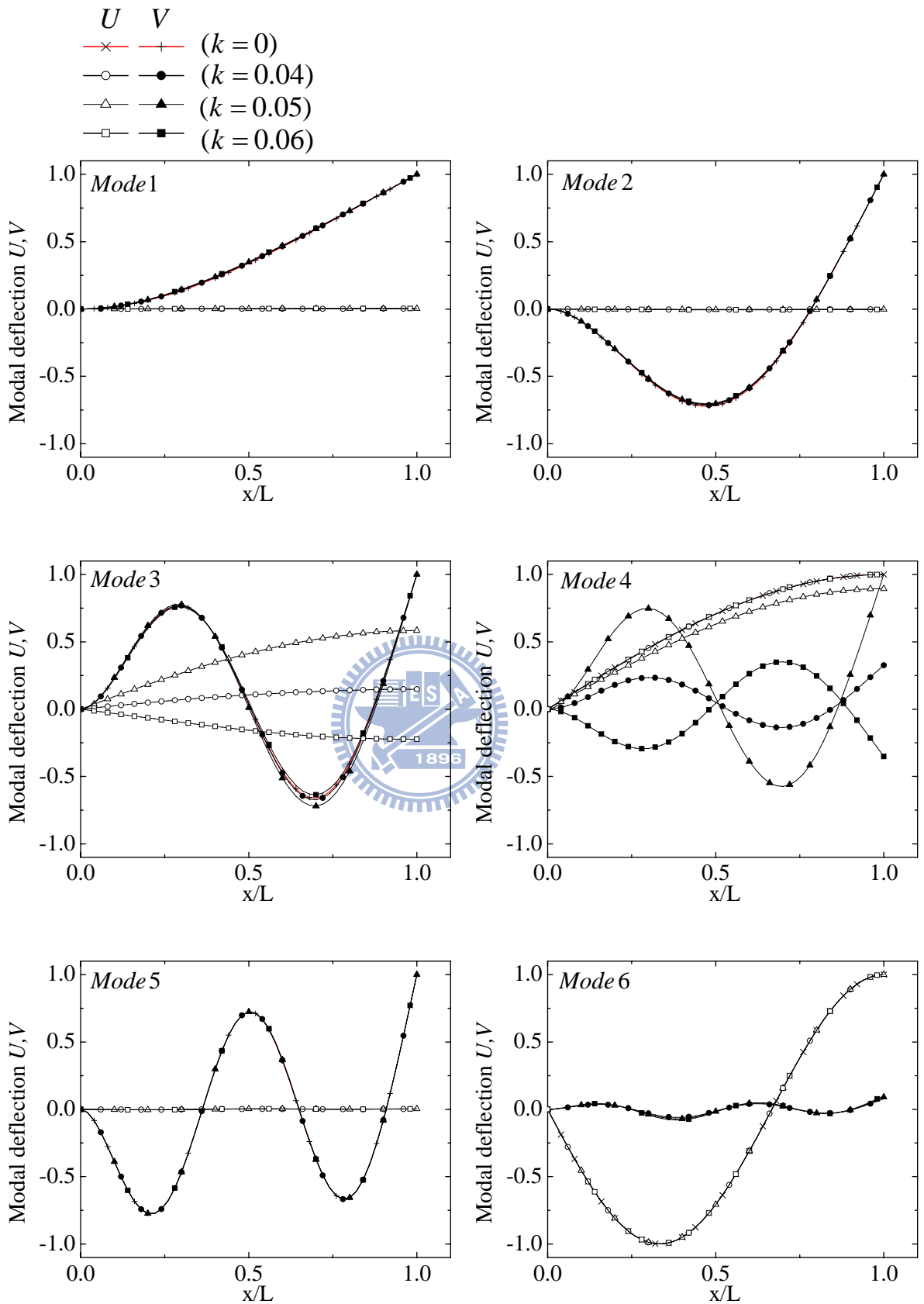
圖三十一 不同轉速下的第一至第六振動模態($\alpha = 5^\circ$, $r = 1$, $\eta = 38$)



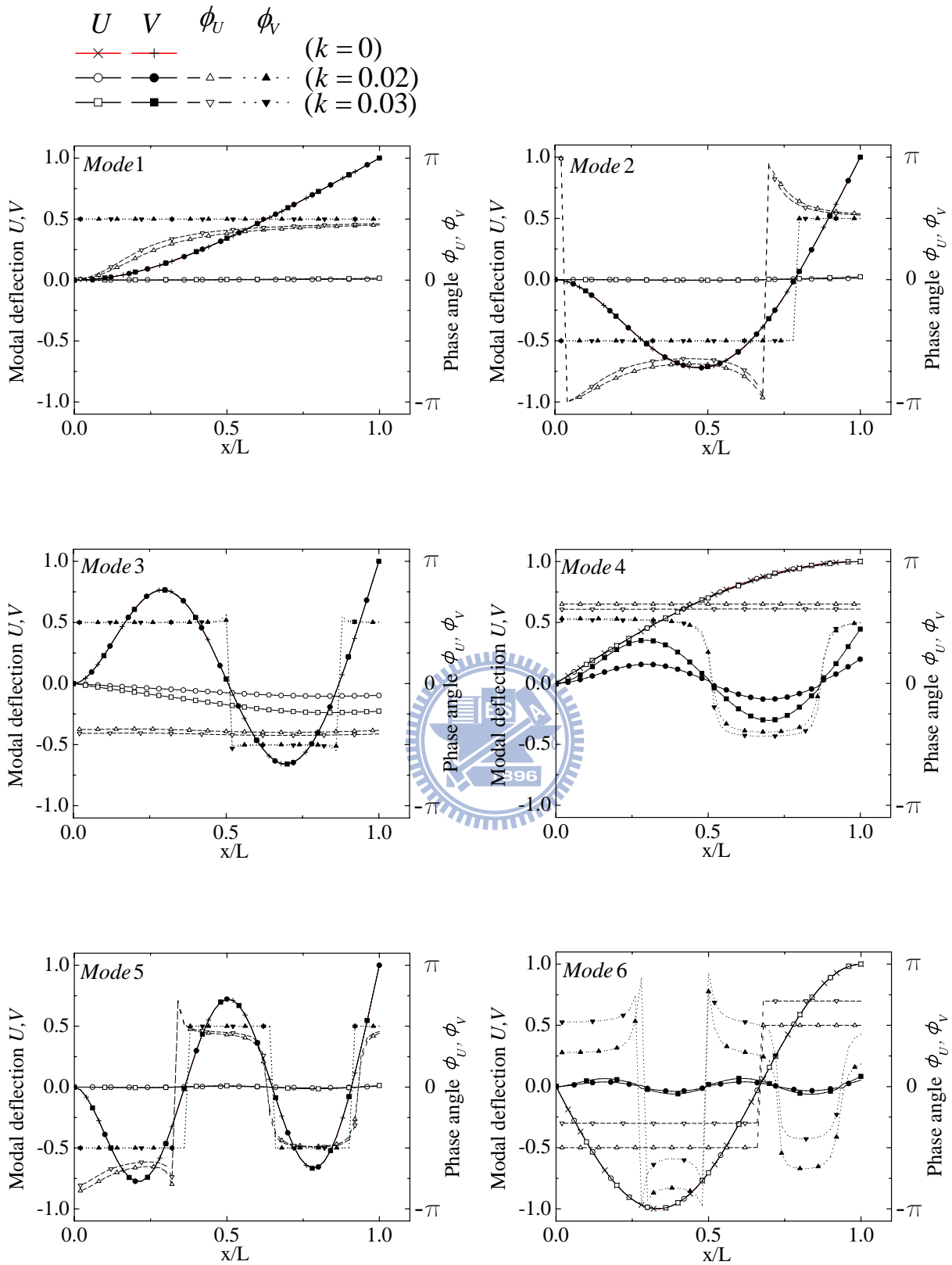
圖三十二 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r = 1$, $\eta = 38$)



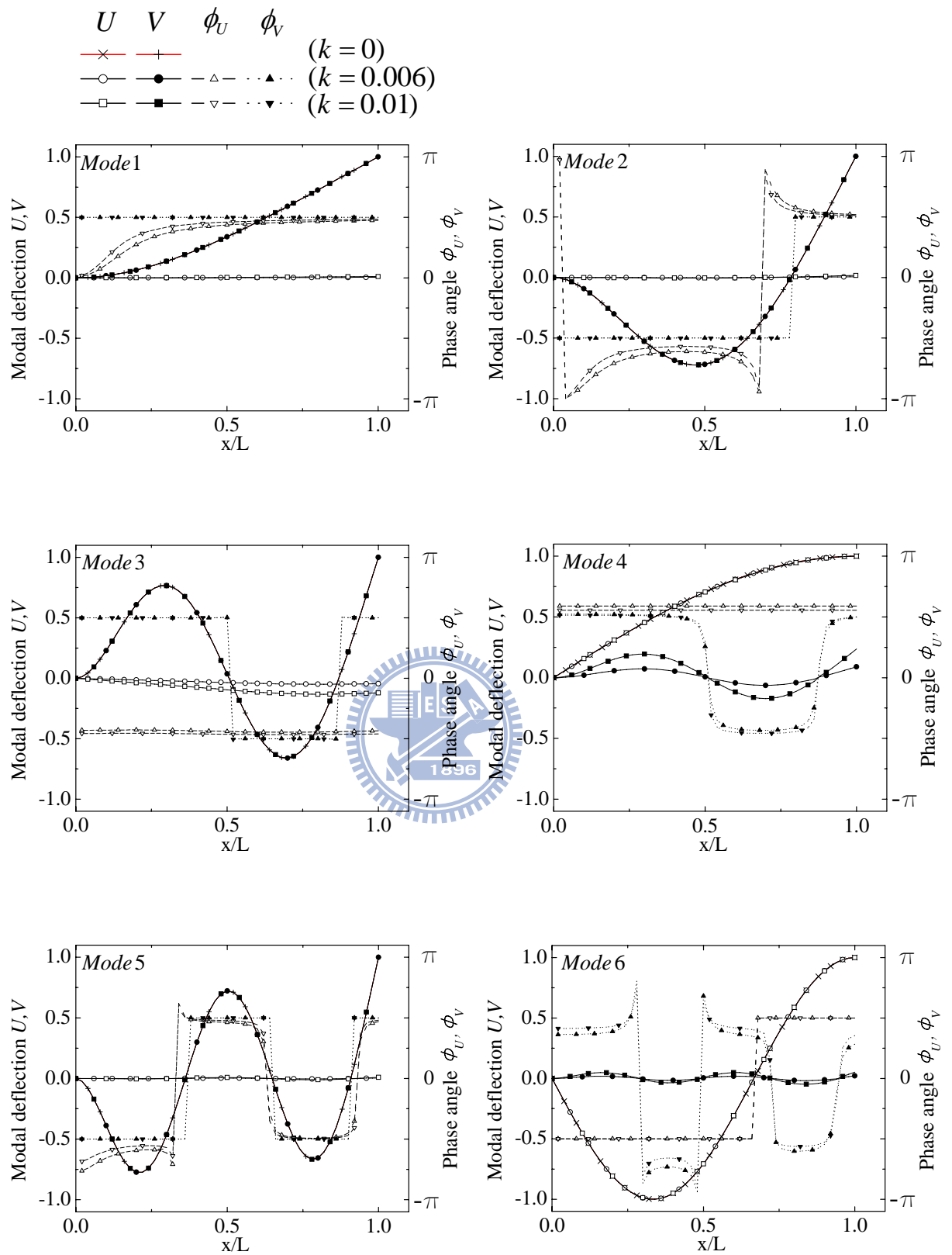
圖三十三 不同轉速下的第一至第六振動模態($\alpha = 90^\circ$, $r = 1$, $\eta = 38$)



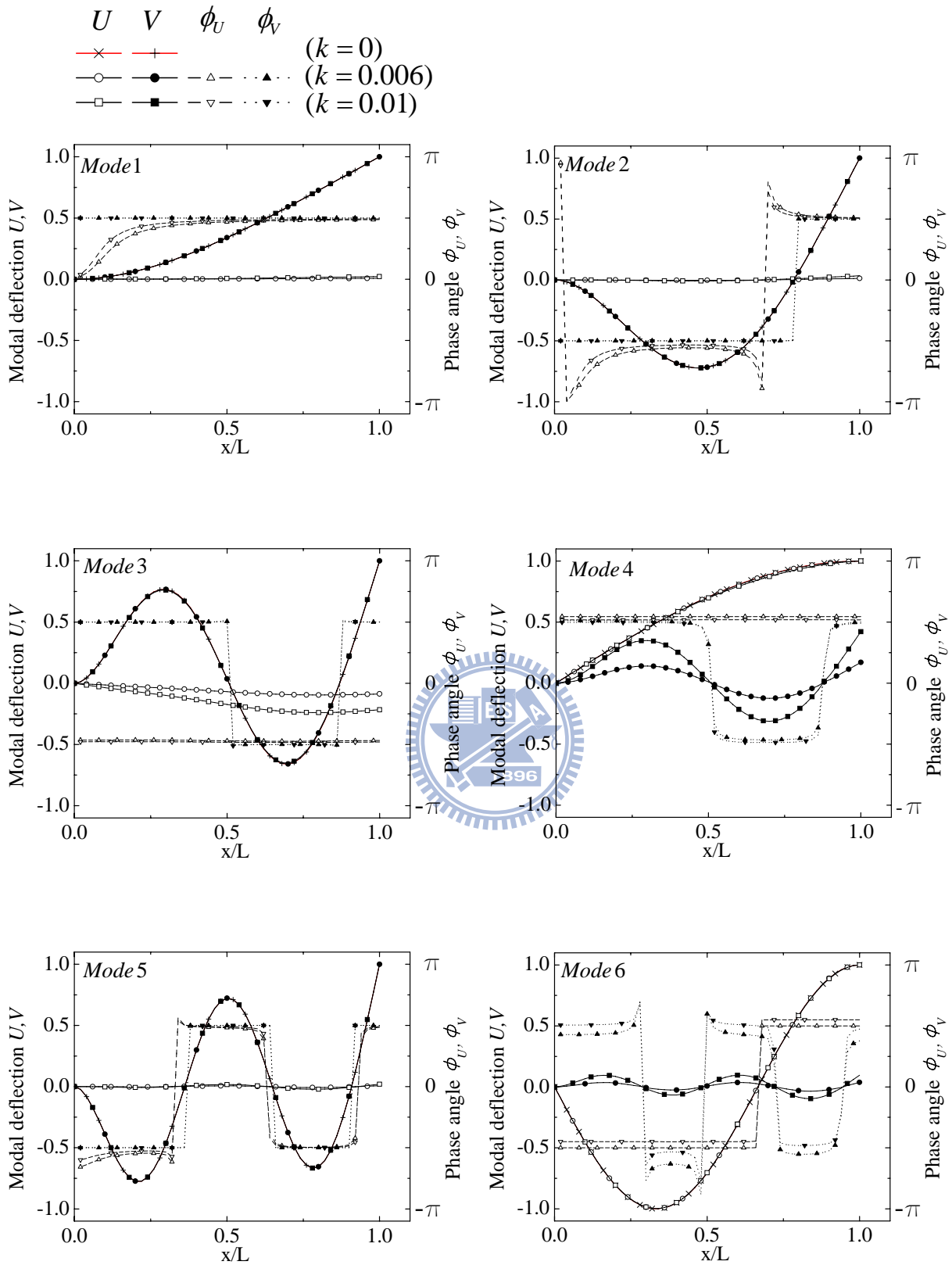
圖三十四 不同轉速下的第一至第六振動模態($\alpha=0^\circ$, $r=1$, $\eta=39$)



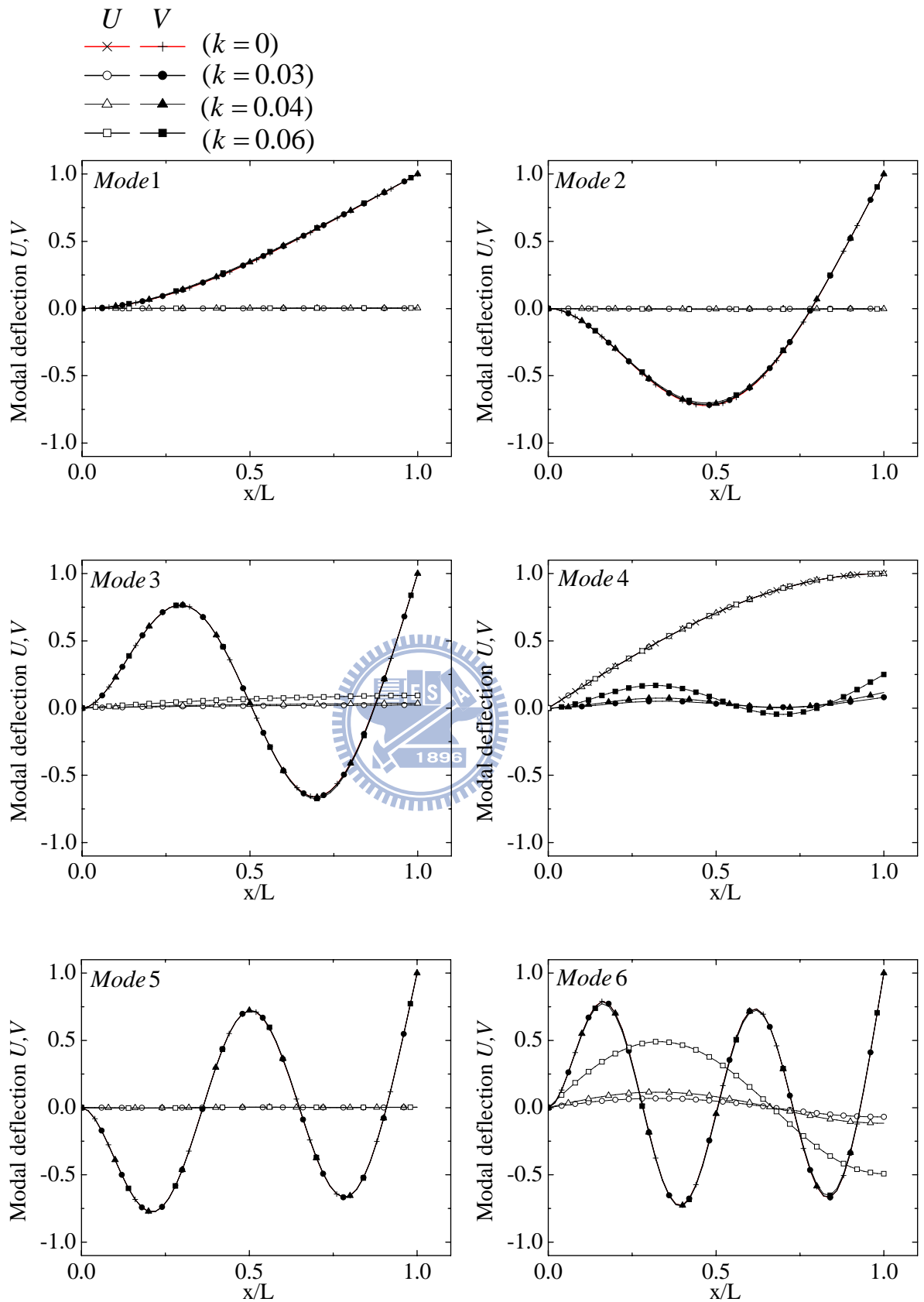
圖三十五 不同轉速下的第一至第六振動模態($\alpha = 5^\circ$, $r = 1$, $\eta = 39$)



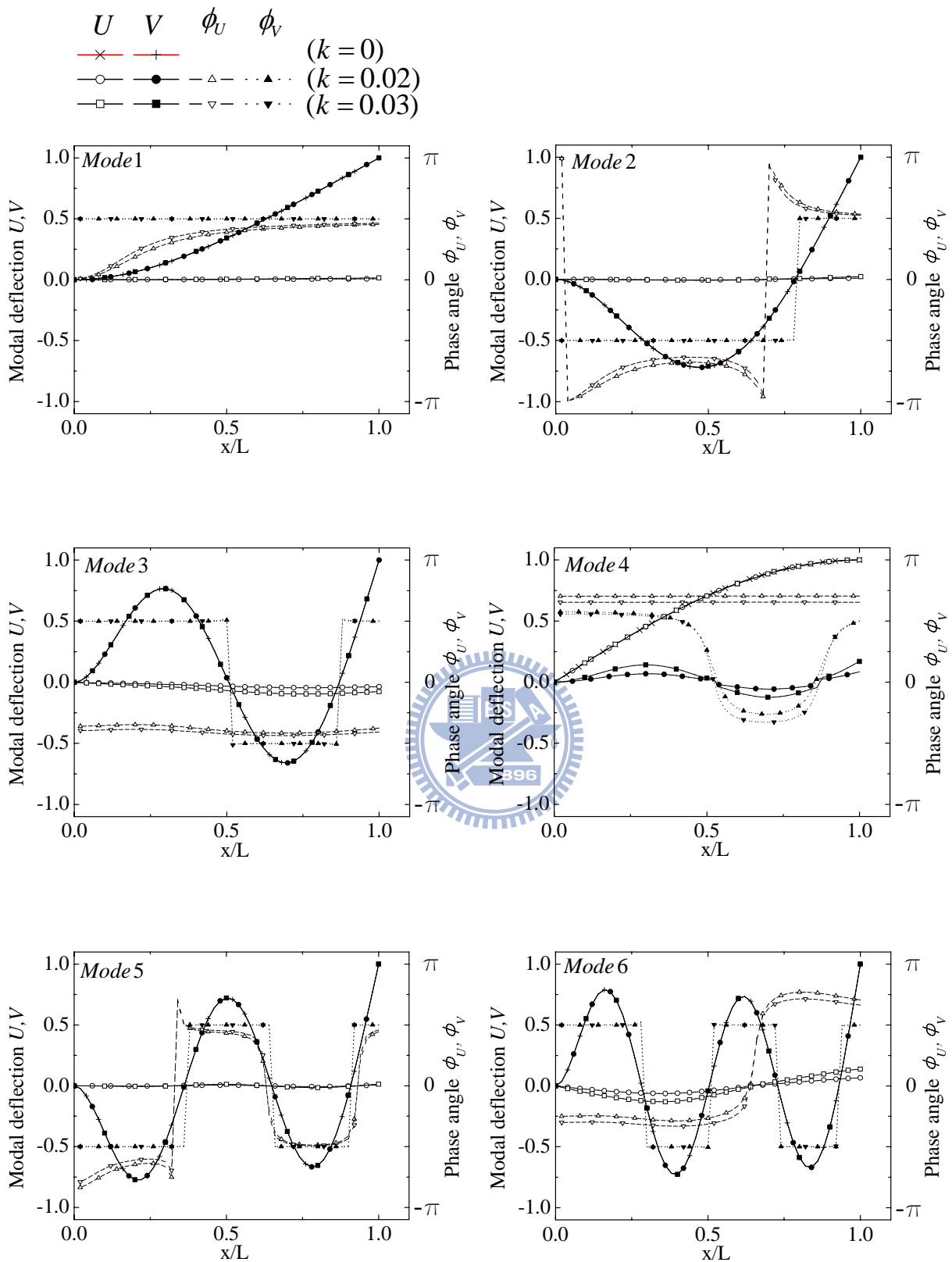
圖三十六 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r = 1$, $\eta = 39$)



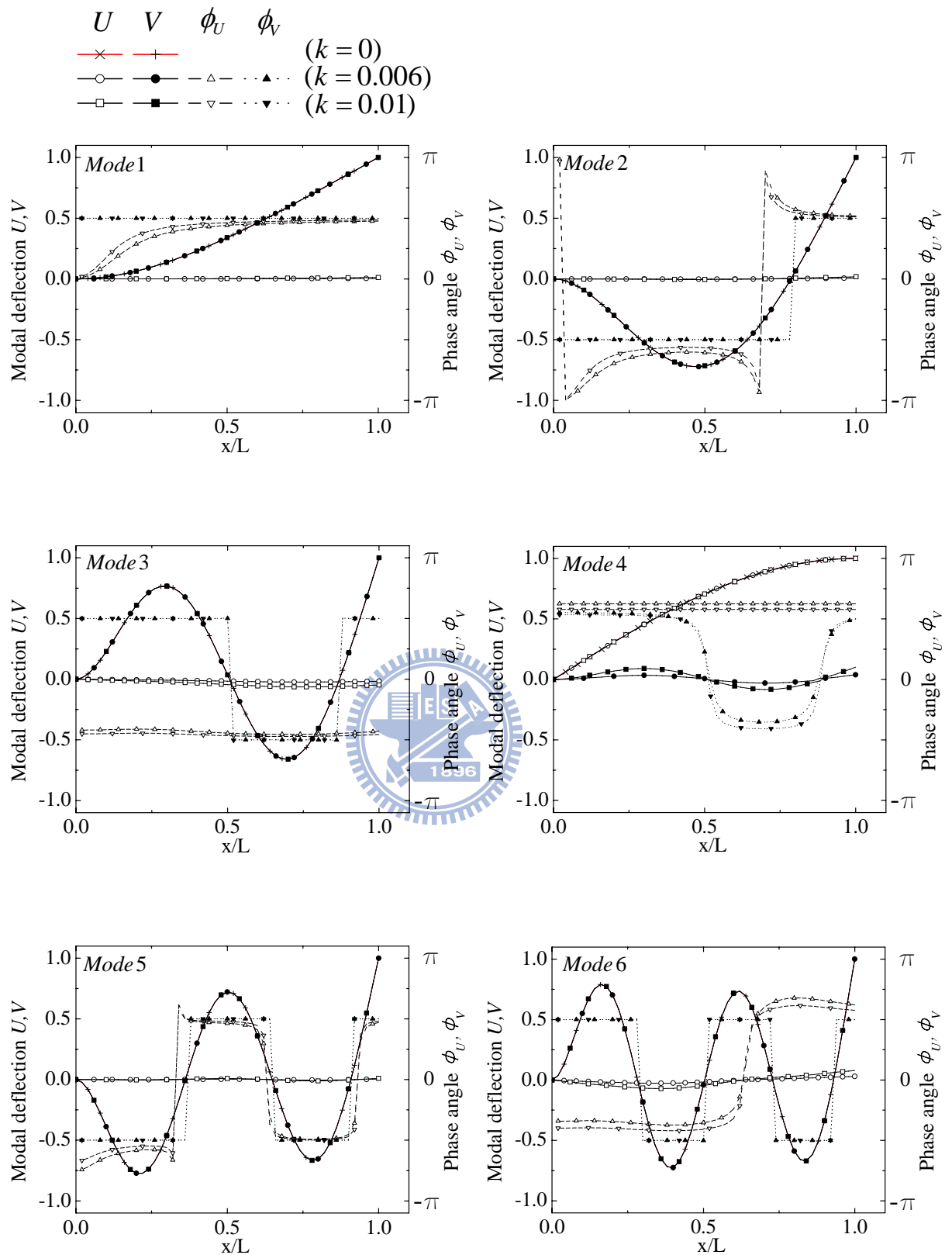
圖三十七 不同轉速下的第一至第六振動模態($\alpha = 90^\circ$, $r=1$, $\eta=39$)



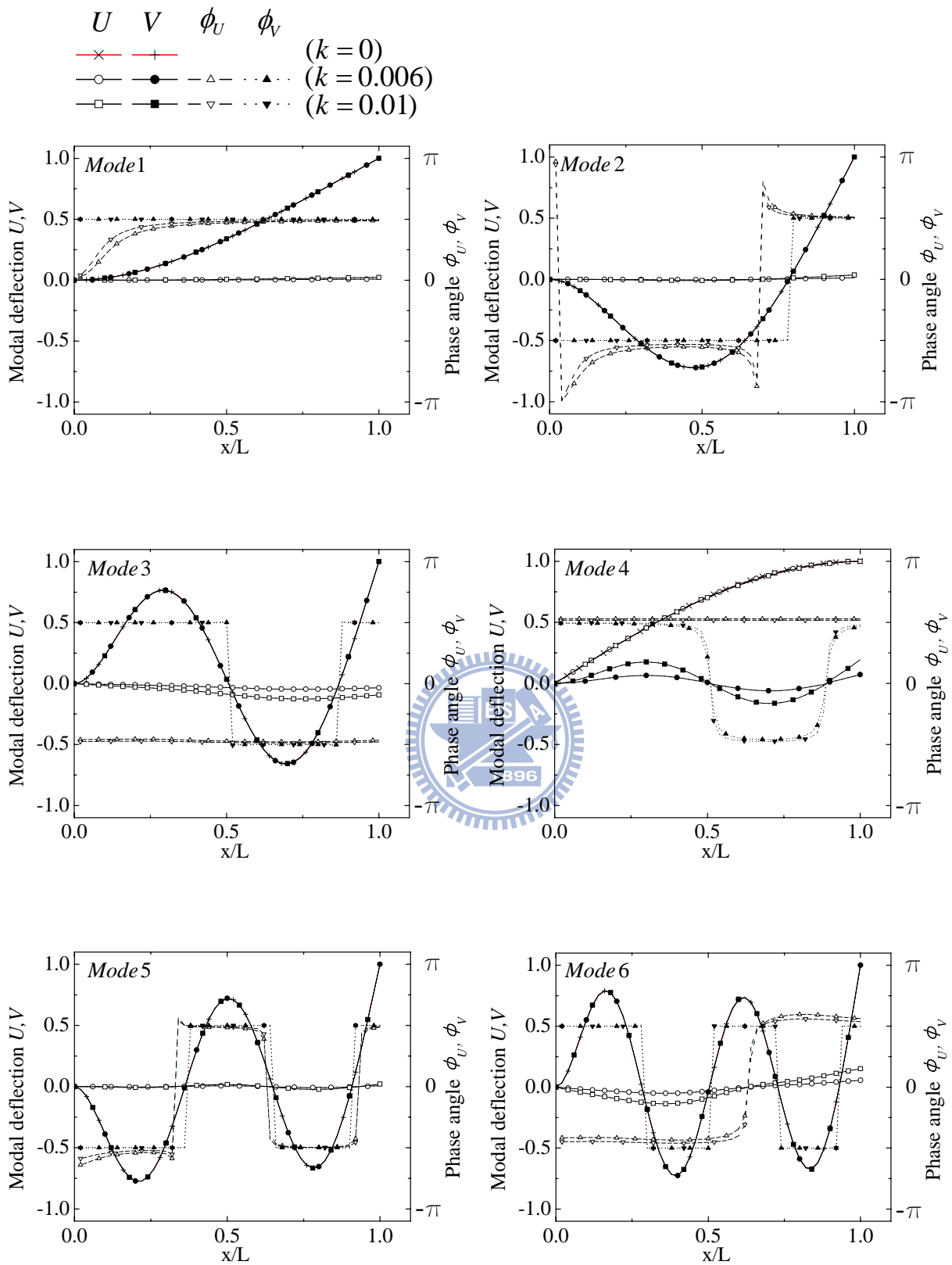
圖三十八 不同轉速下的第一至第六振動模態($\alpha = 0^\circ$, $r = 1$, $\eta = 40$)



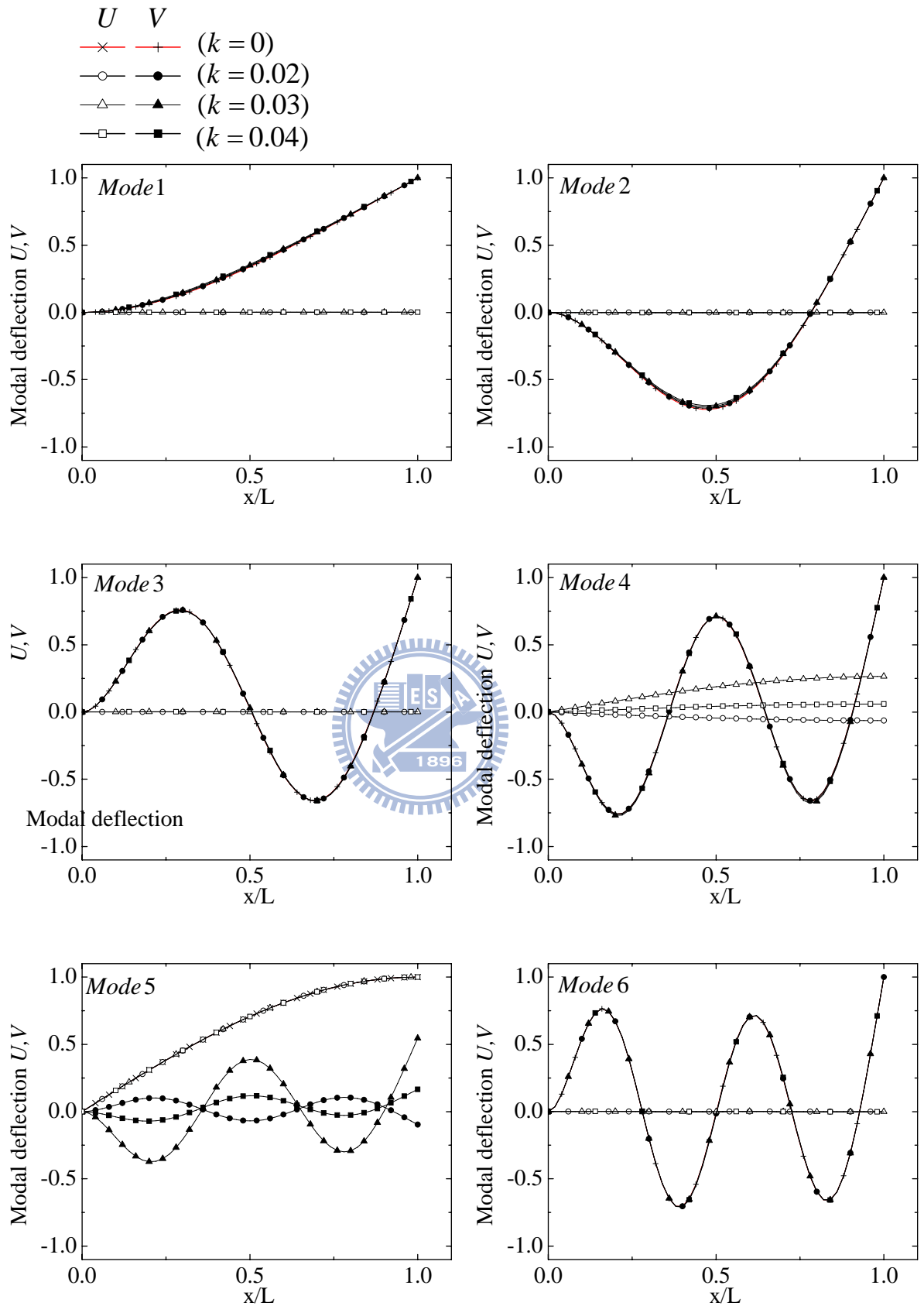
圖三十九 不同轉速下的第一至第六振動模態($\alpha = 5^\circ$, $r = 1$, $\eta = 40$)



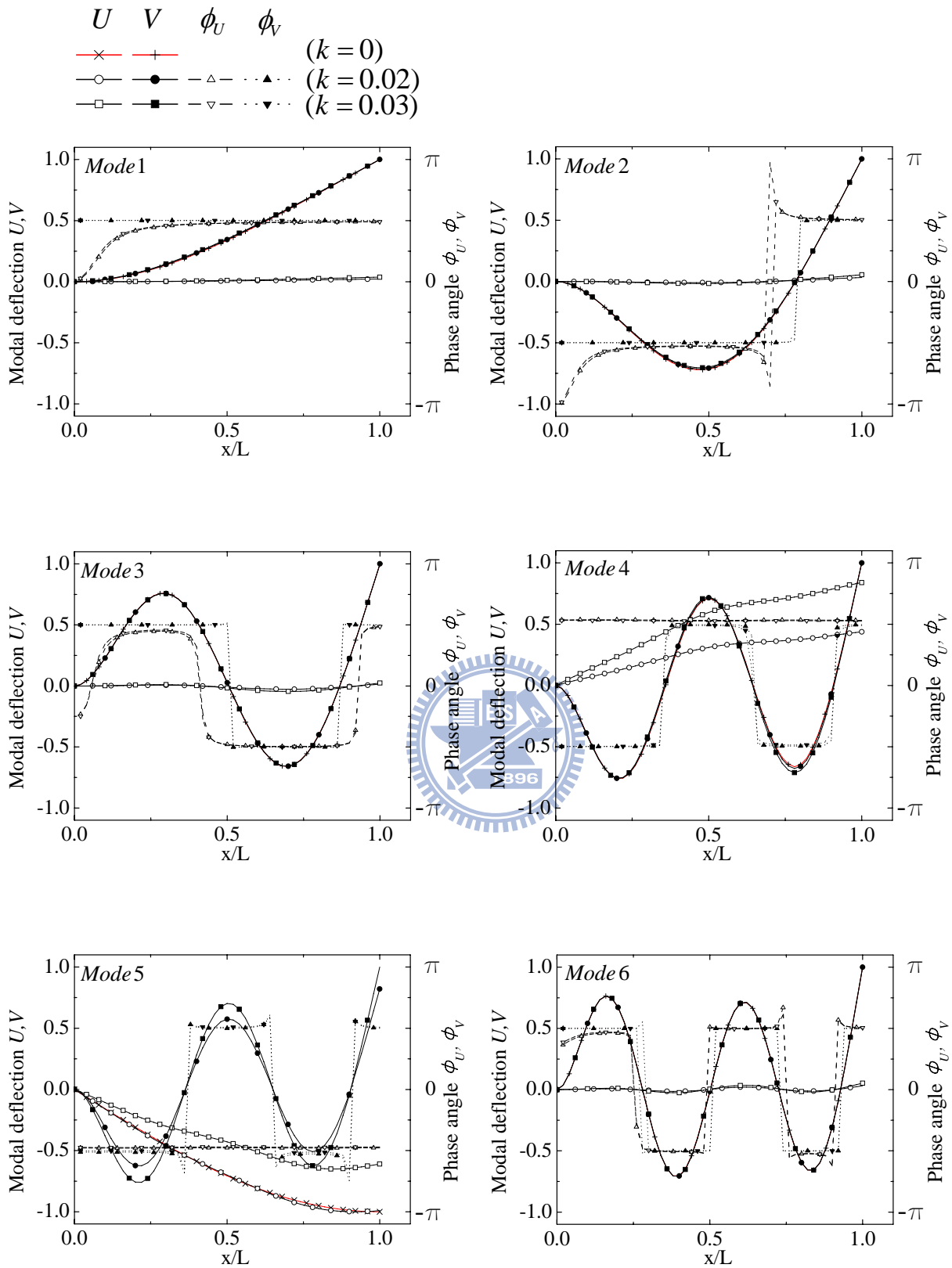
圖四十 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r = 1$, $\eta = 40$)



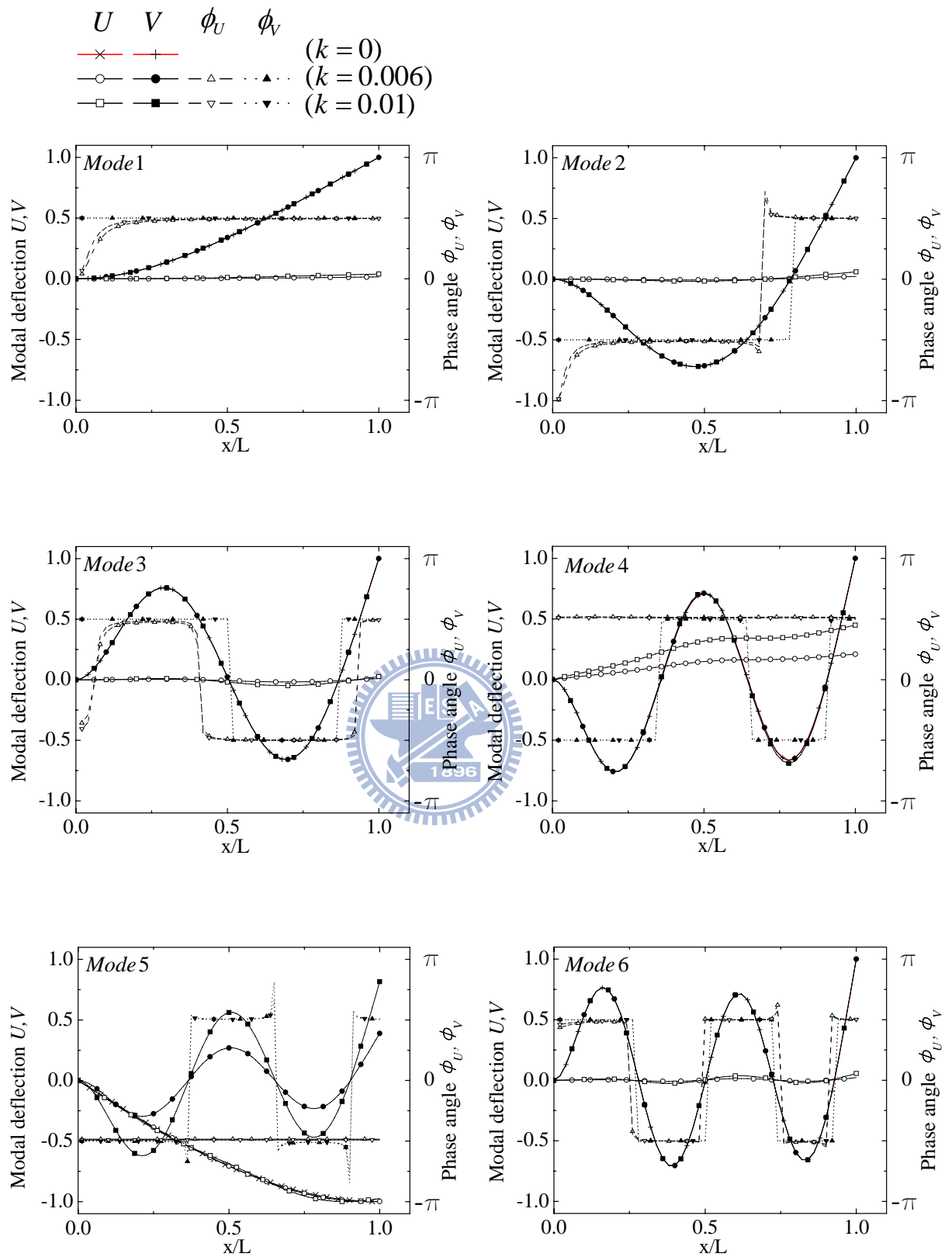
圖四十一 不同轉速下的第一至第六振動模態($\alpha = 90^\circ$, $r = 1$, $\eta = 40$)



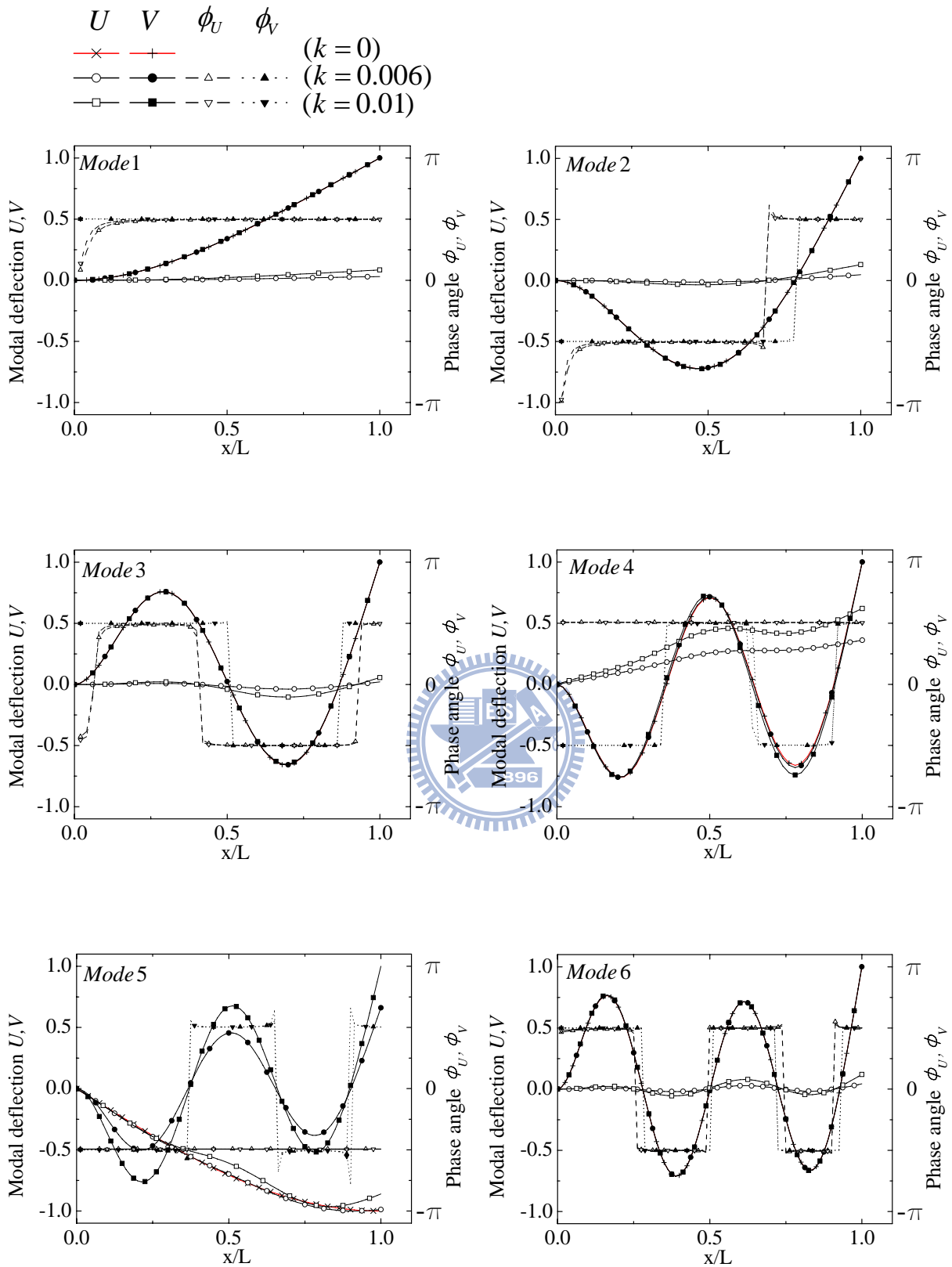
圖四十二 不同轉速下的第一至第六振動模態($\alpha = 0^\circ$, $r = 1$, $\eta = 77$)



圖四十三 不同轉速下的第一至第六振動模態($\alpha = 5^\circ$, $r = 1$, $\eta = 77$)



圖四十四 不同轉速下的第一至第六振動模態($\alpha = 30^\circ$, $r = 1$, $\eta = 77$)



圖四十五 不同轉速下的第一至第六振動模態($\alpha = 90^\circ$, $r = 1$, $\eta = 77$)

附錄A $\mathbf{f}_a^{D\theta}$ 及 $\mathbf{f}_b^{D\theta}$ 的推導

將(2.6.7)及(2.6.10)式代入(2.6.16)式可得

$$f_{12}^{\theta} = -f_{11}^{\theta} = f_{12} + \frac{v_1'}{L}m_1 + \frac{v_2'}{L}m_2 = f_{12} + \bar{f}_{12} \quad (\text{A1})$$

$$f_{21}^{\theta} = -f_{21} + (m_1v_1' + m_2v_2')G_{b1} / L = f_{21} + \bar{f}_{12}G_{b1} \quad (\text{A2})$$

$$f_{22}^{\theta} = -f_{22} + (m_1v_1' + m_2v_2')G_{b3} / L = f_{22} + \bar{f}_{12}G_{b3} \quad (\text{A3})$$

$$m_1^{\theta} = (1 + \varepsilon_0)m_1 + (m_1v_1' + m_2v_2')G_{b2} / L = (1 + \varepsilon_0)m_1 + \bar{f}_{12}G_{b2} \quad (\text{A4})$$

$$m_2^{\theta} = (1 + \varepsilon_0)m_2 + (m_1v_1' + m_2v_2')G_{b4} / L = (1 + \varepsilon_0)m_2 + \bar{f}_{12}G_{b4} \quad (\text{A5})$$

由(2.4.10)、(2.6.28)及(2.6.29)式可得

$$f_{12} = EA\varepsilon_0 - \frac{EI(1 - \varepsilon_0)}{L^2}(4v_1'^2 + 4v_1'v_2' + 4v_2'^2) \quad (\text{A6})$$

$$m_1 \approx (1 - \varepsilon_0)^2 \frac{EI}{L}(4v_1' + 2v_2') \quad (\text{A7})$$

$$m_2 \approx (1 - \varepsilon_0)^2 \frac{EI}{L}(2v_1' + 4v_2') \quad (\text{A8})$$

將(A6) - (A8)式代入(A1)式，並保留到三次項可得

$$f_{12}^{\theta} = EA\varepsilon_0 - \frac{EI}{L}\varepsilon_0 \int v_{,xx}^2 dx \quad (\text{A9})$$

由(A9)及(2.6.20)式可得

$$\mathbf{f}_a^{D\theta} = EA\varepsilon_0 \mathbf{G}_a - \frac{EI}{L}\varepsilon_0 \int v_{,xx}^2 dx \mathbf{G}_a \quad (\text{A10})$$

由(A2) - (A5)式及(2.6.24) - (2.6.27)、(2.6.29)式，保留部份的三次項，可得

$$\begin{aligned} \mathbf{f}_b^{D\theta} = & (\mathbf{T}_{\theta b}^1 + \mathbf{T}_{\theta b}^2)[EA\varepsilon_0 \int \mathbf{N}'_b v_{,x} dx + EI(1 - \varepsilon_0)^2 \int \mathbf{N}''_b v_{,xx} dx - EI(1 - \varepsilon_0) \int v_{,xx}^2 dx \mathbf{G}_b] \\ & \approx \mathbf{T}_{\theta b}^{1t} EI(1 - \varepsilon_0)^2 \int \mathbf{N}''_b v_{,xx} dx + (m_1v_1' + m_2v_2')\mathbf{G}_b + EA\varepsilon_0 \int \mathbf{N}'_b v_{,x} dx \\ & - EI(1 - \varepsilon_0) \int v_{,xx}^2 dx \mathbf{G}_b \approx \mathbf{T}_{\theta b}^{1t} EI(1 - \varepsilon_0)^2 \int \mathbf{N}''_b v_{,xx} dx + \mathbf{f}_{12}^{\theta} \int \mathbf{N}'_b v_{,x} dx \quad (\text{A11}) \end{aligned}$$

附錄 B 旋轉傾斜梁的線性穩態解

若不考慮旋轉傾斜梁穩態變形對慣性力及梁之側向剛度的影響，則可用 d'Alembert 原理及剛體動力學求得慣性分布力，再用材料力學可以求得旋轉傾斜梁的線性穩態解。

令圖 B1 中 \mathbf{r} 為轉軸 A 到旋轉梁上任一點 P 的位置向量， q_1 及 q_2 為作用在 P 點 X_1 及 X_2 方向的慣性分布力，則作用在 P 點之一小段梁 dx 的慣性分布力可以表示成

$$\{q_x, q_y\} dx = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \rho A dx \quad (\text{B.1})$$

$$\boldsymbol{\Omega} = \{0, 0, \Omega\} \quad (\text{B.2})$$

$$\mathbf{r} = \{R \cos \alpha + x, -R \sin \alpha, 0\} \quad (\text{B.3})$$



其中 ρ 、 A 為梁的密度及斷面積， Ω 為轉速。

將(B.2)、(B.3)式代入(B.1)式整理可得

$$q_x = \frac{k^2 EA}{L_T^2} (R \cos \alpha + x) \quad (\text{B.4})$$

$$q_y = -\frac{k^2 EA}{L_T^2} R \sin \alpha \quad (\text{B.5})$$

$$k^2 = \frac{\rho \Omega^2 L_T^2}{E} \quad (\text{3.3})$$

其中 k 為無因次化轉速。由(B.4)及(B.5)式可知旋轉傾斜梁相當於受到如圖 B2 所示的分布力。

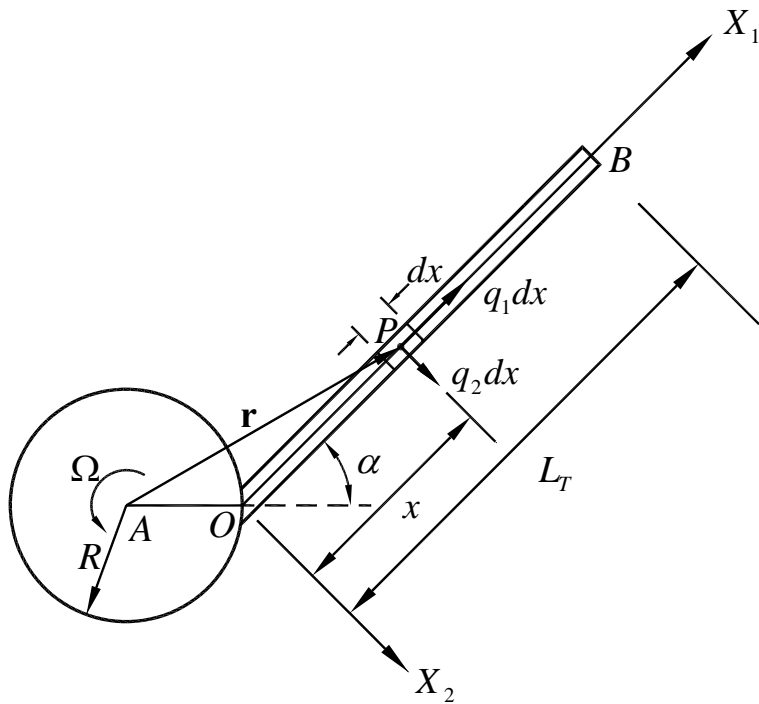


圖 B1

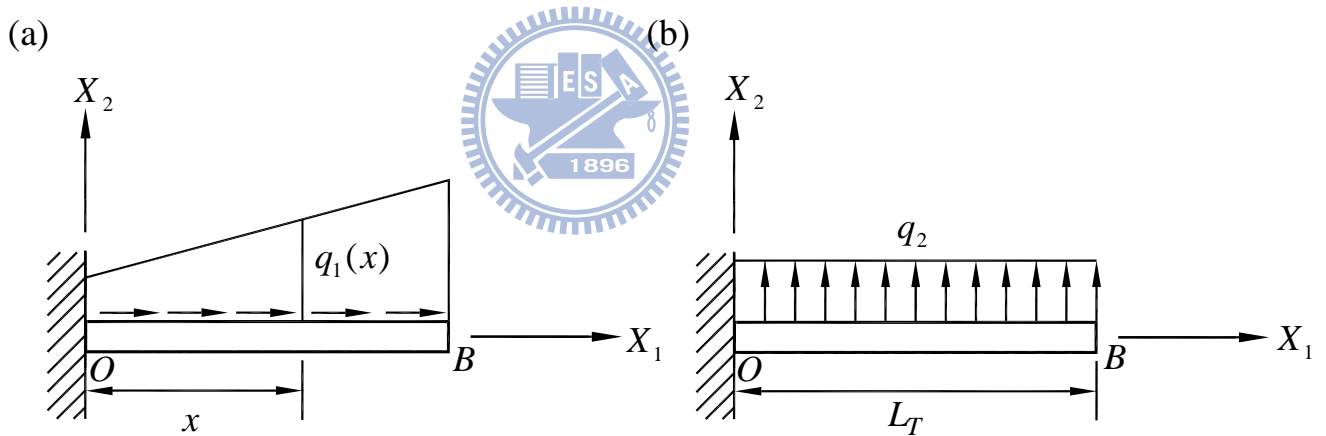


圖 B2

由(B.5)式及圖 B2(a)可得旋轉傾斜梁在距離固定端 x 處的膜應變

(membrane strain) $\varepsilon_0(x)$ 可表示成

$$\varepsilon_0 = \frac{F_x}{AE} = \frac{k^2}{L_T^2} \left[R(L_T - x) \cos \alpha + \frac{1}{2} (L_T^2 - x^2) \right] \quad (\text{B.6})$$

由(B.6)式可知當 $x = 0$ 時，旋轉傾斜梁有最大的膜應變 $\varepsilon_{0\max}$

$$\varepsilon_{0\max} = k^2 \left(r \cos \alpha + \frac{1}{2} \right) \quad (\text{B.7})$$

$$r = \frac{R}{L_T} \quad (\text{3.5})$$

其中 r 為無因次轉軸半徑。

由(B.5)式及圖 B2(b)可得旋轉傾斜梁在自由端的端點位移 δ_B 為[18]

$$\frac{\delta_B}{L_T} = \frac{q_y L_T^4}{8EI L_T} = \frac{\eta^2 k^2 r \sin \alpha}{8} \quad (\text{B.8})$$

$$\eta = \sqrt{\frac{AL_T^2}{I}} \quad (\text{3.4})$$

其中 E 為楊氏係數， I 為梁斷面的二次矩， η 為細長比， r 為無因次轉軸半徑。



若梁的斷面為矩形，厚度為 t ，寬度為 b ，則 $I = \frac{1}{12} bt^3$ 、 $A = bt$ 、

$t = \frac{2\sqrt{3}}{\eta} L_T$ ，由(B.5)式可得梁之任一斷面的最大撓曲應變(flexural strain)為

[18]

$$\varepsilon_b = \frac{M}{EI} \frac{t}{2} = \frac{k^2 EA(L_T - x)^2}{2EI L_T^2} \frac{t}{2} R \sin \alpha = \frac{\sqrt{3}\eta k^2 (L_T - x)^2 r \sin \alpha}{2L_T^2} \quad (\text{B.9})$$

由(B.9)式可知當 $x = 0$ 時， ε_b 有最大值，且可表示成

$$\varepsilon_{b\max} = \frac{\sqrt{3}\eta k^2 r \sin \alpha}{2} \quad (\text{B.10})$$