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The Exact Theory and Perturbation of

the Pendulum Motions

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摘 要

此篇文章主要在探討單擺運動及其擾動現象。首先介紹偏微分方程以及分類，並給予一些波動方程（ hyperbolic equation)的實際例子。接著介紹 Weierstrass 及 Jacobian 橢圓函數以及一些可以用它們來描述的物理現象；並用後者來分析理想的單擺運動，比如算出實際解、週期以及畫出相位圖等。最後對理想單擺運動做擾動進行探討，這部分主要以動態系統的理論為工具來對受擾動的單擺進行質的分析(qualitative analysis) ，我們可以發現在相同的系統裡，即使是二個很接近的初始值，在長時間後它們的位置卻是天差地遠，這就是所謂的混沌現象(Chaos)，是個仍然充滿許多未知結果的領域。

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Abstract

The main topic of this article discusses the motion of the ideal pendulum and its perturbation. First, we introduce the partial differential equations and their classification, and we give some practical problems whose mathematical models are systems of linear hyperbolic equations. Next, we study the classical Elliptic functions and one application in solving a nonlinear equation. Moreover, we use the Jacobian Elliptic function to analyze the Sine-Gordon equation to derive the exact solutions, the periods, and to sketch the phase portraits. Finally, we focus on the perturbed pendulum. We do qualitative analysis by using the tools of dynamical system. We find out that even if two initial conditions are close, their behaviors will have big difference in a later time. The phenomenon is called Chaos, a field which still much open.

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Contents

Chapter 1

Introduction

A partial differential equation represents a relation between an unknown function and its partial derivatives and the parameters of the function. Partial differential equations are not only used in area of physics, but appears in areas such as engineering, biology, chemistry, economic, and etc. The general form of a partial differential equation for a unknown function $u(x_1, x_2, \ldots, x_n)$ can be written as

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, \ldots) = 0, \tag{1.1}$$

where $x_1, x_2, \ldots, x_n$ are the independent variables and u_{x_i} denotes the partial derivative $\dfrac{\partial u}{\partial x_i}$. In general, the equation (1.1) is supplemented by additional conditions such as initial conditions or boundary conditions. In most of case, we just can analyze the properties of the solution rather than solving the equation.

1.1 Classification

At first, we classify the types of the partial differential equations. As a matter of fact, there exist several classifications and we just describe the basic type here.

1. **The order of an equation**
   The first classification is according to the *order* of the equation. The order of an equation is defined to be the order of the highest derivative int the equation. For example, the equation $u_t - u_{xx} = g(x, t)$ is called a second-order equation, while $u_x + u_{yyy} = 0$ is called a third-order equation.

2. **Linear equation**
   Another classification is to determine whether the equation is linear or not. An equation is called *linear* if in (1.1), F is a linear function of the unknown function u and its derivatives. If F is not linear, the equation (1.1) is called nonlinear. For example, the equation $x^3u_x + e^y u_y + sin(xy)u = y^4$ is a linear equation, while $u_x^2 - u_y^2 = 2$ and $u_x u_y = -1$ are nonlinear equations.

3. **Homogeneous equation**
   Mappings between different function sets are called *operators*. The operation of an operator O on a function u will be denoted by $O(u)$. A differential equation can be expressed as $O(u) = f$, where O is an operator and f is a given function. A differential equation is called a homogeneous equation if $f = 0$. For example, $O(u) = u_{tt} - u_{xx} = 0$ is a homogeneous equation, while $O(u) = u_{tt} - u_{xx} = x^3$ is an example of inhomogeneous equation.

1.2 Second-Order Linear Equations in Two Variables

After introducing some basic classification, we focus on the second order linear partial differential equations [5]. Such an equation has the form

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{1.2}$$

where $a, b, \ldots, f, g$ are given functions of x, y and $u(x, y)$ is the unknown function. we assume that a, b, and c do not vanish and define

$$\delta(L)(x, y) = b^2(x, y) - a(x, y)c(x, y). \tag{1.3}$$

Then we classify the equation according to the sign of $\delta(L)$.

Definition 1. Given any point (x_0, y_0), then equation (1.2) is said to be

1. *hyperbolic* at (x, y) if $\delta(L)(x_0, y_0) > 0$.

2. *parabolic* at (x, y) if $\delta(L)(x_0, y_0) = 0$.

3. *elliptic* at (x, y) if $\delta(L)(x_0, y_0) < 0$.

And the equation is hyperbolic (parabolic, elliptic) in Δ if it is hyperbolic (parabolic, elliptic) at all points $(x, y) \in \Delta$. Moreover, the equation (1.2) will be express as

1. $L(u) = w_{\xi\eta} + l(w) = G(\xi, \eta)$ for *hyperbolic equations*,
2. $L(u) = w_{\xi\xi} + l(w) = G(\xi, \eta)$ for *parabolic equations*,
3. $L(u) = w_{\xi\xi} + w_{\eta\eta} + l(w) = G(\xi, \eta)$ for *elliptic equations*,

after the nonsingular transformation where l is a first-order linear differential operator, and G is a function. The difference of them is about the number of characteristics. The hyperbolic equations have two characteristics, the parabolic equations have only one characteristic, and the elliptic equations have no characteristic. The characteristic influence the behavior of the equation. The more content will read the reference [4].

1.3 The Linear Hyperbolic Equations as Mathematical Models

The main topic of this article is to discuss the hyperbolic equation. Now we give some ideal problems which can be represented as a hyperbolic equation in the mathematical model but we just talk about the Telegrapher's equation in detail.

Telegrapher's Equation

Suppose that a transmission lines has a voltage $V(x, t)$ across them and a current $I(x, t)$ at position x and time t. One of the transmission line contains a resistance (R) and a inductance (L). And the two transmission line connected with a capacitance (C) and a leakage resistance (G). Assume the energy is conserved. Now, if the current passes through an inductor, the voltage across the inductor is directly proportional to the time rate of change $V = L\frac{dI}{dt}$. By Kirchhoff's current law and

Kirchhoff's voltage law, we could get

$$I(x+\Delta x,t) = I(x,t) - GV(x,t)\Delta x - C\frac{\partial V(x,t)}{\partial t}\Delta x, \tag{1.4}$$

$$V(x+\Delta x,t) = V(x,t) - RI(x,t)\Delta x - L\frac{\partial I(x,t)}{\partial t}\Delta x. \tag{1.5}$$

Let $\Delta x \to 0$, then (1.4), (1.5) will become

$$I_x(x,t) = -GV(x,t) - CV_t(x,t), \tag{1.6}$$

$$V_x(x,t) = -RI(x,t) - LI_t(x,t). \tag{1.7}$$

Since we assume that there is no energy lost, we get $R = 0$ and $G = 0$. Hence, (1.6) and (1.9) can be deduced as

$$I_x(x,t) = -CV_t(x,t), \tag{1.8}$$

$$V_x(x,t) = -LI_t(x,t). \tag{1.9}$$

After differentiating (1.8) and (1.9) with t and x, respectively, we have

$$I_{xt}(x,t) = -CV_{tt}(x,t), \tag{1.10}$$

$$V_{xx}(x,t) = -LI_{tx}(x,t). \tag{1.11}$$

From (1.10) and (1.11), we could get

$$V_{tt}(x,t) - k^2V_{xx}(x,t) = 0 \text{ where } k^2 = \frac{1}{LC}. \tag{1.12}$$

Similarly, differentiating (1.8) and (1.9) for x and t, respectively, we have

$$I_{xx}(x,t) = -CV_{tx}(x,t), \tag{1.13}$$

$$V_{xt}(x,t) = -LI_{tt}(x,t). \tag{1.14}$$

And we can derive

$$I_{tt}(x,t) - k^2I_{xx}(x,t) = 0 \text{ where } k^2 = \frac{1}{LC}. \tag{1.15}$$

from the equation (1.13) and equation (1.14). The (1.12) and (1.15) shows that the voltage and current will satisfy the wave equation under the ideal (no energy lost) condition.

Next, we introduce the vibrating string. Its mathematical model is also a hyperbolic equation under some ideal assumptions and Newton's second law. The mathematical model can be expressed as the following system:

$$u_{tt} - c^2 u_{xx} = 0, \text{ for } 0 < x < l, \tag{1.16}$$

$$u(x,0) = f(x), \tag{1.17}$$

$$u_t(x,0) = g(x), \tag{1.18}$$

$$u(0,t) = u(l,t) = 0. \tag{1.19}$$

(1.17) and (1.18) are the initial conditions. They limit the shape of position and velocity for the solution at $t = 0$. (1.19) is called the boundary condition. It represent the states at $x = 0$ and $x = l$ for all t. The detail of the derivation can read the reference [4].

There are many other problems can be represented as hyperbolic equation like Maxwell's equation and so on. This shows that the linear hyperbolic equation can be applied in our life. But in most of time, the real problems in our life are corresponding to the nonlinear equation. Thus, we will discuss a nonlinear equation in the whole following contents. It is called Sine-Gordon equation, $u_{tt} - u_{xx} + sin(u) = 0$. On the other hand, it could be transferred to be a ordinary differential equation, $u_{\theta\theta} + sin(u) = 0$, by letting $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$.

Consider a pendulum consisting of a light rod of length l to which is attached a ball of mass m. The position of the mass at time t is described by $u(\theta)$. By Newton's law, the mathematical model of the motion of the pendulum could be represented as

$$ml\frac{d^2u}{d\theta^2} = -bl\frac{du}{d\theta} - mgsin(u), \tag{1.20}$$

where g is the gravitational acceleration and b represents the coefficient of friction with $b > 0$ [8] [9] [10]. Assume that the pendulum is frictionless ($b = 0$) with $m = 1$ and $l = g$. Then the equation (1.20) will become $u_{\theta\theta} + sin(u) = 0$. This implies that the equation $u_{\theta\theta} + sin(u) = 0$ could be regarded as the motion of a ideal pendulum with $m = 1$ and $l = g$. The whole chapter 3 will discuss the problem in detail.

Chapter 2

Elliptic functions

2.1 Definitions and Properties

Before introducing the elliptic functions, we introduce some definitions. We just talk about some important and interesting parts of the elliptic function. The following contents referred to [3] and you could get more information about the elliptic functions from it.

Definition 2. The point z_0 is called the singularity (singular point) of $f(z)$ if $f(z)$ is not analytic at $z = z_0$. If z_0 is a singularity and there exists a neighborhood $N(z_0)$ of z_0 such that the function $f(z)$ is analytic in $N(z_0)\backslash\{z_0\}$, then z_0 is isolated. Moreover, if there is an analytic function $g : N(z_0) \to \mathbb{C}$ such that $g(z) = f(z)$ on $N(z_0)\backslash\{z_0\}$, the point z_0 is called a removable singularity.

Examples

(i) $f(z) = \ln(z)$, $z = 0$ is a non-isolated singularity.

(ii) $f(z) = \dfrac{1}{z}$, $z = 0$ is an isolated singularity, but it is not a removable singularity.

(iii) $f(z) = \dfrac{z^2 - z}{z}$ is analytic except z =0. $z = 0$ is a removable singularity since we can define $g(x) = \begin{cases} z-1, & \text{if } z = 0 \\ \dfrac{z^2-z}{z}, & \text{otherwise.} \end{cases}$ Then $g(z) = g(x)$ on $N_R(0)\backslash\{z_0\}$.

Definition 3. The pole z_0 of the function $f(z)$ satisfies:

1. z_0 is a singularity.

2. z_0 is isolated.

3. $\exists \min k \in \mathbb{N}$ such that $(z - z_0)^k f(z)$ is analytic at z_0.

After knowing the previous definitions, we can define elliptic function now:

Definition 4. Assume that f is a doubly-periodic function with periods $2\omega_1$ and $2\omega_2$.(That is, $f(z+2\omega_1) = f(z+2\omega_2) = f(z)$.) And f is called an **elliptic function** if it is analytic (except poles) and has no singularities other than poles in the finite part of the plane.

Remark 1.

a. The constants $\omega_1, \omega_2 \in \mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$) and $\frac{\omega_1}{\omega_2}$ is not purely real number.

b. If there is no ω inside the parallelograms such that $f(z + \omega) = f(z), \forall z$, the parallelogram constructed by $z, z + 2\omega_1, z + 2\omega_2, z + 2(\omega_1 + \omega_2)$ is called a **fundamental period-parallelogram** for an elliptic function with period $2\omega_1, 2\omega_2$. The points $z, z + 2\omega_1, z + 2\omega_2, z + 2(\omega_1 + \omega_2), \ldots$ will have the same value after transferring by f since $2\omega_1$ and $2\omega_2$ are periods. And any pair of such points are said to be "congruent" to one another. The congruence of two points z, z' is denoted by $z \equiv z' \pmod{2\omega_1, 2\omega_2}$. And the set of poles of an elliptic function in any given cell is called an **irreducible set**.

Some simple properties of elliptic functions

1. The number of poles of an elliptic function in any cell is finite.

2. The number of zeros of an elliptic function in any cell is finite.

3. The sum of the residue of an elliptic function, $f(z)$, at its poles in any cell is zero.

4. Liouville's Theorem:
   An elliptic function ,$f(z)$, with no poles in a cell is merely a constant.

The order of an elliptic function

$f(z)$ is an elliptic function and the number of the roots of the equation $f(z) = c$ (where c is any constant) which lies in any cell depends only on $f(z)$. Then the number is called the order of the elliptic function.

Remark 2.

a. The order of $f(z)$ is the number of poles in the cell.

b. The order of an elliptic function is $\geqq 2$.

c. The simplest elliptic function could be divided into two classes. One is the elliptic functions which have a single irreducible double pole with residue $= 0$. The other is the elliptic functions which have two single poles and the sum of their residues is 0.

After knowing some basic properties of elliptic function. We will introduce the Weierstrass elliptic function which belongs to the former class and the Jacobian elliptic functions in the following sections.

2.2 Weierstrass Elliptic Function

The Weierstrass elliptic function $\wp(z)$ is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq 0} \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right) \tag{2.1}$$

$$= \frac{1}{z^2} + \sum_{m,n\neq 0} \left(\frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right) \tag{2.2}$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$.

Remark 3.

a. When m, n such that $|\Omega_{m,n}|$ is large, the general terms of the series defining $\wp(z)$ is $\mathcal{O}\left(|\Omega_{m,n}|^{-3}\right)$. So $\wp(z)$ converges absolutely and uniformly.

b. $\wp(z)$ is analytic except the poles, namely the points $\Omega_{m,n}$ and the points $\Omega_{m,n}$ are all double poles.

The following contents will introduce some properties and theorem about $\wp(z)$ and $\wp'(z)$.

(1) Periodicity and other properties of $\wp(z)$

Since $\wp(z)$ is uniformly convergent series of analytic function, we could differentiate it term-by-term.Thus,

$$\wp'(z) = -2\sum_{m,n} \frac{1}{(z-\Omega_{m,n})^{-3}} \tag{2.3}$$

and

$$\wp'(-z) = 2\sum_{m,n} \frac{1}{(z+\Omega_{m,n})^{-3}}$$

because the points $-\Omega_{m,n}$ is the same as the set $\Omega_{m,n}$ and $\wp'(z)$ is absolutely convergent,we can get

$$\wp'(-z) = -\wp'(z).$$

This means that $\wp'(z)$ is an **odd** function.We know that $\wp'(z)$ is analytic (except at poles) and which has no singularity other than poles. Moreover,it is not difficult to check that $2\omega_1, 2\omega_2$ are periods of $\wp'(z)$.Then $\wp'(z)$ is an **elliptic function**.Using the same way, we could show that $\wp(z)$ is also an elliptic function, but it is different to $\wp'(z)$. $\wp(z)$ is an **even** function. Given the following table as conclusion:

—	Definition	Periods	Parity	Poles
$\wp(z)$	equation (2.2)	$2\omega_1$,$2\omega_2$	even	$\Omega_{m,n}$
$\wp'(z)$	equation (2.3)	$2\omega_1$,$2\omega_2$	odd	$\Omega_{m,n}$

Table 2.1: The summary of $\wp(z)$ and $\wp'(z)$.

(2) The differential equation satisfied by $\wp(z)$

By the equation (2.2), we know that

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq 0} (\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}).$$

Let $\mathcal{S}(z) = \wp(z) - \frac{1}{z^2}$. Then $\mathcal{S}(z)$ could be represented as

$$\mathcal{S}(z) = \sum_{m,n\neq 0} (\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}). \tag{2.4}$$

Then $\mathcal{S}(z)$ is analytic at $z = 0$ and it is an **even function**. Do Taylor extension for $\mathcal{S}(z)$ for $|z| \to 0$. The (2.4) can be derived as

$$\wp(z) - \frac{1}{z^2} = \frac{z^2}{20}\mathfrak{g}_2 + \frac{z^4}{28}\mathfrak{g}_3 + \mathcal{O}\left(z^6\right), \tag{2.5}$$

where $\mathfrak{g}_2 \equiv \sum_{m,n\neq 0} 60(\Omega_{m,n})^{-4}$ and $\mathfrak{g}_3 \equiv \sum_{m,n\neq 0} 140(\Omega_{m,n})^{-6}$. According to (2.5), the functions $\wp(z)$ and $\wp'(z)$ can be written as

$$\wp(z) = \frac{1}{z^2} + \frac{z^2}{20}\mathfrak{g}_2 + \frac{z^4}{28}\mathfrak{g}_3 + \mathcal{O}\left(z^6\right), \tag{2.6}$$

$$\wp'(z) = \frac{-2}{z^3} + \frac{z}{10}\mathfrak{g}_2 + \frac{z^3}{7}\mathfrak{g}_3 + \mathcal{O}\left(z^5\right). \tag{2.7}$$

By the above two equations, we can derive the following equations

$$\wp^3(z) = \frac{1}{z^6} + \frac{3}{20z^2}\mathfrak{g}_2 + \frac{3}{28}\mathfrak{g}_3 + \mathcal{O}\left(z^2\right), \tag{2.8}$$

$$[\wp'(z)]^2 = \frac{4}{z^6} - \frac{2}{5z^2}\mathfrak{g}_2 + \frac{4}{7}\mathfrak{g}_3 + \mathcal{O}\left(z^2\right). \tag{2.9}$$

Then use (2.8) and (2.9), we get

$$\mathcal{T}(z) \equiv [\wp'(z)]^2 - 4\wp^3(z) + \wp(z)\mathfrak{g}_2 + \mathfrak{g}_3 = \mathcal{O}\left(z^2\right).$$

Since $\mathcal{T}(z)$ is an elliptic function and it is analytic at the origin, the all congruent points of 0 are also analytic. This means that $\mathcal{T}(z)$ is an elliptic function with no singularities. This implies that $\mathcal{T}(z) = c$ where c is a constant by Liouville's Theorem. If we let $z \to 0$, the constant c is zero. This implies that the function $\wp(z)$ satisfies

$$[\wp'(z)]^2 = 4\wp^3(z) - \wp(z)\mathfrak{g}_2 - \mathfrak{g}_3 = \mathcal{O}\left(z^2\right), \tag{2.10}$$

where $\mathfrak{g}_2 = \sum_{m,n\doteq 0} 60(\Omega_{m,n})^{-4}$ and $\mathfrak{g}_3 = \sum_{m,n\neq 0} 140(\Omega_{m,n})^{-6}$.

Conversely, given the equation $(\frac{dy}{dz})^2 = 4y^3 - \mathfrak{g}_2 y - \mathfrak{g}_3$. If ω_1, ω_2 can be determined such that $\mathfrak{g}_2 = \sum'_{m,n} 60(\Omega_{m,n})^{-4}$ and $\mathfrak{g}_3 = \sum'_{m,n} 140(\Omega_{m,n})^{-6}$, then the general solution of the differential equation is $y(z) = \wp(\pm z + \alpha)$, where α is a constant. And the solution can be written as $y(z) = \wp(z + \alpha)$ since $\wp(z)$ is an even function.

Moreover, consider the integral equation

$$z = \int_{\xi}^{\infty} \left(4t^3 - \mathfrak{g}_2 t - \mathfrak{g}_3{}^{-1/2}\right) \mathrm{d}t \tag{2.11}$$

with the path of integration may be any curve which does not pass through a zero of $4t^3 - \mathfrak{g}_2 t - \mathfrak{g}_3$. By the above equation, we differentiate z with respect to ξ, and get

$$(\frac{d\xi}{dz})^2 = 4\xi^3 - \mathfrak{g}_2\xi - \mathfrak{g}_3. \tag{2.12}$$

By the previous result, we know that $\xi = \wp(z+\alpha)$, where α is a constant. Let $\xi \to \infty$, then $z \to 0$. This implies that α is a pole of $\wp(z)$. In other words, $\alpha \in \Omega_{m,n}$ and

$$\xi = \wp(z + \Omega_{m,n}) = \wp(z).$$

So the equation (2.11) is called the **integral formula** for $\wp(z)$ and it is sometimes written as

$$z = \int_{\wp(z)}^{\infty} \left(4t^3 - \mathfrak{g}_2 t - \mathfrak{g}_3{}^{-1/2}\right) dt.$$

(3) Addition Theorem for the function $\wp(z)$

Here, we want to show that $\wp(y+z)$ can be expressed by $\wp(y)$ and $\wp(z)$. If y and z satisfy $y + z = 0$, then we can get the relation as follow:

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(z+y) & \wp'(z+y) & 1 \end{vmatrix} = 0.$$

Therefore, $\wp(z+y)$ can be expressed algebraically in terms of $\wp(z)$ and $\wp(y)$.

Remark 4.

a. If $u + v + w = 0$,then the addition theorem could be extended as

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0.$$

b. The addition theorem could be written in another form as :

$$\wp(z+y) = \frac{1}{4}\left[\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right]^2 - \wp(z) - \wp(y). \tag{2.13}$$

c. Assume that $2z$ is not a period of $\wp(z)$. If we take the limit for the equation (2.13) with $y \to z$, we can get

$$\wp(2z) = \frac{1}{4}\left[\frac{\wp''(z)}{\wp'(z)}\right]^2 - 2\wp(z). \tag{2.14}$$

The result is called **the duplication formula of** $\wp(z)$.

(4) The constants e_1, e_2, e_3

Pick three points w_1, w_2, w_3 in the same cell with $w_1 + w_2 + w_3 = 0$. Set $e_i = \wp(w_i), i = 1, 2, 3$. We could show that w_i is the zero of $\wp(z) - e_i, i = 1, 2, 3$ and $e_1 \neq e_2 \neq e_3$. Furthermore, e_1, e_2, e_3 are roots of $4t^3 - \mathfrak{g}_2 t - \mathfrak{g}_3$ since $\wp(z)$ satisfies the equation $[\wp'(z)]^2 = 4[\wp(z)]^3 - \mathfrak{g}_2\wp(z) - \mathfrak{g}_3$. This means that

$$[\wp'(z)]^2 = 4\prod_{r=1}^{3}(\wp(z) - e_r). \tag{2.15}$$

By roots of equations with their coefficients, we could get

$$\begin{cases} e_1 + e_2 + e_3 = 0, \\ e_1e_2 + e_2e_3 + e_3e_1 = -\dfrac{\mathfrak{g}_2}{4}, \\ e_1e_2e_3 = \dfrac{\mathfrak{g}_3}{4}. \end{cases}$$

Moreover, using the constants and equation (2.13) and equation (2.15). We can derive **the addition theorem of half-period for** $\wp(z)$ and the result is as following:

$$\begin{cases} \wp(z + \omega_1) = e_1 + \dfrac{(e_1 - e_2)(e_1 - e_3)}{\wp(z) - e_1}, \\ \wp(z + \omega_2) = e_2 + \dfrac{(e_2 - e_1)(e_2 - e_3)}{\wp(z) - e_2}, \\ \wp(z + \omega_3) = e_3 + \dfrac{(e_3 - e_1)(e_3 - e_2)}{\wp(z) - e_3}. \end{cases}$$

After introducing the Weierstrass elliptic function and some simple properties about it, we introduce the Jacobian elliptic function in the following. Note that $\wp(z)$ is one of the simplest example for the elliptic function with single double pole. And the next section will show that the Jacobian elliptic function is the elliptic function with two simple poles.

2.3 Jacobian Elliptic Functions

Before starting the Jacobian elliptic functions, we discuss the Theta-functions first. The theta-function $\vartheta(z,q)$ is defined by

$$\vartheta(z,q) \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \tag{2.16}$$

$$= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} cos(2nz) \tag{2.17}$$

$$\equiv \vartheta_4(z,q), \tag{2.18}$$

where $q = e^{\pi i\tau}$, $\tau \in \mathbb{C}$ is constant and its imaginary part is positive and $|q| < 1$.

Remark 5.

a. By the equation (2.16), we could attain the following results

$$\begin{cases} \vartheta(z+\pi,q) = \vartheta(z,q), \\ \vartheta(z+i\pi,q) = \mathcal{K}\vartheta(z,q), \end{cases} \tag{2.19}$$

where $\mathcal{K} = -q^{-1}e^{-2iz}$. This implies that $\vartheta(z)$ is a **quasi doubly-periodic function of z**.

b. By the definition of $\vartheta_4(z)$, other three Theta-functions is defined as:

$$\begin{cases} \vartheta_1(z,q) & \equiv -ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_4(z+\frac{1}{2}\pi\tau,q), \\ \vartheta_2(z,q) & \equiv \vartheta_1(z+\frac{1}{2}\pi,q), \\ \vartheta_3(z,q) & \equiv \vartheta_4(z+\frac{1}{2}\pi,q). \end{cases} \tag{2.20}$$

From (2.17), the definition , (2.20) can be written as series form:

$$\begin{cases} \vartheta_1(z,q) & \equiv 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} sin(2n+1)z, \\ \vartheta_2(z,q) & \equiv 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} cos(2n+1)z, \\ \vartheta_3(z,q) & \equiv 2\sum_{n=0}^{\infty} q^{(n)^2} cos(2nz)+1. \end{cases} \tag{2.21}$$

c. By (2.17) and (2.21) and parity of trigonometric functions, it is easy to see that $\vartheta_1(z,q)$ is an odd function of z, and the other Theta-functions are even functions of z. And the four Theta-functions are all satisfy (2.19).

d. $\vartheta_i(z,q)$ is a two variables function. It is be denoted as $\vartheta_i(z)$ when we just focus on the parameter z. Moreover, the notation ϑ_i is represented $\vartheta_i(0)$, for $i = 1, 2, 3, 4$.

By the relation of (2.19), we know that if z_0 is a zero of $\vartheta(z,q)$, then $z_0+m\pi+n\pi\tau$ is also a zero, $\forall \quad m, n \in \mathbb{N}$. Let us consider the function $\dfrac{\vartheta'(z,q)}{\vartheta(z,q)}$. We could show that it has only one poles in the parallelogram constructed by the points $t, t+\pi, t+\pi+\pi\tau$ and $t+\pi\tau$ by the residue theorem. The we can find the zeros for $\vartheta_i(z,q)$ for $i = 1, 2, 3, 4$ and we discuss $\vartheta_1(z,q)$ first. From equations (2.21), we find out that 0 is the zero of $\vartheta_1(z,q)$. And this means that the all zeros of $\vartheta_1(z,q)$ are congruent to 0 $mod(\pi, \pi\tau)$. Using relation between the Theta-functions, we can find out that the zeros of $\vartheta_2(z,q)$, $\vartheta_3(z,q)$ and $\vartheta_4(z,q)$. The result can be summarized as the following table :

—	zeros	Relation
$\vartheta_1(z,q)$	$z \equiv 0 \quad mod(\pi, \pi\tau)$	By definition
$\vartheta_2(z,q)$	$z \equiv \frac{1}{2}\pi \quad mod(\pi, \pi\tau)$	$\vartheta_2(z,q) = \vartheta_1(z+\frac{1}{2}\pi, q)$
$\vartheta_3(z,q)$	$z \equiv \frac{1}{2}\pi + \frac{1}{2}\pi\tau \quad mod(\pi, \pi\tau)$	$\vartheta_3(z,q) = \vartheta_4(z+\frac{1}{2}\pi, q)$
$\vartheta_4(z,q)$	$z \equiv \frac{1}{2}\pi\tau \quad mod(\pi, \pi\tau)$	$\vartheta_1(z,q) = \mathcal{K}\vartheta_4(z+\frac{1}{2}\pi\tau, q)$

Table 2.2: zeros of Theta-functions.

Next, we will derive the relation between these Theta-functions. We know that $\vartheta_1(z)$, $\vartheta_2(z)$, $\vartheta_3(z)$, and $\vartheta_4(z)$ are analytic and have periodicity factors 1, $-q^{-1}e^{-2\pi\tau}$ with periods π, $\pi\tau$. It is clear that $\vartheta_1^2(z)$, $\vartheta_2^2(z)$, $\vartheta_3^2(z)$, and $\vartheta_4^2(z)$ are analytic and have periodicity factors 1, $-q^{-2}e^{-4\pi\tau}$ and each has a double zero in any cell. If we choose suitable constants a, b, a', b', then we could make

$$\frac{a\vartheta_1^2(z) + b\vartheta_4^2(z)}{\vartheta_2^2(z)} \tag{2.22}$$

and

$$\frac{a'\vartheta_1^2(z) + b'\vartheta_4^2(z)}{\vartheta_3^2(z)} \tag{2.23}$$

will become doubly-periodic function with periods π and $\pi\tau$. By the properties of elliptic function, these two relations are merely a constant and we choose the constant is 1. Hence the equations (2.22) and (2.23) will become

$$\begin{cases} \vartheta_2^2(z) = a\vartheta_1^2(z) + b\vartheta_4^2(z), \\ \vartheta_3^2(z) = a'\vartheta_1^2(z) + b'\vartheta_4^2(z). \end{cases} \tag{2.24}$$

Given z the special values 0 and $\frac{1}{2}\pi\tau$, (2.33) could be represented as

$$\begin{cases} \vartheta_4^2\vartheta_1^2(z) = \vartheta_2^2\vartheta_3^2(z) - \vartheta_3^2\vartheta_2^2(z), \\ \vartheta_4^2\vartheta_2^2(z) = \vartheta_2^2\vartheta_4^2(z) - \vartheta_3^2\vartheta_1^2(z). \end{cases} \tag{2.25}$$

If we replace z with $z + \frac{1}{2}\pi$, we could get other two relations. The relation of square are arranged as following:

$$\begin{cases} \vartheta_4^2\vartheta_1^2(z) = \vartheta_2^2\vartheta_3^2(z) - \vartheta_3^2\vartheta_2^2(z), \\ \vartheta_4^2\vartheta_2^2(z) = \vartheta_2^2\vartheta_4^2(z) - \vartheta_3^2\vartheta_1^2(z), \\ \vartheta_4^2\vartheta_3^2(z) = \vartheta_3^2\vartheta_4^2(z) - \vartheta_2^2\vartheta_1^2(z), \\ \vartheta_4^2\vartheta_4^2(z) = \vartheta_3^2\vartheta_3^2(z) - \vartheta_2^2\vartheta_2^2(z). \end{cases} \tag{2.26}$$

Remark 6.

a. If $z = 0$, the last relation will become $\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4$.

b. It is more clearly to compare the period factors of the four Theta-functions by making a following table:

—	$\vartheta_1(z)$	$\vartheta_2(z)$	$\vartheta_3(z)$	$\vartheta_4(z)$
π	-1	-1	1	1
$\pi\tau$	$-\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$-\mathcal{K}$

Table 2.3: Period factors.

In order to get some relation between the Theta-functions easily, we can represent the Theta-functions as infinite products. The result is derived by Jacobi. Let

$$f(z) = \prod_{n=1}^{\infty}\left(1 - q^{2n-1}e^{2iz}\right)\prod_{n=1}^{\infty}\left(1 - q^{2n-1}e^{-2iz}\right). \tag{2.27}$$

First, we know $\dfrac{\vartheta_4(z)}{f(z)}$ has no poles and no zeros since $f(z)$ has the same zeros as $\vartheta_4(z)$. Second, $f(z)$ has the same periodicity factors as $\vartheta_4(z)$ by calculating $f(z+\pi)$ and $f(z+\pi\tau)$ directly. This means that $\dfrac{\vartheta_4(z)}{f(z)}$ is a doubly-periodic function with no poles. By Liouville's Theorem, $\dfrac{\vartheta_4(z)}{f(z)} = \mathcal{G}$, $\mathcal{G}$ is a constant. Thus, $\vartheta_4(z)$ can be rewritten as

$$\begin{aligned} \vartheta_4(z) &= \mathcal{G}f(z) \\ &= \mathcal{G}\prod_{n=1}^{\infty}\left(1 - 2q^{2n-1}cos(2z) + q^{4n-2}\right) \end{aligned}$$

by using (2.27). Moreover, the relation (2.20) implies that

$$\vartheta_1(z,q) = 2\mathcal{G}q^{\frac{1}{4}}sin(z)\prod_{n=1}^{\infty}\left(1-2q^{2n}cos(2z)+q^{4n}\right), \tag{2.28}$$

$$\vartheta_2(z,q) = 2\mathcal{G}q^{\frac{1}{4}}cos(z)\prod_{n=1}^{\infty}\left(1+2q^{2n}cos(2z)+q^{4n}\right), \tag{2.29}$$

$$\vartheta_3(z,q) = \mathcal{G}\prod_{n=1}^{\infty}\left(1+2q^{2n-1}cos(2z)+q^{4n-2}\right). \tag{2.30}$$

Remark 7.

a. By the expression of infinite product form and given $z = 0$. We can the relation

$$\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4.$$

b. Using the relation of (a). The constant $\mathcal{G}$ can be determined as

$$\mathcal{G} = \prod_{n=1}^{\infty}\left(1-q^{2n}\right).$$

Recall that we found out the differential equation which satisfied by the Weierstrass elliptic function. Similarly, we can derived the differential equation which satisfied by the quotient of Theta-functions. By the table of periodicity factors, it is not hard to see that $\frac{\vartheta_1(z)}{\vartheta_4(z)}$, $\frac{\vartheta_1'(z)\vartheta_4(z)-\vartheta_4'(z)\vartheta_1(z)}{\vartheta_4^2(z)}$, and $\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)}$ have the same periodicity factors -1 and 1 with respect to π, $\pi\tau$ respectively. Ratio the last two functions and define

$$\varphi(z) = \frac{\vartheta_1'(z)\vartheta_4(z)-\vartheta_4'(z)\vartheta_1(z)}{\vartheta_2(z)\vartheta_3(z)}. \tag{2.31}$$

By liouville's Theorem, it shows that $\varphi(z) = c$ where c is a constant since there is no poles of $\varphi(z) = 0$ in the cell. Make $z \to 0$, we can determine $c = \vartheta_4^2$. The we get

$$\left[\frac{\vartheta_1(z)}{\vartheta_4(z)}\right]' = \vartheta_4^2\frac{\vartheta_2(z)}{\vartheta_4(z)}\frac{\vartheta_3(z)}{\vartheta_4(z)}. \tag{2.32}$$

Let $\xi = \frac{\vartheta_1(z)}{\vartheta_4(z)}$ and (2.32) will become

$$\left(\frac{d\xi}{dz}\right)^2 = \left(\vartheta_2^2-\vartheta_3^2\xi^2\right)\left(\vartheta_3^2-\vartheta_2^2\xi^2\ \right). \tag{2.33}$$

The function $\frac{\vartheta_1(z)}{\vartheta_4(z)}$ is a solution of the above equation.

Remark 8.

By the same discussion, we could also find that:

$$\left[\frac{\vartheta_2(z)}{\vartheta_4(z)}\right]' = -\vartheta_3^2 \frac{\vartheta_1(z)}{\vartheta_4(z)} \frac{\vartheta_3(z)}{\vartheta_4(z)}, \tag{2.34}$$

$$\left[\frac{\vartheta_3(z)}{\vartheta_4(z)}\right]' = -\vartheta_2^2 \frac{\vartheta_1(z)}{\vartheta_4(z)} \frac{\vartheta_2(z)}{\vartheta_4(z)}. \tag{2.35}$$

Now, we could introduce the Jacobian elliptic functions. We start from the function $sn(u)$. Let $y = \frac{\vartheta_3}{\vartheta_2}\xi$ and $u = \vartheta_3^2 z$. Then the equation (2.33) will be written as

$$\left(\frac{dy}{du}\right)^2 = \left(1 - y^2\right)\left(1 - \kappa^2 y^2\right), \tag{2.36}$$

where κ is defined by $\kappa = \left(\frac{\vartheta_2}{\vartheta_3}\right)^2$ and it is called the modulus. And the solution y is denoted by $y = sn(u;\kappa)$ or $y = sn(u)$. On the other version, (2.36) can be written as the integral form

$$u = \int_0^y \left(1 - t^2\right)^{-\frac{1}{2}} \left(1 - \kappa^2 t^2\right)^{-\frac{1}{2}} \, \mathrm{d}t \tag{2.37}$$

and $y = sn(u;\kappa)$ satisfies it.

Remark 9.

a. $y(u)$ has periodicity factors $-1, 1$ with the periods $\vartheta_3^2\pi$ and $\vartheta_3\pi\tau$ since $\frac{\vartheta_1(z)}{\vartheta_4(z)}$ with the periods π and $\pi\tau$, respectively. This also implies that $y(u)$ is a doubly-periodic function with periods $2\vartheta_3^2\pi$, $\vartheta_3^2\pi\tau$ and we define them as $4\mathrm{K} \equiv 2\vartheta_3^2\pi$, $2i\mathrm{K}' \equiv \vartheta_3^2\pi\tau$.

b. The poles of $y(u)$ is the zeros of $\vartheta_4(u\vartheta_3^{-2})$. From the definition of $\vartheta_4(z)$, we could know that the poles of $y(u)$ at the points congruent to $\frac{1}{2}\pi\tau\vartheta_3^2$ and $\pi\vartheta_3^2 + \frac{1}{2}\pi\tau\vartheta_3^2$ mod $(4\mathrm{K}, 2i\mathrm{K}')$.Moreover, $y(u)$ has two simple poles in any cell and their residues are equal but opposite sign.

c. Similarly to (b). The zeros of $y(u)$ is the zeros of $\vartheta_1(u\vartheta_3^{-2})$. So the zeros of $y(u)$ at points congruent to 0 and $\pi\vartheta_3^2$ mod$(4\mathrm{K}, 2i\mathrm{K}')$.

Next, we define other two Jacobian elliptic functions $cn(u, \kappa)$ and $dn(u, \kappa)$:

$$\begin{cases} cn(u) & \equiv \dfrac{\vartheta_4 \vartheta_2(\frac{u}{\vartheta_3^2})}{\vartheta_2 \vartheta_4(\frac{u}{\vartheta_3^2})}, \\ dn(u) & \equiv \dfrac{\vartheta_4 \vartheta_3(\frac{u}{\vartheta_3^2})}{\vartheta_3 \vartheta_4(\frac{u}{\vartheta_3^2})}. \end{cases} \tag{2.38}$$

And there are some properties and relation between the three Jacobian elliptic functions as following:

1. From (2.38), we can get some results:

$$sn^2(u) + cn^2(u) = 1. \tag{2.39}$$

$$\kappa^2 sn^2(u) + dn^2(u) = 1. \tag{2.40}$$

$$cn0 = dn0 = 1. \tag{2.41}$$

2. The derivatives of $sn(u)$, $cn(u)$, and $dn(u)$ are as following:

$$\frac{d}{du}\{sn(u)\} = cn(u)dn(u), \tag{2.42}$$

$$\frac{d}{du}\{cn(u)\} = -sn(u)dn(u), \tag{2.43}$$

$$\frac{d}{du}\{dn(u)\} = -\kappa sn(u)cn(u). \tag{2.44}$$

3. By the properties of Theta-functions, the parity of them are

$$sn(-u) = -sn(u),$$

$$cn(-u) = cn(u),$$

$$dn(-u) = dn(u).$$

Similarly, we could find their periods, poles, and zeros just like what we do for $sn(u)$.

——	$sn(u)$	$cn(u)$	$dn(u)$
Periods	$4\mathrm{K},\ 2i\mathrm{K}'$	$4\mathrm{K},\ 2\mathrm{K}+2i\mathrm{K}'$	$2\mathrm{K},\ 4i\mathrm{K}'$
Poles	$i\mathrm{K}', 2\mathrm{K}+i\mathrm{K}'$ mod $(4\mathrm{K}, 2i\mathrm{K}')$	$i\mathrm{K}', 2\mathrm{K}+i\mathrm{K}'$ mod $(4\mathrm{K}, 2\mathrm{K}+2i\mathrm{K}')$	$i\mathrm{K}', 3\mathrm{K}'$ mod $(2\mathrm{K}, 4i\mathrm{K}')$
Zeros	$0 \quad (2\mathrm{K}, 2i\mathrm{K}')$	$\mathrm{K} \quad (2\mathrm{K}, 2i\mathrm{K}')$	$\mathrm{K}+i\mathrm{K}' \quad (2\mathrm{K}, 2i\mathrm{K}')$
Parity	odd	even	even
Derivative	$cn(u)dn(u)$	$-sn(u)dn(u)$	$-\kappa^2 sn(u)cn(u)$

Table 2.4: Summary about $sn(u)$,$cn(u)$ and $dn(u)$.

The above table is a roughly conclusion for $sn(u)$, $cn(u)$, and $dn(u)$. In the end of this chapter, we see the graphs of them with different modulus κ as following:

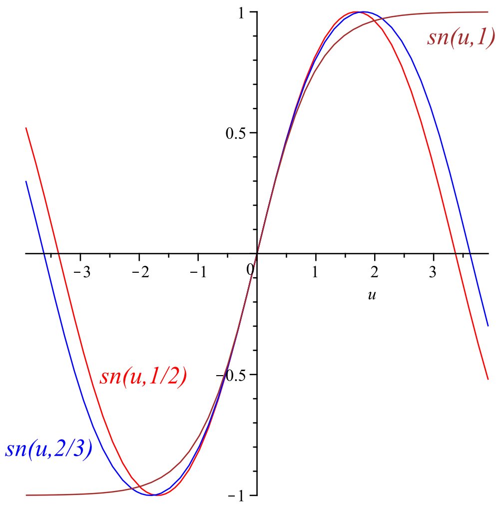

Figure 2.1: $sn(u,\kappa)$ with $\kappa = \frac{1}{2}, \frac{2}{3},$ and 1

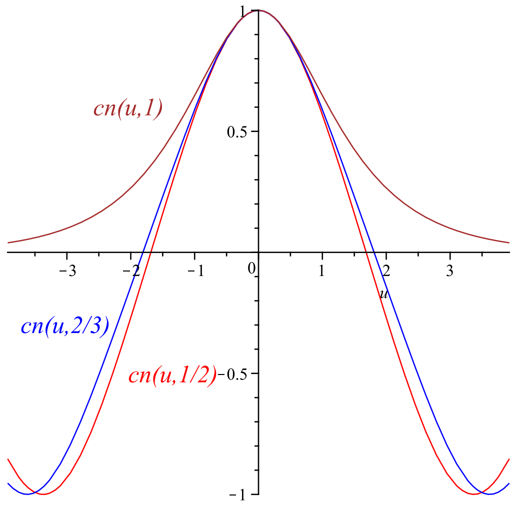

Figure 2.2: $cn(u,\kappa)$ with $\kappa = \frac{1}{2}, \frac{2}{3},$ and 1

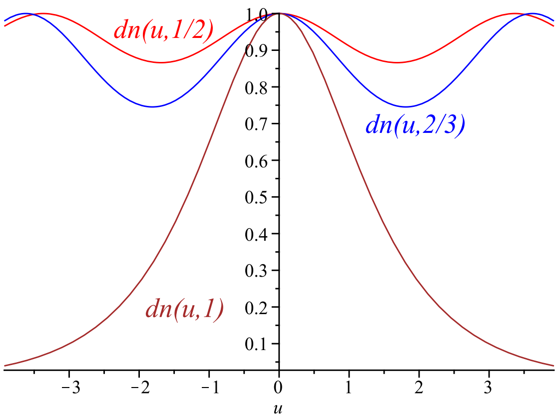

Figure 2.3: $dn(u,\kappa)$ with $\kappa = \frac{1}{2}, \frac{2}{3},$ and 1

2.4 An Application of Elliptic functions

In the last section of this chapter, we will introduce a physical problem which could be solved by elliptic function. That is, its mathematical model can be represented as the elliptic integrals. Most of mathematical models of these problems must derive by the Newton's second law, the famous equation, $\overrightarrow{F} = m\overrightarrow{a}$. Let us see the example of the nonlinear spring vibration [7].

In the before, the mathematical model of the spring motion represented as

$$m\frac{d^2x}{dt^2} = -kx, \tag{2.45}$$

where k is a constant and called the *spring rate*, m is the mass of the body, and x denoted the position. But the linear model only satisfies over a small range of displacements. The spring rate will depend on the position x when the displacement is large. Such spring rate may be represented by a quadratic function $k = k_0 + rx^2$ where r is determined case by case. Applied the Newton's second law, the equation (2.45) can be replaced by

$$m\frac{d^2x}{dt^2} = -\left(k_0 + rx^2\right)x. \tag{2.46}$$

The equation (2.46) can be arranged as

$$\frac{d^2x}{dt^2} = -\left(\frac{k_0}{m}x + \frac{r}{m}x^3\right). \tag{2.47}$$

And the initial conditions are given by

$$x(0) = 0, \quad x'(0) = v_0 > 0.$$

Multiplying equation (2.47) and integrating it, we have

$$\frac{dx}{dt} = \sqrt{v_0^2 - \left(\frac{2k_0}{m}x^2 + \frac{r}{2m}x^4\right)} \tag{2.48}$$

$$= v_0\sqrt{1 - \left(\frac{2k_0}{mv_0^2}x^2 + \frac{r}{2mv_0^2}x^4\right)}. \tag{2.49}$$

Then by the separable equation, the equation (2.49) can be written as

$$t = \frac{1}{v_0}\int_0^x \frac{dz}{\sqrt{1 - \left(\frac{2k_0}{mv_0^2}z^2 + \frac{r}{2mv_0^2}z^4\right)}}. \tag{2.50}$$

The equation (2.50) can be solved by elliptic function. After twice change variables, the solution can be find out as

$$x(t) = \frac{1}{a} cn(\mathrm{K} - v_0\sqrt{a^2 + b^2}t) \tag{2.51}$$

where

$$a^2 = \frac{k_0}{2mv_0^2} + \sqrt{\left(\frac{k_0}{2mv_0^2}\right)^2 + \frac{r}{2mv_0^2}}, \tag{2.52}$$

$$b^2 = \frac{-k_0}{2mv_0^2} + \sqrt{\left(\frac{k_0}{2mv_0^2}\right)^2 + \frac{r}{2mv_0^2}}, \tag{2.53}$$

$$\kappa = \frac{a^2}{a^2 + b^2}. \tag{2.54}$$

The solution shows that this problem could be solved by elliptic function $cn(t, \kappa)$. Not like the pendulum model, this problem is more complicated. In general, most of the problems which involved elliptic problem have to change variables many times. The application of elliptic functions is not only for spring vibrating. There are many problems in engineering, electromagnetics and other science will involve the elliptic functions. The chapter 9 of the reference [7] has more examples.

Chapter 3

The Exact Theory of the Sine-Gordon Equation

3.1 The Exact Theory

Sine-Gordon equation is a partial differential equation with the form

$$u_{tt} - u_{xx} + sin(u) = 0. \tag{3.1}$$

We want to find the traveling wave solution of (3.1). Assume that $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$ and (3.1) will be transferred to be a ordinary differential equation as

$$u_{\theta\theta} + sin(u) = 0 \tag{3.2}$$

which we have introduced in the section 1.3. Multiplying u_θ to (3.2) and integrating it with respect to θ, the equation (3.2) will become

$$\frac{1}{2}u_\theta^2 - cos(u) = E, \quad \text{where} \quad E \text{ is a constant.} \tag{3.3}$$

Then the square roots of u_θ are $\pm\sqrt{2\,(E + cos(u))}$. We focus on the positive sign

$$u_\theta = \sqrt{2\,(E + cos(u))}. \tag{3.4}$$

By $cos(2x) = 1 - 2sin^2(x)$, the relation of trigonometric function. The equation (3.4) can be written as

$$u_\theta = \sqrt{2\left(E + 1 - 2sin^2(\frac{u}{2})\right)}. \tag{3.5}$$

Since the equation is a separable equation, we could get

$$\theta = \int_0^{U(\theta)} \frac{1}{\sqrt{2\left(E+1\right) - 4sin^2(\frac{u}{2})}}\,\mathrm{d}u. \tag{3.6}$$

Our goal is to find the solution of equation (3.6). That is, we must find the representation of $U(\theta)$ in terms of θ. We discuss it in three different cases by given different E.

<u>Case 1</u>. $-1 < E < 1$

If the constant E $\in (-1, 1)$, the equation (3.6) can be written as

$$\theta = \int_0^{U(\theta)} \frac{1}{\sqrt{2\left(E+1\right) - 4sin^2(\frac{u}{2})}}\,\mathrm{d}u \tag{3.7}$$

$$= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(\theta)}{2}} \frac{1}{\sqrt{1 - \frac{2}{E+1} sin^2(\frac{u}{2})}}\,\mathrm{d}\left(\frac{u}{2}\right). \tag{3.8}$$

Let $t = \sqrt{\frac{2}{E+1}} sin\left(\frac{u}{2}\right)$, then $d\left(\frac{u}{2}\right) = \frac{1}{\sqrt{\frac{2}{E+1} - t^2}} dt$. And (3.8) becomes

$$\theta = \int_0^{\sqrt{\frac{2}{E+1}} sin(\frac{U(\theta)}{2})} \frac{1}{\sqrt{1-t^2}} \frac{1}{\sqrt{1 - \left(\frac{E+1}{2}\right) t^2}}\,\mathrm{d}t. \tag{3.9}$$

Let $\kappa = \sqrt{\frac{E+1}{2}}$, then equation (3.9) can be represented as

$$\theta = \int_0^{\kappa^{-1} sin(\frac{U(\theta)}{2})} \frac{1}{\sqrt{1-t^2}} \frac{1}{\sqrt{1-\kappa^2 t^2}}\,\mathrm{d}t. \tag{3.10}$$

By Jacobian elliptic function $sn(u, \kappa)$, the equation (3.10) implies that $sn(\theta, \kappa) = \kappa^{-1} sin(\frac{U(\theta)}{2})$. This means

$$U(\theta) = 2sin^{-1}(\kappa\ sn(\theta, \kappa)), \text{ where } \kappa = \sqrt{\frac{E+1}{2}}. \tag{3.11}$$

The graphs of this case are performed below:

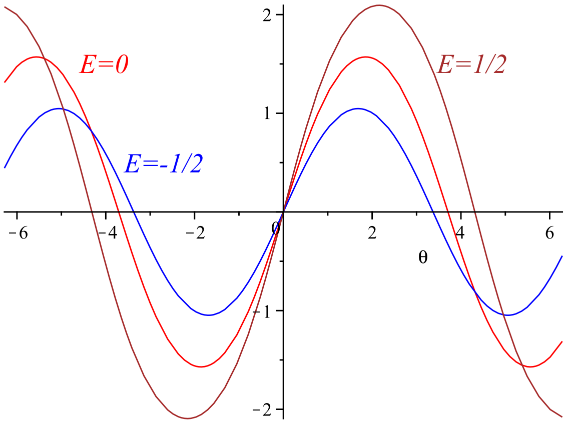

Figure 3.1: Solution curves with $E = -\frac{1}{2}$, $E = 0$, and $E = \frac{1}{2}$

<u>Remark</u> 10.

a. Since $E \in (-1, 1)$, $\sqrt{\frac{E+1}{2}} \in (0, 1)$. That is, $0 < \kappa < 1$. Furthermore, $\kappa \propto E$.

b. See the Figure 3.1. We could find out that the cases which have bigger energy will have bigger period and amplitude.

<u>Case 2</u>. $E = 1$

When the constant $E = 1$, the equation (3.6) becomes

$$\theta = \int_0^{U(\theta)} \frac{1}{\sqrt{4 - 4sin^2(\frac{u}{2})}}\, \mathrm{d}u \tag{3.12}$$

$$= \int_0^{\frac{U(\theta)}{2}} \frac{1}{\sqrt{1 - sin^2(\frac{u}{2})}}\, \mathrm{d}\left(\frac{u}{2}\right). \tag{3.13}$$

Let $sin(\dfrac{u}{2}) = t$, then $d\left(\dfrac{u}{2}\right) = \dfrac{1}{\sqrt{1-t^2}}dt$. And (3.13) will become

$$\theta = \int_0^{sin(\frac{U(\theta)}{2})} \frac{1}{\sqrt{1-t^2}\sqrt{1-t^2}}\, \mathrm{d}t. \tag{3.14}$$

By the Jacobian elliptic function $sn(u, \kappa)$, (3.14) implies that $sin(\dfrac{U(\theta)}{2}) = sn(\theta, 1)$. That is,

$$U(\theta) = 2sin^{-1}\left(sn(\theta, 1)\right). \tag{3.15}$$

Let us see the graph before get a remark.

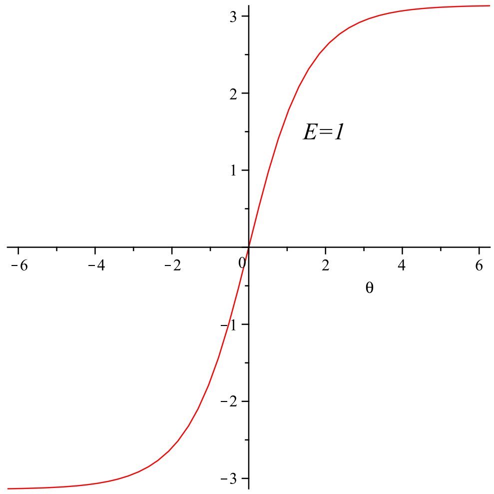

Figure 3.2: Solution curve with $E = 1$

Remark 11.

a. If we do not use Jacobian elliptic function, we can also get the solution by Calculus. The solution is

$$U(\theta) = 2sin^{-1}(\tanh\theta).$$

b. Observe the Figure 3.2. We may guess that $U = \pi$ and $U = -\pi$ are two horizontal asymptotes.

Case 3. $E > 1$

The discussion of this case is similar to the first case. The different is on the modulus κ. From (3.6), we have

$$\theta = \int_0^{U(\theta)} \frac{1}{\sqrt{2\,(E+1) - 4sin^2(\frac{u}{2})}}\,\mathrm{d}u \tag{3.16}$$

$$= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(\theta)}{2}} \frac{1}{\sqrt{1 - \frac{2}{E+1}sin^2(\frac{u}{2})}}\,\mathrm{d}\left(\frac{u}{2}\right). \tag{3.17}$$

Let $\kappa = \sqrt{\frac{2}{E+1}}$, then (3.17) will be written as

$$\theta = \kappa \int_0^{U(\theta)} \frac{1}{\sqrt{1 - \kappa^2 sin^2(\frac{u}{2})}}\,\mathrm{d}\left(\frac{u}{2}\right). \tag{3.18}$$

Let $t = sin(\frac{u}{2})$, and $d\left(\frac{u}{2}\right) = \frac{1}{\sqrt{1-t^2}}dt$. Then (3.18) can be represented as

$$\theta = \kappa \int_0^{sin(\frac{U(\theta)}{2})} \frac{1}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}} \mathrm{dt}. \tag{3.19}$$

By Jacobian elliptic function, (3.19) implies that $sn(\frac{\theta}{\kappa}, \kappa) = sin(\frac{U(\theta)}{2})$. Thus,

$$U(\theta) = 2sin^{-1}(sn(\frac{\theta}{\kappa}, \kappa)) \text{ where } \kappa = \sqrt{\frac{2}{E+1}}. \tag{3.20}$$

The solutions of this case are as following:

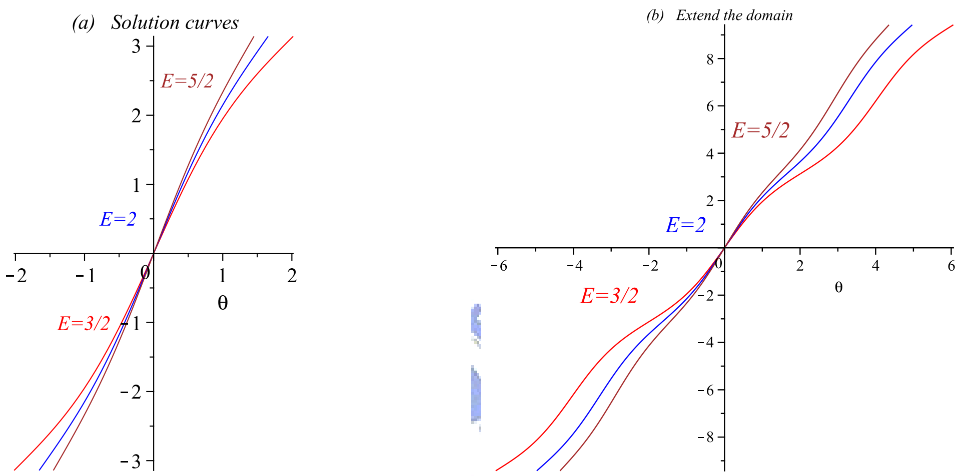

Figure 3.3: Solution curves with $E = \frac{3}{2}, 2, \frac{5}{2}$.

Remark 12.

a. Since $E > 1$, $\sqrt{\frac{2}{E+1}}$ is smaller than 1. That is, $0 < \kappa < 1$. Moreover, $\kappa \propto \frac{1}{E}$.

b. By observing the Figure 3.3 (a), we find that the domain of U is smaller if the E is larger. Moreover, the solutions of this type are not periodic solutions since the position U is increasing as the parameter θ runs.

3.2 The Periods

We had found the solutions for the ordinary differential equation in the form of Jacobian elliptic function with different constant E. Now we want to find out the period of solution if it is a periodic solution. The idea it to find the rest position $U(\theta_0)$ and the period is the four times time of the particle moves from $U(0)$ to $U(\theta_0)$.

Case 1. $-1 < E < 1$

The solution of this case is

$$U(\theta) = 2sin^{-1}(\kappa\ sn(\theta,\kappa)),\ \text{where } \kappa = \sqrt{\frac{E+1}{2}},$$

the equation (3.11). Using (3.5), we could get the velocity of the particle is

$$U_\theta = \sqrt{2\,(E+1) - 4sin^2(\frac{U}{2})}. \tag{3.21}$$

If the equation (3.21) equal to 0, then

$$U(\theta) = \pm 2sin^{-1}(\kappa).$$

Therefore, by (3.10), we know that the period is

$$\begin{aligned} T &= 4\ \theta \\ &= 4\int_0^{\kappa^{-1}sin(sin^{-1}(\kappa))} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-\kappa^2 t^2}}\,\mathrm{d}t \\ &= 4\int_0^{1} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-\kappa^2 t^2}}\,\mathrm{d}t \\ &= 4\mathrm{K}. \end{aligned}$$

Then we find the period for this case.

Remark 13.

a. The constant K here is defined as $\mathrm{K} = \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-\kappa^2t^2)^{-\frac{1}{2}}\,\mathrm{d}t,$ where κ is the modulus. It is the same value to the *Remark* 9. (a).

b. The constant $\mathrm{K} \propto \kappa$. This is implies that the period $T \propto \kappa$. Moreover, it is not difficult to calculate that $T = 2\pi$ if $\kappa = 0$. This means that the period $T > 2\pi, \quad \forall \quad \kappa \in (0,1)$. Furthermore, this also tell us that if $U(\theta) = 2sin^{-1}(\kappa) < 2sin^{-1}(1) = \pi$, it is a periodic solution with period 4K.

Case 2. $E = 1$

The solution of this case is $U(\theta) = 2sin^{-1}(sn(\theta,1))$, the equation (3.15). Putting $E = 1$ into (3.21) and we could get $U_\theta = \sqrt{4 - 4sin^2(\frac{U}{2})}$ and the zeros at $U(\theta) = \pm\pi$. Then using (3.14), the period can be calculated

$$\begin{aligned} T &= 4\ \theta \\ &= 4\int_0^{sin(\pi/2)} \frac{1}{\sqrt{1-t^2}\sqrt{1-t^2}}\,\mathrm{d}t \\ &= 4\int_0^{1} \frac{1}{1-t^2}\,\mathrm{d}t \\ &= \infty. \end{aligned}$$

The period of this case could be regarded as ∞ although it is not a periodic solution. This means that if we release the particle at the position $-\pi$, it needs infinity time to approach the position π.

Case 3. $E > 1$

By the equation (3.21), we know that the velocity is always positive for this case $E > 1$. This means that at each position $U(\theta)$, the pendulum always has velocity, so the pendulum will never stop. This implies that it has no periodicity.

In the end of this section, we construct a table to collect the results we had gotten:

Energy (E)	$E = -1$	$-1 < E < 1$	$E = 1$	$E > 1$
Solution $U(\theta)$	0	$2sin^{-1}(\kappa sn(\theta,\kappa))$	$2sin^{-1}(sn(\theta,1))$	$2sin^{-1}(sn(\frac{\theta}{\kappa},\kappa))$
Modulus (κ)	None	$\sqrt{\frac{E+1}{2}}$	1	$\sqrt{\frac{2}{E+1}}$
Period (T)	$a \in \mathbb{R}$	4K	∞	No periodicity

Table 3.1: Summary about the ideal pendulum model with different E.

3.3 The Phase Portraits

The ordinary differential equation we had discussed is the mathematical model of ideal pendulum. We have plotted the solution curves, the relation between $U(\theta)$ and θ, in the above section. Now we try to plot the relation between U and U_θ

and the graph is called **phase portrait**. Before drawing the phase portrait, we see back to the equation (3.3) first. It shows that $\frac{1}{2}u_\theta^2 - cos(u)$ is a constant. It can be regarded as a conservation law in the view point of mathematics since $-cos(u)$ is not always larger than 0. (But this case can be transferred to the conservation law in the view point of physics by plus a constant $a \geq 1$ for equation (3.3).). This means that its total energy is a constant and the former part $\frac{1}{2}u_\theta^2$ can be regarded as kinetic energy and the latter part $-cos(u)$ can be regarded as potential energy. The following we discuss the potential energy and phase portrait with different cases.

Case 1. $-1 < E < 1$

We set $E = 0$ to analyze the case. By the equation (3.3), we have the equation $u_\theta = \pm\sqrt{2cos(u)}$. The following graphs are potential energy and phase portrait, respectively. This means that they are the relation between u and $cos(u)$ and the relation between u and u_θ.

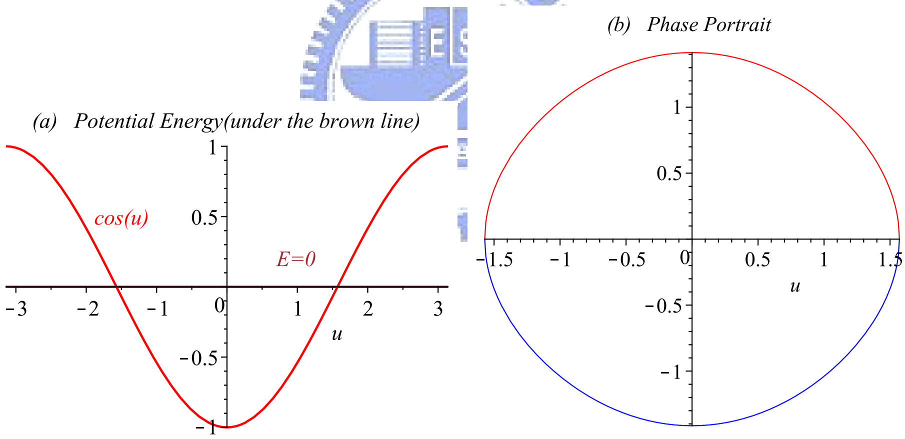

Figure 3.4: The potential energy and phase portrait for E = 0

Remark 14.

a. From the graph of the phase portrait, the red curve means that the velocity at those position are positive and the blue curve means that the velocity at those position are negative. The positive velocity is defined by rotating counterclockwise and the negative velocity is defined by rotating clockwise.

b. By the graph of potential energy, we can find out that the maximum of amplitude, $u(\theta)$, for the pendulum is $\frac{\pi}{2}$ and it oscillates forth and back.

<u>Case 2</u>. $E = 1$

Now we focus on the case with $E = 1$. By the equation (3.3), we have the equation $u_\theta = \pm\sqrt{2\,(1 + cos(u))}$. We see the potential energy and phase portrait as following.

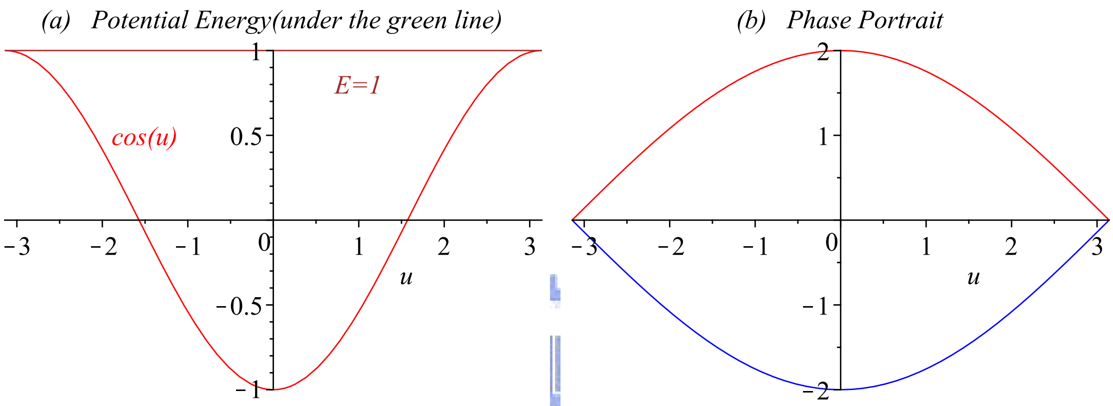

Figure 3.5: The potential energy and phase portrait for E = 1

Note:

By the graph of potential energy, we can find out that the maximum of amplitude, $u(\theta)$, for the pendulum is π. If we release the pendulum at position π, the particle will approach to the position $-\pi$ after infinite time.

<u>Case 3</u>. $E > 1$

Last, we see the case $E > 1$ with $E = \frac{3}{2}$. By the equation (3.3), we have the equation $u_\theta = \pm\sqrt{2\left(\frac{3}{2} + cos(u)\right)}$. We see the potential energy and phase portrait as following.

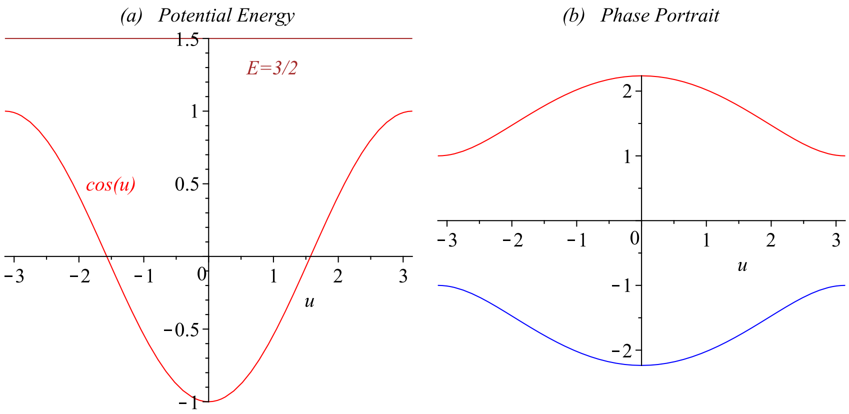

Figure 3.6: The potential energy and phase portrait for E = 3/2

<u>Remark</u> 15.

a. From the graph of the phase portrait, we know that the pendulum of this case will never stop since the phase portrait has no intersection with the $u - axis$.

b. By the graph of potential energy, we observe that the kinetic energy is never equal 0.This implies that the case has no periodic solution and the result is corresponded to the property which we had discussed.

By our discussion, there are three kinds of the phase portraits. Before finishing the section, we combine the three phase portraits and the vector field together.

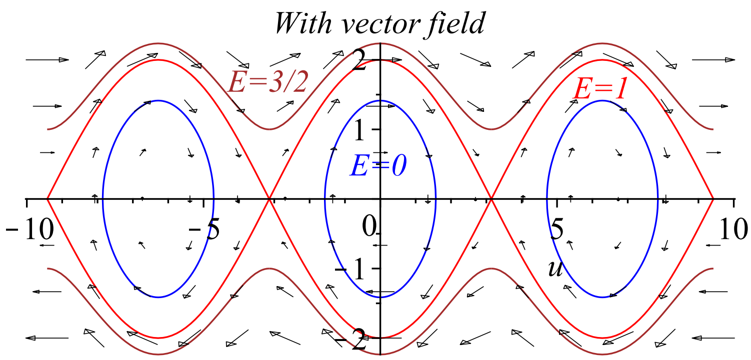

Figure 3.7: Global phase portrait.

We can make some conclusions from the Figure 3.7:

1. There are three different kinds of phase portraits with different energy E. The outer curve corresponds to larger energy E. They are separated by the phase portrait with $E = 1$ and the phase curve is called **the separatrix** with periods ∞. The phase curves outer the separatrix are called **the wave train** and they has no period. The phase curves inside the separatrix are periodic and their period T satisfies $2\pi < T < \infty$.

2. The direction of the phase curves which are upper the $u - axis$ toward the right on the phase plane and it means the pendulum rotates counterclockwise. Similarly, the direction of the phase curves which are below the $u - axis$ toward the left and it means the pendulum rotates clockwise.

3. The points $(n\pi, 0)$ are also the solutions for all $n \in \mathbb{Z}$. They are classified into two classes. The first is the points with n is even. These points are stable and with energy $E = -1$. The other is the points with n is odd and these points are unstable and with energy $E = 1$.

3.4 The Solutions in Terms of (x, t) Variables

At the beginning, we transferred the partial differential equation $u_{tt} - u_{xx} + sin(u) = 0$ to the ordinary different equation $u_{\theta\theta} + sin(u) = 0$ by setting $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$. We solved the ordinary differential equation and got the solution successfully. Now we transfer the solutions $U(\theta)$ to $U(x, t)$ and analyze some properties.

Case 1. $-1 < E < 1$

We know that the solution for $u_{\theta\theta} + sin(u) = 0$ in this case is equation (3.11). Then using $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$, (3.11) will become

$$U(x, t) = 2sin^{-1}(\kappa\ sn(kx - \omega t, \kappa)), \text{ where } \kappa = \sqrt{\frac{E+1}{2}}. \tag{3.22}$$

First, we verify that (3.22) is a solution of $u_{tt} - u_{xx} + sin(u) = 0$. After complicated computing, we have

$$U_{tt} = -2\kappa\omega^2 sn(kx - \omega t, \kappa)\sqrt{1 - \kappa^2 sn^2(kx - \omega t, \kappa)}, \tag{3.23}$$

$$U_{xx} = -2\kappa k^2 sn(kx - \omega t, \kappa)\sqrt{1 - \kappa^2 sn^2(kx - \omega t, \kappa)}, \tag{3.24}$$

$$sin(U) = 2\kappa sn(kx - \omega t, \kappa)\sqrt{1 - \kappa^2 sn^2(kx - \omega t, \kappa)}. \tag{3.25}$$

Then (3.23) $-$ (3.24) $+$ (3.25) $= 0$ since $\omega^2 - k^2 = 1$. Thus, we show that (3.22) is a solution of (3.1).
And we see the graph about (3.22).

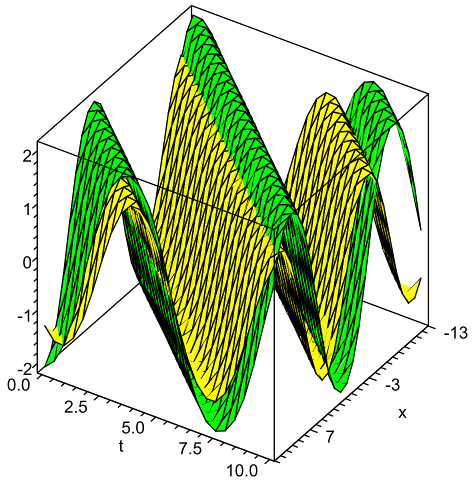

Figure 3.8: The graph of U(x,t) with $E = 0$ and $E = \frac{1}{2}$

The Figure 3.8 displays two surfaces of $U(x,t)$. The green is with $E = \frac{1}{2}$ and the yellow is with $E = 0$. It is obviously that the maximum value of the green surface is larger than the yellow one and the result is corresponding to the property of the Jacobian function $sn(u,k)$ which we have discussed before. Next, we see other two graphs which display the surface $U(x,t)$ and some special curves on the surface.

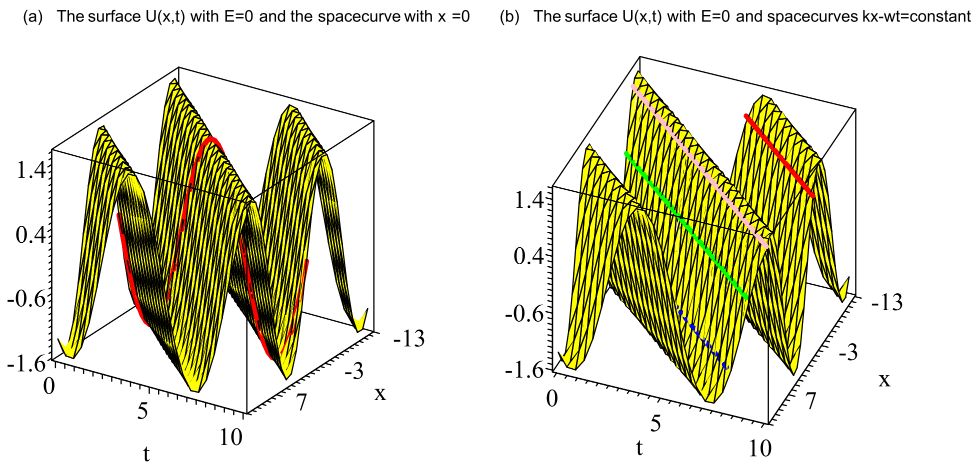

Figure 3.9: $U(x,t)$ with $E = 0$

Case 2. $E = 1$

We found the solution for $u_{\theta\theta} + sin(u) = 0$ in this case is equation (3.15). Using $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$, (3.15) will become

$$U(x,t) = 2sin^{-1}(kx - \omega t, 1). \tag{3.26}$$

It is a special case of (1) with $\kappa = 1$, so (3.26) satisfies equation (3.1). The graph of (3.26) is as following:

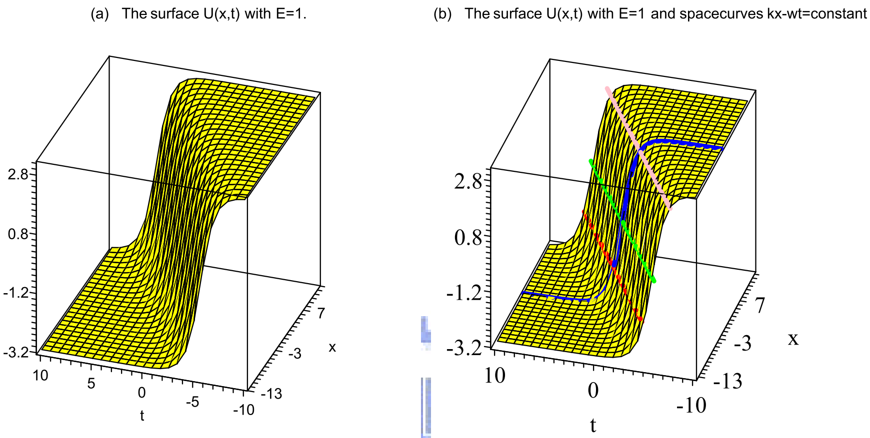

Figure 3.10: $U(x,t)$ with $E = 1$

Case 3. $E > 1$

The solution for $u_{\theta\theta} + sin(u) = 0$ in this case is (3.20). After transferring by $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$, (3.20) will become

$$U(x,t) = 2sin^{-1}(sn(\frac{kx - \omega t}{\kappa}, \kappa)), \text{ where } \kappa = \sqrt{\frac{2}{E+1}}. \tag{3.27}$$

Then computing U_{tt}, U_{xx}, and $sin(U)$ with respect to (3.27), we have

$$U_{tt} = -2\omega^2 sn(\frac{kx - \omega t}{\kappa}, \kappa)\sqrt{1 - sn^2(\frac{kx - \omega t}{\kappa}, \kappa)}, \tag{3.28}$$

$$U_{xx} = -2k^2 sn(\frac{kx - \omega t}{\kappa}, \kappa)\sqrt{1 - sn^2(\frac{kx - \omega t}{\kappa}, \kappa)}, \tag{3.29}$$

$$sin(U) = 2sn(\frac{kx - \omega t}{\kappa}, \kappa)\sqrt{1 - sn^2(\frac{kx - \omega t}{\kappa}, \kappa)}. \tag{3.30}$$

Then $(3.28) - (3.29) + (3.30) = 0$ since $\omega^2 - k^2 = 1$. Thus, we show that (3.27) is a solution of (3.1). Next, we see the graph of (3.27).

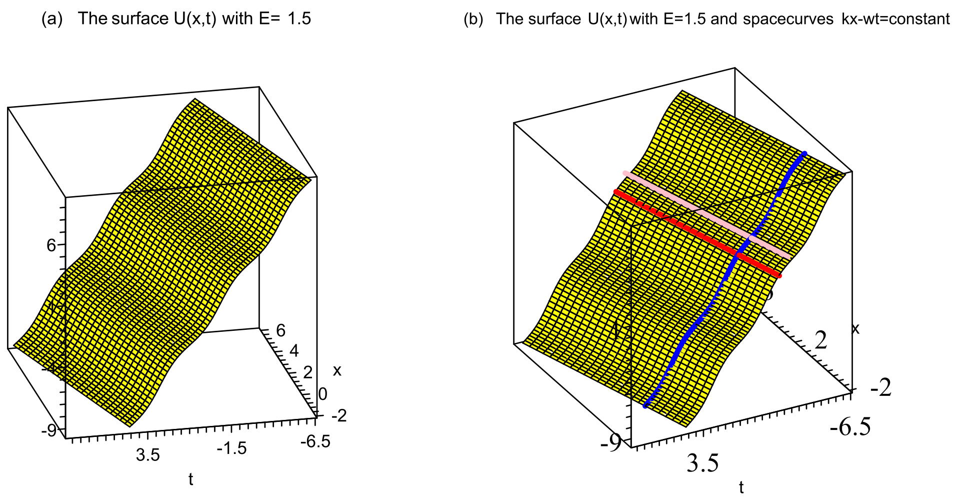

Figure 3.11: $U(x,t)$ with $E = 3/2$

Remark 16.

a. We see the Figure 3.9(a) first. Under the situation $x = 0$, we can get the red curve on the surface. The shape of the red curve is similar to the Figure 3.1 since $U(0,t) = U(kx - \omega t) = U(-\omega t)$. So the equation of red curve has the form like the equation (3.11). Moreover, this implies that if we stay at $x = 0$ to observe the wave, the amplitude at the position is looked like the red curve. Furthermore, the result is satisfied by each fixed point x. The blue curves of the right graphs in the Figure 3.10 and the Figure 3.11 have the same properties as the red curve.

b. The curves on the surface $U(x,t)$ of the Figure 3.9(b) with color blue, green, pink, and red corresponded to $kx - \omega t = -3, -4, -5,$ and -12, respectively. Notice that the curves are all straight lines. This means that the height is the same on the line $kx - \omega t$ is a constant and the lines $kx - \omega t = c$ are called **phase lines**. In the view point of physics, this implies that if we start from any position x_0 with velocity $\frac{\omega}{k}$ to observe a wave, we can always see the same height of the wave. The velocity $\frac{\omega}{k}$ is called the **wave velocity**. Similarly, the straight lines in the right graphs of Figure 3.10 and Figure 3.11 are also phase lines.

c. In our discussion, the ω and k have to satisfy $\omega^2 - k^2 = 1$. In general, $\frac{\omega}{k}$ can be any real number except 0.

Chapter 4

The Perturbation Theory of the Sine-Gordon Equation

We have discussed the solutions, phase portraits, and solution curves of the mathematical model of the ideal pendulum $u_{\theta\theta} + sin(u) = 0$ in the previous chapter. We do the qualitative analysis and quantitative analysis for the equation. In this chapter, we will discuss the perturbed equations. There are many different sources of perturbation. Here, we will mention the damping term and external force and they will have big different on the behavior of their solutions.

4.1 The Pendulum with Friction

First, we start from the equation

$$u_{\theta\theta} + sin(u) = -\varepsilon u_\theta \tag{4.1}$$

with small $\varepsilon > 0$. The term, u_θ, represents the velocity of the pendulum at one position. This means that the source of perturbation of the system is from its velocity. Assume that $v = u_\theta$, then the equation (4.1) will become a system as

$$\begin{cases} \dfrac{du}{d\theta} = v, \\ \dfrac{dv}{d\theta} = -\varepsilon v - sin(u). \end{cases} \tag{4.2}$$

Then we focus on the system (4.2) in this section. Before our analysis, it is necessary to introduce the definition of the **Hamiltonian system** [9].

Definition 5. A system of differential equations is called a **Hamiltonian system** if there exists a real-valued function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y},$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

for all x and y. The function H is called the *Hamiltonian function* for the system.

Remark 17.

a. If $(x(t), y(t))$ is the solution of the system, the function H is always a constant on the solution curve:

$$\begin{aligned}\frac{d}{dt}H(x(t), y(t)) &= \left(\frac{\partial H}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial H}{\partial y}\right)\left(\frac{dy}{dt}\right) \\ &= \left(\frac{\partial H}{\partial x}\right)\left(\frac{\partial H}{\partial y}\right) + \left(\frac{\partial H}{\partial y}\right)\left(-\frac{\partial H}{\partial x}\right) \\ &= 0.\end{aligned}$$

This implies that the solution curves of the system lie on the level curves of the Hamiltonian function $H(x, y)$.

b. If $H(x, y)$ is a Hamiltonian function for a system, then $H(x, y) + a$ is also a Hamiltonian function for the system for any constant $a \in \mathbb{R}$.

c. In some special cases, Hamiltonian function can be regarded as total energy of the system.

Unfortunately, the system (4.1) is not a Hamiltonian system since there does not exist any two variable function $H(u, v)$ satisfy the definition of Hamiltonian system. But we analyze it with the **Lyapunov theorem** [6]. In many case, the Lyapunov function is the Hamiltonian function of the unperturbed system. Hence, we let

$$L(u, v) = \frac{1}{2}v^2 + (1 - cos(u)). \tag{4.3}$$

Next, calculating $\frac{dL}{d\theta}$ with respect to the system (4.2) will get

$$
\begin{aligned}
\frac{d}{d\theta}L(u(\theta), v(\theta)) &= \left(\frac{\partial L}{\partial u}\right)\left(\frac{du}{d\theta}\right) + \left(\frac{\partial L}{\partial v}\right)\left(\frac{dv}{d\theta}\right) \\
&= (sin(u))\, v + v\,(-\varepsilon v - sin(u)) \\
&= -\varepsilon v^2 \\
&\leq 0.
\end{aligned}
$$

This means that if $(u(\theta), v(\theta))$ is a solution of the system (4.2), the total energy is always decrease. In other words, the pendulum will stop swing for a large time no matter how large the total energy for the pendulum. The following is the solution curves with vector field for $\varepsilon = 0.06$.

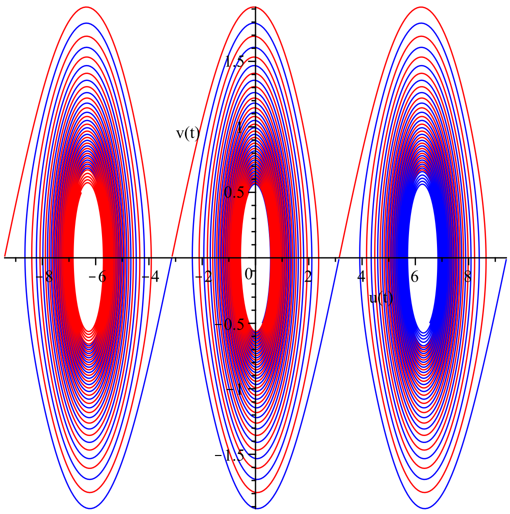

Figure 4.1: Phase portraits for the damped system with $\varepsilon = 0.06$.

By the system (4.2), we could find the equilibrium points are $(n\pi, 0)$ where $n \in \mathbb{Z}$. It is easy to check that the undamped system (derive from equation (3.2)) has the same equilibrium points to the damped system. But we know that the type of equilibrium points are not always the same from their phase portrait Figure 3.7 and Figure 4.1. The equilibrium points for the undamped system is either a center or a saddle. It never has sink since it is a Hamiltonian system. And the equilibrium points for the damped system is either sink or saddle. It never has center because it is not a Hamiltonian system. In other words, the total energy is not conserved for the damped system. As a matter of fact, the total energy is decreasing to 0 as the time runs. This means that the pendulum will must stop no matter how large the energy for the pendulum at the beginning. We make a table to summarize the equilibrium points of the two systems:

Equilibrium Points	$(2(n+1)\pi, 0)$	$(2n\pi, 0)$
(4.2) with $\varepsilon = 0$	saddle	center
(4.2) with $\varepsilon = 0.06$	saddle	sink

Table 4.1: The comparison of the equilibrium points of the undamped system with the damped system.

4.2 The Pendulum with External Force

The previous section we discuss the perturbation with respect to the damped term. In this section, we will focus on the perturbation by giving a external force to the Sine-Gordon equation. We will see that the system will have big different behavior if it is given different external force.

4.2.1 Position

We start from the equation

$$u_{\theta\theta} + sin(u) = -\varepsilon u \text{ where } \varepsilon > 0 \tag{4.4}$$

which is given an little external force related to the position u. Just like the previous section, we transfer (4.4) to a system of two first order differential equations first. Assume that $v = u_\theta$, then the equation (4.4) will become a system as following:

$$\begin{cases} \dfrac{du}{d\theta} = v, \\ \dfrac{dv}{d\theta} = -\varepsilon u - sin(u). \end{cases} \tag{4.5}$$

The following discussions are based on the system (4.5).

Hamiltonian System

By the Definition 5, we could check that (4.5) is a Hamiltonian system. We want to find a function $H(u, v)$ such that

$$\frac{\partial H}{\partial v} = v, \tag{4.6}$$

$$-\frac{\partial H}{\partial u} = -\varepsilon u - sin(u). \tag{4.7}$$

Integrating (4.6) with respect to v, we will get

$$H(u, v) = \frac{1}{2}v^2 + g(u). \tag{4.8}$$

Similarly, integrating (4.7) with respect to u will have

$$H(u, v) = \frac{1}{2}\varepsilon u^2 - cos(u) + h(v). \tag{4.9}$$

Combine (4.8) with (4.9), we could choose a Hamiltonian function

$$H_\varepsilon^*(u, v) = \frac{1}{2}v^2 + \frac{1}{2}\varepsilon u^2 - cos(u). \tag{4.10}$$

To ensure (4.10) is always nonnegative, let $H_\varepsilon = H_\varepsilon^* + 1$. Then

$$H_\varepsilon(u, v) = \frac{1}{2}v^2 + \frac{1}{2}\varepsilon u^2 + (1 - cos(u)). \tag{4.11}$$

Hence, this implies that (4.5) is a Hamiltonian system and H_ε is a Hamiltonian function for this system.

Remark 18.

a. We use the symbol H_ε to represent the Hamiltonian function because the Hamiltonian function $H(u, v)$ is dependent on the ε. This means that the Hamiltonian function varies with ε. Moreover, when $\varepsilon = 0$, it is a Hamiltonian function for the system which derive from the ideal pendulum model $u_{\theta\theta} + sin(u) = 0$.

b. The hamiltonian function H could be regarded as total energy. $\frac{1}{2}v^2$ represents the kinetic energy since it depends on velocity and $\frac{1}{2}\varepsilon u^2 + (1 - cos(u))$ represents the potential energy because it only depends on position. This implies the system (4.5) satisfies the conservation law.

After verifying the system (4.5) is a Hamiltonian system, there are many benefits to analyze the quality of the system. For example, we could draw the phase portraits of the system easily if it is a Hamiltonian system. We will focus on the phase portraits of (4.5) in the end of this subsection. Let us talk about the equilibrium points of the system (4.5) in the following.

Equilibrium points of the system (4.5)

Before discussing the equilibrium points, we introduce the definitions of equilibrium point and linearization system first.

Definition 6. Assume a system of first ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x,y), \\ \dfrac{dy}{dt} = g(x,y). \end{cases}$$

If (x_0, y_0) satisfies $f(x_0, y_0) = g(x_0, y_0) = 0$, the point (x_0, y_0) is called an equilibrium point of the system.

Definition 7. Assume a system of first ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x,y), \\ \dfrac{dy}{dt} = g(x,y). \end{cases}$$

The linearized system of it is $\dfrac{d}{dt}\overrightarrow{Y} = \mathrm{A}\overrightarrow{Y}$ where $\overrightarrow{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and A is called the Jacobian matrix which depends on the point (x,y) with $\mathrm{A} = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x,y) & \dfrac{\partial f}{\partial y}(x,y) \\ \dfrac{\partial g}{\partial x}(x,y) & \dfrac{\partial g}{\partial y}(x,y) \end{pmatrix}$

Remark 19.

a. For the system (4.5), the number of the equilibrium points, N, depends on ε. The smaller ε have more equilibrium points.

b. The linearized system depends on the equilibrium point since the Jacobian matrix is dependent on the equilibrium point. That is, different equilibrium points will correspond to the different linearized system.

c. By discussing the eigenvalues of the Jacobian matrix, we could determine the type of the equilibrium point for linearized system. After deciding the type of equilibrium point for linearized system, we can determine the type of the equilibrium points of the nonlinear system from the linearized system by **Linearization Theorem** [8].

Now, we go back to discuss the system (4.5). Definition 6 tell us that the all equilibrium points of the system (4.5) is the set

$$E = \{(u, 0) : -\varepsilon u = sin(u)\}\,. \tag{4.12}$$

And the linearized system of (4.5) is

$$\frac{d}{d\theta}\overrightarrow{Y} = \begin{pmatrix} 0 & 1 \\ -\varepsilon - cos(u) & 0 \end{pmatrix}\overrightarrow{Y},\ \text{where } \overrightarrow{Y} = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.13}$$

The Jacobian matrix $J(u, v) = \begin{pmatrix} 0 & 1 \\ -\varepsilon - cos(u) & 0 \end{pmatrix}$ and $J(u, v)$ has two eigenvalues

$$\lambda = \pm\sqrt{-\varepsilon - cos(u)}. \tag{4.14}$$

And the set E of (4.12) is divided into two classes E_1 and E_2 by the equation (4.14). $E_1 = \{(u_0, 0) \in E| - \varepsilon - cos(u_0) > 0\}$ and $E_2 = \{(u_0, 0) \in E| - \varepsilon - cos(u_0) < 0\}$. Then we discuss the two classes in the following.

<u>**Case 1.**</u> $(u_0, 0) \in E_1$

The equilibrium point $(u_0, 0)$ in E_1 make the Jacobian matrix $J(u_0, 0)$ has two real-valued eigenvalues which with opposite sign. This implies that the equilibrium point $(0, 0)$ is a saddle for the linearized system (4.13). Moreover, we know that $(u_0, 0)$ is also a **saddle** for the original system (4.5) by linearization theorem. Thus, the equilibrium points in E_1 are all saddle for the system (4.5)

<u>**Case 2.**</u> $(u_0, 0) \in E_2$

The Jacobian matrix $J(u_0, 0)$ has two purely imaginary eigenvalue with opposite sign if the equilibrium point $(u_0, 0) \in E_2$. This means that $(0, 0)$ is a center for the linearized system (4.13). Unfortunately, we can not get any information about the type of the equilibrium point $(u_0, 0)$ for the original system. Hence, we have to use other method to determine the type of equilibrium points in E_2. We use the Lyapunov function to determine the type of these equilibrium points. Consider the function $L(u,v) = \frac{1}{2}v^2 + \frac{1}{2}\varepsilon u^2 + (1 - cos(u)) + C$. It is not difficult to check $L(u,v)$ is a **Lyapunov function** and by the **Lyapunov's stability theorem** [6], we can ensure the equilibrium points in E_2 are all center.

Remark 20.

a. In Hamiltonian system, the Hamiltonian function is a Lyapunov function.

b. The C in the $L(u,v)$ is any constant. We use it to make sure $L(u_0, 0) = 0$ for different $(u_0, 0)$.

c. Using the following table to make a conclusion:

Equilibrium Points	E_1	E_2
Type	Saddle	Center
Method	Linearization	Lyapunov

Table 4.2: The types of the equilibrium points of the system (4.5).

Phase Portraits

In the last, we will see the phase portrait of the system (4.5). Since the ε can not be too large, we discuss the system with $\varepsilon = 0.01$. That is, the following discussion is based on the system

$$\begin{cases} \dfrac{du}{d\theta} = v, \\ \dfrac{dv}{d\theta} = -0.01u - sin(u). \end{cases} \tag{4.15}$$

By equation (4.11), the Hamiltonian function of the system (4.15) is

$$H(u,v) = \frac{1}{2}v^2 + \frac{1}{200}u^2 + (1 - cos(u)). \tag{4.16}$$

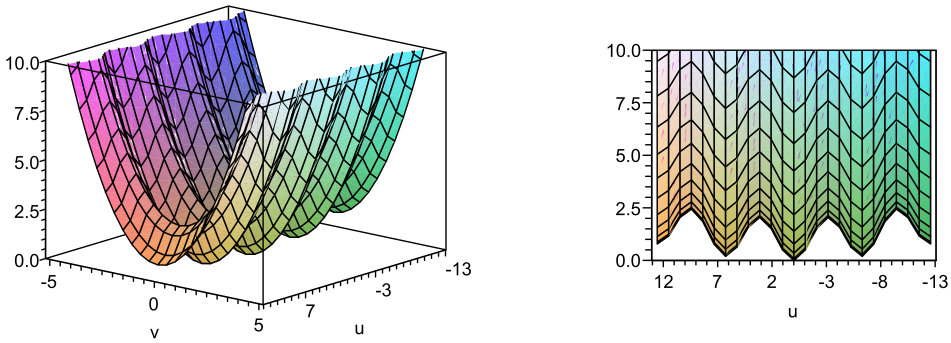

Figure 4.2: The Hamiltonian function of the system (4.15).

The Figure 4.2 is the Hamiltonian function for the system (4.15) with different angle. The Figure 4.3 is the relation between the potential energy and the position u with different velocity v.

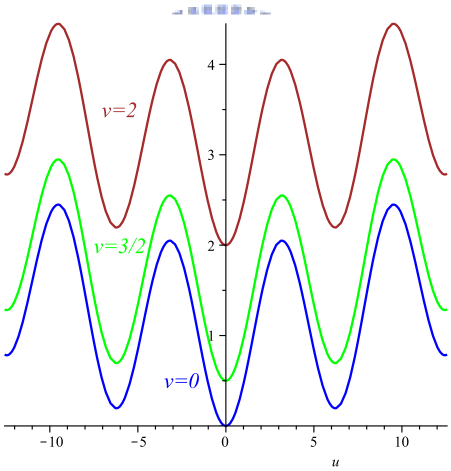

Figure 4.3: Potential energy with $v = 0, 1, 2$

Remark 17 (a) tell us that the phase portraits of a Hamiltonian system is the level curves of its Hamiltonian function. Therefore, we could draw the level curves first (Figure 4.4).

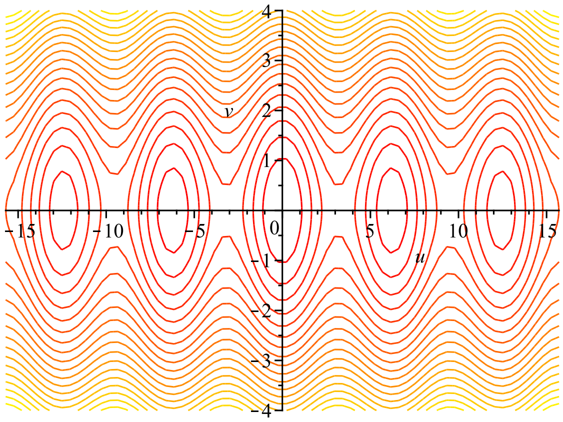

Figure 4.4: Level curves of the system (4.15).

Now we add the vector field to the Figure 4.4, we can know the direction of the level curves. That is, we can get the phase portrait of the perturbed system with $\varepsilon = 0.01$.

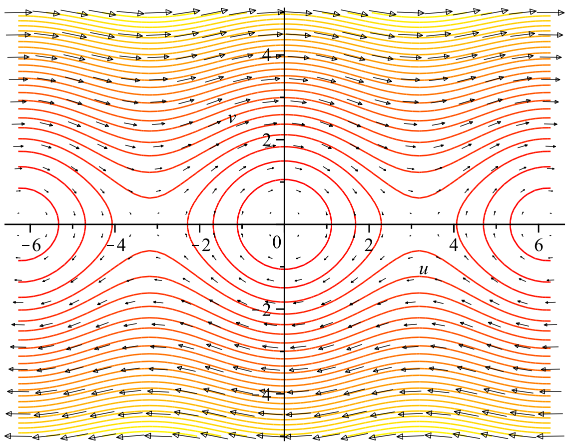

Figure 4.5: Phase portraits of the system (4.15).

Compare Figure 4.5 with Figure 3.7. Figure 3.7 shows that there the ideal pendulum mathematical model has three kinds of phase portrait. This means that there are three different types of solution curves. But in the perturbation case(Figure 4.5), the wave train does not exist anymore since each phase portrait will touch the u-axis. This implies that no matter how large the energy for the system, it will stop at some position. Moreover, the pendulum will not stop forever because the system is a Hamiltonian system. Thus, the solution of the perturbed case corresponds to either the separatrix or the periodic phase portrait.

Last, we use the nullclines to analyze the equilibrium points. We draw the graph of nullclines first.

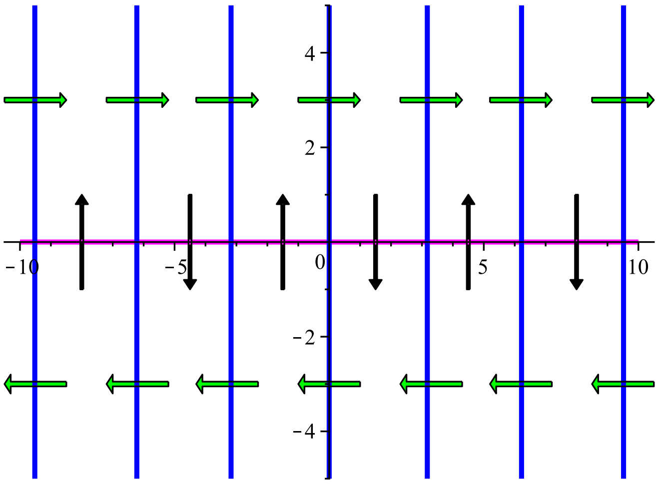

Figure 4.6: Nullclines and the vector field on them.

Notice that the blue lines are x-nullclines so the direction of vectors on them are horizontal. Similarly, the pink line is y-nullcline so the direction of vectors on it are vertical. Moreover, the intersection points are equilibrium points of the perturbed system with $\varepsilon = 0.01$. There two cases of the vector field around the equilibrium points as following.

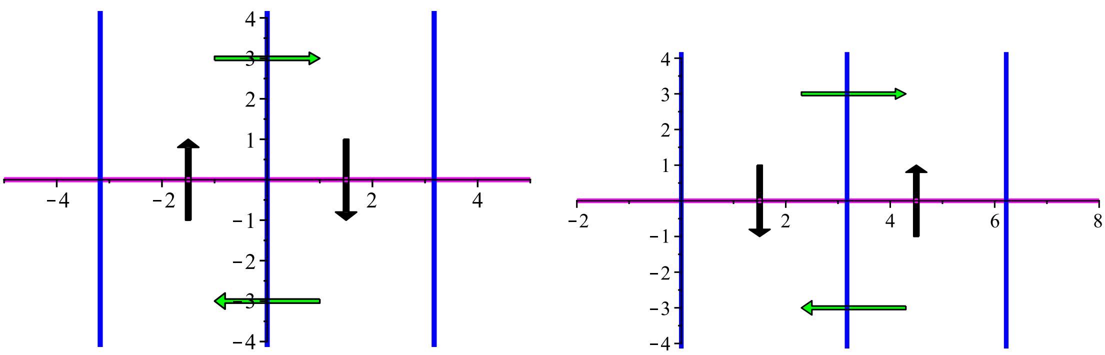

Figure 4.7: Two types of equilibrium.

For the case $\varepsilon = 0.01$, we could use computer to run the approximated value of the equilibrium points as the following table:

$(0, 0)$	$(\pm 3.173331295, 0)$	$(\pm 6.220935755, 0)$	$(\pm 9.520123592, 0)$
$(\pm 12.44163106, 0)$	$(\pm 15.86730985, 0)$	$(\pm 18.66183701, 0)$	$(\pm 22.21516933, 0)$
$(\pm 24.88128684, 0)$	$(\pm 28.56400815, 0)$	$(\pm 31.09968482, 0)$	$(\pm 34.91417424, 0)$
$(\pm 37.31669166, 0)$	$(\pm 41.26607772, 0)$	$(\pm 43.53190452, 0)$	$(\pm 47.62022050, 0)$
$(\pm 49.74482766, 0)$	$(\pm 53.97724185, 0)$	$(\pm 55.95482711, 0)$	$(\pm 60.33799317, 0)$
$(\pm 62.16105599, 0)$	$(\pm 66.70366994, 0)$	$(\pm 68.36232277, 0)$	$(\pm 73.07606663, 0)$
$(\pm 74.55683640, 0)$	$(\pm 79.45813407, 0)$	$(\pm 80.74164906, 0)$	$(\pm 85.85544527, 0)$
$(\pm 86.91119035, 0)$	$(\pm 92.28151203, 0)$	$(\pm 93.05195083, 0)$	$(\pm 98.80544697, 0)$
$(\pm 99.09481991, 0)$			

Table 4.3: The equilibrium points of the system (4.15).

By the Table 4.3, we show that the system has 65 equilibrium points. Notice that the equilibrium points are symmetric to the $v - axis$ and they are either center or saddle from our discussion in section 3.2.. Moreover, the equilibrium points with the vector field around it like the left graphs of Figure 4.7 are center and the other equilibrium points are saddle. Thus, the equilibrium points like the left graph belong to the set E_2 and the equilibrium points like the right graph belong to set E_1. Moreover, the equilibrium points in E_1 will have stable and unstable separatrices. Furthermore, we obeserve that the center and saddle appear alternately by the nullclines, the Figure 4.6. This implies that the equilibrium points in the first and the third columns of the Table 4.3 are centers and the second and the forth columns of the Table 4.3 are saddles.

4.2.2 Periodic Force and Chaos

In this subsection, we will discussed the equation $u_{\theta\theta} + sin(u) = \varepsilon sin(\theta)$ and it can be represented as the system

$$\begin{cases} \dfrac{du}{d\theta} = v, \\ \dfrac{dv}{d\theta} = \varepsilon sin(\theta) - sin(u). \end{cases} \tag{4.17}$$

Notice that (4.17) is a non-autonomous system since the system depends on the parameter θ. We can not analyze the solutions by its vector field as before since the vector field changes with θ. Thus, we need a three dimensional picture with $u-$, $v-$, and $\theta-$ axes to perform the solution curves of this system. But the system (4.17) can be represented as autonomous system by letting $\tau(\theta) = \theta$. We obtain

$$\begin{cases} \dfrac{du}{d\theta} = v, \\ \dfrac{dv}{d\theta} = \varepsilon sin(\tau) - sin(u), \\ \dfrac{d\tau}{d\theta} = 1. \end{cases} \tag{4.18}$$

Thus, we could draw the vector field of the system (4.18) for an ε. Furthermore, if the system given a initial conditions, we can have a solution curve and the solution curve can be projected on the $u\theta-$, $v\theta-$, and $uv-$plane. The Figure 4.8 is the system with $\varepsilon = 0.1$ and the initial conditions given $u(0) = 0.2$, $v(0) = 0$. The Figure 4.9 shows the system with $\varepsilon = 0.1$ and the initial conditions given $u(0) = 3.14$, $v(0) = 0$. The blue, brown and rainbow curves are represented the solution curve projected on the $u\theta-$, $v\theta-$, and $uv-$planes respectively. Compare the Figure 4.8 and the Figure 4.9. Though the two cases have the same perturbed coefficient $\varepsilon = 0.1$, their behaviors are quite different with different initial conditions. The initial condition of the former system is near the point $(0, 0)$, the center of the unperturbed system. The initial condition of the other system is near the point $(\pi, 0)$, the saddle of the unperturbed system. It seems that the initial condition near the different types of equilibrium of the unperturbed system will influence the behavior of the solution. Next, we analyze the two different cases by **return map** [9].

Let us look back to the (4.17). Notice that the external force is $\varepsilon sin(\theta)$, a periodic function. Thus, if any two parameter θ_1, θ_2 with $\theta_1 - \theta_2 = 2n\pi$ where $n \in \mathbb{Z}$, we will get the same system at $\theta = \theta_1$ and $\theta = \theta_2$. If we given an initial condition, $(u(0), v(0))$, there is a solution curve in 3−dimensional space. The solution will intersect the plane $\theta = 2\pi$ at some point $(u_1, v_1) = (u(2\pi), v(2\pi))$. Since the system is the same at $\theta = 0$ and 2π, the solution travels from (u_1, v_1) at plane $\theta = 2\pi$ can be regarded as start from (u_1, v_1) at plane $\theta = 0$. Hence we use a map to translate the point (u_1, v_1) on the plane $\theta = 0$. The map is call the **return map** [9].

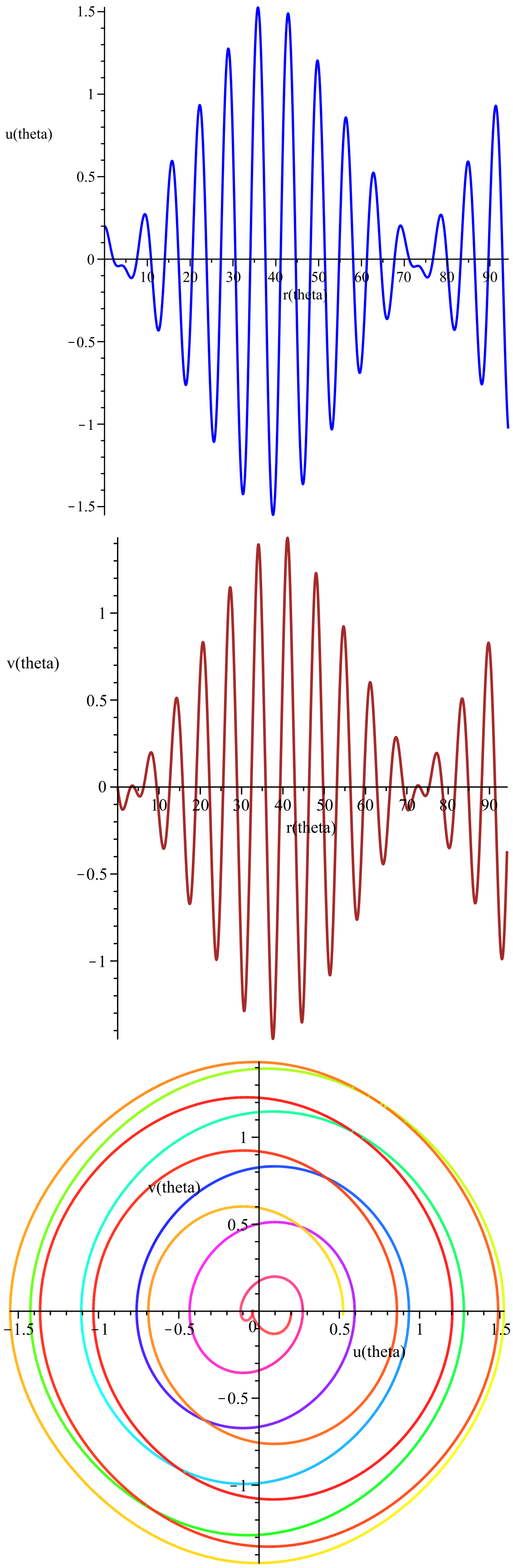

Figure 4.8: $\varepsilon = 0.1$, $u(0) = 0.2$, $v(0) = 0$.

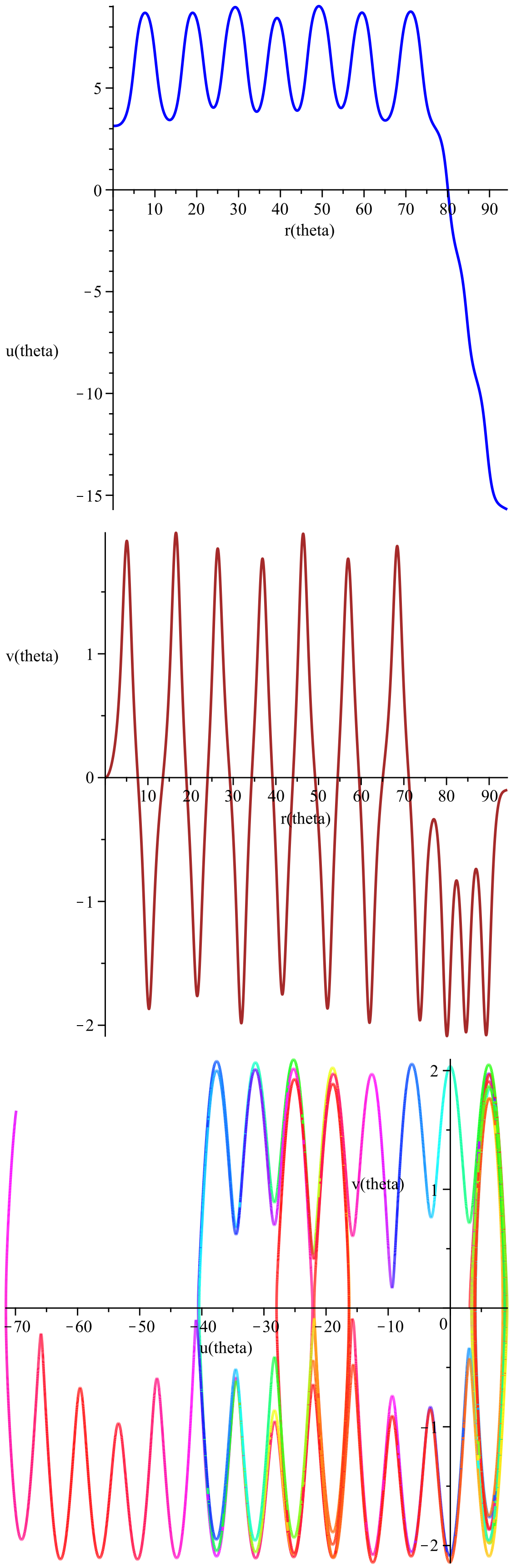

Figure 4.9: $\varepsilon = 0.1$, $u(0) = 3.14$, $v(0) = 0$.

Moreover, the points $(u(2n\pi), v(2n\pi))$ can be translated on the plane $\theta = 0$ for all $n \in \{0\} \cup \mathbb{N}$ if we keep applying the return map. In other words, we choose some particular points on the last graph of the Figure 4.8 (or Figure 4.9) to analyze the behavior of the solution. The Figure 4.10 and the Figure 4.11 are derived by the Figure 4.8 and the Figure 4.9, respectively.

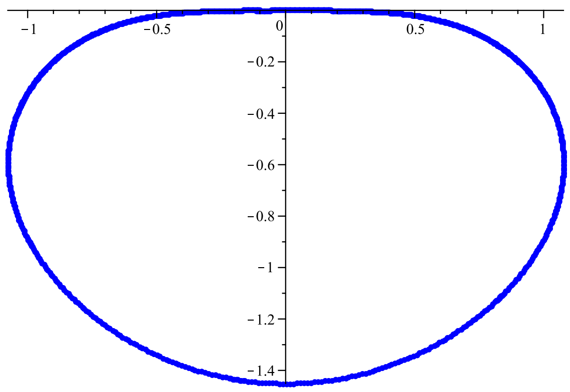

Figure 4.10: Return map with 1000 iterates for $u(0) = 0.2, v(0) = 0$.

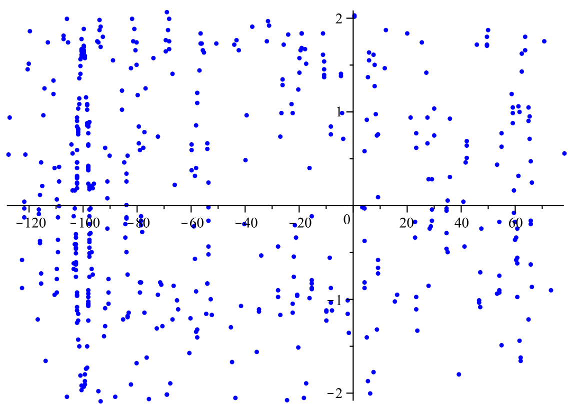

Figure 4.11: Return map with 500 iterates for $u(0) = 3.14, v(0) = 0$.

The Figure 4.10 and the Figure 4.11 tell us that the solution with the initial condition near the center of the unperturbed system has more regular behavior and the behavior of the solution with initial condition near the saddle of the unperturbed system is ***unpredictable***. We can get some clues about the phenomenon from the system (4.17) with $\varepsilon = 0.1$ and the equilibrium points of the unforced system. For the ideal pendulum, the initial condition near the center means that the pendulum given a small displacement and it oscillates forever. Now, we give the pendulum a external force in terms of $\varepsilon sin(\theta)$. The external force will make the pendulum swing higher in some situation and sometimes make it swing less. The result depends on the parameter θ. Since the amplitude of the external force is a small constant ε, the pendulum still oscillate near the center. On the other hand, the initial conditions near the saddle is more complicated since the external force, $\varepsilon sin(\theta)$, plays an important roles. Since when the pendulum swing approach to the saddle, the external force will decide the particle rotate in clockwise or counterclockwise. Thus, the behavior of the solution has closely relation with θ. Hence we can not forecast where the solution will approach in long time. The behavior is called the **chaos** since the solution is incontrollable. Finally, we see some solutions with different initial conditions for the system (4.17) with $\varepsilon = 0.1$ in return map.

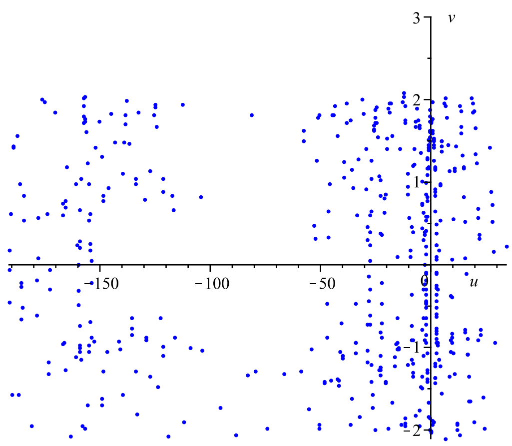

Figure 4.12: Return map with 500 iterates for $u(0) = -3.14, v(0) = 0$.

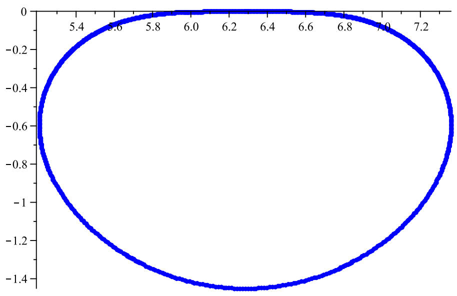

Figure 4.13: Return map with 1000 iterates for $u(0) = 6.28, v(0) = 0$.

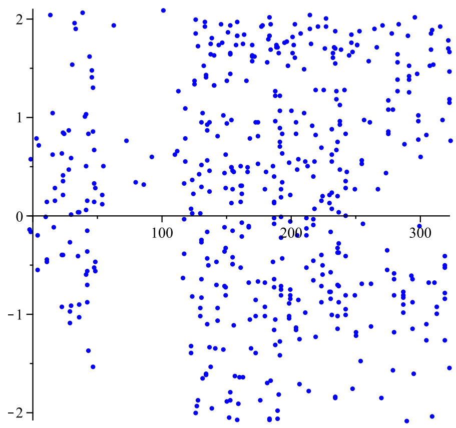

Figure 4.14: Return map with 500 iterates for $u(0) = 9.42, v(0) = 0$.

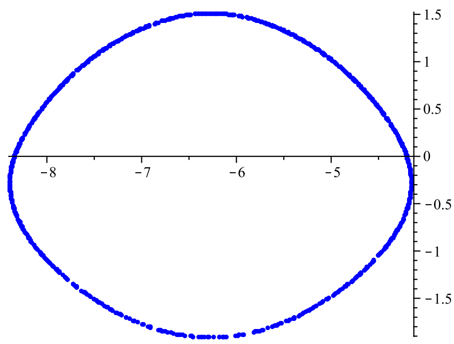

Figure 4.15: Return map with 1000 iterates for $u(0) = -4.21, v(0) = 0$.

Chapter 5

Conclusion

To summarize, at the beginning, we introduced and classified the linear partial differential equations where we then focused on the hyperbolic case. Then we presented certain practical problems whose mathematical models are system of the linear hyperbolic equations and introduced the Sine-Gordon equation roughly. Following, to developed the exact theory of the Sine-Gordon equation, we studied the classical Elliptic functions where one application in solving a nonlinear equation has been presented and gave a mathematical model of the nonlinear vibrating string to practice Jacobian elliptic function. After studying the Jacobian elliptic functions, we applied it to study the systems of the Sine-Gordon equation in detail. In the end, we further studied the perturbations of the Sine-Gordon equation by certain qualitative analysis methods. None of our arguments in this paper are new, yet our effort is. But we will become a more mature researcher for the applied mathematics by engaging a nice and fundamental mathematical model, namely, the Sine-Gordon equation through the hard work in a long period.

Bibliography

[1] David Betounes. *Differential Equations: Thery and Applications with maple*, Springer, 2001.

[2] Darren Redfern and Edgar Chandler. *Maple O.D.E. Lab Book*, Springer, 1996.

[3] E.T. Whittaker and G.N. Watson. *A Course of Modern Analysis*, Cambridge, 1927.

[4] H.F. Weinberger. *A First Course in Partial Differential Equations*, Blaisdell, 1965.

[5] J. Rubinstein and Y. Pinchover. *An Introduction to Partial Differential Equations*, Cambridge, 2005.

[6] Lawrence Perko. *Differential Equations and Dynamical Systems*, Springer, 1991.

[7] Mario G. Salvadori and Ralph J. Schwarz. *Differential Equations in Engineering Problems*, Prentice-Hall, 1954.

[8] Morris W. Hirsch, Stephen Smale, and Robert L. Devaney. *Differential Equations, dynamical Systems and An Introduction to Chaos*, Elsevier, 2004.

[9] Payk Blanchard, Robert L. Devaney, Glen R. Hall, and Jong-Eao Lee. *Differential Equations: A Contemporary Approach*, Thomson, 2007.

[10] Raymond A. Serway and John W. Jewett, Jr. *Physics for Scientists and Engineers with Modern Physics*, Thomson, 2004.