1 Introduction

Cellular Automata formally introduced by John von Neumann in 1951, are mathematical models consisting of a regular lattice of sites which can assume finite number of discrete time steps according to a given local rule. Even in one-dimensional simple rules, they can also exhibit many of the complex dynamical behaviors. In 2002, Stephen Wolfram introduced his work A New Kind of Science, developed a qualitative classification scheme of the $2^{2^3} = 256$ elementary one-dimensional cellular automata rules. Based on Wolfram's work, L. O. Chua provided a rigorous nonlinear dynamical analysis.

As time goes by, many researches about cellular automata have been developed. Shirvani and Rogers [2] show that if f is onto, then it is strongly mixing with respect to uniform Bernoulli measure with only two symbols. Shereshevsky extended this result in [1] that it is also k-mixing for certain kinds of onto function (leftmost and rightmost permutive function), and the number of symbols could be any positive integer.

This paper generalize these results to higher dimension. First introduce notations of polygon and cylinder set. Then extend the definition of left-most and rightmost permutive in one-dimensional to corner-most permutive in higher dimensional. Discussions in this paper separate the domain of f into two cases: rectangle and non-rectangle case. Then provide an algorithm for deciding the conditions of domain. Following a certain orientation while proving non-rectangle case is necessary, or one may not get a general method of proving. The conclusion says that it is k-mixing for any corner-permutive function. This paper also gives an counter-example to explain that it is necessary to permutive at corner.

2 Preliminary

Let $\mathcal{S} = \{0, 1, \cdots, m-1\}, m \in \mathbb{N}$ for some $m > 1$, denote the finite commutative ring of integers modulo m. The space is a d-dimensional space $\mathbf{X} = \mathcal{S}^{\mathbb{Z}^d}$ with its element $x = (x_v)_{v \in \mathbb{Z}^d}$ and $x_v \in \mathcal{S}$.

Each cellular automata map F is based on its local rule f, which is a map from a polygon to a value. First, we introduce how to express a polygon $\mathcal{C}$. If $C = \{c_1, c_2, \cdots, c_k\}$, $c_i = (c_{i1}, c_{i2}, \cdots, c_{id}) \in \mathbb{Z}^d\ \forall i$, are the coordinates of k vertices of a polygon, denote

$$\mathcal{C} = \{(i_1, i_2, \cdots, i_d) \in \mathbb{Z}^d : x \in poly(C)\} = poly(C) \bigcap \mathbb{Z}^d$$

If $f : \mathcal{S}^{\mathcal{C}} \to \mathcal{S}$, then we define the cellular automata map as

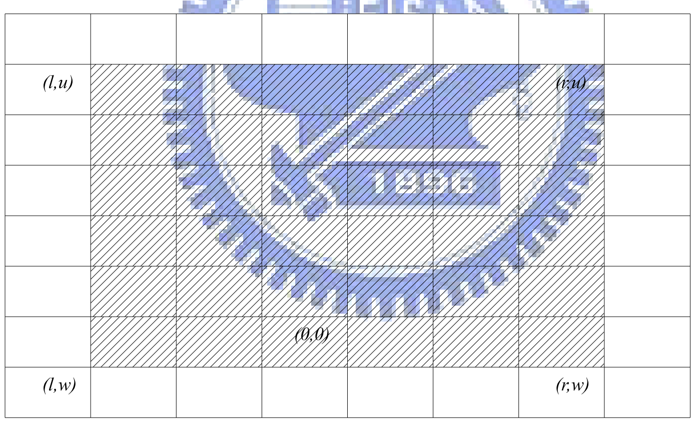
$$(Fx)_v = f(x_{v+c_1}, x_{v+c_2}, \cdots, x_{v+c_k})$$

Here is a two-dimensional example:

Example 2.1. Let $f : \mathcal{S}^{\mathbb{Z}_{(r-l+1)\times(u-w+1)}} \to \mathcal{S}$ be a local rule of F defined on a rectangle with

$$\mathcal{C} = \{(i, j) : l \leq i \leq r, w \leq j \leq u\},$$

and vertices of $\mathcal{C}$ are $C = \{c_1, c_2, c_3, c_4\}$ where $c_1 = (r, u)$, $c_2 = (l, u)$, $c_3 = (l, w)$, $c_4 = (r, w)$



For convenience, define the cellular automata map as

$$(Fx)_{(i,j)} = f \begin{pmatrix} x_{(i+l,j+u)} & \cdots & x_{(i+r,j+u)} \\ \vdots & \ddots & \vdots \\ x_{(i+l,j+w+1)} & \cdots & x_{(i+r,j+w+1)} \\ x_{(i+l,j+w)} & \cdots & x_{(i+r,j+w)} \end{pmatrix}$$

$\square$

Denote d-dimensional cylinder set as

$$\begin{aligned} A &= <(v_1, a_1), (v_2, a_2), \cdots, (v_l, a_l)> \\ &= \{x \in \mathcal{S}^{\mathbb{Z}^d} : x_{v_i} = a_i, \text{ for all } i = 1, 2, \cdots, l\} \end{aligned}$$

where $a_i \in \mathcal{S}$, $v_i = (v_{i1}, v_{i2}, \cdots, v_{id})$. For those d directions, the maximum and minimum coordinate for the cylinder set are defined as: $\forall j = 1, 2, \cdots, d$

$$M_j(A) = max\{v_{ij} : i = 1, 2, \cdots, l\}$$

$$m_j(A) = min\{v_{ij} : i = 1, 2, \cdots, l\}$$

This paper consider uniform Bernoulli measure μ. Here the number of symbol is m, and suppose there are l points in the cylinder set, then

$$\mu(<(v_1, a_1), \cdots, (v_l, a_l)>) = \frac{1}{m^l}$$

Here is a two-dimensional example again:

Example 2.2. Let $\mathcal{S} = \{0, 1\}$.

$$A = <((1,0),0), ((1,1),1), ((1,2),0), ((1,3),1), ((2,1),0), ((2,2),0)>$$

is the cylinder set. The maximum and minimum coordinate for x-direction and y-direction of this cylinder set are

$$M_1(A) = 2, m_1(A) = 1, M_2(A) = 3, m_2(A) = 0$$

and its uniform Bernoulli measure

$$\mu(A) = \frac{1}{2^6}$$

$\square$

Definition 2.3. *The local rule f is called permutive in the variable x_v, $v \in \mathbb{Z}^d$, if f is a permutation at x_v, i.e. fix all element except x_v, there is an one-to-one and onto correspondence between x_v and $f(x_v)$ on their domain $\mathcal{S}$.*

Definition 2.4. *If the local rule for a given cellular automata is defined on a polygon $\mathcal{C}$, then f is said to be corner permutive if f is permutive at either one of its vertices $c_i \in C$.*

Definition 2.5. *Let $(\mathbf{X}, \mathcal{B}, \mu, F)$ be a measure space, and F is a measure preserving transformation(i.e. $\mu(F^{-1}A) = \mu(A)$). Then F is mixing if and only if $\forall A, B \in \mathcal{B}$,*

$$\lim_{n\to\infty} \mu(A \bigcap F^{-n}B) = \mu(A)\mu(B).$$

Definition 2.6. *Let* $(\mathbf{X}, \mathcal{B}, \mu, F)$ *be a measure space, and* F *is a measure preserving transformation. Then* F *is k-mixing if and only if* $\forall A_0, A_1 \cdots A_k \in \mathcal{B}$,

$$\begin{aligned} & \lim_{n_1, n_2, \cdots n_k \to \infty} \mu(A_0 \bigcap F^{-n_1} A_1 \bigcap \cdots \bigcap F^{-(n_1+\ldots+n_k)} A_k) \\ = \quad & \mu(A_0)\mu(A_1)\cdots\mu(A_k) \end{aligned}$$

Proposition 2.7. *Suppose* f *is linear, i.e.* $f(x) = \sum_{i=1}^{k} a_i x_i$, *then* f *is permutive at one variable* x_i *if and only if* $gcd(a_i, m) = 1$

3 Rectangle Rule

In this section, rules from d-dimensional rectangle are considered, whose vertices of domain is $C = \{c_1, c_2, \cdots, c_k\}$, where $c_i = (c_{i1}, c_{i2}, \cdots, c_{id}) \in \mathbb{Z}^d$, for all i.

Define projection map $\pi_j : \mathbb{R}^d \to \mathbb{R}$ as

$$\pi_j(c_i) = c_{ij}$$

where $j = \{1, 2, \cdots, d\}$.

3.1 Two-Dimensional Case

Throughout this section, denote vertices of domain as $c_1 = (r, u)$, $c_2 = (l, u)$, $c_3 = (l, w)$, $c_4 = (r, w)$, where $l \le r$, $w \le u$. It is obvious that $\max_{c \in C}\{\pi_1(c)\} = r$, $\max_{c \in C}\{\pi_2(c)\} = u$.

For convenience, if "vectors" $v_1, v_2, \cdots, v_l$ of two-dimensional cylinder set $A = < (v_1, a_1), \cdots, (v_l, a_l) >$ compose a rectangle, then denote A as

$$\left\langle \begin{array}{ccc} x_{v_{k_1}} & \cdots & x_{v_{k_2}} \\ \vdots & \ddots & \vdots \\ x_{v_{k_3}} & \cdots & x_{v_{k_4}} \end{array} \right\rangle$$

where $x_{v_i} = a_i$ for all $i = 1, 2, \ldots, l$ and $1 \le k_1, k_2, k_3, k_4 \le l$.

Definition 3.1. *The local rule* $f : \mathcal{S}^{\mathbb{Z}_{(r-l+1)\times(u-w+1)}} \to \mathcal{S}$ *for a given cellular automata is said to be:*
- (i) *up-rightmost permutive if* f *is permutive at* $x_{(r,u)}$.
- (ii) *down-rightmost permutive if* f *is permutive at* $x_{(r,w)}$.
- (iii) *up-leftmost permutive if* f *is permutive at* $x_{(l,u)}$.
- (iv) *down-leftmost permutive if* f *is permutive at* $x_{(l,w)}$.

Theorem 3.2. *The local rule* $f : \mathcal{S}^{\mathbb{Z}_{(r-l+1)\times(u-w+1)}} \to \mathcal{S}$ *is from a rectangle to one value. Suppose either of the following condition holds:*
- (i) $r > 0$, $u > 0$ *and* f *is up-rightmost permutive*
- (ii) $r > 0$, $w < 0$ *and* f *is down-rightmost permutive*
- (iii) $l < 0$, $u > 0$ *and* f *is up-leftmost permutive*
- (iv) $l < 0$, $w < 0$ *and* f *is down-leftmost permutive*

then $(\mathbf{X}, \mathcal{B}, \mu, F)$ *is mixing.*

Lemma 3.3. *If the rule $f : \mathcal{S}^{\mathbb{Z}_{(r-l+1)\times(u-w+1)}} \to \mathcal{S}$ is up-rightmost, down-rightmost, up-leftmost or down-leftmost permutive, then so is its k-th iteration $f^k : \mathcal{S}^{\mathbb{Z}_{[k(r-l)+1]\times[k(u-w)+1]}} \to \mathcal{S}$, $\forall k \geq 1$*

Proof. For a down-rightmost permutive rule f. It is obviously true for $k = 1$. Then for $k = 2$,

$$\begin{aligned}(F^2x)_{(i,j)} &= f\begin{pmatrix} y_{(i+l,j+u)} & \cdots & y_{(i+r,j+u)} \\ \vdots & \ddots & \vdots \\ y_{(i+l,j+w)} & \cdots & y_{(i+r,j+w)} \end{pmatrix} \\ &= f^2\begin{pmatrix} x_{(i+2l,j+2u)} & \cdots & x_{(i+2r,j+2u)} \\ \vdots & \ddots & \vdots \\ x_{(i+2l,j+2w)} & \cdots & x_{(i+2r,j+2w)} \end{pmatrix}\end{aligned}$$

where $y_{(s,t)} = f\begin{pmatrix} y_{(s+l,t+u)} & \cdots & y_{(s+r,t+u)} \\ \vdots & \ddots & \vdots \\ y_{(s+l,t+w)} & \cdots & y_{(s+r,t+w)} \end{pmatrix}$.
By definition of down-rightmost permutive, the following three values are permutation: $(Fx)_{(i,j)}$, $x_{(i+r,j+w)}$ and $x_{(i+2r,j+2w)}$. The Lemma is true for $k = 2$. Following the same manner and by induction, we know that if it is true for $k = n$, it must be true for $k = n + 1, \forall n \in \mathbb{N}$. ■

Lemma 3.4. *The k-th iteration F_f^k of the cellular automata map F_f generated by the rule f coincides with the cellular automata map F_{f^k}.*

The proof is omitted.

Proof of Theorem 3.2. This paper only prove part (i) $r > 0$, $u > 0$ and f is up-rightmost permutive, proof of part (ii), (iii) and (iv) are similar.

For any two-dimensional cylinder sets

$$A_0 =< (v_1^0, a_1^0), (v_2^0, a_2^0), \cdots, (v_{l_0}^0, a_{l_0}^0) >$$

$$A_1 =< (v_1^1, a_1^1), (v_2^1, a_2^1), \cdots, (v_{l_1}^1, a_{l_1}^1) >$$

in $\mathbf{X}$. Under the condition $r > 0$, $u > 0$, the order of vectors of A_1 must be specified:

$$\text{for every } v_s, v_t \in \mathbb{Z}^d, s < t \text{ if and only if } \sum_{i=1}^{d} v_{si} < \sum_{i=1}^{d} v_{ti}.$$

Their maximum and minimum coordinate for x-direction and y-direction are $M_1(A_0), m_1(A_0); M_2(A_0), m_2(A_0)$ and $M_1(A_1), m_1(A_1); M_2(A_1), m_2(A_1)$.

Let n_0 be a positive integer greater than both $\frac{M_1(A_0)-m_1(A_1)}{r}$ and $\frac{M_2(A_0)-m_2(A_1)}{u}$. Take arbitrary integer $n > n_0$, easy to see that $m_1(A_1)+nr > M_1(A_0)$ and $m_2(A_1)+nu > M_2(A_0)$.

According to Lemma 3.4, F_f^n is equivalent to F_{f^n}. $F^{-n}A_1$ is the intersection of the following two-dimensional cylinder sets: for all $i = 1, 2, \cdots, l_1$

$$a_i^1 = f^n \begin{pmatrix} x_{(v_{i1}^1+nl, v_{i2}^1+nu)} & \cdots & x_{(v_{i1}^1+nr, v_{i2}^1+nu)} \\ \vdots & \ddots & \vdots \\ x_{(v_{i1}^1+nl, v_{i2}^1+nw)} & \cdots & x_{(v_{i1}^1+nr, v_{i2}^1+nw)} \end{pmatrix}$$

So

$$F^{-n}(<(a_i^1, v_i^1)>) = <f^{-n}(a_i^1)> = \{\left\langle \begin{matrix} x_{(v_{i1}^1+nl, v_{i2}^1+nu)} & \cdots & x_{(v_{i1}^1+nr, v_{i2}^1+nu)} \\ \vdots & \ddots & \vdots \\ x_{(v_{i1}^1+nl, v_{i2}^1+nw)} & \cdots & x_{(v_{i1}^1+nr, v_{i2}^1+nw)} \end{matrix} \right\rangle :$$

$$f^n \begin{pmatrix} x_{(v_{i1}^1+nl, v_{i2}^1+nu)} & \cdots & x_{(v_{i1}^1+nr, v_{i2}^1+nu)} \\ \vdots & \ddots & \vdots \\ x_{(v_{i1}^1+nl, v_{i2}^1+nw)} & \cdots & x_{(v_{i1}^1+nr, v_{i2}^1+nw)} \end{pmatrix} = a_i^1 \text{ for } i = 1, 2, \cdots, l_1\}$$

It is easy to see that

$$F^{-n}A_1 = \bigcap_{i=1}^{l_1} F^{-n}(<(a_i^1, v_i^1)>).$$

Under the condition $r > 0$, $u > 0$, there are total nine cases for the rest two variables:

(i) w and l have the same positive and negative value: w and l both > 0, < 0 or $= 0$;

(ii) $w = 0$ but $l > 0$ or $l < 0$;

(iii) $l = 0$ but $w > 0$ or $w < 0$;

(iv) w and l have opposite sign: $w > 0$, $l < 0$ and $w < 0$, $l > 0$.

This paper only discuss the case $w < 0$, $l < 0$, then the order of index for x direction is

$$i_- = m_1(A_1) + nl < m_1(A_0) \leq M_1(A_0) < m_1(A_1) + nr \leq M_1(A_1) + nr = i_+,$$

and for y direction is

$$j_- = m_2(A_1) + nw < m_2(A_0) \leq M_2(A_0) < m_2(A_1) + nu \leq M_2(A_1) + nu = j_+.$$

Since A_0 is given, $x_{v_i^0} = a_i^0$ for $i = 1, 2, \cdots, l_0$,

$$\mu(A_0) = (\frac{1}{m})^{l_0}.$$

Then consider $F^{-n}(< a_1^1, v_1^1 >)$, all components in

$$B_1 = \left\langle \begin{array}{ccc} x_{(v_{11}^1+nl, v_{12}^1+nu)} & \cdots & x_{(v_{11}^1+nr, v_{12}^1+nu)} \\ \vdots & \ddots & \vdots \\ x_{(v_{11}^1+nl, v_{12}^1+nw)} & \cdots & x_{(v_{11}^1+nr, v_{12}^1+nw)} \end{array} \right\rangle$$

except A_0 and the up-rightmost element $x_{(v_{11}^1+nr, v_{12}^1+nu)}$ can be chosen arbitrary from $\mathcal{S}$. Since a_1^1 is given, f is up-rightmost permutive, and $f^n(B_1) = a_1^1$. By Lemma 3.3, once all components in B_1 except $x_{(v_{11}^1+nr, v_{21}^1+nu)}$ have already chosen, then $x_{(v_{11}^1+nr, v_{21}^1+nu)}$ must fix. Thus

$$\mu(A_0 \bigcap F^{-n}(< (a_1^1, v_1^1) >)) = (\frac{1}{m})^{l_0} \times (\frac{1}{m}).$$

Next consider $F^{-n}(< (a_2^1, v_2^1) >)$, $v_2^1 = (v_{21}^1, v_{22}^1)$, all components in

$$B_2 = \left\langle \begin{array}{ccc} x_{(v_{21}^1+nl, v_{22}^1+nu)} & \cdots & x_{(v_{21}^1+nr, v_{22}^1+nu)} \\ \vdots & \ddots & \vdots \\ x_{(v_{21}^1+nl, v_{22}^1+nw)} & \cdots & x_{(v_{21}^1+nr, v_{22}^1+nw)} \end{array} \right\rangle$$

except those fixed in the previous step and $x_{(v_{21}^1+nr, v_{22}^1+nu)}$ can be chosen arbitrary, and then $x_{(v_{21}^1+nr, v_{22}^1+nu)}$ must fixed. Thus

$$\mu(A_0 \bigcap F^{-n}(< (a_1^1, v_1^1) >) \bigcap F^{-n}(< (a_2^1, v_2^1) >)) = (\frac{1}{m})^{l_0} \times (\frac{1}{m})^2.$$

Continue this manner for all others in the cylinder set A_1,

$$\begin{aligned} \mu(A_0 \bigcap F^{-n} A_1) &= (\tfrac{1}{m})^{l_0} \times (\tfrac{1}{m})^{l_1} \\ &= \mu(A_0)\mu(A_1) \quad \text{for all } n \geq n_0 \end{aligned}$$

Thus $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing. The proof of the other cases can be established analogously. ■

Theorem 3.5. *Under the same condition in Theorem 3.2, we can get a stronger result:* $(\mathbf{X}, \mathcal{B}, \mu, F)$ *is k-mixing for all* $k \geq 1$.

Proof. To prove that F is k-mixing it is sufficient to verify

$$\mu(A_0 \bigcap F^{-n_1} A_1 \bigcap \cdots \bigcap F^{-(n_1+\ldots+n_k)} A_k) = \mu(A_0)\mu(A_1) \cdots \mu(A_k)$$

for any cylinder set $A_0, A_1, \cdots, A_k$. Again we only prove part (i) $r > 0$, $u > 0$ and f is up-rightmost permutive, similar for part (ii), (iii) and (iv).

For any two dimensional cylinder sets

$$A_s =< (v_1^s, a_1^s), (v_2^s, a_2^s), \cdots, (v_{ls}^s, a_{ls}^s) >, s = 0, 1, 2 \cdots, k \text{ in } \mathbf{X}.$$

Let n_0 be a positive number greater than $max\{\frac{M_i(A_{s-1})-m_i(A_s)}{\max_{c\in C}\{\pi_i(c)\}} : 1 \leq i \leq k\}$. Take arbitrary integer $n_1, n_2, \cdots, n_k \geq n_0$, denote

$$N_s = \sum_{j=1}^{s} n_j \text{ for } 1 \leq s \leq k \quad \text{and} \quad N_0 = 0.$$

Put

$$\hat{i} = (i_-, i_+) = (min\{m_1(A_s) + lN_s : 0 \leq s \leq k\}, M_1(A_k) + rN_k)$$

$$\hat{j} = (j_-, j_+) = (min\{m_1(A_s) + wN_s : 0 \leq s \leq k\}, M_2(A_k) + uN_k)$$

Using Lemma 3.3 and 3.4 and the same method as we prove for Theorem 3.2, one checks easily that the set $A_0 \bigcap F^{-n_1}A_1 \bigcap \cdots \bigcap F^{-(n_1+\ldots+n_k)}A_k$ is the intersection of cylinder sets $A(\hat{i},\hat{j}) = \left\langle \begin{matrix} a_{(i_-,j_+)} & \cdots & a_{(i_+,j_+)} \\ \vdots & \ddots & \vdots \\ a_{(i_-,j_-)} & \cdots & a_{(i_+,j_-)} \end{matrix} \right\rangle$ satisfying the following conditions:

$$x_{v_i^0} = x_{(v_{i1}^0, v_{i2}^0)} = a_i^0 \text{ for all } i = 1, 2, ..., l_0$$

$$x_{v_i^s} = F^{-N_s}(a_i^s) = A_{si} \text{ for all } i = 1, 2, ..., l_s \text{ , } s = 1, 2, ..., k$$

Since each N_s are chosen appropriately from the beginning, $x_{(v_{i1}^s+rN_s, v_{i2}^s+uN_s)}$ for $s = 0, 1, ..., k$ do not overlap each other. This leads to the desired equality

$$\begin{aligned} & \mu(A_0 \bigcap F^{-n_1}A_1 \bigcap \cdots \bigcap F^{-(n_1+\ldots+n_k)}A_k) \\ = & \ (\frac{1}{m})^{l_0+l_1+\cdots+l_k} \\ = & \ \mu(A_0)\mu(A_1)\cdots\mu(A_k). \end{aligned}$$

Thus $(\mathbf{X}, \mathcal{B}, \mu, F)$ is k-mixing. The proof is complete. ■

3.2 Multi-Dimensional Case

We can extend the result of 2-dimensional case to multi-dimension:

Theorem 3.6. *Suppose f is permutive at one of the corners $c_n \in C$, with*

$$c_{ni} > 0 \text{ if } \pi_i(c_n) = \max_{c\in C}\{\pi_i(c)\}$$

$$c_{ni} < 0 \text{ if } \pi_i(c_n) = \min_{c\in C}\{\pi_i(c)\}$$

then $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing. Moreover, it is also k-mixing.

Proof of this theorem is similar as 2-dimensional case.

4 Non-Rectangle Rule

This section discuss rules not from a rectangle, but from a polygon. Notations and basic concepts are similar as rectangle case, just a little complicated.

4.1 Two-dimensional Case

Again, we begin the discussion of non-rectangle rule with 2-dimensional case.

Theorem 4.1. *For a two-dimensional local rule f from a polygon $\mathcal{C}$ to one value. The vertices of $\mathcal{C}$ are $C = \{c_1, c_2, \cdots, c_k\}$, $c_i = (c_{i1}, c_{i2}) \in \mathbb{Z}^2$, $\forall i$. Then $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing if f is permutive at $c_n \in C$, and $c_n = (c_{n1}, c_{n2})$ satisfying the following situations:*

$$(I) \begin{cases} \exists j \in \{1,2\} \text{ such that } c_{nj} > c_{ij}, \forall i = \{1,2,\cdots,k\} \backslash \{n\} \Rightarrow c_{nj} > 0 \\ \exists j \in \{1,2\} \text{ such that } c_{nj} < c_{ij}, \forall i = \{1,2,\cdots,k\} \backslash \{n\} \Rightarrow c_{nj} < 0 \end{cases}$$

(II) suppose condition (I) failed, then

(i) $\exists j \in \{1,2\}$ such that $c_{nj} \geq c_{ij}$ (or $c_{nj} \leq c_{ij}$ respectively) $\forall i = \{1,2,\cdots,k\} \backslash \{n\}$, and $c_{nj} = c_{mj}$ for $m \neq n \Rightarrow c_{nj} > 0$ (or $c_{nj} < 0$ respectively)

(ii) Let $\bar{j} = \{1,2\} \backslash \{j\}$, then $c_{n\bar{j}} > c_{m\bar{j}}$ (or $c_{n\bar{j}} < c_{m\bar{j}}$ respectively) $\Rightarrow c_{n\bar{j}} > 0$ (or $c_{n\bar{j}} < 0$ respectively)

Example 4.2. Let $\mathcal{S} = \{0,1,2,3\}$ and the vertices of domain of f is

$$C = \{c_1, c_2, c_3, c_4, c_5\}$$

where $c_1 = (-1,-1), c_2 = (-1,1), c_3 = (0,2), c_4 = (1,1), c_5 = (1,-1)$, then

(1) suppose

$$\begin{aligned} & f(x_{(-1,-1)}, x_{(-1,1)}, x_{(0,2)}, x_{(1,1)}, x_{(1,-1)}) \\ = \;& 2(x_{(-1,-1)} + x_{(-1,1)} + x_{(1,1)} + x_{(1,-1)}) + x_{(0,2)}, \end{aligned}$$

it is obvious that f is permutive at $x_{(0,2)} = x_{c_3}$. It satisfies condition (I) that $c_{32} > c_{i2}$ for $i = \{1,2,4,5\}$. Thus by Theorem 4.1, f permutive at x_{c_3} and $c_{32} = 2 > 0$ implies $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing.

(2) suppose

$$\begin{aligned} & f(x_{(-1,-1)}, x_{(-1,1)}, x_{(0,2)}, x_{(1,1)}, x_{(1,-1)}) \\ = \;& 2(x_{(-1,-1)} + x_{(0,2)} + x_{(1,1)} + x_{(1,-1)}) + x_{(-1,1)}, \end{aligned}$$

it is obvious that f is permutive at $x_{(-1,1)} = x_{c_2}$. It satisfies condition (II) that $c_{21} = c_{11} \leq c_{i1}$ for $i = \{3,4,5\}$ and $c_{22} = 1 > -1 = c_{21}$. Thus by Theorem 4.1, f permutive at x_{c_2} and $c_{21} = -1 < 0$, $c_{22} = 1 > 0$ implies $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing. □

One may be confused that weather conditions in Theorem 4.1 are too easy to be checked. Here follows an example that f is permutive at one vertex but not satisfying condition (I) and (II).

Example 4.3. Let $\mathcal{S} = \{0, 1, 2, 3\}$ and the vertices of domain of f is

$$C = \{c_1, c_2, c_3, c_4, c_5\}$$

where $c_1 = (1, -1), c_2 = (1, 1), c_3 = (2, 2), c_4 = (3, 1), c_5 = (3, -1)$, then suppose

$$\begin{aligned} & f(x_{(1,-1)}, x_{(1,1)}, x_{(2,2)}, x_{(3,1)}, x_{(3,-1)}) \\ = \ & 2(x_{(1,1)} + x_{(2,2)} + x_{(3,1)} + x_{(3,-1)}) + x_{(1,-1)} \end{aligned}$$

it is obvious that f is permutive at $x_{(1,-1)} = x_{c_1}$. But x_{c_1} does not satisfying condition (I)(II) that $c_{11} > c_{i1}$ for $i = \{2, 3, 4, 5\}$, c_{11} must be small than zero. Which does not satisfy conditions in Theorem 4.1. □

Proof of Theorem 4.1. For different c_n, there are different process of proving.
(1) To prove (II), notice that c_n satisfies

$$\begin{cases} c_{n1} \geq c_{i1}, \forall i \in \{1, 2, ..., k\} \backslash \{n\} \text{ but } c_{n2} \text{ is not the extrema} \\ c_{n1} \leq c_{i1}, \forall i \in \{1, 2, ..., k\} \backslash \{n\} \text{ but } c_{n2} \text{ is not the extrema} \\ c_{n2} \geq c_{i2}, \forall i \in \{1, 2, ..., k\} \backslash \{n\} \text{ but } c_{n1} \text{ is not the extrema} \\ c_{n2} \leq c_{i1}, \forall i \in \{1, 2, ..., k\} \backslash \{n\} \text{ but } c_{n1} \text{ is not the extrema} \end{cases}$$

Only prove the case $c_{n1} \geq c_{i1}, \forall i$, but c_{n2} is not the extreme value for c_{i2} . For any two-dimensional cylinder sets

$$A_0 = < (v_1^0, a_1^0), (v_2^0, a_2^0), \cdots, (v_{l_0}^0, a_{l_0}^0) >$$

$$A_1 = < (v_1^1, a_1^1), (v_2^1, a_2^1), \cdots, (v_{l_1}^1, a_{l_1}^1) >$$

in $\mathbf{X}$, using the same method as proof of Theorem 3.2 to choose n properly before computing $\mu(A_0 \bigcap F^{-n} A_1)$. But there is a specific order to compute $\mu(A_0 \bigcap F^{-n} A_1)$: $\forall s, t \in \{1, 2, \cdots, l_1\}$, $F^{-n}(< a_s^1, v_s^1 >)$ must be considered before $F^{-n}(< a_t^1, v_t^1 >)$ if:

- c_n satisfies (I) $c_{n1} > c_{i1}, \forall i$, and $v_{s1}^1 < v_{t1}^1$
- c_n satisfies $(II)(i)$ $c_{n1} = c_{m1} \geq c_{i1} \forall i$, and $\begin{cases} \text{either "}v_{s1}^1 < v_{t1}^1\text{"} \\ \text{or "}v_{s1}^1 = v_{t1}^1 \text{ and } v_{s2}^1 < v_{t2}^1\text{"} \end{cases}$ $(**)$

(2) (I) could be divided into two different kinds of situations: one is the same as what we have proved in (1), and the other is the following:
c_n satisfies

$$\begin{cases} c_{n1} \geq c_{i1} \text{ and } c_{n2} \geq c_{i2} \\ c_{n1} \geq c_{i1} \text{ and } c_{n2} \leq c_{i2} \\ c_{n1} \leq c_{i1} \text{ and } c_{n2} \geq c_{i2} \\ c_{n1} \leq c_{i1} \text{ and } c_{n2} \leq c_{i2} \end{cases}$$

for all $i \in \{1, 2, \cdots, k\} \backslash \{n\}$. That is, c_n is the extreme value both in x-direction and y-direction. It's easy to verify that the proof is just the same as Theorem 3.2. In one word, if the domain of f have been mended to a rectangle by putting their coefficients all zeros, then c_n is a corner of this new-born rectangle and satisfying conditions in Theorem 3.2. This leads to the result $\mu(A_0 \bigcap F^{-n} A_1) = \mu(A_0)\mu(A_1)$. Proof for other cases can be established analogously. ■

Remark 4.4. In the proof of Theorem 4.1(1), the order of computing $\mu(A_0 \bigcap F^{-n} A_1)$ must conform to $(**)$. Suppose we consider $F^{-n}(< a_t^1, v_t^1 >)$ first, then the permutive item for $F^{-n}(< a_s^1, v_s^1 >)$ may be a fixed number such that $F^{-n}(< a_s^1, v_s^1 >) = \emptyset$. Thus it may be very complicated or impossible to have the result $\mu(A_0 \bigcap F^{-n} A_1) = \mu(A_0)\mu(A_1)$.

Here is a example to explain Remark 4.4.

Example 4.5. Let $\mathcal{S} = \{0, 1, 2, 3\}$, vertices of domain of f is

$$C = \{c_1, c_2, c_3, c_4, c_5\}$$

where $c_1 = (-1, -1), c_2 = (-1, 1), c_3 = (0, 1), c_4 = (1, 0), c_5 = (1, -1)$,and the domain of f is

$$\mathcal{C} = \{(-1, -1), (-1, 1), (0, 1), (1, 0), (1, -1), (-1, 0), (-1, 0), (0, 0)\}.$$

The local rule is

$$\begin{aligned} & f(x_{(-1,-1)}, x_{(-1,1)}, x_{(0,1)}, x_{(1,0)}, x_{(1,-1)}, x_{(-1,0)}, x_{(-1,0)}, x_{(0,0)}) \\ = \ & 2(x_{(-1,-1)} + x_{(-1,1)} + x_{(0,1)} + x_{(1,-1)} + x_{(-1,0)} + x_{(-1,0)} + x_{(0,0)}) + x_{(1,0)} \end{aligned}$$

f is permutive at $c_4 = (1, 0)$ obviously.

Pick two cylinder sets $A_0 = \{< (0, 2), 0 >\}$ and $A_1 = \{< (1, 3), 1 >, < (2, 3), 2 >, < (1, 4), 3 >\}$ in $\mathbf{X}$. Thus $M_1(A_0) = 0$, $M_2(A_0) = 2$, $m_1(A_1) = 1$ and $m_2(A_1) = 3$.

Let $n_0 > \frac{M_1(A_0) - m_1(A_1)}{\max_{c \in C}\{\pi_1(c)\}} = \frac{0-1}{1} = -1$ and $n_0 > \frac{M_2(A_0) - m_2(A_1)}{\max_{c \in C}\{\pi_2(c)\}} = \frac{2-3}{1} = 1$. Take $n = 2 > n_0$. In this case $c_{41} = c_{51} > c_{i1}$ for $i = \{1, 2, 3\}$ and $v_{31}^1 < v_{21}^1$. According to the proof of Theorem 4.1, $F^{-2}(< (1, 4), 3 >)$ must be considered before $F^{-2}(< (2, 3), 2 >)$.

Suppose considering $F^{-2}(< (2, 3), 2 >)$ before $F^{-2}(< (1, 4), 3 >)$ instead. First consider $\mu(A_0)$, fix the positions $x_{(0,2)} = 0$, and we denote $\mathcal{C}_1 = poly(C_1) \bigcap \mathbb{Z}^d$, where $C_1 = \{(1 + 2c_{i1}, 3 + 2c_{i2}) : i = 1, 2, 3, 4, 5\}$. For $\mu(A_0 \bigcap F^{-2}(< (1, 3), 1 >))$, fix each $x_{\mathcal{C}_1} = 0$ except $x_{(3,3)} = 1$. For $\mu(A_0 \bigcap F^{-2}(< (1, 3), 1 >) \bigcap F^{-2}(< (2, 3), 2 >))$, except what have been chosen, also choose $x_{(4,1)} = x_{(2,5)} = 0$, $x_{(4,2)} = x_{(4,3)} = x_{(3,4)} = 2$. Then consider $\mu(A_0 \bigcap F^{-2}(< (1, 3), 1 >) \bigcap F^{-2}(< (2, 3), 2 >) \bigcap F^{-2}(< (1, 4), 3 >))$, if components in the above discussion are fixed, $F^{-2}(< (1, 4), 3 >) = \emptyset$. Which is a contradiction. □

4.2 Multi-Dimensional Case

Theorem 4.1 shows the conditions for mixing in two-dimensional. For d-dimensional case, use the following algorithm to check conditions for mixing.

Algorithm.

$$\text{(I)} \begin{cases} \exists j \in \{1,2,\cdots,d\} \text{ such that } c_{nj} > c_{ij}, \forall i = \{1,2,\cdots,k\}\backslash\{n\} \Rightarrow c_{nj} > 0 \\ \exists j \in \{1,2,\cdots,d\} \text{ such that } c_{nj} < c_{ij}, \forall i = \{1,2,\cdots,k\}\backslash\{n\} \Rightarrow c_{nj} < 0 \end{cases}$$

(II) suppose (I) failed, then
$\exists j \in \{1,2,\cdots,d\}$ such that $c_{nj} \geq c_{ij}$ (or $c_{nj} \leq c_{ij}$ respectively) $\forall i \in \{1,2,\cdots,k\}\backslash\{n\}$ and $c_{nj} = c_{i_1 j}$ for some $i_1 \in I_1 \subseteq \{1,2,\cdots,d\}\backslash\{n\} \Rightarrow c_{nj} > 0$ (or $c_{nj} < 0$ respectively).
Let $\sharp(I_1) = k_1$, denote $C^1 = \{c_1^1, c_2^1, \cdots, c_{k_1}^1\}$ as the collection of those c_{i_1} satisfying $c_{i_1 j} = c_{nj}$, and $c_i^1 = (c_{i1}^1, c_{i2}^1, \cdots, c_{i(j-1)}^1, c_{i(j+1)}^1, \cdots, c_{ik_1}^1) \in \mathbb{Z}^{d-1}$.

And runs the algorithm (I),(II) again.

Theorem 4.6. *Let the domain of f be a polygon $\mathcal{C}$ and f is permutive at one of its vertices $c_n \in C$. Suppose $c_n = (c_{n1}, c_{n2}, \cdots, c_{nd})$ satisfying the above algorithm, then $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing.*

Example 4.7. Let vertices of f is

$$C = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}\}$$

where $c_1 = (0,2,-1), c_2 = (-1,0,-1), c_3 = (0,-2,-1), c_4 = (2,-2,-1)$,
$c_5 = (3,0,-1), c_6 = (2,2,-1), c_7 = (0,2,1), c_8 = (-1,0,1), c_9 = (0,-2,1)$,
$c_{10} = (2,-2,1), c_{11} = (3,0,1), c_{12} = (2,2,1)$.

Suppose f is permutive at $x_{(2,2,-1)} = x_{c_6}$. First check that (I) failed, and satisfies (II)(ii) that $\exists 3 \in \{1,2,3\}$ such that $c_{63} \leq c_{i3}$ $\forall i$, and $c_{63} = c_{i_1 3}$ for $i_1 = I_1 =\in \{1,2,3,4,5\}$. This is the first condition $c_{63} = -1 < 0$. Thus $\sharp(I_1) = 5$. Denote

$$C^1 = \{c_1^1, c_2^1, c_3^1, c_4^1, c_5^1\}$$

where $c_1^1 = (0,2), c_2^1 = (-1,0), c_3^1 = (0,-2), c_4^1 = (2,-2), c_5^1 = (3,0)$. Then check the algorithm again.

(I) still failed and satisfies (II)(i) that $\exists 2 \in \{1,2\}$ such that $c_{62} \geq c_{i_1 2}^1$ $\forall i_1$, and $c_{62} = c_{i_2 2}$ for $i_2 = 1$. This is the second condition $c_{62} = 2 > 0$. Finally, check that $c_{61} > c_{11}$, which get the third condition $c_{61} = 2 > 0$. By Theorem 4.6, if f is permutive at x_{c_6}, $(\mathbf{X}, \mathcal{B}, \mu, F)$ is mixing. $\square$

5 Discussion

Previous sections have already shown the sufficient condition for mixing (even k-mixing for all $k \geq 1$). It is nature to consider weather it is also the necessary condition. The following are two examples for one dimensional and two dimensional cases explaining that if it is not permutive at corners, property of mixing failed.

Example 5.1. For one dimensional case, let $\mathcal{S} = \{0, 1, 2, 3\}$, $l = 0$, $r = 2$, and the local rule is

$$f(x_0x_1x_2) = 2x_0 + x_1 + [\frac{2x_2}{3}] + [\frac{x_2}{3}] - x_2 \tag{1}$$

This leads to $f^{-1}(01) = f^{-1}(13) = f^{-1}(21) = \emptyset$ $\square$

Example 5.2. For two dimensional case,

$$\begin{aligned} (Fx)_{00} &= f\begin{pmatrix} x_{02} & x_{12} & x_{22} \\ x_{01} & x_{11} & x_{21} \\ x_{00} & x_{10} & x_{20} \end{pmatrix} \\ &= 2x_{02} + x_{12} + [\tfrac{2x_{22}}{3}] + [\tfrac{x_{22}}{3}] - x_{22} + 2x_{00} + 2x_{20} \end{aligned} \tag{2}$$

This example satisfying $r = 2 > 0$, $u = 2 > 0$, but f is not premutive at up-rightmost component, but permutive at the component on first row, mid column. Under this condition, $f^{-1}(01) = \emptyset$, which contradict the property of mixing. $\square$

References

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