Digital simulation of frequency modulation reticles

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Abstract. Digital simulation is needed for the analysis of the detector signal produced in an optical tracking system by a frequency modulation (FM) reticle that manipulates a target image of any possible shape. By introducing a model in which the image spot is divided into arcuate strips, we develop a system of matrix operations for the digital simulation of the detector signal. By applying this method to examples of various spot shapes, such as disk and coma, we show its capability for handling the digital simulation of FM reticles operated in connection with arbitrarily shaped images.

Subject terms: digital reticle simulation; FM reticles; optical tracking; exposure sampling; cyclic matrices.

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CONTENTS

1. INTRODUCTION

Optical tracking systems operated to aim at a target emanating infrared radiation are widely applied in the fields of astronomical observation, industrial machinery, etc. One ingenious technique in the design of an optical tracking system is the use of reticles[1,2] to modulate the incident light flux. There are two types of reticles, classified according to the characteristic feature of modulation, namely, amplitude modulation (AM) reticles[3,4] and frequency modulation (FM) reticles.[5-8] The reticle modulates the light flux and at the detector output yields a waveform of voltage versus time similar to the AM or FM waveform usually treated in radio engineering. Because an FM signal suffers less noise interference than an AM signal, the FM reticle is of greater interest and therefore is the one considered in this paper.

The performance of a reticle depends not only on its geometrical design but also on the image shape formed within the tracking system. For simple image shapes, e.g., an idealized point[5,7] or an Airy disk of finite radius,[8] analytical results can be obtained. The results can thus be used as a basis for design.

Real cases are seldom so simple because either the target shape is considerably irregular or the optical unit in the tracking system introduces aberrations and thus the synthesized image formed within the tracking system is of complicated shape. In such a case, no analytical method is available to explain the performance of the reticle, and a digital simulation method is needed.

To the authors' knowledge, studies of digital simulation of either AM or FM reticles are rare. Craubner[4] introduced a method of matrix operation for AM reticles, while Anderson and Callary[6] established a method using fast Fourier transformation (FFT) for spatially invariant FM optical tracking systems. The sparsity of research in digital simulation of reticles seems to be due to the great complexity of the modulation process, which is worsened by the lack of well developed tools. For example, the early digital computers were of limited capability. However, the remarkable progress made in recent years in computer technology helps to reduce the difficulty. It is now possible to execute many simulation problems on a personal digital computer quite efficiently.

In this paper we present the results of one phase of research, carried out on an IBM AT personal computer, regarding the straightforward digital simulation of FM reticles.

2. FREQUENCY MODULATION RETICLE SYSTEM

Figure 1 shows a typical layout of an optical tracking system using an FM reticle. The light from a distant target is collected

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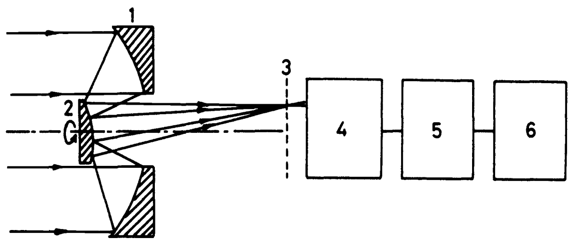

Fig. 1. FM reticle tracking system. (1) Cassegrain objective, (2) rotating mirror, (3) reticle, (4) detector, (5) signal processor, (6) controller.

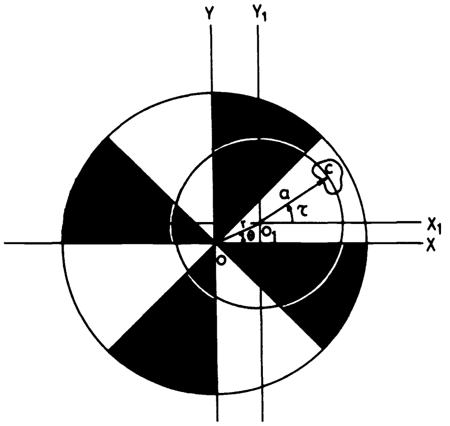

Fig. 2. FM reticle.

by the optical unit, which is usually of the form of a Cassegrain objective with its center mirror tilted slightly with respect to the axis, to form an image spot in the plane of a reticle, the center O of which is on the axis. The light passing through the reticle falls onto the detector behind it.

During operation, the tilted center mirror is kept rotating, causing the image to nutate in the reticle plane along a circular loop whose center O_1 in general does not coincide with O. (It coincides with O only when the target is on the axis.) The reticle (Fig. 2) is a disk having spokes of alternating transmittances 0 (opaque) and 1 (transparent). During nutation the image is manipulated by the reticle spoke(s) that it meets. Thus, the resultant light power transmitted through the reticle and illuminating the detector varies with time, yielding a waveform that not only bears the feature of frequency modulation but also has amplitude distortion. The detector output signal, being the voltage proportional to light power, is thus also frequency modulated and amplitude distorted.

Figure 2 shows a typical FM reticle comprising 2m (where m is a positive integer) equally spaced spokes of alternating transmittances 0 and 1. Suppose that the image spot nutates counterclockwise in the reticle plane; then each spoke is designated counterclockwise by a number j ranging from 1 to 2m. Associated with each spoke are two edges that confine the effective region wherein the image spot can be manipulated, namely, the leading edge, at which the image spot enters the spoke, and the trailing edge, at which the image spot leaves the spoke. Actually, we need concern ourselves with only the leading edge since the trailing edge is merely the leading edge of the next spoke. In the system O-X-Y, the angle made by the axis OX with the leading edge of the *jth* spoke is η_j. The angular span of each spoke is $\eta_{j+1} - \eta_j$, which is equal to the constant π/m over the entire reticle.

As the image spot nutates, each individual point in the spot moves along a different circular loop of different radius common to the point O_1, the loop center, which is at the position (r,θ) with respect to the coordinate system O-X-Y. Among these points the spot centroid C is important for reference and should be noted here especially. It nutates along the circular loop of constant radius a and is at an instantaneous position (a,τ) with respect to the coordinate system O_1-X_1-Y_1. The centroid angle τ is linearly proportional to the time t if the angular speed of nutation is constant.

3. SIMULATION SCHEME

At any time t, i.e., at any value of τ, the light power received by the detector is equal to the integrated light intensity, i.e., the exposure, illuminating the detector front surface. The detector signal u, being proportional to the light power, is thus proportional to the exposure, i.e.,

$$u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(x,y)p(x,y,t)dxdy \ , \tag{1}$$

where x and y are the positional coordinates measured within the O-X-Y system in the reticle plane, r(x,y) is the light intensity transmittance function of the stationary reticle, and p(x,y,t) is the light intensity distribution function of the image spot. It is Eq. (1) that is to be carried out by some approximation technique so that the analysis of u can be performed by digital simulation.

As is usual for digital simulation, a data sampling scheme should be established first. In our problem, two quantities are to be simulated. One is the exposure-dependent signal u. The other is the time t, or the centroid angle τ, within one period of nutation. We therefore should establish a scheme that is twofold: first, the exposure sampling and then the nutation sampling.

3.1. Exposure sampling

To represent exposure by means of sampled data, we first transfer from the O-X-Y coordinate system to the O_1-X_1-Y_1 coordinate system, in which we will stay hereafter, and then approximate Eq. (1) by a summation:

$$u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(x_1,y_1)p(x_1,y_1,t)dx_1dy_1 \ ,$$

$$\approx \sum_{g=G_1}^{G_2}\sum_{h=H_1}^{H_2} r(x_1,y_1)p(x_1,y_1,t)\Delta_g x_1 \Delta_h y_1 \ , \tag{2}$$

where the parameters g and h indicate one of the linear zonal elements we divided in the reticle plane along the x_1 and y_1 directions, respectively. G_1 and G_2 as well as H_1 and H_2 are the marginal values that limit the range of g and h.

If the summation is really to be done going through the whole reticle plane, the ranges G_2 to G_1 and H_2 to H_1 would be very large and the simulation process would not be economic. However, consideration of $p(x_1,y_1,t)$ reveals that it is possible to reduce the summation range by assuming that inside the spot boundary the intensity is uniformly bright ($p = 1$) while outside the spot boundary the intensity vanishes ($p = 0$). In

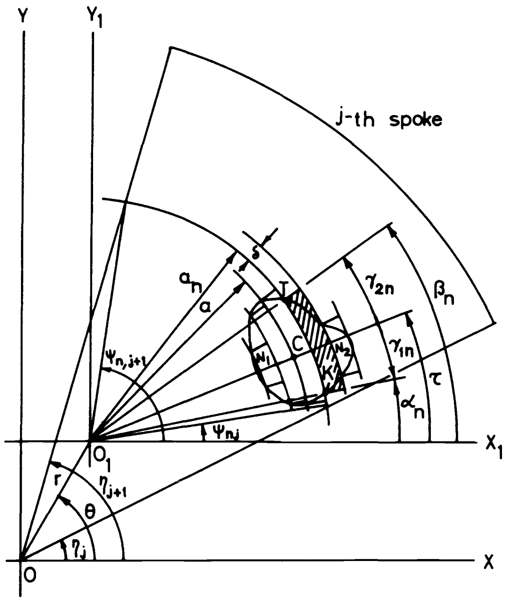

Fig. 3. Image spot divided into arcuate strips.

other words, we have $p\Delta x_I \Delta y_I = \Delta x_I \Delta y_I$ inside the spot boundary and $p\Delta x_I \Delta y_I = 0$ outside the spot boundary. Let $\Delta_n A = \Delta_g x_I \Delta_h y_I$ be the nth area element, ranging from $n = N_I$ to $n = N_2$, divided within the spot boundary. Then from Eq. (2) we have

$$u = \sum_{n=N_I}^{N_2} r(x_I, y_I)\Delta_n A \ , \tag{3}$$

in which N_2 to N_I could be small since the whole area within the spot boundary is usually not large. Therefore, the simulation can be processed very economically.

We now introduce a model in which the area element takes the form of an arcuate strip. Figure 3 shows, in detail, the image spot divided into $(N_2 - N_I + 1)$ arcuate strips, concentric with respect to O_I, with equal radial spacing δ. Each strip, indicated by the parameter n, has an inner radius, an outer radius, a head edge, and a tail edge. The lengths of the two edges are equal to δ. If δ is small, then the two edges are so short that they can be approximately represented by two points: the head point K for the head edge and the tail point T for the tail edge. In this model it is required that one of the strips have its inner radius equal to a and be assigned to be the strip $n = 0$. Based on this feature, a general form for the inner radius of any nth strip can be written

$$a_n = a + n\delta \ . \tag{4}$$

The head point and the tail point of the nth strip are at the positions (a_n, α_n) and (a_n, β_n), respectively, where α_n and β_n are the two angles related to the centroid angle τ by

$$\alpha_n = \tau - \gamma_{1n} \ , \qquad \beta_n = \tau + \gamma_{2n} \ , \tag{5}$$

in which γ_{1n} and γ_{2n} are the constant angles made by the line $O_I C$ with lines $O_I K$ and $O_I T$, respectively. The area of the nth strip is

$$\Delta_n A = a_n(\beta_n - \alpha_n)\delta \ . \tag{6}$$

Substituting Eq. (6) into Eq. (3) gives

$$u = \sum_{n=N_I}^{N_2} \Gamma_n(a_n\delta) \ , \tag{7}$$

where $\Gamma_n = (\beta_n - \alpha_n)r(x_I, y_I)$ is the factor of effective exposure representing the total effect of manipulation of the nth strip by the reticle spoke(s) at time instant t, or when the centroid angle is τ. Although Γ_n is of simple representative form here, it is not simple in actual operational form. The angular span $(\beta_n - \alpha_n)$ will be chopped by $r(x_I, y_I)$ into contributive and noncontributive part(s), thereby becoming of complicated form, which we will show.

Before investigating Γ_n in detail, we note two things. The first regards the description of the spoke leading edge in the coordinate system O_I-X_I-Y_I. The angle made by the axis $O_I X_I$ with the line connecting O_I to the point at which the leading edge of the jth spoke intersects the loop of the nth strip of the inner radius a_n is denoted by ψ_{nj}, which is related to η_j (discussed in Sec. 2) in the following way:

$$\psi_{nj} = \eta_j + \sin^{-1}\left[\frac{r\sin(\eta_j - \theta)}{a_n}\right] , \quad j = 1,2,3,\ldots,2m \ . \tag{8}$$

From this equation we see that angle ψ_{nj} varies not only from spoke to spoke but also from strip to strip. This is the reason we affix two subscripts, n and j, to it.

The second thing to note regards the adjustment of angles α_n and β_n to α_n' and β_n', respectively, as follows:

$$\alpha_n' = \begin{cases} \alpha_n \ , & \psi_{nI} \le \alpha_n < \psi_{nI} + 2\pi \ , \\ \alpha_n + 2\pi \ , & \psi_{nI} - 2\pi \le \alpha_n < \psi_{nI} \ , \\ \alpha_n - 2\pi \ , & \psi_{nI} + 2\pi \le \alpha_n < \psi_{nI} + 4\pi \ , \end{cases} \tag{9}$$

$$\beta_n' = \begin{cases} \beta_n \ , & \psi_{nI} \le \beta_n < \psi_{nI} + 2\pi \ , \\ \beta_n + 2\pi \ , & \psi_{nI} - 2\pi \le \beta_n < \psi_{nI} \ , \\ \beta_n - 2\pi \ , & \psi_{nI} + 2\pi \le \beta_n < \psi_{nI} + 4\pi \ . \end{cases} \tag{10}$$

We now turn to Γ_n. As encountered counterclockwise along the nth strip, there are three parts that contribute to Γ_n: the head part, the central part, and the tail part. For convenience, we discuss the head part first, then the tail part, and finally the central part.

The head part [Fig. 4(a)] of a strip extends from the strip head point to the trailing edge of the $j_I th$ spoke in which the strip head point is temporarily located, i.e., $\psi_{n,j_I+1} > \alpha_n' > \psi_{nj_I}$. The factor of effective exposure due to this part is denoted by $(\Gamma_h)_n$, which is

$$(\Gamma_h)_n = (\psi_{n,j_I+1} - \alpha_n')T_{j_I} \ , \tag{11}$$

or, in a general form,

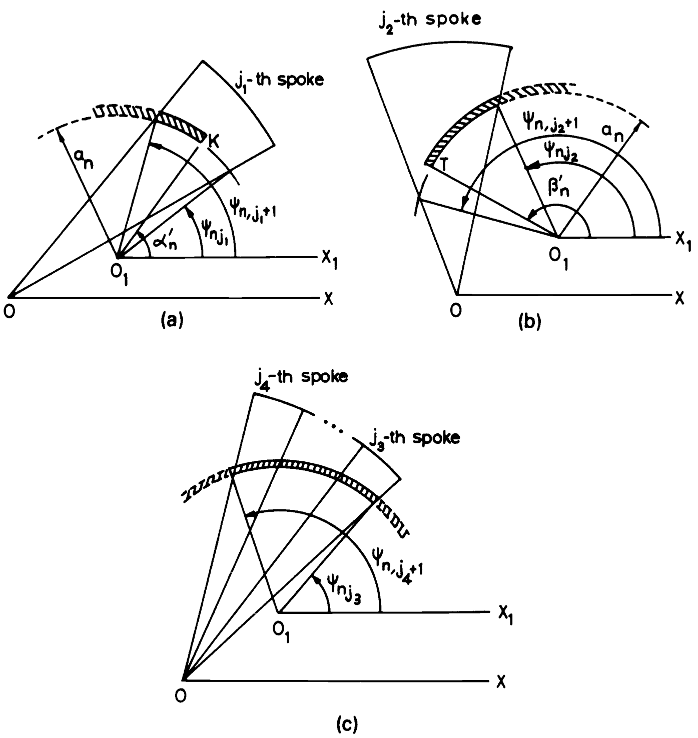

Fig. 4. Manipulation by the reticle spoke of (a) the strip head part, (b) the strip tail part, and (c) the strip central part.

$$(\Gamma_h)_n = \sum_{j=1}^{2m} \mathrm{STP}(\psi_{n,j+1} - \alpha_n')\mathrm{STP}(\alpha_n' - \psi_{nj})(\psi_{n,j+1} - \alpha_n')\mathrm{T}_j \ , \tag{12}$$

where

$$\mathrm{STP}(w) = \begin{cases} 1 \ , & w \geq 0 \ , \\ 0 \ , & w < 0 \ , \end{cases} \tag{13}$$

and

$$\mathrm{T}_j = \mathrm{STP}\left\{\tan\left[\frac{\pi}{4} + (j-1)\frac{\pi}{2}\right]\right\} \ , \tag{14}$$

which is the light intensity transmittance function of the j*th* spoke. For a transparent spoke, $\mathrm{T}_j = 1$. For an opaque spoke, $\mathrm{T}_j = 0$. In addition $\psi_{n,2m+1} \equiv \psi_{n1} + 2\pi$.

The tail part [Fig. 4(b)] of a strip extends from the strip tail point to the leading edge of the j_2*th* spoke in which the strip tail point is temporarily located, i.e., $\psi_{n,j_2+1} > \beta_n' > \psi_{nj_2}$. The factor of effective exposure due to this part is denoted by $(\Gamma_t)_n$, which is

$$(\Gamma_t)_n = (\beta_n' - \psi_{nj_2})\mathrm{T}_{j_2} \ , \tag{15}$$

or, in a general form,

$$(\Gamma_t)_n = \sum_{j=1}^{2m} \mathrm{STP}(\psi_{n,j+1} - \beta_n')\mathrm{STP}(\beta_n' - \psi_{nj})(\beta_n' - \psi_{nj})\mathrm{T}_j \ . \tag{16}$$

The central part [Fig. 4(c)] of a strip extends from the leading edge of the j_3*th* spoke, which is next to the j_1*th* spoke, to the trailing edge of the j_4*th* spoke, which precedes the j_2*th* spoke. The factor of effective exposure due to this part, $(\Gamma_c)_n$, behaves in a way that depends on the situation. There are two situation categories. The first involves the following three situations:

$$\begin{cases} \psi_{n1} - 2\pi \leq \alpha_n < \beta_n \leq \psi_{n1} \ , \\ \psi_{n1} \leq \alpha_n < \beta_n \leq \psi_{n1} + 2\pi \ , \\ \psi_{n1} + 2\pi \leq \alpha_n < \beta_n \leq \psi_{n1} + 4\pi \ . \end{cases} \tag{17}$$

For this category we have

$$(\Gamma_c)_n = \sum_{j=1}^{2m} [\mathrm{STP}(\beta_n' - \psi_{n,j+1}) - \mathrm{STP}(\alpha_n' - \psi_{nj})](\psi_{n,j+1} - \psi_{nj})\mathrm{T}_j \ . \tag{18}$$

The second category involves the following two situations:

$$\begin{cases} \alpha_n < \psi_{n1} < \beta_n \ , \\ \alpha_n < \psi_{n1} + 2\pi < \beta_n \ . \end{cases} \tag{19}$$

For this category we have

$$(\Gamma_c)_n = \sum_{j=1}^{2m} [\mathrm{STP}(\beta_n' - \psi_{n,j+1}) + \mathrm{STP}(\psi_{nj} - \alpha_n')](\psi_{n,j+1} - \psi_{nj})\mathrm{T}_j \ . \tag{20}$$

We can combine Eqs. (18) and (20) to obtain a general form:

$$(\Gamma_c)_n = \sum_{j=1}^{2m} \left\{\mathrm{STP}(\beta_n' - \psi_{n,j+1}) + \mathrm{STP}(\alpha_n' - \psi_{nj})\cos\left[\pi\left(1 - \frac{\xi_n}{2}\right)\right] + \mathrm{STP}(\psi_{nj} - \alpha_n')\sin\left(\pi\frac{\xi_n}{2}\right)\right\}(\psi_{n,j+1} - \psi_{nj})\mathrm{T}_j \ , \tag{21}$$

in which

$$\xi_n = \mathrm{STP}(\psi_{n1} - \alpha_n)\mathrm{STP}(\beta_n - \psi_{n1}) + \mathrm{STP}(\psi_{n1} + 2\pi - \alpha_n)\mathrm{STP}(\beta_n - 2\pi - \psi_{n1}) \ . \tag{22}$$

The factor of total effective exposure of the n*th* strip is

$$\Gamma_n = (\Gamma_h)_n + (\Gamma_t)_n + (\Gamma_c)_n \ . \tag{23}$$

Substituting Eqs. (12), (16), (21), and (23) into Eq. (7), we obtain the detector signal u at the time instant t or, equivalently, when the centroid angle is τ.

3.2. Nutation sampling

The above equations are derived under the assumption that τ can be any arbitrary value in the range $0 \leq \tau \leq 2\pi$. If one now asks how many individual instants, or how many τ values, are necessary for a satisfactory simulation, it is reasonable to argue that one need choose only 2m values of τ for nutation sampling. We suggest that each τ be chosen to be associated with the middle angle of each spoke, i.e.,

$$\tau = \tau_k = \phi_k + \sin^{-1}\left[\frac{r}{a}\sin(\phi_k - \theta)\right] , \quad (24)$$

where

$$\phi_k = \frac{\pi}{2m} + (k-1)\frac{\pi}{m} , \quad k = 1,2,3,\ldots,2m . \quad (25)$$

The simulated result for the signal u, when sampled at τ_k, is considered to be representative of the other u values that come out within the entire *kth* spoke.

4. MATRIX EXPRESSION

4.1. Basic expression

Combining the exposure sampling scheme with the nutation sampling scheme, we can establish a matrix expression for the detector signal u. Since only 2m instantaneous values of angular position τ are required for executing the simulation work in one sampling period, there are only 2m simulated results of the detector signal u in one period. We describe u by a column matrix with elements u_k, $k = 1, 2, 3, \ldots, 2m$, representing the u values corresponding to the centroid angular positions at τ_k, $k = 1, 2, 3, \ldots, 2m$, i.e.,

$$[u] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ u_{2m} \end{bmatrix} . \quad (26)$$

This in turn can be expressed as the product of two matrices:

$$[u] = [U]\times[I] , \quad (27)$$

where [U] is a diagonal matrix of the form

$$[U] = \begin{bmatrix} u_1 & 0 & - & - & - & 0 \\ 0 & u_2 & & & & | \\ | & & u_3 & & & | \\ | & & & \diagdown & & | \\ | & & & & \diagdown & 0 \\ 0 & - & - & - & 0 & u_{2m} \end{bmatrix} \quad (28)$$

and [I] is a $2m\times1$ column matrix with each element equal to unity:

$$[I] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} . \quad (29)$$

4.2. Special matrices

Before developing the representation of [U] in detail, we discuss the construction and the symbolic abbreviations of two special matrices: the diagonal matrix and the cyclic matrix.

4.2.1. Diagonal matrix

A diagonal matrix has nonzero elements along its diagonal only. If the values of these diagonal elements vary with a group of parameters, say, k, j, and l, that range from k_1 to k_2, j_1 to j_2, and l_1 to l_2, respectively, then we use a symbolic abbreviation

$$\mathrm{diag}\langle z_{kjl} | k(k_1,k_2)j(j_1,j_2)l(l_1,l_2)\rangle \quad (30)$$

to denote such a matrix. For example,

$$\mathrm{diag}\langle z_{kj} | k(1,2),j(1,4)\rangle = \begin{bmatrix} z_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_{24} \end{bmatrix} . \quad (31)$$

For a matrix such as

$$\begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_2 \end{bmatrix} , \quad (32)$$

a dummy parameter can be inserted into the symbolic abbreviation in such a way that

$$\mathrm{diag}\langle z_k | k(1,2)j(1,2)\rangle = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_2 \end{bmatrix} . \quad (33)$$

4.2.2. Cyclic matrix of type I

In studying the digital simulation of AM reticles, Craubner[4] introduced a cyclic matrix that we call the cyclic matrix of type I. We describe it briefly for reference.

Given a column matrix of elements z_k, $k = 1,2,3,\ldots,2m$, as a kernel column matrix, we shift the sequence of k one unit each time in a cyclic way until the kernel form is ready to appear again, e.g.,

$$\begin{aligned}
\text{given: } k &= 1,2,3,\ldots,2m \ ; \\
\text{shift to right: } k &= 2m,1,2,\ldots,2m-1 \ , \\
k &= 2m-1,2m,1,2,\ldots,2m-2 \ , \\
&\ \ \cdot \\
&\ \ \cdot \\
&\ \ \cdot \\
k &= 2,3,\ldots,2m,1 \ .
\end{aligned} \tag{34}$$

To each sequence feature of k, there is a corresponding sequence of z_k and hence a corresponding column matrix. The shifting recurrence yields a series of 2m column matrices. Their collection constructs the cyclic matrix of type I, i.e.,

$$\begin{bmatrix}
z_1 & z_{2m} & \cdot & \cdot & \cdot & z_2 \\
z_2 & z_1 & \cdot & \cdot & \cdot & z_3 \\
z_3 & z_2 & \cdot & & & z_4 \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & & \cdot & \cdot \\
z_{2m} & z_{2m-1} & \cdot & \cdot & \cdot & z_1
\end{bmatrix} . \tag{35}$$

In symbolic abbreviation, this is in turn denoted by

$$\mathrm{Zykl}_I <z_j|j(j_1,j_2)> \ , \tag{36}$$

which means that the elements that construct the kernel column are z_j's and the parameter j ranges from j_1 to j_2. For example,

$$\mathrm{Zykl}_I <z_j|j(1,4)> = \begin{bmatrix}
z_1 & z_4 & z_3 & z_2 \\
z_2 & z_1 & z_4 & z_3 \\
z_3 & z_2 & z_1 & z_4 \\
z_4 & z_3 & z_2 & z_1
\end{bmatrix} . \tag{37}$$

4.2.3. Cyclic matrix of type II

We have developed a second kind of cyclic matrix, which we call the cyclic matrix of type II. Given a sequence of elements z_k, $k=1,2,\ldots,2m$, we first construct a kernel column matrix of elements $z_i\mathrm{STP}(2m - i)$, $i=1,2,3,\ldots,(2m\times 2m)$. The kernel matrix is thus a $(2m\times 2m)\times 1$ matrix of the form

$$\begin{bmatrix}
z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_{2m} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0
\end{bmatrix} . \tag{38}$$

Then we shift the sequence of i by 2m units each time in a cyclic way until the kernel form appears again; e.g.,

$$\begin{aligned}
&\text{given: } z_i = z_1,z_2,z_3,\ldots,z_{2m},0,\ldots,0; \\
&\text{1st shift to right: } z_i = \underbrace{0,0,0,\ldots,0}_{2m},\underbrace{z_1,z_2,\ldots,z_{2m}}_{2m},\underbrace{0,\ldots,0}_{(2m\times 2m)-4m} \ ; \\
&\text{2nd shift to right: } z_i = \underbrace{0,\ldots,0}_{2m},\underbrace{0,\ldots,0}_{2m},\underbrace{z_1,z_2,\ldots,z_{2m}}_{2m},\underbrace{0,\ldots,0}_{(2m\times 2m)-6m} \ ; \\
&(2m-1)th \text{ shift to right: } z_i = \underbrace{0,\ldots,0}_{(2m\times 2m)-2m},\underbrace{z_1,z_2,\ldots,z_{2m}}_{2m} \ .
\end{aligned} \tag{39}$$

The recurrence yields a series of 2m column matrices whose elements are shown above. The collection of these column matrices constructs the cyclic matrix of type II, which is denoted by

$$\mathrm{Zykl}_{II} <z_j|j(1,2m)> \ , \tag{40}$$

i.e.,

$$\mathrm{Zykl}_{II} <z_j|j(1,2m)> = \begin{bmatrix}
z_1 & 0 & 0 & - & - & - & 0 \\
z_2 & 0 & 0 & & & & | \\
\vdots & \vdots & \vdots & & & & | \\
z_{2m} & 0 & 0 & & & & | \\
0 & z_1 & 0 & & & & | \\
| & z_2 & 0 & & & & | \\
| & \vdots & \vdots & & & & | \\
| & z_{2m} & 0 & & & & | \\
| & 0 & z_1 & & & & | \\
| & | & z_2 & & & & | \\
| & | & \vdots & & & & | \\
| & | & z_{2m} & & & & | \\
| & | & 0 & \cdot & & & | \\
| & | & | & & \cdot & & | \\
| & | & | & & & \cdot & | \\
| & | & | & & & & 0 \\
| & | & | & & & & z_1 \\
| & | & | & & & & z_2 \\
| & | & | & & & & \vdots \\
0 & 0 & 0 & - & - & 0 & z_{2m}
\end{bmatrix} . \tag{41}$$

This is a matrix of (2m×2m) rows and 2m columns. For example,

$$\mathrm{Zykl}_{II}\langle z_j | j(1,4)\rangle = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 \\ z_3 & 0 & 0 & 0 \\ z_4 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & z_3 & 0 & 0 \\ 0 & z_4 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & z_4 & 0 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & z_2 \\ 0 & 0 & 0 & z_3 \\ 0 & 0 & 0 & z_4 \end{bmatrix} \tag{42}$$

4.2.4. Cyclic matrix of type III

The third kind of cyclic matrix, which we call the cyclic matrix of type III, consists of nonzero elements that occupy the positions in the same manner described in Zykl_{II} but vary in value, i.e., there exists no recurring relations among the nonzero elements. Suppose that these elements vary with the parameters, say, k and j, that range from k_1 to k_2 and j_1 to j_2, respectively; then the cyclic matrix of type III will be denoted by

$$\mathrm{Zykl}_{III}\langle z_{kj} | k(k_1,k_2) j(j_1,j_2)\rangle \ , \tag{43}$$

which is the symbolic abbreviation for the matrix

$$\begin{bmatrix} z_{k_1,j_1} & 0 & - & - & - & 0 \\ z_{k_1,j_1+1} & 0 & & & & 0 \\ \vdots & \vdots & & & & | \\ z_{k_1,j_2} & 0 & & & & | \\ 0 & z_{k_1+1,j_1} & & & & | \\ | & \vdots & \cdot & & & | \\ | & & & \cdot & & | \\ | & z_{k_1+1,j_2} & & & \cdot & 0 \\ | & 0 & & & & z_{k_2,j_1} \\ | & \vdots & & & & \vdots \\ 0 & 0 & - & - & 0 & z_{k_2,j_2} \end{bmatrix} \tag{44}$$

For example,

$$\mathrm{Zykl}_{III}\langle z_{kj} | k(1,4) j(1,4)\rangle = \begin{bmatrix} z_{11} & 0 & 0 & 0 \\ z_{12} & 0 & 0 & 0 \\ z_{13} & 0 & 0 & 0 \\ z_{14} & 0 & 0 & 0 \\ 0 & z_{21} & 0 & 0 \\ 0 & z_{22} & 0 & 0 \\ 0 & z_{23} & 0 & 0 \\ 0 & z_{24} & 0 & 0 \\ 0 & 0 & z_{31} & 0 \\ 0 & 0 & z_{32} & 0 \\ 0 & 0 & z_{33} & 0 \\ 0 & 0 & z_{34} & 0 \\ 0 & 0 & 0 & z_{41} \\ 0 & 0 & 0 & z_{42} \\ 0 & 0 & 0 & z_{43} \\ 0 & 0 & 0 & z_{44} \end{bmatrix} \tag{45}$$

4.3. Full expression

Now that the symbolic abbreviations for certain special matrices are defined, we can develop the full expression of Eq. (28) in detail. With a discrete-nutation-sampling scheme, the instantaneous values of τ are rewritten as τ_k, k=1,2,3,...,2m, as shown in Eq. (24). The α_n and β_n in Eq. (5), the α'_n in Eq. (9), and the β'_n in Eq. (10) should be rewritten as α_{nk}, β_{nk}, α'_{nk}, and β'_{nk}, respectively, which are

$$\alpha_{nk} = \tau_k - \gamma_{1n} \ , \tag{46}$$

$$\beta_{nk} = \tau_k + \gamma_{2n} \ , \tag{47}$$

$$\alpha'_{nk} = \begin{cases} \alpha_{nk} \ , & \psi_{n1} \le \alpha_{nk} < \psi_{n1} + 2\pi \ , \\ \alpha_{nk} + 2\pi \ , & \psi_{n1} - 2\pi \le \alpha_{nk} < \psi_{n1} \ , \\ \alpha_{nk} - 2\pi \ , & \psi_{n1} + 2\pi \le \alpha_{nk} < \psi_{n1} + 4\pi \ , \end{cases} \tag{48}$$

$$\beta'_{nk} = \begin{cases} \beta_{nk} \ , & \psi_{n1} \le \beta_{nk} < \psi_{n1} + 2\pi \ , \\ \beta_{nk} + 2\pi \ , & \psi_{n1} - 2\pi \le \beta_{nk} < \psi_{n1} \ , \\ \beta_{nk} - 2\pi \ , & \psi_{n1} + 2\pi \le \beta_{nk} < \psi_{n1} + 4\pi \ . \end{cases} \tag{49}$$

The subscript of some terms described in the previous sections should accordingly be modified. These terms are $(\Gamma_h)_n$, $(\Gamma_t)_n$, $(\Gamma_c)_n$, ξ_n, and Γ_n. The modifications are as follows:

(1) Modify $(\Gamma_h)_n$ to $(\Gamma_h)_{nk}$, $k=1,2,3,\ldots,2m$ [cf. Eq. (12)]:

$$
\begin{aligned}
(\Gamma_h)_{nk} &= \sum_{j=1}^{2m} \mathrm{STP}(\psi_{n,j+1} - \alpha'_{nk})\,\mathrm{STP}(\alpha'_{nk} - \psi_{nj})(\psi_{n,j+1} - \alpha'_{nk})T_j \\
&= \sum_{j=1}^{2m} Q_{nkj} R_{nkj} S_{nkj} T_j \ ,
\end{aligned}
\tag{50}
$$

where

$$Q_{nkj} = \mathrm{STP}(\psi_{n,j+1} - \alpha'_{nk}) \ , \tag{51}$$

$$R_{nkj} = \mathrm{STP}(\alpha'_{nk} - \psi_{nj}) \ , \tag{52}$$

$$S_{nkj} = \psi_{n,j+1} - \alpha'_{nk} \ . \tag{53}$$

Let us define the matrix $[\Gamma_n]_n$ as the diagonal matrix of the form

$$
[\Gamma_h]_n = \begin{bmatrix}
(\Gamma_h)_{n1} & 0 & - & - & - & 0 \\
0 & (\Gamma_h)_{n2} & & & & | \\
| & & \cdot & & & | \\
| & & & \cdot & & | \\
| & & & & \cdot & 0 \\
0 & - & - & - & 0 & (\Gamma_h)_{n,2m}
\end{bmatrix} \ . \tag{54}
$$

Then we can decompose it into the multiplication of diagonal and cyclic matrices. The derivation is as follows:

$$
\begin{bmatrix}
(\Gamma_h)_{n1} & 0 & - & - & - & 0 \\
0 & (\Gamma_h)_{n2} & & & & | \\
| & & \cdot & & & | \\
| & & & \cdot & & | \\
| & & & & \cdot & 0 \\
0 & - & - & - & 0 & (\Gamma_h)_{n,2m}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{j=1}^{2m}(Q_{n1j}R_{n1j}S_{n1j}T_j) & 0 & - & - & - & 0 \\
0 & \sum_{j=1}^{2m}(Q_{n2j}R_{n2j}S_{n2j}T_j) & & & & | \\
| & \cdot & & & & | \\
| & & \cdot & & & | \\
| & & & \cdot & & 0 \\
0 & - & - & - & 0 & \sum_{j=1}^{2m}(Q_{n,2m,j}R_{n,2m,j}S_{n,2m,j}T_j)
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
Q_{n11}Q_{n12}\cdots Q_{n1,2m} & 0 & - & - & - & - & - & - & - & 0 \\
0 - - - - 0 & Q_{n21}Q_{n22}\cdots Q_{n2,2m} & 0 & - & - & - & - & - & - & 0 \\
| & & \cdot & & & & & & & | \\
| & & & \cdot & & & & & & | \\
| & & & & \cdot & & & & & 0 \\
0 - - - - - - - 0 & & & & & & & & & Q_{n,2m,1}Q_{n,2m,2}\cdots Q_{n,2m,2m}
\end{bmatrix}
\times
\begin{bmatrix}
(R_{n11}S_{n11}T_1) & 0 & - - - & 0 \\
(R_{n12}S_{n12}T_2) & | & & | \\
\vdots & | & & | \\
(R_{n1,2m}S_{n1,2m}T_{2m}) & 0 & & | \\
0 & (R_{n21}S_{n21}T_1) & & | \\
| & (R_{n22}S_{n22}T_2) & & | \\
| & \vdots & & | \\
| & (R_{n2,2m}S_{n2,2m}T_{2m}) & & | \\
| & 0 & \cdot & | \\
| & | & \cdot & | \\
| & | & \cdot & 0 \\
| & | & & (R_{n,2m,1}S_{n,2m,1}T_1) \\
| & | & & (R_{n,2m,2}S_{n,2m,2}T_2) \\
| & | & & \vdots \\
0 & 0 & - - 0 & (R_{n,2m,2m}S_{n,2m,2m}T_{2m})
\end{bmatrix}
$$

$$
= [Zyk1_{III} < Q_{nkj} \,|\, k(1,2m) j(1,2m) >]^{T}
$$

$$
\times \begin{bmatrix}
R_{n11} & 0 & \cdots & & & & & & & & & \cdots & 0 \\
0 & R_{n12} & & & & & & & & & & & \vdots \\
\vdots & & \ddots & & & & & & & & & & \vdots \\
\vdots & & & R_{n1,2m} & & & & & & & & & \vdots \\
\vdots & & & & R_{n21} & & & & & & & & \vdots \\
\vdots & & & & & R_{n22} & & & & & & & \vdots \\
\vdots & & & & & & \ddots & & & & & & \vdots \\
\vdots & & & & & & & R_{n2,2m} & & & & & \vdots \\
\vdots & & & & & & & & \ddots & & & & \vdots \\
\vdots & & & & & & & & & R_{n,2m,1} & & & \vdots \\
\vdots & & & & & & & & & & R_{n,2m,2} & & \vdots \\
\vdots & & & & & & & & & & & \ddots & 0 \\
0 & \cdots & & & & & & & & & \cdots & 0 & R_{n,2m,2m}
\end{bmatrix}
\times \begin{bmatrix}
(S_{n11}T_1) & 0 & \cdots & 0 \\
(S_{n12}T_2) & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
(S_{n1,2m}T_{2m}) & 0 & & \vdots \\
0 & (S_{n21}T_1) & & \vdots \\
\vdots & (S_{n22}T_2) & & \vdots \\
\vdots & \vdots & & \vdots \\
\vdots & (S_{n2,2m}T_{2m}) & & \vdots \\
\vdots & 0 & & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & & 0 \\
\vdots & \vdots & & (S_{n,2m,1}T_1) \\
\vdots & \vdots & & (S_{n,2m,2}T_2) \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots\ 0 & (S_{n,2m,2m}T_{2m})
\end{bmatrix}
$$

$$
= [Zyk1_{III} < Q_{nkj} \,|\, k(1,2m) j(1,2m) >]^{T} \operatorname{diag} < R_{nkj} \,|\, k(1,2m) j(1,2m) \,| >
$$

$$
\times \begin{bmatrix}
S_{n11} & 0 & \cdots & & & & & & & & & \cdots & 0 \\
0 & S_{n12} & & & & & & & & & & & \vdots \\
\vdots & & \ddots & & & & & & & & & & \vdots \\
\vdots & & & S_{n1,2m} & & & & & & & & & \vdots \\
\vdots & & & & S_{n21} & & & & & & & & \vdots \\
\vdots & & & & & S_{n22} & & & & & & & \vdots \\
\vdots & & & & & & \ddots & & & & & & \vdots \\
\vdots & & & & & & & S_{n2,2m} & & & & & \vdots \\
\vdots & & & & & & & & \ddots & & & & \vdots \\
\vdots & & & & & & & & & S_{n,2m,1} & & & \vdots \\
\vdots & & & & & & & & & & S_{n,2m,2} & & \vdots \\
\vdots & & & & & & & & & & & \ddots & 0 \\
0 & \cdots & & & & & & & & & \cdots & 0 & S_{n,2m,2m}
\end{bmatrix}
\times \begin{bmatrix}
T_1 & 0 & \cdots & 0 \\
T_2 & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
T_{2m} & 0 & & \vdots \\
0 & T_1 & & \vdots \\
\vdots & T_2 & & \vdots \\
\vdots & \vdots & & \vdots \\
\vdots & T_{2m} & & \vdots \\
\vdots & 0 & & \vdots \\
\vdots & \vdots & \ddots & 0 \\
\vdots & \vdots & & T_1 \\
\vdots & \vdots & & T_2 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots\ 0 & T_{2m}
\end{bmatrix}
$$

$$= [\mathrm{Zykl}_{III} <Q_{nkj}|k(1,2m)j(1,2m)>]^T \,\mathrm{diag}<R_{nkj}|k(1,2m)j(1,2m)|>$$

$$\times \mathrm{diag}<S_{nkj}|k(1,2m)j(1,2m)|> \mathrm{Zykl}_{II}<T_j|j(1,2m)>$$

$$= [U_0]\times[U_1]\times[U_2]\times[U_3] \ , \tag{55}$$

where, referring to Eqs. (51), (52), and (53), we have

$$[U_0] = [\mathrm{Zykl}_{III}<\mathrm{STP}(\psi_{n,j+1} - \alpha'_{nk})|k(1,2m)j(1,2m)>]^T \ , \tag{56}$$

$$[U_1] = \mathrm{diag}<\mathrm{STP}(\alpha'_{nk} - \psi_{nj})|k(1,2m)j(1,2m)> \ , \tag{57}$$

$$[U_2] = \mathrm{diag}<(\psi_{n,j+1} - \alpha'_{nk})|k(1,2m)j(1,2m)> \ , \tag{58}$$

$$[U_3] = \mathrm{Zykl}_{II}<T_j|j(1,2m)> \ , \tag{59}$$

and the superscript T implies the transpose of the matrix, i.e., $[z_{ij}]^T = [z_{ji}]$. The matrix $[\Gamma_h]_n$ is seen to be decomposed into the multiplication of four matrices: one cyclic matrix of type III, two diagonal matrices, and one cyclic matrix of type II.

(2) Modify $(\Gamma_t)_n$ to $(\Gamma_t)_{nk}$, k=1,2,3,...,2m [cf. Eq. (16)]:

$$(\Gamma_t)_{nk} = \sum_{j=1}^{2m} \mathrm{STP}(\psi_{n,j+1} - \beta'_{nk})\mathrm{STP}(\beta'_{nk} - \psi_{nj})(\beta'_{nk} - \psi_{nj})T_j \ . \tag{60}$$

A diagonal matrix $[\Gamma_t]_n$ can be defined as

$$[\Gamma_t]_n = \begin{bmatrix} (\Gamma_t)_{n1} & 0 & - & - & - & 0 \\ 0 & (\Gamma_t)_{n2} & & & & | \\ | & & \cdot & & & | \\ | & & & \cdot & & | \\ | & & & & \cdot & 0 \\ 0 & - & - & - & 0 & (\Gamma_t)_{n,2m} \end{bmatrix} .$$

$$= [U_4]\times[U_5]\times[U_6]\times[U_3] \ , \tag{61}$$

where

$$[U_4] = [\mathrm{Zykl}_{III}<\mathrm{STP}(\psi_{n,j+1} - \beta'_{nk})|k(1,2m)j(1,2m)>]^T \ , \tag{62}$$

$$[U_5] = \mathrm{diag}<\mathrm{STP}(\beta'_{nk} - \psi_{nj})|k(1,2m)j(1,2m)> \ , \tag{63}$$

$$[U_6] = \mathrm{diag}<(\beta'_{nk} - \psi_{nj})|k(1,2m)j(1,2m)> \ . \tag{64}$$

(3) Modify ξ_n to ξ_{nk}, k=1,2,3,...,2m [cf. Eq. (22)]:

$$\xi_{nk} = \mathrm{STP}(\psi_{n1} - \alpha_{nk})\mathrm{STP}(\beta_{nk} - \psi_{n1}) + \mathrm{STP}(\psi_{n1} + 2\pi - \alpha_{nk})\mathrm{STP}(\beta_{nk} - 2\pi - \psi_{n1}) \ . \tag{65}$$

(4) Modify $(\Gamma_c)_n$ to $(\Gamma_c)_{nk}$, k=1,2,3,...,2m [cf. Eq. (21)]:

$$(\Gamma_c)_{nk} = \sum_{j=1}^{2m} \Big\{ \mathrm{STP}(\beta'_{nk} - \psi_{n,j+1}) + \mathrm{STP}(\alpha'_{nk} - \psi_{nj}) \times \cos\left(\pi - \frac{\pi}{2}\xi_{nk}\right) + \mathrm{STP}(\psi_{nj} - \alpha'_{nk}) \times \sin\left(\frac{\pi}{2}\xi_{nk}\right)\Big\}(\psi_{n,j+1} - \psi_{nj})T_j \ . \tag{66}$$

A diagonal matrix $[\Gamma_c]_n$ can be defined as

$$[\Gamma_c]_n = \begin{bmatrix} (\Gamma_c)_{n1} & 0 & - & - & - & 0 \\ 0 & (\Gamma_c)_{n2} & & & & | \\ | & & \cdot & & & | \\ | & & & \cdot & & | \\ | & & & & \cdot & 0 \\ 0 & - & - & - & 0 & (\Gamma_c)_{n,2m} \end{bmatrix} .$$

$$= ([U_7] + [U_8]\times[U_9] + [U_{10}]\times[U_{11}])\times[U_{12}]\times[U_3] \ , \tag{67}$$

where

$$[U_7] = [\mathrm{Zykl}_{III}<\mathrm{STP}(\beta'_{nk} - \psi_{n,j+1})|k(1,2m)j(1,2m)>]^T \ , \tag{68}$$

$$[U_8] = [\mathrm{Zykl}_{III}<\mathrm{STP}(\alpha'_{nk} - \psi_{nj})|k(1,2m)j(1,2m)>]^T \ , \tag{69}$$

$$[U_9] = \mathrm{diag}<\cos\left(\pi - \frac{\pi}{2}\xi_{nk}\right)\Big| \ k(1,2m)q(1,2m)> \ , \tag{70}$$

$$[U_{10}] = [\mathrm{Zykl}_{III}<\mathrm{STP}(\psi_{nj} - \alpha'_{nk})|k(1,2m)j(1,2m)>]^T \ , \tag{71}$$

$$[U_{11}] = \mathrm{diag}<\sin\left(\frac{\pi}{2}\xi_{nk}\right)\Big| \ k(1,2m)q(1,2m)> \ , \tag{72}$$

$$[U_{12}] = \mathrm{diag}<(\psi_{n,j+1} - \psi_{nj})|j(1,2m)> \ . \tag{73}$$

(5) Modify Γ_n to Γ_{nk}, k=1,2,3,...,2m [cf. Eq. (23)]:

$$\Gamma_{nk} = (\Gamma_h)_{nk} + (\Gamma_t)_{nk} + (\Gamma_c)_{nk} \ . \tag{74}$$

A diagonal matrix $[\Gamma]_n$ can be defined as

$$[\Gamma]_n = \begin{bmatrix} \Gamma_{n1} & 0 & - & - & - & 0 \\ 0 & \Gamma_{n2} & & & & | \\ | & & \cdot & & & | \\ | & & & \cdot & & | \\ | & & & & \cdot & 0 \\ 0 & - & - & - & 0 & \Gamma_{n,2m} \end{bmatrix} .$$

$$= [\Gamma_h]_n + [\Gamma_t]_n + [\Gamma_c]_n \ . \tag{75}$$

Substituting Eqs. (55), (61), and (67) into Eq. (75) gives

$$[\Gamma]_n = \{[U_0]\times[U_1]\times[U_2] + [U_4]\times[U_5]\times[U_6] + ([U_7] + [U_8]\times[U_9] + [U_{10}]\times[U_{11}])\times[U_{12}]\}\times[U_3] \ . \qquad (76)$$

Combining Eqs. (7), (26), (27), (28), (29), and (76), we obtain the full matrix expression for the detector signal u:

$$[u] = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{2m} \end{bmatrix} = \left(\delta \sum_{n=N_1}^{N_2} a_n[\Gamma]_n\right) \times \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

$$= \left\{\delta \sum_{n=N_1}^{N_2} (a + n\delta)\{[U_0]\times[U_1]\times[U_2] + [U_4]\times[U_5]\times[U_6] + ([U_7] + [U_8]\times[U_9] + [U_{10}]\times[U_{11}])\times[U_{12}]\}\times[U_3]\right\} \times \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \ . \qquad (77)$$

5. EXAMPLES

The following examples illustrate the application of the above method to reticle simulation for various kinds of spots. In each case, a = 100, r = 20, and $\theta = \pi/6$. For the strip spot, we consider both 2m = 4 and 2m = 12. For the remaining spots, we consider only 2m = 12.

5.1. Strip spot

Consider a spot consisting of a single narrow arcuate strip. This is the simplest geometry and is discussed here only to show the elementary operation of the matrices. Let us simulate the case in which 2m = 4, $n = N_1 = N_2 = 0$, $\gamma_{1n} = \gamma_{10} = \pi/4$, $\gamma_{2n} = \gamma_{20} = \pi/4$, and $\delta = 1$. From Eq. (8) we have $\psi_{01} = -0.099$, $\psi_{02} = 1.739$, $\psi_{03} = 3.239$, $\psi_{04} = 4.541$, and $\psi_{05} = 6.181$. From Eq. (24) we have $\tau_1 = 0.837$, $\tau_2 = 2.549$, $\tau_3 = 3.876$, and $\tau_4 = 5.300$. Then from Eqs. (46) and (47) we have $\alpha_{01} = 0.052$, $\alpha_{02} = 1.764$, $\alpha_{03} = 3.091$, $\alpha_{04} = 4.515$, $\beta_{01} = 1.622$, $\beta_{02} = 3.334$, $\beta_{03} = 4.661$, and $\beta_{04} = 6.085$, and from Eqs. (48), (49), and (65) we have $\alpha'_{01} = 0.052$, $\alpha'_{02} = 1.764$, $\alpha'_{03} = 3.091$, $\alpha'_{04} = 4.515$, $\beta'_{01} = 1.622$, $\beta'_{02} = 3.334$, $\beta'_{03} = 4.661$, $\beta'_{04} = 6.085$, $\xi_{01} = 0$, $\xi_{02} = 0$, $\xi_{03} = 0$, and $\xi_{04} = 0$. Substituting these data into Eq. (77) gives

$$[u] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 100\left\{\left\{\begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}\right.\right.$$

$$
\times \begin{bmatrix}
1.687 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.187 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.489 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.129 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.475 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2.777 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.417 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.352 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.148 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.450 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.090 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.776 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.276 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.026 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.666
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\times \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\times \begin{bmatrix}
1.721 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.117 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.617 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2.919 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3.433 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.595 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.095 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.207 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.760 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.922 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.422 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.120 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.184 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.346 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.846 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.544
\end{bmatrix}
$$

$$
+\left(\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
+
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}\right.
$$

$$
\times \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
$$

$$
\left. + \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \times \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \right)
$$

$$
\left. \times \begin{bmatrix}
1.838 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.302 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.640 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.838 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.302 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.640 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.838 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.500 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.302 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.640 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.838 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.500 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.302 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.640
\end{bmatrix} \right\}
$$

$$
\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg\} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
= 100 \begin{bmatrix} 1.570 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.475 & 0.095 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.148 & 1.302 & 0.120 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.026 & 1.544 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg\} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 157 \\ 9.5 \\ 130 \\ 2.6 \end{bmatrix} . \qquad (78)
$$

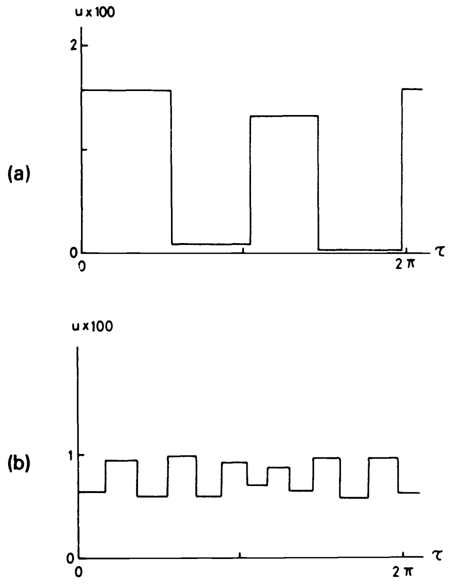


Fig. 5. Simulated waveform for a single strip spot ($\gamma_{10} = \gamma_{20} = \pi/4$, $\delta = 1$) nutating in the reticle of (a) 4 spokes and (b) 12 spokes.

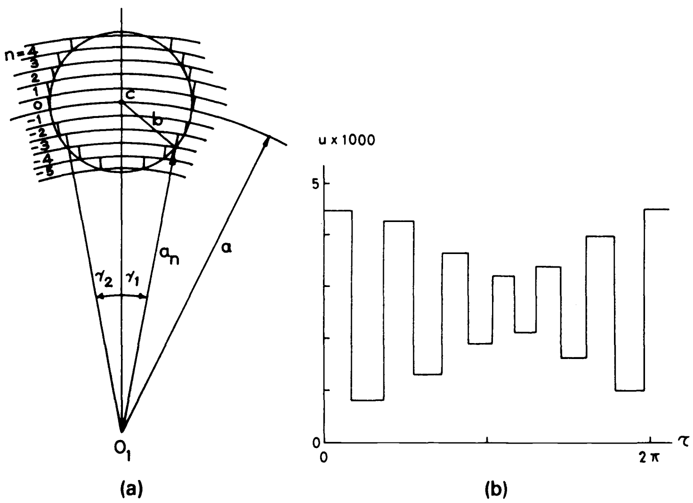


Fig. 6. (a) Circular disk spot divided into arcuate strips. (b) Simulated waveform for a disk spot nutating in the reticle of 12 spokes.

These data are obtained on an IBM AT personal computer, which also gives the waveform shown in Fig. 5(a). The execution time for this single strip for each u_k is 0.06 s. For the same strip manipulated by a reticle of 2m = 12, we obtain the waveform shown in Fig. 5(b); the execution time for each u_k is 0.15 s. It is evident that the greater the number of spokes, the longer the execution time.

5.2. Disk spot

For 2m = 12 and a disk spot of radius b = 42 divided into 10 arcuate strips ($N_1 = -5$, $N_2 = 4$), as shown in Fig. 6(a), the values of the angles γ_{1n} and γ_{2n} for each strip can be calculated by

$$\gamma_{1n} = \gamma_{2n} = \cos^{-1} \frac{a^2 + (a + n\delta)^2 - b^2}{2a(a + n\delta)} . \qquad (79)$$

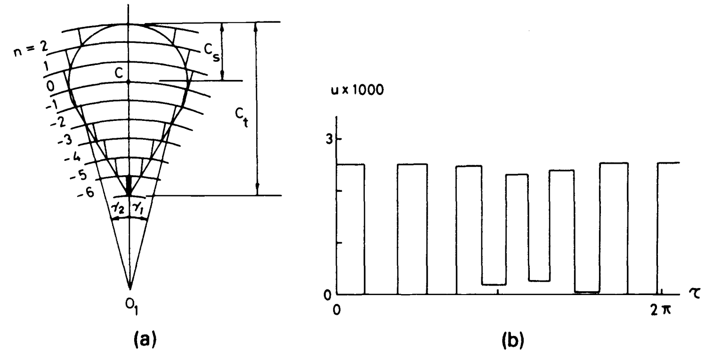


Fig. 7. (a) Comatic spot divided into arcuate strips. (b) Simulated waveform for a comatic spot nutating in the reticle of 12 spokes.

TABLE I. Head point angles (γ_{1n}) and tail point angles (γ_{2n}) of the strips divided in a comatic spot, measured with respect to O_1C.

n	γ_2(deg.)	γ_1(deg.)
2	10	10
1	12.5	12.5
0	15	15
-1	14.5	14.5
-2	13.5	13.5
-3	11.5	11.5
-4	9	9
-5	5	5
-6	0.001	0.001

Substituting these values into Eq. (77), we obtain the waveform of u versus τ shown in Fig. 6(b).

5.3. Comatic spot

For 2m = 12, a comatic spot of $c_s = 27$ and $c_t = 81$, divided into nine arcuate strips ($N_1 = -6$, $N_2 = 2$), as shown in Fig. 7(a), the angles γ_{1n} and γ_{2n} have the measured values listed in Table I. Substituting these values into Eq. (77), we obtain the waveform of u versus τ shown in Fig. 7(b).

5.4. Irregularly shaped spot

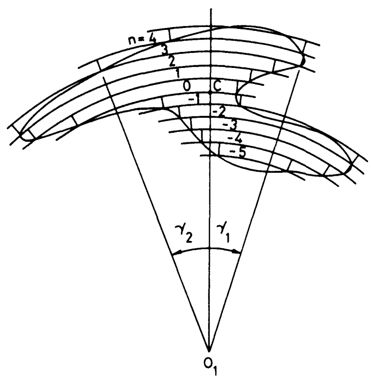
For 2m = 12, an irregularly shaped spot divided into 10 arcuate strips ($N_1 = -5$, $N_2 = 4$), as shown in Fig. 8(a), the angles γ_{1n} and γ_{2n} have the measured values listed in Table II. Substituting these values into Eq. (77), we obtain the waveform of u versus τ shown in Fig. 8(b).

6. CONCLUSION

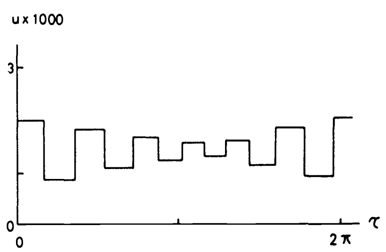
A method of matrix operation for the digital simulation of frequency modulation reticles has been developed. The formula expression employs symbolic abbreviations of the diagonal matrix and of cyclic matrices of types II and III. The application of the method to FM reticles operated in connec-

TABLE II. Head point angles (γ_{1n}) and tail point angles (γ_{2n}) of the strips divided in an irregularly shaped spot, measured with respect to O_1C.

n	γ_2(deg.)	γ_1(deg.)
4	7.5	16
3	21	17
2	41	12.5
1	39.5	8
0	30	5.5
-1	10.5	6.5
-2	7.5	36.5
-3	5	36.5
-4	2.5	32
-5	-2	22.5



(a)



(b)

Fig. 8. (a) Irregularly shaped spot divided into arcuate strips. (b) Simulated waveform for an irregularly shaped spot nutating in the reticle of 12 spokes.

tion with image spots of various shapes gives corresponding waveforms of the detector signal u versus the nutation angle τ, showing the capability of the method in dealing with the analysis of FM reticles.

7. REFERENCES

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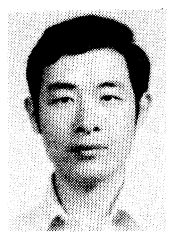

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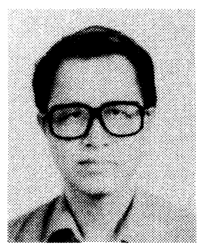

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