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EXISTENCE AND CLASSIFICATION OF SOLUTIONS FOR A PROBLEM OF SURFACE-TENSION DRIVEN FLOWS IN A SLOT WITH AN INSULATED BOTTOM

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Abstract—Existence and classification of solutions of a problem [two-point boundary value problems (TPBVP)]

$$f''' + Q\cdot(ff'' - (f')^2) = \beta, \quad f(0) = f(1) = f'(1) = f''(0) + 1 = 0,$$

which arises from a similarity study for surface-tension driven flows of low Prandtl number fluids in a slot with an insulated bottom are presented. By applying the Leray–Schauder fixed point theorem, we show that for each $Q > 0$ TPVBP has at least one solution. Combined with the one in Ref. [1], the existence property of solutions of TPBVP is completely verified. With the shooting method, we are able to further classify all possible solutions and conclude that TPBVP only possesses either two- or three-cell solutions.

1. INTRODUCTION

We consider the following two-point boundary value problem (TPBVP)

$$f''' + Q\cdot(ff'' - (f')^2) = \beta, \qquad (' = \mathrm{d}/\mathrm{d}\eta), \tag{1}$$

subject to the conditions

$$f(0) = f(1) = f'(1) = f''(0) + 1 = 0. \tag{2}$$

The TPBVP arises from a similarity reduction of boundary layer approximation of Navier–Stokes system in a microgravity environment [2]. The Navier–Stokes system was applied to describe steady states of distributions of velocity profiles of plane flows in a low Prandtl number (Pr) fluid, with the kinematic viscosity ν and density ρ, in a slot with an insulated bottom. Suppose that a constant heat flux with the temperature T_c is imposed at the center of free surface and the given fluid is at temperature T_l with $T_c \neq T_l$. Then, the surface-tension driven flows occur due to the temperature differences. The study of the resultant surface-tension driven flows is important such as in the production of silicon crystal as in Refs [2, 3]. Usually, the striation-free of the crystal is required. Therefore, the onset of oscillatory solutions (u, v) of the Navier–Stokes system in rectangular coordinate system (x, y), $-l \leqslant x \leqslant l$, $0 \leqslant y \leqslant d$, is concerned. Suppose that the quadratic radiation of temperatures between the center $x = 0$ and two ends $x = \pm l$ along the free surface $(-l \leqslant x \leqslant l, y = 0)$ is assumed. Then the similarity variable η is defined by $\eta = y/d$ while $\eta = 1$ represents the position at the insulated bottom $y = d$ and $\eta = 0$ denotes the free surface. Furthermore, the similarity function f in equation (1) is related to the stream function of flows satisfying the correlation $(u, v) = (c_1 x f'(\eta), -c_2\eta f'(\eta) + c_3 f(\eta))$, where constants c_i are properly chosen. Moreover, the dimensionless parameters Q and β are defined by $Q = 2(d/L)^3$ (Ma/Pr) and $\beta = (d^2/\nu\rho c x)\cdot(\partial P/\partial x)$, where Ma denotes the Marangoni number, the constant c is suitably chosen and the pressure distribution $P(x, y)$ is quadratic in x which satisfies the Bernoulli's law.

In Ref. [1], numerical solutions of the TPBVP were found for each $Q \geqslant -49.743$ by applying a multiple shooting code BVPSOL. In fact, a value Q_*, $Q_* \cong 6000$, was observed that the TPBVP has two-cell solutions when Q lies in an interval $[-49.743, Q_*]$ and three-cell solutions for $Q > Q_*$. By means of two-cell solution, the derivative function f' changes sign on $(0, 1)$ only once which corresponds to that the u-component velocity of surface-tension driven flows changes sign once along each x-level line. The sign change in u-component results the occurrence of two cells with one on each half of the slot. Similarly, three-cell solutions are termed if f' changes signs twice while

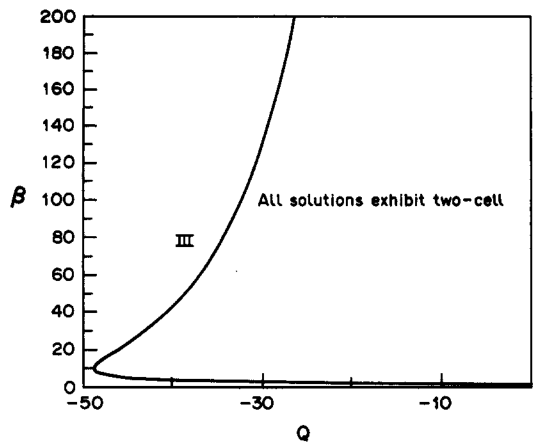

Fig. 1. Bifurcation diagram of solutions of the TPBVP with $Q \leqslant 0$; Q vs β.

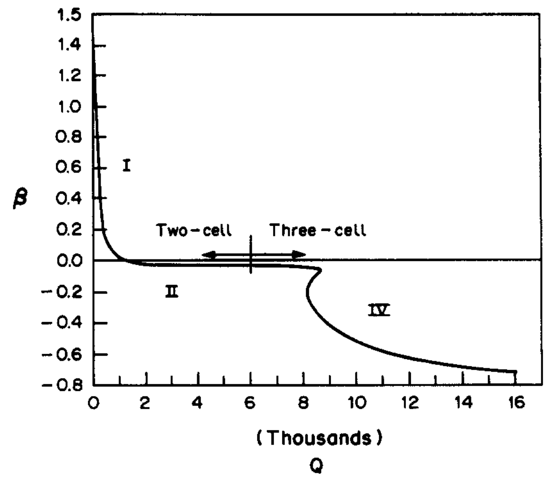

Fig. 2. Bifurcation diagram of solutions of the TPBVP with $Q \geqslant 0$: Q vs β.

the third cell occurs near the midline $y = 0$ of the slot. Also, multiple solutions of the types two- and three-cell were reported at some negative and positive values of Q, respectively. Moreover, Hwang *et al.* [1] also presented mathematical result that for each $\beta > 0$, the TPBVP has a nonnegative solution which exhibits two-cell. The result has verified only a small portion of the observed result in the bifurcation diagram Figs 1 and 2.

In this paper, we shall verify existence of solutions of the TPBVP for each $Q > 0$ by applying the Leray–Schauder fixed point theorem. Also, by applying the shooting method, we are able to classify all possible solutions of the TPBVP and conclude that the TPBVP can only possess either two- or three-cell solutions.

2. GENERAL EXISTENCE RESULT

In this section we shall prove the following theorem.

Theorem 2.1

For each $Q > 0$, there exists at least one real number β such that the TPBVP has a solution.

To prove Theorem 2.1, the qualitative property and *a priori* estimate of solutions for TPBVP are required. Differentiating equation (1) yields

$$f'''' + Q[ff''' - f'f''] = 0. \tag{3}$$

Hence, a lemma which described qualitative property in Ref. [3] is quoted as follows.

Lemma 2.2

If f is a solution of TPBVP, then $Qf'''' < 0$ and $ff''' - f'f'' > 0$ on [0, 1). Moreover, $f'(0) > (0)$ and $0 \leqslant (f')^2 - ff'' \leqslant [f'(0)]^2$ on [0, 1].

By directly integrating equation (1) and applying condition (2), we see the TPBVP is equivalent to the following integral equations:

$$f(\eta) = f_0(\eta) + Q\int_0^1 [h(\eta)j(1,s) - J(\eta,s)][f'^2 - ff'']\,ds \tag{4}$$

and

$$\beta = \tfrac{3}{2} - 3Q\int_0^1 J(1,s)[f'^2 - ff'']\,ds, \tag{5}$$

where $J(\eta, s)$ is given by

$$J(\eta, s) = \begin{cases} \eta - \dfrac{\eta^2}{2} - \dfrac{s^2}{2}, & 0 \leqslant s < \eta \leqslant 1, \\ \eta(1-s), & 0 \leqslant \eta \leqslant s \leqslant 1, \end{cases}$$

where $f_0(\eta) = \frac{1}{4}\eta(\eta-1)^2$ and $h(\eta) = 3\int_0^1 J(\eta, s)\,ds = \frac{3}{2}\eta - (\eta^3/2)$. Then, by differentiating equation (4) twice, one obtains

$$f''(\eta) = -1 + \tfrac{3}{2}\eta - 3\eta Q \int_0^1 J(1, s)[f'^2 - ff''] \, ds + Q \int_0^\eta [f'^2 - ff''] \, ds. \tag{6}$$

From Lemma 2.2, we conclude that

$$-1 + \tfrac{3}{2}\eta - Q[f'(0)]^2\eta \leqslant f''(\eta) \leqslant -1 + \tfrac{3}{2}\eta + Q[f'(0)]^2\eta, \tag{7}$$

since Q is positive. Therefore, from equation (7), *a priori* estimate of $f'(0)$ will lead to *a priori* estimate of f''. Hence, we have the following lemma.

Lemma 2.3

Given any $Q > 0$, then $f'(0) \in (0, \frac{1}{4})$ and $\|f''\|_\infty \leqslant 1 + (Q/16)$, provided that f is a solution of the TPBVP.

Proof. Let

$$G(\eta, s) = \frac{1}{12} \begin{cases} \eta(s-1)^2[(3-\eta)^2 s - 2\eta^2], & 0 \leqslant \eta \leqslant s \leqslant 1, \\ s(\eta-1)^2[(3-s^2)\eta - 2s^2], & 0 \leqslant s \leqslant \eta \leqslant 1, \end{cases}$$

be the Green's function of $v'''' = 0$ satisfying $v(0) = v(1) = v'(1) = v''(0) = 0$. Then a solution of equation (3) with boundary condition (2) can be written as

$$f(\eta) = f_0(\eta) - Q \int_0^1 G(\eta, s)[ff''' - f'f''] \, ds. \tag{8}$$

Differentiating equation (8) and setting $\eta = 0$, we find

$$f'(0) = \tfrac{1}{4} - \tfrac{1}{4}Q \int_0^1 s(1-s)[ff''' - f'f''] \, ds.$$

Since $Q > 0$, Lemma 2.2 and a solution of equation (4) is also a solution of equation (8), we then have $0 < f'(0) < \frac{1}{4}$. Therefore, combining with equation (6), one obtains $\|f''\|_\infty \leqslant 1 + (Q/16)$.

Let $B = \{u \in C^2[0, 1]\colon u(0) = u(1) = u'(1) = 0\}$. Then B is a Banach space if B is given the C^2-norm of u, i.e., $\|u\|_B = \|u\|_\infty + \|u'\| + \|u''\|_\infty$. Moreover, consider a u in B and integrate u'' twice, one has

$$u'(t) = -\int_t^1 u''(s) \, ds$$

and

$$u(t) = \int_0^t u'(s) \, ds.$$

This shows that $\|u\|_\infty \leqslant \|u'\|_\infty \leqslant \|u''\|_\infty$ and $\|u\|_B \leqslant 3\|u''\|_\infty$. Now, we define an open set D and operator T by

$$D = \{u \in B\colon \|u\|_B < 4 + \tfrac{3}{16}Q\}$$

and

$$(T(f))(\eta) = f_0(\eta) + Q \int_0^1 [h(\eta)J(1, s) - J(\eta, s)][f'^2 - ff''] \, s, \quad f \in \bar{D}.$$

Then, the compactness and continuity of T can be described in the next lemma.

Lemma 2.4

The operator T is compact, continuous from $\bar{D}$ into B. Moreover, λT has no fixed point on the boundary of D for $\lambda \in [0, 1]$.

The desired property of the operator T is similar to the one in Lemma 1.4 in Ref. [1], we omit the proof here. With the preparation, the proof of Theorem 2.1 can be given as follows.

Proof of Theorem 2.1. Suppose that the topological degree $\deg(\cdot, 0, D)$ is defined as in Ref. [4]. By Lemma 2.4 and the homotopy invariant, we conclude that for each λ in [0, 1] $\deg(I, 0, D) = \deg(I - \lambda T, 0, D)$. It is clear that $\deg(I, 0, D) = 1$. Therefore, $\deg(I - \lambda T, 0, D) = 1 \neq 0$. Hence the Leray–Schauder fixed point theorem implies that T has a fixed point in D. Thus, from equation (4), the TPBVP has a solution. Q.E.D.

3. CLASSIFICATION OF SOLUTIONS

Although we have established existence of solutions of the TPBVP, the behavior of solutions is still unknown. In particular, the mathematical result gives no indication of existence of three-cell solutions. For this reason, we shall study the qualitative properties of solutions of the TPBVP in this section.

Note that the TPBVP has a unique solution $f_0(\eta) = \frac{1}{4}\eta(1 - \eta)^2$, for $Q = 0$ and $\beta = \frac{3}{2}$. Therefore, we are interested in the case of $Q \neq 0$. To study the solutions of the TPBVP, we set

$$g(\eta) = \frac{Q}{b} f\left(1 - \frac{\eta}{b}\right),$$

where b is a positive constant and to be determined. Then equations (1) and (3) tend to

$$g''' + g'^2 - g'' = B, \tag{9}$$

$$g'''' + g'g'' - gg''' = 0, \tag{10}$$

respectively, where $B = -Q\beta/b^4$. Moreover, we obtain the corresponding conditions:

$$g(0) = g'(0) = g(b) = g''(b) + \frac{Q}{b^3} = 0. \tag{11}$$

By further assuming values to $g''(0)$ and B, one can treat equation (9) as an initial value problem (IVP) subject to conditions $g(0) = g'(0) = 0$. Now we may integrate the IVP until (if possible) a point $\eta^* > 0$ is found, such that $g(\eta^*) = 0$ and $g''(\eta^*) \neq 0$. Then by setting

$$b = \eta^*, \qquad Q = -(\eta^*)^3 g''(\eta)^* \quad \text{and} \quad f(t) = \frac{b}{Q} g(b(1 - t)),$$

a solution of the TPBVP is found. Therefore, we shall classify all possible solutions of TPBVP by studying the solution of the IVP directly.

Let $g''(0) = A$. Now if $A = B = 0$, then the IVP has only a trivial solution $g_0(\eta) = 0$ for $\eta \geqslant 0$ which is not a desired solution of the TPBVP. Therefore, under the assumption $A^2 + B^2 > 0$, we shall present some analytical results as follows.

Lemma 3.1

If g is the solution of the IVP, then $g'''' < 0$ and $g'g'' - gg''' > 0$, for all $\eta > 0$.

Due to the similarity of Lemma 1.1 in Ref. [1], we omit the proof of Lemma 3.1 here. Now the qualitative properties of $g(\eta)$ will be discussed by examining the following four cases which are corresponding to the combinations of signs of A and B.

Theorem 3.2

If $A \leqslant 0$ and $B \leqslant 0$, then $g(\eta)$ has no positive zero.

Proof. From equation (9), we have

$$g(\eta) = \frac{A}{2}\eta^2 + \frac{B}{6}\eta^3 - \frac{1}{2}\int_0^\eta (\eta - s)^2 F(s)\,ds,$$

where $F(s) = g'^2(s) - g(s)g''(s)$. It is clear that, $F(0) = 0$, and, by Theorem 3.1, $F'(s) = g'(s)g''(s) - g(s)g'''(s) > 0$. Thus $F(s)$ must be positive for each $s > 0$ and $g(\eta)$ is negative for all $\eta > 0$, provided that $A \leqslant 0$ and $B \leqslant 0$. Q.E.D.

Theorem 3.3

If $A > 0$ and $B \leqslant 0$ then $g(\eta)$ has exactly one positive zero.

Proof. Since $g'''(0) = B \leqslant 0$ and by Theorem 3.1, g'' is concave and decreasing for $\eta \geqslant 0$. Hence g'' must cross the η-axis and, consequently, g' is concave and decreasing on $(0, \infty)$. Thus g' must cross the η-axis making g concave and decreasing. Thus g should cross the η-axis. This shows that there exists unique $\eta^* > 0$ such that $g(\eta^*) = 0$. Moreover g' changes sign once for all $\eta > 0$ and $g''(\eta^*) < 0$, this gives $Q > 0$ and $\beta \geqslant 0$. The solutions of the TPBVP corresponding to this zero of g will be designated as Section I solutions for the two-cell as in Fig. 2. Q.E.D.

Theorem 3.4

If $A \geqslant 0$ and $B > 0$ then g has exactly one positive zero.

Proof. Suppose $g''' > 0$ for all $\eta \geqslant 0$. Then g, g' and g'' are all positive in $(0, \infty)$. Now by differentiating equation (10), we obtain

$$g''''' = g''''g - (g'')^2. \tag{12}$$

Hence, we conclude that $g''''' < 0$ in $(0, \infty)$. This shows that g''' is concave and decreasing in $(0, \infty)$, thus g''' must cross η-axis. This leads to a contradiction. Hence g''' has exactly one positive zero and it makes g concave and decreasing for large η. Thus there exists unique $\eta_1 > 0$ such that $g(\eta_1) = 0$. Moreover, g' changes sign once and $g''(\eta_1) < 0$. This gives $Q > 0$ and $\beta < 0$. The solutions of the TPBVP corresponding to this zero of g will be designated as Section II solutions for the two-cell in Fig. 2. Q.E.D.

Theorem 3.5

If $A < 0$ and $B > 0$ then g has either no positive zero or two positive zeros.

Proof. We shall consider the following cases.

Case 1 ($g''' > 0$ for $\eta > 0$). Suppose that g''' crosses the η-axis at some $c > 0$, then g' is strictly increasing and convex for $\eta > c$. Since $g'(c) < 0$, then $g'(\eta)$ must cross the η-axis and it makes g be convex and increasing. Thus g crosses the η-axis at a point, say η_2. Now equation (12) and Theorem 3.1 show that $g''''' < 0$ for $\eta > \eta_2$. Thus g''' is concave, decreasing for $\eta > \eta^*$ and must cross the η-axis. This contradicts $g''' > 0$ for all $\eta > 0$. Hence we obtain $g'' < 0$ for $\eta > 0$. This gives that both g' and g are negative for all $\eta > 0$. Thus g has no positive zero.

Case 2 (g''' crosses the η-axis at point $c > 0$). By Theorem 3.1, we have $g'(c)g''(c) > g(c)g'''(c) = 0$. If $g''(c) < 0$, then properties $g'''(c) = 0$ and $g''''(c) < 0$ give that $g''(\eta)$ takes the maximum at c on $(0, \infty)$. Hence $g'' < 0$ for $\eta > 0$ and this yields that g has no zero.

If $g''(c) > 0$, then we have that $g''(0) = A < 0$ and g'' is concave, decreasing for $\eta > c$, g'' has exactly two positive zeros, says $d < \bar{d}$. Moreover, g'' has the following properties:

$$g''(\eta) < 0, \quad \text{for } \eta \in (0, d) \cup (\bar{d}, \infty)$$

and

$$g''(\eta) > 0, \quad \text{for } \eta \in (d, \bar{d}).$$

This gives $g'(d) < 0$. Thus, $g'(c) > 0$ implies that there exists a unique $e \in (d, c)$ such that $g'(e) = 0$ and $g(e) < 0$. Also, $g'(\bar{d}) > g'(e) = 0$ and g' is concave, decreasing in $(\bar{d}, \infty)$. Hence g' has exactly one zero $\bar{e} > \bar{d}$. This gives that g' has the following properties:

$$g'(\eta) < 0, \quad \text{for } \eta \in (0, e) \cup (\bar{e}, \infty)$$

and

$$g'(\eta) > 0, \quad \text{for } \eta \in (e, \bar{e}).$$

Moreover, $g(\bar{e})g'''(\bar{e}) < g'(\bar{e})g''(\bar{e}) = 0$ and $g'''(\bar{e}) < 0$, $g(\bar{e}) > 0$. Thus there exists a unique $\zeta \in (e, \bar{e})$ such that $g(\zeta) = 0$. Moreover, $g(\eta)$ and its derivatives have the following properties:

$$g(\eta) < 0, \quad \text{for } \eta \in (0, \zeta),$$

$$g'(\eta) \text{ changes sign once in } (0, \zeta),$$

and

$$g''(\zeta) > 0.$$

This corresponds to $Q < 0$ and $\beta > 0$. The solutions of the TPBVP corresponding to this zero of g will be designated as Section III solutions for the two-cell in Fig. 1.

On the other hand, since $g(\bar{e}) > 0$ and g is concave, decreasing in $(\bar{e}, \infty)$, g must have exactly one zero $\xi > \bar{e}$. In this case, $g(\eta)$ and its derivatives have the following properties:

$$g(\eta) \text{ changes sign twice in } (0, +\infty),$$

$$g'(\eta) \text{ changes sign twice in } (0, +\infty)$$

and

$$g''(\xi) < 0.$$

This shows that $Q > 0$ and $\beta < 0$. The solutions of the TPBVP corresponding to this zero of g will be designated as Section IV solutions for three-cell in Fig. 2.

Let $g(\eta; A, B)$ be the solution of equation (10) with $g''(0) = A$ and $g'''(0) = \text{B}$. Then we have the following lemma.

Lemma 3.6

$$g(\eta; A, B) = \lambda g\left(\lambda\eta; \frac{A}{\lambda^3}, \frac{B}{\lambda^4}\right), \text{ for all } \lambda > 0.$$

Proof. Given $\lambda > 0$ and set

$$h(t) = \frac{1}{\lambda} g\left(\frac{t}{\lambda}; A, B\right).$$

If we can show that $h(t)$ is the solution of equation (10) with

$$h''(0) = \frac{A}{\lambda^3} \quad \text{and} \quad h'''(0) = \frac{B}{\lambda^4},$$

then the lemma follows immediately. In fact, the kth derivatives of $h(t)$ yield that

$$h^{(k)}(t) = \left(\frac{1}{\lambda}\right)^{k+1} g^{(k)}\left(\frac{t}{\lambda}; A, B\right) \quad \text{for all } k = 1, 2, \ldots.$$

Hence we have

$$h(0) = h'(0) = h''(0) - \frac{A}{\lambda^3} = h'''(0) - \frac{B}{\lambda^4} = 0$$

and

$$\lambda^4 h'''' + \lambda^4 h'h'' - \lambda^4 hh''' = 0.$$

This yields the desired result. Q.E.D.

Theorem 3.7

Under the hypotheses of Theorem 3.5, there are two positive constants δ and M such that $\delta < M$ and

(i) if $B < \delta |A|^{4/3}$ then g has no zero
and
(ii) if $B > M|A|^{4/3}$ then g has two zeros.

Proof. It is clear that, from Theorem 3.2, the function $g(\eta; -1, 0)$ has the following properties:

$$g(\eta; -1, 0) \leqslant 0, \quad g'(\eta; -1, 0) \leqslant 0, \quad g''(\eta; -1, 0) \leqslant -1$$

and

$$g'''(\eta; -1, 0) \leqslant 0,$$

for all $\eta \geqslant 0$. Let $\bar{\eta} > 0$ be the value of η satisfying $g''(\bar{\eta}; -1, 0) = -2$ and ϵ_1 be defined by $\epsilon_1 = \frac{1}{2}\min\{|g^{(k)}(\bar{\eta})| 0 \leqslant k \leqslant 3\}$. Then, by continuity of solutions in initial values, there is a sufficiently small constant $\delta > 0$, more precisely one can choose $\delta < 1$, such that

$$\sum_{k=0}^{3} |g^{(k)}(\eta; -1, \bar{B}) - g^{(k)}(\eta; -1, 0)| < \epsilon_1, \quad \text{for all } \eta \in [0, \bar{\eta}],$$

provided that $0 < \bar{B} < \delta$. Hence we have

$$g''(\eta; -1, \bar{B}) < -1 + \epsilon_1 < 0, \quad \text{for all } \eta \in [0, \bar{\eta}]$$

and

$$g'''(\bar{\eta}; -1, \bar{B}) < g'''(\bar{\eta}; -1, 0) + \epsilon_1 < 0.$$

This shows that $g''(\eta; -1, \bar{B}) < 0$, for all $\eta \geqslant \bar{\eta}$. Thus $g(\eta; -1, \bar{B}) < 0$, for all $\eta > 0$. Now if $B < \delta |A|^{4/3}$ then, since $g(\eta; A, B) = |A|^{1/3} g(|A|^{1/3}\eta; -1, B|A|^{-4/3})$, $g(\eta; A, B)$ has no positive zero. This proves the first assertion.

To prove assertion (ii), we consider the function $g(\eta; 0, 1)$ which by Theorem 3.4, has the property that $g'''(\eta; 0, 1)$ has exactly one zero. Let $\xi_1 > 0$ be the zero of $g'''(\eta; 0, 1)$ and $\epsilon_2 = \frac{1}{2}\min\{|g'''(\xi_1 + 1; 0, 1)|, g''(\xi_1; 0, 1)\}$. Then by continuity of solutions in initial conditions again, there is a sufficiently small constant $\bar{\delta} > 0$, in fact we can choose $\bar{\delta} < 1$, such that

$$\sum_{k=0}^{3} |g^{(k)}(\eta; 0, 1) - g^{(k)}(\eta; \bar{A}, 1)| < \epsilon_2, \text{ for all } \eta \text{ on } [0, \xi_1 + 1],$$

provided that $0 < -\bar{A} < \bar{\delta}$. Hence we obtain that

$$g'''(\xi_1 + 1; \bar{A}, 1) < g'''(\xi_1 + 1; 0, 1) + \epsilon_2 < 0$$

and

$$g''(\xi_1; \bar{A}, 1) > g''(\xi_1; 0, 1) - \epsilon_2 > 0.$$

Thus, the properties $g'''(0; \bar{A}, 1) = 1 > 0$ and $g''(0; \bar{A}, 1) = \bar{A} < 0$ imply that $g'''(\eta; \bar{A}, 1)$ and $g''(\eta; \bar{A}, 1)$ have exactly one and two positive zeros, respectively. If

$$B > \left[\frac{|A|}{\bar{\delta}}\right]^{4/3}$$

then, $g(\eta; A, B) = |B|^{1/4} g(|B|^{1/4}\eta; A|B|^{-3/4}, 1)$ and $g(\eta; A, B)$ has two positive zeros. This completes the proof of this theorem. Q.E.D.

4. CONCLUSION

The results here and in Ref. [1], when combined, we verify the existence of solutions of the TPBVP for each $\beta > 0$ and $Q > 0$. The result of classification for all possible solutions implies that the TPBVP can only possess either two- or three-cell solutions. A similar result has also been reported by Wang and Chen [5] for the surface-tension driven flows in floating disks and slots. For either problem, the exact multiplicity of solutions is open at this moment. The rigorous verification of the multiplicity may rely on the further delicate numerical investigation on the classified regions presented in Section 3 and Ref. [5].

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