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碩 士 論 文

有限體下正規Laurent級數上的Duffin-Schaeffer猜想

The Duffin-Schaeffer Conjecture for Formal Laurent Series over A Finite Base Field

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前言

我們將在本論文中探討有限體下正規Laurent級數上的Diophantine逼近。更進一步地，在此深入研究Duffin-Schaeffer猜想，並證明一些特殊情況下的結果。本文的主要架構如下：在第一章，我們將介紹Diophantine逼近及賦距數論，並藉由實數中一些已知的結果，作爲我們研究動機。

在第二章，我們將介紹在正規Laurent級數體上的Diophantine逼近。第一節羅列了一些定義及我們即將使用的符號。在這一節，我們發展一些重要的基本性質，例如：廣義Borel-Cantelli引理及數論中類似的結果等。第二節是近來有關這個主題研究結果的整理。在這節裡，我們探討各種情況下的Diophantine逼近並給出相對應的結果。除此之外，在第二節也明確指出Duffin-Schaeffer猜想的目標。第三節，則是我們對這個猜想所貢獻的結果。

在三章，我們給出第二章第三節所談主要結果的詳細證明。簡短地說，我們先估計任兩事件交集發生的機率大小，接著應用廣義的Borel-Cantelli引理及零一律。主要證明的架構，模仿實數情況中Vaaler的證法。

在第四章，我們改變探討猜想的觀點，與實數情況中Harman的部分結果相似。

在第五章，論文的尾聲，我們將針對整體工作做一個總結。

Preface

This thesis is concerned with metric Diophantine approximation for formal Laurent series over a finite base field. More precisely, we will discuss an analogue of the famous Duffin-Schaeffer conjecture for formal Laurent series and prove it in some special cases.

An outline of the thesis is as follows. In Chapter 1, we will briefly introduce Diophantine approximation and metric Diophantine approximation over the real number field and state some results which are important for our work. In Chapter 2, we will give an introduction into the theory of Diophantine approximation for formal Laurent series over a finite base field. More precisely, Section 2.1 will collect the definitions, notations and results we are going to use throughout this work. Then, in Section 2.2 we will give a survey on recent research activities in Diophantine approximation for formal Laurent series. Apart from such results, this section will also be used to state the Duffin-Schaeffer conjecture in our context and explain the goal of this thesis in more details. Finally, Section 2.3 will contain our findings concerning this conjecture.

In Chapter 3, we will give details of the proof of our main result. Roughly speaking, we will follow the classical path which is concerned with estimating the measure of the intersection of two events, and applying the generalized Borel-Cantelli lemma.

In Chapter 4, we will consider the conjecture from a different angle and prove some analogous results of Harman.

Finally, we will end the thesis with a conclusion in Chapter 5.

致　謝　辭

時光飛逝，從彰師滿懷夢想地來到交大，轉眼間已是驪歌再度響起的時候。回顧而過的兩年，有歡笑、有淚水。期間受到很多老師的提攜、學長姐的照顧、同學們的支持與鼓勵以及大學部可愛的學生們。學到很多，也改變很多；知道自己不足的部分，處事不夠圓融有待改進的地方。改變的、不足的，我會好好檢討，使自己能成長。感謝我的指導老師—符麥克博士，使論文得以出產。感謝兩位口試委員，針對論文給出寶貴的建議及通過碩士資格的審核。

何其幸運，有你們來豐富我的生活，陪我成長！容我引用 陳之藩 謝天一文中所提「因爲需要感謝的人太多了，就感謝天罷。」

夏　2009

奕伸

Contents

Chapter 1

Introduction

We know that the set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. That is for each $a \in \mathbb{R}$ and for each $n \in \mathbb{N}$, there exists rational number r_n such that $|a - r_n| < \frac{1}{n}$. An important task both in theory and praxis is to approximate real numbers by rational numbers with good accuracy, where the accuracy is measured in terms of the size of the denominator. The area which is concerned with such investigations is called *Diophantine approximation*. The following is one typical result.

Theorem 1.1 (Dirichlet)**.** *Let α be an irrational number. Then*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}, \ m, n \in \mathbb{Z}, \ n \neq 0, \ (m, n) = 1$$

has infinitely many solutions $\frac{m}{n}$.

A subarea called *metric Diophantine approximation* asks for properties which hold for almost all real numbers (in the sense of Lebesgue measure). Again, we give a typical result.

Theorem 1.2 (Khintchine)**.** *Let $\psi(x)$ be a positive continuous function and suppose that $x\psi(x)$ is non-increasing. Then*

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, \ m, n \in \mathbb{Z}, \ n \neq 0, \ (m, n) = 1 \tag{1.1}$$

has infinitely many solutions $\frac{m}{n}$ for almost all $\alpha \in \mathbb{R}$ if and only if $\sum_{n=1}^{\infty} \psi(n) = \infty$.

The inequality (1.1) is called *Diophantine inequality*. Finding which conditions make this inequality have infinitely many solutions for almost all numbers is the main goal of metric Diophantine approximation. By using the Borel-Cantelli lemma, one can easily obtain:

Theorem 1.3. *Let $\psi(n)$ be a non-negative-valued function such that*

$$\sum_{n=1}^{\infty} \psi(n) \frac{\varphi(n)}{n} < \infty.$$

Then

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, \ m, n \in \mathbb{Z}, \ n \neq 0, \ (m, n) = 1$$

has only finitely many solutions $\frac{m}{n}$ *for almost all* $\alpha \in \mathbb{R}$.

The most famous unsolved conjecture in metric Diophantine approximation is:

Conjecture (Duffin-Schaeffer Conjecture (1941))**.** *Let $\psi(n)$ be a non-negative-valued function such that*

$$\sum_{n=1}^{\infty} \psi(n) \frac{\varphi(n)}{n} = \infty. \tag{1.2}$$

Then

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, \ m, n \in \mathbb{Z}, \ n \neq 0, \ (m, n) = 1$$

has infinitely many solutions $\frac{m}{n}$ *for almost all* $\alpha \in \mathbb{R}$.

Vaaler made an important contribution to this conjecture.

Theorem 1.4 (Vaaler)**.** *The Duffin-Schaeffer conjecture is true when $\psi(n) = \mathcal{O}\left(\frac{1}{n}\right)$ and $\psi(n)$ satisfies* (1.2).

In this thesis, we will study this kind of problems for the formal Laurent series field and give some analogous results.

Chapter 2

Metric Diophantine Approximation for Formal Laurent Series

In this chapter, we will give the precise definitions of the notations we are going to use, and some historical discussion and recent results about this research.

2.1 Fundamental Properties

Let $q = p^n$ where p is a prime number and $n \in \mathbb{N}$. We use the standard notation $\mathbb{F}_q$ from algebra to denote the (unique) finite field with q elements. Moreover, we denote by $\mathbb{F}_q[X]$ the set of polynomials with coefficients in $\mathbb{F}_q$, and by $\mathbb{F}_q(X)$ the quotient field of $\mathbb{F}_q[X]$. Finally, we denote by $\mathbb{F}_q((X^{-1}))$ the set of formal Laurent power series, that is

$$\mathbb{F}_q((X^{-1})) = \left\{ f = a_l X^l + a_{l-1} X^{l-1} + \cdots : l \in \mathbb{Z},\ \text{each } a_i \in \mathbb{F}_q,\ a_l \neq 0 \right\} \cup \{0\}$$

Next, we equip $\mathbb{F}_q((X^{-1}))$ with an addition and multiplication, where both operations are defined as for polynomials. With these rules, we have the following property:

Proposition 2.1. *$(\mathbb{F}_q((X^{-1})), +, \cdot)$ is a field.*

Proof. First, it is easy to see that $\left(\mathbb{F}_q\left((X^{-1})\right), +, \cdot\right)$ is a commutative ring with identity element. Hence, we only need to check that every $f \in \mathbb{F}_q((X^{-1}))$ of the form

$$f = \sum_{i=-l}^{\infty} a_{-i} X^{-i}, \ a_l \neq 0$$

has a multiplicative inverse.

Therefore, we claim that

$$f^{-1} := \sum_{i=l}^{\infty} b_{-i} X^{-i} \ \in \mathbb{F}_q((X^{-1})),$$

where b_i can be determined recursively as follows:

$$\begin{cases} b_{-l} = a_l^{-1} \\ b_{-l-k} = -\sum_{j=1}^{k} a_l^{-1} \cdot (a_{l-j} b_{-l-k+j}), \ k \geq 1. \end{cases}$$

The claim is easily checked. This concludes the proof. □

For $f = a_l X^l + a_{l-1} X^{l-1} + \cdots \in \mathbb{F}_q((X^{-1})), \ a_l \neq 0$, consider the field $\mathbb{F}_q((X^{-1}))$ with the following

$$\nu(f) = \deg f = l$$

and $\deg 0 = -\infty$ as usual.

Define $|f| = q^{\nu(f)}$. We have the following properties:

Proposition 2.2. *For $f, g \in \mathbb{F}_q((X^{-1}))$, $|\cdot|$ satisfies the following*

(1) $|f| = 0 \Leftrightarrow f = 0$

(2) $|fg| = |f||g|$

(3) $|f - g| \leq \max\{|f|, |g|\}$ (*ultra-metric property*).

That is, $|\cdot|$ is a valuation on $\mathbb{F}_q((X^{-1}))$.

Proof.

(1) $|f| = 0 \Leftrightarrow \nu(f) = -\infty \Leftrightarrow f = 0$.

(2) $|fg| = q^{\nu(fg)} = q^{\deg f + \deg g} = q^{\deg f} \cdot q^{\deg g} = q^{\nu(f)} \cdot q^{\nu(g)} = |f| \cdot |g|$.

(3) $|f - g| = q^{\nu(f-g)} = q^{\deg(f-g)} \leq q^{\max\{\deg f, \deg g\}} = \max\left\{q^{\nu(f)}, q^{\nu(g)}\right\} = \max\{|f|, |g|\}$.

This concludes the proof. □

Remark 2.1. *As it is well-known from the theory of evaluated fields, the function* $d : \mathbb{F}_q((X^{-1})) \times \mathbb{F}_q((X^{-1})) \to \mathbb{R}$ *defined by* $d(f, g) = |f - g|$ *is a metric on* $\mathbb{F}_q((X^{-1}))$*.*

The following subset of $\mathbb{F}_q((X^{-1}))$ can be viewed as the analogue of the interval $[0, 1)$ in the field of formal Laurent series

$$\mathbb{L} = \left\{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : \ a_i \in \mathbb{F}_q \text{ for } i \leq -1\right\}.$$

By restriction of the valuation of $\mathbb{F}_q((X^{-1}))$ on $\mathbb{L}$ one gets a compact Abelian topological group.

Proposition 2.3. $\mathbb{L}$ *is a compact Abelian topological group with the metric* $d(f, g) = |f - g|$*.*

Proof. We need to prove the following two things:

Abelian topological group:

First, it can be easily checked that $\mathbb{L}$ is an Abelian subgroup of $\mathbb{F}_q((X^{-1}))$. Moreover, if $f_n, \ g_n, \ f, \ g$ are in $\mathbb{L}$ for each n and $f_n \to f, \ g_n \to g$, then by the properties of the product topology, we have:

$$|f_n+g_n-(f+g)| \leq \max\{|f_n - f|, |g_n - g|\} = \max\{d(f_n, f), d(g_n, g)\} = d\left((f_n, g_n), (f, \ g)\right).$$

That is, $(f, g) \mapsto f + g$ is continuous.

On the other hand, since $f_n \to f$,

$$|-f_n - (-f)| = |f_n - f| \to 0.$$

Hence, $f \mapsto -f$ is continuous as well. Overall, we have proved that $\mathbb{L}$ is a topological group.

Compact group with the metric d:

Since a metric space is compact if and only if it is sequentially compact, we only have to show the latter. Therefore, let f_n be a sequence in $\mathbb{L}$. We have to show that there exists a convergent subsequence. First, denote by A_1 an infinity subset of $\{f_n\}$ with the same first digit (the coefficient of X^{-1}). This is possible due to the finiteness of the base field. Now, assume that $A_1, A_2, \ldots, A_i$ are already defined such that the first i digits of all elements in A_i are the same and that every A_i contains infinitely many elements. Define A_{i+1} as the infinity set of elements from A_i with the same first $i+1$ digits (again this is possible due to the finiteness of the base field). This recursively defines a sequence of non-empty sets $A_1, A_2, A_3, \ldots$. Now, pick one element from every set (which is possible by the axiom of choice). This obviously gives a sequence that converges. □

Remark 2.2. *The part in proposition* 2.3 *that* $\mathbb{L}$ *is compact with* d *can also be proved as follows:*

Proof. Consider the finite base field equipped with the discrete topology. Form the infinity product space. Since the product topology on $\mathbb{F}_q^{\mathbb{N}}$ has as basis all sets of the form $\prod U_\alpha$, where U_α is a basis of $\mathbb{F}_q$ for finitely many α (which equals singleton due to the discrete topology) and U_α equals $\mathbb{F}_q$ for infinitely many values of α, it is easy to see that the resulting topology is the same as the topology on $\mathbb{L}$ induced by our valuation. Next, note that the finite base field with the discrete topology is trivially compact. Hence, the infinity product space is compact as well by Tychonoff's theorem. □

Remark 2.3. *Compact topological groups are important because they have a unique, normalized, translation-invariant measure called the Haar measure.*

We denote by m the normalized Haar measure on $\mathbb{L}$ and for $g \in \mathbb{L},\ r \geq 1$ define

$$B(g, q^{-r}) = \left\{ f \in \mathbb{L} : \ |g - f| < q^{-r} \right\}.$$

Then, we have the following important property:

Proposition 2.4. *For any $b_1, b_2, \ldots, b_r \in \mathbb{F}_q,\ g \in \mathbb{L}$ and $r \geq 1$,*

$$m\left(\left\{ f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_{-1} = b_1, a_{-2} = b_2, \cdots, a_{-r} = b_r \right\}\right) = \frac{1}{q^r}$$

and

$$m\left(B\left(g, q^{-r}\right)\right) = q^{-r}.$$

Proof. Assume

$$g = b_1X^{-1} + b_2X^{-2} + \cdots + b_rX^{-r}.$$

By the translation-invariant property of m,

$$m\left(B\left(g, q^{-r}\right)\right) = m\left(B\left(h, q^{-r}\right)\right), \text{ for any } h \in \mathbb{L}.$$

Clearly, for $f \in \mathbb{L}$

$$f \in B\left(g, q^{-r}\right) \Leftrightarrow a_{-j} = b_j, \text{ for } j = 1, 2, \ldots, r.$$

Thus,

$$\left\{ f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_{-1} = b_1, a_{-2} = b_2, \cdots, a_{-r} = b_r \right\} = B\left(g, q^{-r}\right)$$

which inherits the translation-invariant property. Consequently,

$$m\left(\left\{ f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_{-1} = b_1, a_{-2} = b_2, \cdots, a_{-r} = b_r \right\}\right) = C(r) \text{ a function of } r.$$

As a result,

$$
\begin{aligned}
1 &= m(\mathbb{L}) \\
&= m\left(\bigcup_{b_1,b_2,\ldots,b_r\in\mathbb{F}_q} \{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_{-1} = b_1, a_{-2} = b_2, \cdots, a_{-r} = b_r\}\right) \\
&= \sum_{b_1,b_2,\ldots,b_r\in\mathbb{F}_q} m\left(\{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_{-1} = b_1, a_{-2} = b_2, \cdots, a_{-r} = b_r\}\right) \\
&= q^r \cdot C(r).
\end{aligned}
$$

Note that the disjoint property was used to obtain the third equality. Hence, for any $b_1, b_2, \ldots, b_r \in \mathbb{F}_q$ and $g \in \mathbb{L}$, $r \geq 1$

$$
\begin{aligned}
\frac{1}{q^r} &= m\left(\{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_{-1} = b_1, a_{-2} = b_2, \cdots, a_{-r} = b_r\}\right) \\
&= m\left(B\left(g, q^{-r}\right)\right)
\end{aligned}
$$

which completes the proof. □

Remark 2.4. *Let ψ be a $\{q^{-n} : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$-valued function defined on the set of normed (monic) polynomials in $\mathbb{F}_q[X]$ of the form*

$$X^l + a_{l-1}X^{l-1} + \cdots + a_1X + a_0,\ \ a_i \in \mathbb{F}_q,\ \ i = 0, 1, \ldots, l-1.$$

With this setting, for any $f \in \mathbb{L}$ and $Q \in \mathbb{F}_q[X]$ with Q normed, we have

$$m\left(B\left(f, \frac{\psi(Q)}{|Q|}\right)\right) = \frac{\psi(Q)}{|Q|}.$$

So far, we have established some properties of $\mathbb{F}_q((X^{-1}))$. Next, we also need some analogue definitions and results of number theory, where now $\mathbb{F}_q[X]$, $\mathbb{F}_q(X)$, $\mathbb{F}_q((X^{-1}))$ play the roles of integers, rational numbers and real numbers, respectively.

As usual in algebra, we use capital letters to denote polynomials of $\mathbb{F}_q[X]$ and "I" for irreducible normed polynomials.

Definition 2.1 (Euler's phi function)**.** *The function* $\varphi(Q) : \mathbb{F}_q[X] \setminus \{0\} \to \mathbb{N} \cup \{0\}$ *of* Q *is defined to be the number of polynomials in* $\mathbb{F}_q[X]$ *with degree less than* Q *that are coprime to* Q*.*

Remark 2.5. *We use* $(P, Q) = 1$ *to denote that* P, Q *are coprime, whereas* $\langle \cdot, \cdot \rangle$ *will be reserved for pairs.*

Definition 2.2 (Möbius function)**.** $\mu(Q)$ *is defined for all* $Q \in \mathbb{F}_q[X] \setminus \{0\}$ *and has its values in* $\{-1, 0, 1\}$ *depending on the factorization of* Q *into distinct normed irreducible factors. More precisely, write* $Q = c \cdot I_1^{a_1} I_2^{a_2} \cdots I_k^{a_k}$ *for some* $c \in \mathbb{F}_q$ *and* $I_j \neq I_k$ *for all* $j \neq k$*. Then*

$$\mu(Q) = \begin{cases} (-1)^k, & \text{if } a_1 = a_2 = \cdots = a_k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

As in classical number theory, we have the following results for Euler's phi function and Möbius function in $\mathbb{F}_q[X]$.

Proposition 2.5. *All capital letters in the following denote normed polynomials in* $\mathbb{F}_q[X]$*. We have*

$$|D| = \sum_{L|D} \varphi(L),$$

$$\sum_{M|U} \mu(M) = \begin{cases} 1, & \text{if } |U| = 1; \\ 0, & \text{if } |U| > 1, \end{cases}$$

$$\varphi(N) = |N| \cdot \prod_{I|N} \left(1 - \frac{1}{|I|}\right).$$

Proof. The claims follow with a similar method as in classical number theory. □

Proposition 2.6. *For* $N \in \mathbb{F}_q[X]$*,* N *normed, we have*

$$\frac{1}{2}\sqrt{|N|} \leq \varphi(N) \leq |N|. \tag{2.1}$$

Proof. First,

$$\varphi(N) = |N| \cdot \prod_{I|N} \left(1 - \frac{1}{|I|}\right) \leq |N|,$$

we get one side of (2.1). On the other hand, assume

$$N = I_1^{k_1} I_2^{k_2} \cdots I_l^{k_l}, \quad k_i \in \mathbb{N},\ i = 1, 2, \cdots, l,$$

where $I_j \neq I_k$ for all $j \neq k$. Then,

$$\varphi(N) = |I_1|^{k_1 - 1} \cdot |I_2|^{k_2 - 1} \cdots |I_l|^{k_l - 1} \cdot (|I_1| - 1|) \cdots (|I_l| - 1).$$

Case 1: $q > 2$.
Since $|I| - 1 > |I|^{\frac{1}{2}}$ and $k - \dfrac{1}{2} \geq \dfrac{k}{2}$,

$$\varphi(N) \geq |I_1|^{\frac{k_1}{2}} \cdot |I_2|^{\frac{k_2}{2}} \cdots |I_l|^{\frac{k_l}{2}} = \sqrt{|N|}.$$

Case 2: $q = 2$. In this case,

$$\begin{aligned}
\varphi(N) &= \left(\prod_{\deg I_j = 1} |I_j|^{k_j - 1}\right) \left(\prod_{\deg I_s \geq 2} |I_s|^{k_s - 1} \left(|I_s| - 1\right)\right) \\
&\geq \prod_{\deg I_j = 1} |I_j|^{k_j - 1} \prod_{\deg I_s \geq 2} |I_s|^{k_s/2} \\
&= \prod_{\substack{\deg I_j = 1 \\ k_j > 1}} |I_j|^{k_j - 1} \prod_{\deg I_s \geq 2} |I_s|^{k_s/2} \\
&\geq \prod_{\substack{\deg I_j = 1 \\ k_j > 1}} |I_j|^{k_j/2} \prod_{\deg I_s \geq 2} |I_s|^{k_s/2} \\
&= \left(2^{-1/2}\right)^{\#\{k_j:\ \deg I_j = 1,\ k_j = 1\}} \sqrt{|N|} \\
&\geq \frac{1}{2} \sqrt{|N|}.
\end{aligned}$$

Note that for $q = 2$, the number of irreducible normed polynomials with degree 1 is 2. This proves the desired result. $\square$

Later on, we will need a result about the number of irreducible normed polynomials:

Proposition 2.7. *Let N_n denote the number of irreducible normed polynomials of degree n over $\mathbb{F}_q$. Then $N_1 = q$ and*

$$N_n \leq \frac{q^n - q}{n}, \ \textit{for } n \geq 2.$$

Proof. See [1, Chapter 3]. $\square$

Now, let $\omega(Q)$ denote the number of distinct normed irreducible divisors of the normed polynomial Q. That is,

$$\omega(Q) = \sum_{I|Q} 1.$$

We obtain a bound of $\omega(Q)$ in the following proposition:

Proposition 2.8. *For $Q \in \mathbb{F}_q[X]$ be normed, we have*

$$\omega(Q) \leq \log_q |Q|.$$

Proof. Assume

$$Q = I_1^{l_1} I_2^{l_2} \cdots I_n^{l_n}, \ l_j \geq 1, \ j = 1, 2, \ldots, n,$$

where $I_j \neq I_k$ for all $j \neq k$. Then, it follows that

$$|Q| = |I_1^{l_1} I_2^{l_2} \cdots I_n^{l_n}| \geq q^n \Rightarrow \omega(Q) = n \leq \log_q |Q|.$$

We obtain the desired result. $\square$

Finally, we introduce a useful notation and a well-known theorem from probability theory (or measure theory) which plays a crucial role in this research.

Definition 2.3. *Suppose $f(x)$ and $g(x)$ are defined for x large enough. We say*

$$f(x) = \mathcal{O}\left(g\left(x\right)\right), \text{ as } x \to \infty,$$

or

$$f(x) \ll g(x)$$

if there exists a positive real number M and a real number x_0 such that

$$|f(x)| \leq M \cdot |g(x)| \text{ for } x > x_0.$$

Lemma 2.1 (Borel-Cantelli Lemma)**.** *Let E_n be a sequence of events in a probability space. The Borel-Cantelli lemma states:*

$$(1) \ \sum_{n=1}^{\infty} P\left(E_n\right) < \infty \Rightarrow P\left(\limsup_{n\to\infty} E_n\right) = P\left(E_n \ i.o.\right) = 0.$$

(2) *If the sequence E_n is independent, then*

$$\sum_{n=1}^{\infty} P\left(E_n\right) = \infty \Rightarrow P\left(\limsup_{n\to\infty} E_n\right) = P\left(E_n \ i.o.\right) = 1.$$

Proof. See [4, Lemma 3.14, p.41] □

Moreover, we have a generalized Borel-Cantelli lemma without the condition of independence in part (2) above:

Lemma 2.2. *Let Ω be a measure space with measure m such that $m(\Omega)$ is finite. Let E_n be a sequence of measurable subsets of Ω such that*

$$\sum_{n=1}^{\infty} m(E_n) = \infty.$$

Then the set E of points belonging to infinitely many sets E_n satisfies

$$m(E) \geq \limsup_{N\to\infty} \left(\sum_{n=1}^{N} m(E_n)\right)^2 \left(\sum_{n,l=1}^{N} m(E_n \cap E_l)\right)^{-1}.$$

Proof. See [10, Chapter 2, Lemma 2.3]. □

Remark 2.6. *For any subsequence* $(n_k)_{k\in\mathbb{N}}$ *of* $(n)_{n\in\mathbb{N}}$ *in Lemma 2.2, due to the property of* lim sup*, we still have*

$$m(E) \geq \limsup_{N\to\infty} \left(\sum_{k=1}^{N} m(E_{n_k})\right)^2 \left(\sum_{j,k=1}^{N} m(E_{n_j} \cap E_{l_k})\right)^{-1}.$$

2.2 Previous Results

In the sequel, we will consider the following inequality

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|},\ P, Q \in \mathbb{F}_q[X],\ Q \neq 0,\ (P,Q) = 1,\ Q \text{ is normed.} \tag{2.2}$$

By Fuchs [7], we know that Khintchine's theorem has an analogue in the field of formal Laurent series.

Theorem 2.1 (Analogue of Khintchine's Theorem for Formal Laurent Series)**.** *Let* ψ *be a* $\{q^{-n} : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$*-valued function defined on* $\mathbb{F}_q[X]$*, such that* $\psi(Q)$ *depends only on the degree of* $Q \in \mathbb{F}_q[X]$ *with* $|Q|\psi(Q)$ *non-increasing. Then the inequality* (2.2) *has infinitely many solutions* $\langle P, Q\rangle$ *for almost all* $f \in \mathbb{L}$*, if and only if*

$$\sum_{k=0}^{\infty} q^k \psi\left(X^k\right) = \infty. \tag{2.3}$$

Remark 2.7. *By "*$|Q|\psi(Q)$ *non-increasing" we mean that if* $\deg Q_1 \leq \deg Q_2$*, then* $|Q_1|\psi(Q_1) \geq |Q_2|\psi(Q_2)$*.*

The proof of Khintchine's theorem for the convergence case of (2.3) is just a simple application of the Borel-Cantelli lemma. For the other case, the result can be obtained by using the theory of continued fraction. See [7, Theorem 3.7].

In [11], Inoue and Nakada gave a refinement of Khintchine's theorem:

Theorem 2.2. *Let* ψ *be a* $\{q^{-n} : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$*-valued function defined on* $\mathbb{F}_q[X]$ *such that* $\psi(Q)$ *depends only on the degree of* $Q \in \mathbb{F}_q[X]$*. For any set* S *of positive* *integers, the inequality* (2.2) *with* $\deg Q \in S$ *has infinitely many solutions* $\langle P, Q\rangle$ *for almost every* $f \in \mathbb{L}$ *if and only if*

$$\sum_{k\in S} q^k \psi\left(X^k\right) = \infty. \tag{2.4}$$

We point out that the main improvement is that we can remove the monotonicity condition "$|Q|\psi(Q)$ non-increasing" from Khintchine's theorem. Moreover, the result holds now for any set of positive integers while (2.3) only holds for $\mathbb{N}$. This is an interesting result which has a different flavor from the real number case.

For the case that $\psi(Q)$ does not only depend on the degree of Q, Inoue and Nakada also gave a related result which is an analogue of the Duffin-Schaeffer theorem in the field of formal Laurent series:

Theorem 2.3 (Analogue of Duffin-Schaeffer Theorem for Formal Laurent Series)**.** *Let* ψ *be a* $\{q^{-n} : n \in \mathbb{N}\cup\{0\}\}\cup\{0\}$*-valued function which satisfies*

$$\sum_{n=0}^{\infty} \sum_{\substack{\deg Q = n\\ Q:normed}} \psi(Q) = \infty. \tag{2.5}$$

Suppose there are infinitely many positive integers n *such that*

$$\sum_{\substack{\deg Q\le n\\ Q:normed}} \psi(Q) < C \sum_{\substack{\deg Q\le n\\ Q:normed}} \psi(Q)\frac{\varphi(Q)}{|Q|} \tag{2.6}$$

for a constant $C > 0$*. Then the inequality* (2.2) *has infinitely many solutions* $\langle P, Q\rangle$ *for almost all* $f \in \mathbb{L}$*.*

Again, the convergence of (2.5) implies that (2.2) has only finitely many solutions. Thus, the Duffin-Schaeffer conjecture focus on the divergent part:

Conjecture (Duffin-Schaeffer Conjecture for Formal Laurent Series)**.** *Let* ψ *be a non-*

negative function which takes values on $\{q^{-n} : n \in \mathbb{Z}\} \cup \{0\}$ *such that*

$$\sum_{n=0}^{\infty} \sum_{\substack{\deg Q = n \\ Q:normed}} \psi(Q) \frac{\varphi(Q)}{|Q|} = \infty.$$

Then, (2.2) *has infinitely many solutions* $\langle P, Q \rangle$ *for almost all* $f \in \mathbb{L}$.

Note that the approximation function ψ in this conjecture does not only depend on the degree of Q. The main goal of the Duffin-Schaeffer conjecture is to find necessary and sufficient condition for the solution set of (2.2) to be infinite. In some sense, Theorem 2.2 confirms the Duffin-Schaeffer conjecture (for the situation where $\psi(Q)$ depends only on the degree of Q). We will further discuss this conjecture later in Section 2.3.

There are also some deeper results about the number of solutions of Diophantine inequalities. Now, we consider the following,

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_n}}, \ \deg Q = n, \ Q \neq 0, \ (P, Q) = 1, \ Q \text{ is normed}, \tag{2.7}$$

where (l_n) is a sequence of positive integers, that is, ψ has form $\psi(Q) = \dfrac{1}{q^{n+l_n}}$ if $\deg Q = n$. Again, we are interested in studying the solution set. Results of different strengths made necessary different restrictions on the set of sequences (l_n). The sets which have been considered are as follows:

$$\mathcal{A} = \left\{ (l_n)_{n \geq 0} \, | l_n > 0 \text{ and non-decreasing} \right\}$$

$$\mathcal{B} = \left\{ (l_n)_{n \geq 0} \, \middle| l_n > 0 \text{ and either (C1): } \lim_{n \to \infty} l_n = l < \infty, \right.$$

$$\left. \text{or (C2): } \lim_{n \to \infty} l_n = \infty, \lim_{i \to \infty} \sum_{i < j \leq i + l_i} q^{-l_j} \text{ exists} \right\}$$

$$\mathcal{C} = \left\{ (l_n)_{n \geq 0} \, | l_n > 0 \right\}.$$

Note that we have the following chain of proper inclusions $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$.

Define a sequence of random variables as

$$Z_N(f) := \#\left\{\frac{P}{Q} : \langle P, Q\rangle \text{ is a solution of (2.7)}, \;\deg Q \leq N\right\}.$$

Assuming that $(l_n) \in \mathcal{A}, \sum_{n=0}^{\infty} q^{-l_n} = \infty$ and under some further technical conditions on (l_n), Fuchs [7] proved the central limit theorem for (Z_N). His approach was based on continued fraction expansions. With a new approach, not relying on continued fraction expansions, Deligero and Nakada [6] proved a central limit theorem for the number of coprime solutions in the setting of the classical theorem of Khintchine, that is, for all sequences $(l_n) \in \mathcal{A}, \sum_{n=0}^{\infty} q^{-l_n} = \infty$ but without the additional conditions in Fuchs' version. Note that a similar result for the real number case has not been proved yet.

Remark 2.8. *The meaning of "in the setting of Khintchine" or "Khintchine's setting" in the following is that the function ψ in* (2.7) *satisfies the condition of Khintchine's theorem (Theorem* 2.1*). Note, however that the inequality* (2.7) *we consider is not equivalent to* (2.2)*.*

Besides, in [7], Fuchs obtained the invariance principle for sequence $(l_n) \in \mathcal{A}$ that satisfy $\sum_{n=0}^{\infty} q^{-l_n} = \infty$ and some technical extra conditions. In [5], Deligero, Fuchs and Nakada explored further the approach of [6] in order to extend Fuchs' result to all sequences $(l_n) \in \mathcal{B}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$. Therefore, set

$$F(N) := \begin{cases} q^{-2l-2}\left(q^{l+1}(q-1) - (2l+1)(q-1)^2\right) N, & \text{if } (C1); \\ q^{-1}(q-1)\sum_{n\leq N} q^{-l_n}, & \text{if } (C2) \end{cases}$$

and for $t \geq 0$,

$$N_t := \begin{cases} \max\{n \,|\, F(n) \leq t\}, & \text{if } t \geq F(0); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{\mathcal{B}}$ be the set of Borel sets on $[0,1]$ and λ the Lebesgue measure. Define on

$(\mathbb{L},\mathcal{L},m)\times\big([0,1],\bar{\mathcal{B}},\lambda\big)$ the following stochastic process

$$Z(t) := Z(t;f,x) := Z_{N_t}(f) - \left(1-\frac{1}{q}\right)\sum_{n=0}^{N} q^{-l_n}.$$

Note that the definition does not depend on the second variable.

Theorem 2.4 (Invariance Principle for Formal Laurent Series)**.** *There exists a sequence* $(Y_n)_{n\geq 0}$ *of independent, standard normal random variables on* $(\mathbb{L},\mathcal{L},m)\times\big([0,1],\bar{\mathcal{B}},\lambda\big)$ *such that, for all* $\epsilon>0$,

$$\left|Z(N) - \sum_{n\leq N} Y_n\right| = o\left((N\log\log N)^{1/2}\right), \ \ a.s.$$

and

$$(m\times\lambda)\left[\frac{1}{\sqrt{N}}\max_{n\leq N}\left|Z(n)-\sum_{k\leq n}Y_k\right|\geq\epsilon\right]\to 0 \ \ as\ N\to\infty.$$

The above result implies the functional central limit theorem which generalizes the result of Deligero and Nakada [6].

Corollary 2.1 (Functional Central Limit Theorem for Formal Laurent Series)**.** *Let* $W(t)$ *denote the standard Brownian motion. Then,*

$$\left\{\frac{Z(F(N)t)}{\sqrt{F(N)}}, 0\leq t\leq 1\right\}\to\{W(t),0\leq t\leq 1\},$$

as $N\to\infty$.

Moreover, we have the functional law of the iterated logarithm.

Corollary 2.2 (Functional Law of the Iterated Logarithm for Formal Laurent Series)**.** *The sequence of functions*

$$\left\{\frac{Z(F(N)t)}{(2F(N)\log\log F(N))^{1/2}}, 0\leq t\leq 1\right\}_{N\geq 0}$$

is a.s. relatively compact in the topology of uniform convergence and has Strassen's set as its set of limit points.

Remark 2.9. *Invariance principles, functional central limit theorems and functional laws of the iterated logarithm are deeper results in probability theory. See [3, Section 37] for reference.*

Since the set of sequences (l_n) here contains the sequences of Khintchine's theorem, we note the following consequence of the latter result which is a refinement of Khintchine's theorem for formal Laurent series.

Corollary 2.3 (Law of the Iterated Logarithm for Khintchine's setting)**.** *Assume that* $(l_n) \in \mathcal{A}$ *and* $\sum_{n=0}^{\infty} q^{-l_n} = \infty$. *Then, for almost all* f,

$$\limsup_{N\to\infty} \frac{\left|Z_N(f) - (1-q^{-1})\sum_{n\le N} q^{-l_n}\right|}{\sqrt{2F(N)\log\log F(N)}} = 1.$$

Note that a similar result for the real number field has so far not been established. Moreover, the above result also gives the optimal bound in the strong law of large numbers:

Corollary 2.4 (Strong Law of Large Numbers for Formal Laurent Series)**.** *Let* $(l_n) \in \mathcal{A}$. *Then, for almost all f,*

$$Z_N(f) = \left(1-q^{-1}\right)\sum_{n\le N} q^{-l_n} + \mathcal{O}\left(\left(G(N)\log\log G(N)\right)^{1/2}\right),$$

where $G(N) = \displaystyle\sum_{n=1}^{N} \frac{1}{q^{l_n}}$.

For the more general case $(l_n) \in \mathcal{C}$, Nakada and Natsui [13] gave the following result about the strong law of large numbers

Theorem 2.5. *Let* $(l_n) \in \mathcal{C}$. *Define*

$$G(N) = \sum_{n=1}^{N} \frac{1}{q^{l_n}}.$$

Then, we have for almost all f

$$Z_N(f) = \left(1 - q^{-1}\right) \sum_{n \leq N} q^{-l_n} + \mathcal{O}\left(G(N)^{1/2} \log^{3/2+\epsilon} G(N)\right).$$

Other types of Diophantine approximation problems have been discussed as well. In [2], Berthe, Nakada and Natsui consider specific inequalities with restricted denominators (powers of irreducible polynomials) with an approximation function which does not only depend on the degree of the denominator.

More precisely, they consider an inequality with restricted denominators that are supposed to be normed irreducible polynomials and a function $\psi : \mathbb{F}_q[X] \to \mathbb{R}$ of the form $Q \mapsto \left(|Q| q^{l_Q}\right)^{-1}$ where l_Q takes nonnegative integer values for Q normed irreducible, and infinite value otherwise, that is

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_Q}}, \ (P,Q) = 1, \ \deg Q = n, \ l_Q = \infty \text{ whenever } Q \text{ is not normed irreducible.} \tag{2.8}$$

Theorem 2.6. *Denote*

$$H(N) = \sum_{n=1}^{N} \sum_{\deg Q = n} \frac{1}{q^{n+l_Q}}.$$

Then, for almost all $f \in \mathbb{L}$, the number of solutions of (2.8) *with* $\deg Q \leq N$ *satisfies*

$$H(N) + \mathcal{O}\left(H^{1/2}(N) \log^{3/2+\epsilon} H(N)\right)$$

for any $\epsilon > 0$.

Note that Theorem 2.6 means that there exist at most finitely many solutions for almost all $f \in \mathbb{L}$ whenever $H(N)$ does not diverge.

Moreover, two more cases are discussed in [2]: Q is a fixed power of a normed irreducible polynomial, and Q is some power of a normed irreducible polynomial, see [2] for more details. The case we mentioned above is quite interesting, since it corresponds to the approximation of irrational numbers by rational numbers with prime denominators.

One other line of research was concerned with the same inequality like (2.7) but without the coprimeness of P and Q. In [13], Nakada and Natsui proved that the strong law of large numbers also holds in the setting of Khintchine theorem plus some additional conditions.

Yet, another line of research investigate the inhomogeneous Diophantine approximation problem: for $f, g \in \mathbb{L}$ consider the Diophantine inequality

$$|Qf - g - P| < \frac{1}{q^{n+l_n}}, \text{ Q is normed, } \deg Q = n \tag{2.9}$$

whose solutions are pairs of polynomials $\langle P, Q\rangle \in \mathbb{F}_q[X] \times \mathbb{F}_q[X]$ with $Q \neq 0$. Here, l_n is a sequence of non-negative integers. In particular, note that l_n just depends on the degree of Q like before. In a recent paper, C.Ma and W.-Y.Su [12] investigated the above problem and proved a Khintchine type 0-1 law for the number of solutions if both f and g are chosen randomly (with respect to m) from $\mathbb{L}$. This situation is sometimes called the "double-metric" case. Moreover, fixing f and choosing a random $g \in \mathbb{L}$ or fixing g and choosing a random $f \in \mathbb{L}$ gives two "single-metric" cases. Fuchs has lately done some research on this topic, see [8].

2.3 Main Results

Recall the Duffin-Schaeffer Conjecture from Section 2.2. Our main result is an analogue of Vaaler's theorem, namely that the conjecture is true when $\psi(Q) = \mathcal{O}\left(\frac{1}{|Q|}\right)$. This condition means that there exists $M > 0$ such that $\psi(Q) \leq M\frac{1}{|Q|}$ for $|Q|$ large enough. We can prove that

Theorem 2.7. *Let ψ be a non-negative function which takes values on $\{q^{-n} : n \in \mathbb{Z}\} \cup \{0\}$ with $\psi(Q) = \mathcal{O}\left(\frac{1}{|Q|}\right)$ such that*

$$\sum_{n=0}^{\infty} \sum_{\substack{\deg Q = n \\ Q:\ normed}} \psi(Q)\frac{\varphi(Q)}{|Q|} = \infty.$$

Then for almost all $f \in \mathbb{L}$, there are infinitely many solutions $\langle P, Q\rangle$ to

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|},\ \ P, Q \in \mathbb{F}_q[X],\ \ Q \neq 0,\ \ (P, Q) = 1,\ \ Q \textit{ is normed.}$$

Note that the approximation function ψ here does not only depend on the degree of Q.

This result is an analogue of Vaaler's theorem in the field of formal Laurent series. It is a refinement of Theorem 2.2 from the introduction, since the approximation function of this result does not only depend on the degree of Q.

In Chapter 3, we will prove this theorem. The proof will follow along the lines of the classical proof given by Harman (see [10]). It will involve a technical lemma (Lemma 3.1) that is obtained from Selberg's sieve method (see [9], Chapter 3). Moreover, suitable estimates of the measure of certain intersections of events (see Lemma 3.4) will turn out to be crucial. Apart from this, we will use the generalized Borel-Cantelli lemma (Lemma 2.2) and an analogue of Gallagher's theorem for the field of formal Laurent series.

In Chapter 4, we will give an equivalent statement of the Duffin-Schaeffer conjecture and develop some further results which are similar to the real number case as in [10, Section 2.6].

First, we will show that the Duffin-Schaeffer conjecture is equivalent to this one:

Conjecture. *Let F_n be a sequence of distinct normed polynomials of $\mathbb{F}_q[X]$, and $\psi(F_n)$ a $\{q^{-n} : n \in \mathbb{Z}\} \cup \{0\}$-valued function. Then if*

$$\sum_{n=1}^{\infty} \psi(F_n) \frac{\varphi(F_n)}{|F_n|} = \infty,$$

there are infinitely many solutions $\langle P, F_n \rangle$ to

$$\left| f - \frac{P}{F_n} \right| < \frac{\psi(F_n)}{|F_n|}, \ P \in \mathbb{F}_q[X], \ (P, F_n) = 1$$

for almost all $f \in \mathbb{L}$.

Then, we will prove that the latter conjecture is true for the following sequences:

Theorem 2.8. *Let F_n be a sequence of distinct normed polynomials of $\mathbb{F}_q[X]$. If there is an absolute bound $c \in \mathbb{N}$ such that*

$$\#\{F_n \mid \deg F_n = j\} \leq c, \text{ for all } j \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} \psi(F_n) \frac{\varphi(F_n)}{|F_n|} = \infty,$$

then there are infinitely many solutions $\langle P, F_n \rangle$ to

$$\left| f - \frac{P}{F_n} \right| < \frac{\psi(F_n)}{|F_n|}, \ P \in \mathbb{F}_q[X], \ (P, F_n) = 1$$

for almost all $f \in \mathbb{L}$.

Chapter 3

Analogue of Vaaler's Theorem

3.1 Preliminary Lemmas

Lemma 3.1. *For $Q \in \mathbb{F}_q[X]$ be a normed polynomial, denote by $I(Q)$ an irreducible normed factor of Q with the highest degree. If $N \in \mathbb{F}_q[X]$ with $|I(Q)| < |N|$, then we have*

$$\sum_{\substack{|P|<|N| \\ (P,Q)=1}} 1 \ll |N| \cdot \prod_{\substack{I|Q \\ I:normed}} \left(1 - \frac{1}{|I|}\right)$$

where the sum is over $P \in \mathbb{F}_q[X]$.

Remark 3.1. *The constant implied in $\ll$ above is independent of any quantity involved in the lemma. In the sequel, the $\ll$ notation will always have this property.*

Proof. All polynomials in this proof are assumed to be normed, we will mention it for those which are not. First, we define $I(n) = \prod_{\substack{|I|<q^n \\ I|Q}} I$, $n \in \mathbb{N}$. Moreover, let $\lambda_1 = 1$ and λ_D be arbitrary real numbers corresponding to the polynomial D with $\deg D \geq 1$. Finally, $[D_1, D_2]$ will denote the least common multiple of D_1, D_2, and

$$R_{[D_1,D_2]} = \sum_{\substack{|P|<|N| \\ [D_1,D_2]|P}} 1 - \frac{|N|}{|[D_1, D_2]|} = \max\left\{1, \frac{|N|}{|[D_1, D_2]|}\right\} - \frac{|N|}{|[D_1, D_2]|}.$$

Then, note that

$$\sum_{\substack{|P|<|N| \\ (P,Q)=1}} 1 \le |\{P \in \mathbb{F}_q[X] : |P| < |N|, (P, \prod_{\substack{|I|<q^n \\ I|Q}} I) = 1\}|$$

$$\le \sum_{|P|<|N|} \Big(\sum_{\substack{D|P \\ D|I(n)}} \lambda_D \Big)^2$$

$$= \sum_{\substack{D_i|I(n) \\ i=1,2}} \lambda_{D_1}\lambda_{D_2} \sum_{\substack{|P|<|N| \\ [D_1,D_2]|P}} 1$$

$$= |N| \sum_{\substack{D_i|I(n) \\ i=1,2}} \lambda_{D_1}\lambda_{D_2} \frac{1}{|[D_1,D_2]|} + \sum_{\substack{D_i|I(n) \\ i=1,2}} \lambda_{D_1}\lambda_{D_2} |R_{[D_1,D_2]}|$$

$$:= |N| \sum_1 + \sum_2$$

where P, N need not to be normed. Note that the second inequality is true without any further conditions on the numbers λ_D; for if $|P| < |N|$ and $(P, I(n)) = 1$, $D = 1$ is the only divisor appearing on the right and it makes a contribution 1 since $\lambda_1 = 1$; moreover, all the other terms on the right, namely those associated with $|P| < |N|$, $(P, I(n)) > 1$, are non-negative because of the square. Since $|D| = \sum_{L|D} \varphi(L)$,

$$\sum_1 = \sum_{D_1|I(n)} \sum_{D_2|I(n)} \lambda_{D_1}\lambda_{D_2} \frac{|(D_1,D_2)|}{|D_1D_2|}$$

$$= \sum_{D_1|I(n)} \sum_{D_2|I(n)} \frac{\lambda_{D_1}}{|D_1|}\frac{\lambda_{D_2}}{|D_2|} \sum_{\substack{L|D_1 \\ L|D_2}} \varphi(L)$$

$$= \sum_{\substack{|L|<q^n \\ L|I(n)}} \Big(\sum_{\substack{D|I(n) \\ L|D}} \frac{\lambda_D}{|D|} \Big)^2.$$

Now, choose

$$\lambda_D = \frac{\mu(D)}{\prod\limits_{I|D}\left(1-\frac{1}{|I|}\right)} \frac{\sum\limits_{\substack{|S|<q^n/|D| \\ (S,D)=1}} \frac{\mu^2(S)}{\varphi(S)}}{\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}}.$$

Then, for $|L| < q^n, L|I(n)$

$$\sum_{\substack{D|I(n) \\ L|D}} \frac{\lambda_D}{|D|} = \sum_{\substack{M|I(n) \\ (M,L)=1}} \frac{\mu(LM)}{\varphi(LM)} \frac{\sum\limits_{\substack{|S|<q^n/|LM| \\ (S,LM)=1}} \frac{\mu^2(S)}{\varphi(S)}}{\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}}$$

$$= \frac{\mu(L)}{\varphi(L)} \frac{1}{\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}} \sum_{\substack{M|I(n) \\ (M,L)=1}} \frac{\mu(M)}{\varphi(M)} \sum_{\substack{|S|<q^n/|LM| \\ (S,LM)=1}} \frac{\mu^2(S)}{\varphi(S)}$$

$$= \frac{\mu(L)}{\varphi(L)} \frac{1}{\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}} \sum_{\substack{|U|<q^n/|L| \\ (U,L)=1 \\ U|I(n)}} \frac{\mu(U)}{\varphi(U)} \sum_{M|U} \mu(M)$$

$$= \frac{\mu(L)}{\varphi(L)} \frac{1}{\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}}.$$

We used the fact $\sum\limits_{M|U} \mu(M) = \begin{cases} 1 & \text{if } |U| = 1, \\ 0 & \text{if } |U| > 1 \end{cases}$ to obtain the last equality.

Thus, with the λ_D we have

$$\sum_1 = \frac{1}{\left(\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}\right)^2} \sum_{\substack{|L|<q^n \\ L|I(n)}} \frac{\mu^2(L)}{\varphi(L)} = \frac{1}{\sum\limits_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)}}.$$

Note that

$$\sum_{|W|<q^n} \frac{\mu^2(W)}{\varphi(W)} = \sum_{L|K} \sum_{\substack{|W|<q^n \\ (W,K)=L}} \frac{\mu^2(W)}{\varphi(W)}$$

$$= \sum_{L|K} \sum_{\substack{|H|<q^n/|L| \\ (H,K/L)=1 \\ (H,L)=1}} \frac{\mu^2(LH)}{\varphi(LH)}$$

$$\leq \sum_{L|K} \frac{\mu^2(L)}{\varphi(L)} \sum_{\substack{|W|<q^n \\ (W,K)=1}} \frac{\mu^2(W)}{\varphi(W)}$$

$$\leq \Big(\sum_{L|K} \frac{\mu^2(L)}{\varphi(L)} \Big) \sum_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)} \qquad \Big(\text{choose } \; K = \prod_{\substack{|I|<q^n \\ I\nmid Q}} I\Big).$$

Let $K(M)$ denote a squarefree divisor of M with the highest degree.

Since

$$\sum_{L|K} \frac{\mu^2(L)}{\varphi(L)} = \prod_{I|K} \left(1 + \frac{1}{|I|-1}\right) = \prod_{I|K} \left(1 - \frac{1}{|I|}\right)^{-1}$$

and

$$\sum_{|W|<q^n} \frac{\mu^2(W)}{\varphi(W)} = \sum_{|W|<q^n} \frac{\mu^2(W)}{|W|} \prod_{I|W} \left(1 - \frac{1}{|I|}\right)^{-1} \geq \sum_{|K(M)|<q^n} \frac{1}{|M|}$$

$$\geq \sum_{|M|<q^n} \frac{1}{|M|} = \sum_{a=0}^{n-1} \sum_{\deg M=a} \frac{1}{|M|} = n.$$

Therefore,

$$\sum_1 = \Big(\sum_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)} \Big)^{-1} \leq \Big(n \cdot \prod_{\substack{|I|<q^n \\ I\nmid Q}} \left(1 - \frac{1}{|I|}\right) \Big)^{-1}.$$

On the other hand, note that

$$\sum_{\substack{|W|<q^n \\ W|I(n)}} \frac{\mu^2(W)}{\varphi(W)} = \sum_{L|D} \sum_{\substack{|M|<q^n \\ (M,D)=L}} \frac{\mu^2(M)}{\varphi(M)}$$

$$= \sum_{L|D} \sum_{\substack{|H|<q^n/|L| \\ (H,D/L)=1 \\ (H,L)=1}} \frac{\mu^2(LH)}{\varphi(LH)}$$

$$= \sum_{L|D} \frac{\mu^2(L)}{\varphi(L)} \sum_{\substack{|H|<q^n/|L| \\ (H,D)=1}} \frac{\mu^2(H)}{\varphi(H)}$$

$$\geq \Big(\sum_{L|D} \frac{\mu^2(L)}{\varphi(L)} \Big) \sum_{\substack{|H|<q^n/|L| \\ (H,D)=1}} \frac{\mu^2(H)}{\varphi(H)}$$

$$= \frac{\displaystyle\sum_{\substack{|H|<q^n/|L| \\ (H,D)=1}} \frac{\mu^2(H)}{\varphi(H)}}{\displaystyle\prod_{I|D} \left(1 - \frac{1}{|I|}\right)}.$$

So, the λ_D we have chosen satisfy $\lambda_1 = 1,\ |\lambda_D| \leq 1$ and $\lambda_D = 0$ for $|D| > q^n$.

As a result,

$$\sum_2 \leq \sum_{\substack{D_i|I(n) \\ |D_i|<q^n \\ i=1,2}} |R_{[D_1,D_2]}| \leq \Big(q^n\Big)^2.$$

Thus,

$$\sum_{\substack{|P|<|N| \\ (P,Q)=1}} 1 \leq \frac{|N|}{n \cdot \displaystyle\prod_{\substack{|I|<q^n \\ I\nmid Q}} \left(1 - \frac{1}{|I|}\right)} + q^{2n}.$$

Since $|I(Q)| < |N|$, we have

$$\frac{|Q|}{\varphi(Q)} \frac{\sum\limits_{\substack{|P|<|N| \\ (P,Q)=1}} 1}{|N|} \leq \frac{1}{\prod\limits_{|I|<|N|} \left(1 - \frac{1}{|I|}\right)} \Big(\frac{1}{n} + \frac{q^{2n}}{|N|}\Big).$$

And

$$\begin{aligned}
\prod_{|I|<|N|} \left(1 - \frac{1}{|I|}\right)^{-1} &= \exp\left\{ - \sum_{|I|<|N|} \ln\left(1 - \frac{1}{|I|}\right) \right\} \\
&= \exp\left\{ \sum_{|I|<|N|} \frac{1}{|I|} + \mathcal{O}\Big(\sum_{|I|<|N|} \frac{1}{|I|^2} \Big) \right\} \\
&\ll \exp\left\{ \sum_{a=1}^{\log_q |N|-1} \sum_{degI=a} \frac{1}{|I|} \right\} \\
\text{Proposition 2.7} \quad &\ll \exp\left\{ \ln(\log_q |N|) \right\} \ll \ln |N|.
\end{aligned}$$

Choose $n = \log_q |N|^{\frac{1}{3}}$. Then, we have

$$\frac{|Q|}{\varphi(Q)} \frac{\sum\limits_{\substack{|P|<|N| \\ (P,Q)=1}} 1}{|N|} \ll \ln |N| \Big(\frac{3}{\ln |N|} + \frac{1}{|N|^{\frac{1}{3}}} \Big) \ll 1.$$

Hence, for $|I(Q)| < |N|$,

$$\sum_{\substack{|P|<|N| \\ (P,Q)=1}} 1 \ll |N| \cdot \prod_{\substack{I|Q \\ I:normed}} \left(1 - \frac{1}{|I|}\right).$$

We obtain the desired result. □

Corollary 3.1. *For $Q \in \mathbb{F}_q[X]$ be normed , P and $N \in \mathbb{F}_q[X]$,*

$$\sum_{\substack{|P|<|N| \\ (P,Q)=1}} 1 \ll |N| \cdot \prod_{\substack{I|Q \\ |I|<|N| \\ I:normed}} \left(1 - \frac{1}{|I|}\right).$$

Proof. Say, $Q = Q_1Q_2$, where for $I|Q_1,\ |I| \le |N|$ and $I|Q_2,\ |I| \ge |N|$. Then

$$\sum_{\substack{|P|<|N| \\ (P,Q)=1}} 1 \le \sum_{\substack{|P|<|N| \\ (P,Q_1)=1}} 1$$

$$\overset{\text{Lemma 3.1}}{\ll} |N| \cdot \prod_{\substack{I|Q_1 \\ I:normed}} \left(1 - \frac{1}{|I|}\right)$$

$$= |N| \cdot \prod_{\substack{I|Q \\ |I|<|N| \\ I:normed}} \left(1 - \frac{1}{|I|}\right).$$

This gives the desired result. $\square$

In the following, we define

$$E_Q = \bigcup_{\substack{(P,Q)=1 \\ \deg P < \deg Q}} \left\{ f \in \mathbb{L} : \left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|} \right\}$$

and $\psi(Q)$ be a $\{q^{-n} : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$-valued function.

Lemma 3.2. *Let $Q \in \mathbb{F}_q[X]$ be normed with $|Q|$ large enough. Then*

$$E_Q = \bigcup_{\substack{(P,Q)=1 \\ \deg P < \deg Q}} B\left(\frac{P}{Q}, \frac{\psi(Q)}{|Q|}\right)$$

where the union is disjoint.

Moreover,

$$m(E_Q) = \varphi(Q)\frac{\psi(Q)}{|Q|}.$$

Proof. Let Q be fixed with $|Q|$ large enough. Suppose there are $P_1, P_2 \in \mathbb{F}_q[X]$ such that

$$\left| f - \frac{P_1}{Q} \right| < \frac{\psi(Q)}{|Q|}, \quad \left| f - \frac{P_2}{Q} \right| < \frac{\psi(Q)}{|Q|}, \quad \deg P_1 < \deg Q, \quad \deg P_2 < \deg Q.$$

Then, by the ultra-metric property, we have

$$\left|\frac{P_1}{Q}-\frac{P_2}{Q}\right|<\max\left\{\left|f-\frac{P_1}{Q}\right|,\ \left|f-\frac{P_2}{Q}\right|\right\}=\frac{\psi(Q)}{|Q|}$$

$$\Rightarrow |P_1-P_2|<\psi(Q)\leq 1\ \Rightarrow P_1=P_2.$$

This implies that such balls with different centers are disjoint. Consequently,

$$m(E_Q)=m\left(\bigcup_{\substack{(P,Q)=1\\ \deg P<\deg Q}} B\left(\frac{P}{Q},\frac{\psi(Q)}{|Q|}\right)\right)=\sum_{\substack{(P,Q)=1\\ \deg P<\deg Q}}\frac{\psi(Q)}{|Q|}=\varphi(Q)\frac{\psi(Q)}{|Q|}.$$

□

Lemma 3.3. *For* $Q,\ Q'\in\mathbb{F}_q[X]$ *be normed,*

$$m(E_Q\cap E_{Q'})=K(Q,Q')\cdot\min\left\{\frac{\psi(Q)}{|Q|},\ \frac{\psi(Q')}{|Q'|}\right\}$$

where

$$K(Q,Q')=\#\left\{\langle P,P'\rangle:\left|\frac{P}{Q}-\frac{P'}{Q'}\right|<\max\left\{\frac{\psi(Q)}{|Q|},\ \frac{\psi(Q')}{|Q'|}\right\},\ (P,Q)=1,\ (P',Q')=1\right\}.$$

Proof. First, assume $\frac{\psi(Q')}{|Q'|}<\frac{\psi(Q)}{|Q|}$.
Since

$$E_Q=\bigcup_{\substack{(P,Q)=1\\ \deg P<\deg Q}} B\left(\frac{P}{Q},\frac{\psi(Q)}{|Q|}\right),\ E_{Q'}=\bigcup_{\substack{(P',Q')=1\\ \deg P'<\deg Q'}} B\left(\frac{P'}{Q'},\frac{\psi(Q')}{|Q'|}\right)$$

by Lemma 3.2, the balls in E_Q and $E_{Q'}$ are disjoint respectively.

Note that by the ultra-metric property we have for

$$B\left(\frac{P}{Q},\frac{\psi(Q)}{|Q|}\right)\cap B\left(\frac{P'}{Q'},\ \frac{\psi(Q')}{|Q'|}\right)\neq\emptyset$$

$$
\begin{aligned}
f \in B\left(\frac{P'}{Q'},\ \frac{\psi(Q')}{|Q'|}\right) \Rightarrow \left|f-\frac{P}{Q}\right| &= \left|f-\frac{P'}{Q'}-\left(\frac{P}{Q}-\frac{P'}{Q'}\right)\right| \\
&\le \max\left\{\left|f-\frac{P'}{Q'}\right|,\ \left|\frac{P}{Q}-\frac{P'}{Q'}\right|\right\} \\
&= \max\left\{\left|f-\frac{P'}{Q'}\right|,\ \left|\frac{P}{Q}-f-\left(\frac{P'}{Q'}-f\right)\right|\right\} \\
&\le \max\left\{\left|f-\frac{P'}{Q'}\right|,\ \max\left\{\left|\frac{P}{Q}-f\right|,\ \left|\frac{P'}{Q'}-f\right|\right\}\right\} \\
&= \max\left\{\frac{\psi(Q)}{|Q|},\ \frac{\psi(Q')}{|Q'|}\right\} = \frac{\psi(Q)}{|Q|} \Rightarrow f \in B\left(\frac{P}{Q},\ \frac{\psi(Q)}{|Q|}\right).
\end{aligned}
$$

Thus, any ball in $E_{Q'}$ is either contained in one of the balls in E_Q or disjoint with all the balls in E_Q. The intersections of two balls is no more than the radius of the smaller. Hence, we need to count how many balls with smaller radius are contained in those balls with larger radius. This concludes the first case. The other case $\frac{\psi(Q)}{|Q|} < \frac{\psi(Q')}{|Q'|}$ is similar. □

Lemma 3.4. *For $Q,\ Q' \in \mathbb{F}_q[X]$ be normed, we have*

$$m(E_Q \cap E_{Q'}) \ll P(Q,Q')m(E_Q)m(E_{Q'})$$

where

$$P(Q,Q') = \prod_{\substack{I \left| \frac{QQ'}{(Q,Q')^2} \right. \\ |I| > L(Q,Q')}} \left(1-\frac{1}{|I|}\right)^{-1},\ L(Q,Q') = \frac{\max\{|Q'|\psi(Q), |Q|\psi(Q')\}}{|(Q,Q')|}.$$

Proof. All polynomials in this proof are assumed to be normed, we will mention it for those which are not.

Let

$$A = \max\{|Q'|\psi(Q), |Q|\psi(Q')\},\ T = (Q,Q'),\ \bar{A} = \frac{A}{|T|},\ \bar{Q} = \frac{Q}{|T|},\ \bar{Q}' = \frac{Q'}{|T|}.$$

By Lemma 3.3, we wish to bound $K(Q,Q')$, that is the number of pairs $\langle P,P'\rangle$ satisfying

$$|Q'P-QP'| < A \text{ with } (Q,P)=(Q',P')=1,\ \deg P<\deg Q,\ \deg P'<\deg Q' \tag{3.1}$$

$$\Rightarrow \left|\bar{Q}'P-\bar{Q}P'\right| < \bar{A} \text{ with } (Q,P)=(Q',P')=1,\ \deg P<\deg Q,\ \deg P'<\deg Q' \tag{3.2}$$

where P,P' need not to be normed.

Say,

$$\bar{Q}'P-\bar{Q}P'=B,\quad \text{for some } B\in\mathbb{F}_q[X] \text{ with } |B|>1 \text{ need not to be normed.} \tag{3.3}$$

Let ${\bar{Q}'}^{-1}$ denote the inverse of $\bar{Q}'$ mod $\bar{Q}$, that is ${\bar{Q}'}^{-1}\bar{Q}'\equiv 1(\bar{Q})$.

We obtain,

$$P=B{\bar{Q}'}^{-1}+D\bar{Q},\quad \text{with } \left|D+\frac{B{\bar{Q}'}^{-1}}{\bar{Q}}\right|<|T|. \tag{3.4}$$

Let

$$W=\frac{{\bar{Q}'}^{-1}\bar{Q}'-1}{\bar{Q}}.$$

Since we must have $(B,\bar{Q}\bar{Q}')=1=(PP',T)$ to satisfy the conditions of (3.2), we find that the number of solutions of (3.3) is

$$\sum_{E|T}\mu(E)\sum_{PP'\equiv 0(E)}1=\sum_{E|T}\mu(E)\sum_{\substack{|D|\\(B{\bar{Q}'}^{-1}+D\bar{Q})(BW+D\bar{Q}')\equiv 0(E)}}1 \tag{3.5}$$

where the sum over D satisfies the conditions of (3.4).

Note that

$$(B{\bar{Q}'}^{-1}+D\bar{Q})(BW+D\bar{Q}')=D^2\bar{Q}\bar{Q}'+D(\bar{Q}BW+B\bar{Q}'{\bar{Q}'}^{-1})+B^2\bar{Q}'{\bar{Q}'}^{-1}\equiv 0(E). \tag{3.6}$$

Let $E=I$ be some irreducible normed polynomial. Then, there are three cases to discuss in (3.6).

Case 1: If $I \mid \bar{Q}\bar{Q}'$, then (3.6) is not quadratic. Therefore, there is only one solution.

Case 2: If $I \nmid \bar{Q}\bar{Q}'$ but $I|B$, then (3.6) becomes $D^2\bar{Q}\bar{Q}' \equiv 0(I) \Rightarrow D = 0$. So, again we have only one solution.

Case 3: If $I \nmid \bar{Q}\bar{Q}'$ and $I \nmid B$, then $B\bar{Q}'^{-1} + D\bar{Q} \equiv 0(I)$, $BW + D\bar{Q}' \equiv 0(I)$ have distinct solutions. If not, then $\bar{Q}'B\bar{Q}'^{-1} - \bar{Q}'BW \equiv 0(I)$. (where $\bar{Q}'\bar{Q} \equiv 1(I)$)

Since $I \nmid B$, we have

$$\bar{Q}'\bar{Q}'^{-1} \equiv \bar{Q}'W \equiv \bar{Q}'\frac{\bar{Q}'\bar{Q}'^{-1} - 1}{\bar{Q}}(I)$$

$$\Rightarrow \bar{Q}'^{-1} \equiv \bar{Q}'^{-1} - \bar{Q}'(I)$$

$$\Rightarrow \bar{Q}' \equiv 0(I) \Rightarrow 1 \equiv 0(I)$$

which is a contradiction.

Thus, (3.5) becomes

$$|T|\prod_{I|T}\left(1 - \frac{\rho(I)}{|I|}\right) \quad \text{where} \quad \rho(I) = \begin{cases} 1 & \text{if } I|B\bar{Q}\bar{Q}', \\ 2 & \text{otherwise.} \end{cases}$$

Moreover

$$\begin{aligned} |T|\prod_{I|T}\left(1 - \frac{\rho(I)}{|I|}\right) &= |T| \prod_{\substack{I|T \\ I\nmid B\bar{Q}\bar{Q}'}} \left(1 - \frac{2}{|I|}\right) \prod_{I|(T,B)} \left(1 - \frac{1}{|I|}\right) \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \\ &\leq |T| \prod_{\substack{I|T \\ I\nmid B\bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,B)} \left(1 - \frac{1}{|I|}\right) \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \\ &= |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,B)} \left(1 - \frac{1}{|I|}\right)^{-1} \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right). \end{aligned}$$

Hence, the number of solutions to (3.2) is

$$\leq |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \sum_{\substack{|B|=1 \\ (B,\bar{Q}\bar{Q}')=1}}^{\bar{A}} \prod_{I|(T,B)} \left(1 - \frac{1}{|I|}\right)^{-1}$$

(Here B is not necessarily normed.)

$$= |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \sum_{\substack{|B|=1 \\ (B,\bar{Q}\bar{Q}')=1}}^{\bar{A}} \sum_{S|(T,B)} \frac{\mu^2(S)}{\varphi(S)}$$

$$= |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \sum_{S|T} \frac{\mu^2(S)}{\varphi(S)} \sum_{\substack{|B|=1 \\ (B,\bar{Q}\bar{Q}')=1 \\ B\equiv 0(S)}}^{\bar{A}} 1$$

$$\leq |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \sum_{S|T} \frac{\mu^2(S)}{\varphi(S)} \sum_{\substack{|B'|\leq \bar{A}/|S| \\ (B',\bar{Q}\bar{Q}')=1}}^{\bar{A}} 1$$

(Here $B = B'S$ with B' is not necessarily normed.)

Corollary 3.1
$$\ll |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \sum_{S|T} \frac{\mu^2(S)}{\varphi(S)} \frac{\bar{A}}{|S|} \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\leq \bar{A}/|S|}} \left(1 - \frac{1}{|I|}\right)$$

Proposition 2.6
$$\ll |T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \sum_{S|T} \frac{1 \cdot \bar{A} \cdot \ln \bar{A}}{|S|^{1/2} \cdot |S|} \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\leq \bar{A}}} \left(1 - \frac{1}{|I|}\right)$$

$$\ll \bar{A}|T| \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right) \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\leq \bar{A}}} \left(1 - \frac{1}{|I|}\right)$$

$$= A \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\leq \bar{A}}} \left(1 - \frac{1}{|I|}\right) \prod_{\substack{I|T \\ I\nmid \bar{Q}\bar{Q}'}} \left(1 - \frac{1}{|I|}\right)^2 \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1 - \frac{1}{|I|}\right)$$

$$= A \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|>\bar{A}}} \left(1-\frac{1}{|I|}\right)^{-1} \prod_{I|\bar{Q}\bar{Q}'} \left(1-\frac{1}{|I|}\right) \prod_{\substack{I|T \\ I\nmid\bar{Q}\bar{Q}'}} \left(1-\frac{1}{|I|}\right)^{2} \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1-\frac{1}{|I|}\right)$$

$$= A \cdot P(Q,Q') \prod_{I|\bar{Q}\bar{Q}'} \left(1-\frac{1}{|I|}\right) \prod_{\substack{I|T \\ I\nmid\bar{Q}\bar{Q}'}} \left(1-\frac{1}{|I|}\right)^{2} \prod_{I|(T,\bar{Q}\bar{Q}')} \left(1-\frac{1}{|I|}\right)$$

$$= A \cdot P(Q,Q') \frac{\varphi(Q)}{|Q|} \frac{\varphi(Q')}{|Q'|}.$$

As a result,

$$K(Q,Q') \ll A \cdot P(Q,Q') \frac{\varphi(Q)}{|Q|} \frac{\varphi(Q')}{|Q'|}.$$

Hence, by Lemma 3.1

$$m(E_Q \cap E_{Q'}) \ll P(Q,Q') \frac{\psi(Q)\varphi(Q)}{|Q|} \frac{\psi(Q')\varphi(Q')}{|Q'|} = P(Q,Q') m(E_Q) m(E_{Q'}).$$

□

Remark 3.2. *Note that*

$$\prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\le\bar{A}/|S|}} \left(1-\frac{1}{|I|}\right) = \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\le\bar{A}}} \left(1-\frac{1}{|I|}\right) \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\le\bar{A}/|S|}} \left(1-\frac{1}{|I|}\right) \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\le\bar{A}}} \left(1-\frac{1}{|I|}\right)^{-1}$$

$$\le \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\le\bar{A}}} \left(1-\frac{1}{|I|}\right) \prod_{\substack{I|\bar{Q}\bar{Q}' \\ |I|\le\bar{A}}} \left(1-\frac{1}{|I|}\right)^{-1}$$

$$\le \prod_{|I|\le\bar{A}} \left(1-\frac{1}{|I|}\right)^{-1} \ll \ln \bar{A}.$$

We use the same claim as in Lemma 3.1.

Remark 3.3. *If $L(Q,Q') < 1$, then by the definition of $\bar{A}$ we have $E_Q \cap E_{Q'} = \emptyset$, that is $m(E_Q \cap E_{Q'}) = 0$. Therefore, the result of Lemma* 3.3 *is trivial.*

Write $g(S)$ for the least positive integer such that $\sum_{\substack{I|S \\ |I|>q^{g(S)}}} \frac{1}{|I|} < \frac{1}{q}$.

Then we have,

Lemma 3.5. *For* $S \in \mathbb{F}_q[X]$ *be normed,*

$$\prod_{\substack{I|S \\ |I|>q^{g(S)}}} \left(1 - \frac{1}{|I|}\right)^{-1} \ll 1 \quad and \quad \frac{|S|}{\varphi(S)} \ll g(S).$$

Proof.

$$\prod_{\substack{I|S \\ |I|>q^{g(S)}}} \left(1 - \frac{1}{|I|}\right)^{-1} = \exp\left\{ - \sum_{\substack{I|S \\ |I|>q^{g(S)}}} \ln\left(1 - \frac{1}{|I|}\right) \right\}$$

$$= \exp\left\{ \sum_{\substack{I|S \\ |I|>q^{g(S)}}} \frac{1}{|I|} + \mathcal{O}\left(\sum_{\substack{I|S \\ |I|>q^{g(S)}}} \frac{1}{|I|^2} \right) \right\}$$

$$\ll \exp\left\{ \frac{1}{q} \right\} \ll 1.$$

Since,

$$\sum_{\substack{I|S \\ |I|\leq q^{g(S)}}} \frac{1}{|I|} \leq \sum_{|I|\leq q^{g(S)}} \frac{1}{|I|} = \sum_{k=1}^{g(S)} \sum_{\deg I=k} \frac{1}{|I|} = 1 + \sum_{k=2}^{g(S)} \frac{1}{k} \leq \ 1 + \ln g(S) \tag{3.7}$$

we have

$$\prod_{\substack{I|S \\ |I|\leq q^{g(S)}}} \left(1 - \frac{1}{|I|}\right)^{-1} = \exp\left\{ \sum_{\substack{I|S \\ |I|>\leq q^{g(S)}}} \frac{1}{|I|} + \mathcal{O}\left(\sum_{\substack{I|S \\ |I|\leq q^{g(S)}}} \frac{1}{|I|^2} \right) \right\} \ll g(S).$$

Thus,

$$\frac{|S|}{\varphi(S)} = \prod_{I|S} \left(1 - \frac{1}{|I|}\right)^{-1} = \prod_{\substack{I|S \\ |I|>q^{g(S)}}} \left(1 - \frac{1}{|I|}\right)^{-1} \prod_{\substack{I|S \\ |I|\leq q^{g(S)}}} \left(1 - \frac{1}{|I|}\right)^{-1} \ll g(S).$$

□

Corollary 3.2. *For $Q, Q' \in \mathbb{F}_q[X]$ be normed. Let*

$$w = w(Q,Q') = \max\left\{ g\left(\frac{Q}{(Q,Q')}\right), g\left(\frac{Q'}{(Q,Q')}\right) \right\}.$$

Then,

$$m(E_Q \cap E_{Q'}) \ll \begin{cases} m(E_Q)m(E_{Q'})w^2 & ,\text{if } \ 1 < L(Q,Q') < q^w \\ m(E_Q)m(E_{Q'}) & ,\text{otherwise.} \end{cases}$$

Proof. By Remark 3.3, we apply Lemma 3.5 to the following two cases:

Case 1: $q^w \leq L(Q,Q')$

$$P(Q,Q') = \prod_{\substack{I \left| \frac{QQ'}{(Q,Q')^2} \right. \\ |I| > L(Q,Q')}} \left(1 - \frac{1}{|I|}\right)^{-1} \leq \prod_{\substack{I \left| \frac{QQ'}{(Q,Q')^2} \right. \\ |I| > q^w}} \left(1 - \frac{1}{|I|}\right)^{-1}$$

$$\ll \prod_{\substack{I \left| \frac{Q}{(Q,Q')} \right. \\ |I| > q^w}} \left(1 - \frac{1}{|I|}\right)^{-1} \prod_{\substack{I \left| \frac{Q'}{(Q,Q')} \right. \\ |I| > q^w}} \left(1 - \frac{1}{|I|}\right)^{-1} \ll 1.$$

Case 2: $q^w > L(Q,Q') > 1$

$$P(Q,Q') = \frac{\left|\frac{QQ'}{(Q',Q)^2}\right|}{\varphi\left(\frac{QQ'}{(Q',Q)^2}\right)} \prod_{\substack{I \left| \frac{QQ'}{(Q,Q')^2} \right. \\ |I| \leq L(Q,Q')}} \left(1 - \frac{1}{|I|}\right)^{-1} \ll \frac{\left|\frac{Q}{(Q',Q)}\right|}{\varphi\left(\frac{Q}{(Q',Q)}\right)} \frac{\left|\frac{Q'}{(Q',Q)}\right|}{\varphi\left(\frac{Q'}{(Q',Q)}\right)}$$

$$\ll g\left(\frac{Q}{(Q,Q')}\right) g\left(\frac{Q'}{(Q,Q')}\right) \ll w^2.$$

This completes the proof. □

Lemma 3.6. *Let $L \in \mathbb{F}_q[X]$ and $t \in \mathbb{N}$ be given. Then,*

$$\sum_{\substack{|S| < |L| \\ g(S) \geq \ t}} 1 \ll \frac{|L|}{q^{2t}}.$$

Proof. Consider those polynomials with degree smaller than the degree of L which have at least t distinct irreducible normed factors with degree in $(t, 2t]$. Using the same argument as in (3.7), we obtain that the number of such polynomials is no more than

$$\frac{|L|}{t!}\left(\sum_{q^t<|I|\le q^{2t}}\frac{1}{|I|}\right)^t \ll \frac{|L|(\ln 2)^t}{t!} \ll \frac{|L|}{t!} \ll \frac{|L|}{q^{2t}}.$$

Note that

$$\sum_{\substack{q^t<|I|\le q^{2t}\\ \#I=t-1}}\frac{1}{|I|} \le t\cdot\frac{1}{q^t}. \tag{3.8}$$

Hence, for all large t, the left-hand side of (3.8) is smaller than $\dfrac{1}{2q}$.

Since,

$$\sum_{\substack{I|S\\ |I|>q^{g(S)}}}\frac{1}{|I|} = \frac{1}{q} + \mathcal{O}\left(\sum_{\substack{I|S\\ |I|=q^{g(S)}}}\frac{1}{|I|}\right) = \frac{1}{q} + \mathcal{O}\left(\frac{1}{t}\right)$$

and,

$$\sum_{\substack{I|S\\ |I|>q^{g(S)}}}\frac{1}{|I|} = \sum_{\substack{I|S\\ q^{g(S)}<|I|\le q^{2t}}}\frac{1}{|I|} + \sum_{\substack{I|S\\ |I|>q^{2t}}}\frac{1}{|I|}.$$

which yields that for a polynomial $|S|<|L|$ with $g(S)\ge t$, but not having t distinct irreducible normed factors with degree in $(t,2t]$, we obtain

$$\sum_{\substack{I|S\\ |I|>q^{2t}}}\frac{1}{|I|} > \frac{1}{2q}.$$

Therefore,

$$\sum_{\substack{|S|<|L|\\ g(S)\ge t}} 1 \ll \frac{|L|}{q^{2t}} + 2q\sum_{\substack{|S|<|L|\\ g(S)\ge t}}\ \sum_{\substack{I|S\\ |I|>q^{2t}}}\frac{1}{|I|}$$

$$\ll \frac{|L|}{q^{2t}} + 2q\sum_{\substack{I|S\\ |I|>q^{2t}}}\frac{1}{|I|}\cdot\frac{|L|}{|I|}$$

$$\ll \frac{|L|}{q^{2t}} + 2q|L| \sum_{\substack{I|S \\ |I|>q^{2t}}} \frac{1}{|I|^2}$$

$$\ll \frac{|L|}{q^{2t}} + 2q|L| \int_{q^{2t+1}}^{\infty} \frac{1}{x^2} dx \ll \frac{|L|}{q^{2t}}.$$

We obtain the desired result. □

Lemma 3.7. *For $D, Q \in \mathbb{F}_q[X]$ normed, Q is fixed and $t \in \mathbb{N}$, we have*

$$\sum_{\substack{D|Q \\ g\left(\frac{Q}{D}\right) \leq t}} \frac{1}{|D|} \ll \; t.$$

Proof. By Lemma 3.5 we have

$$\frac{\varphi\left(\frac{Q}{D}\right)}{\left|\frac{Q}{D}\right|} \cdot g\left(\frac{Q}{D}\right) \gg 1 \Rightarrow \frac{\varphi\left(\frac{Q}{D}\right)}{\left|\frac{Q}{D}\right|} \cdot t \gg 1 \quad \text{whenever} \quad g\left(\frac{Q}{D}\right) \leq t.$$

Hence,

$$\sum_{\substack{D|Q \\ g\left(\frac{Q}{D}\right) \leq t}} \frac{1}{|D|} \ll \sum_{D|Q} \frac{1}{|D|} \frac{\varphi\left(\frac{Q}{D}\right)}{\left|\frac{Q}{D}\right|} t = \frac{t}{|Q|} \sum_{D|Q} \varphi\left(\frac{Q}{D}\right) = t.$$

This gives the desired result. □

Lemma 3.8 (Analogue of Gallagher's Theorem for Formal Laurent Series)**.** *For any ψ,*

$$m\left(\bigcap_{n=1}^{\infty} \bigcup_{degQ \geq n} E_Q\right) = 0 \text{ or } 1.$$

This means that the Diophantine inequality (2.2) has infinitely many solutions $\langle P, Q \rangle$ for either almost all f or almost no f in $\mathbb{L}$.

Proof. See [11, Theorem 4]. □

3.2 Proof of Theorem 2.7

Proof. In the following assume that Q, Q' are normed. Our aim is to prove that, for all N,

$$\sum_{\deg Q,\ \deg Q' \le N} m(E_Q \cap E_{Q'}) \ll \left(\sum_{\deg Q' \le N} m(E_{Q'}) \right)^2 = \left(\sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \right)^2$$

and then to apply Lemma 3.8 and Remark 2.6 of Lemma 2.2. In view of Corollary 3.2, it suffices to demonstrate that

$$\sum^{*} \frac{\psi(Q)\psi(Q')\varphi(Q)\varphi(Q')w^2}{|Q||Q'|} \ll \left(\sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \right)^2$$

where $*$ represents the conditions $\deg Q,\ \deg Q' \le N, 1 < L(Q, Q') < q^w$.

There are four cases to consider. We write $T = (Q, Q')$ and $S = \dfrac{Q}{T}$ in the following.

Case (i): $w = g\left(\dfrac{Q}{T}\right)$, $L(Q, Q') = \dfrac{|Q|\psi(Q')}{|T|}$. The sum we are dealing with is thus

$$\le \sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \sum_{\substack{T|Q' \\ g\left(\frac{Q'}{T}\right) \le w}} \sum_{\substack{1<|S|<q^w \\ g(S)=w}} \psi(TS)w^2$$

$$\ll \sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \sum_{\substack{T|Q' \\ g\left(\frac{Q'}{T}\right) \le w}} \sum_{\substack{1<|S|<q^w \\ g(S)=w}} \frac{1}{|TS|} w^2$$

$$\text{Lemma 3.6} \ll \sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \sum_{\substack{T|Q' \\ g\left(\frac{Q'}{T}\right) \le w}} \frac{w^2}{|T|} \frac{1}{q^w}$$

$$\text{Lemma 3.7} \ll \frac{w^3}{q^w} \sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \ll \sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|}.$$

Here, we used

$$\sum_w \frac{w^3}{q^w} < \infty.$$

Case (ii): $w = g\left(\frac{Q'}{T}\right)$, $L(Q,Q') = \frac{|Q'|\psi(Q)}{|T|}$. The proof for this case is analogous to Case (i), with the roles of Q and Q' reversed.

Case (iii): $w = g\left(\frac{Q}{T}\right)$, $L(Q,Q') = \frac{|Q'|\psi(Q)}{|T|}$. For every integer h we put

$$B(h) = \left\{Q : q^h \le |Q|\psi(Q) < q^{h+1}\right\}.$$

Since $\psi(Q) = \mathcal{O}\left(\frac{1}{|Q|}\right)$, we have

$$\sum_{\substack{h \\ B(h)\neq\emptyset}} q^{h+1} \ll 1.$$

We commence the proof for this case by fixing w and h and restricting our attention to $Q \in B(h)$. We then have

$$\le \sum_{\deg Q' \le N} \psi(Q')\frac{\varphi(Q')}{|Q'|} \sum_{\substack{T|Q' \\ g\left(\frac{Q'}{T}\right)\le w}} \sum_{\substack{1<\psi(TS)|Q'|/|T|<q^w \\ TS\in B(h),\ g(S)=w}} \psi(TS)w^2$$

$$\ll \sum_{\deg Q' \le N} \psi(Q')\frac{\varphi(Q')}{|Q'|} \sum_{\substack{T|Q' \\ g\left(\frac{Q'}{T}\right)\le w}} \sum_{\substack{|Q'|q^h/|T|^2q^w<|S|<|Q'|q^{h+1}/|T|^2 \\ g(S)=w}} \frac{q^h w^2}{|TS|}$$

Lemma 3.6 $$\ll \frac{q^h w^2}{q^w} \sum_{\deg Q' \le N} \psi(Q')\frac{\varphi(Q')}{|Q'|} \sum_{\substack{T|Q' \\ g\left(\frac{Q'}{T}\right)\le w}} \frac{1}{|T|}$$

Lemma 3.7 $$\ll \frac{q^h w^2}{q^w} \sum_{\deg Q' \le N} \psi(Q')\frac{\varphi(Q')}{|Q'|} \ll \sum_{\deg Q' \le N} \psi(Q')\frac{\varphi(Q')}{|Q'|}.$$

Again, after summing over h and w we obtain the last estimate above.

Case (iv): $w = g\left(\frac{Q'}{T}\right)$, $L(Q,Q') = \frac{|Q|\psi(Q')}{|T|}$. This case is similar to Case (iii), with the roles of Q and Q' reversed.

Combining our estimates, we get

$$\sum_{\deg Q,\ \deg Q' \le N} m(E_Q \cap E_{Q'}) \ll \left(\sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \right)^2 + \sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|}$$

$$\ll \left(\sum_{\deg Q' \le N} \psi(Q') \frac{\varphi(Q')}{|Q'|} \right)^2 .$$

Since $\mathbb{F}_q[X]$ is countable, we can order the polynomials, and then apply Remark 2.6 of Lemma 2.2. Finally, by the zero-one law (Lemma 3.8) we obtain the desired result. □

Remark 3.4. *Note that the condition $\psi(Q) = \mathcal{O}\left(\frac{1}{|Q|}\right)$ in our theorem means that there exists $M > 0$ such that $\psi(Q) \le M \frac{1}{|Q|}$ for $|Q|$ large enough which also implies that $\psi(Q) \le 1$ for $|Q|$ large enough. Hence, the previous lemmas can be applied legally in the above proof.*

Chapter 4

Further Results

4.1 Preliminary Lemmas

Lemma 4.1. *The Duffin-Schaeffer conjecture is equivalent to the following statement: Let F_n be a sequence of distinct normed polynomials of $\mathbb{F}_q[X]$, and $\psi(F_n)$ a $\{q^{-n} : n \in \mathbb{Z}\} \cup \{0\}$-valued function. Then if*

$$\sum_{n=1}^{\infty} \psi(F_n) \frac{\varphi(F_n)}{|F_n|} = \infty,$$

there are infinitely many solutions $\langle P, F_n \rangle$ to

$$\left| f - \frac{P}{F_n} \right| < \frac{\psi(F_n)}{|F_n|}, \; P \in \mathbb{F}_q[X], \; (P, F_n) = 1$$

for almost all $f \in \mathbb{L}$.

Proof. Define

$$\bar{\psi}(Q) = \begin{cases} \psi(F_n), & \text{if } Q = F_n; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{Q:normed} \bar{\psi}(Q) \frac{\varphi(Q)}{|Q|} = \infty \Leftrightarrow \sum_{n=1}^{\infty} \psi(F_n) \frac{\varphi(F_n)}{|F_n|} = \infty,$$

and

$$\left|f - \frac{P}{F_n}\right| < \frac{\psi(F_n)}{|F_n|},\ P \in \mathbb{F}_q[X],\ (P, F_n) = 1 \Leftrightarrow \left|f - \frac{P}{Q}\right| < \frac{\bar{\psi}(Q)}{|Q|},\ P \in \mathbb{F}_q[X],\ (P, Q) = 1.$$

The above argument and the freedom of ψ in the Duffin-Schaeffer conjecture completes the proof. □

Lemma 4.2 (0-1 Law). *Let $\psi(F_n)$ be a $\{q^{-n} : n \in \mathbb{Z}\} \cup \{0\}$-valued function. Then, the inequality*

$$\left|f - \frac{P}{F_n}\right| < \frac{\psi(F_n)}{|F_n|},\ P \in \mathbb{F}_q[X],\ (P, F_n) = 1$$

has infinitely many solutions $\langle P, F_n \rangle$ for either almost all f or almost no f in $\mathbb{L}$.

Proof. As in Lemma 4.1, we define

$$\bar{\psi}(Q) = \begin{cases} \psi(F_n), & \text{if } Q = F_n; \\ 0, & \text{otherwise.} \end{cases}$$

Then the inequality is equivalent to

$$\left|f - \frac{P}{Q}\right| < \frac{\bar{\psi}(Q)}{|Q|},\ P \in \mathbb{F}_q[X],\ (P, Q) = 1.$$

Then Lemma 3.8 guarantees that it has infinitely many solutions for either almost all f or almost no f in $\mathbb{L}$. □

We begin with the simple observation that we can assume $\psi(F_n) \leq c$ for some suitable $c > 0$ small enough. To see this, note that at least one of

$$\sum_{\psi(F_n) > c} \psi(F_n) \frac{\varphi(F_n)}{|F_n|},\ \sum_{\psi(F_n) \leq c} \psi(F_n) \frac{\varphi(F_n)}{|F_n|}$$

must diverge. If the former sum diverges, we discard those F_n for which $\psi(F_n) \leq c$

and relabel the remaining sequence. Since $(ab)^{1/2} \le \max(a,b)$,

$$\begin{aligned} P(F_n, F_k) &= \sum_{\substack{I \mid \frac{F_n F_k}{(F_n,F_k)^2} \\ |I| > L(F_n,F_k)}} \frac{1}{|I|} \\ &< \sum_{\substack{I \mid \frac{F_n F_k}{(F_n,F_k)^2} \\ |I| > \ c\left(\frac{|F_n F_k|}{|(F_n,F_k)|^2}\right)^{1/2}}} \frac{1}{|I|} \\ &\le c \cdot \left(\frac{|(F_n,F_k)|^2}{|F_n F_k|}\right)^{1/2} \cdot \sum_{I \left| \frac{F_n F_k}{(F_n,F_k)^2}\right.} 1 \\ &\le 2c. \end{aligned}$$

We use the fact that for $D, Q \in \mathbb{F}_q[X]$ normed, $\sum_{D|Q} 1 \le 2\sqrt{|Q|}$ to obtain the last inequality. This property can be easily checked.

Therefore, $P(F_n, F_k)$ given in Lemma 3.4 is $\ll 1$. If the latter sum diverges, we can replace $\psi(F_n)$ by $\min(\psi(F_n), 1/2)$.

We shall use Lemma 3.4 with the simplification of replacing $L(F_n, F_k)$ by

$$E(n,k) = \left(\frac{|F_n F_k|}{|(F_n,F_k)^2|}\psi(F_n)\psi(F_k)\right)^{1/2}$$

which is no larger than $L(F_n, F_k)$. We write

$$L_1(n) = \ln\left(\frac{1}{\psi(F_n)}\right), \ L_2(n,k) = \ln(-2\ln(\psi(F_n)\psi(F_k)),$$

$$A(n,k) = \frac{|(F_n,F_k)^2|}{|F_n F_k|}\frac{L_1(n)L_1(k)}{\psi(F_n)\psi(F_k)}.$$

Note that $L_2(n,k) \ll L_1(n)L_1(k)$, since we can choose a suitable c. With this setting, we have

Lemma 4.3. *For* $Q \in \mathbb{F}_q[X]$ *be normed and* $\deg Q \geq 1$*,*

$$\prod_{\substack{I|Q \\ |I|>\ln|Q|}} \left(1 - \frac{1}{|I|}\right)^{-1} \ll 1.$$

Proof. As in the proof of Lemma 3.1, we know that

$$\begin{aligned}
\prod_{\substack{I|Q \\ |I|>\ln|Q|}} \left(1 - \frac{1}{|I|}\right)^{-1} &= \exp\left(- \sum_{\substack{I|Q \\ |I|>\ln|Q|}} \ln\left(1 - \frac{1}{|I|}\right)\right) \\
&\ll \exp\left(\sum_{\substack{I|Q \\ |I|>\ln|Q|}} \frac{1}{|I|}\right) \\
&\leq \frac{1}{\ln|Q|} \sum_{I|Q} 1 \ll 1.
\end{aligned}$$

Proposition 2.8 was used to obtain the last estimate. This completes the proof. □

The next lemma establishes the required property of $A(n,k)$.

Lemma 4.4. *If*

$$A(n,k) < 1, \tag{4.1}$$

then

$$m(E_{F_n} \cap E_{F_k}) \ll m(E_{F_n})m(E_{F_k}). \tag{4.2}$$

If (4.1) *fails, then*

$$m(E_{F_n} \cap E_{F_k}) \ll m(E_{F_n})m(E_{F_k})L_2(n,k) \tag{4.3}$$

$$\ll m(E_{F_n})m(E_{F_k})L_1(n)L_1(k) \tag{4.4}$$

Proof. If

$$E(n,k) \geq \ln\left(\frac{|F_nF_k|}{|(F_n,F_k)^2|}\right) \tag{4.5}$$

then (4.2) follows from Lemma 3.4 with Lemma 4.3. We may suppose that (4.5) fails. Since, for $s < \ln|F|$ with $\deg F \geq 1$,

$$\prod_{\substack{I|F\\|I|>s}}\left(1-\frac{1}{|I|}\right)^{-1} \ll \exp\left(\sum_{\substack{I|F\\s<|I|\leq \ln|F|}}\frac{1}{|I|}\right) \text{ by Lemma 4.3}$$

$$\leq \exp\left(\sum_{s<|I|\leq \ln|F|}\frac{1}{|I|}\right)$$

$$= \exp\left(\sum_{n=\max\left\{1,\lceil \log_q s\rceil\right\}}^{\lfloor \log_q \ln|F|\rfloor}\sum_{\deg I=n}\frac{1}{|I|}\right)$$

$$\ll \frac{\ln\ln|F|}{\max\{1,\ln s\}},$$

by choosing $F = \dfrac{F_nF_k}{(F_n,F_k)^2}$, $s = E(n,k)$, we have

$$P(F_n,F_k) \ll \frac{\ln\ln\left|\dfrac{F_nF_k}{(F_n,F_k)^2}\right|}{\max\{1,\ln E(n,k)\}}. \tag{4.6}$$

Now suppose that (4.1) fails. Then

$$\left|\frac{F_nF_k}{(F_n,F_k)^2}\right| \leq \frac{L_1(n)L_1(k)}{\psi(F_n)\psi(F_k)} \leq \psi(F_n)^{-2}\psi(F_k)^{-2}.$$

Hence, (4.6) gives $P(F_n,F_k) \ll L_2(n,k)$ and so (4.3) follows from Lemma 3.4. Of course, (4.4) is an immediate consequence of (4.3).

Now suppose that (4.1) holds. We must distinguish two cases:

(i) If

$$\left|\frac{F_nF_k}{(F_n,F_k)^2}\right| \geq \left(\psi(F_n)\psi(F_k)\right)^{-2}, \tag{4.7}$$

then

$$E(n,k)^2 \geq \left(\left|\frac{F_nF_k}{(F_n,F_k)^2}\right|\right)^{1/2} \geq \ln\left(\left|\frac{F_nF_k}{(F_n,F_k)^2}\right|\right).$$

It then follows from (4.6) that $P(F_n, F_k) \ll 1$, which gives (4.2).

(ii) If (4.7) does not hold, then

$$\begin{aligned} \ln\left(\left|\frac{F_nF_k}{(F_n,F_k)^2}\right|\right) &< -2\ln\left(\psi(F_n)\psi(F_k)\right) \\ &\ll L_1(n)L_1(k) \\ \text{by (4.1)} \quad &\leq \left|\frac{F_nF_k}{(F_n,F_k)^2}\right| \psi(F_n)\psi(F_k) \\ &= E(n,k)^2. \end{aligned}$$

Hence $P(F_n, F_k) \ll 1$ from (4.6), and this completes the proof. □

Lemma 4.5. *Let $c > 1$ be given. Then, for any non-negative function $f(x)$, we have*

$$\sum_{1\leq k<n\leq N} f(n)f(k)c^{k-n} \leq \sum_{1\leq n\leq N} f(n)^2(c-1)^{-1}.$$

Proof.

$$\begin{aligned} \sum_{1\leq k<n\leq N} f(n)f(k)c^{k-n} &\leq \frac{1}{2}\sum_{1\leq k<n\leq N}\left(f(k)^2+f(n)^2\right)c^{k-n} \\ &= \left(\sum_{k=1}^{N-1}\sum_{n=k+1}^{N} f(k)^2c^{k-n} + \sum_{n=2}^{N}\sum_{k=1}^{n-1} f(n)^2c^{k-n}\right) \\ &\leq \left(\sum_{k=1}^{N-1} f(k)^2\sum_{n=k+1}^{\infty} c^{k-n} + \sum_{n=2}^{N} f(n)^2\sum_{k=1}^{\infty} c^{-k}\right) \\ &\leq \sum_{n=1}^{N} f(n)^2\sum_{j=0}^{\infty} c^{-j} = \sum_{n=1}^{N} f(n)^2(c-1)^{-1}. \end{aligned}$$

This gives the desired result. □

4.2 Proof of Theorem 2.8

Proof. Without loss of generality, we assume that the degree of $\{F_n\}$ is nondecreasing. For $\sigma > 0$, define

$$\sum_{\sigma} = \sum_{1\le k<n\le N} A^{\sigma}(n,k)\frac{\varphi(F_n)\varphi(F_k)}{|F_nF_k|}\psi(F_n)\psi(F_k)L_2(n,k).$$

It follows from Lemma 4.4 that

$$\sum_{1\le k<n\le N} m\left(E_{F_n}\cap E_{F_k}\right) \ll \left(\sum_{n=1}^{N}\psi(F_n)\frac{\varphi(F_n)}{|F_n|}\right)^2 + \sum_{\sigma}.$$

To prove Theorem 2.8, we take $\sigma = \frac{1}{4}$, which gives

$$\sum_{\frac{1}{4}} \ll \sum_{1\le k<n\le N}\left(\frac{|(F_n,F_k)|^2}{|F_nF_k|}\right)^{1/4}\frac{\varphi(F_n)\varphi(F_k)}{|F_nF_k|}\left(L_1(n)L_1(k)\right)^{5/4}\left(\psi(F_n)\psi(F_k)\right)^{3/4} \quad (4.8)$$

$$\ll \sum_{1\le k<n\le N}\left(\frac{|(F_n,F_k)|^2}{|F_nF_k|}\right)^{1/4}\left(\psi(F_n)\psi(F_k)\right)^{1/2}\frac{\varphi(F_n)\varphi(F_k)}{|F_nF_k|}. \quad (4.9)$$

Note that

$$\frac{|(F_n,F_k)|^2}{|F_nF_k|} \le \frac{|F_k|^2}{|F_nF_k|} = \frac{|F_k|}{|F_n|} \le q^{-\lfloor (n-k+1)/c\rfloor} \ll q^{(k-n)/c}.$$

By Lemma 4.5, the sum (4.9) is

$$\ll \sum_{n=1}^{N}\psi(F_n)\left(\frac{\varphi(F_n)}{|F_n|}\right)^2 \ll \left(\sum_{n=1}^{N}\psi(F_n)\frac{\varphi(F_n)}{|F_n|}\right)^2.$$

Again, by Remark 2.6 of Lemma 2.2 and Lemma 4.2 we complete the proof. □

Chapter 5

Conclusion

We conclude this thesis with some remarks.

Theorem 2.2 of Inoue and Nakada was the first result in the field of formal Laurent series concerning the Duffin-Schaeffer conjecture without the monotonicity condition "$|Q|\psi(Q)$ non-increasing". In particular, most of the results in real number case also involve some monotonicity condition. Hence, from the result of Inoue and Nakada one might guess that the Duffin-Schaeffer conjecture is maybe easier to solve in the field of formal Laurent series. Therefore, we began our exploration by obtaining some analogous results of Vaaler (Theorem 2.7) and Harman (Theorem 2.8).

However, in the proof of the above results, no real simplifications arising from the more simpler metric structure of the formal Laurent series were achieved. Hence, it seems that our original guess was wrong and the Duffin-Schaeffer conjecture in the formal Laurent series field is as hard as the original one. Apart from the results of the thesis from Chapter 3 and Chapter 4, this observations is also one of the contributions of this work.

Overall, the quest for proving the conjecture unfortunately does not end with this thesis. It seems that there is still a long road in front of us.

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