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碩士論文

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Topics on Linear Parabolic Partial Differential Equations

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摘要

這篇論文中我們討論線性拋物偏微分方程。首先舉出關於此方程的一個實際例子。其次，我們介紹不同方法來解析此類型偏微分方程，包含一維度和多維度的問題。當我們用不同方法解決同一個方程時，將會證明這些方法所得之結果會是一致的。

還有，在數值分析方面，我們用有限差分法來求數值解。我們用數學軟體 Mathematica 6 來展示用有限差分法所得的解以及探討實際解之趨近值，最後比較兩者之的差距。

Topics on Linear Parabolic Partial Differential Equations

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Abstract

We study the linear parabolic partial differential equations. First, we give a practical example whose mathematical model is a parabolic PDE. Next, we apply some classical methods to solve the linear parabolic PDEs in one and higher dimension. For the same equation, we will identify the solutions if they are derived by different methods.

In numerical analysis, we use finite difference method to obtain the numerical solution of the parabolic PDE. We will use Mathematica 6 to find the numerical solutions, to approach the exact solutions, and to compare between those solutions.

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Contents

Ⅰ. Introduction

The parabolic equations are often the mathematical models in applied science. Before we introduce the models, we classify the partial differential equation first.

1-1. Classification of Linear Partial Differential Equations

A linear second-order partial differential equation has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, and F are function of (x, y). The equation is called homogeneous if $G = 0$. We classify the equations into three types, depending on the second-order coefficients A, B, and C:

If $AC - B^2 > 0$, the equation is *elliptic*.
If $AC - B^2 < 0$, the equation is *hyperbolic*.
If $AC - B^2 = 0$, the equation is *parabolic*.

The standard homogeneous equations are

1. $u_{xx} + u_{yy} = 0$ (the elliptic equation),

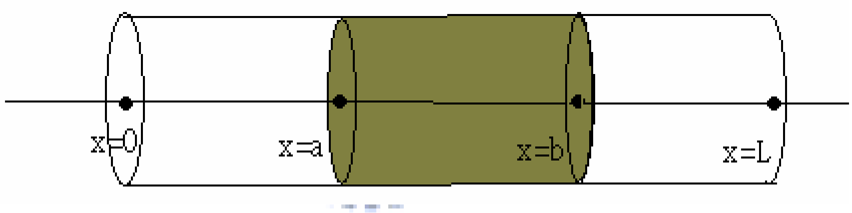
2. $u_{tt} - u_{xx} = 0$ (the hyperbolic equation),

3. $u_t - K u_{xx} = 0$ (the parabolic equation).

1-2. The Linear Parabolic PDE as model for the Practical Problem

This model is excerpted from Mathematical Models and Their Analysis [2].

Consider a straight rod of length *L*, mass density (per unit volume) ρ, and uniform cross-sectional area *A*. The cylindrical surface of the rod is insulated form heat flow. The rod is sufficiently slender so that the temperature distribution is uniform over a cross section at any point along the central axis of the rod. Let the central axis be positioned along the *x* extending from $x = 0$ to $x = L$. To discuss the temperature distribution $u(x,t)$ in the rod, we look at a segment of the rod between $x = a(>0)$ and $x = b(<L)$ at time *t*. Let c_0 be the *specific heat* of the rod material defined to be the amount of heat energy needed to raise the temperature of a unit mass of material by one degree Kelvin (1 K). The specific heat is a material property and can be measured experimentally.



The rate of change in the heat (thermal) energy $H(x,t)$ in this segment is given by

$$\frac{dH}{dt} = \frac{d}{dt}\int_a^b c_0 u(x,t)\rho A dx \,. \tag{1.2.1}$$

There is another way of calculating the same rate of change of heat energy. Let $\phi(x,t)$ be the amount of heat per unit time flowing from left to right across a unit area of the cross section at location *x* and at time *t*. Let $F(x,t)$ be the rate of heat generated by a unit mass of the rod within the insulated cylindrical surface by chemical, electrical, or other processes. In that case, we have

$$\frac{dH}{dt} = A[\phi(a,t) - \phi(b,t)] + \int_a^b F(x,t)\rho A dx . \tag{1.2.2}$$

The two expressions (1.2.1) and (1.2.2) for $\frac{dH}{dt}$ must be identical. The rate of increase in heat in the segment (calculated by way of the temperature of the segment) must be equal to the rate of heat gained (calculated by the net heat flow across the two cross sections of the segment plus the rate of heat generated inside the segment. We have

$$\frac{d}{dt}\int_a^b c_0 u(x,t)\rho A dx = A[\phi(a,t) - \phi(b,t)] + \int_a^b F(x,t)\rho A dx . \tag{1.2.3}$$

We suppose u and ϕ are both smooth functions and ρ is independent of t. In that case, the equation (1.2.3) becomes into

$$\int_a^b [c_0 u_t(x,t)\rho + \phi_x(x,t) - F(x,t)\rho] dx = 0 . \tag{1.2.4}$$

If the integrand is continuous in x and the segment (a,b) is arbitrary, we have the conclusion that

$$c_0 \rho u_t = -\phi_x + \rho F . \tag{1.2.5}$$

Equation (1.2.5) holds for all x in $(0,L)$ and all $t > 0$. Equation (1.2.5) is a equation for two unknowns: u and ϕ. And by the *Fourier law:*

$$\phi = -K_0 u_x . \tag{1.2.6}$$

where the coefficient of thermal conductivity K_0 is a measure of the ability of a material to diffuse or conduct heat. K_0 has to be determined experimentally and it varies from material to material. Substituting (1.2.6) into (1.2.5), we obtain a second order PDE for $u(x,t)$:

$$c_0 \rho u_t = (K_0 u_x)_x + \rho F . \tag{1.2.7}$$

If the rod is homogeneous in its ability to conduct heat, K_0 is a constant and (1.2.7) becomes to

$$c_0 \rho u_t = K_0 u_{xx} + \rho F \ .$$

Ⅱ. Methods of Solving the Linear Parabolic PDE

In this chapter, we will use some methods to solve one-dimensional heat equations and multiple dimensional heat equations. After the methods, we will apply these methods to the same problem.

2-1. <u>One-Dimensional Heat Equation</u>

In one-dimensional equations, we use methods to solve the heat equation with different initial conditions, boundary conditions and domains. If we solve the same equation by two or more different methods, we will identify the answers.

2-1-1 <u>The Method of Separation of Variables</u>

The method of *<u>separation of variables</u>* is a fundamental technique for obtaining solutions of homogeneous partial differential equation when the solutions in the form $u(x, y) = a(x)b(y)$.

We consider a homogenous one-dimensional heat equation problem

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < \pi \;,\; t > 0 \;, \qquad (2.1.1a)$$

$$u(x,0) = f(x) \qquad \text{for} \quad 0 \le x \le \pi \;, \qquad (2.1.1b)$$

$$u(0,t) = 0 \qquad \text{for} \quad t \ge 0 \;, \qquad (2.1.1c)$$

$$u(\pi,t) = 0 \qquad \text{for} \quad t \ge 0 \;. \qquad (2.1.1d)$$

Suppose that $u(x,t) = X(x)T(t)$, then the equation (2.1.1a) has the form

$$\frac{T'}{T} - K\frac{X''}{X} = 0$$

or

$$\frac{T'}{T} = K\frac{X''}{X} \quad . \qquad (2.1.2)$$

The left-hand side of the equation (2.1.2) depends only on *t*. But the right-hand side of the equation depends only on *x*. We say the equation (2.1.1a) is separable.

Now, we take the partial derivative with respect to *t* in two sides. We obtain

$$\frac{d}{dt}\left[\frac{T'}{T}\right] = 0 \quad . \qquad (2.1.3)$$

That means the both sides are constant. Then we have two ordinary differential equations.

$$\frac{T'}{T} = -\lambda \quad , \quad K\frac{X''}{X} = -\lambda \quad , \quad X(0) = X(\pi) = 0 \qquad (2.1.4)$$

where λ is a constant .

If $\lambda > 0$, the general solutions of these ordinary differential equations are

$$X(x) = A_1 \sin\sqrt{\frac{\lambda}{\kappa}}x + A_2 \cos\sqrt{\frac{\lambda}{\kappa}}x \,,$$

$$T(t) = A_3 e^{-\lambda t} \,.$$

If $\lambda = 0$, the general solutions of these ordinary differential equations are

$$X(x) = A_1 x + A_2\ ,$$

$$T(t) = A_3\ .$$

If $\lambda < 0$, the general solutions of these ordinary differential equations are

$$X(x) = A_1 e^{-\sqrt{\frac{\lambda}{\kappa}}x} + A_2 e^{\sqrt{\frac{\lambda}{\kappa}}x}\ ,$$

$$T(t) = A_3 e^{\lambda t}\ .$$

And we have the boundary condition

$$X(0) = X(1) = 0\ .$$

For $\lambda < 0$, $X(0) \neq 0$ and $X(1) \neq 0$ since A_1 、 $A_2 \neq 0$. For $\lambda = 0$, by the boundary condition, we have $X \equiv 0$. It is a trivial solution. For $\lambda > 0$, by the boundary condition, we obtain $A_2 = 0$ and

$$\sin\sqrt{\frac{\lambda}{\kappa}} = 0\ , \tag{2.1.5}$$

for X is not identically zero .That is

$$\lambda = n^2\kappa \quad , \quad n = 1,2,3\ldots\ldots\ .$$

The λ is called the eigenvalues of this problem (2.1.1). And the functions

$$X_n(x) = \sin nx \tag{2.1.6}$$

are the eigenfuctions corresponding to eigenvalues $\lambda = n^2\kappa$.

Now, the equation (2.1.4) becomes to

$$T' + n^2KT = 0 \quad , \tag{2.1.7}$$

then

$$T_n(t) = e^{-n^2Kt}\ . \tag{2.1.8}$$

By equation (2.1.6) and (2.1.8), we have the particular solutions of equation (2.1.1a)

$$u_n(x,t) = e^{-n^2 Kt}\sin nx\,. \tag{2.1.9}$$

The $u_n(x, y)$ satisfies the problem (2.1.1), and the any finite linear combination of $u_n(x, y)$ also satisfies the problem (2.1.1). We attempt to represnt the solution of (2.1.1) has the following form:

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin nx\,. \tag{2.1.10}$$

We need to determine the coefficients b_n by the initail condition

$$u(x,0) = f(x)\,.$$

We have

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx\,.$$

The b_n are the Fourier coefficients determined by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

The Fourier series of $f(x)$ converges to $f(x)$ uniformly if $f(x)$ is continuous for $-\pi \le x \le \pi$, $f(-\pi) = f(\pi)$, and $\int_{-\pi}^{\pi} f'^2\, dx$ is finite. Then we have the solution

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

If $f(x) \in L^1$, by the Weierstrass M-test the series converges absolutely and unifomly.

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin nx$$

In (2.1.1), we only talk about the one kind of boundary condition. We want to discuss some different boundary conditions. First we talk about

$$u(0,t)=u_x(\pi,t)=0\,. \tag{2.1.11}$$

Then the eigenvalue problem (2.1.4) becomes

$$\begin{aligned} &X''+\lambda X=0\ , \\ &X(0)=0\ , \\ &X'(\pi)=0\,. \end{aligned} \tag{2.1.12}$$

We can find the eigenvalues $\lambda=(n+\frac{1}{2})^2\kappa$, $n=0,1,2,\cdots$ with the corresponding eigenfuctions $X_n=\sin(n+\frac{1}{2})x$. Then we get the solution

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin(n+\frac{1}{2})x \tag{2.1.13}$$

of problem (2.1.1) with the boundary condition (2.1.11)

Then we look the another boundary condition

$$u_x(0,t)=u(\pi,t)=0\,, \tag{2.1.14}$$

then the eigenvalue problem (2.1.4) becomes

$$\begin{aligned} &X''+\lambda X=0\ , \\ &X'(0)=0\ , \\ &X(\pi)=0\ . \end{aligned} \tag{2.1.15}$$

We can find the eigenvalues $\lambda=(n+\frac{1}{2})^2\kappa$, $n=0,1,2,\cdots$ with the corresponding eigenfuctions $X_n=\cos(n+\frac{1}{2})x$ Then the solution

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\cos(n+\frac{1}{2})x \qquad (2.1.16)$$

of problem (2.1.1) with the boundary condition (2.1.14)

At last, we look the boundary condition

$$u_x(0,t)=u_x(\pi,t)=0\,. \qquad (2.1.17)$$

Then the eigenvalue problem (2.1.4) becomes

$$X''+\lambda X=0\ , \qquad (2.1.18)$$
$$X'(0)=0\ ,$$
$$X'(\pi)=0\ .$$

We can find the eigenvalues $\lambda=n^2\kappa$, $n=0,1,2,\cdots$ with the corresponding eigenfuctions $X_n=\cos nx$ Then the solution

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\cos nx \qquad (2.1.19)$$

of problem (2.1.1) with the boundary condition (2.1.17) .

Three things are needed to apply the method to the problem in two variables *x* and *y* [1]:

(1) The differential operator *L* must be separable. That is, there must be a function $\phi(x,y)$ such that the expression

$$\frac{L[X(x)Y(y)]}{\phi(x,y)X(x)Y(y)}=P(x)+Q(y)$$

where $P(x)$ is a function of *x* only and $Q(y)$ is a function of *y* only.

(2) All initial and boundary conditions must be on lines *x* = constant and *y* = constant.

(3) The boundary conditions at *x* = constant must involve no partial derivatives of *u* with respect to *y*, and their coefficients must be independent of *y*. Those at *y* =

constant must involve no partial derivatives of u with respect to x, and their coefficients must be independent of x.

The method of *separation of variables* also can be used in two dimensional problems as shown in the next chapter.

Here we give one example solving by *separation of variables*.

Example1.1 (Using *separation of variables*)

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$
$$u(0,t) = 0 , \quad t \ge 0 ,$$
$$u(\pi,t) = 0 , \quad t \ge 0 ,$$
$$u(x,0) = \sin^3 x , \quad 0 \le x \le \pi .$$

Solution

By the equation (2.1.10), we obtain

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \kappa t} \sin nx .$$

Now, we have to determine b_n

$$\sin^3 x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx .$$

We have b_n

$$b_1 = \frac{3}{4}, \quad b_3 = -\frac{1}{4}, \text{ other } \ b_n = 0 .$$

Then the solution is

$$u(x,t) = \frac{3}{4} e^{-kt} \sin x - \frac{1}{4} e^{-9kt} \sin 3x .$$

2-1-2 Finite Fourier Transform with Nonhomogeneous Problem

In the 1-1, the heat equations are homogeneous. Here we use *finite Fourier transform* to solve the nonhomogeneous problems.

Now, we look the nonhomogeneous problem corresponding to (2.1.1),

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{for} \quad 0 < x < \pi,\ t > 0, \tag{2.2.1a}$$

$$u(x,0) = f(x) \qquad \text{for} \quad 0 \le x \le \pi, \tag{2.2.1b}$$

$$u(0,t) = 0, \quad t \ge 0 \tag{2.2.1c}$$

$$u(\pi,t) = 0, \quad t \ge 0. \tag{2.2.1d}$$

First, we expand the solution into a Fourier sine series for fixed t

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\sin nx \tag{2.2.2}$$

due to the boundary conditions $u(0,t)=0$ and $u(1,t)=0$. For other boundary conditions we have different Fourier series forms:

$$u(0,t) = u_x(\pi,t) = 0 \quad \Rightarrow \quad u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\sin(n+\frac{1}{2})x,$$

$$u_x(0,t) = u(\pi,t) = 0 \quad \Rightarrow \quad u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\cos(n+\frac{1}{2})x,$$

$$u_x(0,t) = u_x(\pi,t) = 0 \quad \Rightarrow \quad u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\cos nx.$$

In (2.2.2), the $\{b_n(t)\}$ are the sine coefficients

$$b_n(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\sin nx dx. \tag{2.2.3}$$

These integrals determine the $u(x,t)$ uniquely called the *finite sine transform* of $u(x,t)$.

If $\dfrac{\partial^2 u}{\partial x^2}$ is integrable, its finite sine transform is

$$\frac{2}{\pi}\int_0^{\pi}\frac{\partial^2 u}{\partial x^2}\sin nx dx=\frac{2}{\pi}\left[\frac{\partial u}{\partial x}\sin nx-un\cos nx\right]_0^{\pi}-n^2\frac{2}{\pi}\int_0^{\pi}u\sin nx dx$$

$$=-n^2 b_n(t)\,. \qquad (2.2.4)$$

Let

$$B_n(t)=\frac{2}{\pi}\int_0^{\pi}F(x,t)\sin nx dx\,.$$

Then the equation (2.2.1a) becomes

$$b_n^{'}(t)+n^2Kb_n(t)=B_n(t)\,. \qquad (2.2.5)$$

Multiply a integrating factor $p(t)$ defined as $p(t)=e^{n^2Kt}$ on (2.2.5)

$$e^{n^2Kt}b_n^{'}(t)+e^{n^2Kt}n^2Kb_n(t)=e^{n^2Kt}B_n(t)\,.$$

Solving the ODE. (2.2.7), we obtain

$$b_n(t)=\int_0^t e^{-n^2K(t-\tau)}B_n(\tau)d\tau+e^{-n^2Kt}C_0 \qquad (2.2.6)$$

where the C_0 is determined by the initial condition

$$C_0=b_n(0)=\frac{2}{\pi}\int_0^{\pi}u(x,0)\sin nx dx\,.$$

so

$$C_0=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx dx\,.$$

Substituting the $b_n(t)$ to the equation (2.2.2)

$$u(x,t) \sim \sum_{n=1}^{\infty}\left(\int_0^t e^{-n^2K(t-\tau)}B_n(\tau)d\tau + e^{-n^2Kt}\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx\right)\sin nx \,. \quad (2.2.7)$$

There are still two things to be verified:

1. The sum converges in entire domain , that means $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx$.
2. The sum can be term by term differentiated , that means

   $u_{xx}(x,t) = \sum (b_n(t)\sin nx)_{xx}$ and $u_t(x,t) = \sum (b_n(t)\sin nx)_t$.

First we show 1. By the Schwarz's Inequality for integrals we have

$$|b_n(t)|^2 \le \left(\int_0^t e^{-2n^2(t-\tau)}d\tau\right)\left(\int_0^t B_n^{\,2}(\tau)d\tau\right) = \frac{1}{2n^2}\int_0^t B_n^{\,2}d\tau \,.$$

By Schwarz's inequality for sums and Parseval's equation

$$\left|\sum_{n=M+1}^{N} b_n(t)\sin nx\right|^2 \le \frac{1}{2}\left(\sum_{n=M+1}^{N}\frac{1}{n^2}\right)\left(\int_0^t \sum_{n=M+1}^{N} B_n^{\,2}d\tau\right)$$
$$\le \frac{1}{\pi}\sum_{n=M+1}^{N}\frac{1}{n^2}\int_0^t\int_0^{\pi} F(x,\tau)^2 dx d\tau \,.$$

For some $t_0 > 0$, $\int_0^t\int_0^{\pi} F(x,\tau)^2 dx d\tau$ converges , and by M-test $\sum_{n=1}^{\infty} b_n(t)\sin nx$ converges uniformly in domain $[0,\pi]\times[0,\infty]$. That means it is independent of (x,t) . So (2.2.2) becomes

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx \,.$$

Next, we show 2. Here, $F(x,t)$, F , F_X are continuous , $F(0,t) = F(\pi,t) = 0$, and $\int_0^{\pi}\frac{\partial^2 u}{\partial x^2}dx$ uniformly bounded in t . Then the sum can be term by term differentiated.

<u>Example1.2</u> (Using *finite Fourier transform*)

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = t\sin^3 x \qquad \text{for} \qquad 0 < x < \pi \ , \ t > 0,$$
$$u(0,t) = 0, \qquad t \ge 0,$$
$$u(\pi,t) = 0, \qquad t \ge 0,$$
$$u(x,0) = 0, \quad 0 \le x \le \pi.$$

Solution

We can find

$$B_n(t) = \frac{2}{\pi}\int_0^\pi t\sin^3 x\sin nx dx.$$

$$B_1(t) = \frac{3}{4} \ , \ B_3(t) = -\frac{1}{4}t \ , \ B_n(t) = 0.$$

By the equation (2.2.7)

$$u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t e^{-n^2K(t-\tau)}B_n(\tau)d\tau\right)\sin nx$$
$$= \frac{3}{4}(t-1+e^{-Kt})\sin x - \frac{1}{36}(t-\frac{1}{9}+\frac{1}{9}e^{-9Kt})\sin 3x.$$

2-1-3 Fourier Transform

A 2π - periodic function $f(x)$ in $[-\pi,\pi]$ can be expanded in Fourier series As

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx), \tag{2.3.1}$$

where

$$a_n = \frac{<f(x),\cos nx>}{<\cos nx,\cos nx>} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,$$
$$b_n = \frac{<f(x),\sin nx>}{<\sin nx,\sin nx>} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

Using

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}) \ , \ \cos nx = \frac{1}{2}(e^{inx} + e^{-inx}).$$

We define

$$C_n = \begin{cases} \frac{1}{2}(a_n + ib_n) & ,n \ge 0 \\ \frac{1}{2}(a_{-n} - ib_{-n}) & ,n < 0 \end{cases}.$$

Then

$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-inx}, \tag{2.3.2}$$

where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx.$$

If $f(x)$ is continuous and piecewise continuously differentiable on $[-\pi,\pi]$, and if $f(-\pi) = f(\pi)$, then

$$f'(x) \sim \sum d_n e^{-inx}, \tag{2.3.3}$$

where

$$\begin{aligned} d_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{inx}dx \\ &= f(x)e^{inx}\Big]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(in)e^{inx}dx \\ &= -in\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\ &= -inC_n. \end{aligned}$$

Then the (2.3.3) becomes

$$f'(x) \sim \sum_{n=-\infty}^{\infty} -inC_n e^{-inx}.$$

If $f(x)$ is twice continuously differentiable, then

$$\int_{-\pi}^{\pi} f''(x)e^{inx}dx = (-in)^2 C_n = -n^2 C_n.$$

We obtain

$$f''(x) \sim \sum_{n=-\infty}^{\infty} (-n^2) C_n e^{-inx} .$$

Next, we consider the $f(x)$ in $[-L, L]$ and $g(\bar{x})$ in $[-\pi, \pi]$ where L is a constant. We let $\bar{x} = \frac{\pi}{L} x$.

$$[-L, L] \underset{1-1}{\rightarrow} [-\pi, \pi] \overset{g(\bar{x})}{\rightarrow} \boldsymbol{R} \text{ or } \boldsymbol{C}$$

$$x \quad \rightarrow \quad \bar{x}$$

Then we have

$$f(x) = g(\bar{x}) \sim \sum_{-\infty}^{\infty} C_n e^{-in\bar{x}} ,$$

where

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\bar{x}) e^{in\bar{x}} d\bar{x} = \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{i\frac{n\pi}{L}x} \left(\frac{\pi}{L}\right) dx$$

$$= \frac{1}{2L} \int_{-L}^{L} f(x) e^{i\frac{n\pi}{L}x} dx . \qquad (2.3.4)$$

So, we have

$$f(x) \sim \sum_{-\infty}^{\infty} C_n e^{-i\frac{n\pi}{L}x} . \qquad (2.3.5)$$

Next, we consider the $f(x)$ in $(-\infty, \infty)$. We can determine the $f(x)$ on any subinterval $(-L, L)$ in terms of the coefficients C_n in (2.3.4).We extend the $f(x)$ on the whole interval $(-\infty, \infty)$ by taking limit on $L \rightarrow \infty$.
The (2.3.4) can be written as

$$2LC_n = \int_{-L}^{L} f(x) e^{i\frac{n\pi}{L}x} dx .$$

For any fixed n

$$\lim_{L\to\infty} 2LC_n = \lim_{L\to\infty}\int_{-L}^{L} f(x)e^{i\frac{n\pi}{L}x}dx = \int_{-\infty}^{\infty} f(x)dx\,.$$

The series $\{\frac{n\pi}{L}\}$ dense on ***R*** if L is large enough where $n = 0, \pm 1, \pm 2\cdots$. So we may replace discontinuous $\{\frac{n\pi}{L}\}$ by continuous variable $w \in R$

$$\frac{n\pi}{L} \sim w\,,$$

$$C_n \sim C_{\frac{wL}{\pi}}\,.$$

And we define

$$C_{\frac{wL}{\pi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{iwx}dx\,.$$

For each w we have an unique $C_{\frac{wL}{\pi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{iwx}dx$. Suppose that $f(x)$ is absolutely integrable, then the integral

$$\int_{-\infty}^{\infty} |\, f(x)\, |dx$$

converges, and

$$|\,C_n\,| = \frac{1}{2L}\,|\int_{-L}^{L} f(x)e^{i\frac{n\pi}{L}x}dx\,|$$

$$\leq \frac{1}{2L}\int_{-\infty}^{\infty} |\, f(x)\,|\, dx\,.$$

Now we define

$$\hat{f}(w) = \lim_{L\to\infty} 2LC_{\frac{wL}{\pi}} = \lim_{L\to\infty}\int_{-L}^{L} f(x)e^{iwx}dx$$

$$= \int_{-\infty}^{\infty} f(x)e^{iwx}dx\,. \qquad (2.3.6)$$

For $f \in L^1(-\infty,\infty)$, $\hat{f}(w)$ exists for $\forall w \in R$. The integral (2.3.6) is called the *Fourier Transform* of $f(x)$, and denoted as $\Im[f]$.

Here, we want to see how to impose conditions for $f(x)$, $x \in (-\infty,\infty)$ such that $\hat{f}(w)$ has same good property as the Fourier coefficients. The property we hope is

$$f'(x) \sim \Im[f'(x)] = (-iw)\Im[f(x)]. \qquad (2.3.7)$$

Suppose that $f(x)$ is continuous and piecewise continuously differentiable, then

$$\int_{-\infty}^{\infty} f(x)e^{iwx}dx$$

converges for each *w*, and

$$|f(x)e^{iwx}| = f(x) \to 0 \qquad \text{as} \qquad x \to \pm\infty,$$

$$\lim_{x\to\pm\infty} f(x) = 0.$$

By definition

$$\begin{aligned}\Im[f'(x)] &= \int_{-\infty}^{\infty} f'(x)e^{iwx}dx \\ &= f(x)e^{iwx}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(iw)e^{iwx}dx \\ &= (-iw)\Im[f(x)]. \qquad (2.3.8)\end{aligned}$$

Hence $f(x)$ has the *Fourier Transform* $-iw\Im[f]$.

There are some other operational formulas for *Fourier Transform* [1]:

1. $\Im[ixf(x)] = \dfrac{d}{dw}\Im[f]$.

2. $\Im[f(ax-b)] = \dfrac{1}{|a|}e^{iw\frac{b}{a}}\hat{f}(\dfrac{w}{a})$. (shift formula)

3. $\Im[e^{icx} f] = \hat{f}(w+c)$.

4. $\Im[\cos cx f(x)] = \frac{1}{2}[\hat{f}(w+c) + \hat{f}(w-c)]$.

5. $\Im[\sin cx f(x)] = \frac{1}{2i}[\hat{f}(w+c) - \hat{f}(w-c)]$.

Inversion formula is [1]

$$f(x) = \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(w) e^{-iwx} dw .$$

$\hat{f}(w)$ determines $f(x)$ uniquely.

The convolution theorem for Fourier transform is useful tool.

Convolution Theorem

If $f(x)$ and $h(x)$ are both absolutely integrable and square integrable, and if $\hat{f}(w)$ and $\hat{h}(w)$ are their *Fourier transforms*, then the product $\hat{f}(w)\hat{h}(w)$ is the *Fourier transform* of the convolution product $f * h$.[1]

$$\Im[f * h(x)](w) = \hat{f}(w)\hat{h}(w) .$$

The following, we solve the infinite-slab heat flow problem.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad -\infty < x < \infty \;,\; t > 0,$$

$$u(x,0) = f(x) \qquad \text{for} \quad -\infty < x < \infty ,$$

$$u(x,t) \quad \text{bounded.}$$

If $f(x)$ is absolutely integrable. Making the hypothesis that u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$ are continuous in *x* and *t*, and absolutely integrable in *x*, uniformly in *t*. Then u and

$$\frac{\partial u}{\partial t} \to 0 \quad \text{as} \quad x \to \infty .$$

By (2.3.6) and (2.3.8), we obtain

$$\hat{u}_t(w) = \int_{-\infty}^{\infty} u_t(x,t) e^{iwx} dx = \frac{\partial \hat{u}}{\partial t}(w,t),$$

$$\hat{u}_{xx}(w) = \int_{-\infty}^{\infty} u_{xx}(x,t) e^{iwx} dx = -w^2 \hat{u}(x,t).$$

Taking the *Fourier transform* with respect to x to the problem. We have an ODE. in t

$$\hat{u}_t + w^2 \hat{u} = 0,$$

$$\hat{u}(w,0) = \hat{f}(w).$$

The solution of this ODE is

$$\hat{u} = \hat{f} e^{-w^2 t}.$$

Take the inversion formula

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{-w^2 t} e^{-iwx} dw. \qquad (2.3.9)$$

$e^{-w^2 t}$ is the *Fourier transform* of

$$\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t},$$

which is absolutely integrable and bounded for $t > 0$. By the convolution theorem of *Fourier transform*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/4t} dy.$$

<u>Example1.3</u> (Using *Fourier transform*)

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0 \qquad \text{for} \quad -\infty<x<\infty\,, \quad t>0\,,$$
$$u(x,0)=e^{-x^2} \qquad \text{for} \quad -\infty<x<\infty\,,$$
$$u(x,t) \ \text{ bounded.}$$

<u>Solution</u>

We take the *Fourier transform* with respect to x to e^{-x^2}

$$\int_{-\infty}^{\infty} e^{-x^2} e^{iwx}\,dx=\sqrt{\pi}\,e^{-w^2/4}\,.$$

By (2.3.9) we have

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\pi}\,e^{-w^2/4}e^{-w^2 t}e^{-iwx}\,dw$$
$$=(1+4t)^{-\frac{1}{2}}e^{-x/1+4t}\,.$$

2-1-4 <u>Sine and Cosine Transform</u>

If $f(x)$ is given for $0\le x<\infty$. The *Sine transform* of $f(x)$ defined as

$$\Im_s[f]\equiv\int_0^{\infty} f(x)\sin wx\,dx\,.$$

We extend the $f(x)$ to the domain $-\infty<x<\infty$ as an odd function, i.e. $f(-x)=-f(x)$. The <u>*Fourier transform*</u> of $f(x)$ can be written as

$$\hat{f}(w)=\int_{-\infty}^{\infty} f(x)e^{iwx}dx=\int_{-\infty}^{\infty} f(x)(\cos wx+i\sin wx)dx$$
$$=i\int_{-\infty}^{\infty} f(x)\sin wx\,dx$$
$$=2i\int_0^{\infty} f(x)\sin wx\,dx$$
$$=2i\Im_s[f]\,.$$

By the inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwx} 2i\Im_s[f]dw. \qquad (2.4.1)$$

Since the *Sine transform* is an odd function of w. The (2.4.1) becomes to

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{\infty} 4\sin wx \Im_s[f]dw \\ &= \frac{2}{\pi}\int_0^{\infty} \sin wx \Im_s[f]dw \\ &= \frac{2}{\pi}\Im_s[\Im_s[f]]. \end{aligned}$$

We define the *Cosine transform* as

$$\Im_c[f] \equiv \int_0^{\infty} f(x)\cos wxdx.$$

We extend the $f(x)$ to the domain $-\infty < x < \infty$ as an even function, i.e. $f(-x) = f(x)$. Similarly, the *Fourier transform* of $f(x)$ becomes to

$$\hat{f}(w) = 2\Im_c[f].$$

By the inversion formula

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^{\infty} \cos wx \Im_c[f]dw \\ &= \frac{2}{\pi}\Im_c[\Im_c[f]]. \end{aligned}$$

Sine and *Cosine transform* are useful in solving problems with the boundary condition only at $x = 0$. We note that

$$\Im_s[f'] = \int_0^{\infty} f'(x)\sin wxdx$$

$$= f(x)\sin wx]_0^{\infty} - w\int_0^{\infty} f(x)\cos wxdx$$

$$= -w\Im_c[f],$$

$$\Im_s[f''] = f(0)w - w^2\Im_s[f], \tag{2.4.2a}$$

$$\Im_c[f''] = -f'(0) - w^2\Im_c[f], \tag{2.4.2b}$$

provides $f(x)$ and $f'(x) \to 0$ as $x \to \infty$. By (2.4.2a) and (2.4.2b)**,** the *sine transform* is particularly useful when $f(0)$ is given. The *cosine transform* is useful when $f'(0)$ is given.

The following is a heat conduction problem in a semi-infinite slab and use the *sine transform* to solve it.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < \infty \;,\; t > 0,$$
$$u(0,t) = 0, \quad t \geq 0,$$
$$u(x,0) = f(x), \quad 0 \leq x < \infty,$$
$$u(x,t) \quad \text{bounded.} \tag{2.4.3}$$

Suppose that $f(x)$ is absolutely integrable, and that u, $\partial u/\partial t$, $\partial u/\partial x$, and $\partial^2 u/\partial x^2$ are continuous and absolutely integrable in *x* for any fixed *t*. Taking the *sine transform* with respect to *x* since the $u(0,t) = 0$ is given and put $U(w,t) = \Im_s[u]$. The problem (2.4.3) becomes to

$$\frac{\partial U}{\partial t} + w^2 U = 0,$$
$$U(w,0) = \Im_s[f].$$

The solution

$$U(w,t) = \Im_s[f](w)e^{-w^2 t}.$$

By the inversion formula (2.4.1)

$$u(x,t)=\frac{2}{\pi}\int_0^\infty \Im_s[f](w)e^{-w^2t}\sin wxdx. \qquad (2.4.4)$$

<u>Example1.4</u> (Using *sine or cosine transform*)

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0 \qquad \text{for} \qquad 0<x<\infty, \quad t>0,$$

$u(0,t)=0, \quad t\geq 0,$

$u(x,0)=e^{-x}, \quad t\geq 0,$

$u(x,t)$ bounded.

<u>Solution</u>

The initial condition $u(0,t)=0$ is given. Using the *<u>sine transform</u>* to solve this problem. Putting $\Im_s[f]=U(w,t)$,

$$\frac{\partial U}{\partial t}+w^2U=0,$$

$$U(w,0)=\int_0^\infty e^{-x}\sin wxdx=\frac{w}{1+w^2}.$$

The solution of this ODE

$$U(w,t)=\frac{w}{1+w^2}e^{-w^2t}.$$

By the inversion formula

$$u(x,t)=\frac{2}{\pi}\int_0^\infty \frac{w}{1+w^2}e^{-w^2t}\sin wxdw.$$

2-1-5 <u>Laplace Transform</u>

We consider the function $f(x)$ which becomes to zero for any negative value of *x*. That means

$$f(x)=0 \qquad \text{for} \qquad x<0.$$

If $e^{-s_1 x} f(x)$ is absolutely integrable, so is $e^{-sx} f(x)$ for $s \ge s_1$. It follows that the *Fourier transform* $\hat{f}(\xi)$ of such a function (if it ever exists) is analytic in a half-plane $\operatorname{Im}\xi > s_1$.

We define the *Laplace transform*

$$L[f](s) \equiv \Im[f](is)$$

or

$$L[f] \equiv \int_0^\infty e^{-sx} f(x)dx.$$

By the integration by parts, we have some properties of *Laplace transform*:

$$L[f'] = sL[f] - f(0),$$

$$L[f''] = s^2 L[f] - sf(0) - f'(0)$$

If $f(x)$ and $g(x)$ vanish when $x < 0$,

$$f * g(x) = \int_0^\infty f(y)g(x-y)dy \quad \text{for} \quad x \ge 0,$$

and

$$f * g(x) = 0 \qquad \text{for} \qquad x \le 0.$$

The convolution theorem for *Laplace transform* follows from that for the *Fourier transform*.

$$L[f * g] = L[f] \cdot L[g].$$

By inversion theorem for the *Fourier transform*, we have the inversion formula for *Laplace transform* [1].

$$f(x) = \frac{1}{2\pi i} \lim_{L \to 0} \int_{-iL+s}^{iL+s} L[f](\sigma) e^{\sigma x} d\sigma \qquad (2.5.1)$$

where $s > s_1$. $L[f](\sigma)$ is analytic for $\operatorname{Re}\sigma \ge s_1$, and the path is vertical. We call $f(x)$ is the *inverse Laplace transform* of $L[f](s)$.

Now, we consider the problem of heat conduction in an infinite slab.

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0 \qquad \text{for } -\infty<x<\infty \ , \ t>0, \tag{2.5.2a}$$

$$u(x,0)=f(x), \quad -\infty<x<\infty, \tag{2.5.2b}$$

$$u(x,t) \text{ bounded.} \tag{2.5.2c}$$

Taking the *Laplace transform* with respect to *t* and let

$$U(x,s)=\int_0^\infty e^{-ts}u(x,t)dt \, .$$

We have

$$\int_0^\infty e^{-ts}\frac{\partial u}{\partial t}dt = sU(x,s)-u(x,0)=sU-f(x) \, .$$

Suppose that $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ are continuous and bounded, we have

$$\int_0^\infty e^{-ts}\frac{\partial^2 u}{\partial x^2}dt=\frac{\partial^2 U}{\partial x^2} \, .$$

The equation (2.5.2a) becomes to an ODE with fixed *t*.

$$sU-f(x)-\frac{\partial^2 U}{\partial x^2}=0 \, . \tag{2.5.3}$$

Solving it by means of the *Fourier transform*. Letting

$$\hat{U}(w,s)=\int_{-\infty}^\infty U(x,s)e^{iwx}dx \, ,$$

$$\Im[U_{xx}](w,s)=\int_{-\infty}^\infty U_{xx}(x,s)e^{iwx}dx$$

$$=-w^2\hat{U}(w,s) \, .$$

The ODE (2.5.3) becomes to

$$s\hat{U}(w,s) - \hat{f}(w) + w^2\hat{U}(w,s) = 0 \quad \text{for} \quad -\infty < x < \infty\,, \quad s > 0\,. \qquad (2.5.4)$$

The solution of (2.5.4) is

$$\hat{U}(w,s) = \frac{\hat{f}(w)}{s+w^2}\,.$$

The *inverse Fourier transform* of $\frac{1}{s+w^2}$ is $\frac{1}{2\sqrt{s}}e^{-\sqrt{s}|x|}$. By the convolution theorem of *Fourier transform*, the solution of (2.5.3) is

$$U(x,s) = \frac{1}{2\sqrt{s}}\int_{-\infty}^{\infty} e^{-\sqrt{s}|x-y|} f(y)dy\,.$$

Using the *inverse Laplace transform* (2.5.1), we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\lim_{L\to 0}\int_{s-iL}^{s+iL}\Big(\frac{1}{2\sqrt{\sigma}}\int_{-\infty}^{\infty} e^{-\sqrt{\sigma}|x-y|} f(y)dy\Big)e^{\sigma t}d\sigma \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{2\pi i}\lim_{L\to\infty}\int_{s-iL}^{s+iL}\frac{1}{\sqrt{\sigma}}e^{\sqrt{\sigma}|x-y|}e^{\sigma t}d\sigma\,)\, f(y)dy\,. \qquad (2.5.5) \end{aligned}$$

Now, we need the *inverse Laplace transform* of $\frac{1}{\sqrt{s}}e^{-\sqrt{s}|x-y|}$. The function $g(\sigma) = \frac{1}{\sqrt{\sigma}}e^{-\sqrt{\sigma}|x-y|}$ is a multiple-valued function. We choose a particular branch which is cut along the negative real axis: $-\pi < \arg\sigma < \pi$. To solve

$$L^{-1}\Big[\frac{e^{-\sqrt{s}|x-y|}}{\sqrt{s}}\Big] = \frac{1}{2\pi i}\lim_{L\to\infty}\int_{s-iL}^{s+iL} e^{\sigma t} g(\sigma)d\sigma \quad , \quad s > 0\,.$$

We apply Cauchy's theorem to the integral of $e^{\sigma t}g(\sigma)$ over the contour C which is shown in Figure (2.5.1)

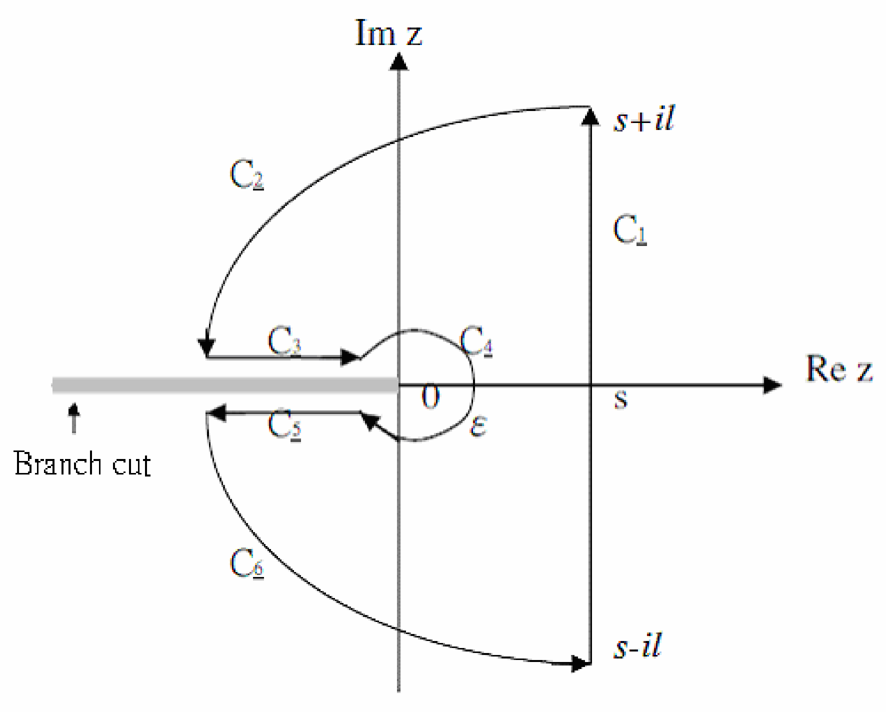


Figure (2.5.1)   Contour C.

We know that $g(\sigma)$ is analytic inside this contour. We obtain

$$\oint_c e^{\sigma t} g(\sigma) d\sigma = \int_{s-iL}^{s+iL} e^{\sigma t} g(\sigma) d\sigma + \sum_{n=2}^{6} \oint_{C_n} e^{\sigma t} g(\sigma) d\sigma = 0,$$

$$\int_{s-iL}^{s+iL} e^{\sigma t} g(\sigma) d\sigma = -\sum_{n=2}^{6} \oint_{C_n} e^{\sigma t} g(\sigma) d\sigma. \qquad (2.5.6)$$

Contour C_2

$$\sigma = s + Le^{i\theta} \quad \text{from} \quad \frac{\pi}{2} \quad \text{to} \quad \pi,$$

$$d\sigma = iLe^{i\theta} d\theta.$$

Then

$$\oint_{C_2} e^{\sigma t} g(\sigma) d\sigma = \int_{\frac{\pi}{2}}^{\pi} e^{(s+Le^{i\theta})t} g(s+Le^{i\theta}) iLe^{i\theta} d\theta.$$

Since

$$|g(s+Le^{i\theta})|=|\frac{1}{s+Le^{i\theta}}|\le\frac{1}{\sqrt{s-L}}\to 0 \quad \text{as} \quad L\to\infty.$$

By the Jordan's lemma, the integral over C_2 becomes zero.

<u>Contour</u> C_6

$$\sigma = s+Le^{i\theta} \quad \text{from} \quad \pi \text{ to } \frac{3\pi}{2},$$

$$d\sigma = iLe^{i\theta}d\theta.$$

It's similarly to contour C_2. We have that the integral over C_6 becomes zero.

<u>Contour</u> C_4

$$\sigma = \varepsilon e^{i\theta} \quad \text{from} \quad \pi \text{ to } -\pi,$$

$$d\sigma = i\varepsilon e^{i\theta}d\theta,$$

$$\oint_{C_4} e^{\sigma t}g(\sigma)d\sigma = \int_{\pi}^{-\pi} e^{\varepsilon e^{i\theta}} g(\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta. \tag{2.5.7}$$

We know

$$|g(\varepsilon e^{i\theta})|=|\frac{1}{\sqrt{\varepsilon e^{i\theta}}}|\le\frac{1}{\sqrt{\varepsilon}}.$$

and

$$|e^{\varepsilon e^{i\theta}t}|=e^{\varepsilon t}.$$

(2.5.7) becomes to

$$|\oint_{C_4} e^{\sigma t}g(\sigma)d\sigma|\le 2\pi\varepsilon e^{\varepsilon t}\frac{1}{\sqrt{\varepsilon}}. \tag{2.5.8}$$

We let $\varepsilon\to 0$, then the integral in **(1.5.8)** approaches zero.

<u>Contour</u> C_3 and C_5

We put $\sigma = -\gamma^2$, then

$$g(\sigma)=\frac{1}{i\gamma}e^{-i\gamma|x-y|} \qquad \text{for} \qquad \arg\sigma=\pi\ ,$$

$$g(\sigma)=\frac{1}{-i\gamma}e^{-i\gamma|x-y|} \qquad \text{for} \qquad \arg\sigma=-\pi\ ,$$

$$d\sigma=-2\gamma d\gamma\ .$$

The integrals over C_3 and C_5 become to

$$\oint_{C_3} e^{\sigma t}g(\sigma)d\sigma=\int_{\infty}^{0}\frac{e^{-\gamma^2 t-i\gamma|x-y|}}{i\gamma}(-2\gamma d\gamma)\,,$$

$$\oint_{C_5} e^{\sigma t}g(\sigma)d\sigma=\int_{0}^{\infty}\frac{e^{-\gamma^2 t+i\gamma|x-y|}}{-i\gamma}(-2\gamma d\gamma)\,.$$

By (2.5.6) and letting $L\to\infty$, $\varepsilon\to 0$

$$\begin{aligned}\lim_{L\to\infty}\int_{s-iL}^{s+iL} e^{\sigma t}g(\sigma)d\sigma &= -\int_{\infty}^{0}\frac{e^{-\gamma^2 t-i\gamma|x-y|}}{i\gamma}(-2\gamma d\gamma)-\int_{0}^{\infty}\frac{e^{-\gamma^2 t+i\gamma|x-y|}}{-i\gamma}(-2\gamma d\gamma)\\ &= -\int_{-\infty}^{0}\frac{e^{-\gamma^2 t+i\gamma|x-y|}}{-i\gamma}(-2\gamma d\gamma)-\int_{0}^{\infty}\frac{e^{-\gamma^2 t+i\gamma|x-y|}}{-i\gamma}(-2\gamma d\gamma)\\ &= -\int_{-\infty}^{\infty}\frac{e^{-\gamma^2 t+i\gamma|x-y|}}{-i\gamma}(-2\gamma d\gamma)\\ &= \frac{-4}{i}\int_{0}^{\infty}e^{-\gamma^2 t}\cos\gamma\,|x-y|\,d\gamma\,.\end{aligned}$$

The *inverse Laplace transform* of $\dfrac{e^{-\sqrt{s}|x-y|}}{\sqrt{s}}$

$$L^{-1}[\frac{e^{-\sqrt{s}|x-y|}}{\sqrt{s}}]=\frac{2}{\pi}\int_{0}^{\infty}e^{-\gamma^2 t}\cos\gamma(x-y)d\gamma\,.$$

Recalling that the *Fourier transform* of $e^{-\gamma^2 t}$ is $\sqrt{\frac{\pi}{t}}e^{\frac{-w}{4t}}$. Then

$$\begin{aligned}\frac{2}{\pi}\int_{0}^{\infty}e^{-\gamma^2 t}\cos\gamma(x-y)d\gamma &= \int_{-\infty}^{\infty}e^{-\gamma^2 t}\cos\gamma(x-y)d\gamma\\ &= \int_{-\infty}^{\infty}e^{-\gamma^2 t}e^{i\gamma|x-y|}d\gamma\end{aligned}$$

$$= \frac{1}{\sqrt{\pi t}} e^{-(x-y)^2/4t}.$$

Hence,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/4t} dy. \tag{2.5.9}$$

2-1-6 Comparison of Separation of Variables and Finite Fourier Transform

We use *separation of variables* to solve homogeneous problem and use *finite Fourier transform* to solve nonhomogeneous problem. If $F(x,t)=0$, we identify the answers.

Here we consider a homogeneous problem.

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < \pi \ , \ t > 0,$$

$$u(x,0) = f(x) \qquad \text{for} \quad 0 \le x \le \pi,$$

$$u(0,t) = 0, \qquad t \ge 0,$$

$$u(\pi,t) = 0, \qquad t \ge 0.$$

The answer by *separation of variables* in 1-1

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t} \sin nx,$$

where b_n determined by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

Using *finite Fourier transform* to solve this problem by letting $F(x,t)=0$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t e^{-n^2\kappa(t-\tau)} B_n(\tau) d\tau + e^{-n^2\kappa t} \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \right) \sin nx. \tag{2.6.1}$$

where

$$B_n(t) = \frac{2}{\pi}\int_0^{\pi} F(x,t)\sin nx dx .$$

Since $F(x,t) = 0$. We have $B_n(t) = 0$. Then (2.6.1) becomes to

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\kappa t} \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \sin nx \cdot$$

Let

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx .$$

Then

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin nx ,$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx .$$

2-1-7 Comparison of Fourier Transform and Sine and Cosine Transform

In general, we use *sine and cosine transform* to solve half-infinity slab heat conduction problem. But if we extend the $f(x)$ to the full-line domain $-\infty < x < \infty$ as an odd or even function., we can also use the *Fourier transform* to solve it.

Here we consider an half-infinity problem, and extend the $f(x)$ to $-\infty < x < \infty$ as an odd function.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad -\infty < x < \infty \ , \ t > 0 ,$$

$u(x,0) = f(x)$, $-\infty < x < \infty$,

$f(x)$ is an odd function.

Since $f(x)$ is an odd function, we have

$\hat{f}(w) = 2i\Im_s[f](w)$.

By (2.3.9)

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} i\Im_s[f](w)e^{-w^2t}(\cos wx - i\sin wx)dw$$
$$= \frac{2}{\pi}\int_0^{\infty} \Im_s[f](w)e^{-w^2t}\sin wx dw.$$

Since $\Im_s[f](w)$ and $\sin wx$ are odd functions of w and $\cos wx$ is even function of w.

2-1-8 Comparison of Fourier Transform and Laplace Transform

In 1-3 and 1-5, we use *Fourier transform* and *Laplace transform* to solve the full-line slab heat conduction problem. Here we want to identify the answers

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{-w^2t}e^{-iwx}dw, \qquad (2.3.9)$$

and

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy \qquad (2.5.9)$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{-w^2t}e^{-iwx}dw$$
$$= \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} e^{-iwx}e^{-w^2t}\left(\int_{-\infty}^{\infty} f(y)e^{iwy}dy\right)dw$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left(\lim_{L\to\infty}\int_{-L}^{L} e^{-w^2t}e^{iw(y-x)}dw\right)dy.$$

We have shown the *Fourier transform* of e^{-ax^2} is $\sqrt{\frac{\pi}{a}}e^{\frac{-w^2}{4a}}$ in 1-5.

Then

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\sqrt{\frac{\pi}{t}}e^{-(x-y)^2/4t}dy$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy .$$

For *Fourier transform* we need the domain is full-line. So we usually take *Fourier transform* on P.D.E. with respect to x since the domain of x is $-\infty < x < \infty$. And for *Laplace transform* we need the domain is half-line. So we usually take on P.D.E. with respect to t since the domain of t is $t > 0$.

2-2. Higher-Dimensional Heat Equation

In this chapter, we use two methods: *separation of variables* and *Fourier transform*.

2-2-1 The Method of Separation of Variables in Cylindrical Coordinates

In 2-1-1 we have shown using the method of *separation of variables* in one dimensional problem. Here we won't do it in detail. Now we consider the heat equation in cylindrical coordinates, independent of z [3].

$$u_t = k\nabla^2 u = k(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi}) ,$$

$$0 < \rho < 1, \quad 0 \le \varphi \le 2\pi , \quad t > 0 .$$

$$u(\rho,\phi,0) = f(\rho,\phi) \tag{2.9.1}$$

We let

$$u(\rho,\varphi,t) = f_1(\rho) f_2(\varphi) f_3(t) .$$

Substituting into (2.9.1) and dividing by $Ku(\rho,\varphi,t)$, we obtain

$$\frac{f_3'(t)}{Kf_3(t)} = \frac{f_1'' + (1/\rho) f_1'}{f_1} + \frac{(1/\rho^2) f_2''}{f_2}. \tag{2.9.2}$$

By (2.9.2) we can obtain three ODEs.

$$f_3'(t) + \lambda K f_3(t) = 0, \tag{2.9.3}$$

$$f_2'' + \mu f_2 = 0, \tag{2.9.4}$$

$$f_1'' + \frac{1}{\rho} f_1' + (\lambda - \frac{\mu}{\rho^2}) f_1 = 0, \tag{2.9.5}$$

where λ and μ are constant. The solution of (2.9.3) is

$$f_3(t) = e^{-\lambda K t}.$$

Since $f_2(\varphi)$ is a 2π -periodic function, the solution of (2.9.4). The μ must be positive.

$$f_2(\varphi) = A\cos m\varphi + B\sin m\varphi \quad , \quad m = 0,1,2.....$$

where $\mu = m^2$.

The general Bessel's equation form is

$$y'' + (d-1)\frac{y'}{x} + (\lambda - \frac{\mu}{x^2}) y = 0$$

And the equation (2.9.5) is a Bessel's equation in two dimensional form.

$$f_1'' + \frac{1}{\rho} f_1' + (\lambda - \frac{\mu}{\rho^2}) f_1 = 0$$

We use Frobenius method. Using power series to solve the Bessel's equation with singular point $\rho = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+\gamma}$$

And we have the solution is the Bessel's function

$$J_m(\rho\sqrt{\lambda}).$$

where

$$J_m(\rho\sqrt{\lambda}) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n \rho^{2n+m}}{2^{m+2n}(m+n)!n!}.$$

The separated solutions of the heat equation [3].

$$u(\rho,\varphi,t) = J_m(\rho\sqrt{\lambda})(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt}.$$

If $\int\int |f(\rho,\phi)|\, d\rho d\phi$ is finite. The series converges uniformly.

$$u\ (\rho,\varphi,t) = \sum_{m,n}^{\infty} J_m(\rho\sqrt{\lambda_n})(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)e^{-\lambda_n Kt}$$

<u>Example 2.1</u> (Using *separation of variables*)

$$u_t = K\nabla^2 u \qquad \text{for} \qquad 0 \le \rho < 1,$$
$$u(1,\varphi,t) = 0, \quad t \ge 0,$$
$$u(\rho,\varphi,0) = 1-\rho^2, \quad 0 \le \rho \le 1.$$

<u>Solution</u>

The required separated solutions are

$$J_m(\rho\sqrt{\lambda})(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt}.$$

The boundary condition requires that $J_m(R\sqrt{\lambda}) = 0$. We let $R\sqrt{\lambda} = x_n^{(m)}$, a positive zero of the Bessel function J_m. The solution becomes to

$$\sum_{m,n} J_m(\frac{\rho x_n^{(m)}}{R})(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)e^{\frac{-(x_n^{(m)})^2 Kt}{R^2}} .$$

To satisfy the initial condition

$$1 = \sum_{m,n} J_m(\frac{\rho x_n^{(m)}}{R})(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi) .$$

This is a Fourier series in ($\cos m\varphi, \sin m\varphi$), whose coefficients are Fourier-Bessel expansion with $\beta = \frac{\pi}{2}$. Here we use the Fourier-Bessel expansion. We have

$$1 - x^2 = \sum_{n=1} A_n J_0(xx_n)$$

where $J_0(x_n) = 0$ and

$$\int_0^1 (1-x^2)xJ_0(xx_n)dx = A_n \int_0^1 J_0(xx_n)^2 xdx, \qquad n = 1, 2.... $$

We compute the left side integral by letting $t = xx_n$ and use integration by parts twice.

$$\int_0^1 (1-x^2)xJ_0(xx_n)dx = \frac{1}{x_n^{\ 4}} \int_0^{x_n} (x_n^{\ 2} - t^2)tJ_0(t)dt$$

$$= \frac{4}{x_n^{\ 3}} J_1(x_n) .$$

The right side integral

$$A_n \int_0^1 J_0(xx_n)^2 xdx = \frac{A_n}{2} J_1(x_n) .$$

The required expansion is

$$1-x^2 = 8\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n{}^3 J_1(x_n)}.$$

Therefore the solution of the initial-value problem is

$$u(\rho,\varphi,t) = 8\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n{}^3 J_1(x_n)}e^{-x_n{}^2Kt/R^2}.$$

2-2-2 <u>Multiple Fourier Transform</u>

We use *<u>Fourier transform</u>* to solve heat conduction problem in one dimension. Here we use it in two dimension. We consider an initial value problem.

$$\frac{\partial u}{\partial t}-(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})=0 \quad \text{for} \quad -\infty < x, y < \infty \quad , \quad t>0, \tag{2.10.1a}$$

$$u(x,y,0) = f(x,y), \quad -\infty < x, y < \infty, \tag{2.10.1b}$$

$$u(x,y,t) \quad \text{bounded.} \tag{2.10.1c}$$

Suppose that u, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, and $\frac{\partial^2 u}{\partial y^2}$ are continuous and absolutely integrable with respect to x and uniformly in *y* and *t*. Taking the *<u>Fourier transform</u>* with respect to *x* into (2.10.1a). We obtain

$$\frac{\partial \hat{u}}{\partial t}-(-w^2\hat{u}+\frac{\partial^2 \hat{u}}{\partial y^2})=0, \tag{2.10.2a}$$

$$\hat{u}(w,y,0) = \hat{f}(w,y). \tag{2.10.2b}$$

In one dimension, we take the *<u>Fourier transform</u>* one time and have the problem become into an ODE. But here it is still a PDE. We take the *<u>Fourier transform</u>* again with respect to *y*. The equation (2.10.2) goes to be

$$\frac{\partial \hat{\hat{u}}}{\partial t}-(-w_1{}^2\hat{\hat{u}}-w_2{}^2\hat{\hat{u}})=0, \tag{2.10.4a}$$

$$\hat{\hat{u}}(w_1, w_2, 0) = \hat{\hat{f}}(w_1, w_2)\,. \tag{2.10.4b}$$

After taking twice *Fourier transform*, we obtain an O.D.E.. The solution of the (2.10.4) is

$$\hat{\hat{u}}(w_1, w_2, t) = \hat{\hat{f}}(w_1, w_2) e^{-({w_1}^2 + {w_2}^2)t}\,. \tag{2.10.5}$$

The function $e^{-({w_1}^2 + {w_2}^2)t}$ is the *Fourier transform* with respect to y of

$$e^{-w_1^2 t} \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}\,.$$

We keep the w_1 and *t* fixed. Taking inversion formula on the equation (2.10.5) with respect to w_2. And use the convolution. We have

$$\hat{u}(w_1, y, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \hat{f}(w_1, \xi) e^{-w_1^2 t - \frac{(y-\xi)}{4t}}\, d\xi\,.$$

By the step again.

$$u(x, y, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta, \xi) e^{-\frac{(x-\eta)^2 + (y-\xi)^2}{4t}}\, d\xi d\eta\,.$$

<u>Example2.2</u> (Using *multiple Fourier transform*)

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad \text{for} \quad -\infty < x, y < \infty \;\;,\;\; t > 0\,,$$

$$u(x, y, 0) = e^{-x^2 - y^2}\,, \quad -\infty < x, y < \infty\,,$$

$$u(x, y, t) \quad \text{bounded.}$$

<u>Solution</u>

Taking *Fourier transform* with respect to *x*. We obtain

$$\frac{\partial \hat{u}}{\partial t} - (-w_1{}^2 \hat{u} + \frac{\partial^2 u}{\partial y^2}) = 0 ,$$

$$\hat{u}(w_1, y, 0) = e^{-y^2} \sqrt{\pi} e^{\frac{-w_1{}^2}{4}} .$$

Taking *Fourier transform* with respect to *y*.

$$\frac{\partial \hat{\hat{u}}}{\partial t} - (-w_1{}^2 \hat{\hat{u}} - w_2{}^2 \hat{\hat{u}}) = 0 ,$$

$$\hat{\hat{u}}(w_1, w_2, 0) = \pi e^{\frac{-(w_1{}^2 + w_2{}^2)}{4}} .$$

The solution

$$\hat{\hat{u}}(w_1, w_2, t) = \pi e^{\frac{-(w_1{}^2 + w_2{}^2)}{4}} e^{-(w_1{}^2 + w_2{}^2)t} .$$

By the convolution

$$u(x, y, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{\xi^2 + \eta^2}{4}} e^{-\frac{(x-\eta)^2 + (y-\xi)^2}{4t}} d\xi d\eta .$$

Ⅲ. Numerical Method for Parabolic Equation

In this chapter, we use *finite difference methods* to have the data of problem by Mathematica 6. We can draw the graph of the heat conduction problem by Mathematica 6. The following are two problems: one-dimensional and two-dimensional problems.

3-1 Finite Difference Method of one-dimensional problem

Here we consider a one-dimensional heat conduction problem.

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < 1, \quad t > 0, \tag{3.1.1a}$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{3.1.1b}$$

$$u(x,0) = x(1-x), \qquad 0 \leq x \leq 1. \tag{3.1.1c}$$

The definition of the partial derivative $\frac{\partial u}{\partial x}$ is the limit of a difference quotient

$$\frac{\partial u}{\partial x}(x,t) = \lim_{h \to 0} \frac{u(x+h,t) - u(x,t)}{h}.$$

We have

$$\frac{\partial u}{\partial x}(x,t) \sim \frac{u(x+h,t) - u(x,t)}{h}$$

if h is small enough. And it's similar to $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial t}$. We have

$$\frac{\partial u}{\partial t}(x,t) \sim \frac{u(x,t+l) - u(x,t)}{l} \tag{3.1.2}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x,t) \sim \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}. \qquad (3.1.3)$$

We let the domain of t is $0 \le t \le 1$. Dividing the domain into some partitions by letting

$$x_i = ih, \quad t_j = jl, \quad i = 0,1.....,m+1, \quad j = 0,1......,n,$$

where

$$h = \frac{1}{m+1}, \quad l = \frac{1}{n}.$$

Now, we fix the *i and j*. Substituting (3.1.2) and (3.1.3) into (3.1.1a).

$$\frac{u_{i,j+1}-u_{i,j}}{l} = \kappa \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} \qquad (3.1.4)$$

where

$$u_{i,j} = u(ih, jl).$$

The relation

$$u_{i,j+1} = u_{i,j} + \frac{l}{h^2}\kappa(u_{i+1,j}-2u_{i,j}+u_{i-1,j}).$$

That means $u_{i,j+1}$ can be obtained by linear combination of $u_{i+1,j}$, $u_{i,j}$, and $u_{i-1,j}$.

We have the initial condition $u(x,0) = x(1-x)$. So we can obtain each $u_{i,j}$.

Method:

We make a linear system $f = Tu$

$$f=\begin{pmatrix}(1-K\frac{2(m+1)^2}{n})u_{1,0}+K\frac{(m+1)^2}{n}(u_{2,0}+u_{0,0})\\ 0\\ \vdots\\ (1-K\frac{2(m+1)^2}{n})u_{2,0}+K\frac{(m+1)^2}{n}(u_{3,0}+u_{1,0})\\ 0\\ \vdots\\ (1-K\frac{2(m+1)^2}{n})u_{m-1,0}+K\frac{(m+1)^2}{n}(u_{m,0}+u_{m-2,0})\\ 0\\ \vdots\\ 0\end{pmatrix}_{mn\times 1}$$

$$u=\begin{pmatrix}u_{1,1}\\ u_{1,2}\\ \vdots\\ u_{1,n}\\ u_{2,1}\\ \vdots\\ u_{m-1,n}\\ u_{m,1}\\ \vdots\\ u_{m,n}\end{pmatrix}_{mn\times 1}$$

$$T = \begin{pmatrix} A & B & 0 \\ B & \ddots & B \\ 0 & B & A \end{pmatrix}_{mn \times mn}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2K\dfrac{(m+1)^2}{n} - 1 & \ddots & 0 \\ 0 & 2K\dfrac{(m+1)^2}{n} - 1 & 1 \end{pmatrix}_{m \times m}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \dfrac{-(m+1)^2}{n}K & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{-(m+1)^2}{n}K & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \dfrac{-(m+1)^2}{n}K & 0 \end{pmatrix}_{m \times m}$$

We have each $u_{i,j}$ from $u = T^{-1} f$ with the initial condition $u(x,0) = x(1-x)$.

Here we put $m=4$, $n=40$, and $k=1$ for this problem. Here are some figures of solution to problem (3.1.1) by Mathematica 6..

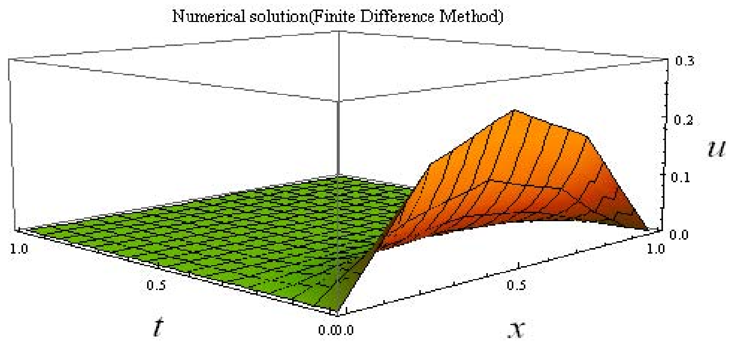


Figure 3.1 Numerical solution of equation (3.1.1) with k=1.

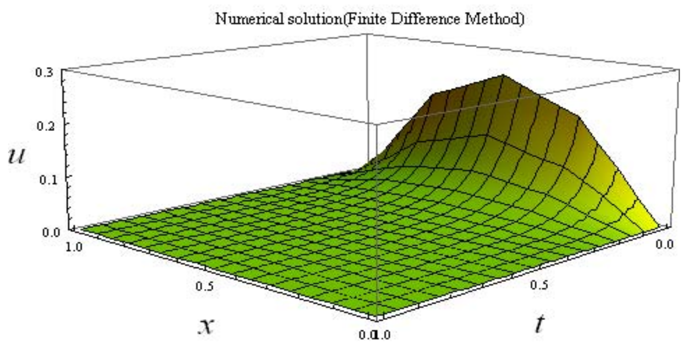


Figure 3.2 Numerical solution of equation (3.1.1) with k=1.

We have the exact solution of this problem (3.1.1)

$$u(x,t)=\frac{8}{\pi^3}(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2\pi^2\kappa t}\sin(2n-1)\pi x).$$

The following are some figures of truncation of exact solution up to 1000 and the campare between numerical solution and truncation of exact solution up to 1000 of problem (3.1.1)

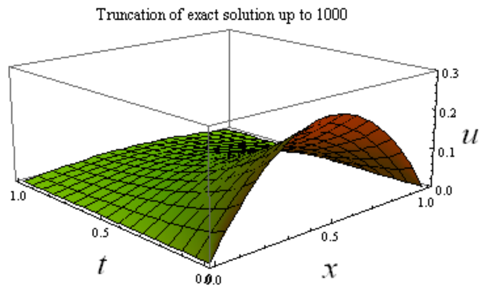


Figure 3.3 Truncation of exact solution of problem (3.1.1) up to 1000 with k=1.

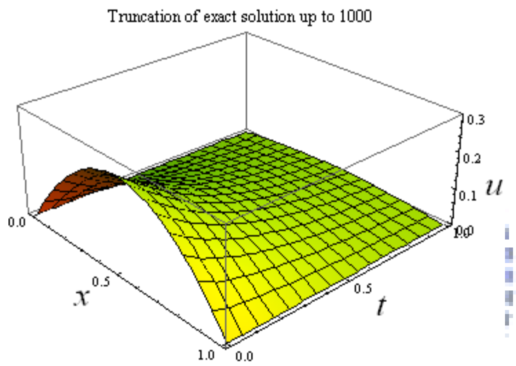


Figure 3.4 Truncation of exact solution of problem (3.1.1) up to 1000 with k=1.

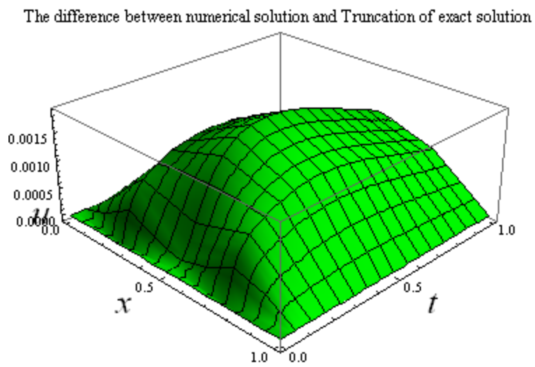


Figure 3.5 The difference of Numerical solution of problem (3.1.1) with Truncation of exact solution of problem (3.1.1) up to 1000 with k=1.

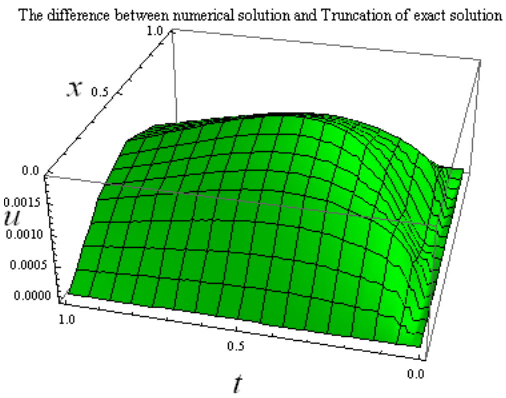


Figure 3.6 The difference of Numerical solution of problem (3.1.1) with Truncation of exact solution of problem (3.1.1) up to 1000 with k=1.

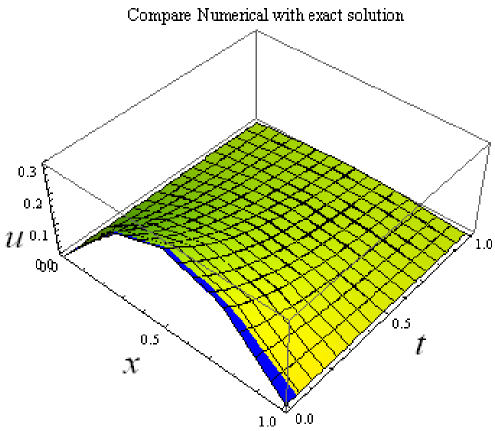


Figure 3.7 Comparison between Numerical with Truncation of Exact solution up to 1000. Yellow is numerical solution. Blue is Truncation of exact solution.

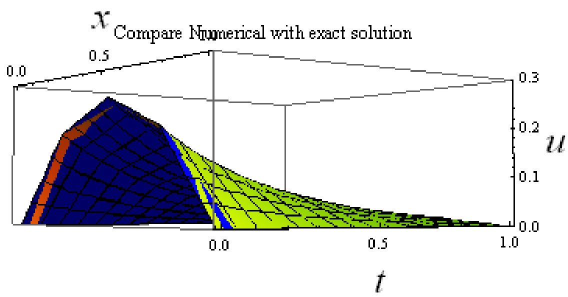


Figure 3.8 Comparison between Numerical with Truncation of Exact solution up to 1000. Yellow is numerical solution. Blue is Truncation of exact solution.

Here is the table for the data of the difference of numerical solution of problem (3.1.1) and truncation of exact solution of problem (3.1.1) up to 1000.

	$u_{0,j}$	$u_{1,j}$	$u_{2,j}$	$u_{3,j}$	$u_{4,j}$
$u_{i,0}$	0	$0. \times 10^{-8}$	$0. \times 10^{-8}$	$0. \times 10^{-8}$	0
$u_{i,1}$	0	-0.0000704261	-1.57581×10^{-8}	-0.0000704261	0
$u_{i,2}$	0	0.000324565	-0.0000109346	0.000324565	0
$u_{i,3}$	0	0.000616433	0.000113594	0.000616433	0
$u_{i,4}$	0	0.000794568	0.000336591	0.000794568	0
$u_{i,5}$	0	0.0009071	0.000592818	0.0009071	0
$u_{i,6}$	0	0.000986792	0.000841198	0.000986792	0
$u_{i,7}$	0	0.00105048	0.00106289	0.00105048	0
$u_{i,8}$	0	0.00110557	0.00125198	0.00110557	0
$u_{i,9}$	0	0.00115477	0.00140887	0.00115477	0
$u_{i,10}$	0	0.00119867	0.00153658	0.00119867	0
$u_{i,11}$	0	0.00123713	0.00163884	0.00123713	0
$u_{i,12}$	0	0.00126986	0.00171933	0.00126986	0
$u_{i,13}$	0	0.00129667	0.00178131	0.00129667	0
$u_{i,14}$	0	0.00131754	0.00182755	0.00131754	0
$u_{i,15}$	0	0.0013326	0.00186036	0.0013326	0
$u_{i,16}$	0	0.00134211	0.00188169	0.00134211	0
$u_{i,17}$	0	0.00134641	0.00189313	0.00134641	0
$u_{i,18}$	0	0.00134592	0.00189605	0.00134592	0
$u_{i,19}$	0	0.00134104	0.00189159	0.00134104	0
$u_{i,20}$	0	0.00133221	0.00188075	0.00133221	0
$u_{i,21}$	0	0.00131986	0.00186438	0.00131986	0
$u_{i,22}$	0	0.00130439	0.00184323	0.00130439	0
$u_{i,23}$	0	0.00128618	0.00181797	0.00128618	0
$u_{i,24}$	0	0.0012656	0.00178919	0.0012656	0
$u_{i,25}$	0	0.00124298	0.00175741	0.00124298	0
$u_{i,26}$	0	0.00121861	0.00172309	0.00121861	0
$u_{i,27}$	0	0.00119279	0.00168667	0.00119279	0
$u_{i,28}$	0	0.00116576	0.00164851	0.00116576	0
$u_{i,29}$	0	0.00113775	0.00160895	0.00113775	0
$u_{i,30}$	0	0.00110899	0.00156829	0.00110899	0
$u_{i,31}$	0	0.00107964	0.0015268	0.00107964	0
$u_{i,32}$	0	0.00104988	0.00148473	0.00104988	0
$u_{i,33}$	0	0.00101986	0.00144229	0.00101986	0
$u_{i,34}$	0	0.000989722	0.00139967	0.000989722	0
$u_{i,35}$	0	0.000959576	0.00135704	0.000959576	0
$u_{i,36}$	0	0.000929531	0.00131455	0.000929531	0
$u_{i,37}$	0	0.000899682	0.00127234	0.000899682	0
$u_{i,38}$	0	0.000870109	0.00123052	0.000870109	0
$u_{i,39}$	0	0.000840885	0.00118919	0.000840885	0
$u_{i,40}$	0	0.000812072	0.00114844	0.000812072	0

Table 3.1 The difference of numerical solution of problem (3.1.1) and truncation of exact solution of equation (3.1.1) up to 1000.

We change the thermal conduction coefficient K from 1 to $\frac{1}{4}$.

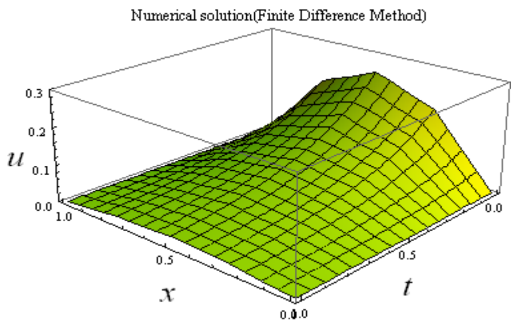


Figure 3.9 Numerical solution of problem (3.1.1) with k=1/4

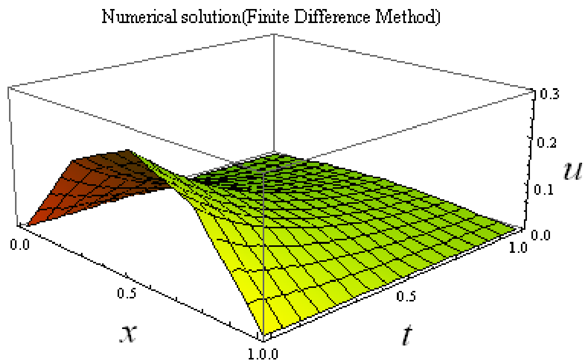


Figure 3.10 Numerical solution of problem (3.1.1) with k=1/4.

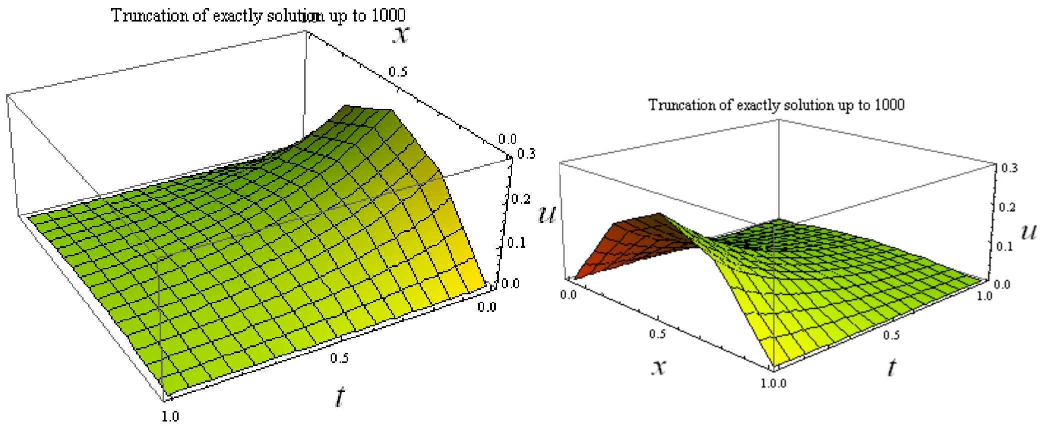


Figure 3.11 Truncation of exact solution of problem (3.1.1) up to 1000 with k=1/4 .

We can find that if the thermal coefficient k becomes smaller, $u(x,t)$ is decreasing slowly. And if k becomes bigger, $u(x,t)$ is decreasing fast.

3-2 Finite Difference Method of two-dimensional problem

Here we consider a two-dimensional heat conduction problem.

$$\frac{\partial u}{\partial t}-k(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})=0 \quad \text{for} -\pi<x<\pi\,,\ -\pi<y<\pi\,,\ t>0\,, \tag{3.2.1a}$$

$$u(0,y,t)=u(\pi,y,t)=0 \quad \text{for} \quad -\pi\le y\le\pi\,,\ t>0\,, \tag{3.2.1b}$$

$$u(x,0,t)=u(x,\pi,t)=0 \quad \text{for} \quad -\pi\le x\le\pi\,,\ t>0\,, \tag{3.2.1c}$$

$$u(x,y,0)=x(\pi-x)y(\pi-y)\,. \tag{3.2.1d}$$

The definition of the partial derivative $\frac{\partial u}{\partial x}$ is the limit of a difference quotient

$$\frac{\partial u}{\partial x}(x,y,t)=\lim_{h\to 0}\frac{u(x+h,y,t)-u(x,y,t)}{h}\,.$$

We have

$$\frac{\partial u}{\partial x}(x,y,t)\sim\frac{u(x+h,y,t)-u(x,y,t)}{h}\,.$$

if *h* is small enough. And it's similar to $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial u}{\partial t}$. We have

$$\frac{\partial u}{\partial t}(x,y,t)\sim\frac{u(x,y,t+r)-u(x,y,t)}{r}\,, \tag{3.2.2}$$

$$\frac{\partial^2 u}{\partial x^2}(x,y,t)\sim\frac{u(x+h,y,t)-2u(x,y,t)+u(x-h,y,t)}{h^2}\,, \tag{3.2.3}$$

$$\frac{\partial^2 u}{\partial y^2}(x,y,t)\sim\frac{u(x,y+l,t)-2u(x,y,t)+u(x,y-l,t)}{l^2}\,. \tag{3.2.4}$$

Since the domain is half-infinity with respect to *t*. We let the domain of t is $0\le t\le 1$. Dividing the domain into some partitions by letting

$$x = ih\pi, \quad y = jl\pi, \quad t = pr,$$
$$i = 0,1,\ldots,m+1, \quad j = 0,1,\ldots.,n+1, \quad p = 0,1,..,s.$$

where $h = \dfrac{1}{m+1}$, $l = \dfrac{1}{n+1}$, and $r = \dfrac{1}{s}$.

We fix *i*, *j*, and *p*. Substituting (3.2.2), (3.2.3), and (3.2.4) into (3.2.1a), we have

$$\frac{u_{i,j,p+1} - u_{i,j,p}}{r} = \kappa\left(\frac{u_{i+1,l,p} - 2u_{i,j,p} + u_{i-1,j,p}}{h^2} + \frac{u_{i,j+1,p} - 2u_{i,j,p} + u_{i,j-1,p}}{l^2}\right).$$

where

$$u_{i,j,p} = u(ih\pi, jl\pi, pr).$$

The relation

$$u_{i,j,p+1} = u_{i,j,p}\left(1 - \frac{2\kappa}{s}((m+1)^2 + (n+1)^2)\right) + \frac{\kappa}{s}(m+1)^2(u_{i+1,j,p} + u_{i-1,l,p})$$
$$+ \frac{\kappa}{s}(n+1)^2(u_{i,j+1,p} + u_{i,j-1,p}).$$

That means $u_{i,j,p+1}$ can be obtained by linear combination of $u_{i+1,j,p}$, $u_{i,j,p}$, $u_{i-1,j,p}$, $u_{i,j+1,p}$, and $u_{i,j-1.p}$. We have the initial condition (3.2.1d). So we can obtain each $u_{i,j,p}$.

Method:

We make a linear system *f=Tu*

$$
f=\begin{pmatrix}
(1-2K\dfrac{(m+1)^2+(n+1)^2}{s})u_{1,1,0}+\dfrac{K}{s}(m+1)^2(u_{2,1,0}+u_{0,1,0})+\dfrac{K}{s}(n+1)^2(u_{1,2,0}+u_{1,0,0})\\
0\\
\vdots\\
(1-2K\dfrac{(m+1)^2+(n+1)^2}{s})u_{1,2,0}+\dfrac{K}{s}(m+1)^2(u_{2,2,0}+u_{0,2,0})+\dfrac{K}{s}(n+1)^2(u_{1,3,0}+u_{1,1,0})\\
0\\
\vdots\\
(1-2K\dfrac{(m+1)^2+(n+1)^2}{s})u_{1,n,0}+\dfrac{K}{s}(m+1)^2(u_{2,n,0}+u_{0,n,0})+\dfrac{K}{s}(n+1)^2(u_{1,n+1,0}+u_{1,n-1,0})\\
0\\
\vdots\\
(1-2K\dfrac{(m+1)^2+(n+1)^2}{s})u_{2,1,0}+\dfrac{K}{s}(m+1)^2(u_{3,1,0}+u_{1,1,0})+\dfrac{K}{s}(n+1)^2(u_{2,2,0}+u_{2,0,0})\\
0\\
\vdots\\
(1-2K\dfrac{(m+1)^2+(n+1)^2}{s})u_{m,n,0}+\dfrac{K}{s}(m+1)^2(u_{m+1,n,0}+u_{0,n,0})+\dfrac{K}{s}(n+1)^2(u_{m,n+1,0}+u_{m,n-1,0})\\
0\\
\vdots\\
0
\end{pmatrix}_{mns\times 1}
$$

$$
u=\begin{pmatrix}
u_{1,1,1}\\
u_{1,1,2}\\
\vdots\\
u_{1,1,s}\\
u_{1,2,1}\\
\vdots\\
u_{1,n,s}\\
u_{2,1,1}\\
\vdots\\
u_{m,n,s}
\end{pmatrix}_{mns\times 1}
$$

$$T = \begin{pmatrix} A & B & 0 \\ B & \ddots & B \\ 0 & B & A \end{pmatrix}_{mns \times mns}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \frac{-(m+1)^2}{s}K & 0 & 0 & \cdots & 0 \\ 0 & \frac{-(m+1)^2}{s}K & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{-(m+1)^2}{s}K & 0 \end{pmatrix}_{ns \times ns}$$

$$A = \begin{pmatrix} C & D & & & & \\ D & C & D & & & \\ & D & C & D & & \\ & & & \ddots & & \\ & & & D & C & D \\ & & & & D & C \end{pmatrix}_{ns \times ns}$$

$$C=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1-\frac{2K}{s}[(m+1)^2+(n+1)^2] & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1-\frac{2K}{s}[(m+1)^2+(n+1)^2] & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1-\frac{2K}{s}[(m+1)^2+(n+1)^2] & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1-\frac{2K}{s}[(m+1)^2+(n+1)^2] & 1 \end{pmatrix}_{s\times s}$$

$$D=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \frac{-(n+1)^2}{s}K & 0 & 0 & \cdots & 0 \\ 0 & \frac{-(n+1)^2}{s}K & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{-(n+1)^2}{s}K & 0 \end{pmatrix}_{s\times s}$$

We have the each $u_{i,j,p}$ from $u=T^{-1}f$ with the initial condition $u(x,y,0)=x(\pi-x)y(\pi-y)$.

Here we put $m=4$, $n=4$, $s=80$, and $k=1$ for this problem. We have the exactly solution of this problem

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{4}{\pi}\int_0^{\pi}\int_0^{\pi}x(\pi-x)y(\pi-y)dxdy\right)e^{-(n^2+m^2)t}\sin nx\sin my$$

.

The following are some figures of solution to this problem by Mathematica 6. Since this solution $u(x,y,t)$ is three dimensional type. We fix the t to show the solution.

We fix $t=0$.

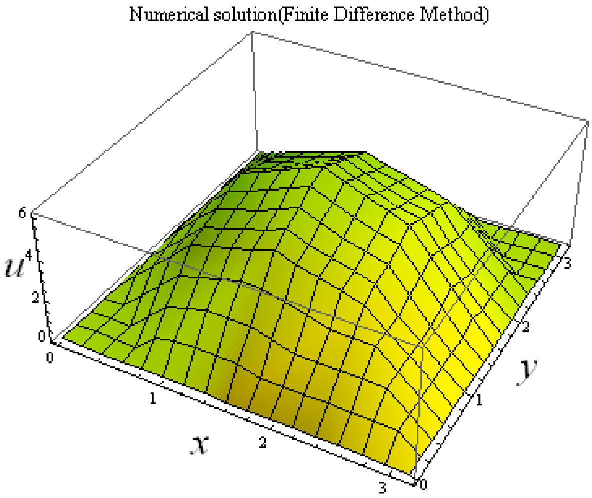


Figure 3.13 The numerical solution of problem (3.2.1) at t = 0.

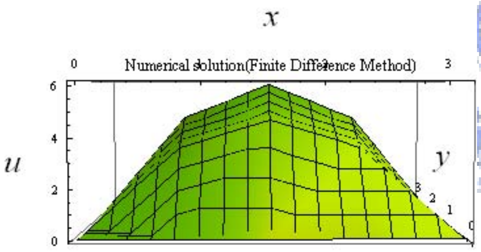


Figure 3.14 The numerical solution of problem (3.2.1) at t = 0.

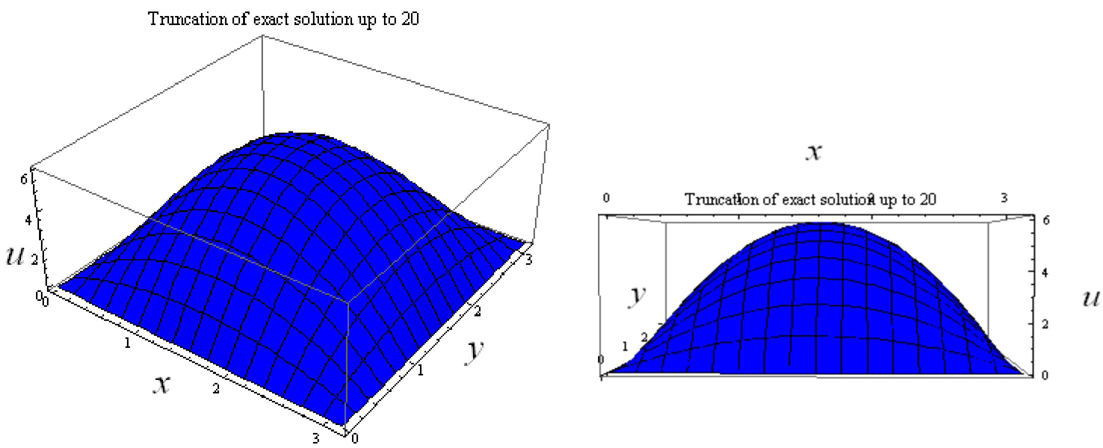


Figure 3.15 The truncation of exact solution of problem (3.2.1) up to 20 at t=0.

Here is the table for the data of the difference of numerical solution of problem (3.2.1) and truncation of exact solution of problem (3.2.1) up to 20 at t =0

	$u_{i,0}$	$u_{i,1}$	$u_{i,2}$	$u_{i,3}$	$u_{i,4}$
$u_{0,j}$	0	0	0	0	0
$u_{1,j}$	0	-0.000799293451	-0.000242560062	-0.000799293451	0
$u_{2,j}$	0	-0.000242560062	0.000773941482	-0.000242560062	0
$u_{3,j}$	0	-0.000799293451	-0.000242560062	-0.000799293451	0
$u_{4,j}$	0	0	0	0	0

Table 3.17 The difference between numerical solution and truncation of exact solution of problem (3.2.1) up to 20 at t=0.

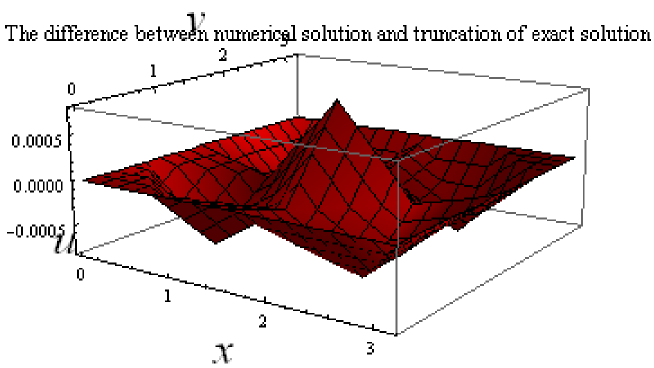


Figure 3.17 The difference between numerical solution and truncation of exact solution of problem (3.2.1) up to 20 at t=0.

We change t from 0 to $\frac{1}{10}$

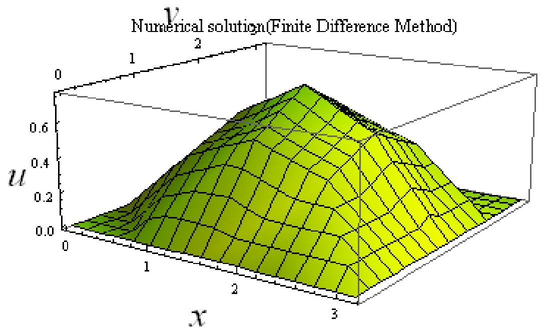


Figure 3.18 The numerical solution of problem (3.2.1) at *t* =1/10

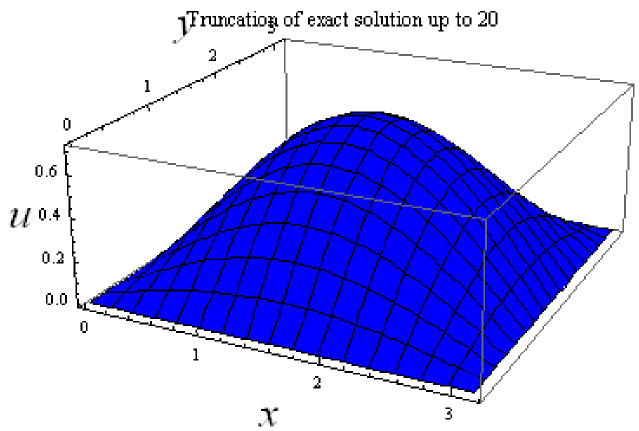


Figure 3.19 The truncation of exact solution of problem (3.2.1) up to 20 at t=1/10.

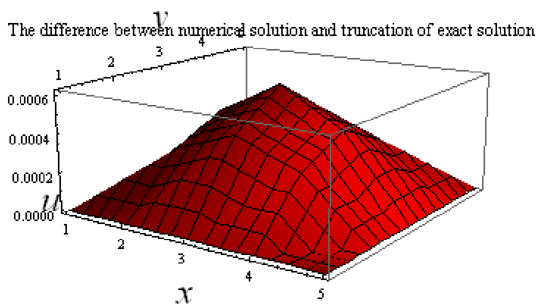


Figure 3.20 The difference between numerical solution and truncation of exact solution of problem (3.2.1) up to 20 with t = 1/10.

Here is the table for the data of the difference of numerical solution of problem (3.2.1) and truncation of exact solution of problem (3.2.1) up to 20 at t =1/10.

	$u_{i,0}$	$u_{i,1}$	$u_{i,2}$	$u_{i,3}$	$u_{i,4}$
$u_{0,j}$	0	0	0	0	0
$u_{1,j}$	0	0.000337067	0.000481116	0.000337067	0
$u_{2,j}$	0	0.000481116	0.000693833	0.000481116	0
$u_{3,j}$	0	0.000337067	0.000481116	0.000337067	0
$u_{4,j}$	0	0	0	0	0

Table 3.20 The difference between numerical solution and truncation of exact solution of problem (3.2.1) up to 20 with t = 1/10.

Ⅳ. Summary

We have some conclusions of the methods we show above

Similarity:

For the all method, its make the PDE become to one or more ODEs. And we solve the simple ODEs to obtain the solution. In the method of *separation of variable* we have two or three solutions from ODE systems and obtain the particular solution. About the transform, we take the transform to the PDE and make it become an ODE. To obtain the solutions of PDEs by taking the inversion formula on the solution of ODE.

Difference:

Different methods apply to different domain and equation. *Separation of variables* applies to bounded domain and the equation is homogeneous. The *finite Fourier transform* applies to bounded domain and the equation is nonhomogeneous. The *Fourier transform* applies to the full-line domain. The *sine and cosine transform* applies to the half-line domain. The finite difference method applies to irregular domain.

The method of *separation of variables* in cylindrical coordinates is harder than in one dimensional. It generate the Bessel's equation.

There are some restrictions on boundary condition for the *sine and cosine transform*. *Sine and cosine transform* are only useful in solving problem with the boundary condition with $x=0$

References

[1] H. F. Weinberger, A First Course in Partial Differential Equations with Complex Variables and Transform Methods, Blaisdell, 1965.

[2] Frederic Y. M. Wan, Mathematical Models and Their Analysis, Harper & Row, 1989.

[3] Mark A. Pinsky, Partial Differential Equations and Boundary-Value Problems with Applications, Second Edition, McGraw-Hill, 1911.

[4] Gordon D. Smith, Numerical Solution of Partial Differential Equations: Finite Difference Methods, Oxford University Press, 1985.

Appendix

The mathematica 6 code of problem (3.1.1)

```
Clear[m, n, k, A, B];
m = 39;
n = 3;
k = 1/4;
S = Table[If[i + (m + 1) ≤ (m + 1) * n && i - (m + 1) ≤ 0 && Mod[i, (m + 1)] == 1, {{i, i} → 1},
        If[i + (m + 1) ≤ (m + 1) * n && i - (m + 1) ≤ 0 && Mod[i, (m + 1)] ≠ 1,
    {{i, i} → 1, {i, i - 1} → -(1 - 2 * (n + 1)^2/(m + 1) * k), {i, i + m} → (-(n + 1)^2)/(m + 1) * k},
          If[i + (m + 1) ≤ (m + 1) * n && i > (m + 1) && Mod[i, (m + 1)] == 1, {{i, i} → 1 },
            If[i + (m + 1) ≤ (m + 1) * n && i > (m + 1) && Mod[i, (m + 1)] ≠ 1,
     {{i, i} → 1, {i, i - 1} → -(1 - 2 (n + 1)^2/(m + 1) * k), {i, i + m} → (-(n + 1)^2)/(m + 1) * k, {i, i - m - 2} → (-(n + 1)^2)/(m + 1) * k },
     If[i + (m + 1) > (m + 1) * n && i > (m + 1) && Mod[i, (m + 1)] == 1, {{i, i} → 1},
       If[i + (m + 1) > (m + 1) * n && i > (m + 1) && Mod[i, (m + 1)] ≠ 1,
       {{i, i} → 1, {i, i - 1} → -(1 - 2 (n + 1)^2/(m + 1) * k), {i, i - m - 2} → (-(n + 1)^2)/(m + 1) * k}, 0]]]]]], {i, 1, (m + 1) * n}];
A = {};
For[i = 1, i < Length[S] + 1, i++, A = Join[A, S[[i]]]];
L = Inverse[SparseArray[A]];
B =
  Table[If[Mod[r, (m + 1)] == 1, ((n + 1)^2/(m + 1)) * k * ((r - 1)/(m + 1) + 2)/(n + 1) * (1 - ((r - 1)/(m + 1) + 2)/(n + 1)) + (1 - 2 (n + 1)^2/(m + 1) * k) * ((r - 1)/(m + 1) + 1)/(n + 1) * (1 - ((r - 1)/(m + 1) + 1)/(n + 1)) +
    ((n + 1)^2/(m + 1) * k) * ((r - 1)/(m + 1))/(n + 1) * (1 - ((r - 1)/(m + 1))/(n + 1)), 0], {r, (m + 1) * n}];
T = N[L.B];
U[x_, t_] := If[x == 0 || x == n + 1, 0,
        If[t == 0, x/(n + 1) (1 - x/(n + 1)), T[[(m + 1) * (x - 1) + t]]]];
U1[x_, t_] := U[(n + 1) * x, (m + 1) * t];
pic = Table[{x, y, N[U1[x, y], 5]}, {x, 0, 1, 1/(n + 1)}, {y, 0, 1, 1/(m + 1)}];
picJoin = {};
For[i = 1, i < 6, i++, picJoin = Join[picJoin, pic[[i]]]];
ListPlot3D[picJoin, PlotStyle → Yellow, PlotRange → {0, 0.3}
, AxesLabel → {Style[x, Large], Style[t, Large], Style[u, Large]}, PlotLabel → "Numerical solution(Finite Difference Method)"];
U2[q_, r_] := 8/π^3 (Sum[1/(2 n - 1)^3 * 1/Exp[(2 n - 1)^2 * π^2 * k * r] * Sin[(2 n - 1) * π * q], {n, 1, 1000}]);
pic2 = Table[{x, y, N[U2[x, y], 5]}, {x, 0, 1, 1/(n + 1)}, {y, 0, 1, 1/(m + 1)}];
picJoin2 = {};
For[i = 1, i < 6, i++, picJoin2 = Join[picJoin2, pic2[[i]]]];
ListPlot3D[picJoin2, PlotStyle → Yellow, PlotRange → {0, 0.3}
, AxesLabel → {Style[x, Large], Style[t, Large], Style[u, Large]}, PlotLabel → "Truncation of exact solution up to 1000"];
pic3 = Table[{x, y, N[U1[x, y], 7] - N[U2[x, y], 7]}, {x, 0, 1, 1/(n + 1)}, {y, 0, 1, 1/(m + 1)}];
pic3Join = {};
For[i = 1, i < 6, i++, pic3Join = Join[pic3Join, pic3[[i]]]];
ListPlot3D[pic3Join, PlotStyle → Green, AxesLabel → {Style[x, Large], Style[t, Large], Style[u, Large]},
  PlotLabel → "The difference between numerical solution and Truncation of exact solution "];
ListPlot3D[{picJoin, picJoin2}, PlotStyle → {Yellow, Blue}, PlotRange → {0, 0.3}
, AxesLabel → {Style[x, Large], Style[t, Large], Style[u, Large]}, PlotLabel → "Compare Numerical with exact solution"]
Clear[i, j];
Tn = TableForm[Table[N[U1[x, y], 7] - N[U2[x, y], 7], {y, 0, 1, 1/(m + 1)}, {x, 0, 1, 1/(n + 1)}],
  TableHeadings → {Table[Subscript[u, i, j], {j, 0, m + 1}], Table[Subscript[u, i, j], {i, 0, n + 1}]}, TableSpacing → {1, 2}]
```

The mathematica 6 code of problem (3.2.1)

```
Clear[m, n, s, k];
m = 3;
n = 3;
s = 79;
k = 1;
W = Table[If[i ≤ (s + 1) && Mod[i, (s + 1)] == 1, {{i, i} → 1},
        If[i ≤ (s + 1) && Mod[i, (s + 1)] ≠ 1, {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i + s} → -k/(s + 1) * (n + 1)^2,
      {i, i + n * (s + 1) - 1} → -k/(s + 1) (m + 1)^2},
          If[i > s + 1 && i ≤ (n - 1) * (s + 1) && Mod[i, (s + 1)] == 1, {{i, i} → 1},
          If[i > s + 1 && i ≤ (n - 1) * (s + 1) && Mod[i, (s + 1)] ≠ 1,
      {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i + s} → -k/(s + 1) * (n + 1)^2, {i, i - s - 2} → -k/(s + 1) * (n + 1)^2,
       {i, i - 1 + n * (s + 1)} → -k/(s + 1) (m + 1)^2},
             If[i > (n - 1) (s + 1) && i ≤ n (s + 1) && Mod[i, (s + 1)] == 1, {{i, i} → 1},
             If[i > (n - 1) (s + 1) && i ≤ n (s + 1) && Mod[i, (s + 1)] ≠ 1,
        {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i - s - 2} → -k/(s + 1) * (n + 1)^2, {i, i - 1 + n (s + 1)} → -k/(s + 1) (m + 1)^2},
                If[i > n (s + 1) && i ≤ (m - 1) * n * (s + 1) && Mod[Ceiling[i/(s + 1)], n] == 1 && Mod[i, (s + 1)] == 1, {{i, i} → 1},
                If[i > n (s + 1) && i ≤ (m - 1) * n * (s + 1) && Mod[Ceiling[i/(s + 1)], n] == 1 && Mod[i, (s + 1)] ≠ 1,
         {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i + s} → -k/(s + 1) * (n + 1)^2, {i, i - 1 + n (s + 1)} → -k/(s + 1) (m + 1)^2,
          {i, i - 1 - n * (s + 1)} → -k/(s + 1) (m + 1)^2},
                  If[i > n (s + 1) && i ≤ (m - 1) * n * (s + 1) && Mod[Ceiling[i/(s + 1)], n] == 0 && Mod[i, (s + 1)] == 1, {{i, i} → 1},
                  If[i > n (s + 1) && i ≤ (m - 1) * n * (s + 1) && Mod[Ceiling[i/(s + 1)], n] == 0 && Mod[i, (s + 1)] ≠ 1,
         {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i - 1 + n (s + 1)} → -k/(s + 1) (m + 1)^2,
          {i, i - 1 - n * (s + 1)} → -k/(s + 1) (m + 1)^2, {i, i - s - 2} → -k/(s + 1) * (n + 1)^2},
          If[i > n (s + 1) && i ≤ (m - 1) * n * (s + 1) && Mod[Ceiling[i/(s + 1)], n] ≠ 1 && Mod[Ceiling[i/(s + 1)], n] ≠ 0 && Mod[i, (s + 1)] == 1,
          {{i, i} → 1},
          If[i > n (s + 1) && i ≤ (m - 1) * n * (s + 1) && Mod[Ceiling[i/(s + 1)], n] ≠ 1 && Mod[Ceiling[i/(s + 1)], n] ≠ 0 &&
           Mod[i, (s + 1)] ≠ 1, {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i - 1 + n (s + 1)} → -k/(s + 1) (m + 1)^2,
           {i, i - 1 - n * (s + 1)} → -k/(s + 1) (m + 1)^2, {i, i - s - 2} → -k/(s + 1) * (n + 1)^2, {i, i + s} → -k/(s + 1) * (n + 1)^2},
           If[i > (m - 1) * n * (s + 1) && i ≤ (((m - 1) * n) + 1) * (s + 1) && Mod[i, (s + 1)] == 1, {{i, i} → 1},
           If[i > (m - 1) * n * (s + 1) && i ≤ (((m - 1) * n) + 1) * (s + 1) && Mod[i, (s + 1)] ≠ 1, {{i, i} → 1,
             {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i + s} → -k/(s + 1) * (n + 1)^2, {i, i - 1 - n * (s + 1)} → -k/(s + 1) (m + 1)^2},
             If[i > (((m - 1) * n) + 1) * (s + 1) && i ≤ (m * n - 1) (s + 1) && Mod[i, (s + 1)] == 1, {{i, i} → 1},
             If[i > (((m - 1) * n) + 1) * (s + 1) && i ≤ (m * n - 1) (s + 1) && Mod[i, (s + 1)] ≠ 1,
             {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2), {i, i - 1 - n * (s + 1)} → -k/(s + 1) (m + 1)^2,
              {i, i - s - 2} → -k/(s + 1) * (n + 1)^2, {i, i + s} → -k/(s + 1) * (n + 1)^2},
               If[i > (m * n - 1) * (s + 1) && Mod[i, (s + 1)] == 1, {{i, i} → 1},
               If[i > (m * n - 1) * (s + 1) && Mod[i, (s + 1)] ≠ 1, {{i, i} → 1, {i, i - 1} → -1 + 2 * k/(s + 1) * ((m + 1)^2 + (n + 1)^2),
                {i, i - 1 - n * (s + 1)} → -k/(s + 1) (m + 1)^2, {i, i - s - 2} → -k/(s + 1) * (n + 1)^2}
             ]]]]]]]]]]]]]]]]]], {i, 1, m * n * (s + 1)}];
```

```
A = {};
For[i = 1, i < Length[W] + 1, i++, A = Join[A, W[[i]]]];
L = Inverse[SparseArray[A]];
B =
  Table[
   If[Mod[i, (s + 1)] == 1, (k/(s + 1)) * (m + 1)^2 * ((Ceiling[i/((s+1)*n)] + 1) π)/(m + 1) * (π - ((Ceiling[i/((s+1)*n)] + 1) π)/(m + 1)) * (((Mod[(i-1)/(s+1), n] + 1) π)/(n + 1)) *
     (π - ((Mod[(i-1)/(s+1), n] + 1) π)/(n + 1))    + (k/(s + 1)) * (m + 1)^2 * ((Ceiling[i/((s+1)*n)] - 1) π)/(m + 1) * (π - ((Ceiling[i/((s+1)*n)] - 1) π)/(m + 1)) *
     (((Mod[(i-1)/(s+1), n] + 1) π)/(n + 1)) * (π - ((Mod[(i-1)/(s+1), n] + 1) π)/(n + 1))    +
     k/(s + 1) * (n + 1)^2 * ((Ceiling[i/((s+1)*n)]) π)/(m + 1) * (π - ((Ceiling[i/((s+1)*n)]) π)/(m + 1)) * (((Mod[(i-1)/(s+1), n] + 2) π)/(n + 1)) * (π - ((Mod[(i-1)/(s+1), n] + 2) π)/(n + 1)) +
     k/(s + 1) * (n + 1)^2 * ((Ceiling[i/((s+1)*n)]) π)/(m + 1) * (π - ((Ceiling[i/((s+1)*n)]) π)/(m + 1)) * (((Mod[(i-1)/(s+1), n]) π)/(n + 1)) * (π - ((Mod[(i-1)/(s+1), n]) π)/(n + 1)) +
     (1 - 2 k/(s + 1) ((m + 1)^2 + (n + 1)^2)) * (((Ceiling[i/((s+1)*n)]) π)/(m + 1)) * (π - (Ceiling[i/((s+1)*n)]/(m + 1)) * π) * ((Mod[(i-1)/(s+1), n] + 1)/(n + 1)) * π *
     (π - ((Mod[(i-1)/(s+1), n] + 1) π)/(n + 1)) , 0], {i, m * n * (s + 1)}];

T = N[L.B];

U[x_, y_, t_] := If[x == 0 || x == m + 1 || y == 0 || y == n + 1, 0,
   If[t == 0, (x*π)/(m + 1) * (π - (x*π)/(m + 1)) * (y*π)/(n + 1) * (π - (y*π)/(n + 1)), T[[(x - 1) * n * (s + 1) + (y - 1) * (s + 1) + t]]]];
U1[x_, y_, t_] := U[(m + 1)/π * x, (n + 1)/π * y, (s + 1) * t];
pict = Table[{x, y, N[U1[x, y, 1/10], 7]}, {x, 0, π, π/(m + 1)}, {y, 0, π, π/(n + 1)}];
picJoin = {};
For[i = 1, i < 6, i++, picJoin = Join[picJoin, pict[[i]]]];
ListPlot3D[picJoin, PlotStyle → Yellow
, AxesLabel → {Style[x, Large], Style[y, Large], Style[u, Large]}, PlotLabel → "Numerical solution(Finite Difference Method)"]
U2[x_, y_, t_] :=
  Sum[1/(m^3 n^3 π^2) 8 (-2 + 2 Cos[m π] + m π Sin[m π]) (n π Cos[n π/2] - 2 Sin[n π/2]) Sin[n π/2] * Exp[-(n^2 + m^2) t] * Sin[n * x] * Sin[m * y], {n, 1, 20}, {m, 1, 20}];
Plot3D[U2[x, y, 1/10], {x, 0, π}, {y, 0, π}, PlotStyle → Blue, AxesLabel → {Style[x, Large], Style[y, Large], Style[u, Large]},
 PlotLabel → "Truncation of exact solution up to 20  "]
pic3 = Table[U1[x, y, 1/10] - U2[x, y, 1/10], {x, 0, π, π/(m + 1)}, {y, 0, π, π/(n + 1)}];
ListPlot3D[pic3, PlotStyle → Red, PlotRange → All, AxesLabel → {Style[x, Large], Style[y, Large], Style[z, Large]},
 PlotLabel → "The difference between numerical solution and truncation of exact solution"]
Clear[i, j];
Tn = TableForm[Table[U1[x, y, 1/10] - U2[x, y, 1/10], {x, 0, π, π/(m + 1)}, {y, 0, π, π/(n + 1)}],
  TableHeadings → {Table[Subscript[u, i, j], {i, 0, m + 1}], Table[Subscript[u, i, j], {j, 0, m + 1}]}, TableSpacing → {1, 2}]

pict = Table[{x, y, N[U1[x, y, 0], 7]}, {x, 0, π, π/(m + 1)}, {y, 0, π, π/(n + 1)}];
picJoin = {};
```

```
For[i = 1, i < 6, i++, picJoin = Join[picJoin, pict[[i]]]];
ListPlot3D[picJoin, PlotStyle → Yellow
, AxesLabel → {Style[x, Large], Style[y, Large], Style[u, Large]}, PlotLabel → "Numerical solution(Finite Difference Method)"]
Plot3D[U2[x, y, 0], {x, 0, π}, {y, 0, π}, PlotStyle → Blue, AxesLabel → {Style[x, Large], Style[y, Large], Style[u, Large]},
 PlotLabel → "Truncation of exact solution up to 20  "]
pic3 = Table[{x, y, U1[x, y, 0] - U2[x, y, 0]}, {x, 0, π, π/(m + 1)}, {y, 0, π, π/(n + 1)}];
picJoin3 = {};
For[i = 1, i < 6, i++, picJoin3 = Join[picJoin3, pic3[[i]]]];
ListPlot3D[picJoin3, PlotStyle → Red, PlotRange → All, AxesLabel → {Style[x, Large], Style[y, Large], Style[z, Large]},
 PlotLabel → "The difference between numerical solution and truncation of exact solution"]
Tn = TableForm[Table[N[U1[x, y, 0] - U2[x, y, 0], 9], {x, 0, π, π/(m + 1)}, {y, 0, π, π/(n + 1)}],
  TableHeadings → {Table[Subscript[u, i, j], {i, 0, m + 1}], Table[Subscript[u, i, j], {j, 0, m + 1}]}, TableSpacing → {1, 2}]
```