國立交通大學

應用數學系

碩 士 論 文

黎曼空間的理論和其在微分方程上的應用

Theory of Riemann Surfaces and Its Applications to Differential Equations

研 究 生：吳昀庭

指導教授：李榮耀　教授

中 華 民 國 九 十 九 年 四 月

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A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

April 2010

Hsinchu, Taiwan, Republic of China

中華民國九十九年四月

誌　　謝

在交大研究所的日子裡，經歷了許多心境的轉折，曾經徬徨迷失不知所措過，忘記當初想來交大的目的。幸好在做研究寫論文的日子裡，不只找回了目的，更找回能吃苦肯努力且認真有目標的態度面對人生的自己。接下來的人生也將會全力以赴。

非常感謝李榮耀教授，對我的指導和鼓勵，不論是做學問或是做研究的方法，都給了學生許多教導，讓我對做研究產生了興趣和自信。不僅僅課業上，還有許多各方面，像是生涯規劃、做學問的態度等都給了學生許多教誨。謝謝林松山教授，題點了學生許多人生的道理。感謝李志豪教授，對於我的疑惑都耐心解答。

感謝父母對於我人生的夢想圖給予很大的尊重和支持，不只讓我專心的完成學業。並在我失敗，遇到挫折和錯誤時鼓勵我，點醒我。還要感謝弟不少的幫忙，教我如何使用繪圖軟體。謝謝朋友們的陪伴，讓我在交大的日子多了很多精彩。最後感謝交通大學的老師、系辦的人員及學長姐們，在我交大求學階段帶給我一個很棒的經歷，學到了很多，成長了很多。真的由衷地感謝。

黎曼空間的理論和其在微分方程上的應用

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摘 要

此篇文章主要在探討擁有

$$\frac{d^2u}{dt^2} + P_N(u) = 0$$

形式的非線性二階微分方程，其解的函數理論，其中 $P_N(u)$ 是 2N 或 2N-1 次多項式。此方程的解存在於 N-1 相黎曼空間上。我們要利用正確的代數結構來建構這些黎曼空間。以此為基準，無論是理論或數值上我們可以在黎曼空間執行路徑的積分，並在此原則上獲得其解。其中 $P_N(u)$ 的根扮演了重要的角色，而複數分析是我們主要的工具。

中 華 民 國 九 十 九 年 四 月

Theory of Riemann Surfaces and Its Applications to Differential Equations

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Abstract

In this paper, we study the function theory of the solutions of the nonlinear second-order equations which have the following forms,

$$\frac{d^2u}{dt^2} + P_N(u) = 0$$

where $P_N(u)$ is a polynomial of degree 2N-1 or 2N. Solutions of such equations reside on Riemann surfaces of genus N-1. We construct those Riemann surfaces with the correct algebraic structures. From which, we are able to perform path integrals on the Riemann surfaces theoretically and numerically, and, in principle, solutions can be derived. The roots of $P_N(u)$ play the essential roles in every aspects, and complex analysis is our main tool.

Contents

1 Introduction the Riemann Surface

$u'' + P_N(u) = 0$ is a nonlinear second-order equation where $P_N(u)$ is a polynomial of u with degree 2N-1 or 2N. From

$$u'' + P_N(u) = 0,$$

we have

$$u''u' + P_N(u)u' = 0$$

and

$$\frac{1}{2}(u')^2 + P_{N+1}(u) = E$$

where E is the integration constant. It is a conservation law. So the function theory of solutions u of the equation involve $\sqrt{\prod_{k=1}^{N+1}(u-u_k)}$, we must investigate the space where u reside.

Indeed, $f(z) = \sqrt{\prod_{k=1}^{n}(z-z_k)}$ is a two-valued function of z on complex plane $\mathbb{C}$. We use algebra and analysis to develop a new surface such that f becomes a single-valued and analytic function on this surface, namely, a Riemann Surface.

1.1 Construct the corresponding Riemann Surface

When $w, z \in \mathbb{C}$ and $w^m = z$, we use polar form to find the solution

$$w^m = z = |z|e^{i\theta} = |z|e^{i\theta+2n\pi}, \theta \in [-\pi, \pi), n \in \mathbb{Z}$$
$$\text{then } w = |z|^{\frac{1}{m}} e^{\frac{(\theta+2n\pi)i}{m}}, \theta \in [-\pi, \pi), n \in \mathbb{Z}$$

First, take $f(z) = \sqrt{z}$ for example, $f : \mathbb{C} \to \mathbb{C}$. Using polar form, let $z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)}$, $n \in \mathbb{Z}$

$$\begin{aligned} f(z) &= \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{\theta+2n\pi}{2}i} \\ &= \begin{cases} |z|^{\frac{1}{2}} e^{\frac{\theta}{2}i} & \text{if n:even,} \\ -|z|^{\frac{1}{2}} e^{\frac{\theta}{2}i} & \text{if n:odd.} \end{cases} \end{aligned} \tag{1}$$

is a two-valued function. Now we want to let $f(z)$ becomes a single valued function, so we modify its domain $\mathbb{C}$ to develop the corresponding Riemann Surface such that f becomes a single-valued and analytic function on this surface.

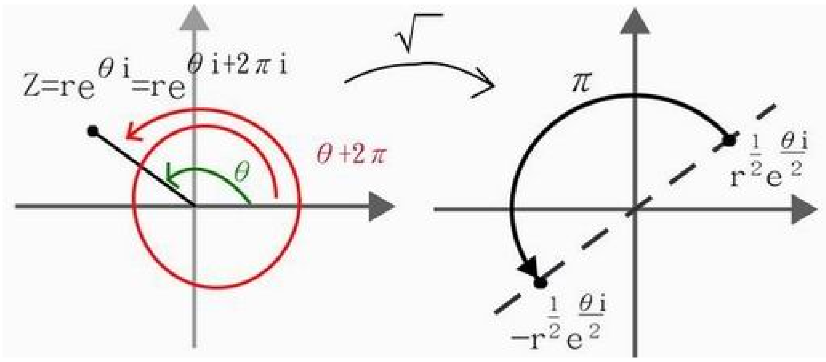

Figure 1: The idea of two sheets

Starting at $z = re^{i\theta}$, we have $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{\theta}{2}i}$, $r \neq 0$. Fixing r and continuing along a closed path once around the origin so that θ increases by 2π, $f(z)$ comes to the value $\sqrt{r}e^{\frac{\theta+2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}i}$ which is just the negative of its original value. Continuing above way then θ increases by 2π and $f(z)$ comes to original value. First, image two sheets lying over the complex plane and cut the plane along negative real axis (i.e. from 0 to infinite) and restrict ourselves so as never to continue $f(z)$ over this cuts, we get single-valued branches of $f(z)$. Define that

$$f(z) = |z|^{\frac{1}{2}}e^{\frac{i\theta}{2}}, -\pi \leq \theta < \pi \quad (2)$$

$$f(z) = |z|^{\frac{1}{2}}e^{\frac{i\theta}{2}}, \pi \leq \theta < 3\pi \quad (3)$$

called sheet-I and sheet-II, respectively. The cut on each sheet has two edges, label the edge of starting edge with $-$ and the edge of terminal edge with $+$. (Show in Figure 1.) Moreover, we cross the cut, we pass from one sheet to another. Second we extend the plane of complex numbers with one additional point at infinity constitute a number system known as the extended complex numbers. Use stereographic projection, we can consider the two sheets to be a spheres.

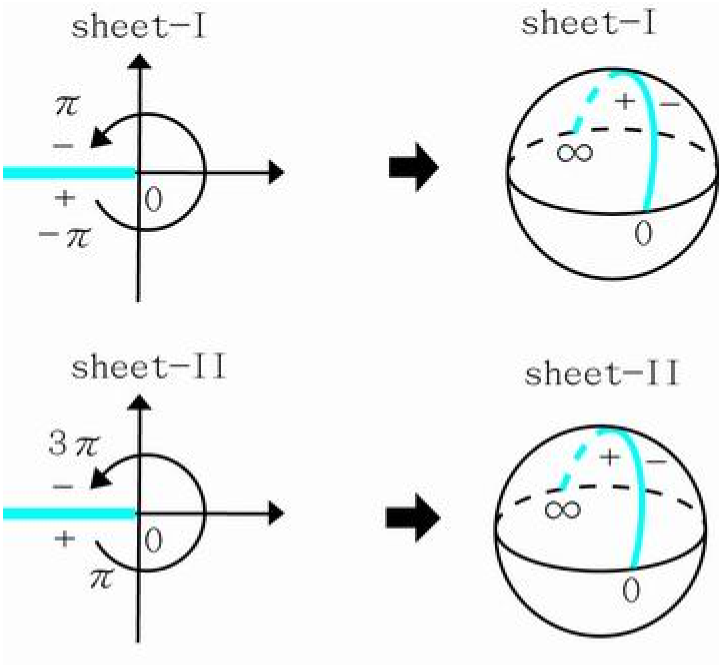

Figure 2: complex plane and extended complex plane

Next, image that the spheres are made of rubber and stretch each cut into circular holes.

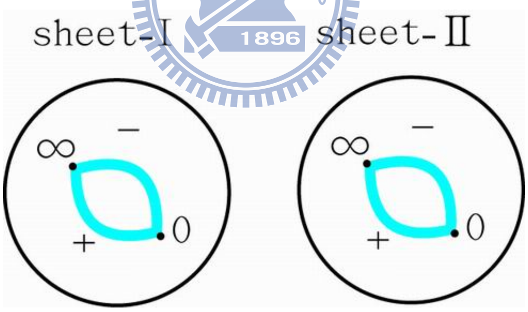

Figure 3: place the cuts open

Rotate the spheres until the holes face each other, and paste two cuts together (+)edge of sheet-I with (−)edge of sheet-II and (−)edge of sheet-I with (+)edge of sheet-II. We can derive a sphere. We called this sphere, Riemann surface of genus 0 , denoted R_0. Show in Figure 4. Notice that in Riemann Surface (+)edge of sheet-I is equivalent to (−)edge of sheet-II and (−)edge of sheet-I is equivalent to (+)edge of sheet-II.

We could using similar way to develop the corresponding Riemann surface for $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$.

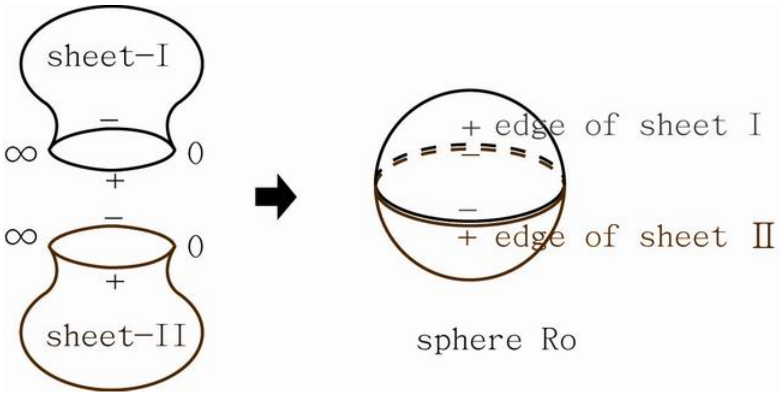

Figure 4: construct R_0

Example 1: There are 7 roots of f(z). Construct the Riemann Surface of $f(z) = \sqrt{\prod_{k=1}^{7}(z-z_k)} = \prod_{k=1}^{7}\sqrt{(z-z_k)}$, $z_k \in \mathbb{Z}$, $z_1 > z_2 > \ldots > z_7$, we cut plane starts from z_k to $-\infty$, $k = 1, \ldots, 7$.

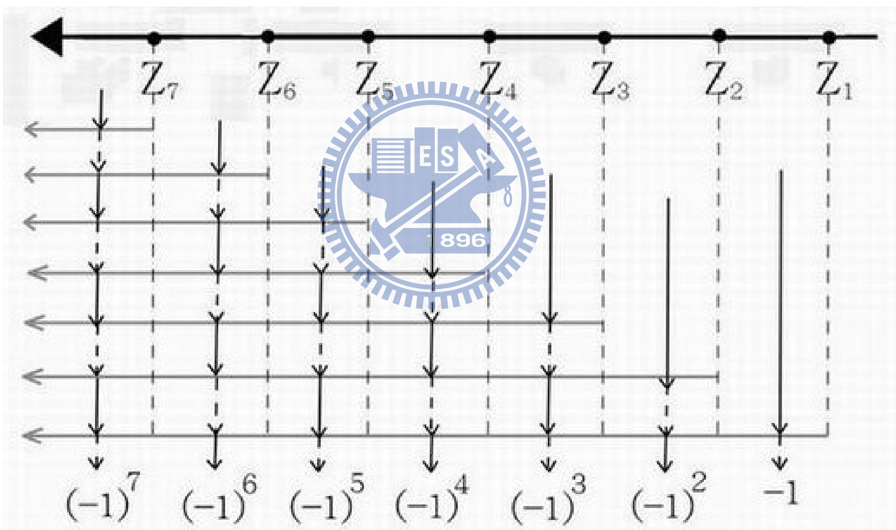

Figure 5: Cut plane start from z_k to $-\infty$

Since cross one cut, we pass from one sheet to another, the argument of z increases by 2π then argument of f(z) increases by π which is just the negative of its original value. We have crossing one cut need to change the sign, using -1 represent that. So crossing odd times will change sign and even times will no change.

Figure 6: The cut structure

There are branch cuts in $[z_1, z_2], [z_3, z_4], [z_5, z_6], [z_7, -\infty)$ and then using same idea to construct the corresponding Riemann Surface.

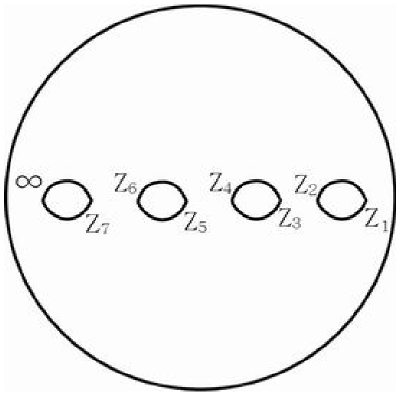

Figure 7: Placing the cuts open

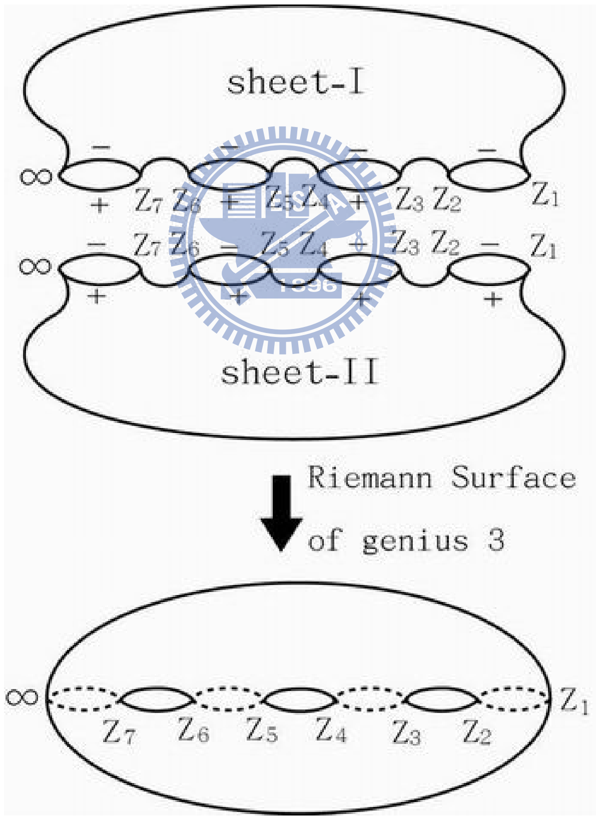

Figure 8: Geometric graph of R_3

Finally, to place the cuts open and put two sheet together with the rule (+)edge with (−)edge and then we obtain corresponding Riemann Surface of genus 3.

Example 2: Construct the Riemann Surface of $f(z) = \sqrt{\prod_{k=1}^{8}(z - z_k)} = \prod_{k=1}^{8} \sqrt{(z - z_k)}$, $z_k \in \mathbb{R}$, $z_1 > z_2 > ...... > z_8$. Similarly we cut plane start from z_k to $-\infty$, $k = 1, ..., 8$.

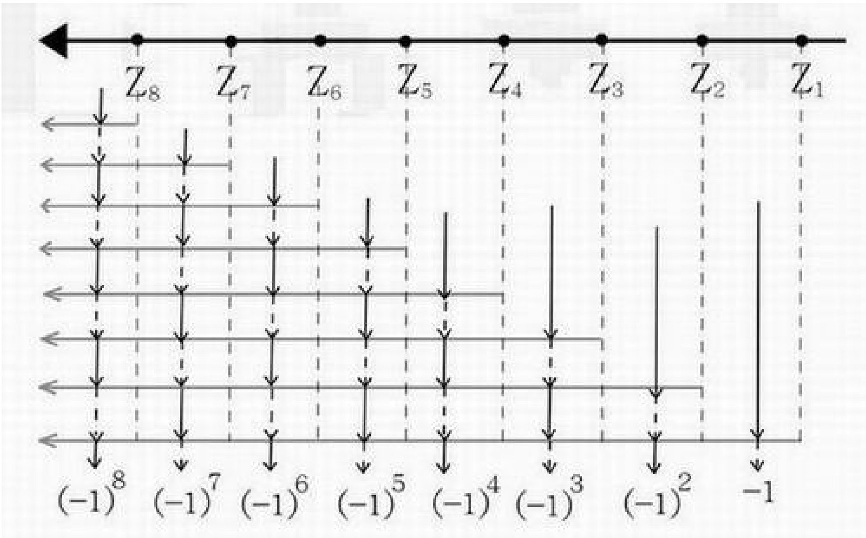

Figure 9: cut start from z_k to $-\infty$

As same as example 1, so there are branch cuts in $[z_1, z_2], [z_3, z_4], [z_5, z_6], [z_7, z_8]$.

Figure 10: The cut plane of example 2

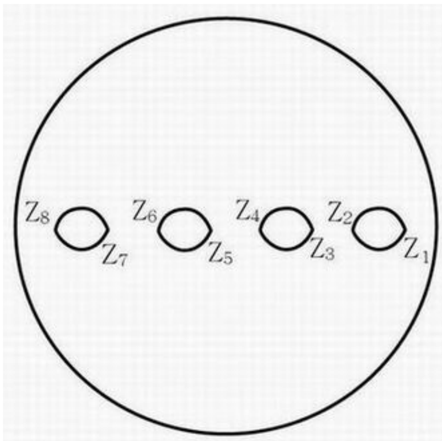

Figure 11: Same way to place the cut open

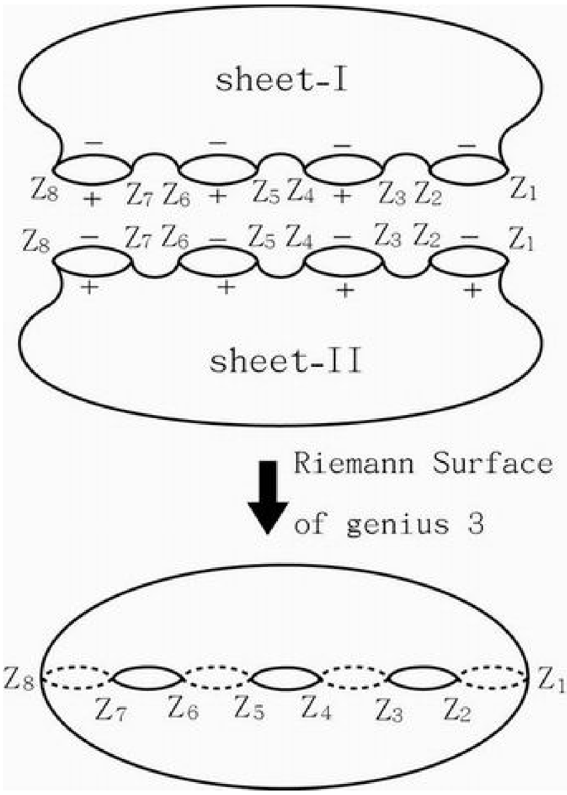

Figure 12: R_3

We found $f(z)$ of 7 or 8 roots have different algebraic structures but same geometric graph with 3 holes. That is no matter 7 or 8 points, we can construct corresponding Riemann Surface of genus 3.

In general situation, using same idea to construct Riemann surface of $f(z)$ where $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)} = \prod_{k=1}^{n}\sqrt{(z - z_k)}$, $z_k \in \mathbb{R}$ and $z_1 > z_2 > \ldots\ldots > z_n$ for horizontal cut. First, we cut plane starts from z_k to $-\infty$. If the curve cross even cuts it will no change that is becomes no cut. If the curve cross odd cuts, it will has a branch cut.

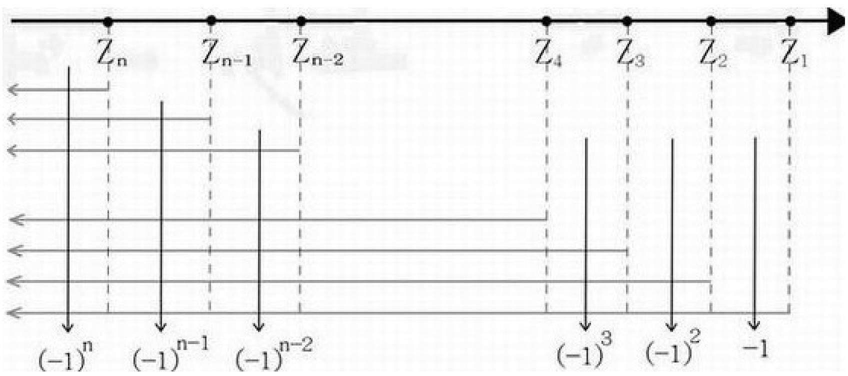

Figure 13: $n = 2N - 1$ or $2N$

So the cuts plane structure is

Case1. If n=2N-1. There are cuts along $[z_1, z_2], [z_3, z_4], \cdots, [z_{2j-1}, z_{2j}], \cdots, [z_{2N-1}, -\infty)$

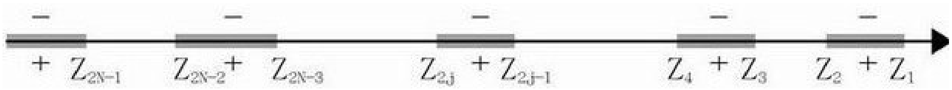

Figure 14: $n = 2N - 1$

Case2. If n=2N. There are cuts along $[z_1, z_2], [z_3, z_4], \cdots, [z_{2j-1}, z_{2j}], \cdots, [z_{2N-1}, z_{2N}]$

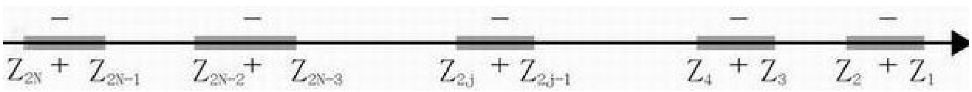

Figure 15: $n = 2N$

We use same idea to construct the corresponding Riemann Surface:

(1) n=2N-1

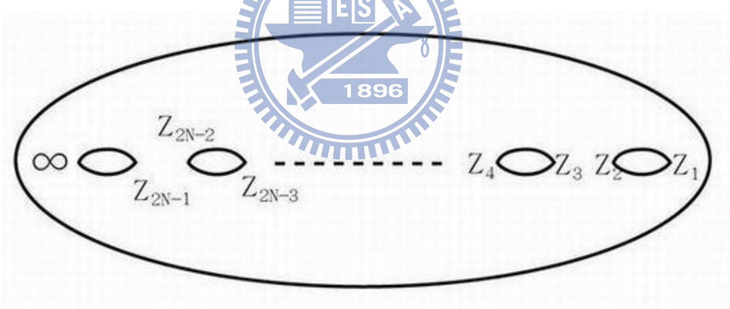

Figure 16: Placing cuts open in both sheets

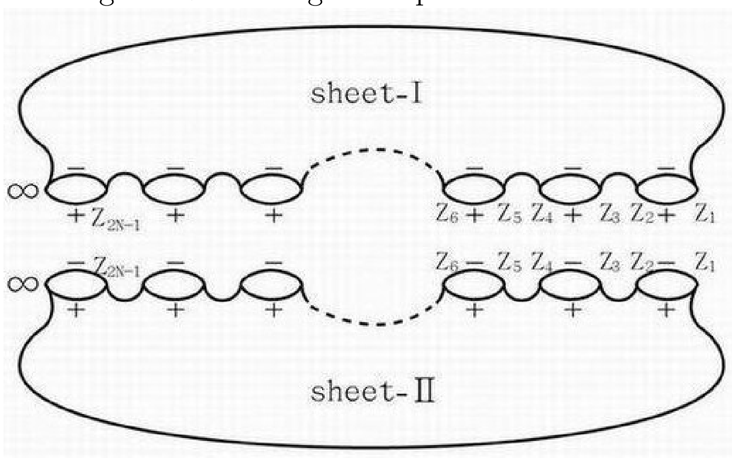

Figure 17: Together two sheets

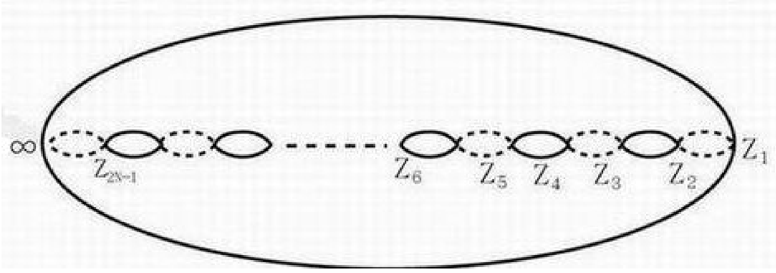

Figure 18: $N-1$ holes for $n = 2N-1$

becomes Riemann Surface with N-1 holes, that is R_{N-1} (2) n=2N

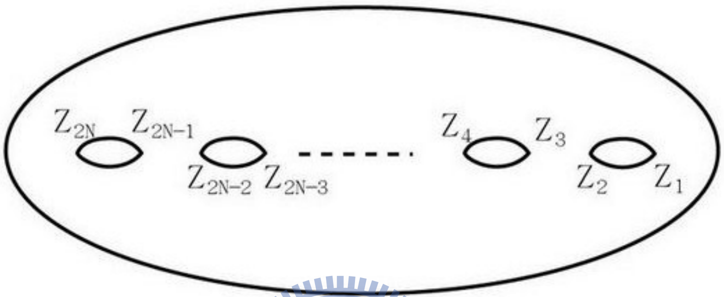

Figure 19: do this in two sheets

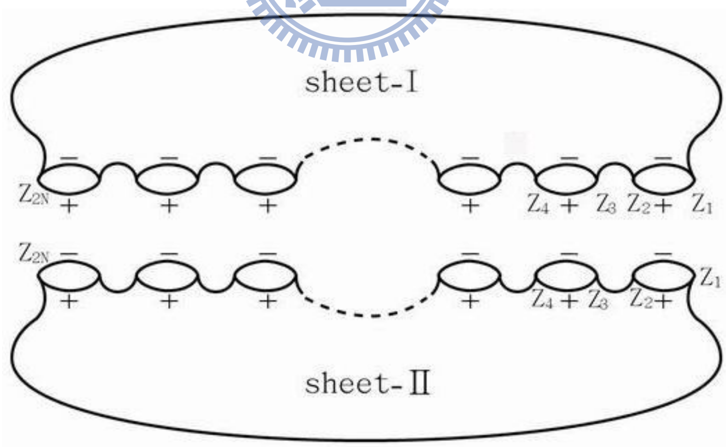

Figure 20: together two sheets

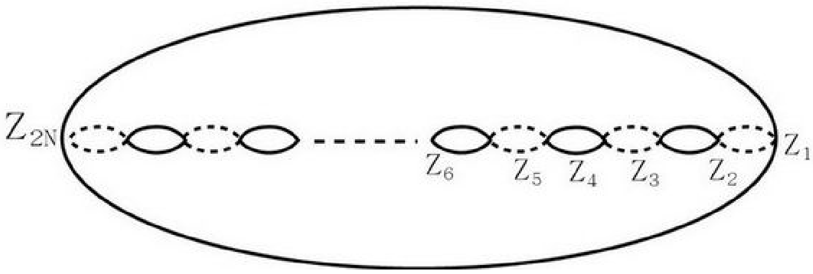

Figure 21: $N-1$ holes for $n=2N$

So

$$f(z)=\sqrt{\prod_{k=1}^{2N-1}(z-z_k)}\overset{or}{=}\sqrt{\prod_{k=1}^{2N}(z-z_k)}$$

will make N cuts and construct Riemann surface of genus N-1. (There is N-1 holes of geometric graph.)

1.2 The curve in algebraic and geometric structure

For convenience, we use algebraic to discuss and compute the integrals later. We already know the relation of algebraic and geometric structure with $f(z)=\sqrt{\prod_{k=1}^{n}(z-z_k)}$ and how to create the Riemann surface. Here give some examples to show that the curve in algebraic structure and its corresponding in geometric structure.

We defined something as follow:

1. The curve in sheet-I is solid line and the curve in sheet-II is dash line in algebraic structure.
2. The curve in overhead Riemann Surface is solid line and the curve in ventral Riemann Surface is dash line in geometric structure.

Example 1.

r_1 is the curve from a point at (I,+) to (I,−) in sheet-I and r_2 is the curve from a point at (II,+) to (II,−) in sheet-II.

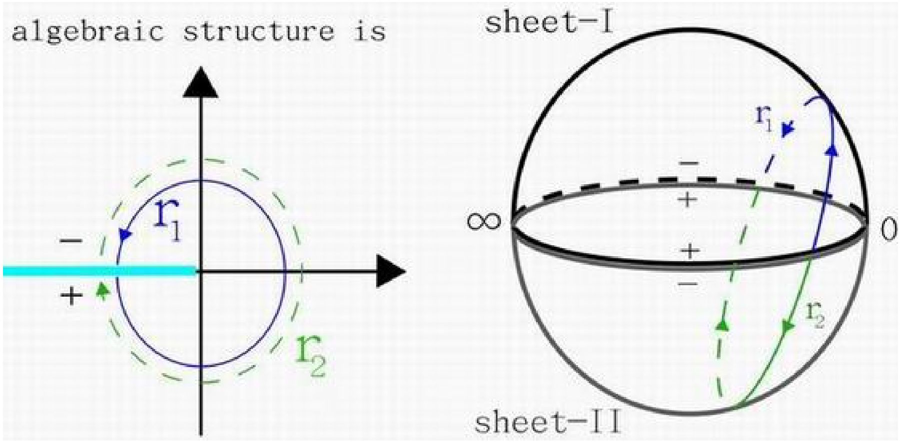

Figure 22: Example 1.

Example 2. The curve r is start from point A in sheet-I and cross the cut to point B on sheet-II

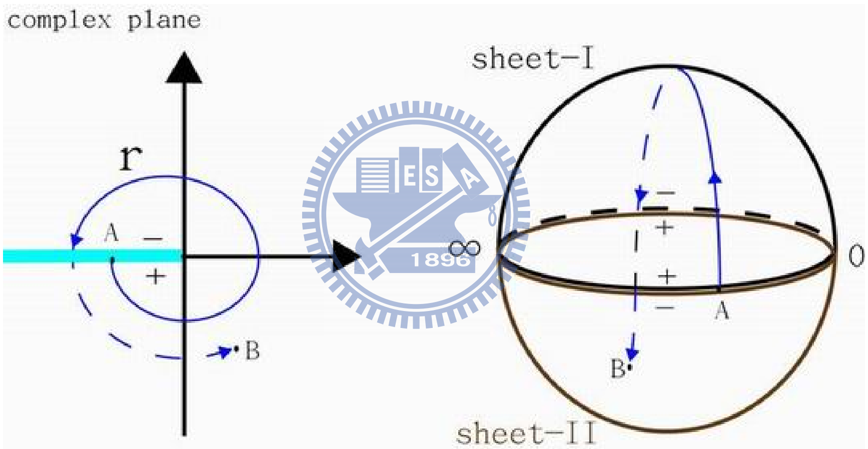

Figure 23: Another example

1.3 The a,b cycles and its equivalent paths

We know every closed curve on Riemann Surface R_N can be deformed into an integral combination of the loop-cut a_i and b_i, i=1,2,...,N. So in this paper, we will consider the integrals of $f(z)$ over a, b-cycles help us to obtain the integrals easier. Example $f(z) = \sqrt{z(z-1)(z-2)(z-3)}$

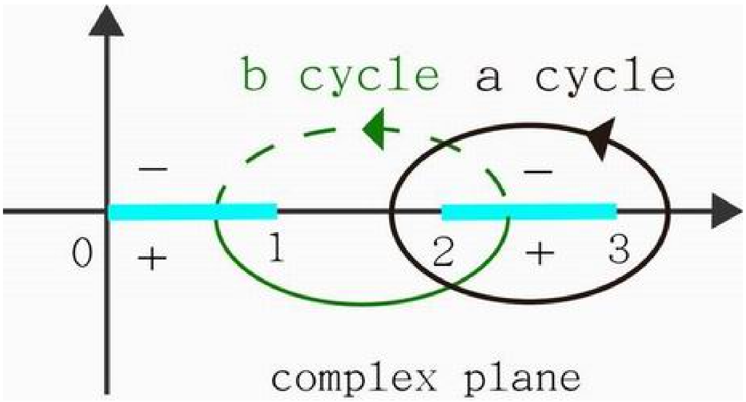

Figure 24: Cut plane and a,b cycle of $f(z) = \sqrt{z(z-1)(z-2)(z-3)}$

If f(z) has four roots and then construct two cuts and one a, b cycle. Notice a, b cycles have the same amount.

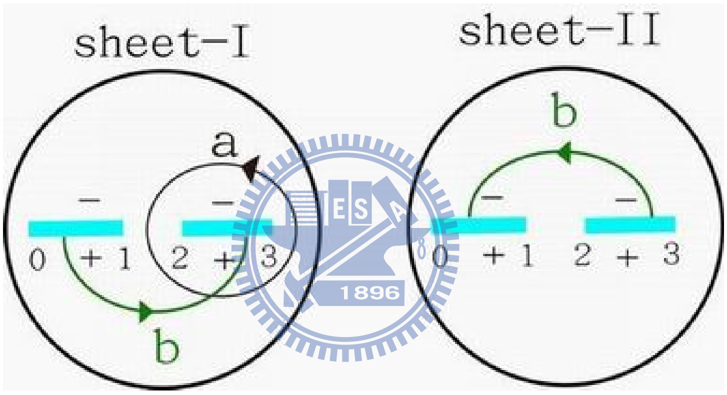

Figure 25: Step 1

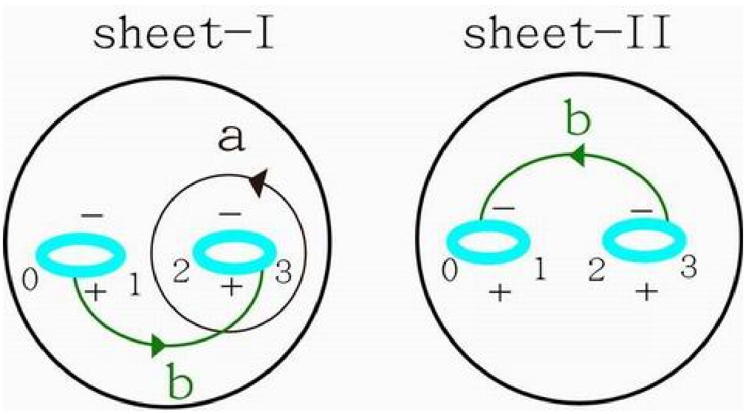

Figure 26: Step 2

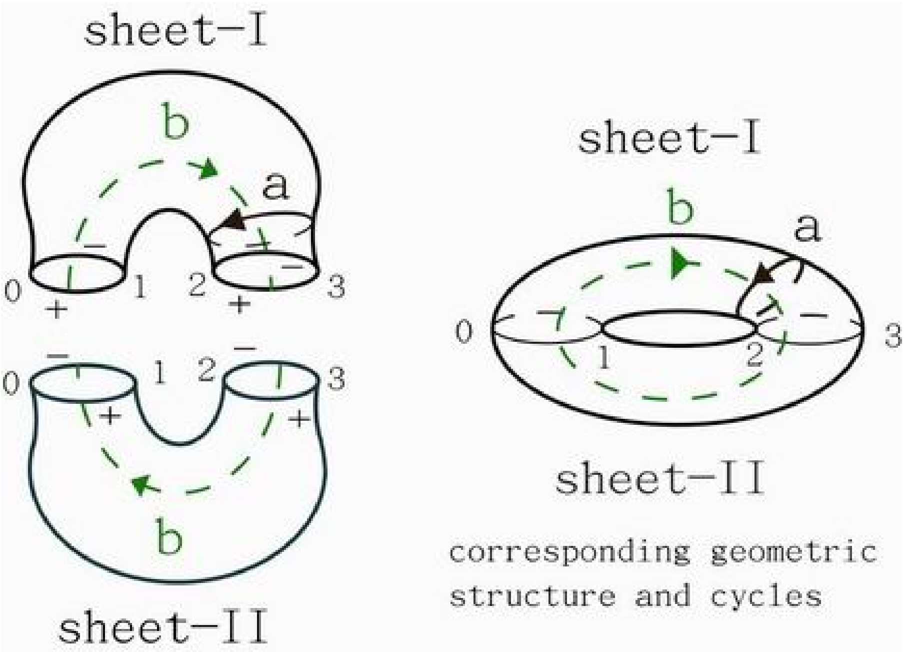

Figure 27: Step 3 and Step 4

Now if $f(z)$ has 2N-1 or 2N roots, there are loop-cuts a_i and b_i, i=1,2,...,N-1.

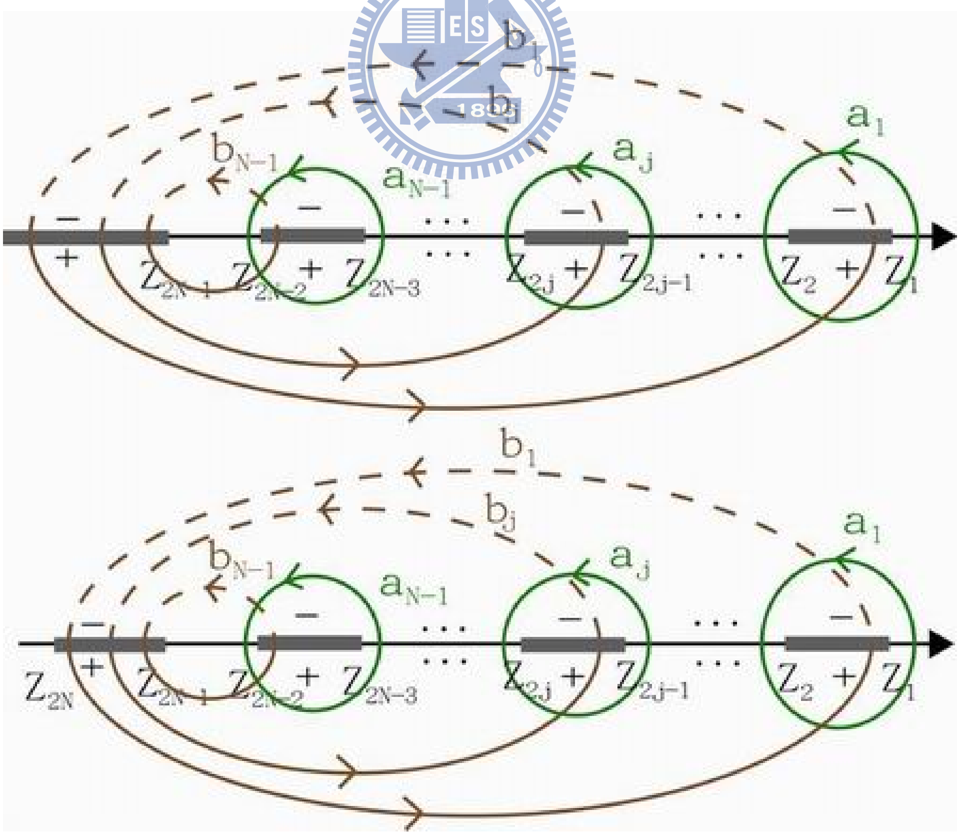

Figure 28: a,b cycles on complex plane

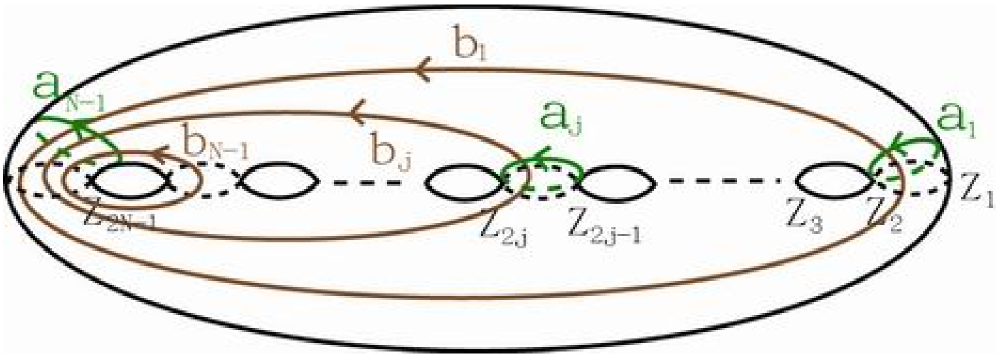

Figure 29: a,b cycles on Riemann Surface

Each a cycles are non-overlapping and each b cycles are non-overlapping. Also a, b cycles have the same number.

Sometimes the curves are difficult to write out their parameters, but straight lines are ease to write out their parameters. It could help us quicker and easier to obtain the integrals over the curves. So now using homotopic of curves to find the equivalent paths of curves. Take an example to explain.

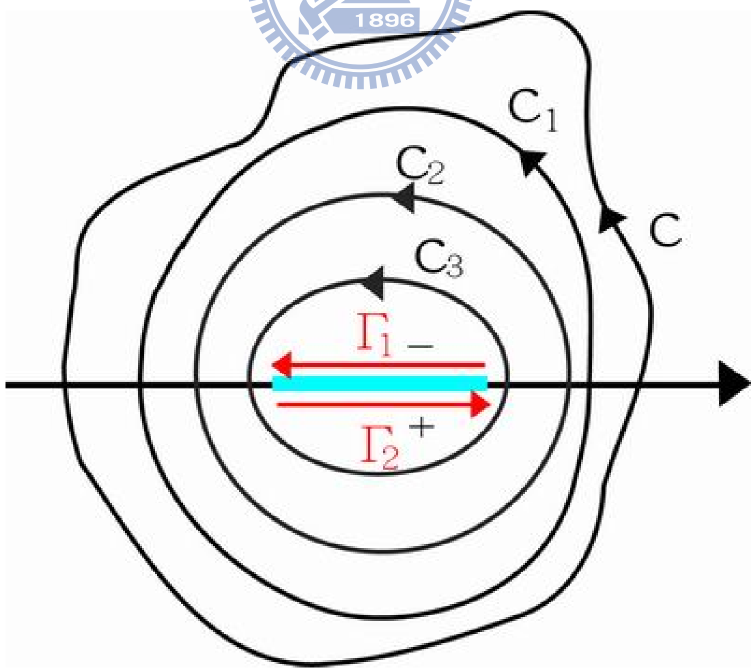

Figure 30: Homotopic

From C is homotopic to C_1, denotes $C \approx C_1$. We have

$$\int_C f(z)^{-1} dz = \int_{C_1} f(z)^{-1} dz + \int_{\Gamma_2} f(z)^{-1} dz$$

In figure 30, $C \approx C_1 \approx C_2 \approx C_3$ and finally we compression the curve C until we find the equivalent paths of curves $C \approx \Gamma_1 \bigcup \Gamma_2$. So

$$\int_C f(z)^{-1} dz = \int_{\Gamma_1} f(z)^{-1} dz + \int_{\Gamma_2} f(z)^{-1} dz$$

We will use this tool in hole paper.

1.4 Conclusion of Riemann Surface

The above statement and result all in horizontal cut, but the way is the same in any cut. Take $w^2 = a(z - z_1)(z - z_2)(z - z_3)$, where z_1, z_2, z_3 are distinct for example. We let $f(z) = \sqrt{(z - z_1)}\sqrt{(z - z_2)}\sqrt{(z - z_3)}$ and discuss $f(z)$. Cause $\sqrt{a}$ does not influence the cuts. For $f(z)$ the factor $\sqrt{(z - z_k)}$ change the sign when when $\arg(z - z_k)$ changes by 2π. We cut complex plane from z_1 to z_2 and from z_3 to ∞. Label left of cut with (+)edge and right of cut with (−)edge

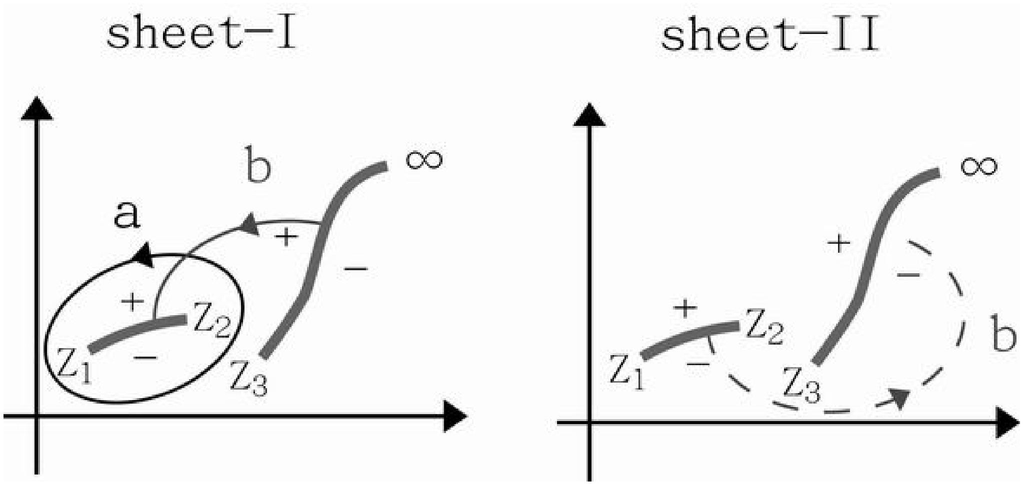

Figure 31: The cut-plane and a, b cycle in sheets

Using similarly way to construct the Riemann Surface. Image this two sheets are made of rubber, and together the (+)edges of sheet-I with the (−)edges of sheet-II. We get correspond Riemann Surface R_1. The curve a,b correspond to the meridian curve a and latitude curve b on Riemann Surface R_1, respectively.

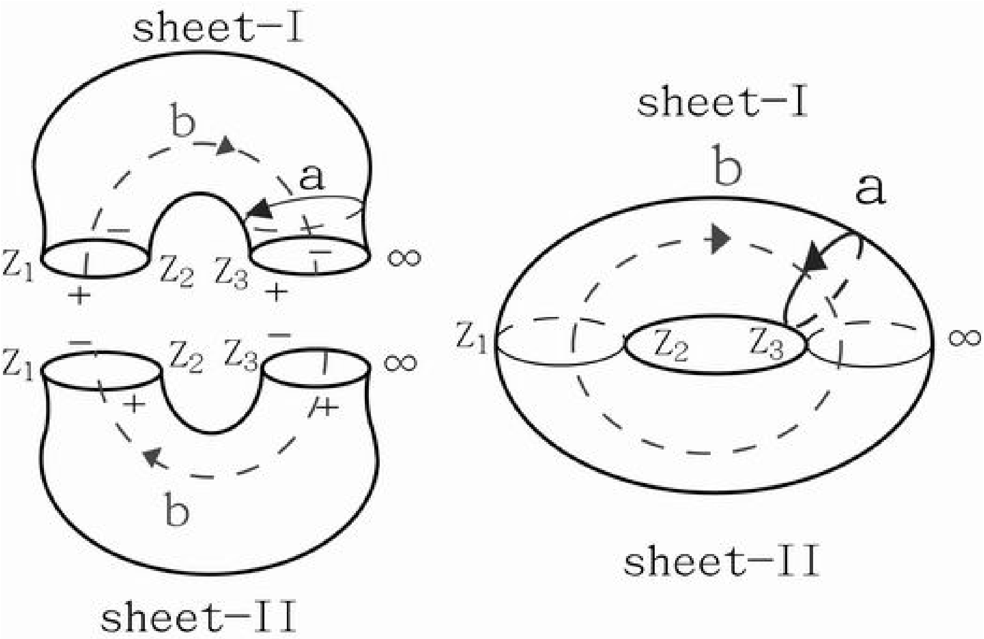

Figure 32: Corresponding Riemann Surface

For arbitrary cut, if f(z) has 2N-1 or 2N roots, then

1. There are N cuts in complex plane.

2. It's geometric graph has N-1 holes, that is construct corresponding Riemann Surface of genus N-1.

3. There are N-1 a-cycles and N-1 b-cycles.

2 The integrals of $\frac{1}{f(z)}$ over a,b cycles for horizontal cut

We will use Mathematica help us to obtain the values of integrals of $\frac{1}{f(z)}$ over a,b cycles. First, discuss the values in sheet-I, sheet-II and Mathematica for horizontal cuts. $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$, using polar form $\prod_{k=1}^{n}(z - z_k) = re^{\theta i}$. Let θ_1 denotes θ in sheet-I and θ_2 denotes in sheet-II. So

$$\theta_2 = \theta_1 + 2\pi$$

We have

$$\begin{aligned} f(z)|_{(II)} &= \sqrt{r}e^{\frac{\theta_2}{2}i} \\ &= \sqrt{r}e^{\frac{\theta_1+2\pi}{2}i} \\ &= \sqrt{r}e^{\frac{\theta_1}{2}i}e^{\pi i} \\ &= -\sqrt{r}e^{\frac{\theta_1}{2}i} = -f(z)|_{(I)} \end{aligned} \tag{4}$$

where $f(z)|_{(I)}$ denote the value of $f(z)$ with z in sheet-I and $f(z)|_{(II)}$ means z in sheet-II. Because the difference of argument between z in sheet-I and in sheet-II is 2π , that is the difference between $f(z)|_{(I)}$ and $f(z)|_{(II)}$ is π . So $f(z)|_{(I)} = -f(z)|_{(II)}$.

Now discuss the difference in sheet-I of theory and in Mathematica. First, $\sqrt{-1}$. From the definition of argument in sheet-I, $\sqrt{-1} = -i$, but we compute $\sqrt{-1}$ in Mathematica obtain $\sqrt{-1} \overset{Math.}{=} i$. Why? We found that $\theta \in (-\pi, \pi]$ of $re^{i\theta}$ in Mathematica, actually.

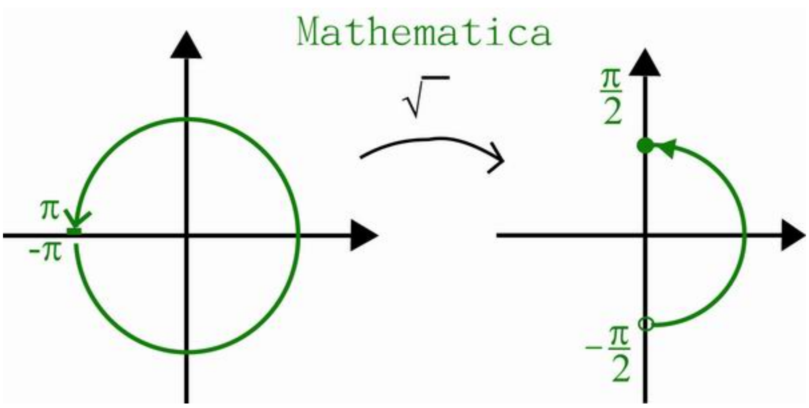

Figure 33: Domain and range in Mathematica

For any other θ of $re^{i\theta}$ which does not belong to $(-\pi,\pi]$, Mathematica will conversion $re^{i\theta}$ into $re^{i\theta^*}$, $\theta^* \in (-\pi,\pi]$ where $re^{i\theta} = re^{i\theta^*}$.

Compare the value of $f(z)$ with z in sheet-I and in Mathematica, we discover that

Lemma 1. *If* $\prod_{k=1}^{n}(z-z_k) = re^{i\theta}$ *in sheet-I for horizontal cut*

$$f(z)|_{(I)} = \begin{cases} f(z)|_{Mathematica} & \text{if } \theta \in (-\pi,\pi), \\ -f(z)|_{Mathematica} & \text{if } \theta = -\pi \end{cases}$$

Proof.

Since$-\pi$ does not in $(-\pi,\pi]$, Mathematica will conversion $re^{-\pi i}$ into $re^{\pi i}$ and $re^{-\pi i} = re^{\pi i}$ but $f(z)$ will different.

In theory: $\quad -1 = e^{-\pi i} \quad \xrightarrow{\sqrt{}} \sqrt{-1} = e^{-\frac{\pi i}{2}} = -i.$

In Mathematica: $\quad -1 = e^{-\pi i} \overset{Math.}{=} e^{\pi i} \quad \xrightarrow{\sqrt{}} \sqrt{-1} = e^{\frac{\pi i}{2}} = i$

So $f(z) \overset{Math.}{=} -f(z)$ if $\theta = -\pi$ in Mathematica. ∎

In hole paper, $f(z) \overset{Math.}{=} -f(z)$ denotes the polynomial $f(z)$ in front of $\overset{Math.}{=}$ is the value of $f(z)$ in theory and the polynomial f(z) behind the $\overset{Math.}{=}$ is the value of $f(z)$ in Mathematica.

After we known the state above, we must modify the computation when we want to use Mathematica to calculate the value. Take example to explain: evaluate $\int_r \frac{1}{f(z)}dz$ where $f(z) = \sqrt{z(z-1)(z-2)}$, $z \in R$ and $r = r_1 \bigcup r_2$ where $r_1 =$ the path on a horizontal cut from 1 to 2 with (+)edge of sheet-I and $r_2 =$ the path on a horizontal cut from 2 to 1 with (−)edge of sheet-I.

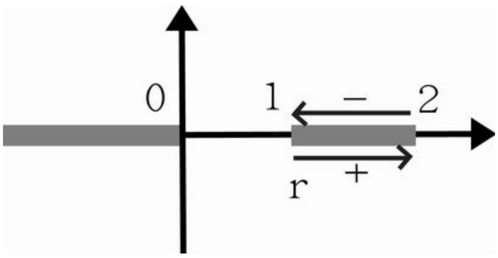

Figure 34: cuts in complex plane of $f(z) = \sqrt{z(z-1)(z-2)}$

Analysis the integrals: $f(z) = \sqrt{z(z-1)(z-2)} = \sqrt{z}\sqrt{z-1}\sqrt{z-2}$

1. If $z \in r_1$.

(1) In theory: $z \geq 0$ then $\sqrt{z} = |z|^{\frac{1}{2}}$ and $z-1 \geq 0$ then $\sqrt{z-1} = |z-1|^{\frac{1}{2}}$

$z-2 < 0$ then $z-2 = |z-2|e^{-\pi i}$, so $\sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z-2|^{\frac{1}{2}}$

$$\int_{r_1} \frac{1}{f(z)} dz = i \int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} dz$$

(2) In Mathematica: $z \geq 0$ then $\sqrt{z} = |z|^{\frac{1}{2}}$, $z-1 \geq 0$ then $\sqrt{z-1} = |z-1|^{\frac{1}{2}}$

$z-2 < 0$ then $z-2 = |z-2|e^{\pi i}$. We have $\sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{\frac{\pi}{2}i} = i|z-2|^{\frac{1}{2}}$

$$\int_{r_1} \frac{1}{f(z)} dz = -i \int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} dz$$

Compare (1) and (2), we found there a difference of a minus sign with the value in sheet-I and in Mathematica.

2. $z \in r_2$

(1) In theory: $z \geq 0$ then $\sqrt{z} = |z|^{\frac{1}{2}}$, $z-1 \geq 0 \Rightarrow \sqrt{z-1} = |z-1|^{\frac{1}{2}}$

$z-2 < 0$ then $z-2 = |z-2|e^{\pi i}$ then $\sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{\frac{\pi}{2}i} = i|r-2|^{\frac{1}{2}}$

$$\int_{r_2} \frac{1}{f(z)} dz = -i \int_2^1 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} dz$$

(2) In Mathematica: $z \geq 0$ then $\sqrt{z} = |z|^{\frac{1}{2}}$, $z-1 \geq 0$ then $\sqrt{z-1} = |z-1|^{\frac{1}{2}}$

$z-2 < 0$ then $z-2 = |z-2|e^{\pi i}$ then $\sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{\frac{\pi}{2}i} = i|r-2|^{\frac{1}{2}}$

$$\int_{r_2} \frac{1}{f(z)} dz = -i \int_2^1 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} dz$$

Compare (1) and (2), it is the same.

By 1,2 we have

$$\begin{aligned} \int_r \frac{1}{f(z)} dz &= \begin{cases} 2i \int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} dz & \text{in sheet-I ,} \\ 0 & \text{in Mathematica} \end{cases} \\ &= \begin{cases} 0. + 5.24412i & \text{in sheet-I ,} \\ 0 & \text{in Mathematica} \end{cases} \end{aligned}$$

Clearly, there is a mistake when $\theta = -\pi$. When we use Mathematica to get the value of integration we want, we need modify some range where the value will wrong.

Determine the difference of sign(f) (same or negative) and then modify the computation of Mathematica to get right value. Because sometimes the form of integration is complex, if we could simplify the way about modify the difference of sigh(f), it will help us to get right value easier.

Example: Same f(z) as the example before, using lemma1 to modify.

1. If $z \in r_1$, $z : 1 \to 2$

$$z \geq 0 \quad \text{then} \quad \arg(z) = 0 \quad \text{then} \quad \sqrt{z} \overset{Math.}{=} \sqrt{z}$$
$$z-1 \geq 0 \quad \text{then} \quad \arg(z-1) = 0 \quad \text{then} \quad \sqrt{z-1} \overset{Math.}{=} \sqrt{z-1}$$
$$z-2 < 0 \quad \text{then} \quad \arg(z) = -\pi \quad \text{then} \quad \sqrt{z-2} \overset{Math.}{=} -\sqrt{z-2}$$

$$\int_{r_1} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^2 \frac{1}{\sqrt{z}} \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z-2}} dz$$

2. If $z \in r_2$, $z : 2 \to 1$

$$z \geq 0 \quad \text{then} \quad \arg(z) = 0 \quad \text{then} \quad \sqrt{z} \overset{Math.}{=} \sqrt{z}$$
$$z-1 \geq 0 \quad \text{then} \quad \arg(z-1) = 0 \quad \text{then} \quad \sqrt{z-1} \overset{Math.}{=} \sqrt{z-1}$$
$$z-2 < 0 \quad \text{then} \quad \arg(z) = \pi \quad \text{then} \quad \sqrt{z-2} \overset{Math.}{=} \sqrt{z-2}$$

$$\int_{r_2} \frac{1}{f(z)} dz \overset{Math.}{=} \int_2^1 \frac{1}{\sqrt{z}} \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z-2}} dz$$

By 1,2 we have

$$\begin{aligned} \int_{r} \frac{1}{f(z)} dz &\overset{Math.}{=} -2\int_1^2 \frac{1}{\sqrt{z}} \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z-2}} dz \\ &= 0. + 5.24412i \end{aligned}$$

Take another example to consider how to modify computation in Mathematica such that numeral result is right for horizontal cuts. And discuss the difference between the value in theory and in Mathematica.

Example 2 : Evaluate $\int \frac{1}{f(z)} dz$ over a_1, a_2, a_3, b_1, b_2 and b_3 cycles. where $f(z) = \sqrt{(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)(z-9)}$. We analysis the integral in

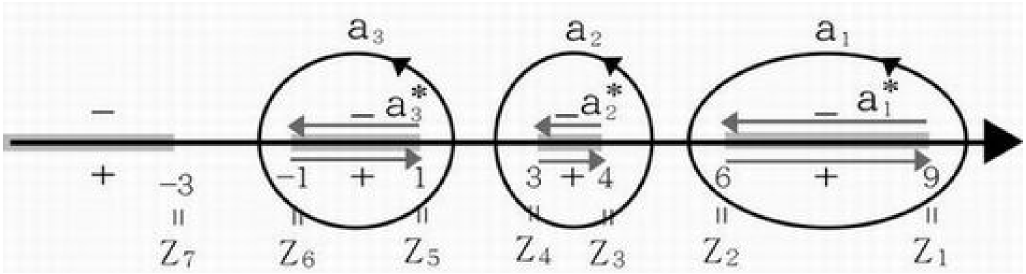

Figure 35: a-cycles and their equivalent path a^*

Mathematica and in theory to compare the result and using the result of angle to modify the computation to get value. Let $z_1 = 9, z_2 = 6, z_3 = 4, z_4 = 3, z_5 = 1, z_6 = -1, z_7 = -3$

Solution:

1. Let a_1 is a cycle center at $\frac{15}{2}$ with radius 2 and enclosed the cut $[6, 9]$. So let $z = \frac{15}{2} + 2e^{i\theta}$, we have

$$\begin{aligned}\int_{a_1} \frac{1}{f(z)} dz &= \int_{-\pi}^{\pi} \frac{2ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{\frac{15}{2} + 2e^{i\theta} - z_k}} d\theta \\ &= 1.0842 \times 10^{-19} + 0.0776642i\end{aligned}$$

By Cauchy Theorem. Since a_k cycle is simple connected, we can use some equivalent paths, say a_k^*, to easily compute the integrals for a_k cycle.

(1) If $z \in a_1^*$ of theory in sheet-I where $a_1^* = 6 \overset{+}{\to} 9 \bigcup 6 \overset{-}{\leftarrow} 9$

(a) $6 \overset{+}{\to} 9$: the path along x-axis from 6 to 9 on (+)edge of sheet-I.

$$z - 9 = -|z - 9| = |z - 9|e^{-\pi i} \text{ then } \frac{1}{\sqrt{z-9}} = |z - 9|^{-\frac{1}{2}} e^{\frac{\pi}{2}i} = |z - 9|^{-\frac{1}{2}} i$$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}},\ k = 2, 3, 4, 5, 6, 7$$

$$\int_{6 \overset{+}{\to} 9} \frac{1}{f(z)} dz = \int_6^9 i \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $6 \overset{-}{\leftarrow} 9$: the path along x-axis from 9 to 6 on sheet-I with (−)edge.

$$z - 9 = -|z - 9| = |z - 9|e^{\pi i} \text{ then } \frac{1}{\sqrt{z-9}} = |z - 9|^{-\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z - 9|^{-\frac{1}{2}}$$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}},\ k = 2, 3, 4, 5, 6, 7$$

$$\int_{6\overset{-}{\leftarrow}9}\frac{1}{f(z)}dz=\int_9^6(-i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

So by (a) and (b) we have

$$\begin{aligned}\int_{a_1^*}\frac{1}{f(z)}dz &= -2i\int_9^6\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz\\ &= 0.0776642i\end{aligned}$$

(2) Analysis the integral over a_1^* in Mathematica

(a) $6\overset{+}{\rightarrow}9$: the path along x-axis from 6 to 9 on sheet-I with (+)edge

$z-9=-|z-9|=|z-9|e^{\pi i}$ then $\frac{1}{\sqrt{z-9}}=|z-9|^{-\frac{1}{2}}e^{-\frac{\pi}{2}i}=-i|z-9|^{-\frac{1}{2}}$

$z-z_k=|z-z_k|$ then $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}},\ k=2,3,4,5,6,7$

$$\int_{6\overset{+}{\rightarrow}9}\frac{1}{f(z)}dz=\int_6^9(-i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

A difference of a minus sign with in sheet-I

(b) $6\overset{-}{\leftarrow}9$: the path along x-axis from 9 to 6 of sheet-I with (−)edge

$z-9=-|z-9|=|z-9|e^{\pi i}$ then $\frac{1}{\sqrt{z-9}}=|z-9|^{-\frac{1}{2}}e^{-\frac{\pi}{2}i}=-i|z-9|^{-\frac{1}{2}}$

$z-z_k=|z-z_k|$ then $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}},\ k=2,3,4,5,6,7$

$$\int_{6\overset{-}{\leftarrow}9}\frac{1}{f(z)}dz=\int_9^6(-i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

as same as in sheet-I.

But in Mathematica

$$\int_{a_1^*}\frac{1}{f(z)}dz=0$$

(3) Using the results before to modify

(a) $6\overset{+}{\rightarrow}9$: the path along x-axis from 6 to 9 on sheet-I with (+)edge

$\arg(z-z_1)=-\pi$ then $\sqrt{z-z_1}\overset{Math.}{=}-\sqrt{z-z_1}$

$\arg(z-z_k)=0$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k},\ k=2,\ldots,7$

So $f(z)\overset{Math.}{=}-f(z)$

(b) $6 \overset{-}{\leftarrow} 9$: the path along x-axis from 9 to 6 on sheet-I with $(-)$edge

$\arg(z - z_1) = \pi$ then $\sqrt{z-z_1} \overset{Math.}{=} \sqrt{z-z_1}$

$\arg(z - z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2, \ldots, 7$

So $f(z) \overset{Math.}{=} f(z)$

We have

$$\begin{aligned} \int_{a_1^*} \frac{1}{f(z)} dz &\overset{Math.}{=} -2 \int_6^9 \frac{1}{f(z)} dz \\ &= 0.0776642i \end{aligned}$$

2. Let a_2 is a cycle center at $\frac{7}{2}$ with radius 1 and enclosed the cut $[3, 4]$. So let $z = \frac{7}{2} + e^{i\theta}$, we have

$$\begin{aligned} \int_{a_2} \frac{1}{f(z)} dz &= \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{\frac{7}{2} + e^{i\theta} - z_k}} d\theta \\ &= 0. - 0.200969i \end{aligned}$$

Same as a_1 , by Cauchy Theorem to compute equivalent path a_2^* where $a_2^* = 3 \overset{+}{\rightarrow} 4 \bigcup 3 \overset{-}{\leftarrow} 4$

(1) Analysis the integral of a_2^* in sheet-I

(a) $3 \overset{+}{\rightarrow} 4$: the path along x-axis from 3 to 4 on sheet-I with $(+)$edge

$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i, \ k = 1, 2, 3$$

$$z - z_k = |z - z_k| \quad \text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4, 5, 6, 7$$

$$\int_{3 \overset{+}{\rightarrow} 4} \frac{1}{f(z)} dz = \int_3^4 i^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $3 \overset{-}{\leftarrow} 4$: the path along x-axis from 4 to 3 on sheet-I with $(-)$edge

$$z - z_k = -|z - z_k| = |z - z_k| e^{\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z - z_k|^{-\frac{1}{2}}, \ k = 1, 2, 3$$

$$z - z_k = |z - z_k| \quad \text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4, 5, 6, 7$$

$$\int_{3\overset{-}{\leftarrow}4} \frac{1}{f(z)}dz = \int_4^3 (-i)^3 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

So we have

$$\begin{aligned}\int_{a_2^*} \frac{1}{f(z)}dz &= -2\int_4^3 (-i)^3 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz \\ &= 0. - 0.200969i\end{aligned}$$

(2) Analysis the integral of a_2^* in Mathematica

(a) $3 \overset{+}{\rightarrow} 4$: the path along x-axis from 6 to 9 on sheet-I with (+)edge

$$z - z_k = -|z-z_k| = |z-z_k|e^{\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}e^{-\frac{\pi}{2}i} = -i|z-z_k|^{-\frac{1}{2}}, \ k = 1,2,3$$

$$z - z_k = |z-z_k| \quad \text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}, \ k = 4,5,6,7$$

$$\int_{3\overset{+}{\rightarrow}4} \frac{1}{f(z)}dz = \int_3^4 (-i)^3 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

(b) $3 \overset{-}{\leftarrow} 4$: the path along x-axis from 4 to 3 on sheet-I with (−)edge

$$z - z_k = -|z-z_k| = |z-z_k|e^{\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}e^{-\frac{\pi}{2}i} = -i|z-z_k|^{-\frac{1}{2}}, \ k = 1,2,3$$

$$z - z_k = |z-z_k| \quad \text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}, \ k = 4,5,6,7$$

$$\int_{3\overset{-}{\leftarrow}4} \frac{1}{f(z)}dz = \int_4^3 (-i)^3 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

But by (a),(b) we obtain different value in Mathematica

$$\int_{a_2^*} \frac{1}{f(z)}dz = 0$$

(3) Using lemma 1 to modify

(a) $3 \overset{+}{\rightarrow} 4$: the path along x-axis from 3 to 4 on sheet-I with (+)edge

$\arg(z-z_k) = -\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2,3$

$\arg(z-z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4,5,6,7$

So $f(z) \overset{Math.}{=} -f(z)$

(b) $3 \overset{-}{\leftarrow} 4$: the path along x-axis from 4 to 3 on sheet-I with $(-)$edge

$\arg(z-z_k)=\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k=1,2,3$

$\arg(z-z_k)=0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k=4,5,6,7$

So $f(z) \overset{Math.}{=} f(z)$

We have

$$\int_{a_3^*} \frac{1}{f(z)}dz \overset{Math.}{=} -2\int_3^4 \frac{1}{f(z)}dz = 0.-0.200969i$$

3. a_3 : Let a_3 is a cycle center at 0 with radius 2 and enclosed the cut $[-1,1]$. So let $z=2e^{i\theta}$, we have

$$\begin{aligned}\int_{a_3} \frac{1}{f(z)}dz &= \int_{-\pi}^{\pi} \frac{2ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{2e^{i\theta}-z_k}} d\theta \\ &= 3.46945 \times 10^{-18} + 0.151409i\end{aligned}$$

Same as a_1 , by Cauchy Theorem to compute equivalent path a_3^* where $a_3^* = -1 \overset{+}{\rightarrow} 1 \bigcup 1 \overset{-}{\leftarrow} -1$

(1) Analysis of a_3^* in theory

(a) $-1 \overset{+}{\rightarrow} 1$: the path along x-axis from -1 to 1 on sheet-I with $(+)$edge

$$z-z_k = -|z-z_k| = |z-z_k|e^{-\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}e^{\frac{\pi}{2}i} = i|z-z_k|^{-\frac{1}{2}},\ k=1,2,3,4,5$$

$$z-z_k = |z-z_k| \quad \text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}},\ k=6,7$$

$$\int_{-1\overset{+}{\rightarrow}1} \frac{1}{f(z)}dz = \int_{-1}^{1} i^5 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}}dz$$

(b) $-1 \overset{-}{\leftarrow} 1$: the path along x-axis from 1 to -1 on $(-)$edge of sheet-I

$$z-z_k = -|z-z_k| = |z-z_k|e^{\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}e^{-\frac{\pi i}{2}} = -i|z-z_k|^{-\frac{1}{2}},\ k=1,2,3,4,5$$

$$z-z_k = |z-z_k| \quad \text{then} \quad \frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}},\ k=6,7$$

$$\int_{-1\overset{-}{\leftarrow}1} \frac{1}{f(z)}dz = \int_{1}^{-1} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

By (a), (b), we obtain the value

$$\begin{aligned}\int_{a_3^*} \frac{1}{f(z)}dz &= 2\int_{-1}^{1} i^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &= 2\int_{-1}^{1} i \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &\overset{Math.}{=} 0.0151409i\end{aligned}$$

(2) Consider a_3^* in Mathematica

(a) $-1 \overset{+}{\rightarrow} 1$:

$$z - z_k = -|z - z_k| = |z - z_k|e^{\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi i}{2}} = -i|z - z_k|^{-\frac{1}{2}}, \ k = 1, 2, 3, 4, 5$$

$$z - z_k = |z - z_k| \quad \text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 6, 7$$

$$\int_{-1\overset{+}{\rightarrow}1} f(z)dz = \int_{-1}^{1} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $-1 \overset{-}{\leftarrow} 1$:

$$z - z_k = -|z - z_k| = |z - z_k|e^{\pi i}$$

$$\text{then} \quad \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi i}{2}} = -i|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5$$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 6, 7$$

$$\int_{-1\overset{-}{\leftarrow}1} \frac{1}{f(z)}dz = \int_{-1}^{1} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

But we obtain different value in Mathematica

$$\int_{a_3^*} \frac{1}{f(z)}dz = 0$$

(3) Using lemma 1 to modify

(a) $-1 \overset{+}{\to} 1$: the path along x-axis from -1 to 1 on sheet-I with $(+)$edge

$\arg(z-z_k) = -\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2,3,4,5$

$\arg(z-z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 6,7$

So $f(z) \overset{Math.}{=} -f(z)$

(b) $1 \overset{-}{\leftarrow} -1$: the path along x-axis from 1 to -1 on sheet-I with $(-)$edge

$\arg(z-z_k) = \pi$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 1,2,3,4,5$

$\arg(z-z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 6,7$

So $f(z) \overset{Math.}{=} f(z)$

$$\begin{aligned} \int_{a_3} \frac{1}{f(z)} dz &= \int_{a_3^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} -2 \int_{-1}^{1} \frac{1}{f(z)} dz \\ &= 0.151409i \end{aligned}$$

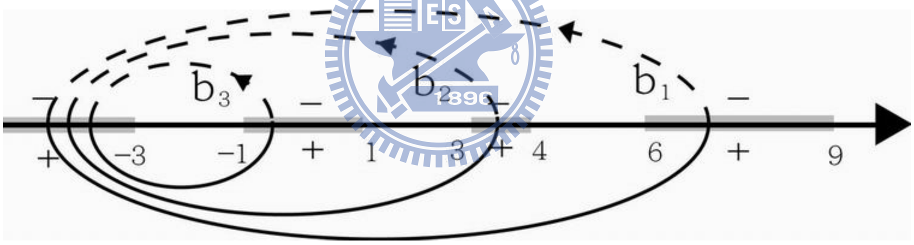

Figure 36: b-cycles

4. b_3 : Let b_3 is a cycle which center at -2 with radius 2. We could write down the parameter, let $z = -2 + 2e^{i\theta}$ and $\theta \in [-\pi, 0) \bigcup [2\pi, 3\pi)$. Notice that $f(z)|_{(II)} = -f(z)|_{(I)}$, so we have

$$\begin{aligned} \int_{b_3} \frac{1}{f(z)} dz &= \int_{-\pi}^{0} \frac{2ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{-2+2e^{i\theta}-z_k}} d\theta - \int_{0}^{\pi} \frac{2ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{-2+2e^{i\theta}-z_k}} d\theta \\ &= -0.0765026 + 6.93889 \times 10^{-18} i \end{aligned}$$

Since b_k cycle is simple connected, we can use some equivalent paths, say b_k^*, such that $b_k \approx b_k^*$ to easily compute the integrals for b_k cycle. Here $b_3 \approx b_3^*$.

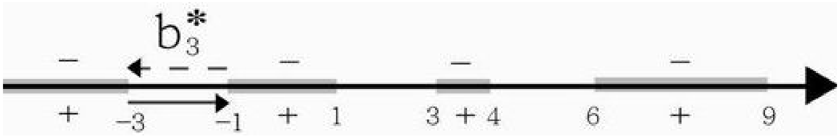

Figure 37: b_3's equivalent path b_3^*

(1) Consider b_3^* of theory in sheet-I

(a) $-3 \to -1$

$$z+z_3=|z+z_3| \text{ then } \frac{1}{\sqrt{z+z_3}}=|z+z_3|^{-\frac{1}{2}}$$

$$z-z_k=-|z-z_k|=|z-z_k|e^{-\pi i}$$

$$\text{then } \frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}e^{\frac{\pi}{2}i}=|z-z_k|^{-\frac{1}{2}}i,\ k=1,2,3,4,5,6$$

$$\int_{-3\to-1}\frac{1}{f(z)}dz=-\int_{-3}^{-1} i^6\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

(b) $-1 \dashrightarrow -3$: the path along x-axis from -1 to -3 of sheet-II. We known that $f(z)|_{(I)}=-f(z)|_{(II)}$, so consider $-1\to-3$

$$z+z_3=|z+z_3| \text{ then } \frac{1}{\sqrt{z+z_3}}=|z+z_3|^{-\frac{1}{2}}$$

$$z-z_k=-|z-z_k|=|z-z_k|e^{-\pi i}$$

$$\text{then } \frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}e^{\frac{\pi}{2}i}=|z-z_k|^{-\frac{1}{2}}i,\ k=1,2,3,4,5,6$$

$$\int_{-3\dashleftarrow-1}\frac{1}{f(z)}dz=-\int_{-3\leftarrow-1}\frac{1}{f(z)}dz=\int_{-1}^{-3} i^6\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

By (1), (2), we obtain

$$\begin{aligned}\int_{b_3^*}\frac{1}{f(z)}dz &= 2\int_{-1}^{-3} i^6\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz\\ &= -2\int_{-1}^{-3}\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz\\ &= -0.0765026\end{aligned}$$

(2) Consider b_3^* in Mathematica

(a) $-3 \to -1$

$$z + z_3 = |z + z_3| \text{ then } \frac{1}{\sqrt{z+z_3}} = |z+z_3|^{-\frac{1}{2}}$$

$$z - z_k = -|z - z_k| = |z - z_k| e^{\pi i}$$

$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z - z_k|^{-\frac{1}{2}}, \; k = 1,2,3,4,5,6$$

$$\int_{-3\to -1} \frac{1}{f(z)} dz = \int_{-3}^{-1} (-i)^6 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz = -\int_{-3}^{-1} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $-1 \dashrightarrow -3$: the path along x-axis from -1 to -3 on sheet-II

$$z + z_3 = |z + z_3| \text{ then } \frac{1}{\sqrt{z+z_3}} = |z+z_3|^{-\frac{1}{2}}$$

$$z - z_k = -|z - z_k| = |z - z_k| e^{\pi i}$$

$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\pi i} = |z - z_k|^{-\frac{1}{2}} i, \; k = 1,2,3,4,5,6$$

$$\int_{-3 \dashleftarrow -1} \frac{1}{f(z)} dz = -\int_{-1}^{-3} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

But in Mathematica

$$\int_{b_3^*} \frac{1}{f(z)} dz = 0$$

(3) Using Lemma1 to modify

(a) $-3 \to -1$: the path along x-axis from -3 to -1 of sheet-I

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1,2,3,4,5,6$

$\arg(z - z_7) = 0$ then $\sqrt{z - z_7} \overset{Math.}{=} \sqrt{z - z_7}$

So $f(z) \overset{Math.}{=} f(z)$

(b) $-1 \dashrightarrow -3$: the path along x-axis from -1 to -3 of sheet-II

We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we consider $-1 \to -3$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, k=1,...,6

$\arg(z - z_7) = 0$ then $\sqrt{z - z_7} \overset{Math.}{=} \sqrt{z - z_7}$

From $f(z)|_{-1\to -3} \overset{Math.}{=} f(z)$. We have

$$f(z)|_{-1 \dashrightarrow -3} = -f(z)|_{-1\to -3} \overset{Math.}{=} -f(z)$$

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)} dz &= \int_{b_3^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} 2 \int_{-3}^{-1} \frac{1}{f(z)} dz \\
&= -0.0765026
\end{aligned}$$

5. b_2 : Let b_2 is a cycle center at 0 with radius $\frac{7}{2}$. So we could write down the parameter, let $z = \frac{7}{2}e^{i\theta}$ and $\theta \in [-\pi, 0) \bigcup [2\pi, 3\pi)$. Notice that $f(z)|_{(II)} = -f(z)|_{(I)}$, so we have

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)} dz &= \int_{-\pi}^{0} \frac{\frac{7}{2}ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{\frac{7}{2}e^{i\theta} - z_k}} d\theta - \int_{0}^{\pi} \frac{\frac{7}{2}ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{\frac{7}{2}e^{i\theta} - z_k}} d\theta \\
&= 0.157328
\end{aligned}$$

Using same way in (4). Consider equivalent path $b_2^* = b_3^* \bigcup -1 \overset{+}{\rightarrow} 1 \bigcup -1 \overset{-}{\dashleftarrow} 1 \bigcup 1 \rightarrow 3 \bigcup 1 \dashleftarrow 3$

Figure 38: The equivalent path b_2^*

(1) Consider b_2^* of theory in sheet-I

(a) $-1 \overset{+}{\rightarrow} 1$:

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 6, 7$$

$$z - z_k = -|z - z_k| = |z - z_k|e^{-\pi i}$$

$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2}i} = i|z - z_k|^{-\frac{1}{2}}, \ k = 1, 2, 3, 4, 5$$

$$\int_{-1 \overset{+}{\rightarrow} 1} \frac{1}{f(z)} dz = \int_{-1}^{1} i^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $-1 \overset{-}{\dashleftarrow} 1 \equiv -1 \overset{+}{\leftarrow} 1$ i.e. the path on horizontal cut from -1 to 1 on $(-)$edge in sheet-II equals the path on horizontal cut from -1 to 1 of $(+)$edge in sheet-I. So consider $z \in -1 \overset{+}{\leftarrow} 1$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}},\ k = 6, 7$$
$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$
$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = i|z - z_k|^{-\frac{1}{2}},\ k = 1, 2, 3, 4, 5$$

$$\int_{-1 \overset{-}{\dashleftarrow} 1} \frac{1}{f(z)} dz = \int_{-1 \overset{+}{\leftarrow} 1} \frac{1}{f(z)} dz = \int_{1}^{-1} i^5 \prod_{k=1}^{5} |z - z_k|^{-\frac{1}{2}} dz$$

(c) $1 \to 3$:

$$z - z_k = |z - z_k| \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}},\ k = 5, 6, 7$$
$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$
$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i,\ k = 1, 2, 3, 4$$

$$\int_{1 \to 3} \frac{1}{f(z)} dz = \int_{1}^{3} i^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(d) $1 \dashleftarrow 3$: we known that $f(z)|_{(II)} = -f(z)|_{(I)}$, so we first consider $1 \leftarrow 3$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}},\ k = 5, 6, 7$$
$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$
$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i,\ k = 1, 2, 3, 4$$

$$\int_{1 \dashleftarrow 3} \frac{1}{f(z)} dz = -\int_{1 \leftarrow 3} \frac{1}{f(z)} dz = -\int_{3}^{1} {}^{47}_{k=1} |z - z_k|^{-\frac{1}{2}} dz$$

By (a), (b), (c) and (d), we have

$$\begin{aligned} \int_{b_2^*} \frac{1}{f(z)} dz &= 2 \int_{-1}^{1} i^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz + 2 \int_{1}^{3} i^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &= 0.157328 \end{aligned}$$

(2) Analysis integral over b_2^* in Mathematica

(a) $-1 \xrightarrow{+} 1$

$$
\begin{aligned}
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z - z_k|^{-\frac{1}{2}},\ k = 1,2,3,4,5 \\
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 6,7
\end{aligned}
$$

$$
\int_{-1 \xrightarrow{+} 1} \frac{1}{f(z)} dz = \int_{-1}^{1} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
$$

(b) $-1 \overset{-}{\dashleftarrow} 1$

$$
\begin{aligned}
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z - z_k|^{-\frac{1}{2}},\ k = 1,2,3,4,5 \\
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 6,7
\end{aligned}
$$

$$
\int_{-1 \overset{-}{\dashleftarrow} 1} \frac{1}{f(z)} dz = \int_{1}^{-1} (-i)^5 \prod_{k=1}^{5} |z - z_k|^{-\frac{1}{2}} dz
$$

(c) $1 \rightarrow 3$

$$
\begin{aligned}
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 5,6,7 \\
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z - z_k|^{-\frac{1}{2}}, k = 1,2,3,4
\end{aligned}
$$

$$
\int_{1 \rightarrow 3} \frac{1}{f(z)} dz = \int_{1}^{3} (-i)^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
$$

(d) $1 \dashleftarrow 3$

$$
\begin{aligned}
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 5,6,7 \\
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z - z_k|^{-\frac{1}{2}}, k = 1,2,3,4
\end{aligned}
$$

$$
\int_{1 \dashleftarrow 3} \frac{1}{f(z)} dz = \int_{3}^{1} (-i)^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
$$

But in Mathematica we obtain different value

$$\int_{b_2^*} \frac{1}{f(z)} dz = 0$$

(3) Using Lemma1 to modify

(a) $-1 \overset{+}{\to} 1$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, \ldots, 5$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 6, 7$

So $f(z) \overset{Math.}{=} -f(z)$

(b) $-1 \overset{-}{\dashleftarrow} 1 \equiv -1 \overset{+}{\leftarrow} 1$ that is the path on horizontal cut from -1 to 1 of $(-)$edge in sheet-II is equal the path on horizontal cut from -1 to 1 of $(+)$edge in sheet-I. So consider $z \in -1 \overset{+}{\leftarrow} 1$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, \ldots, 5$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 6, 7$

So $f(z) \overset{Math.}{=} -f(z)$

(c) $1 \to 3$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, 3, 4$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 5, 6, 7$

So $f(z) \overset{Math.}{=} f(z)$

(d) $1 \dashleftarrow 3$: we known that $f(z)|_{(II)} = -f(z)|_{(I)}$, so we first consider $1 \leftarrow 3$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, 3, 4$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 5, 6, 7$

We have $f(z)|_{1 \leftarrow 3} \overset{Math.}{=} f(z)$ then

$$f(z)|_{1 \dashleftarrow 3} = -f(z)|_{1 \leftarrow 3} \overset{Math.}{=} -f(z)$$

By (1), (2), (3) and Cauchy Integral Theorem

$$\begin{aligned} \int_{b_2} \frac{1}{f(z)} dz &= \int_{b_2^*} \frac{1}{f(z)} dz \\ &\overset{M.}{=} -2 \int_{-1}^{1} \prod_{k=1}^{7} \frac{1}{f(z)} dz + 2 \int_{1}^{3} \prod_{k=1}^{7} \frac{1}{f(z)} dz \\ &= 0.157328 \end{aligned}$$

6. b_1 : Let b_1 is a cycle center at 3 with radius 3 . So we could write down the parameter, let $z = 3 + 3e^{i\theta}$ and $\theta \in [-\pi, 0) \bigcup [2\pi, 3\pi)$. Notice that $f(z)|_{(II)} = -f(z)|_{(I)}$, so we have

$$\begin{aligned}\int_{b_1} \frac{1}{f(z)} dz &= \int_{-\pi}^{0} \frac{3ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{3+3e^{i\theta} - z_k}} d\theta - \int_{0}^{\pi} \frac{3ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{3+3e^{i\theta} - z_k}} d\theta \\ &= 0.0565161\end{aligned}$$

Consider equivalent path $b_1^* = b_2^* \bigcup b_3^* \bigcup 3 \overset{+}{\to} 4 \bigcup 3 \overset{-}{\leftarrow\!\!-\!-} 4 \bigcup 4 \to 6 \bigcup 4 \leftarrow\!\!-\!- 6$

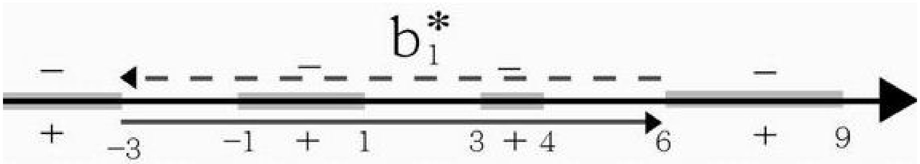

Figure 39: The equivalent path b_1^*

(1) Analysis the integration over b_1^* in sheet-I

(a) $3 \overset{+}{\to} 4$:

$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$

$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i, \ k = 1, 2, 3$$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4, 5, 6, 7$$

$$\int_{3 \overset{+}{\to} 4} \frac{1}{f(z)} dz = \int_4^3 i^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $3 \overset{-}{\leftarrow\!\!-\!-} 4 \equiv 3 \overset{+}{\leftarrow} 4$ that is the path on horizontal cut from 3 to 4 of $(-)$edge in sheet-II is equal to the path on horizontal cut from 3 to 4 of $(+)$edge in sheet-I. So we consider $z \in -1 \overset{+}{\leftarrow} 1$

$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$

$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i, \ k = 1, 2, 3$$

$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4, 5, 6, 7$$

$$\int_{3 \overset{-}{\dashleftarrow} 4} \frac{1}{f(z)} dz = \int_{3 \overset{+}{\leftarrow} 4} \frac{1}{f(z)} dz = \int_4^3 i^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(c) $4 \to 6$

$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$
$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i, \ k = 1, 2$$
$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 3, 4, 5, 6, 7$$

$$\int_{4 \to 6} \frac{1}{f(z)} dz = \int_4^6 i^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(d) $4 \dashleftarrow 6$: we known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we first consider $4 \leftarrow 6$

$$z - z_k = -|z - z_k| = |z - z_k| e^{-\pi i}$$
$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{\pi}{2} i} = |z - z_k|^{-\frac{1}{2}} i, \ k = 1, 2$$
$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 3, 4, 5, 6, 7$$

$$\int_{4 \dashleftarrow 6} \frac{1}{f(z)} dz = -\int_{4 \leftarrow 6} \frac{1}{f(z)} dz = -\int_6^4 i^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

$$\int_{b_1^*} \frac{1}{f(z)} dz = 2 \int_{-1}^{-3} i^6 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz + 2 \int_1^3 i^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$
$$- 2 \int_4^6 i^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(2) Consider b_1^* in Mathematica

(a) $3 \overset{+}{\to} 4$

$$z - z_k = -|z - z_k| = |z - z_k| e^{\pi i}$$
$$\text{then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i |z - z_k|^{-\frac{1}{2}}, \ k = 1, 2, 3$$
$$z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4, 5, 6, 7$$

$$\int_{-1 \overset{+}{\to} 1} \frac{1}{f(z)} dz = \int_{-1}^{1} (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $3 \overset{-}{\dashleftarrow} 4$

$$
\begin{aligned}
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i |z - z_k|^{-\frac{1}{2}},\ k = 1, 2, 3 \\
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 4, 5, 6, 7
\end{aligned}
$$

$$
\int_{3 \overset{-}{\dashleftarrow} 4} \frac{1}{f(z)} dz = \int_4^3 (-i)^4 \prod_{k=1}^{7} (-i)^4 |z - z_k|^{-\frac{1}{2}} dz
$$

(c) $4 \rightarrow 6$

$$
\begin{aligned}
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i |z - z_k|^{-\frac{1}{2}},\ k = 1, 2 \\
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 3, 4, 5, 6, 7
\end{aligned}
$$

$$
\int_{4 \rightarrow 6} \frac{1}{f(z)} dz = \int_4^6 (-i)^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
$$

(d) $4 \dashleftarrow 6$

$$
\begin{aligned}
z - z_k &= -|z - z_k| = |z - z_k| e^{\pi i} \\
\text{then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i |z - z_k|^{-\frac{1}{2}},\ k = 1, 2 \\
z - z_k = |z - z_k| \text{ then } \frac{1}{\sqrt{z - z_k}} &= |z - z_k|^{-\frac{1}{2}},\ k = 3, 4, 5, 6, 7
\end{aligned}
$$

$$
\int_{4 \dashleftarrow 6} \frac{1}{f(z)} dz = \int_6^4 (-i)^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
$$

(3) Using Lemma1 to modify

(a) $3 \overset{+}{\rightarrow} 4$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, 3$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 4, 5, 6, 7$

So $f(z) \overset{Math.}{=} -f(z)$

(b) $3 \overset{-}{\dashleftarrow} 4 \equiv 3 \overset{+}{\leftarrow} 4$ = the path on horizontal cut from 3 to 4 of $(-)$edge in sheet-II is equal to the path on horizontal cut from 3 to 4 of $(+)$edge in sheet-I. So consider $z \in 3 \overset{+}{\leftarrow} 4$.

$\arg(z-z_k) = -\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2,3$

$\arg(z-z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4,5,6,7$

So $f(z) \overset{Math.}{=} -f(z)$

(c) $4 \rightarrow 6$

$\arg(z-z_k) = -\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2$

$\arg(z-z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 5,6,7$

So $f(z) \overset{Math.}{=} f(z)$

(d) $4 \leftarrow\!-\!-\, 6$: we known that $f(z)|_{(II)} = -f(z)|_{(I)}$, so we first consider $4 \leftarrow 6$

$\arg(z-z_k) = -\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2$

$\arg(z-z_k) = 0$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 3,4,5,6,7$

From $f(z)|_{4\leftarrow 6} \overset{Math.}{=} f(z)$, we have

$$f(z)|_{4\leftarrow -- 6} = -f(z)|_{(4\leftarrow 6)} \overset{Math.}{=} -f(z)$$

By (1), (2), (3) and Cauchy Integral Theorem

$$\begin{aligned} \int_{b_1} \frac{1}{f(z)} dz &= \int_{b_1^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} -2\int_{-3}^{-1} \frac{1}{f(z)} dz + 2\int_{1}^{3} \frac{1}{f(z)} dz - 2\int_{6}^{9} \frac{1}{f(z)} dz \end{aligned}$$

Discuss in general situation:

Compute $\int \frac{1}{f(z)} dz$ over a,b cycles for horizontal cut where $f(z) = \sqrt{\prod_{k=1}^{m} (z-z_k)}$, $z_k \in R$, $\forall k = 1 \sim m$ and $z_1 > z_2 > ...... > z_m$

1. a-cycle

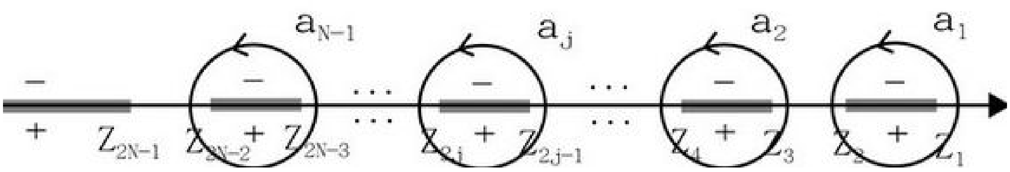

Figure 40: a-cycles for 2N-1 points

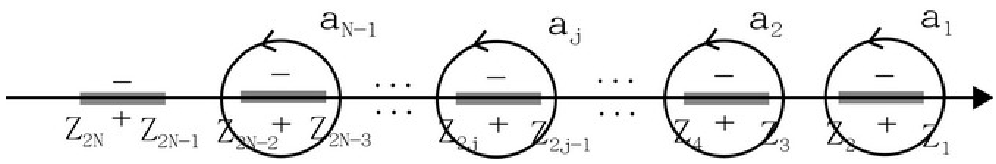

Figure 41: a-cycles for 2N points

There are N cuts (N-1 holes), we give that a_j is a cycle center at x with radius r enclosed $[z_{2j}, z_{2j-1}]$ and doesn't intersect with other cuts.

If $z \in a_j$, let $z = x + re^{i\theta}$ where $\theta \in [-\pi, \pi)$

$$\int_{a_j} \frac{1}{f(z)} dz = \int_{a_j} \frac{1}{\sqrt{\prod_{k=1}^{m} (z - z_k)}} dz \tag{5}$$

$$= \int_{-\pi}^{\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta \tag{6}$$

2. Consider $\int_{a_j^*} \frac{1}{f(z)} dz$ where a_j^* is an equivalent path for a_j and it's from z_{2j} to z_{2j-1} in (+)edge and then from z_{2j-1} to z_{2j} in (−)edge.

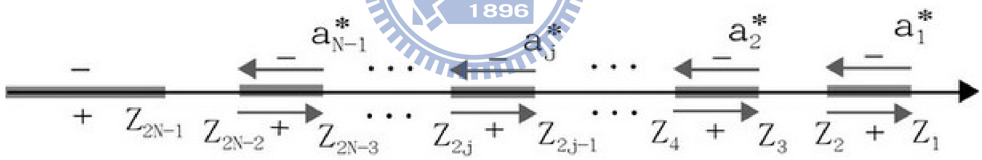

Figure 42: a^*-cycles for 2N-1 points

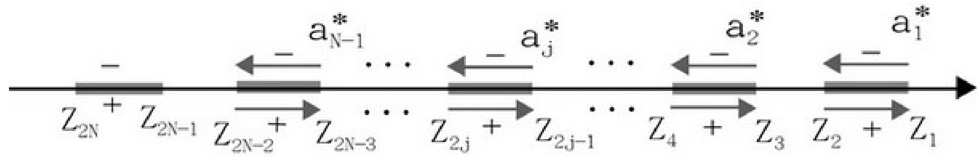

Figure 43: a^*-cycles for 2N points

By Cauchy theorem, we can get that

$$\int_{a_j} \frac{1}{f(z)} dz = \int_{a_j^*} \frac{1}{f(z)} dz \tag{7}$$

Similarly, we use some ideas of complex number to analysis the integrations, first.

(1) $z_{2j} \stackrel{+}{\to} z_{2j-1}$: That is consider the path from z_{2j} to z_{2j-1} on (+)edge

(a) Analysis in theory

$z - z_k > 0$ then $\arg(z - z_k) = 0,\ k = 2j, 2j+1, ..., m$

$z - z_k < 0$ then $\arg(z - z_k) = -\pi,\ k = 1, 2, ..., 2j-1$

So we have $z - z_k = |z - z_k| e^{i\theta_k}$ where $\theta_k = \arg(z - z_k)$ then

$$\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{\frac{i\theta_k}{2}}$$

$$\begin{aligned}
\int_{z_{2j} \overset{+}{\rightarrow} z_{2j-1}} \frac{1}{f(z)} dz &= \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{\frac{1}{2}} e^{-(-\frac{i\theta_k}{2})(2j-1)} dz \\
&= \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{\frac{1}{2}} e^{j\pi i} (-i) dz \\
&= (-1)^{j+1} i \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{\frac{1}{2}} dz
\end{aligned}$$

We can use this with Mathematica to get value, and we compare with the result below to know the difference.

(b) Analysis in Mathematica:

$z - z_k > 0$ then $\arg(z - z_k) = 0,\ k = 2j, 2j+1, ..., m$

$z - z_k < 0$ then $\arg(z - z_k) = \pi,\ k = 1, 2, ..., 2j-1$

So

$$\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{\frac{i\theta_k}{2}}$$

$$\begin{aligned}
\int_{z_{2j} \overset{+}{\rightarrow} z_{2j-1}} \frac{1}{f(z)} dz &= \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{\frac{1}{2}} e^{-(\frac{i\theta_k}{2})(2j-1)} dz \\
&= \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{\frac{1}{2}} e^{j\pi i} i dz \\
&= (-1)^{j} i \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{\frac{1}{2}} dz
\end{aligned}$$

We can find that the difference of value between theory in sheet-I and Mathematica. is a minus and it must be pure imaginary number.

(2) $z_{2j} \overset{-}{\leftarrow} z_{2j-1}$: That is consider the path from z_{2j-1} to z_{2j} in $(-)$ edge

Same as above

(a) In theory

$z - z_k > 0$ then $\arg(z - z_k) = 0,\ k = 2j, 2j+1, ..., m$

$z - z_k < 0$ then $\arg(z - z_k) = \pi,\ k = 1, 2, ..., 2j-1$

$$\begin{aligned}\int_{z_{2j} \overset{-}{\leftarrow} z_{2j-1}} \frac{1}{f(z)} dz &= \int_{z_{2j-1}}^{z_{2j}} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} e^{-(\frac{i\theta_k}{2})(2j-1)} dz \\ &= \int_{z_{j-1}}^{z_{2j}} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} e^{j\pi i} i dz\end{aligned}$$

(b) In Mathematica, same as above

But if we can modify the computation it will more quick and easier.

(3) Using Lemma 1 to modify the computation.

(a) $z_{2j} \overset{+}{\rightarrow} z_{2j-1}$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2j, 2j+1, \ldots, m$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, \ldots, 2j-1$

So

$$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z) = -f(z)$$

$$\begin{aligned}\int_{z_{2j} \overset{+}{\rightarrow} z_{2j-1}} \frac{1}{f(z)} dz &\overset{Math.}{=} \int_{z_{2j}}^{z_{2j-1}} \frac{1}{(-1)^{2j-1} f(z)} dz \\ &= -\int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz\end{aligned}$$

(b) $z_{2j} \overset{-}{\leftarrow} z_{2j-1}$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2j, 2j+1, ..., m$

$\arg(z - z_k) = \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 1, 2j, ..., 2j-1$

So

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{z_{2j} \overset{+}{\rightarrow} z_{2j-1}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz$$

Conclusion: We obtain

$$\int_{a_j^*} \frac{1}{f(z)} dz = -2ie^{j\pi i} \int_{z_j}^{z_{2j-1}} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} dz \tag{8}$$

$$\overset{Math.}{=} -2 \int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz \tag{9}$$

3. b-cycles

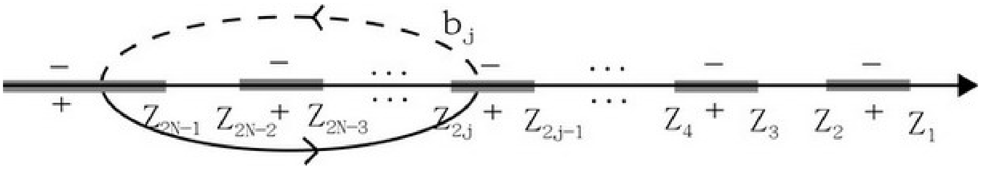

Figure 44: b_j-cycle for 2N-1 points

Give b_j is a circle centered at x with radius r and enclosed the $[z_{2N-1}, z_{2j}]$ and intersect at the points on $[z_{2j}, z_{2j-1}]$ and $[z_{2N-1}, z_{2N}]$.

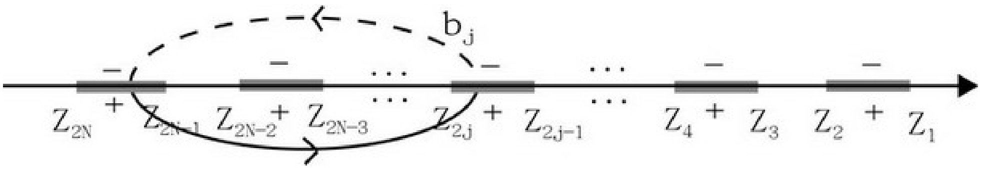

Figure 45: b_j-cycle for 2N points

Give b_j is a circle centered at x with radius r and enclosed the $[z_{2N-1}, z_{2j}]$ and intersect at the points on $[z_{2j}, z_{2j-1}]$ and $[z_{2N-1}, \infty)$. If $z \in b_j$, $z = x + re^{i\theta}$ where $\theta \in [-\pi, 0) \bigcup [2\pi, 3\pi)$. From

$$f(z)|_{(II)} = -f(z)|_{(I)}$$

$$\int_{b_j} \frac{1}{f(z)} dz = \int_{-\pi}^{0} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta + \int_{2\pi}^{3\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta \quad (10)$$

$$= \int_{-\pi}^{0} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta - \int_{0}^{\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta \quad (11)$$

4. The equivalent path b_j^* :

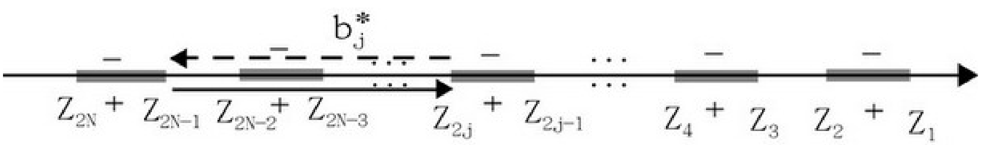

Figure 46: b_j^*-cycle for 2N-1 points

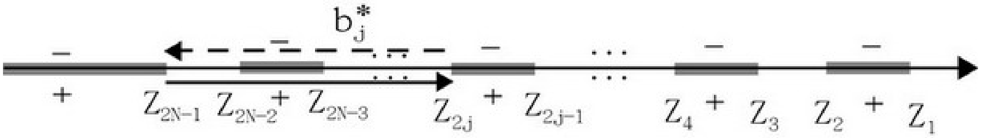

Figure 47: b_j^*-cycle for 2N points

From Cauchy Theorem, we have

$$\int_{b_j} \frac{1}{f(z)} dz = \int_{b_j^*} \frac{1}{f(z)} dz \tag{12}$$

where b_j^* is a path from z_m to z_{2j} in sheet-I and then from z_{2j} to z_m in sheet-II

(1) the path on the cut that is the path from z_{2s+2} to z_{2s+1}, $s = j, j+1, ..., N-2$ on (+)edge in sheet-I and the path from z_{2s+1} to z_{2s+2}, $s = j, j+1, \cdots, N-2$ on (−)edge in sheet-II.

(a) In theory

(i) $z_{2s+2} \xrightarrow{+} z_{2s+1}$:

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}$, $k = 2s+2, 2s+3, \cdots, m$

$z - z_k < 0$ then $\arg(z - z_k) = -\pi$ then $\arg(\sqrt{z - z_k}) = -\frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, \cdots, 2s+1$

So we have

$$f(z) = i^{2s+1} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(ii) $z_{2s+2} \overset{-}{\dashleftarrow} z_{2s+1}$ in (−) edge of sheet-II is as same as in (+)edge of sheet-I, so consider $z_{2s+2} \xrightarrow{+} z_{2s+1}$.

$z - z_k > 0$ then $\arg(z - z_k) = 0 \Rightarrow \arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}$, $k = 2s+2, 2s+1, ..., m$

$z - z_k < 0$ then $\arg(z - z_k) = -\pi$ then $\arg(\sqrt{z - z_k}) = -\frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, ..., 2s+1$

We found that same as above

$$f(z) = i^{2s+1} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(b) In Mathematica:

(i) $z_{2s+2} \xrightarrow{+} z_{2s+1}$

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}$, $k = 2s + 2, \cdots, m$

$z - z_k < 0$ then $\arg(z - z_k) = \pi$ then $\arg(\sqrt{z - z_k}) = \frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{\frac{\pi}{2} i} = i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, \cdots, 2s + 1$

We have

$$f(z) = (-i)^{2s+1} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(ii) $z_{2s+2} \overset{-}{\dashleftarrow} z_{2s+1}$: $z_{2s+2} \xrightarrow{+} z_{2s+1}$.

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}$, $k = 2s + 2, 2s + 1, \ldots\ldots, m$

$z - z_k < 0$ then $\arg(z - z_k) = \pi$ then $\arg(\sqrt{z - z_k}) = \frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{\frac{\pi}{2} i} = i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, \ldots\ldots, 2s + 1$

So we obtain same value of $f(z)$ as (i), but different with in theory

$$f(z) = (-i)^{2s+1} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(c) Using Lemma 1 to modify the computation.

(i) $z \in z_{2s+2} \xrightarrow{+} z_{2s+1}$:

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2s + 2, \ldots\ldots, m$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, \ldots\ldots, 2s + 1$

We obtain

$$f(z) \overset{Math.}{=} (-1)^{2s+1} f(z) = -f(z)$$

(ii) $z_{2s+2} \xrightarrow{+} z_{2s+1}$: in $(-)$ edge of sheet-II is as same as in $(+)$edge of sheet-I, $z_{2s+2} \xrightarrow{+} z_{2s+1}$

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2s + 2, \ldots\ldots, m$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, \ldots\ldots, 2s + 1$

We have

$$f(z) \overset{Math.}{=} (-1)^{2s+1} f(z) = -f(z)$$

(2) In no cuts that is the path from z_{2s+1} to z_{2s}, $s = j, j+1, ..., N-2$ in sheet-I and the path from z_{2s} to z_{2s+1}, $s = j, j+1, ..., N-2$ in sheet-II.

(a) In theory:

(i) $z_{2s+1} \rightarrow z_{2s}$:

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}, k = 2s+1, \ldots, m$

$z - z_k < 0$ then $\arg(z - z_k) = -\pi$ then $\arg(\sqrt{z - z_k}) = -\frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, \ldots, 2s$

Then we have

$$f(z) = i^{2s} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} = (-1)^s \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(ii) The path $z_{2s+1} \dashleftarrow z_{2s}$ is in sheet-II. We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we first consider $z_{2s+1} \leftarrow z_{2s}$

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}, k = 2s+1, \ldots, m$

$z - z_k < 0$ then $\arg(z - z_k) = -\pi$ then $\arg(\sqrt{z - z_k}) = -\frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, \ldots, 2s$

We have $f(z)|_{z_{2s+1} \leftarrow z_{2s}} = i^{2s} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} = (-1)^s \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$. So

$$f(z)|_{z_{2s+1} \dashleftarrow z_{2s}} = (-1)^{s+1} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(b) In Mathematica:

(i) $z_{2s+1} \rightarrow z_{2s}$:

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}$, $k = 2s+1, \ldots, m$

$z - z_k < 0$ then $\arg(z - z_k) = \pi$ then $\arg(\sqrt{z - z_k}) = \frac{\pi}{2}$

then $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{-\frac{\pi}{2}i} = -i|z - z_k|^{-\frac{1}{2}}$, $k = 1, 2, ..., 2s$

So

$$f(z) = (-i)^{2s} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} = (-1)^s \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(ii) $z_{2s+1} \dashleftarrow z_{2s}$:

$z - z_k > 0$ then $\arg(z - z_k) = 0$ then $\arg(\sqrt{z - z_k}) = 0$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}}$, $k = 2s + 1, ..., m$

$z - z_k < 0$ then $\arg(z - z_k) = \pi$ then $\arg(\sqrt{z - z_k}) = \frac{\pi}{2}$

then $\sqrt{z - z_k} = |z - z_k|^{\frac{1}{2}} e^{\frac{\pi}{2} i} = i|z - z_k|^{\frac{1}{2}}$, $k = 1, 2, ..., 2s$

So we have same value of $f(z)$ as (i), but different with in theory

$$f(z) = (-i)^{2s} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} = (-1)^s \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}}$$

(c) Using Lemma 1 to modify the computation in Mathematica:

(i) If $z \in z_{2s+1} \to z_{2s}$:

$\arg(z - z_k) = 0$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2s + 1, ..., m$

$\arg(z - z_k) = -\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, ..., 2s$

So

$$f(z) \overset{Math.}{=} (-1)^{2s} f(z) \overset{Math.}{=} f(z)$$

(ii) If $z_{2s+1} \dashleftarrow z_{2s}$, since $f(z)|_{(II)} = -f(z)|_{(I)}$ consider $z_{2s+1} \leftarrow z_{2s}$

$\arg(z - z_k) = 0 \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2s + 1, ..., m$

$\arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, ..., 2s$

So if $z \in z_{2s+1} \dashleftarrow z_{2s}$

$$f(z) \overset{Math.}{=} -(-1)^{2s} f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_j^*} \frac{1}{f(z)} dz = \sum_{s=j}^{N-1} [(-1)^s 2 \int_{z_{2s+1}}^{z_{2s}} \prod_{k=1}^{m} |z - z_k|^{-\frac{1}{2}} dz] \tag{13}$$

$$\overset{Math.}{=} \sum_{s=j}^{N-1} (2 \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz) \tag{14}$$

3 The integrals of $\frac{1}{f(z)}$ over a,b cycles for vertical cut

After knowing the integrals in horizontal cut, we will discuss the integrals for vertical cuts. In this case, we define that

$$z - z_k = \begin{cases} re^{i\theta}, \theta \in [-\frac{3\pi}{2}, \frac{\pi}{2}) & \text{iff z in sheet-I} \\ re^{i\theta}, \theta \in [\frac{\pi}{2}, \frac{5\pi}{2}) & \text{iff z in sheet-II} \end{cases} \tag{15}$$

the cut in each sheet has two edges, label the starting edge with "+" and the terming edge with "−" and z_k is the end point of the vertical cut.

Analysis the value of f(z) in sheet-I and sheet-II of theory.

Example $f(z) = \sqrt{z}$. If $z = ri \subset$ sheet-I

$$z = |z|e^{i\theta}, \theta \in [-\frac{3\pi}{2}, \frac{\pi}{2}) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \frac{\theta}{2} \in [-\frac{3\pi}{4}, \frac{\pi}{4}) \tag{16}$$

If $z = ri \subset$ sheet-II

$$z = |z|e^{i\theta}, \theta \in [\frac{\pi}{2}, \frac{5\pi}{2}) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \frac{\theta}{2} \in [\frac{\pi}{4}, \frac{5\pi}{4}) \tag{17}$$

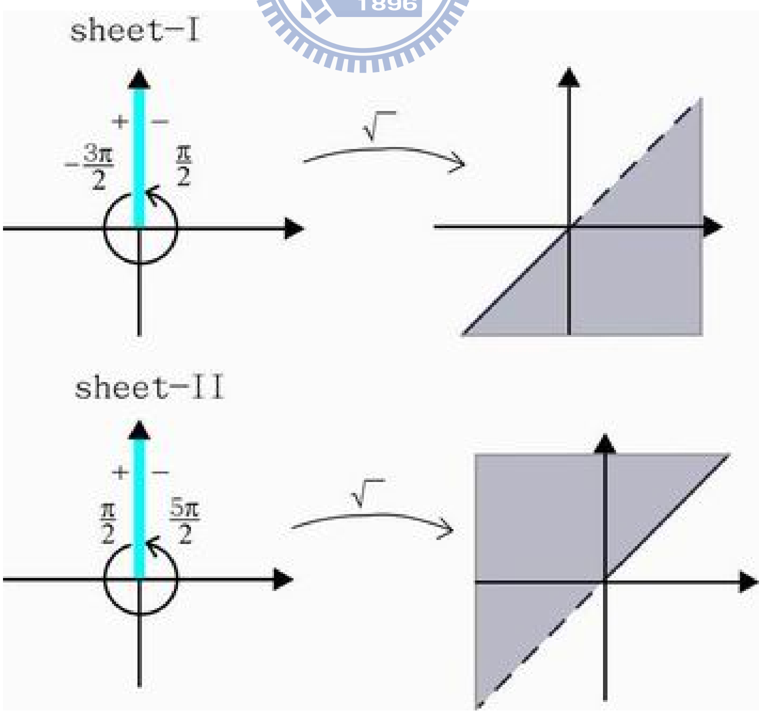

Figure 48: Example of $f(z) = \sqrt{z}$

If $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$, then

$$\begin{aligned}\prod_{k=1}^{n}(z - z_k) &= re^{i\theta_1}, \theta_1 \in [-\frac{3\pi}{2}, \frac{\pi}{2}) \text{ in sheet-I} \\ \prod_{k=1}^{n}(z - z_k) &= re^{i\theta_2}, \theta_2 \in [\frac{\pi}{2}, \frac{5\pi}{2}) \text{ in sheet-II}\end{aligned}$$

From the idea of definition, $re^{i\theta_1} = re^{i\theta_2}$ and $\theta_2 = \theta_1 + 2\pi$.

$$f(z)|_{(II)} = \sqrt{r}e^{\frac{\theta_2}{2}i} = \sqrt{r}e^{\frac{\theta_1+2\pi}{2}i} = \sqrt{r}e^{\frac{\theta_1}{2}i}e^{\pi i} = -f(z)|_{(I)} \tag{18}$$

Discuss the difference between the value in theory and in Mathematica and find out how to modify the computation.

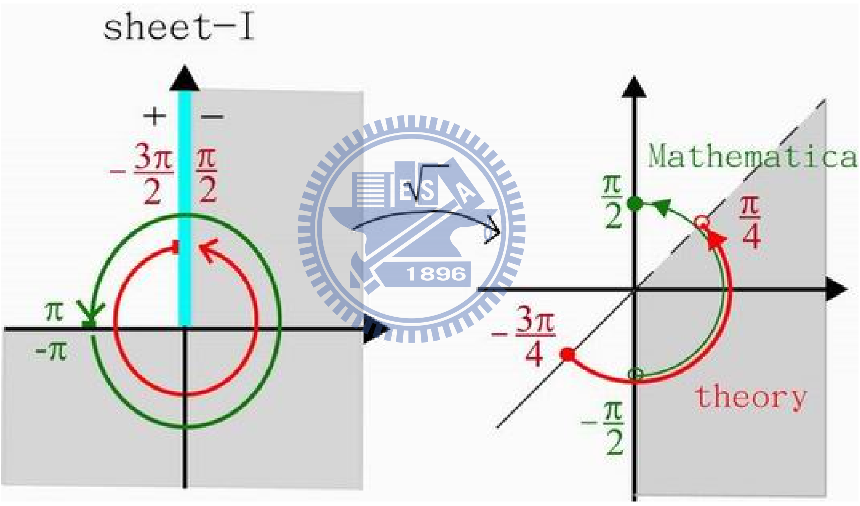

Figure 49: The value in sheet-I and Mathematica of $\sqrt{z}$

So we need to modify the computation in Mathematica s.t. the numerical result of Mathematica is identical to the numerical result of theory when $\theta \in [-\frac{3\pi}{2}, -\pi]$.

Lemma 2. *When z in sheet-I for vertical cut whose one of the end points is* z_k

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z - z_k} & \text{if } \arg(z - z_k) \in [-\frac{3\pi}{2}, -\pi], \\ \sqrt{z - z_k} & \text{if } \arg(z - z_k) \in (-\pi, \frac{\pi}{2}) \end{cases}$$

Proof.

Let z in sheet-I and using polar form $z - z_k = re^{i\theta}$. When $\theta \in (-\pi, \frac{\pi}{2})$, the argument in theory or Mathematica is the same. When $\theta \in [-\frac{3\pi}{2}, -\pi]$, Mathematica will conversion

θ into $\theta + 2\pi \in [\frac{\pi}{2}, \pi]$ where $\theta + 2\pi \in [\frac{\pi}{2}, \pi]$ and $re^{\theta i} = re^{\theta i + 2\pi i}$, but

$$\text{In theory:} \qquad \sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta}{2}i}$$

$$\text{In Mathematica:} \qquad \sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta+2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}}$$

So if $\theta \in [-\frac{3\pi}{2}, -\pi]$

$$\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$$

∎

As same as horizontal cut. We first discuss the difference between the value in theory and the value in Mathematica. Compare their sign(f) is different or not? Using statement before about modify and get value, the result will be the same or not?

Example: The integrals of $\frac{1}{f(z)}$ over a,b cycles for vertical cut where $f(z) = \sqrt{(z-i)(z-2i)(z-3i)(z-4i)(z-5i)(z-6i)}$. Let $f(z) = \prod_{k=1}^{6} \sqrt{z - z_k}$ where $z_k = ki$, $k = 1, 2, 3, 4, 5, 6$

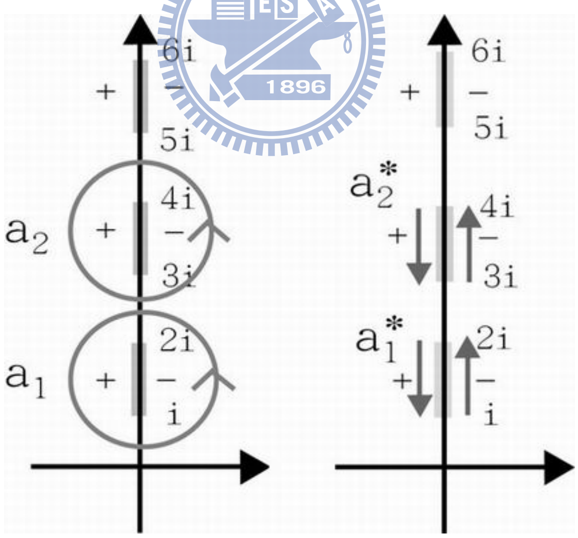

Figure 50: a and its equivalent path a^*

1. Compute $\int_{a_1^*} \frac{1}{f(z)} dz$ where a_1^* is equivalent path for a_1 and a_1^* = the path along vertical cut from i to $2i$ on (+)edge of sheet-I (called a_{11}^*) and then back from $2i$ to i on (−)edge of sheet-I (a_{12}^*)

   (1) a_{11}^*: Let $z = ri, r : 2 \to 1, dz = idr$

(a) Analysis in theory:

Since $z - ki = |z - ki| e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - i) = -\frac{3}{2}\pi$ then $\arg\sqrt{z-i} = -\frac{3}{4}\pi$

$\arg(z - ki) = -\frac{\pi}{2}$ then $\arg\sqrt{z-ki} = -\frac{\pi}{4}$, $k = 2, \ldots, 6$

$$\begin{aligned} f(z) &= (\prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}}) e^{-\frac{3}{4}\pi} (e^{-\frac{1}{4}\pi})^5 = (\prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}}) e^{-\frac{8}{4}\pi} \\ &= \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} = R \end{aligned}$$

(Let $\prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} = R$)

(b) Analysis in mathematica (no matter in which sheet):

Since $z - ki = |z - ki| e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - i) = \frac{\pi}{2}$ then $\arg\sqrt{z-i} = \frac{\pi}{4}$

$\arg(z - ki) = -\frac{\pi}{2}$ then $\arg\sqrt{z-ki} = -\frac{\pi}{4}$, k=2,3,4,5,6

$$\begin{aligned} f(z) &= \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} e^{\frac{1}{4}\pi} (e^{-\frac{\pi}{4}})^5 = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} e^{-\frac{4}{4}\pi} \\ &= -\prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} = -R \end{aligned}$$

Compare with (a) and (b), we find that when you want to get true value, the value which we get from Mathematica should multiply -1 that is sign($f(z)|_{(I)}$)=-sign($f(z)|_{Math.}$)

(c) Using the Lemma 2 to modify:

$\arg(z - i) = -\frac{3}{2}\pi$ than $\sqrt{z-i} \overset{Math.}{=} -\sqrt{z-i}$

$\arg(z - ki) = -\frac{\pi}{2}$ than $\sqrt{z-ki} \overset{Math.}{=} \sqrt{z-ki}$, k=$2, 3, 4, 5, 6$

So $f(z) \overset{Math.}{=} -f(z)$, the same result as above difference between theory and Mathematica, the difference is a minus.

(2) a_{12}^{*} : Let $z = ri, r : 1 \to 2$ and then $dz = idr$

(a) Analysis in theory:

Since $z - ki = |z - ki| e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z-i)=\frac{\pi}{2}$ than $\arg(\sqrt{z-i})=\frac{\pi}{4}$

$\arg(z-ki)=-\frac{\pi}{2}$ than $\arg(\sqrt{z-ki})=-\frac{\pi}{4}$, k=2,3,4,5,6

$$f(z)=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}e^{\frac{\pi}{4}}(e^{-\frac{\pi}{4}})^5=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}e^{-\frac{4}{4}\pi}$$
$$=-\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}$$

(b) Analysis in Mathematica (no matter in which sheet):

Since $z-ki=|z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-i)=\frac{\pi}{2}$ than $\arg(\sqrt{z-i})=\frac{\pi}{4}$

$\arg(z-ki)=-\frac{\pi}{2}$ than $\arg(\sqrt{z-ki})=-\frac{\pi}{4}$, k=2,3,4,5,6

$$f(z)=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}e^{\frac{\pi}{4}}(e^{-\frac{\pi}{4}})^5=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}e^{-\frac{4}{4}\pi}$$
$$=-\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find the value is same

(c) Using Lemma 2 to modify:

$\arg(z-i)=\frac{\pi}{2}$ than $\sqrt{z-i}\overset{Math.}{=}\sqrt{z-i}$

$\arg(z-ki)=-\frac{\pi}{2}$ than $\sqrt{z-ki}\overset{Math.}{=}\sqrt{z-ki},\ k=2,3,4,5,6$

Here we obtain $f(z)\overset{Math.}{=}f(z)$, the same result as above.

$$\int_{a_1}\frac{1}{f(z)}dz=\int_{a_1^*}\frac{1}{f(z)}dz$$
$$=-2\int_1^2 i\prod_{k=1}^{6}|ri-ki|^{-\frac{1}{2}}dr$$
$$\overset{Math.}{=}2\int_1^2\frac{i}{f(ri)}dr$$
$$=0.+0.871563i$$

2. Compute $\int_{a_2^*}\frac{1}{f(z)}dz$ where a_2^* is equivalent path for a_2 and a_2^* = the path along vertical cut from $4i$ to $3i$ on (+)edge of sheet-I (called a_{21}^*) and then back from $3i$ to $4i$ on $(-)$edge of sheet-I (a_{22}^*)

(1) a^*_{21} : Let $z = ri, r : 4 \to 3, dz = idr$

(a) Analysis in theory:

Since $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = -\frac{3}{2}\pi$ than $\arg(\sqrt{z - ki}) = -\frac{3}{4}\pi, k = 1, 2, 3$

$\arg(z - ki) = -\frac{\pi}{2}$ than $\arg(\sqrt{z - ki}) = -\frac{\pi}{4}, k = 4, 5, 6$

$$f(z) = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}} (e^{-\frac{3}{4}\pi})^3 (e^{-\frac{\pi}{4}})^3 = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}} e^{-\frac{12}{4}\pi}$$
$$= -\prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}$$

(b) analysis in Mathematica (no matter in which sheet):

Since $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = \frac{\pi}{2}$ than $\arg(\sqrt{z - i}) = \frac{\pi}{4}$, k=1,2,3

$\arg(z - ki) = -\frac{\pi}{2}$ than $\arg(\sqrt{z - ki}) = -\frac{\pi}{4}$, k=4,5,6

$$f(z) = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}} (e^{\frac{\pi}{4}})^3 (e^{-\frac{\pi}{4}})^3 = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find that when you want to obtain true value, the value which we have from Mathematica should multiply -1, $sign(f(z)|_{(I)}) = -sign(f(z)|_{Mathewatica})$

(c) Using Lemma 2 to modify:

$\arg(z - i) = -\frac{3}{2}\pi$ than $\sqrt{z - i} \overset{Math.}{=} -\sqrt{z - i}$, $k = 1, 2, 3$

$\arg(z - ki) = -\frac{\pi}{2}$ than $\sqrt{z - ki} \overset{Math.}{=} \sqrt{z - ki}$, $k = 4, 5, 6$

$$f(z) \overset{Math.}{=} -f(z)$$

same as the above result

(2) a^*_{22} : Let $z = ri, r : 1 \to 2, dz = idr$

(a) Analysis in theory:

Using polar form $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = \frac{\pi}{2}$ than $\arg(\sqrt{z - ki}) = \frac{\pi}{4}$, k=1,2,3

$\arg(z-ki)=-\frac{\pi}{2}$ than $\arg(\sqrt{z-ki})=-\frac{\pi}{4}$, k=4,5,6

$$f(z)=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}(e^{\frac{\pi}{4}})^3(e^{-\frac{\pi}{4}})=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}$$

(b) Analysis in Mathematica (no matter in which sheet):

Using polar form $z-ki=|z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-ki)=\frac{\pi}{2}$ than $\arg(\sqrt{z-ki})=\frac{\pi}{4}$, k=1,2,3

$\arg(z-ki)=-\frac{\pi}{2}$ than $\arg(\sqrt{z-ki})=-\frac{\pi}{4}$, k=4,5,6

$$f(z)=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}(e^{\frac{\pi}{4}})^3(e^{-\frac{\pi}{4}})^3=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find the value is same

(c) Using Lemma 2 to modify:

$\arg(z-i)=\frac{\pi}{2}$ then $\sqrt{z-i}\overset{Math.}{=}\sqrt{z-i}$, k=1,2,3

$\arg(z-ki)=-\frac{\pi}{2}$ then $\sqrt{z-ki}\overset{Math.}{=}\sqrt{z-ki}$, k=4,5,6

$$f(z)\overset{Math.}{=}f(z)$$

same as the above result

$$\begin{aligned}\int_{a_2}\frac{1}{f(z)}dz&=\int_{a_2^*}\frac{1}{f(z)}dz\\&=2\int_3^4 i\prod_{k=1}^{6}|ri-ki|^{-\frac{1}{2}}dr\\&\overset{Math.}{=}2\int_3^4\frac{i}{f(ri)}dr\\&=0.+1.74313i\end{aligned}$$

3. Compute $\int_{b_2^*}\frac{1}{f(z)}dz$ where b_2^* is an equivalent path for b_2 and $b_2^*=$ the path along vertical line from 5i to 4i on sheet-I (called b_{21}^*) and then back from 5i to 4i on sheet-II (b_{22}^*)

(1) b_{21}^* : Let $z=ri, r:5\to 4$ and then $dz=idr$

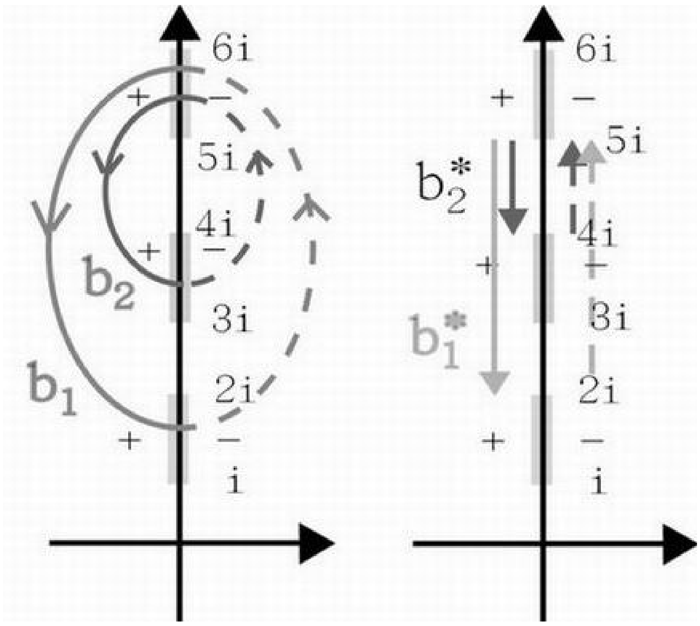

Figure 51: b and b^*

(a) Analysis in theory:

Using polar form $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = -\frac{3}{2}\pi$ then $\arg(\sqrt{z - ki}) = -\frac{3}{4}\pi$, k=1,2,3,4

$\arg(z - ki) = -\frac{\pi}{2}$ then $\arg(\sqrt{z - ki}) = -\frac{\pi}{4}$, k=5,6

$$f(z) = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}(e^{-\frac{3}{4}\pi})^4(e^{-\frac{\pi}{4}})^2 = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}e^{-\frac{14}{4}\pi}$$
$$= i\prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}$$

(b) Analysis of Mathematica (no matter in which sheet):

Using polar form $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = \frac{\pi}{2}$ then $\arg(\sqrt{z - ki}) = \frac{\pi}{4}$, k=1,2,3,4

$\arg(z - ki) = -\frac{\pi}{2}$ then $\arg(\sqrt{z - ki}) = -\frac{\pi}{4}$, k=5,6

$$f(z) = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}(e^{-\frac{\pi}{4}})^2(e^{\frac{\pi}{4}})^4 = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}e^{\frac{2}{4}\pi}$$
$$= i\prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find the value is same.

(c) Using Lemma 2 to modify:

$\arg(z-i)=\frac{\pi}{2}$ then $\sqrt{z-i} \overset{Math.}{=} \sqrt{z-i}$, k=1, 2, 3, 4

$\arg(z-ki)=-\frac{\pi}{2}$ then $\sqrt{z-ki} \overset{Math.}{=} \sqrt{z-ki}$, k=5, 6

$$f(z) \overset{Math.}{=} f(z)$$

same as the above result

(2) b_{22}^{*}: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{22}^{**} is the path along vertical line from 4i to 5i on sheet-I.

Let $z = ri, r: 4 \to 5$ then $dz = idr$

(a) Analysis in theory:

Using polar form $z-ki = |z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-ki) = -\frac{3}{2}\pi$ then $\arg(\sqrt{z-ki}) = -\frac{3}{4}\pi$, k=1,2,3,4

$\arg(z-ki) = -\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki}) = -\frac{1}{4}\pi$, k=5,6

$$\begin{aligned} f(z) &= \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} (e^{-\frac{3}{4}\pi})^4 (e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} e^{-\frac{14}{4}\pi} \\ &= i \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} \end{aligned}$$

So

$$f(z)|_{b_{22}^{*}} = -f(z)|_{b_{22}^{**}} = -i \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}}$$

(b) Analysis in Mathematica (no matter in which sheet):

Using polar form $z-ki = |z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-ki) = \frac{1}{2}\pi$ then $\arg(\sqrt{z-ki}) = \frac{1}{4}\pi$, k=1,2,3,4

$\arg(z-ki) = -\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki}) = -\frac{1}{4}\pi$, k=5,6

$$\begin{aligned} f(z) &= \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} (e^{\frac{1}{4}\pi})^4 (e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} e^{\frac{2}{4}\pi} \\ &= i \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} \end{aligned}$$

Compare with (a) and (b) we find when you want get true value, the value which we obtain from Mathematica should multiply -1

(c) Using Lemma 2 to modify:

$\arg(z-i)=\frac{1}{2}\pi$ then $\sqrt{z-i} \overset{Math.}{=} \sqrt{z-i}$, k=1,2,3,4

$\arg(z-ki)=-\frac{1}{2}\pi$ then $\sqrt{z-ki} \overset{Math.}{=} \sqrt{z-ki}$, k=5,6

$f(z)|_{b_{22}^{**}} \overset{Math.}{=} f(z)$ then

$$f(z)|_{b_{22}^{*}} = -f(z)|_{b_{22}^{**}} \overset{Math.}{=} -f(z)$$

same as the above result

$$\begin{aligned}\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\ &= -2\int_4^5 i\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}dr \\ &\overset{Math.}{=} -2\int_4^5 \frac{i}{f(ri)}dr \\ &= -1.48065\end{aligned}$$

4. Compute $\int_{b_1^*} \frac{1}{f(z)}dz$ where b_1^* is equivalent path for b_1 and $b_1^* = b_2^* \bigcup b_{11}^* \bigcup b_{13}^* \bigcup b_{14}^* \bigcup b_{12}^*$ where b_{11}^* = the path along vertical cut from $4i$ to $3i$ on (+)edge of sheet-I b_{12}^* = the path along vertical cut from $3i$ to $4i$ on (−)edge of sheet-II, b_{13}^* = the path along vertical line from $3i$ to $2i$ on sheet-I, b_{14}^* = the path along vertical line from $2i$ to $3i$ on sheet-II.

(1) $b_{11}^* \equiv a_{21}^*$: Done

(2) $b_{12}^* \equiv$ the path along vertical cut from $3i$ to $4i$ on (+)edge of sheet-I

Let $z = ri, r: 3 \to 4$, so $dz = idr$

(a) In theory:

Using polar form $z-ki = |z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-ki)=-\frac{3}{2}\pi$ then $\arg(\sqrt{z-ki})=-\frac{3}{4}\pi$, k=1,2

$\arg(z-ki)=-\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki})=-\frac{1}{4}\pi$, k=3,4,5,6

$$\begin{aligned}f(z) &= \prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}(e^{-\frac{3}{4}\pi})^4(e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}e^{-\frac{10}{4}\pi} \\ &= -i\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}\end{aligned}$$

(b) In Mathematica (no matter in which sheet):

Using polar form $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = \frac{1}{2}\pi$ then $\arg(\sqrt{z - ki}) = \frac{1}{4}\pi$, k=1,2

$z - i = -|z - ki|i$ $\arg(z - ki) = -\frac{1}{2}\pi$ then $\arg(\sqrt{z - ki}) = -\frac{1}{4}\pi$, k=3,4,5,6

$$f(z) = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}(e^{-\frac{1}{4}\pi})^4(e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}e^{-\frac{2}{4}\pi}$$
$$= -i\prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find that the value is same

(c) Using Lemma 2 to modify:

$\arg(z - ki) = \frac{1}{2}\pi$ then $\sqrt{z - ki} \overset{Math.}{=} \sqrt{z - ki}$, k=1,2

$\arg(z - ki) = -\frac{1}{2}\pi$ then $\sqrt{z - ki} \overset{Math.}{=} \sqrt{z - ki}$, k=3,4,5,6

$$f(z) \overset{Math.}{=} f(z)$$

same as the above result

(3) b_{13}^{*}: Let $z = ri, r : 3 \to 2$, so $dz = idr$

(a) In theory:

Using polar form $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = -\frac{3}{2}\pi$ then $\arg(\sqrt{z - ki}) = -\frac{3}{4}\pi$, k=1,2

$\arg(z - ki) = -\frac{1}{2}\pi$ then $\arg(\sqrt{z - ki}) = -\frac{1}{4}\pi$, k=3,4,5,6

$$f(z) = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}(e^{-\frac{3}{4}\pi})^4(e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}e^{-\frac{10}{4}\pi}$$
$$= -i\prod_{k=1}^{6} |z - ki|^{-\frac{1}{2}}$$

(b) In Mathematica:

Using polar form $z - ki = |z - ki|e^{\arg(z-ki)i}$, consider $\arg(z - ki)$.

$\arg(z - ki) = \frac{1}{2}\pi$ then $\arg(\sqrt{z - ki}) = \frac{1}{4}\pi$, k=1,2

$\arg(z-ki) = -\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki}) = -\frac{1}{4}\pi$, k=3,4,5,6

$$f(z) = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} (e^{-\frac{1}{4}\pi})^4 (e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} e^{-\frac{2}{4}\pi}$$
$$= -i \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find the value is same

(c) Using Lemma 2 to modify:

$\arg(z-i) = \frac{1}{2}\pi$ then $\sqrt{z-i} \overset{Math.}{=} \sqrt{z-i}$, k=1,2

$\arg(z-ki) = -\frac{1}{2}\pi$ then $\sqrt{z-ki} \overset{Math.}{=} \sqrt{z-ki}$, k=3,4,5,6

$$f(z) \overset{Math.}{=} f(z)$$

same as the above result

(4) b^*_{14} : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{14}= the path along vertical line from 2i to 3i on sheet-I.

Let $z = ri, r: 2 \to 3$, so $dz = idr$

(a) Analysis in theory:

Using polar form $z - ki = |z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-ki) = \frac{1}{2}\pi$ then $\arg(\sqrt{z-ki}) = \frac{1}{4}\pi$, k=1,2

$\arg(z-ki) = -\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki}) = -\frac{1}{4}\pi$, k=3,4,5,6

$$f(z) = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} (e^{\frac{1}{4}\pi})^4 (e^{-\frac{1}{4}\pi})^2 = \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} e^{-\frac{10}{4}\pi}$$
$$= -i \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}}$$

So

$$f(z)|_{b^*_{14}} = -f(z)|_{b^{**}_{14}} = i \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}}$$

(b) In Mathematica (no matter in which sheet):

Using polar form $z - ki = |z-ki|e^{\arg(z-ki)i}$, consider $\arg(z-ki)$.

$\arg(z-ki)=\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki})=\frac{1}{4}\pi$, k=1,2

$\arg(z-ki)=-\frac{1}{2}\pi$ then $\arg(\sqrt{z-ki})=-\frac{1}{4}\pi$, k=3,4,5,6

$$f(z)=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}(e^{-\frac{1}{4}\pi})^4(e^{-\frac{1}{4}\pi})^2=\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}e^{-\frac{2}{4}\pi}$$
$$=-i\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}$$

Compare with (a) and (b) we find when you want get true value, the value which we got from Mathematica should multiply -1

(c) Using Lemma 2 to modify:

$\arg(z-i)=\frac{1}{2}\pi$ then $\sqrt{z-i}\overset{Math.}{=}\sqrt{z-i}$, k=1,2

$\arg(z-ki)=-\frac{1}{2}\pi$ then $\sqrt{z-ki}\overset{Math.}{=}\sqrt{z-ki}$, k=3,4,5,6

$f(z)|_{b_{14}^{**}}\overset{Math.}{=}f(z)$ then $f(z)|_{b_{14}^{*}}=-f(z)|_{b_{14}^{**}}\overset{Math.}{=}-f(z)$

same as the above result

By (1), (2), (3), (4) and Cauchy Integral Theorem

$$\int_{b_1}\frac{1}{f(z)}dz=\int_{b_1^*}\frac{1}{f(z)}dz$$
$$=-2i\int_4^5\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}dr-2i\int_2^3\prod_{k=1}^{6}|z-ki|^{-\frac{1}{2}}dr$$
$$\overset{Math.}{=}-2\int_4^5\frac{i}{f(ri)}dr-2\int_2^3\frac{i}{f(ri)}dr$$
$$=-2.9613$$

We find that when compute the integral by Mathematica, the difference of integral form between theory and Mathematica can help us know how to modify. We also can use angle to decide how to modify. It will the same so we only use angle to decide how to modify below.

In general situation: Compute $\int\frac{1}{f(z)}dz$ over a^*,b^* for vertical cut where $f(z)=\sqrt{\prod_{k=1}^{m}(z-z_k)}$ and $z_k=a_ki$, $a_k\in R$.

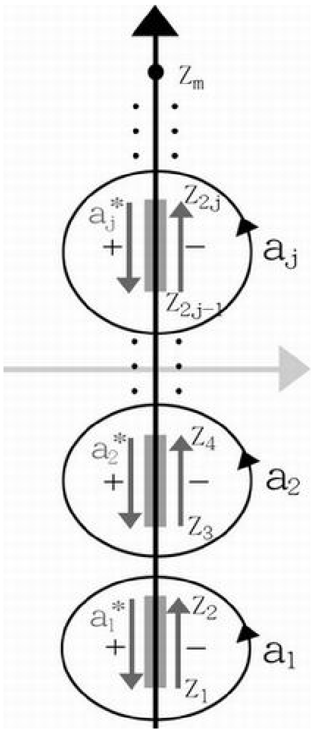

Figure 52: a-cycle and their equivalent path a^*

1. a-cycles: a_j cycle is a cycle center at x with radius r enclosed $[z_{2j}, z_{2j-1}]$ and doesn't intersect with other cuts. $\int_{a_j} \frac{1}{f(z)} dz = \int_{a_j^*} \frac{1}{f(z)} dz$ in sheet-I. The equivalent path a_j^* =the path on a vertical cut from z_{2j} to z_{2j-1} with $(+)$edge of sheet-I and then from z_{2j-1} to z_{2j} with $(-)$edge of sheet-I.

   Using lemma to modify:

   (a) $z \in z_{2j} \overset{+}{\to} z_{2j-1}$: Let $z = ri, r : Im(z_{2j}) \to Im(z_{2j-1})$, so $dz = idr$

   $\arg(z - z_k) = -\frac{3\pi}{2}$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2, 3, ...2j - 1$

   $\arg(z - z_k) = -\frac{\pi}{2}$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2j, 2j + 1, ..., m$

   $$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z) \overset{M.}{=} -f(z)$$

   (b) $z \in z_{2j-1} \overset{-}{\to} z_{2j}$: Let $z = ri, r = Im(z_{2j}) \to Im(z_{2j-1})$, so $dz = idr$

   $\arg(z - z_k) = -\frac{\pi}{2}$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$,k=1,2,3,......2j-1

   $\arg(z - z_k) = -\frac{\pi}{2}$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 2j, 2j + 1, ..., m$

   $$f(z) \overset{Math.}{=} f(z)$$

By (a),(b) and Cauchy Theory:

$$\begin{aligned}\int_{a_j} \frac{1}{f(z)} dz &= \int_{a_j^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_{Im(z_{2j-1})}^{Im(z_{2j})} \prod_{k=1}^{m} \frac{1}{\sqrt{z-z_k}} idr \end{aligned} \tag{19}$$

2. b-cycles :

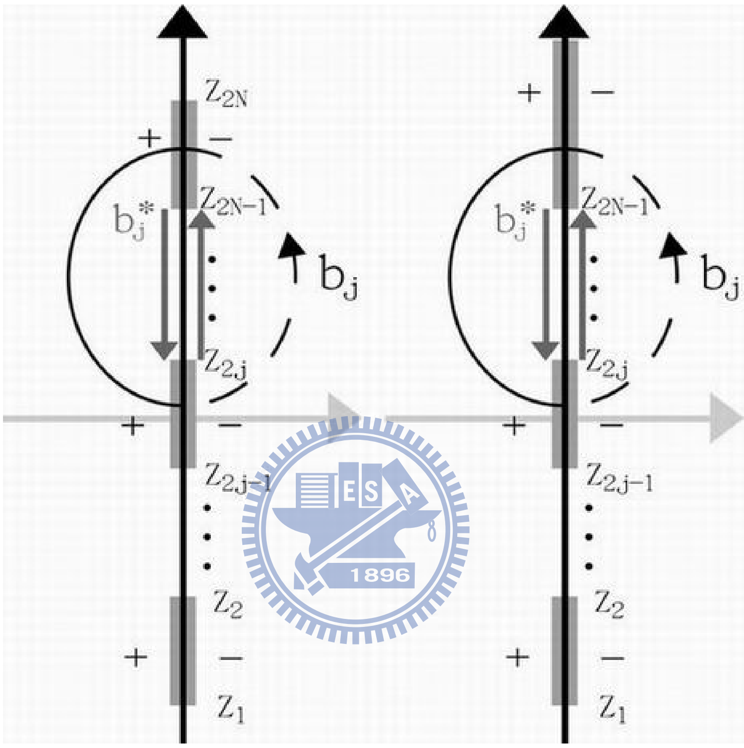

Figure 53: b_j and b_j^* of $2N-1$ points and $2N$ points

b_j is a circle centered at x with radius r and enclosed the $[z_{2N-1}, z_{2j}]$ and intersect at the points on $[z_{2j}, z_{2j-1}]$ and $[z_{2N-1}, z_{2N}]$ or b_j is a circle centered at x with radius r and enclosed the $[z_{2N-1}, z_{2j}]$ and intersect at the points on $[z_{2j}, z_{2j-1}]$ and $[z_{2N-1}, \infty)$

By Cauchy Theorem, we know that

$$\int_{b_j} \frac{1}{f(z)} dz = \int_{b_j^*} \frac{1}{f(z)} dz \tag{20}$$

Equivalent path b_j^* is a path from z_m to z_{2j} in sheet-I and then from z_{2j} to z_m in sheet-II

(1) the path in cut i.e. the path from z_{2s+2} to z_{2s+1}, $s = j, j+1, ..., N-2$ on (+)edge in sheet-I and path from z_{2s+1} to z_{2s+2}, $s = j, j+1, \cdots, N-2$ on (−)edge in sheet-II.

(a) $z_{2s+2} \xrightarrow{+} z_{2s+1}$

$$\begin{aligned}\arg(z-z_k) &= -\frac{\pi}{2} \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 2s+2, ..., m \\ \arg(z-z_k) &= -\frac{3}{2}\pi \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1, 2, ..., 2s+1\end{aligned}$$

So

$$f(z) \overset{Math.}{=} (-1)^{2s+1} f(z) \overset{Math.}{=} -f(z)$$

(b) if $z_{2s+2} \xrightarrow{-} z_{2s+1}$

$$\arg(z-z_k) = -\tfrac{\pi}{2} \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 2s+2, ..., m$$
$$\arg(z-z_k) = \tfrac{1}{2}\pi \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 1, 2, ..., 2s+1$$

So

$$f(z) \overset{Math.}{=} f(z)$$

(2) In no cuts that is the path from z_{2s+1} to z_{2s}, $s = j, j+1, ..., N-2$ in sheet-I and the path from z_{2s} to z_{2s+1}, $s = j, j+1, ..., N-2$ in sheet-II.

(a) $z \in z_{2s+1} \to z_{2s}$

$$\arg(z-z_k) = -\frac{\pi}{2} \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 2s+1, ..., m$$
$$\arg(z-z_k) = -\frac{3\pi}{2} \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1, 2, ..., 2s$$

So

$$f(z) \overset{Math.}{=} (-1)^{2s} f(z) \overset{Math.}{=} f(z)$$

(b) If $z_{2s+1} \dashleftarrow z_{2s}$: Since $f(z)|_{(II)} = -f(z)|_{(I)}$, we consider $z_{2s+1} \leftarrow z_{2s}$

$$\arg(z-z_k) = -\frac{\pi}{2} \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 2s+1, ..., m$$
$$\arg(z-z_k) = -\frac{3\pi}{2} \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1, 2, ..., 2s$$

So

$$z \in z_{2s+1} \dashleftarrow z_{2s} \text{ then } f(z) \overset{Math.}{=} -(-1)^{2s} f(z) \overset{Math.}{=} -f(z)$$

From (1), (2) we have

$$\begin{aligned} \int_{b_j^*} \frac{1}{f(z)} dz &= \int_{b_j^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \sum_{s=j}^{N-1} (2 \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz) \end{aligned} \tag{21}$$

When we want to modify the computation of $f(z)$ and $f(z)$ has m roots. We needs to consider $\sqrt{z-z_k}$, k=1,...,m. There are m steps of modifying the computation, if m is large it will become troublesome. Here we provide a way to reduce the step. We can divided domain R into many areas to discuss the way to modify on vertical cuts We call it the area to modify. Take a example to explain $f(z) = \sqrt{(z-i)(z-2i)} = \sqrt{z-i}\sqrt{z-2i}$

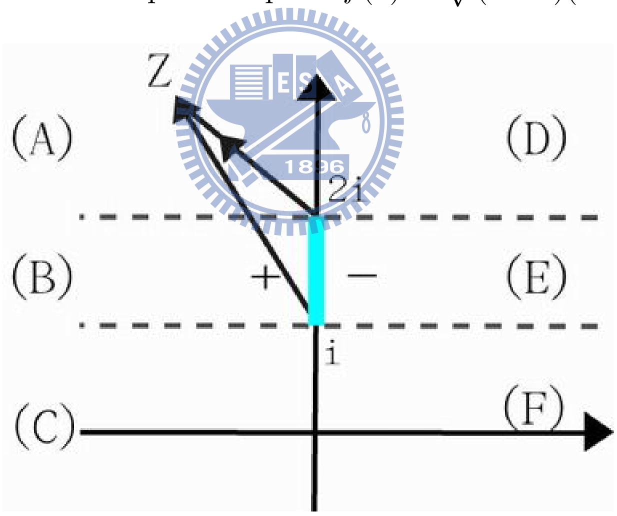

Figure 54: 6 blocks of domain

Only discuss in sheet-I and divided it into six blocks (A),(B),(C),(D),(E) and (F) where

$(A) = \{z = x + yi : x < 0,\ 2 \le y\}$, $(B) = \{z = x + yi : x < 0,\ 1 \le y < 2\}$,

$(C) = \{z = x + yi : x < 0,\ y < 1\}$, $(D) = \{z = x + yi : x > 0,\ 2 \le y\}$,

$(E) = \{z = x + yi : x > 0,\ 1 \le y < 2\}$, $(F) = \{z = x + yi : x > 0,\ y < 1\}$

1. $z \in (A)$

   $\arg(z-i), \arg(z-2i) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

   So $\sqrt{z-i} \overset{Math.}{=} -\sqrt{z-i}$ and $\sqrt{z-2i} \overset{Math.}{=} -\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

2. $z \in (B)$

   $\arg(z-i) \in (-\frac{3\pi}{2}, -\pi)$ then $\sqrt{z-i} \overset{Math.}{=} -\sqrt{z-i}$

   $\arg(z-2i) \in (-\pi, -\frac{\pi}{2})$ then $\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-2i}$

$$f(z) \overset{Math.}{=} -f(z)$$

3. $z \in (C), (D), (E), (F)$

   $\arg(z-i), \arg(z-2i) \in (-\pi, \pi)$

   So $\sqrt{z-i} \overset{Math.}{=} \sqrt{z-i}$ and $\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

4. $z \in (+)$edge of the cut $[i, 2i]$.

   $\arg(z-i) = -\frac{3\pi}{2}$ then $\sqrt{z-i} \overset{Math.}{=} -\sqrt{z-i}$

   $\arg(z-2i) = -\frac{\pi}{2}$ then $\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-2i}$

$$f(z) \overset{Math.}{=} -f(z)$$

5. $z \in (+)$edge of the cut $[i, 2i]$.

   $\arg(z-i) = \frac{\pi}{2}$ then $\sqrt{z-i} \overset{Math.}{=} \sqrt{z-i}$

   $\arg(z-2i) = -\frac{\pi}{2}$ then $\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

Conclusion:

$$f(z) \overset{Math.}{=} \begin{cases} f(z) & \text{if } z \in (B) \bigcup (+)\text{edge of the cut [i,2i]}, \\ -f(z) & \text{otherwise.} \end{cases} \tag{22}$$

If $f(z)=\sqrt{\prod_{k=1}^{n}(z-z_k)}$ for vertical cut.

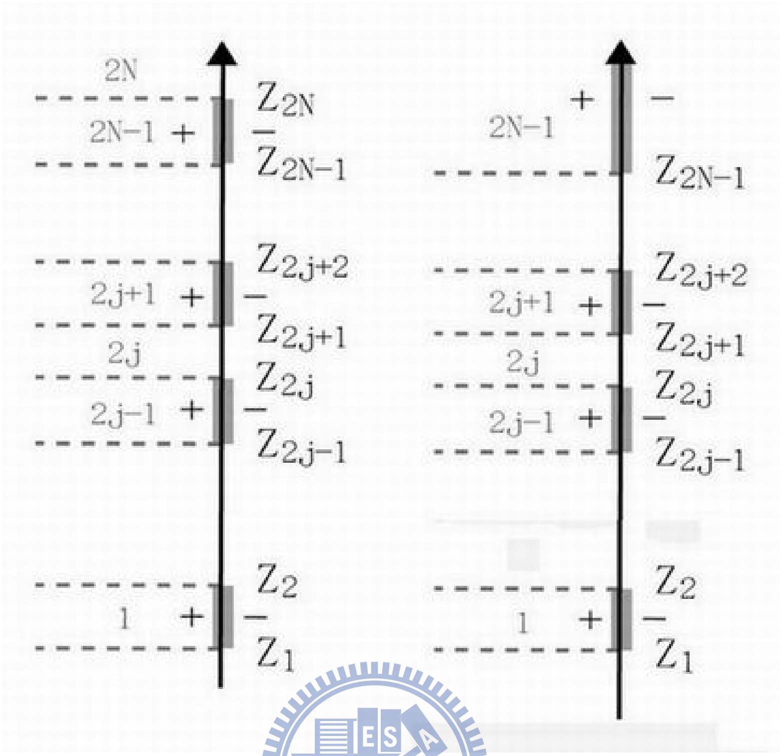

Figure 55: The areas with points $2N-1$ and $2N$ in vertical cuts

In each case we can find the z-region where $f(z)\overset{Math.}{=}f(z)$ and where $f(z)\overset{Math.}{=}-f(z)$.

Case 1. $z_k=a_k i,\ a_k\in R,\ k=1,2,......,2N-1=m$

$$\text{region}-(1)=\{(x,y):x<0,a_1\le y<a_2\}\bigcup\{\overline{z_1z_2}\text{ cut in }(+)\text{edge}\}$$

$$\text{region}-(2)=\{(x,y):x<0,a_2\le y<a_3\}\bigcup\{\overline{z_2z_3}\text{ cut in }(+)\text{ edge }\}$$

$$\text{region}-(2j-1)=\{(x,y):x<0,a_{2j-1}\le y<a_{2j}\}\bigcup\{\overline{z_{2j-1}z_{2j}}\text{ cut in }(+)\text{ edge}\}$$

$$\text{region}-(2N-1)=\{(x,y):x<0,a_{2N-1}\le y\}\bigcup\{z_{2N-1}\to-\infty\text{ cut in }(+)\text{ edge}\}$$

1. $z\in$ region$-(2j-1)$

   $\arg(z-z_k)\in[-\frac{3\pi}{2},-\pi]$ then $\sqrt{z-z_k}\overset{Math.}{=}-\sqrt{z-z_k}$, k=$1,2,...,2j-1$

   $\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}$, k=$2j-1,2j,......,2N-1$

   So

$$f(z)\overset{M.}{=}(-1)^{2j-1}f(z)\overset{Math.}{=}-f(z)$$

2. $z \in$ region$-(2j)$

$\arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi]$ $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $\quad k = 1, 2, ..., 2j$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $\quad k = 2j, 2j+1, ..., 2N-1$

So

$$f(z) \overset{M.}{=} (-1)^{2j} f(z) \overset{Math.}{=} f(z)$$

Case 2. $z_k = a_k i$, $a_k \in R$, $k = 1, 2, ..., 2N = m$

region $-$ $(1) = \{(x, y) : x < 0, a_1 \leq y < a_2\} \bigcup \{\overline{z_1 z_2}$ cut in $(+)$edge$\}$

region $-$ $(2) = \{(x, y) : x < 0, a_2 \leq y < a_3\} \bigcup \{\overline{z_2 z_3}$ cut in $(+)$ edge $\}$

region $-$ $(2j-1) = \{(x, y) : x < 0, a_{2j-1} \leq y < a_{2j}\} \bigcup \{\overline{z_{2j-1} z_{2j}}$ cut in $(+)$ edge$\}$

region $-$ $(2N-1) = \{(x, y) : x < 0, a_{2N-1} \leq y\} \bigcup \{z_{2N-1} \to -\infty$ cut in $(+)$ edge$\}$

region $-$ $(2N) = \{(x, y) : x < 0, a_{2N-1} < y\}$

1. $z \in$ region$-(2j-1)$, $j = 1, 2, ..., N$

$\arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi]$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, k=$1, 2, ..., 2j-1$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2j-1,2j,......,2N-1

So

$$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z) = -f(z)$$

2. $z \in$ region$-(2j)$, $j = 1, 2, ..., N$

$\arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi]$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1, 2, ..., 2j$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2j, 2j+1, ..., 2N-1$

So

$$f(z) \overset{Math.}{=} (-1)^{2j} f(z) = f(z)$$

Conclusion:

Case 1.

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (2j-1), \text{ j=1,2,...,N-1}, \\ f(z) & \text{otherwise.} \end{cases} \tag{23}$$

Case 2.

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (2j-1), \text{ j=1,2,...,N}, \\ f(z) & \text{otherwise.} \end{cases} \tag{24}$$

Take an example to show this way how to help us modify : Compute $\int \frac{1}{f(z)}dz$ over a_1, a_2, a_3, b_1, b_2 and b_3 cycles where $f(z)=$

$\sqrt{(z-2+2i)(z-2-2i)(z-1+3i)(z-1-3i)(z+i)(z-i)(z+1+3i)(z+1-3i)}$

Let $z_1 = 2-2i$, $z_2 = 2+2i$, $z_3 = 1-3i$, $z_4 = 1+3i$, $z_5 = -i$, $z_6 = i$, $z_7 = -1-3i$, $z_8 = -1+3i$.

Using the way of blocks to modify the computation in Mathematica to get right value.

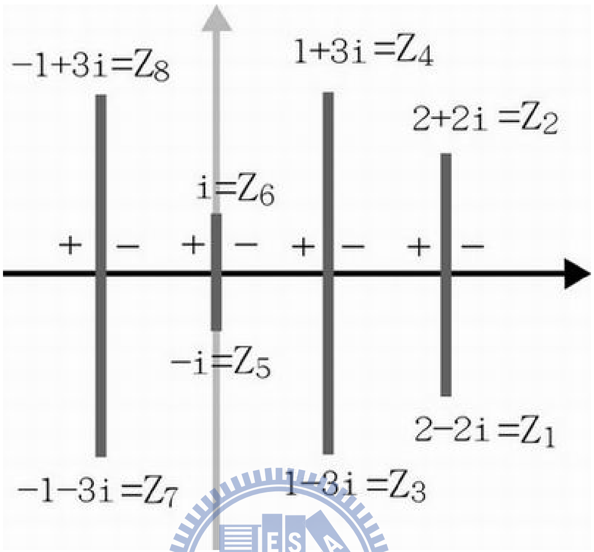

Figure 56: cut plane

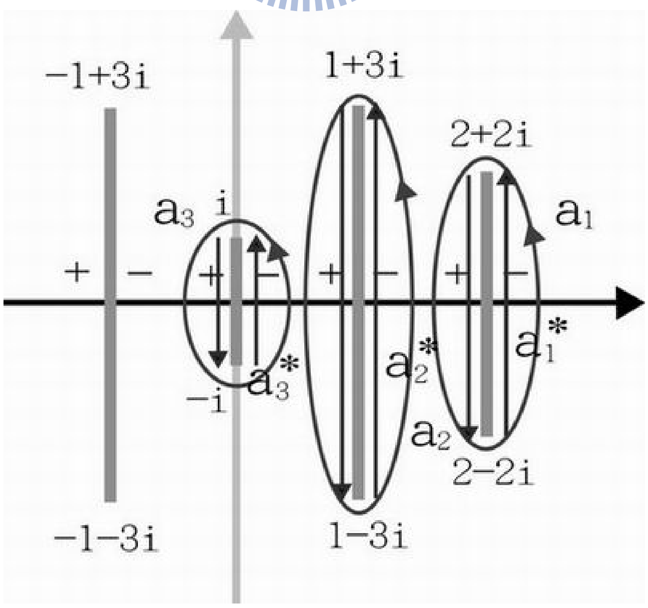

Figure 57: a-cycles and their equivalent path

1. Compute $\int_{a_1^*} \frac{1}{f(z)}dz$ where a_1^* is equivalent path for a_1 and $a_1^* = a_{11}^* \bigcup a_{12}^*$

(1) $z \in a_{11}^{*}$= the path for vertical cut from $2+2i$ to $2-2i$ on (+)edge in sheet-I

Let $z = 2 + ri, r : 2 \to -2 \Rightarrow dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$
$$\int_{a_{11}^{*}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{2}^{-2} \frac{i}{f(2+ri)}dr$$

(2) $z \in a_{12}^{*}$ = the path on vertical cut from $2-2i$ to $2+2i$ in ($-$)edge of sheet-I

Let $z = 2 + ri, r : -2 \to 2 \Rightarrow dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$
$$\int_{a_{12}^{*}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-2}^{2} \frac{i}{f(2+ri)}dr$$

by (1), (2) and Cauchy Integrate Theorem

$$\begin{aligned} \int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^{*}} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_{-2}^{2} \frac{i}{f(2+ri)}dr \\ &= 1.04083 \times 10^{-17} + 0.119738i \end{aligned}$$

2. Compute $\int_{a_2^{*}} \frac{1}{f(z)}dz$ where a_2^{*}=$a_{21}^{*} \bigcup a_{22}^{*} \bigcup a_{23}^{*} \bigcup a_{24}^{*} \bigcup a_{25}^{*} \bigcup a_{26}^{*}$ is the equivalent path for a_2 and a_{21}^{*} = the path on vertical cut from $1+3i$ to $1+2i$ on (+)edge in sheet-I, a_{22}^{*} = the path on vertical cut from $1+2i$ to $1+3i$ on ($-$)edge in sheet-I, a_{23}^{*} = the

path on vertical cut from $1+2i$ to $1-2i$ on $(+)$edge in sheet-I, a_{24}^{*} = the path on vertical cut from $1-2i$ to $1+2i$ on $(-)$edge in sheet-I, a_{25}^{*} = the path on vertical cut from $1-2i$ to $1-3i$ on $(+)$edge in sheet-I and a_{26}^{*} = the path on vertical cut from $1-3i$ to $1-2i$ on $(-)$edge in sheet-I.

(1) $z \in a_{21}^{*}$: Let $z = 1 + ri, r : 3 \to 2$ and $dz = idr$

$$
\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\
\sqrt{z-z_7}\sqrt{z-z_8} &\overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8} \\
f(z) &\overset{Math.}{=} -f(z)
\end{aligned}
$$

(2) $z \in a_{22}^{*}$: Let $z = 1 + ri, r : 2 \to 3$ and $dz = idr$

$$
\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\
\sqrt{z-z_7}\sqrt{z-z_8} &\overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8} \\
f(z) &\overset{Math.}{=} f(z)
\end{aligned}
$$

(3) $z \in a_{23}^{*}$: Let $z = 1 + ri, r : 2 \to -2$ and $dz = idr$

$$
\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\
\sqrt{z-z_7}\sqrt{z-z_8} &\overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8} \\
f(z) &\overset{Math.}{=} f(z)
\end{aligned}
$$

(4) $z \in a_{24}^* :$ Let $z = 1 + ri, r : -2 \to 2$ and $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(5) $z \in a_{25}^* :$ Let $z = 1 + ri, r : -2 \to -3$ and $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(6) $z \in a_{26}^* :$ Let $z = 1 + ri, r : -3 \to -2$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

By (1), (2),......,(6) and Cauchy Integrate Theorem

$$\begin{aligned}\int_{a_2} \frac{1}{f(z)} dz &= \int_{a_2^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_2^3 \frac{i}{f(1-ri)} dr - 2\int_{-2}^2 \frac{i}{f(1-ri)} dr \\ &\quad +2\int_{-3}^{-2} \frac{i}{f(1-ri)} dr \\ &= 0. - 0.103156i\end{aligned}$$

3. Compute $\int_{a_3^*} \frac{1}{f(z)} dz$ where a_3^* is equivalent path for a_3 and $a_3^* = a_{31}^* \bigcup a_{32}^*$ where a_{31}^* is the path on vertical cut from i to $-i$ on (+)edge in sheet-I and a_{32}^* is the path on vertical cut from $-i$ to i on (−)edge in sheet-I

   (1) $z \in a_{31}^*$: Let $z = ri, r : 1 \to -1$ and $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

   (2) $z \in a_{32}^*$: Let $z = ri, r : -1 \to 1$ and $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$
$$\int_{a_{32}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{-1}^{1} \frac{i}{f(2+ri)} dr$$

   By (1), (2) and Cauchy Integrate Theorem

$$\begin{aligned} \int_{a_3} \frac{1}{f(z)} dz &= \int_{a_3^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_{-1}^{1} \frac{i}{f(ri)} dr \\ &= -1.73472 \times 10^{-18} + 0.227188i \end{aligned} \tag{25}$$

In order to compute b_k cycles, we compress and deform b_k to a equivalent closed path b_k^* such that

$$\int_{b_k} \frac{1}{f(z)} dz = \int_{b_k^*} \frac{1}{f(z)} dz$$

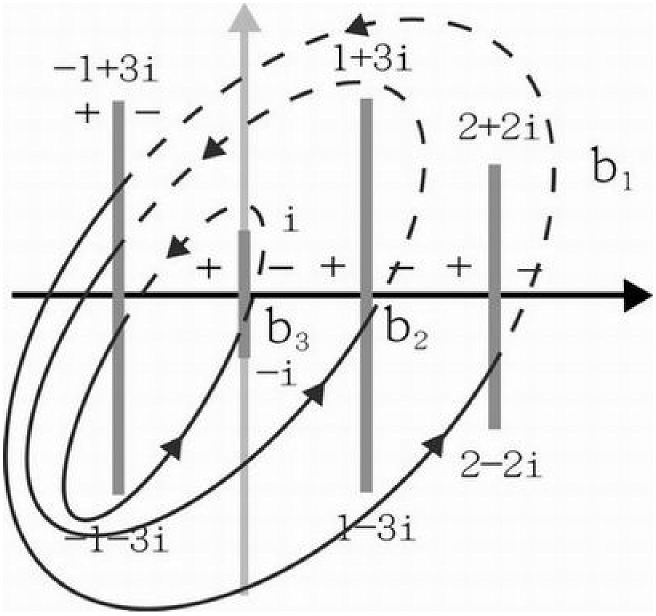

Figure 58: The b-cycles

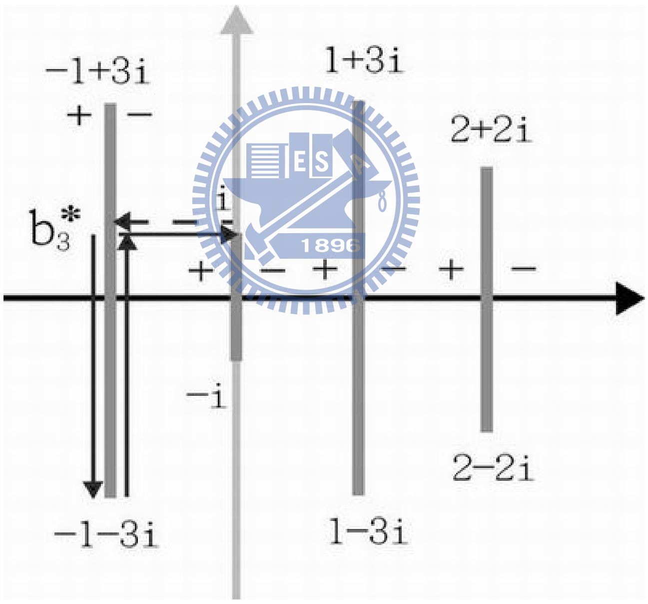

Figure 59: The equivalent path b_3^*

4. The equivalent path $b_3^* = b_{31}^* \bigcup b_{32}^* \bigcup b_{33}^* \bigcup b_{34}^* \bigcup b_{35}^* \bigcup b_{36}^* \bigcup b_{37}^* \bigcup b_{38}^*$ where b_{31}^* = the path on horizontal line from i to $-1+i$ in sheet-I, b_{32}^* is the path on horizontal line from $-1+i$ to i in sheet-II, b_{33}^* is the path on vertical cut from $-1+i$ to $-1-i$ in (+)edge of sheet-I, b_{34}^* is the path on vertical cut from $-1-i$ to $-1+i$ in (−)edge of sheet-I, b_{35}^* is the path on vertical cut from $-1-i$ to $-1-2i$ in (+)edge of sheet-I, b_{36}^* is the path on vertical cut from $-1-2i$ to $-1-i$ in (−)edge of sheet-I, b_{37}^* is

the path on vertical cut from $-1-2i$ to $-1-3i$ in $(+)$edge of sheet-I and b_{38}^* is the path on vertical cut from $-1-3i$ to $-1-2i$ in $(-)$edge of sheet-I.

(1) $z \in b_{31}^*$: Let $z = r+i, r : -1 \to 0$ and then $dz = dr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

(2) $z \in -1+i \dashleftarrow i$: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider $z \in -1+i \leftarrow i$,first. Let $z = r+i, r : 0 \to -1$ and then $dz = dr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

So

$$\int_{-1+i \dashleftarrow i} \frac{1}{f(z)}dz = -\int_{-1+i\leftarrow i} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^{-1} \frac{1}{f(r+i)}dr$$

(3) $z \in b_{33}^*$: Let $z = -1+ri, r : 1 \to -1$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} -\sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

(4) $z \in b_{34}^{*}$: Let $z = -1 + ri, r : -1 \to 1$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \overset{Math.}{=} -f(z)$$

(5) $z \in b_{35}^{*}$: Let $z = -1 + ri, r : -1 \to -2$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} -\sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \overset{Math.}{=} -f(z)$$

(6) $z \in b_{36}^{*}$: Let $z = -1 + ri, r : -2 \to -1$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \overset{Math.}{=} f(z)$$

(7) $z \in b_{37}^{*}$: Let $z = -1 + ri, r : -2 \to -3$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} -\sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \overset{Math.}{=} f(z)$$

(8) $z \in b_{38}^*$: Let $z = -1 + ri, r : -3 \to -2$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \stackrel{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \stackrel{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \stackrel{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \stackrel{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \stackrel{Math.}{=} -f(z)$$

By (1), (2),......,(8) and Cauchy Integrate Theorem

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)} dz &= \int_{b_3^*} \frac{1}{f(z)} dz \\
&\stackrel{Math.}{=} 2\int_{-1}^{0} \frac{i}{f(r+i)} dr - 2\int_{-1}^{1} \frac{i}{f(-1+ri)} dr \\
&\quad +2\int_{-2}^{-1} \frac{i}{f(-1+ri)} dr - 2\int_{-3}^{-2} \frac{i}{f(-1+ri)} dr \\
&= 0.0944734 + 0.0000942992i
\end{aligned}$$

5. b_2: consider equivalent path $b_2^* = b_3^* \bigcup a_3^* \bigcup b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup b_{24}^* \bigcup b_{25}^* \bigcup b_{26}^*$

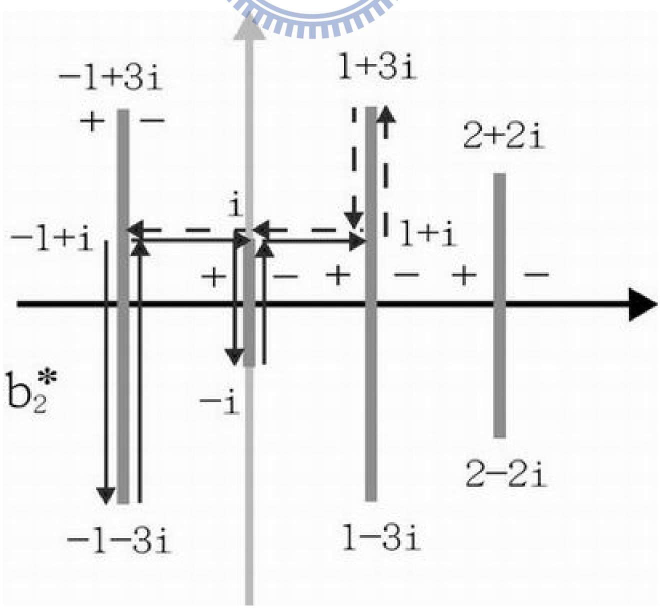

Figure 60: The equivalent path b_2^*

b_{21}^* is the path on horizontal line from i to $1+i$ in sheet-I, b_{22}^* is the path on horizontal line from $1+i$ to i in sheet-II, b_{23}^* is the path on vertical cut from $1+3i$ to $1+2i$ in

(+)edge of sheet-II, b_{24}^* is the path on vertical cut from $1+2i$ to $1+i$ in (+)edge of sheet-II, b_{25}^* is the path on vertical cut from $1+2i$ to $1+3i$ in (−)edge of sheet-II and b_{26}^* is the path on vertical cut from $1+i$ to $1+2i$ in (−)edge of sheet-II.

(1) $z \in b_{21}^* = i \to 1+i$: Let $z = r+i, r : 0 \to 1$ and $dz = dr$

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\
\sqrt{z-z_7}\sqrt{z-z_8} &\overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8} \\
f(z) &\overset{Math.}{=} f(z)
\end{aligned}$$

(2) $z \in b_{22}^* = i \dashleftarrow 1+i$: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider $z \in b_{22}^{**} = i \leftarrow 1+i$,first. Let $z = r+i, r : 1 \to 0$ and then $dz = dr$

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\
\sqrt{z-z_7}\sqrt{z-z_8} &\overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8} \\
f(z) &\overset{Math.}{=} f(z)
\end{aligned}$$

So

$$\begin{aligned}
\int_{b_{22}^*} \frac{1}{f(z)}dz &= \int_{i \dashleftarrow 1+i} \frac{1}{f(z)}dz = -\int_{i \leftarrow 1+i} \frac{1}{f(z)}dz \\
&= -\int_{b_{22}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0 \frac{1}{f(r+i)}dr
\end{aligned}$$

(3) Since the (+)edge of sheet-II is equivalent to the (−)edge of sheet-I and the (−)edge of sheet-II is equivalent to the (+)edge of sheet-I. We have $b_{23}^* \equiv$ the path on vertical cut from $1+3i$ to $1+2i$ in (−)edge of sheet-I, $b_{24}^* \equiv$ the path on vertical cut from $1+2i$ to $1+i$ in (−)edge of sheet-I, $b_{25}^* \equiv$ the path on vertical cut from $1+2i$ to $1+3i$ in (+)edge of sheet-I and $b_{26}^* \equiv$ the path on vertical cut from $1+i$ to $1+2i$ in (+)edge of sheet-I. So we consider in sheet-I.

(a) $z \in b^*_{23}$: Let $z = 1 + ri, r : 3 \to 2$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

(b) $z \in b^*_{24}$: Let $z = 1 + ri, r : 2 \to 1$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(c) $z \in b^*_{25}$: Let $z = 1 + ri, r : 2 \to 3$ and then $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(d) $z \in b^*_{26}$: Let $z = 1 + ri, r : 1 \to 2$ and $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

By (1), (2), (3) and Cauchy Integrate Theorem

$$
\begin{aligned}
\int_{b_2} \frac{1}{f(z)} dz &= \int_{b_2^*} \frac{1}{f(z)} dz \\
&= 2\int_{-1}^{1} \frac{i}{f(ri)} dr + 2\int_{-1}^{0} \frac{i}{f(r+i)} dr \\
&\quad -2\int_{-1}^{1} \frac{i}{f(-1+ri)} dr + 2\int_{-2}^{-1} \frac{i}{f(-1+ri)} dr \\
&\quad -2\int_{-3}^{-2} \frac{i}{f(-1+ri)} dr + 2\int_{0}^{1} \frac{i}{f(r+i)} dr \\
&\quad -2\int_{2}^{3} \frac{i}{f(1+ri)} dr + 2\int_{1}^{2} \frac{i}{f(1+ri)} dr \\
&= 0.189492 + 0.165266i
\end{aligned}
$$

6. $b_1^* = b_2^* \bigcup_{k=1}^{8} b_{1k}^*$:

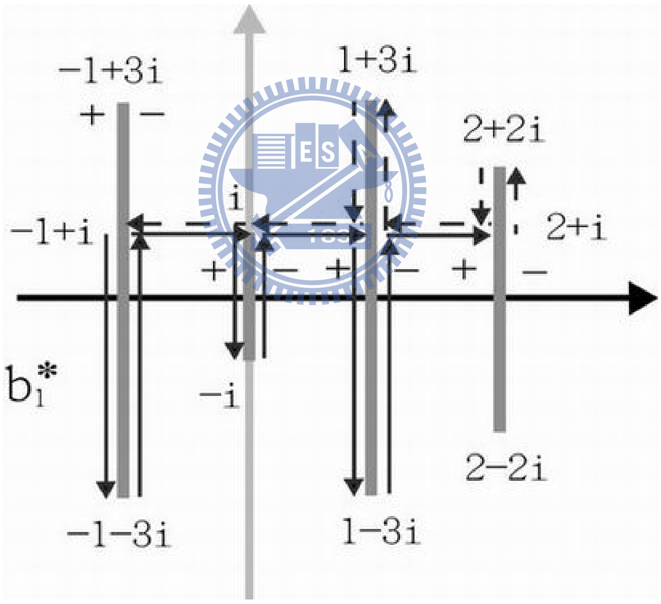

Figure 61: The equivalent path b_1^*

(1) b_{11}^* = the path on a vertical cut from $1+i$ to $1-2i$ with (+)edge of sheet-I.

Let $z = 1 + ri, r : 1 \to -2$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

(2) b^*_{12} = the path on a vertical cut from $1-2i$ to $1+i$ with $(-)$edge of sheet-I.

Let $z = 1 + ri, r : -2 \to 1$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(3) b^*_{13} = the path on a vertical cut from $1-2i$ to $1-3i$ with $(+)$edge of sheet-I.

Let $z = 1 + ri, r : -2 \to -3$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$
$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(4) b^*_{14} = the path on a vertical cut from $1-3i$ to $1-2$ with $(-)$edge of sheet-I.

Let $z = 1 + ri, r : -3 \to -2$, so $dz = idr$

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{Math.}{=} \sqrt{z - z_1}\sqrt{z - z_2}$$
$$\sqrt{z - z_3}\sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_3}\sqrt{z - z_4}$$
$$\sqrt{z - z_5}\sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_5}\sqrt{z - z_6}$$
$$\sqrt{z - z_7}\sqrt{z - z_8} \overset{Math.}{=} \sqrt{z - z_7}\sqrt{z - z_8}$$
$$f(z) \overset{Math.}{=} f(z)$$

(5) b^*_{15} = the path on a horizontal line from $1 + i$ to $2 + i$ on sheet-I.

Let $z = r + i, r : 1 \to 2$, so $dz = dr$

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{Math.}{=} -\sqrt{z - z_1}\sqrt{z - z_2}$$
$$\sqrt{z - z_3}\sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_3}\sqrt{z - z_4}$$
$$\sqrt{z - z_5}\sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_5}\sqrt{z - z_6}$$
$$\sqrt{z - z_7}\sqrt{z - z_8} \overset{Math.}{=} \sqrt{z - z_7}\sqrt{z - z_8}$$
$$f(z) \overset{Math.}{=} -f(z)$$

(6) b^*_{16} = the path on a horizontal line from $2 + i$ to $1 + i$ on sheet-II. We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{16} = the path on a horizontal line from $2 + i$ to $1 + i$ on sheet-I. Let $z = r + i, r : 2 \to 1$, so $dz = dr$

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{Math.}{=} -\sqrt{z - z_1}\sqrt{z - z_2}$$
$$\sqrt{z - z_3}\sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_3}\sqrt{z - z_4}$$
$$\sqrt{z - z_5}\sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_5}\sqrt{z - z_6}$$
$$\sqrt{z - z_7}\sqrt{z - z_8} \overset{Math.}{=} \sqrt{z - z_7}\sqrt{z - z_8}$$
$$f(z)|_{(b^{**}_{16})} \overset{Math.}{=} -f(z)$$

So $f(z)|_{(b^*_{16})} = -f(z)|_{(b^{**}_{16})} \overset{Math.}{=} f(z)$

(7) b^*_{17} = the path on a vertical cut from $2 + 2i$ to $2 + i$ with $(+)$edge of sheet-II $\equiv$ the path on a vertical cut from $2 + 2i$ to $2 + i$ with $(-)$edge of sheet-I.

Let $z = 2 + ri, r : 2 \to 1$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \overset{Math.}{=} f(z)$$

(8) b_{18}^* = the path on a vertical cut from $2+i$ to $2+2i$ with $(-)$edge of sheet-II

$\equiv$ the path on a vertical cut from $2+i$ to $2+2i$ with $(+)$edge of sheet-I

Let $z = 2 + ri, r : 2 \to 1$ and $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

$$\sqrt{z-z_7}\sqrt{z-z_8} \overset{Math.}{=} \sqrt{z-z_7}\sqrt{z-z_8}$$

$$f(z) \overset{Math.}{=} -f(z)$$

By (1)-(8) and Cauchy Integrate Theorem

$$\int_{b_1} \frac{1}{f(z)} dz = \int_{b_1^*} \frac{1}{f(z)} dz = -0.108259 - 0.187169i$$

After look the way of area, now let us discuss the integrals of $\frac{1}{\sqrt{\prod_{k=1}^{n}(z-z_k)}}$ over a,b cycles for other kind of vertical cut (general and special cuts like figure below) by using two ways. Consider $f(z) = \prod_{k=1}^{n}(z-z_k)^{\frac{1}{2}}$, where $n = 2N$, $z_{2k-1} = \overline{z_{2k}}$, k=1,...,N and $Re(z_1) > Re(z_2) > \cdots > Re(z_{2N-1})$, $Im(z_1) = Im(z_3) = \cdots = Im(z_{2N-1})$ and $Im(z_2) = Im(z_4) = \cdots = Im(z_{2N})$.

1. Compute $\int_{a_j^*} \frac{1}{f(z)} dz$ where a_j^* is an equivalent path for a_j

(1) Compute by using argument of complex number to modify

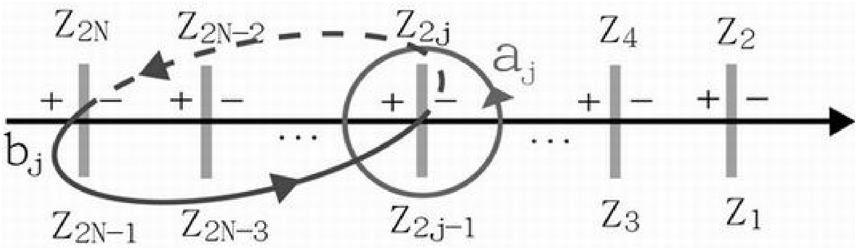

Figure 62: a_j and b_j cycles in other special kinds of vertical cuts

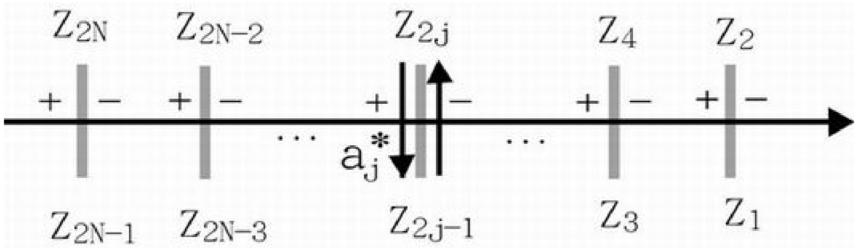

Figure 63: a_j^* cycles in other special kinds of vertical cuts

(a) the path from z_{2j} to z_{2j-1} on (+) edge of sheet-I.

Let $z = Re(z_{2j}) + ri$, $r : Im(z_{2j}) \to Im(z_{2j-1})$

$\arg(z - z_1) \in [-\frac{3\pi}{2}, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, ..., 2j - 1$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k =$ otherwise

So

$$f(z) \overset{Math.}{=} (-1)^j f(z)$$

(b) The path from z_{2j-1} to z_{2j} on (−)edge of sheet-I

Let $z = Re(z_{2j}) + ri$, $r : Im(z_{2j}) \to Im(z_{2j-1})$

$\arg(z - z_1) \in [-\frac{3\pi}{2}, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 3, ..., 2j - 3$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k =$ otherwise

So

$$f(z) \overset{Math.}{=} (-1)^{j-1} f(z)$$

$$\int_{a_j^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{Im(z_{2j})}^{Im(z_{2j-1})} \frac{(-1)^j i}{f(Re(z_{2j}+)ri)} dr + \int_{Im(z_{2j-1})}^{Im(z_{2j})} \frac{(-1)^{j-1} i}{f(Re(z_{2j}) + ri)} dr \tag{26}$$

$$\overset{Math.}{=} 2i(-1)^{j-1} \int_{Im(z_{2j})}^{Im(z_{2j-1})} \frac{1}{f(Re(z_{2j} + ri))} dr \tag{27}$$

(2) Using the result about modify of blocks and then we can compute $\int_{a_k^*} \frac{1}{f(z)} dz$

(a) The path from z_{2j} to z_{2j-1} in (+) edge of sheet I

Let $z = Re(z_{2j}) + ri$, $r : Im(z_{2j}) \to Im(z_{2j-1})$

$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \stackrel{Math.}{=} -\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}$, $k = 1, 2, \cdots, j$

$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \stackrel{Math.}{=} \sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}$, $k = j+1, j+2, \cdots, N$

So we have

$$f(z) \stackrel{Math.}{=} (-1)^j \prod_{k=1}^{m} \sqrt{z - z_k} = (-1)^j f(z)$$

(b) The path from z_{2j-1} to z_{2j} in (−) edge of sheet-I

Let $z = Re(z_{2j}) + ri$, $r : Im(z_{2j}) \to Im(z_{2j-1})$

$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \stackrel{Math.}{=} -\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}$, $k = 1, 2, \cdots, j-1$

$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \stackrel{Math.}{=} \sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}$, $k = j, j+1, \cdots, N$

So we have

$$f(z) \stackrel{Math.}{=} (-1)^{j-1} \prod_{k=1}^{m} \sqrt{z - z_k} = (-1)^{j-1} f(z)$$

$$\int_{a_j^*} \frac{1}{f(z)} dz \stackrel{Math.}{=} \int_{Im(z_{2j})}^{Im(z_{2j-1})} \frac{(-1)^j i}{f(Re(z_{2j}) + ri)} dr + \int_{Im(z_{2j-1})}^{Im(z_{2j})} \frac{(-1)^{j-1} i}{f(Re(z_{2j}) + ri)} dr \quad (28)$$

$$\stackrel{Math.}{=} 2i(-1)^{j-1} \int_{Im(z_{2j})}^{Im(z_{2j-1})} \frac{1}{f(Re(z_{2j}) + ri)} dr \quad (29)$$

2. Compute $\int_{b_j^*} \frac{1}{f(z)} dz$ where b_j^* is equivalent path for b_j and $b_j^* = \bigcup_{j+1}^{N} a_s^* \bigcup z_{2N} \dashrightarrow z_{2j} \bigcup z_{2N} \leftarrow z_{2j}$

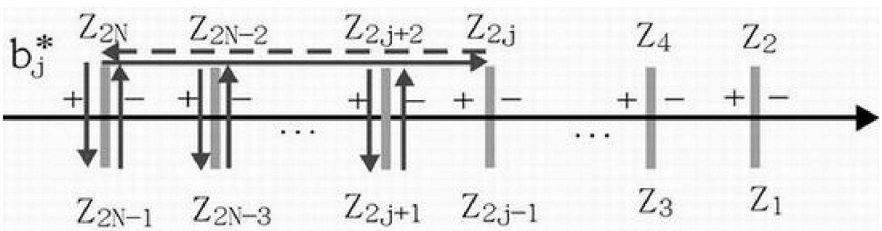

Figure 64: b_j^* cycles in other special kinds of vertical cuts

(1) $z \in a_{j+1}^*$: as above

(2) $z_{2s+2} \leftarrow z_{2s} \bigcup z_{2s+2} \dashrightarrow z_{2s}$, $s = j, \cdots, N-1$

(A) Compute by using the argument of complex number to modify

(a) $z_{2s+2} \leftarrow z_{2s}$ (That is b_{s1}^*), $s = j, \cdots, N-1$

Let $z = r + iIm(z_{2s})$, $r : Re(z_{2s}) \rightarrow Re(z_{2s+2})$

$\arg(z - z_{2k-1}) \in (-\frac{3\pi}{2}, -\pi)$

then $\sqrt{z - z_{2k-1}} \overset{Math.}{=} -\sqrt{z - z_{2k-1}}$, $k = 1, 2, \cdots, s$

$\arg(z - z_{2k-1}) \in (-\pi, \pi)$

then $\sqrt{z - z_{2k-1}} \overset{Math.}{=} \sqrt{z - z_{2k-1}}$, $k = s+1, \cdots, N$

$\arg(z - z_{2k}) = -\pi$ then $\sqrt{z - z_{2k}} \overset{Math.}{=} -\sqrt{z - z_{2k}}$, $k = 1, 2, \cdots, s$

$\arg(z - z_{2k}) = 0$ then $\sqrt{z - z_{2k}} \overset{Math.}{=} \sqrt{z - z_{2k}}$, $k = s+1, \cdots, N$

$$\prod_{k=1}^{N} \sqrt{z - z_{2k-1}} \overset{Math.}{=} (-1)^s \prod_{k=1}^{N} \sqrt{z - z_{2k-1}}$$

$$\prod_{k=1}^{N} \sqrt{z - z_{2k}} \overset{Math.}{=} (-1)^s \prod_{k=1}^{N} \sqrt{z - z_{2k}}$$

$$f(z) \overset{Math.}{=} (-1)^{2s} f(z) = f(z)$$

$$\int_{b_{s1}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{Re(z_{2s})}^{Re(z_{2s+2})} \frac{1}{f(r + iIm(z_{2s}))} dr \qquad (30)$$

(b) $z_{2s+2} \dashrightarrow z_{2s}$ (That isb_{s2}^*). We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we first consider $z_{2s+2} \rightarrow z_{2s}(b_{s2}^{**})$, $s = j, \cdots, N-1$

Let $z = r + iIm(z_{2s})$, $r : Re(z_{2s}) \rightarrow Re(z_{2s+2})$

$\arg(z - z_{2k-1}) \in (-\frac{3\pi}{2}, -\pi)$

then $\sqrt{z - z_{2k-1}} \overset{Math.}{=} -\sqrt{z - z_{2k-1}}$, $k = 1, 2, \cdots, s$

$\arg(z - z_{2k-1}) \in (-\pi, \pi)$

then $\sqrt{z - z_{2k-1}} \overset{Math.}{=} \sqrt{z - z_{2k-1}}$, $k = s+1, \cdots, N$

$\arg(z - z_{2k}) = -\pi$ then $\sqrt{z - z_{2k}} \overset{Math.}{=} -\sqrt{z - z_{2k}}$, $k = 1, 2, \cdots, s$

$\arg(z - z_{2k}) = 0$ then $\sqrt{z - z_{2k}} \overset{Math.}{=} \sqrt{z - z_{2k}}$, $k = s+1, \cdots, N$ If $z \in b_{s2}^{**}$, $f(z) \overset{Math.}{=} f(z)$, so we have if $z \in b_{s2}^*$,

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{s2}^*}\frac{1}{f(z)}dz \overset{Math.}{=} -\int_{b_{s2}^{**}}\frac{1}{f(z)}dz$$

$$\overset{Math.}{=} -\int_{Re(z_{2s+2})}^{Re(z_{2s})}\frac{1}{f(r+iIm(z_{2s}))}dr \tag{31}$$

By (a), (b), we have

$$\int_{b_s^*}\frac{1}{f(z)}dz \overset{Math.}{=} 2\int_{Re(z_{2s})}^{Re(z_{2s+2})}\frac{1}{f(r+iIm(z_{2s}))}dr \tag{32}$$

(B) Compute by using areas of domain to modify the computation in Mathematica

(a) $z_{2s+2} \leftarrow z_{2s}(b_{s1}^*)$, $s = j, \cdots, N-1$

$\sqrt{z-z_{2k-1}}\sqrt{z-2k} \overset{Math.}{=} \sqrt{z-z_{2k-1}}\sqrt{z-2k}$, $\forall k$

$$f(z) \overset{Math.}{=} f(z)$$

(b) $z_{2s+2} \dashrightarrow z_{2s}(b_{s2}^*)$ and then we known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we first consider $z_{2s+2} \rightarrow z_{2s}(b_{s2}^{**})$, $s = j, \cdots, N-1$

$\sqrt{z-z_{2k-1}}\sqrt{z-2k} \overset{Math.}{=} \sqrt{z-z_{2k-1}}\sqrt{z-2k}$, $\forall k$

If $z \in b_{s2}^{**}$, $f(z) \overset{Math.}{=} f(z)$, so if $z \in b_{s2}^*$ then

$$f(z) \overset{Math.}{=} -f(z)$$

Reduced by half steps to modify the computation of $f(z)$ than (A).

Conclusion: By (1), (2) and Cauchy Integrate Theorem

$$\int_{b_j}\frac{1}{f(z)}dz = \int_{b_j^*}\frac{1}{f(z)}dz$$

$$= \sum_{s=j+1}^{N}\int_{a_j^*}\frac{1}{f(z)}dz + \sum_{s=j}^{N-1}\int_{b_j^*}\frac{1}{f(z)}dz \tag{33}$$

$$\overset{Math.}{=} \sum_{s=j+1}^{N}(-1)^{j-1}2\int_{Im(z_{2j-1})}^{Im(z_{2j})}\frac{i}{Re(z_{2j})+ri}dr \tag{34}$$

$$+\sum_{s=j+1}^{N}2\int_{Re(z_{2s})}^{Re(z_{2s+2})}\frac{1}{f(r+iIm(z_{2s}))}dr$$

No matter what way we use, way of areas or way of the arguments of complex number, the modifying is the same. We could according to different situation chose one way which let us modify easier.

Example: Compute $\int \frac{1}{f(z)}dz$ over a_1, a_2, a_3, b_1, b_2 and b_3 cycles where
$f(z) = \sqrt{(z-2-2i)(z-2+2i)(z-1-3i)(z-1-5i)(z-0)(z-2i)(z+1-i)}$.
Let $z_1 = 2+2i, z_2 = 2-2i, z_3 = 1+3i, z_4 = 1+5i, z_5 = 0, z_6 = 2i, z_7 = -1+i$

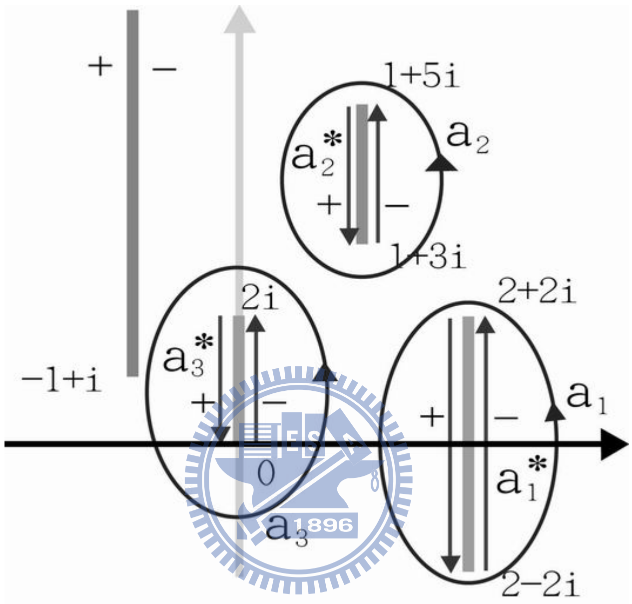

Figure 65: a_j and their equivalent path

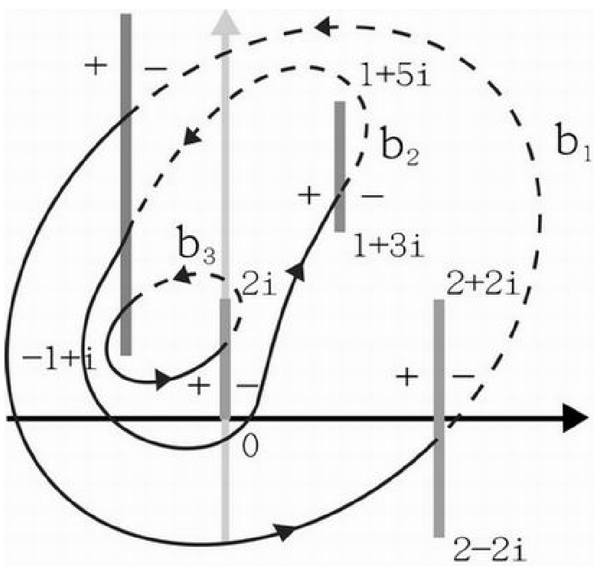

Figure 66: b_j

1. The equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on a vertical cut from $2+2i$ to $2-2i$ with (+)edge of sheet-I and a_{12}^* = the path on a vertical cut from $2-2i$ to $2+2i$ with (−)edge of sheet-I.

   (1) a_{11}^* : Let $z = 2 + ri, r : 2 \to -2$ and $dz = idr$

   $\arg(z-2+2i) = -\frac{3\pi}{2}$ then $\sqrt{z-2+2i} \overset{Math.}{=} -\sqrt{z-2+2i}$.

   $\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2, ..., 7$

   So we have $f(z) \overset{Math.}{=} -f(z)$

   (2) a_{12}^* : Let $z = 2 + ri, r : -2 \to 2$ and $dz = idr$

   $\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=$1, 2, ..., 7$

   So we have $f(z) \overset{Math.}{=} f(z)$

   By (1), (2) and Cauchy Integrate Theorem

$$\begin{aligned} \int_{a_1} \frac{1}{f(z)} dz &= \int_{a_1^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_{-2}^{2} \frac{i}{f(2+ri)} dr \\ &= -0.281519 + 0.0877839i \end{aligned}$$

2. The equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^*$ where a_{21}^* = the path on a vertical cut from $1+5i$ to $1+3i$ with (+)edge of sheet-I and a_{22}^* = the path on a vertical cut from $1+3i$ to $1+5i$ with (−)edge of sheet-I.

   (1) a_{21}^*: Let $z = 1 + ri, r : 5 \to 3$ and $dz = idr$

   $\arg(z-z_1)$ and $\arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$

   then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

   $\arg(z-1-3i) = -\frac{3\pi}{2}$ then $\sqrt{z-1-3i} \overset{Math.}{=} -\sqrt{z-1-3i}$

   $\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4, 5, 6, 7$

   So we have

$$f(z) \overset{Math.}{=} -f(z)$$

   (2) a_{22}^*= the path on a vertical cut from $1+3i$ to $1+5i$ with (−)edge of sheet-I.

   Let $z = 1 + ri, r : 3 \to 5$ and $= idr$

$\arg(z-z_1)$ and $\arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$

then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-1-3i) = \frac{\pi}{2}$ then $\sqrt{z-1-3i} \overset{Math.}{=} \sqrt{z-1-3i}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4, 5, 6, 7$

So we have

$$f(z) \overset{Math.}{=} f(z)$$

By (1), (2) and Cauchy Integrate Theorem

$$\begin{aligned} \int_{a_2} \frac{1}{f(z)} dz &= \int_{a_2^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2i \int_3^5 \frac{1}{f(1+ri)} dr \\ &= -0.137892 - 0.320934i \end{aligned}$$

3. The equivalent path $a_3^* = a_{31}^* \bigcup a_{32}^*$ where a_{31}^* = the path on a vertical cut from $2i$ to 0 on (+)edge of sheet-I and a_{32}^* = the path on a vertical cut from 0 to $2i$ on (−)edge of sheet-I.

(1) a_{31}^*: Let $z = ri, r : 2 \to 0$ and $dz = idr$

$\arg(z-z_1) \in (-\frac{3\pi}{2}, -\pi)$ then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$

$\arg(z-z_5) = -\frac{3\pi}{2}$ then $\sqrt{z-z_5} \overset{Math.}{=} -\sqrt{z-z_5}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2, 3, 4, 6, 7

So we have

$$f(z) \overset{Math.}{=} f(z)$$

(2) a_{32}^*: Let $z = ri, r : 0 \to 2$ and $dz = idr$

$\arg(z-z_1) \in (-\frac{3\pi}{2}, -\pi)$ then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$

$\arg(z-z_5) = \frac{\pi}{2}$ then $\sqrt{z-z_5} \overset{Math.}{=} -\sqrt{z-z_5}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2, 3, 4, 6, 7

So we have

$$f(z) \overset{Math.}{=} -f(z)$$

By (1), (2) and Cauchy Integrate Theorem

$$\begin{aligned}\int_{a_3}\frac{1}{f(z)}dz &= \int_{a_3^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} -2i\int_0^2\frac{1}{f(ri)}dr\\ &= 0.545309+0.41295i\end{aligned}$$

4. The equivalent path $b_3^* = b_{31}^* \bigcup b_{32}^* \bigcup b_{33}^* \bigcup b_{34}^*$ where b_{31}^* = the path on a vertical cut from $-1+2i$ to $-1+i$ with (+)edge of sheet-I, b_{32}^* = the path on a vertical cut from $-1+i$ to $-1+2i$ with (−)edge of sheet-I, b_{33}^* = the path on a horizontal line from $1+2i$ to $2i$ on sheet-I and b_{34}^* = the path on a horizontal line from $2i$ to $1+2i$ on sheet-II.

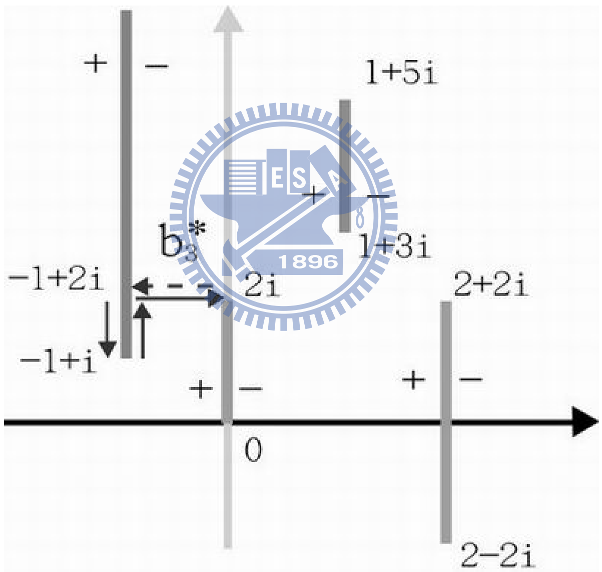

Figure 67: b_3^*

(1) b_{31}^*: Let $z=-1+ri, r:2\to 1$ and then $dz=idr$. Here using area to modify (cause we can choose idea which help us modify the sign easier.)

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

Now only one point need to discuss:

$\arg(z-z_7) = -\frac{3\pi}{2}$ then $\sqrt{z-z_7} \overset{Math.}{=} -\sqrt{z-z_7}$

So we have

$$f(z) \overset{Math.}{=} -f(z)$$

(2) b_{32}^*: Let $z = -1 + ri, r : 1 \to 2$, so $dz = idr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

Now only one point need to discuss,

$\arg(z-z_7) = \frac{\pi}{2}$ then $\sqrt{z-z_7} \overset{Math.}{=} \sqrt{z-z_7}$

So we have

$$f(z) \overset{Math.}{=} f(z)$$

(3) b_{33}^*: Let $z = r + 2i, r : -1 \to 0$, so $dz = dr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

Now only one point need to discuss

$\arg(z-z_7) \in (-\pi, \pi)$ then $\sqrt{z-z_7} \overset{Math.}{=} \sqrt{z-z_7}$

So $f(z) \overset{Math.}{=} f(z)$

(4) b_{34}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{34}^{**} =the path on a horizon line from $2i$ to $-1+2i$ on sheet-I.

Let $z = r + 2i, r : 0 \to -1$, so $dz = dr$

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}$$

Now only one point need to discuss

$\arg(z-z_7)\in(-\pi,\pi)$ then $\sqrt{z-z_7}\overset{Math.}{=}\sqrt{z-z_7}$

So

$$f(z)|_{b_{34}^*}=-f(z)|_{b_{34}^{**}}\overset{Math.}{=}-f(z)$$

By (1), (2), (3), (4) and Cauchy Integrate Theorem

$$\begin{aligned}\int_{b_3}\frac{1}{f(z)}dz &= \int_{b_3^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} 2i\int_1^2\frac{1}{f(-1+ri)}dr-2\int_0^{-1}\frac{1}{f(r+2i)}dr\\ &= -0.527726-0.0295031i\end{aligned}$$

5. The equivalent path $b_2^*=b_3^*\bigcup a_3^*\bigcup b_{21}^*\bigcup b_{22}^*\bigcup b_{23}^*\bigcup b_{24}^*\bigcup b_{25}^*\bigcup b_{26}^*$ where b_{21}^* = the path on a vertical line from $2i$ to $3i$ on sheet-I, b_{22}^* = the path on a horizon line from $3i$ to $1+3i$ on sheet-I, b_{23}^* = the path on a vertical cut from $1+3i$ to $1+5i$ with $(-)$edge of sheet-II, b_{24}^* = the path on a vertical cut from $1+5i$ to $1+3i$ with $(+)$edge of sheet-II, b_{25}^* = the path on a horizon line from $1+3i$ to $3i$ on sheet-II and b_{26}^* = the path on a vertical line from $3i$ to $2i$ on sheet-II.

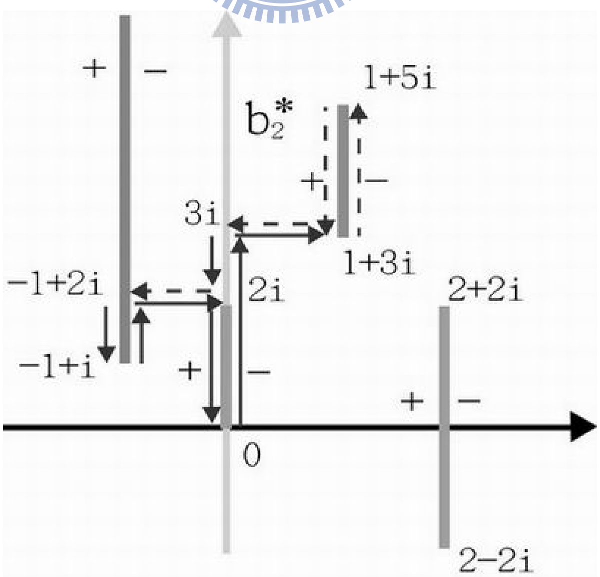

Figure 68: b_2^*

(1) b_{21}^* = the path on a vertical line from $2i$ to $3i$ on sheet-I

Let $z=ri, r:2\to3$ and $dz=idr$

$\arg(z-z_1), \arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z-z_1} \stackrel{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \stackrel{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_5) = \arg(z-z_6) = \frac{\pi}{2}$

then $\sqrt{z-z_5} \stackrel{Math.}{=} \sqrt{z-z_5}$ and $\sqrt{z-z_6} \stackrel{Math.}{=} \sqrt{z-z_6}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k}, k = 3, 4, 7$

So

$$f(z) \stackrel{Math.}{=} f(z)$$

(2) $b_{22}^* =$ the path on a horizontal line from $3i$ to $1+3i$ on sheet-I

Let $z = r + 3i, r : 0 \to 1$ and $dz = dr$

$\arg(z-z_1), \arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z-z_1} \stackrel{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \stackrel{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_3) = -\pi$ then $\sqrt{z-z_3} \stackrel{Math.}{=} -\sqrt{z-z_3}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k}, k = 4, 5, 6, 7$

So

$$f(z) \stackrel{Math.}{=} -f(z)$$

(3) $b_{23}^* \equiv$ the path on a vertical cut from $1+3i$ to $1+5i$ with $(+)$edge of sheet-I.

Let $z = 1 + ri, r : 3 \to 5$ and $dz = dr$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k}, k = 1, 2, 5, 6, 7$

$\arg(z-z_3) = -\frac{3\pi}{2}$ then $\sqrt{z-z_3} \stackrel{Math.}{=} -\sqrt{z-z_3}$

$\arg(z-z_4) = -\frac{\pi}{2}$ then $\sqrt{z-z_4} \stackrel{Math.}{=} \sqrt{z-z_4}$

So

$$f(z) \stackrel{Math.}{=} -f(z)$$

(4) $b_{24}^* \equiv$ the path on a vertical cut from $1+5i$ to $1+3i$ with $(-)$edge of sheet-I.

Let $z = 1 + ri, r : 5 \to 3$ and $dz = dr$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k}, k = 1, 2, 5, 6, 7$

$\arg(z-z_3) = \frac{\pi}{2}$ then $\sqrt{z-z_3} \stackrel{Math.}{=} \sqrt{z-z_3}$

$\arg(z-z_4) = -\frac{\pi}{2}$ then $\sqrt{z-z_4} \stackrel{Math.}{=} \sqrt{z-z_4}$

So

$$f(z) \overset{Math.}{=} f(z)$$

(5) b^*_{25} = the path on a horizontal line from $1+3i$ to $3i$ on sheet-II

We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{25} = the path on a horizontal line from $1+3i$ to $3i$ on sheet-I. Let $z = r+3i, r : 1 \to 0$ and $dz = dr$

$\arg(z-z_1), \arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_3) = -\pi$ then $\sqrt{z-z_3} \overset{Math.}{=} -\sqrt{z-z_3}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 4, 5, 6, 7$

So $f(z)|_{(b^{**}_{25})} \overset{Math.}{=} -f(z)$ then

$$f(z)|_{(b^*_{25})} = -f(z)|_{(b^{**}_{25})} \overset{Math.}{=} f(z)$$

(6) b^*_{26} = the path on a vertical line from $3i$ to $2i$ on sheet-II

We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{26} = the path on a horizontal line from $3i$ to $2i$ on sheet-I. Let $z = ri, r : 3 \to 2$ and $dz = idr$

$\arg(z-z_1), \arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_5) = \arg(z-z_6) = \frac{\pi}{2}$

then $\sqrt{z-z_5} \overset{Math.}{=} \sqrt{z-z_5}$ and $\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_6}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 3, 4, 7$

So $f(z)|_{(b^{**}_{26})} \overset{Math.}{=} f(z)$ then

$$f(z)|_{(b^*_{26})} = -f(z)|_{(b^{**}_{26})} \overset{Math.}{=} -f(z)$$

By (1), (2), (3), (4), (5), (6) and Cauchy Integrate Theorem

$$\begin{aligned}\int_{b_2}\frac{1}{f(z)}dz &= \int_{b_2^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} 2i\int_1^2\frac{1}{f(-1+ri)}dr-2\int_0^1\frac{1}{f(r+2i)}dr\\ &\quad -2i\int_0^2\frac{1}{f(ri)}dr+2i\int_2^3\frac{1}{f(ri)}dr\\ &\quad -2\int_0^1\frac{1}{f(r+3i)}dr+2i\int_3^5\frac{1}{f(1+ri)}dr\\ &= 0.211319-0.45172i\end{aligned}$$

6. The equivalent path $b_1^* = b_2^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^*$ where b_{11}^* = the path on a vertical line from $1+3i$ to $1+2i$ on sheet-I, b_{12}^* = the path on a horizontal line from $1+2i$ to $2+2i$ on sheet-I, b_{13}^* = the path on a horizontal line from $2+2i$ to $1+2i$ on sheet-II and b_{14}^* = the path on a vertical line from $1+2i$ to $1+3i$ on sheet-II.

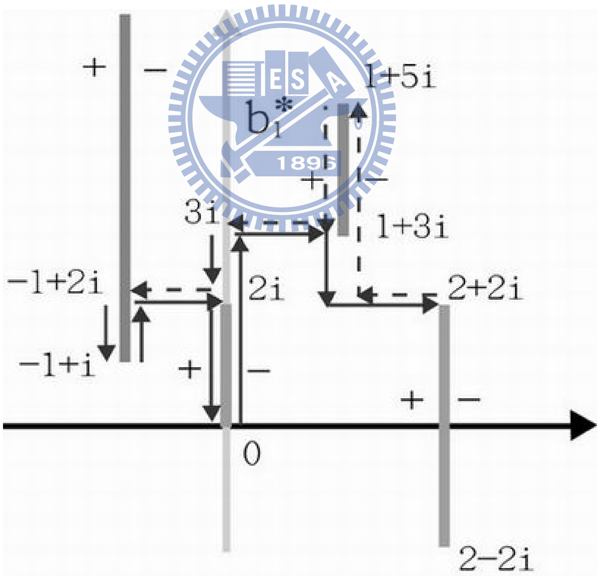

Figure 69: b_1^*

(1) b_{11}^* : Let $z = 1+ri, r : 3 \to 2$, so $dz = idr$

$\arg(z-z_1), \arg(z-z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 3,4,5,6,7$

So $f(z)|_{(b_{11}^*)} \overset{Math.}{=} f(z)$

(2) b^*_{12} : Let $z = r + 2i, r : 1 \to 2$, so $dz = dr$

$\arg(z - z_1), \arg(z - z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$ and $\sqrt{z - z_2} \overset{Math.}{=} -\sqrt{z - z_2}$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 3, 4, 5, 6, 7$

So $f(z)|_{(b^*_{12})} \overset{Math.}{=} f(z)$

(3) b^*_{13} : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{13} = the path on a horizontal line from $2 + 2i$ to $1 + 2i$ on sheet-I

Let $z = r + 2i, r : 2 \to 1$, so $dz = dr$

$\arg(z - z_1), \arg(z - z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$ and $\sqrt{z - z_2} \overset{Math.}{=} -\sqrt{z - z_2}$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, k=3,4,5,6,7

So $f(z)|_{(b^{**}_{13})} \overset{Math.}{=} f(z)$ then

$$f(z)|_{(b^*_{13})} = -f(z)|_{(b^{**}_{13})} \overset{Math.}{=} -f(z)$$

(4) b^*_{14}: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{13} = the path on a vertical line from $1 + 2i$ to $1 + 3i$ on sheet-I

Let $z = 1 + ri, r : 2 \to 3$, so $dz = idr$

$\arg(z - z_1), arg(z - z_2) \in (-\frac{3\pi}{2}, -\pi)$ (where we need to modify)

then $\sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$ and $\sqrt{z - z_2} \overset{Math.}{=} -\sqrt{z - z_2}$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 3, 4, 5, 6, 7$

So $f(z)|_{(b^{**}_{11})} \overset{Math.}{=} f(z)$ then

$$f(z)|_{(b^*_{11})} = -f(z)|_{(b^{**}_{11})} \overset{Math.}{=} -f(z)$$

By (1), (2), (3), (4) and Cauchy Integrate Theorem

$$\begin{aligned} \int_{b_1} \frac{1}{f(z)} dz &= \int_{b^*_1} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{b^*_2} \frac{1}{f(z)} - 2\int_2^3 \frac{i}{f(1 + ri)} dr + 2\int_1^2 \frac{1}{f(r + 2i)} dr \\ &= -0.220099 - 0.652467i \end{aligned}$$

4 The integrals of $\frac{1}{f(z)}$ over a,b cycles for slant cut

Same as usual, before compute the integrals of $\int \frac{1}{f(z)}dz$ over a, b-cycles for slant cut where $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$. We have to consider the definition of the sheet-I and sheet-II and discuss how to modify the computation in Mathematica for slant cut. We know all cuts on a straight line and the slope of the line is $m = \tan\alpha$, $0 < \alpha \leq \pi$.

Definition 3. *The cut with α means the slope of the straight line where the cut on is $m = \tan\alpha$, $0 < \alpha \leq \pi$.*

First, take a special angle: 1. The cut with $\alpha = 45° = \frac{\pi}{4}$ and $f(z) = \sqrt{z}$

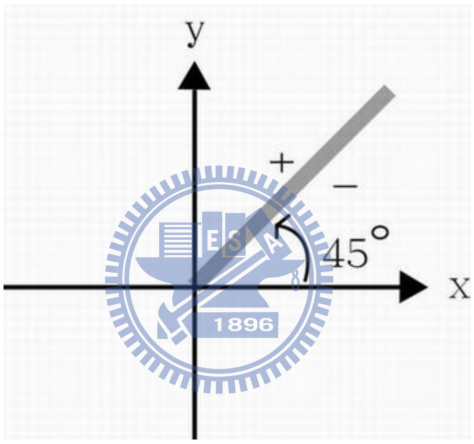

Figure 70: $\alpha = 45° = \frac{\pi}{4}$

In this case, we define two sheets first,

$$z = \begin{cases} re^{i\theta}, \theta \in [-\frac{7\pi}{4}, \frac{\pi}{4}) & \text{if z in sheet-I} \\ re^{i\theta}, \theta \in [\frac{\pi}{4}, \frac{9\pi}{4}) & \text{if z in sheet-II} \end{cases} \tag{35}$$

The cut in each sheet has two edges, label the starting edge of the cut with "+" and the terminal edge of the cut with "−". Analysis in theory

$$z = |z|e^{i\theta},\ \theta \in [-\frac{7\pi}{4}, \frac{\pi}{4}) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}}e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [-\frac{7\pi}{8}, \frac{\pi}{8}) \tag{36}$$

$$z = |z|e^{i\theta},\ \theta \in [\frac{\pi}{4}, \frac{9\pi}{4}) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}}e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [\frac{\pi}{8}, \frac{9\pi}{8}) \tag{37}$$

Discuss the difference between the value in theory and in Mathematica. and find out how to modify the computation. Since in Mathematica the value is always single-value, no matter in which sheet.

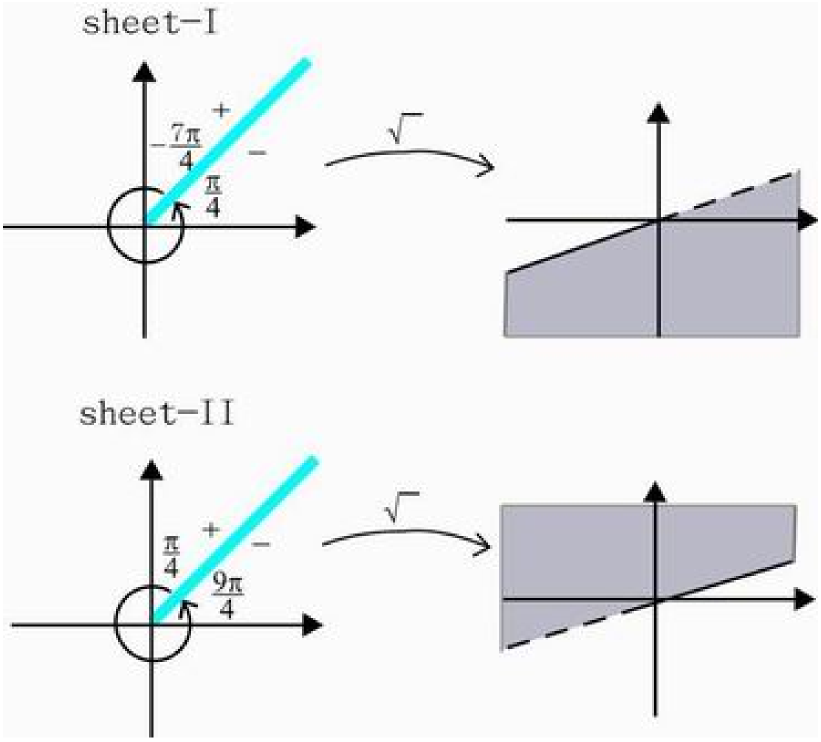

Figure 71: The value in sheet-I and sheet-II

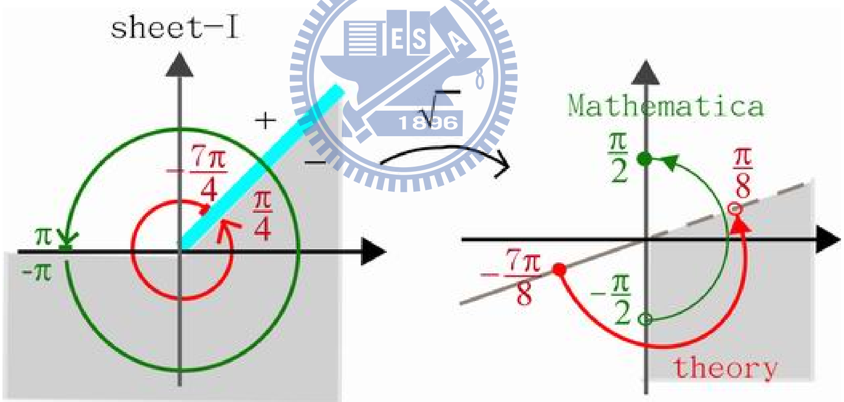

Figure 72: The value in sheet-I and Mathematica

Compare the range of θ then we get that if $\theta \in [-\frac{7\pi}{4}, -\pi]$ we need to modify. Let

$$z = \begin{cases} |z|e^{i\theta},\ \theta \in [-\frac{7\pi}{4}, \frac{\pi}{4}) & \text{if z in sheet-I} \\ |z|e^{i\theta + 2\pi},\ \theta + 2\pi \in [\frac{\pi}{4}, \frac{9\pi}{4}) & \text{if z in Mathematica} \end{cases} \tag{38}$$

Do the following to compare the value:

$$\begin{aligned} \sqrt{z} &= |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [-\frac{7\pi}{8}, \frac{\pi}{8}) && \text{if z in sheet-I} \\ \sqrt{z} &= |z|^{\frac{1}{2}} e^{\frac{i\theta+2\pi}{2}} \\ &= -|z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \theta \in [-\frac{7\pi}{4}, \frac{\pi}{4}) && \text{if z in Mathematica} \end{aligned}$$

When z in sheet-I and one of the end points is z_k of slant cut with $\frac{\pi}{4}$. The value of $\sqrt{z-z_k}$ in theory and in Mathematica is

$$\sqrt{z-z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z-z_k} & \text{if } \arg(z-z_k) \in [-\frac{7\pi}{4}, -\pi], \\ \sqrt{z-z_k} & \text{if } \arg(z-z_k) \in (-\pi, \frac{\pi}{4}) \end{cases} \tag{39}$$

Example 2: The cut with $\alpha = 120° = \dfrac{2\pi}{3}$ and $f(z) = \sqrt{z}$

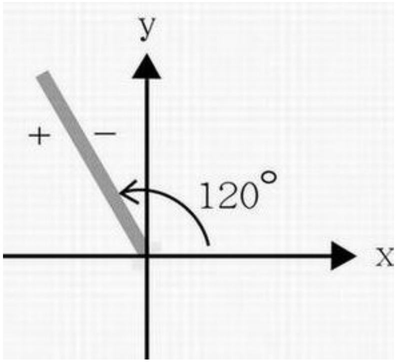

Figure 73: $\alpha = 120°$

In this case, we define that

$$z = \begin{cases} re^{i\theta}, \theta \in [-\frac{4\pi}{3}, \frac{2\pi}{3}) & \text{if z in sheet-I} \\ re^{i\theta}, \theta \in [\frac{2\pi}{3}, \frac{8\pi}{3}) & \text{if z in sheet-II} \end{cases} \tag{40}$$

the cut in each sheet has two edges, label the edge of the left of the cut with "+" and the right of the cut with "−"

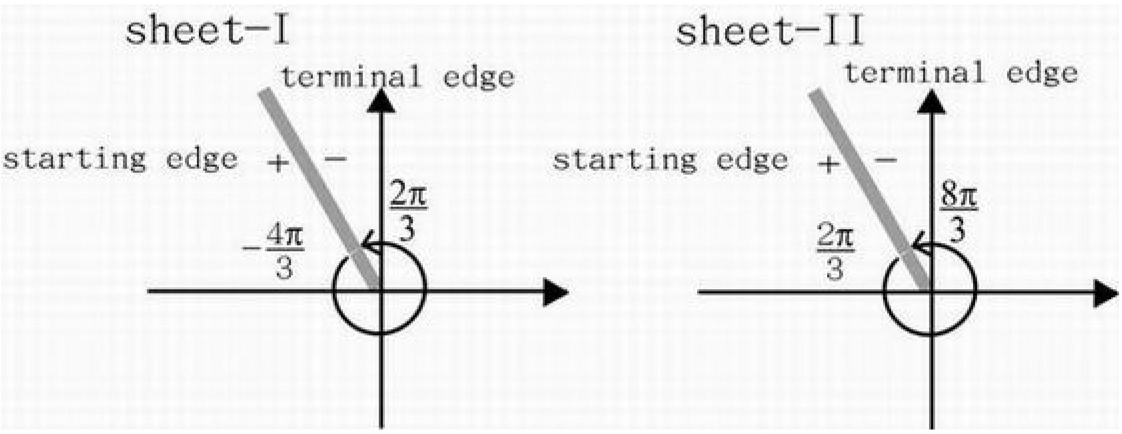

Figure 74: value of θ

$$z = |z|e^{i\theta},\ \theta \in [-\frac{4\pi}{3}, \frac{2\pi}{3}) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [-\frac{2\pi}{3}, \frac{\pi}{3}) \tag{41}$$

$$z = |z|e^{i\theta},\ \theta \in [\frac{2\pi}{3}, \frac{8\pi}{3})\ \text{then}\ \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [\frac{\pi}{3}, \frac{4\pi}{3}) \tag{42}$$

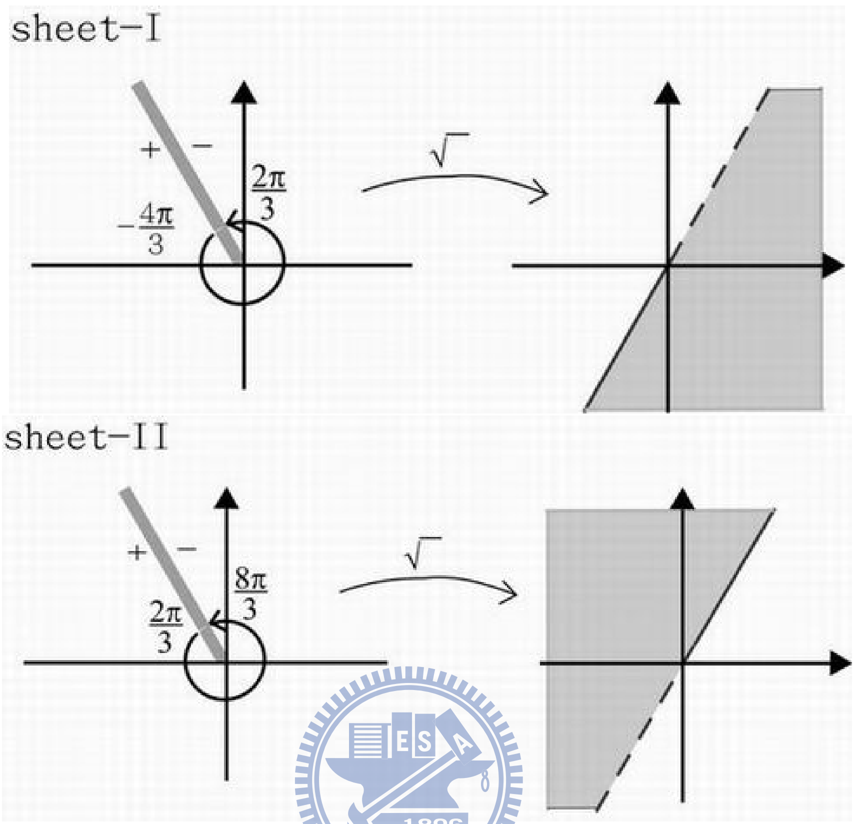

Figure 75: The value in sheet-I and sheet-II

Discuss the difference between the value in theory and in Mathematica. and find out how to modify the computation. Since in Mathematica the value is always single-value, no matter in which sheet.

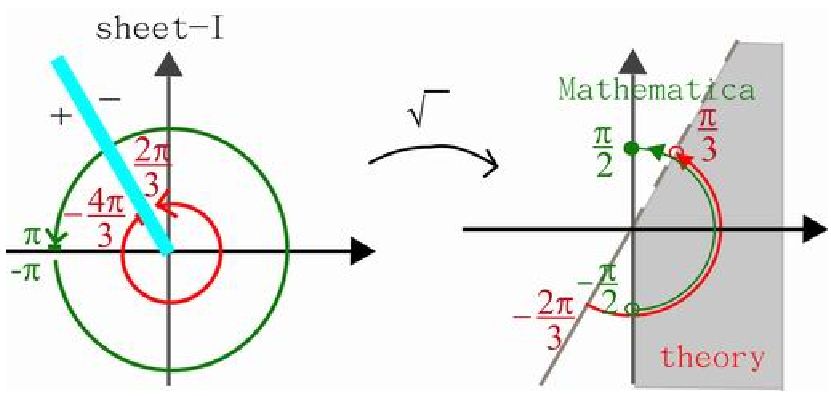

Figure 76: The value in sheet-I and Mathematica

The argument of z in $\frac{\theta}{2} \in [-\frac{2\pi}{3}, \frac{\pi}{3})$ in sheet-I will change when we use Mathematica to compute in mathematica $\frac{\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$

The difference is π then we get that if $\theta \in [-\frac{2\pi}{3}, -\pi]$ we need to modify

$$z = \begin{cases} |z|e^{i\theta},\ \theta \in [-\frac{4\pi}{3}, \frac{2\pi}{3}) & \text{if z in sheet-I} \\ |z|e^{i\theta+2\pi},\ \theta + 2\pi \in [\frac{2\pi}{3}, \frac{8\pi}{3}) & \text{if z in Mathematica} \end{cases} \quad (43)$$

$$\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [-\frac{2\pi}{3}, \frac{\pi}{3}) \qquad \text{if z in sheet-I.} \quad (44)$$

$$\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta+2\pi}{2}} \qquad \text{if z in Mathematica}$$

$$= -|z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \theta \in [-\frac{4\pi}{3}, \frac{2\pi}{3}) \quad (45)$$

That is when $z \in$sheet-I,

$$\text{If}\ \ \arg(z) \in [-\frac{2\pi}{3}, -\pi]\ \text{then}\ \ \sqrt{z} \overset{Math.}{=} -\sqrt{z} \quad (46)$$

After discuss four different cuts, we could sum up a conclusion.

Definition 4. *Consider any cut on the line and the slope of line is $m = \tan\alpha$, $0 < \alpha \leq \pi$. Define that*

$$z = \begin{cases} re^{i\theta}, \theta \in [\alpha - 2\pi, \alpha) & \textit{if z in sheet-I} \\ re^{i\theta}, \theta \in [\alpha, \alpha + 2\pi) & \textit{if z in sheet-II} \end{cases}$$

the cut in each sheet has two edges, label the starting edge with "+" and the terminal edge with "−"

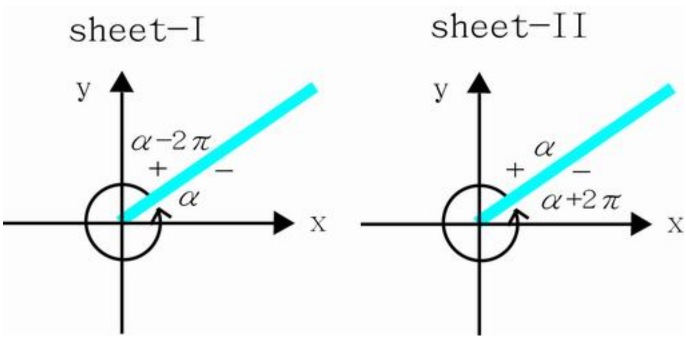

Figure 77: In two sheets

$$z = |z|e^{i\theta},\ \theta \in [\alpha - 2\pi, \alpha)\ \text{then}\ \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [\frac{\alpha - 2\pi}{2}, \frac{\alpha}{2}) \quad (47)$$

$$z = |z|e^{i\theta},\ \theta \in [\alpha, \alpha + 2\pi)\ \text{then}\ \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}},\ \frac{\theta}{2} \in [\frac{\alpha}{2}, \frac{\alpha + 2\pi}{2}) \quad (48)$$

Since in Mathematica the value is always single-value, no matter in which sheet. Compare the range of θ.

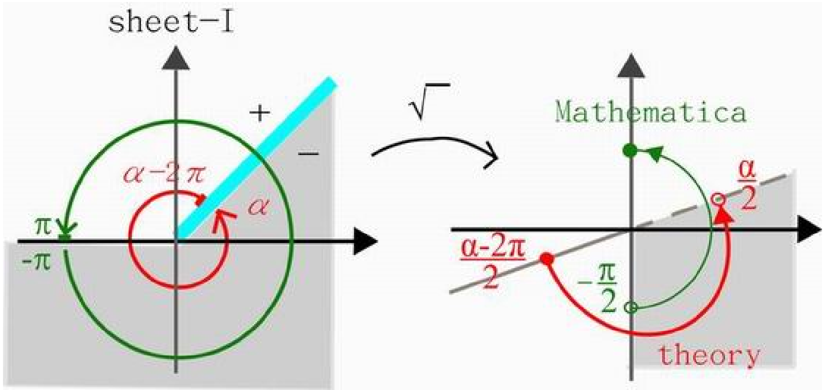

Figure 78: Example of $f(z) = \sqrt{z}$

Theorem 5. *If the cut with z_k on the line where the slope of line is $m = \tan\alpha$, $0 < \alpha \le \pi$. If z in sheet-I then*

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z - z_k} & \textit{if } \arg(z - z_k) \in [\alpha - 2\pi, -\pi], \\ \sqrt{z - z_k} & \textit{if } \arg(z - z_k) \in (-\pi, \alpha) \end{cases}$$

Proof.

Let z in sheet-I and using polar form $z - z_k = re^{i\theta}$ where $\arg(z - z_k) = \theta$. When $\theta \in (-\pi, \alpha)$, the argument in theory and in Mathematica is the same. When $\theta \in [\alpha - 2\pi, -\pi]$, Mathematica will conversion θ into $\theta + 2\pi$ where $\theta + 2\pi \in [\alpha, \pi]$ and $re^{\theta i} = re^{\theta i + 2\pi i}$ but

$$\text{In theory:} \qquad \sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta}{2}i}$$

$$\text{In Mathematica:} \qquad \sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta + 2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}}$$

So

$$\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k} \text{ if } \theta \in [\alpha - 2\pi, -\pi]$$

∎

Example: Compute $\int \frac{1}{f(z)} dz$ over a,b cycles where $f(z) = \sqrt{(z - 3 + i)(z - 4 - \sqrt{3}i + i)(z - 2 - i)(z - 2 - \sqrt{3} - 2i)z(z - 2 - 2i)(z + i)}$. Let $z_1 = 3 - i, z_2 = 4 + (\sqrt{3} - 1)i, z_3 = 2 + i, z_4 = (2 + \sqrt{3} + 2i), z_5 = 0, z_6 = 2 + 2i, z_7 = -i$

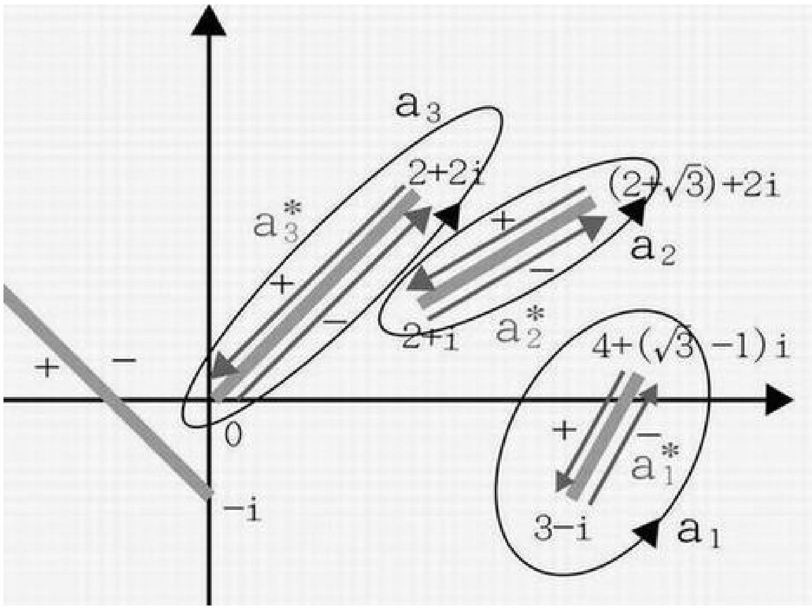

Figure 79: a cycles and their equivalent path

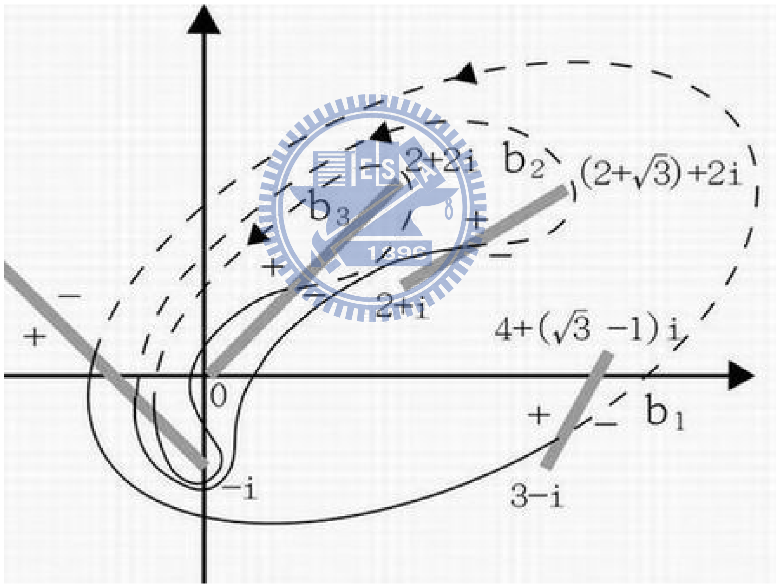

Figure 80: b cycles

1. The equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* =the path along slant cut from $4 + (\sqrt{3}-1)i$ to $3-i$ on (+)edge of sheet-I and a_{12}^* =the path along slant cut from $3-i$ to $4+(\sqrt{3}-1)i$ on (−)edge of sheet-I

(1) a_{11}^*: Let $z = 3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$, $r : 2 \to 0$ and $dz = \frac{1}{2} + \frac{\sqrt{3}}{2}idr$

$$\arg(z - z_1) = -\frac{5\pi}{3} \text{ then } \sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$$
$$\arg(z - z_2) = -\frac{2\pi}{3} \text{ then } \sqrt{z - z_2} \overset{Math.}{=} \sqrt{z - z_2}$$
$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k},\ k = 3, 4, 5, 6, 7$$
$$f(z) \overset{Math.}{=} -f(z)$$
$$\int_{a_{11}^*} f(z)dz \overset{Math.}{=} -\int_2^0 \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{f(3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i))}dr$$

(2) a_{12}^*: Let $z = 3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$, $r : 0 \to 2$ and $dz = \frac{1}{2} + \frac{\sqrt{3}}{2}idr$

$$\arg(z - z_1) = \frac{\pi}{3} \text{ then } \sqrt{z - z_1} \overset{Math.}{=} \sqrt{z - z_1}$$
$$\arg(z - z_2) = -\frac{2\pi}{3} \text{ then } \sqrt{z - z_2} \overset{Math.}{=} \sqrt{z - z_2}$$
$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k},\ k = 3, 4, 5, 6, 7$$
$$f(z) \overset{Math.}{=} f(z)$$
$$\int_{a_{12}^*} f(z)dz \overset{Math.}{=} \int_0^2 \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{f(3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i))}dr$$

By (1), (2) and Cauchy Theorem

$$\int_{a_1} \frac{1}{f(z)}dz = \int_{a_1^*} \frac{1}{f(z)}dz \overset{Math.}{=} 2\int_0^2 \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{f(3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i))}dr$$
$$= -0.226932 + 0.0125601i$$

2. The equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^*$ where a_{21}^* =the path along slant cut from $2 + \sqrt{3} + 2i$ to $2 + i$ on (+)edge of sheet-I and a_{22}^* =the path along slant cut from $2 + i$ to $2 + \sqrt{3} + 2i$ on (−)edge of sheet-I

(1) a_{21}^* : Let $z = 2 + i + r(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$, $r : 2 \to 0$, so $dz = \frac{\sqrt{3}}{2} + \frac{1}{2}idr$

$$\arg(z - z_3) = -\frac{11\pi}{6} \text{ then } \sqrt{z - z_3} \overset{Math.}{=} -\sqrt{z - z_3}$$
$$\arg(z - z_4) = -\frac{5\pi}{6} \text{ then } \sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_4}$$
$$\arg(z - z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k},\ k = 1, 2$$
$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k},\ k = 5, 6, 7$$
$$f(z) \overset{Math.}{=} (-1)^3 f(z) = -f(z)$$

$$\int_{a_{21}^*} f(z)dz \stackrel{Math.}{=} -\int_2^0 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr$$

(2) a_{22}^* : Let $z = 2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i),\ r: 0 \to 2$ then $dz = \frac{\sqrt{3}}{2}+\frac{1}{2}idr$

$$\arg(z-z_3) = \frac{\pi}{6} \text{ then } \sqrt{z-z_3} \stackrel{Math.}{=} \sqrt{z-z_3}$$

$$\arg(z-z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_k} \stackrel{Math.}{=} -\sqrt{z-z_k},\ k = 1,2$$

$$\arg(z-z_k) \in (-\pi,\pi) \text{ then } \sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k},\ k = 4,5,6,7$$

$$f(z) \stackrel{Math.}{=} (-1)^2 f(z) = f(z)$$

$$\int_{a_{22}^*} f(z)dz \stackrel{Math.}{=} \int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_2} \frac{1}{f(z)}dz &= \int_{a_2^*} \frac{1}{f(z)}dz \\ &\stackrel{Math.}{=} 2\int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr \\ &= -0.0584232+0.842766i\end{aligned}$$

3. The equivalent path $a_1^* = a_{31}^* \bigcup a_{32}^* \bigcup a_{33}^* \bigcup a_{34}^* \bigcup a_{35}^* \bigcup a_{36}^*$ where a_{31}^* = the path along slant cut from $2+2i$ to $1+i$ on (+)edge of sheet-I, a_{32}^* = the path along slant cut from $1+i$ to $\sqrt{3}-1(\sqrt{3-1}i)$ on (+)edge of sheet-I, a_{33}^* = the path along slant cut from $\sqrt{3}-1(\sqrt{3-1}i)$ to 0 on (+)edge of sheet-I, a_{34}^* = the path along slant cut from 0 to $\sqrt{3}-1(\sqrt{3-1}i$ on (−)edge of sheet-I, a_{35}^* = the path along slant cut from $\sqrt{3}-1(\sqrt{3-1}i$ to $1+i$ on (−)edge of sheet-I and a_{36}^* = the path along slant cut from $1+i$ to $2+2i$ on (−)edge of sheet-I.

(1) a_{31}^* : Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)$, $r : 2\sqrt{2} \to \sqrt{2}$, so $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z - z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k},\ k = 1, 2$$

$$\arg(z - z_4) \in [-\frac{11}{6}\pi, -\pi] \text{ then } \sqrt{z - z_4} \overset{Math.}{=} -\sqrt{z - z_4}$$

$$\arg(z - z_5) = -\frac{7}{4}\pi \text{ then } \sqrt{z - z_5} \overset{Math.}{=} -\sqrt{z - z_5}$$

$$\arg(z - z_6) = -\frac{3}{4}\pi \text{ then } \sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_6}$$

$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k},\ k = 3, 7$$

$$f(z) \overset{Math.}{=} (-1)^4 f(z) = f(z)$$

$$\int_{a_{31}^*} f(z)dz \overset{Math.}{=} \int_{2\sqrt{2}}^{\sqrt{2}} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))} dr$$

(2) a_{32}^*: Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)$, $r : \sqrt{2} \to \sqrt{2}(\sqrt{3} - 1)$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z - z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k},\ k = 1, 2$$

$$\arg(z - z_5) = -\frac{7}{4}\pi \text{ then } \sqrt{z - z_5} \overset{Math.}{=} -\sqrt{z - z_5}$$

$$\arg(z - z_6) = -\frac{3}{4}\pi \text{ then } \sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_6}$$

$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 3, 4, 7$$

$$f(z) \overset{Math.}{=} (-1)^3 f(z) = -f(z)$$

$$\int_{a_{32}^*} f(z)dz \overset{Math.}{=} -\int_{\sqrt{2}}^{\sqrt{2}(\sqrt{3}-1)} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))} dr$$

(3) a_{33}^* : Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r : \sqrt{2}(\sqrt{3} - 1) \to 0$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z - z_1) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$$

$$\arg(z - z_5) = -\frac{7}{4}\pi \text{ then } \sqrt{z - z_5} \overset{Math.}{=} -\sqrt{z - z_5}$$

$$\arg(z - z_6) = -\frac{3}{4}\pi \text{ then } \sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_6}$$

$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 2, 3, 4, 7$$

$$f(z) \overset{Math.}{=} (-1)^2 f(z) = f(z)$$

$$\int_{a_{33}^*} f(z)dz \overset{Math.}{=} \int_{\sqrt{2}(\sqrt{3}-1)}^{0} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))} dr$$

(4) a_{34}^*: Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r: 0 \to \sqrt{2}(\sqrt{3}-1)$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_1) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$$

$$\arg(z-z_5) = \frac{1}{4}\pi \text{ then } \sqrt{z-z_5} \overset{Math.}{=} \sqrt{z-z_5}$$

$$\arg(z-z_6) = -\frac{3}{4}\pi \text{ then } \sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_6}$$

$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 2,3,4,7$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{34}^*} f(z)dz \overset{Math.}{=} -\int_0^{\sqrt{2}(\sqrt{3}-1)} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr$$

(5) a_{35}^*: Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)$, $r: \sqrt{2}(\sqrt{3}-1) \to \sqrt{2}$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1,2$$

$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 3,4,5,6,7$$

$$f(z) \overset{Math.}{=} (-1)^2 f(z) = f(z)$$

$$\int_{a_{35}^*} f(z)dz \overset{Math.}{=} \int_{\sqrt{2}(\sqrt{3}-1)}^{\sqrt{2}} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr$$

(6) a_{36}^*: Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r: \sqrt{2} \to 2\sqrt{2}$, so $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1,2$$

$$\arg(z-z_3) \in [-\frac{11}{6}\pi, -\pi] \text{ then } \sqrt{z-z_3} \overset{Math.}{=} -\sqrt{z-z_3}$$

$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 4,5,6,7$$

$$f(z) \overset{Math.}{=} (-1)^3 f(z) = -f(z)$$

$$\int_{a_{36}^*} f(z)dz \overset{Math.}{=} -\int_{\sqrt{2}}^{2\sqrt{2}} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr$$

By (1), (2),...,(6) and Cauchy Theorem

$$\begin{aligned}
\int_{a_3} \frac{1}{f(z)}dz &= \int_{a_3^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} -2\int_0^{\sqrt{2}(\sqrt{3}-1)} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr \\
&\quad +2\int_{\sqrt{2}(\sqrt{3}-1)}^{\sqrt{2}} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr - 2\int_{\sqrt{2}}^{2\sqrt{2}} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr \\
&= 0.291554 - 0.388271i
\end{aligned}$$

4. The equivalent path $b_3^* = b_{31}^* \bigcup b_{32}^* \bigcup b_{33}^* \bigcup b_{34}^* \bigcup b_{35}^* \bigcup b_{36}^*$ where

b_{31}^* =the path along slant cut from $-\frac{1}{2}-\frac{1}{2}i$ to $-i$ on $(+)$edge of sheet-I,

b_{32}^* =the path along slant cut from $-i$ to $-\frac{1}{2}-\frac{1}{2}i$ on $(-)$edge of sheet-I,

b_{33}^* =the path along slant line from $-\frac{1}{2}-\frac{1}{2}i$ to 0 on sheet-I,

b_{34}^* =the path along slant cut from 0 to $2+2i$ on $(-)$edge of sheet-II,

b_{35}^* =the path along slant cut from $2+2i$ to 0 on $(+)$edge of sheet-II and

b_{36}^* =the path along slant line from 0 to $-\frac{1}{2}-\frac{1}{2}i$ on sheet-II

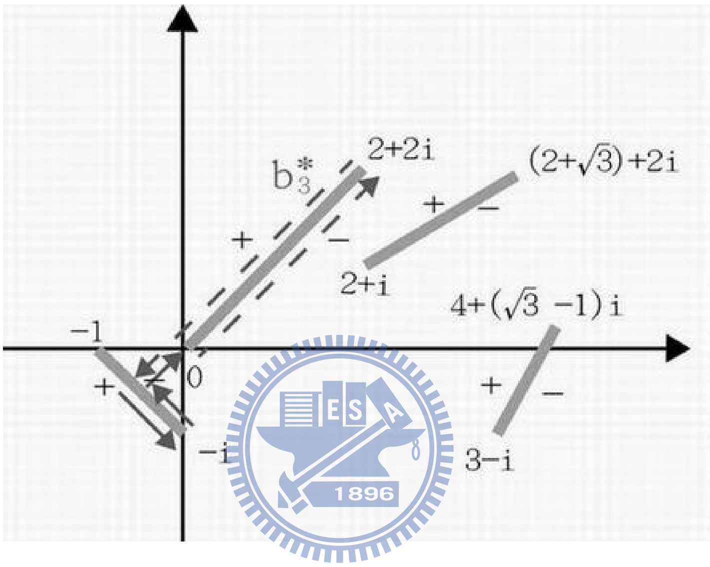

Figure 81: The equivalent path b_3^*

(1) b_{31}^*: Let $z = -i + (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r, r : \frac{1}{\sqrt{2}} \to 0 \Rightarrow dz = (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_1) \in (-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$$
$$\arg(z-z_7) = -\frac{5}{4}\pi \text{ then } \sqrt{z-z_7} \overset{Math.}{=} -\sqrt{z-z_7}$$
$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k},\ k = 2,3,4,5,6$$
$$f(z) \overset{Math.}{=} (-1)^2 f(z) = f(z)$$

$$\int_{b_{31}^*} f(z)dz \overset{Math.}{=} \int_{\frac{1}{\sqrt{2}}}^{0} \frac{-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(-i + (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r)} dr$$

(2) b_{32}^{*} : Let $z=-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r, r: 0 \rightarrow \frac{1}{\sqrt{2}}$, so $dz=(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_1) \in [-\frac{5}{3}\pi,-\pi] \text{ then } \sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$$

$$\arg(z-z_7) = \frac{3}{4}\pi \text{ then } \sqrt{z-z_7} \overset{Math.}{=} \sqrt{z-z_7}$$

$$\arg(z-z_k) \in (-\pi,\pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k},\ k=2,3,4,5,6$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{32}^{*}} f(z)dz \overset{Math.}{=} -\int_0^{\frac{1}{\sqrt{2}}} \frac{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

(3) b_{33}^{*}: Let $z=r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)$, $r: -\frac{1}{\sqrt{2}} \rightarrow 0$ and $dz=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_1) \in [-\frac{5}{3}\pi,-\pi] \text{ then } \sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$$

$$\arg(z-z_k) = -\frac{3}{4}\pi \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k=5,6$$

$$\arg(z-z_k) \in (-\pi,\pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k=2,3,4,7$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{33}^{*}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-\frac{1}{\sqrt{2}}}^{0} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

(4) $b_{34}^{*} \equiv$ the path along slant cut from 0 to $2+2i$ on (+)edge of sheet-I and $b_{35}^{*} \equiv$ the path along slant cut from $2+2i$ to 0 on (−)edge of sheet-I

$$\int_{b_{34}^{*} \bigcup b_{35}^{*}} \frac{1}{f(z)}dz = -\int_{a_3^{*}} \frac{1}{f(z)}dz$$

(5) b_{36}^{*} : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider $b_{36}^{**} =$ the path along slant line from 0 to $-\frac{1}{2}-\frac{1}{2}i$ on sheet-I

Let $z=r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i), r: 0 \rightarrow -\frac{1}{\sqrt{2}}$, so $dz=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\arg(z-z_1) \in [-\frac{5}{3}\pi,-\pi] \text{ then } \sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$$

$$\arg(z-z_k) = -\frac{3}{4}\pi \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k=5,6$$

$$\arg(z-z_k) \in (-\pi,\pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k=2,3,4,7$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{36}^{*}} \frac{1}{f(z)}dz = -\int_{b_{36}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^{-\frac{1}{\sqrt{2}}} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

By (1), (2),...,(5) and Cauchy Theorem

$$\begin{aligned}\int_{b_3}\frac{1}{f(z)}dz &= \int_{b_3^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} 2\int_{\frac{1}{\sqrt{2}}}^{0}\frac{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr\\ &\quad+\int_0^{-\frac{1}{\sqrt{2}}}\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr-\int_{a_3^*}\frac{1}{f(z)}dz\\ &= 0.219078-0.138721i\end{aligned}$$

5. b_2: Consider equivalent path $b_2^* = b_{31}^*\bigcup b_{32}^*\bigcup b_{34}^*\bigcup a_{32}^*\bigcup b_{21}^*\bigcup b_{22}^*\bigcup b_{23}^*\bigcup b_{24}^*\bigcup b_{35}^*\bigcup b_{36}^*$ where b_{21}^* =the path along vertical line from $2+2i$ to $2+i$ on sheet-I, b_{22}^* =the path along slant cut from $2+i$ to $2+\sqrt{3}+2i$ on $(-)$edge of sheet-II, b_{23}^* =the path along slant cut from $2+\sqrt{3}+2i$ to $2+i$ on $(+)$edge of sheet-II and b_{24}^* =the path along vertical line from $2+i$ to $2+2i$ on sheet-II.

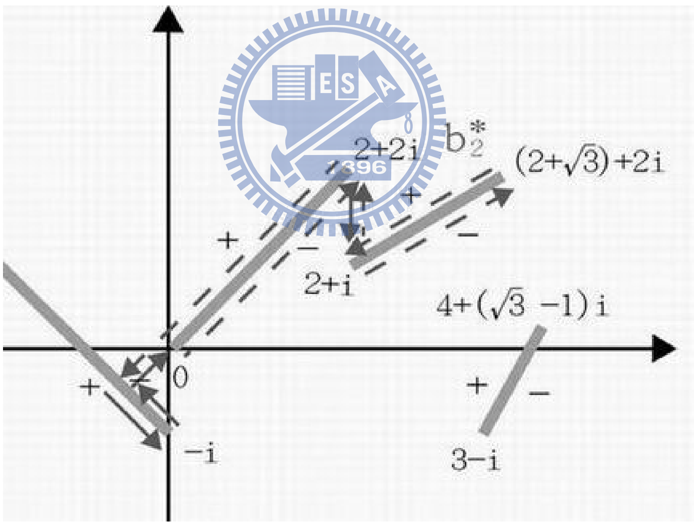

Figure 82: The equivalent path b_2^*

(1) b_{21}^*: Let $z=2+ri, r: 2\to 1$, so $dz = idr$

$$\arg(z-z_k)\in[-\frac{5}{3}\pi,-\pi] \text{ then } \sqrt{z-z_k}\overset{Math.}{=}-\sqrt{z-z_k}, k=1,2$$

$$\arg(z-z_k)\in[-\frac{11}{6}\pi,-\pi] \text{ then } \sqrt{z-z_k}\overset{Math.}{=}-\sqrt{z-z_k}, k=3,4$$

$$\arg(z-z_k)\in(-\pi,\pi) \text{ then } \sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}, k=5,6,7$$

$$f(z)\overset{Math.}{=}(-1)^4f(z)=f(z)$$

$$\int_{b_{21}^*} f(z)dz \overset{Math.}{=} -\int_2^1 \frac{i}{f(2+ri)}dr$$

(2) $b_{22}^* \equiv$ the path along slant cut from $2+i$ to $2+\sqrt{3}+2i$ on $(+)$edge of sheet-I and $b_{23}^* \equiv$ the path along slant cut from $2+\sqrt{3}+2i$ to $2+i$ on $(-)$edge of sheet-I

$$\int_{b_{22}^* \bigcup b_{23}^*} \frac{1}{f(z)}dz = -\int_{a_2^*} \frac{1}{f(z)}dz$$

(3) b_{24}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{24}^{**} = the path along vertical line from $2+i$ to $2+2i$ on sheet-I

Let $z = 2+ri, r: 1 \to 2$, so $dz = idr$

$$\arg(z-z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1, 2$$
$$\arg(z-z_k) \in [-\frac{11}{6}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 3, 4$$
$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 5, 6, 7$$
$$f(z) \overset{Math.}{=} (-1)^4 f(z) = f(z)$$

$$\int_{b_{24}^*} \frac{1}{f(z)}dz = -\int_{b_{24}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^2 \frac{i}{2+ri}dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{b_3^*} \frac{1}{f(z)}dz + 2\int_1^2 \frac{i}{2+ri}dr \\
&\quad +2\int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr \\
&= -0.383089 + 1.06343i
\end{aligned}$$

6. b_1: consider equivalent path $b_1^* = b_{31}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^* \bigcup b_{15}^* \bigcup b_{35}^* \bigcup b_{36}^*$ where b_{11}^* = the path along horizontal line from $-i$ to $3-i$ on sheet-I, b_{12}^* = the path along slant cut from $3-i$ to $4+(\sqrt{3}-1)i$ on $(+)$edge of sheet-I, b_{13}^* = the path along slant line from $4+(\sqrt{3}-1)i$ to $3+\sqrt{3}+2i$ on sheet-II, b_{14}^* = the path along horizontal line from $3+\sqrt{3}+2i$ to $2\sqrt{3}+2i$ on sheet-II and b_{15}^* = the path along horizontal line from $2\sqrt{3}+2i$ to $2+2i$ on sheet-II.

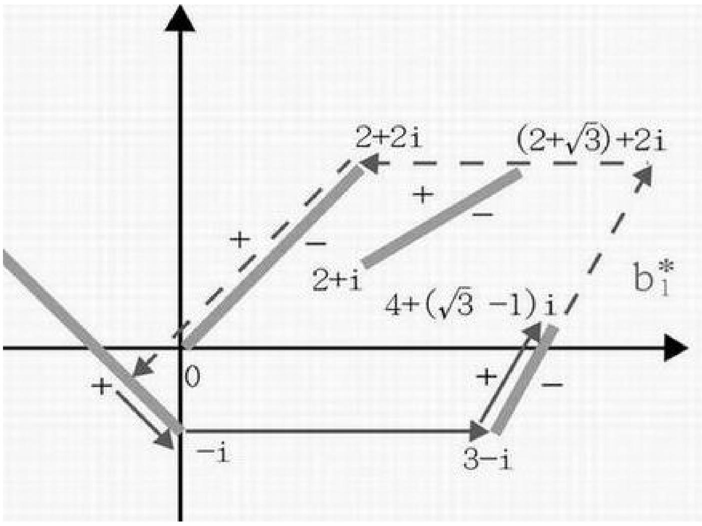

Figure 83: The equivalent path b_1^*

(1) b_{11}^* : Let $z = r - i, r : 0 \to 3$, so $dz = dr$

$$\arg(z - z_1) = -\pi \text{ then } \sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$$

$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 2, 3, 4, 5, 6, 7$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{11}^*} f(z)dz \overset{Math.}{=} -\int_0^3 \frac{1}{f(r-i)}dr$$

(2) $b_{12}^* \equiv -a_{11}^*$

$$\int_{b_{12}^*} \frac{1}{f(z)}dz = -\int_{a_{11}^*} \frac{1}{f(z)}dz$$

(3) b_{13}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{13}^{**} = the path along horizontal line from $-i$ to $3 - i$ on sheet-I Let $z = 3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i), r : 2 \to 2\sqrt{3}$ and $dz = \frac{1}{2} + \frac{\sqrt{3}}{2}idr$

$$\arg(z - z_k) = -\frac{5}{3}\pi \text{ then } \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2$$

$$\arg(z - z_k) \in (-\pi, \pi) \text{ then } \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 3, 4, 5, 6, 7$$

$$f(z) \overset{Math.}{=} (-1)^2 f(z) = f(z)$$

$$\int_{b_{13}^*} f(z)dz = -\int_{b_{13}^{**}} f(z)dz \overset{Math.}{=} -\int_2^{2\sqrt{3}} \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{f(3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i))}dr$$

(4) b_{14}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{14}^{**} = the path along horizontal line from $3+\sqrt{3}+2i$ to $2+\sqrt{3}+2i$ on sheet-I

Let $z = r+2i, r : 3+\sqrt{3} \to 2+\sqrt{3}$ and $dz = dr$

$$\arg(z-z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1,2$$

$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 3,4,5,6,7$$

$$f(z) \overset{Math.}{=} (-1)^2 f(z) = f(z)$$

$$\int_{b_{14}^*} f(z)dz = -\int_{b_{14}^{**}} f(z)dz \overset{Math.}{=} -\int_{3+\sqrt{3}}^{2+\sqrt{3}} \frac{1}{f(r+2i)}dr$$

(5) b_{15}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{15}^{**} = the path along horizontal line from $2\sqrt{3}+2i$ to $2+2i$ on sheet-I

Let $z = r+2i, r : 2+\sqrt{3} \to 2$ and $dz = dr$

$$\arg(z-z_k) \in [-\frac{5}{3}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 1,2$$

$$\arg(z-z_k) \in [-\frac{11}{6}\pi, -\pi] \text{ then } \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, k = 3,4$$

$$\arg(z-z_k) \in (-\pi, \pi) \text{ then } \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 5,6,7$$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{15}^*} f(z)dz = -\int_{b_{15}^{**}} f(z)dz \overset{Math.}{=} -\int_{2+\sqrt{3}}^{2} \frac{1}{f(r+2i)}dr$$

By (1),..., (5) and Cauchy Theorem

$$\begin{aligned}\int_{b_1} \frac{1}{f(z)}dz &= \int_{b_1^*} \frac{1}{f(z)}dz \overset{Math.}{=} \frac{1}{2}\int_{b_3^*} \frac{1}{f(z)}dz - \int_0^3 \frac{1}{f(r-i)}dr \\ &\quad - \int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i))}dr - \int_2^{2\sqrt{3}} \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i))}dr \\ &\quad - \int_{3+\sqrt{3}}^{2+\sqrt{3}} \frac{1}{f(r+2i)}dr - \int_{2+\sqrt{3}}^{2} \frac{1}{f(r+2i)}dr \\ &= -0.0281258 - 0.494596i\end{aligned}$$

Similarly we divided domain C into many blocks to discuss way to modify on slant cuts. It can help us reduce the steps of modifying $f(z)$.

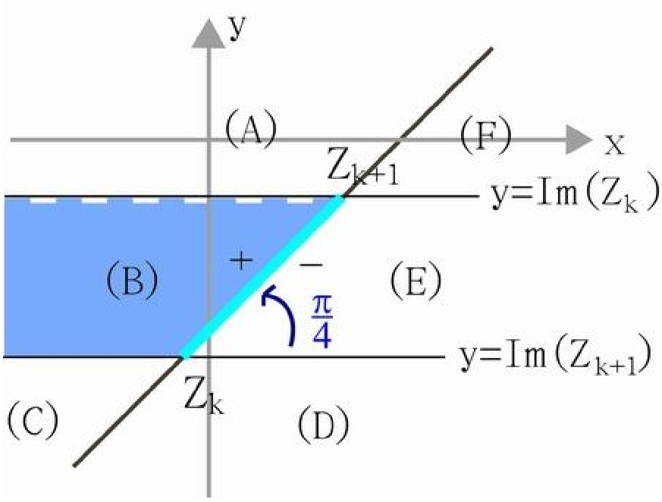

Figure 84: Divided domain C into 6 blocks

Example 1. $\alpha = 45° = \frac{\pi}{4}$ and $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}}$

$$(A) = \{(x, y) : x - y < 0 \text{ and } y \geq Im(z_{k+1})\}$$

$$(B) = \{(x, y) : x - y < 0 \text{ and } Im(z_k) \leq y < Im(z_{k+1})\}$$

$$(C) = \{(x, y) : x - y < 0 \text{ and } y < Im(z_k)\}$$

$$(G) = (D)\bigcup(E)\bigcup(F) = \{(x, y) : x - y > 0\}$$

(1) $z \in (A)$: $\arg(z - z_k), \arg(z - z_{k+1}) \in [-\frac{7\pi}{4}, -\pi]$

$\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

(2) $z \in (B)$: $\arg(z - z_k) \in [-\frac{7\pi}{4}, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$

$\arg(z - z_{k+1}) \in (-\pi, \pi)$ then $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$

$f(z) \overset{Math.}{=} -f(z)$

(3) $z \in (C)\bigcup(G)$: $\arg(z - z_k), \arg(z - z_{k+1}) \in (-\pi, \pi)$

$\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

(4) $z \in$ {the cut with (+)edge of sheet-I}:

$\arg(z - z_k) = -\frac{7\pi}{4}$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$

$\arg(z-z_{k+1}) = -\frac{3\pi}{4}$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} -f(z)$

(5) $z \in$ {the cut with $(-)$edge of sheet-I}:

$\arg(z-z_k) = \frac{\pi}{4}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$

$\arg(z-z_{k+1}) = -\frac{3\pi}{4}$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

So

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (B)\bigcup\{\text{the cut with } (+)\text{edge of sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases} \tag{49}$$

Example 2. $\alpha = 120° = \frac{2\pi}{3}$ and $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}}$

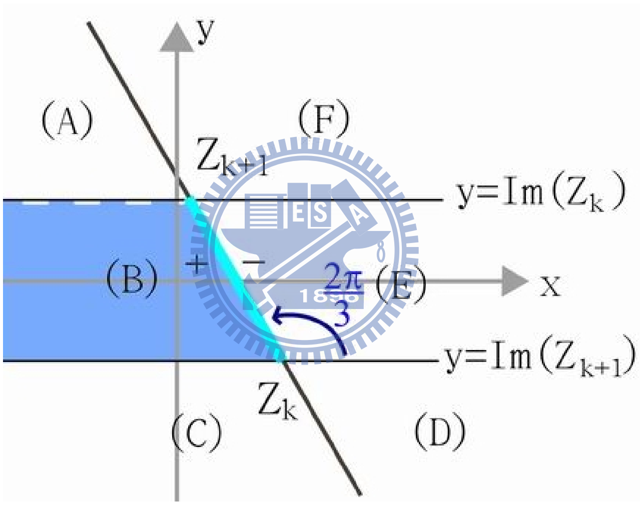

Figure 85: Divided domain C into 6 blocks

$$(A) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } y \geq Im(z_{k+1})\}$$

$$(B) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } Im(z_k) \leq y < Im(z_{k+1})\}$$

$$(C) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } y < Im(z_k)\}$$

$$(G) = (D)\bigcup(E)\bigcup(F) = \{(x,y) : \sqrt{3}x + y > 0\}$$

(1) $z \in (A)$: $\arg(z-z_k), \arg(z-z_{k+1}) \in [-\frac{4\pi}{3}, -\pi]$

$\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

(2) $z \in (B)$: $\arg(z-z_k) \in [-\frac{4\pi}{3}, -\pi]$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$

$\arg(z-z_{k+1}) \in (-\pi, \pi)$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} -f(z)$

(3) $z \in (C) \bigcup (G)$: $\arg(z-z_k), \arg(z-z_{k+1}) \in (-\pi, \pi)$

$\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

(4) $z \in$ {the cut with (+)edge of sheet-I}:

$\arg(z-z_k) = -\frac{4\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$

$\arg(z-z_{k+1}) = -\frac{\pi}{3}$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} -f(z)$

(5) $z \in$ {the cut with (−)edge of sheet-I}:

$\arg(z-z_k) = \frac{2\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$

$\arg(z-z_{k+1}) = -\frac{\pi}{3}$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

So

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (B) \bigcup \{\text{the cut with } (+)\text{edge of sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases} \tag{50}$$

We sum up the conclusion.

Definition 6. *Any slant cut whose slope of line is $m = \tan\alpha$, $0 < \alpha \leq \pi$ and the end points of cut are $z_k = x_k + iy_k$ and $z_{k+1} = x_{k+1} + iy_{k+1}$. We define the condition C is that*

$$(x,y) \in C \equiv \textit{ if } \tan\alpha > 0 \textit{ then } y - y_k > \tan\alpha(x-x_k) \textit{ and} \tag{51}$$
$$\textit{if } \tan\alpha < 0 \textit{ then } y - y_k < \tan\alpha(x-x_k)$$

Theorem 7. *If* $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}}$ *and the cut of slope is* $m = \tan\alpha$. *We divided domain* C *into* 6 *areas*

$$(A) = \{(x,y) : (x,y) \in C \textit{ and } y \geq y_{k+1}\}$$

$$(B) = \{(x,y) : (x,y) \in C \textit{ and } y_k \leq y < y_{k+1}\}$$

$$(C) = \{(x,y) : (x,y) \in C \textit{ and } y < y_k\}$$

$$(G) = (D)\bigcup(E)\bigcup(F) = \{(x,y) : (x,y) \in R \setminus C\}\bigcup\{(x,y) : y - y_k = \tan\alpha(x - x_k)\}$$

then we have

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \textit{if } z \in (B)\bigcup\{\textit{the cut with } (+)\textit{edge of sheet-I}\}, \\ f(z) & \textit{otherwise.} \end{cases} \tag{52}$$

Proof.

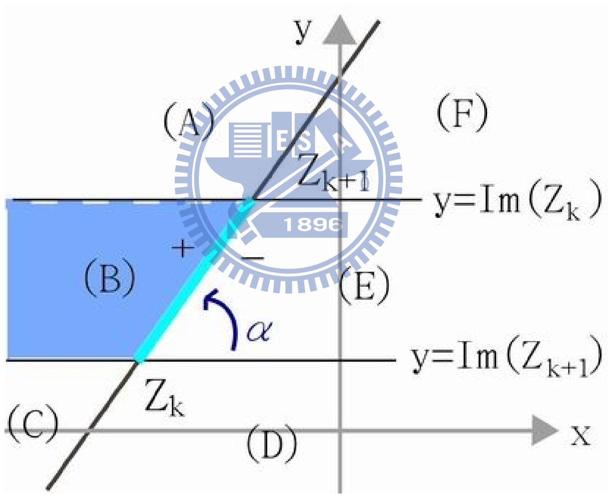

Figure 86: Divided domain C into 6 blocks

(1) $z \in (A)$: $\arg(z - z_k), \arg(z - z_{k+1}) \in [\alpha - 2\pi, -\pi]$

$\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

(2) $z \in (B)$: $\arg(z - z_k) \in [\alpha - 2\pi, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$

$\arg(z - z_{k+1}) \in (-\pi, \pi)$ then $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$

$f(z) \overset{Math.}{=} -f(z)$

(3) $z \in (C) \bigcup (G)$: $\arg(z-z_k), \arg(z-z_{k+1}) \in (-\pi, \pi)$

$\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

(4) $z \in$ {the cut with (+)edge of sheet-I}:

$\arg(z-z_k) = \alpha - 2\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$

$\arg(z-z_{k+1}) = \alpha - \pi$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} -f(z)$

(5) $z \in$ {the cut with (−)edge of sheet-I}:

$\arg(z-z_k) = \alpha$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$

$\arg(z-z_{k+1}) = \alpha - \pi$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$

$f(z) \overset{Math.}{=} f(z)$

∎

Example: $\alpha = 120^\circ = \frac{2\pi}{3}$ and $f(z) = \sqrt{\prod_{k=1}^{5}(z-z_k)}$, $z_k = x_k + iy_k$ where $y_k = \tan\frac{2\pi}{3}x_k = -\sqrt{3}x_k$

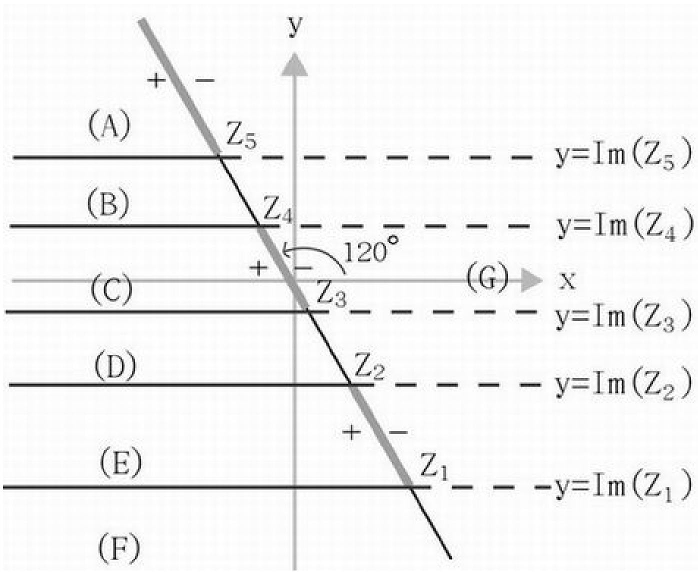

Figure 87: regions of domain

$$(A) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } y > Im(z_5)\}$$

$$(B) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } Im(z_5) > y > Im(z_4)\}$$

$$(C) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } Im(z_4) > y > Im(z_3)\}$$

$$(D) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } Im(z_3) > y > Im(z_2)\}$$

$$(E) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } Im(z_2) > y > Im(z_1)\}$$

$$(F) = \{(x,y) : \sqrt{3}x + y < 0 \text{ and } Im(z_1) > y\}$$

$$(G) = \{(x,y) : \sqrt{3}x + y > 0\}$$

Solution:

(1) $z \in (A)$:

$\arg(z - z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 5$

$$f(z) \overset{Math.}{=} (-1)^5 f(z) = -f(z)$$

(2) $z \in (B)$:

$\arg(z - z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, \cdots, 4$

$\arg(z - z_5) \in (-\pi, \pi)$ then $\sqrt{z - z_5} \overset{Math.}{=} \sqrt{z - z_5}$

$$f(z) \overset{Math.}{=} (-1)^4 f(z) \overset{Math.}{=} f(z)$$

(3) $z \in (C)$:

$\arg(z - z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, \cdots, 3$

$\arg(z - z_4) \in (-\pi, \pi)$ then $\sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_4}$

$\arg(z - z_5) \in (-\pi, \pi)$ then $\sqrt{z - z_5} \overset{Math.}{=} \sqrt{z - z_5}$

$$f(z) \overset{Math.}{=} (-1)^3 f(z) = -f(z)$$

(4) $z \in (D)$:

$\arg(z - z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $k = 1, 2$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $k = 3, 4, 5$

$$f(z) \overset{Math.}{=} f(z)$$

(5) $z \in (E)$:

$\arg(z-z_1) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2,3,4,5$

$$f(z) \overset{Math.}{=} (-1)f(z)$$

(6) $z \in (F)\bigcup(G)$:

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 1,2,3,4,5$

$$f(z) \overset{Math.}{=} f(z)$$

(7) $z \in \{y = Im(z_5)$ and $\sqrt{3}x - y < 0\}$:

$\arg(z-z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2,3,4$

$\arg(z-z_5) = -\pi$ then $\sqrt{z-z_5} \overset{Math.}{=} -\sqrt{z-z_5}$

$$f(z) \overset{Math.}{=} (-1)^5 f(z)$$

(8) $z \in \{y = Im(z_4)$ and $\sqrt{3}x - y < 0\}$:

$\arg(z-z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1,2,3$

$\arg(z-z_4) = -\pi$ then $\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_4}$

$\arg(z-z_5) \in (-\pi, \pi)$ then $\sqrt{z-z_5} \overset{Math.}{=} \sqrt{z-z_5}$

$$f(z) \overset{Math.}{=} (-1)^4 f(z) \overset{Math.}{=} f(z)$$

(9) $z \in \{y = Im(z_3)$ and $\sqrt{3}x - y < 0\}$:

$\arg(z-z_k) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$,$k = 1,2$

$\arg(z-z_3) = -\pi$ then $\sqrt{z-z_3} \overset{Math.}{=} -\sqrt{z-z_3}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4,5$

$$f(z) \overset{Math.}{=} (-1)^3 f(z)$$

(10) $z \in \{y = Im(z_2)$ and $\sqrt{3}x - y < 0\}$:

$\arg(z-z_1) \in (-\frac{4\pi}{3}, -\pi)$ then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$

$\arg(z-z_2) = -\pi$ then $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 3, 4, 5$

$$f(z) \overset{Math.}{=} (-1)^2 f(z) \overset{Math.}{=} f(z)$$

(11) $z \in \{y = Im(z_1)$ and $\sqrt{3}x - y < 0\}$:

$\arg(z-z_1) = -\pi$ then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2, 3, 4, 5$

$$f(z) \overset{Math.}{=} -f(z)$$

(12) $z \in \sqrt{3}x - y = 0$:

(a) $z \in$\{the slant cut from z_5 to $-\infty$ with (+)edge of sheet-I\}:

$\arg(z-z_k) = -\frac{4}{3}\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1, 2, 3, 4, 5$

$$f(z) \overset{Math.}{=} (-1)^5 f(z) \overset{Math.}{=} -f(z)$$

(b) $z \in$\{the cut from z_5 to $-\infty$ with (−)edge of sheet-I\}:

$\arg(z-z_k) = \frac{2}{3}\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 1, 2, 3, 4, 5$

$$f(z) \overset{Math.}{=} f(z)$$

(c) $z \in \overline{z_4 z_5}$:

$\arg(z-z_k) = -\frac{4}{3}\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1, 2, 3, 4$

$\arg(z-z_5) = -\frac{\pi}{3}$ then $\sqrt{z-z_5} \overset{Math.}{=} \sqrt{z-z_5}$

$$f(z) \overset{Math.}{=} (-1)^4 f(z) \overset{Math.}{=} f(z)$$

(d) $z \in$\{the cut from z_3 to z_4 with (+)edge of sheet-I\}:

$\arg(z-z_k) = -\frac{4}{3}\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$, $k = 1, 2, 3$

$\arg(z-z_k) = \arg(z-z_k) = -\frac{\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4, 5$

$$f(z) \overset{Math.}{=} (-1)^3 f(z)$$

(e) $z \in$ {the cut from z_3 to z_4 with $(-)$edge of sheet-I}:

$\arg(z-z_k) = \frac{2}{3}\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 1,2,3$

$\arg(z-z_k) = \arg(z-z_k) = -\frac{\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 4,5$

$$f(z) \overset{Math.}{=} f(z)$$

(f) $z \in \overline{z_4 z_5}$: $\arg(z-z_1) = \arg(z-z_2) = -\frac{4}{3}\pi$

then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$ and $\sqrt{z-z_2} \overset{Math.}{=} -\sqrt{z-z_2}$

$\arg(z-z_k) = -\frac{\pi}{3}$ then $\sqrt{z-z_5} \overset{Math.}{=} \sqrt{z-z_5}$, $k = 3,4,5$

$$f(z) \overset{Math.}{=} (-1)^2 f(z) \overset{Math.}{=} f(z)$$

(g) $z \in$ {the cut from z_1 to z_2 with $(+)$edge of sheet-I}:

$\arg(z-z_1) = -\frac{4}{3}\pi$ then $\sqrt{z-z_1} \overset{Math.}{=} -\sqrt{z-z_1}$

$\arg(z-z_k) = -\frac{\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2,3,4,5$

$$f(z) \overset{Math.}{=} (-1) f(z)$$

(h) $z \in$ {the cut from z_3 to z_4 with $(-)$edge of sheet-I}:

$\arg(z-z_1) = \frac{2}{3}\pi$ then $\sqrt{z-z_1} \overset{Math.}{=} \sqrt{z-z_1}$

$\arg(z-z_k) = -\frac{\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 2,3,4,5$

$$f(z) \overset{Math.}{=} f(z)$$

(i) $z \in \bar{[}z_4, \infty]$: $\arg(z-z_k) = -\frac{\pi}{3}$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, $k = 1,2,3,4,5$

$$f(z) \overset{Math.}{=} f(z)$$

Conclusion:

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } \{(x,y) : \sqrt{3}x - y < 0 \text{ and } y \geq Im(z_5)\} \\ & \bigcup\{(x,y) : \sqrt{3}x - y < 0 \text{ and } Im(z_4) > y \geq Im(z_3)\} \\ & \bigcup\{(x,y) : \sqrt{3}x - y < 0 \text{ and } Im(z_2) > y \geq Im(z_1)\} \\ & \bigcup\{\text{the cuts of any } (+)\text{edge in sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases} \tag{42}$$

In this special case (only consider in sheet-I): Any slant cut which has slope $m = \tan\alpha$, $0 < \alpha \leq \pi$ and $f(z) = \sqrt{\prod_{k=1}^{m}(z - z_k)}$, $z_k = x_k + iy_k$ on the same line.

Case1. $m = 2N - 1$

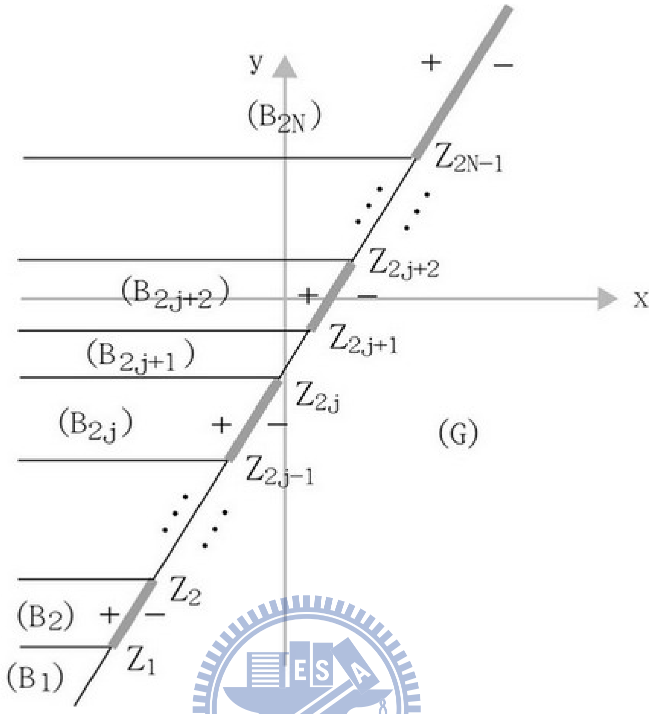

Figure 88: Case1. $m = 2N - 1$ for slant cuts

$$(B_1) = \{(x, y) : (x, y) \in C \text{ and } y < y_1\}$$

$$(B_2) = \{(x, y) : (x, y) \in C \text{ and } y_1 \leq y < y_2\}$$

$$(B_{2j}) = \{(x, y) : (x, y) \in C \text{ and } y_{2j-1} \leq y < y_{2j}\}$$

$$(B_{2j+1}) = \{(x, y) : (x, y) \in C \text{ and } y_{2j} \leq y < y_{2j+1}\}$$

$$(B_{2N-1}) = \{(x, y) : (x, y) \in C \text{ and } y_{2N-2} \leq y < y_{2j-1}\}$$

$$(B_{2N}) = \{(x, y) : (x, y) \in C \text{ and } y_{2N-1} \leq y\}$$

$$(G) = \{(x, y) : (x, y) \in R \setminus C\} \bigcup \{(x, y) : y - y_1 = \tan\alpha(x - x_1)\}$$

(1) $z \in (B_1)$: $\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, m$

$$f(z) \overset{Math.}{=} f(z)$$

(2) $z \in (B_2)$: $\arg(z - z_1) \in (\alpha - 2\pi, -\pi]$ then $\sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2, \cdots, m$

$$f(z) \overset{Math.}{=} (-1)f(z)$$

(3) $z \in (B_{2j})$:

$\arg(z - z_k) \in (\alpha - 2\pi, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j - 1$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j, \cdots, m$

$$f(z) \overset{Math.}{=} (-1)^{2j-1}f(z)$$

(4) $z \in (B_{2j+1})$: $\arg(z-z_k) \in (\alpha-2\pi, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j$

$\arg(z - z_k) \in (-\pi, \pi)$ then $sqrt z - z_k \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j + 1, \cdots, m$

$$f(z) \overset{Math.}{=} (-1)^{2j}f(z)$$

(5) $z \in (G)$: $\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, m$

$$f(z) \overset{Math.}{=} f(z)$$

(6) $z \in L : y - y_1 = \tan\alpha(x - x_1)$:

(a) No cuts, $z \in L \bigcap \{(x, y) : y < y_1\}$:

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, 2N - 1$

$$f(z) \overset{Math.}{=} f(z) \overset{Math.}{=} f(z)$$

(b) No cuts, $z \in L \bigcap \{(x, y) : y_{2j} \le y < y_{2j+1}\}, j = 1, 2, ..., N - 1$

$\arg(z - z_k) = \alpha - 2\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j + 1, \cdots, 2N - 1$

$$f(z) \overset{Math.}{=} (-1)^{2j}f(z) \overset{Math.}{=} f(z)$$

(c) On cuts of $z \in L \bigcap \{(x, y) : y_{2j-1} \le y < y_{2j}\}$ with (+)edge, $j = 1, 2, ..., N - 1$

$\arg(z - z_k) = \alpha - 2\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j - 1$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j, \cdots, 2N - 1$

$$f(z) \overset{Math.}{=} (-1)^{2j-1}f(z)$$

(d) On cuts with $z \in L \bigcap \{(x,y) : y_{2N-1} \leq y\}$ with (+) edge

$\arg(z - z_k) = \alpha - 2\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2N-1$

$$f(z) \overset{Math.}{=} (-1)^{2N-1} f(z)$$

(e) On cuts of $z \in L \bigcap \{(x,y) : y_{2j-1} \leq y < y_{2j}\}$ with (−) edge , $j = 1, 2, ..., N-1$

$\arg(z - z_k) = \alpha$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j-1$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j, \cdots, 2N-1$

$$f(z) \overset{Math.}{=} f(z)$$

(f) On cuts with $z \in L \bigcap \{(x,y) : y_{2N-1} \leq y\}$ with (+) edge

$\arg(z - z_k) = \alpha$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2N-1$

$$f(z) \overset{Math.}{=} f(z)$$

Case 2. $m = 2N$

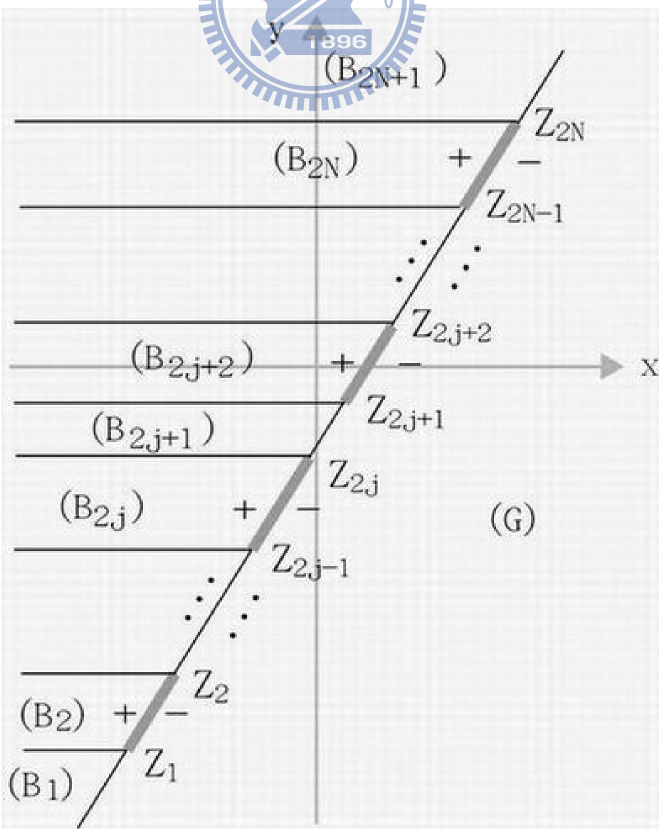

Figure 89: Case 2. $m = 2N$ for slant cuts

$$(B_1) = \{(x,y) : (x,y) \in C \text{ and } y < y_1\}$$

$$(B_2) = \{(x,y) : (x,y) \in C \text{ and } y_1 \le y < y_2\}$$

$$(B_{2j}) = \{(x,y) : (x,y) \in C \text{ and } y_{2j-1} \le y < y_{2j}\}$$

$$(B_{2j+1}) = \{(x,y) : (x,y) \in C \text{ and } y_{2j} \le y < y_{2j+1}\}$$

$$(B_{2N}) = \{(x,y) : (x,y) \in C \text{ and } y_{2N-1} \le y < y_{2N}\}$$

$$(B_{2N+1}) = \{(x,y) : (x,y) \in C \text{ and } y_{2N} \le y\}$$

$$(G) = \{(x,y) : (x,y) \in R \setminus C\} \bigcup \{(x,y) : y - y_1 = \tan\alpha(x - x_1)\}$$

Solution:

(1) $z \in (B_{2j})$, $j = 1, ..., N$

$\arg(z - z_k) \in (\alpha - 2\pi, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j-1$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j, \cdots, 2N$

$$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z)$$

(2) $z \in (B_{2j+1}), j = 1, ..., N-1$

$\arg(z - z_k) \in (\alpha - 2\pi, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j$

$\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j+1, \cdots, 2N$

$$f(z) \overset{Math.}{=} (-1)^{2j} f(z)$$

(3) $z \in (B_{2N+1})$:

$\arg(z - z_k) \in (\alpha - 2\pi, -\pi]$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2N$

$$f(z) \overset{Math.}{=} (-1)^{2N} f(z)$$

(4) $z \in (G)$: $\arg(z - z_k) \in (-\pi, \pi)$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, m$

$$f(z) \overset{Math.}{=} f(z)$$

(5) $z \in L : y - y_1 = \tan\alpha(x - x_1)$

(a) no cuts, $z \in L \bigcap \{(x,y) : y < y_1\}$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, 2N - 1$

$$f(z) \overset{Math.}{=} f(z) \overset{Math.}{=} f(z)$$

(b) no cuts, $z \in L \bigcap \{(x,y) : y_{2j} \leq y < y_{2j+1}\}$

$\arg(z - z_k) = \alpha - 2\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j + 1, \cdots, 2N - 1$

$$f(z) \overset{Math.}{=} (-1)^{2j} f(z) \overset{Math.}{=} f(z)$$

(c) no cuts, $z \in L \bigcap \{(x,y) : y_{2N} \leq y\}$

$\arg(z - z_k) = \alpha - 2\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2N$

$$f(z) \overset{Math.}{=} (-1)^{2N} f(z) \overset{Math.}{=} f(z)$$

(d) on cuts of $z \in L \bigcap \{(x,y) : y_{2j-1} \leq y < y_{2j}\}$ with $(+)$ edge , $j = 1, 2, ..., N$

$\arg(z - z_k) = \alpha - 2\pi$ then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j - 1$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j, \cdots, 2N$

$$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z)$$

(e) on cuts of $z \in L \bigcap \{(x,y) : y_{2j-1} \leq y < y_{2j}\}$ with $(-)$ edge , $j = 1, 2, ..., N$

$\arg(z - z_k) = \alpha$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 1, \cdots, 2j - 1$

$\arg(z - z_k) = \alpha - \pi$ then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$, $\forall k = 2j, \cdots, 2N$

$$f(z) \overset{Math.}{=} f(z)$$

Conclusion:

1. m=2N-1:

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if} \bigcup_{j=1}^{N-1} \{(x,y) \in C \text{ and } y_{2j-1} \leq y < y_{2j}\} \\ & \bigcup\{(x,y) : (x,y) \in C \text{ and } y_{2N-1} \leq y\} \\ & \bigcup_{j=1}^{N-1} \{(x,y) : (x,y) \in L \text{ and } y_{2j-1} \leq y < y_{2j} \text{with } (+) \text{ edge}\} \\ & \bigcup\{(x,y) : (x,y) \in L \text{ and } y_{2N-1} \leq y \text{with } (+) \text{ edge}\} \\ f(z) & \text{otherwise.} \end{cases} \tag{43}$$

2. m=2N:

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } \bigcup_{j=1}^{N} \{(x,y) \in C \text{ and } y_{2j-1} \leq y < y_{2j}\} \\ & \bigcup_{j=1}^{N} \{(x,y) : (x,y) \in L \text{ and } y_{2j-1} \leq y < y_{2j} \text{with } (+) \text{ edge}\} \\ f(z) & \text{otherwise.} \end{cases} \tag{44}$$

Example: Compute $\displaystyle\int \frac{1}{f(z)} dz$ over a,b cycles where

$f(z) = \sqrt{(z-3+i)(z-4-\sqrt{3}i+i)(z-2-i)(z-2-\sqrt{3}-2i)z(z-2-2i)(z+i)(z+1)}$.

Using the way of areas what we discuss before to modify. Let $z_1 = 3-i$, $z_2 = 4+(\sqrt{3}-1)i$, $z_3 = 2+i$, $z_4 = 2+\sqrt{3}+2i$, $z_5 = 0$, $z_6 = 2+2i$, $z_7 = -i$, $z_8 = -1$.

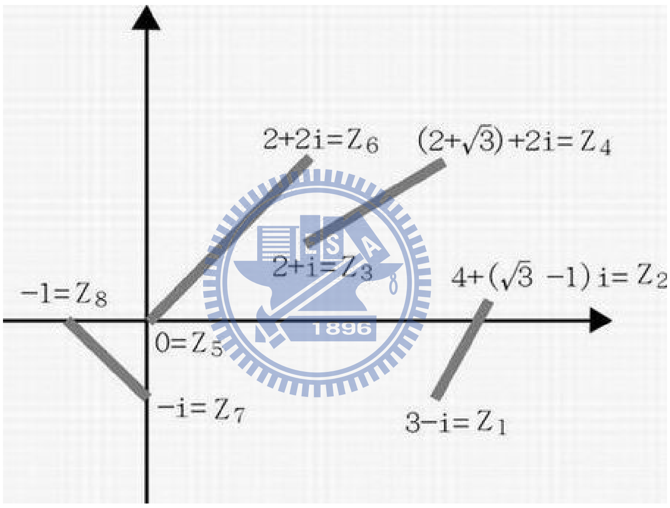

Figure 90: The cut plane

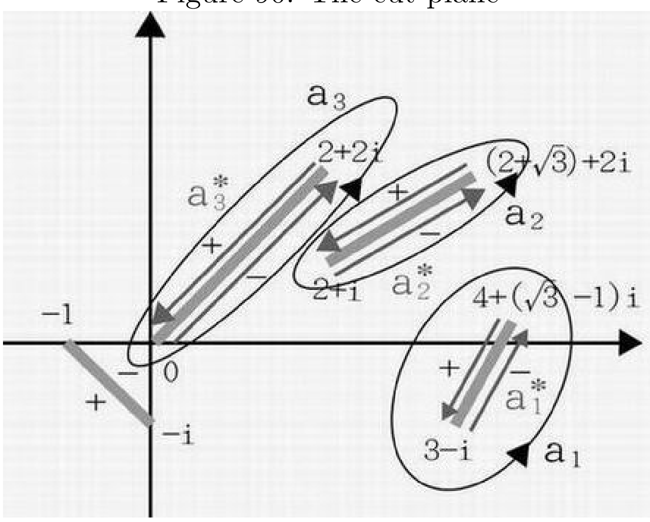

Figure 91: a cycles and their equivalent path

1. a_1 : consider the equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* =the path along slant cut from $4+(\sqrt{3}-1)i$ to $3-i$ on $(+)$edge of sheet-I and a_{12}^* =the path along slant cut from $3-i$ to $4+(\sqrt{3}-1)i$ on $(-)$edge of sheet-I

   (1) a_{11}^* : Let $z = 3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i), r: 2 \to 0$ and then $dz = \frac{1}{2}+\frac{\sqrt{3}}{2}idr$

   $\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

   $\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

   $\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

   $\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{11}^*} f(z)dz \overset{Math.}{=} -\int_2^0 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i))}dr$$

   (2) a_{12}^*: Let $z = 3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i), r: 0 \to 2$ and $dz = \frac{1}{2}+\frac{\sqrt{3}}{2}idr$

   $\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

   $\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

   $\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

   $\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{12}^*} f(z)dz \overset{Math.}{=} \int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i))}dr$$

   By (1), (2) and Cauchy Theorem

$$\begin{aligned} \int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i)}dr \\ &= -0.104573-0.000693869i \end{aligned}$$

2. a_2: consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^*$ where a_{21}^* =the path along slant cut from $2+\sqrt{3}+2i$ to $2+i$ on $(+)$edge of sheet-I and a_{22}^* =the path along slant cut from $2+i$ to $2+\sqrt{3}+2i$ on $(-)$edge of sheet-I.

(1) a_{21}^*: Let $z = 2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i), r: 2 \to 0$ and $dz = \frac{\sqrt{3}}{2}+\frac{1}{2}idr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} -\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{21}^*} f(z)dz \overset{Math.}{=} -\int_2^0 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr$$

(2) a_{22}^*: Let $z = 2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i), r: 0 \to 2$ and $dz = \frac{\sqrt{3}}{2}+\frac{1}{2}idr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{22}^*} f(z)dz \overset{Math.}{=} \int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned} \int_{a_2} \frac{1}{f(z)}dz &= \int_{a_2^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr \\ &= 0.0384459+0.422154i \end{aligned}$$

3. a_3: consider the equivalent path $a_1^* = a_{31}^* \bigcup a_{32}^* \bigcup a_{33}^* \bigcup a_{34}^* \bigcup a_{35}^* \bigcup a_{36}^*$ where $a_{31}^* =$ the path along slant cut from $2+2i$ to $1+i$ on (+)edge of sheet-I, $a_{32}^* =$ the path along slant cut from $1+i$ to $\sqrt{3}-1(\sqrt{3}-1i)$ on (+)edge of sheet-I, $a_{33}^* =$ the path along slant cut from $\sqrt{3}-1(\sqrt{3}-1i)$ to 0 on (+)edge of sheet-I, $a_{34}^* =$ the path along slant cut from 0 to $\sqrt{3}-1(\sqrt{3}-1i$ on (−)edge of sheet-I, $a_{35}^* =$ the path along slant cut from $\sqrt{3}-1(\sqrt{3}-1i$ to $1+i$ on (−)edge of sheet-I and $a_{36}^* =$ the path along slant cut from $1+i$ to $2+2i$ on (−)edge of sheet-I.

(1) a_{31}^{*}: Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r : 2\sqrt{2} \to \sqrt{2}$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{-} = \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} -\sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{31}^{*}} f(z)dz \overset{Math.}{=} \int_{2\sqrt{2}}^{\sqrt{2}} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr$$

(2) a_{32}^{*} : Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r : \sqrt{2} \to \sqrt{2}(\sqrt{3}-1)$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} -\sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{32}^{*}} f(z)dz \overset{Math.}{=} -\int_{\sqrt{2}}^{\sqrt{2}(\sqrt{3}-1)} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr$$

(3) a_{33}^{*} : Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r : \sqrt{2}(\sqrt{3}-1) \to 0$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} -\sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{33}^{*}} f(z)dz \overset{Math.}{=} \int_{\sqrt{2}(\sqrt{3}-1)}^{0} \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i))}dr$$

(4) a_{34}^{*} : Let $z = r(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i), r : 0 \to \sqrt{2}(\sqrt{3}-1)$ and $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$

$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{34}^*} f(z)dz \overset{Math.}{=} -\int_0^{\sqrt{2}(\sqrt{3}-1)} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

(5) a_{35}^*: Let $z = r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i), r : \sqrt{2}(\sqrt{3}-1) \to \sqrt{2}$ and $dz = (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$

$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$

$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$

$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{35}^*} f(z)dz \overset{Math.}{=} \int_{\sqrt{2}(\sqrt{3}-1)}^{\sqrt{2}} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

(6) a_{36}^* : Let $z = r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i), r : \sqrt{2} \to 2\sqrt{2}$ and $dz = (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$

$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} -\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$

$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$

$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{36}^*} f(z)dz \overset{Math.}{=} -\int_{\sqrt{2}}^{2\sqrt{2}} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

By (1), (2),..., (6) and Cauchy Theorem

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)}dz &= \int_{b_3^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} 2\int_{2\sqrt{2}}^{0} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr \\
&\quad -2\int_{\sqrt{2}}^{\sqrt{2}(\sqrt{3}-1)} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr \\
&\quad +2\int_{\sqrt{2}(\sqrt{3}-1)}^{0} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr \\
&= 0.153675 - 0.222745i
\end{aligned}$$

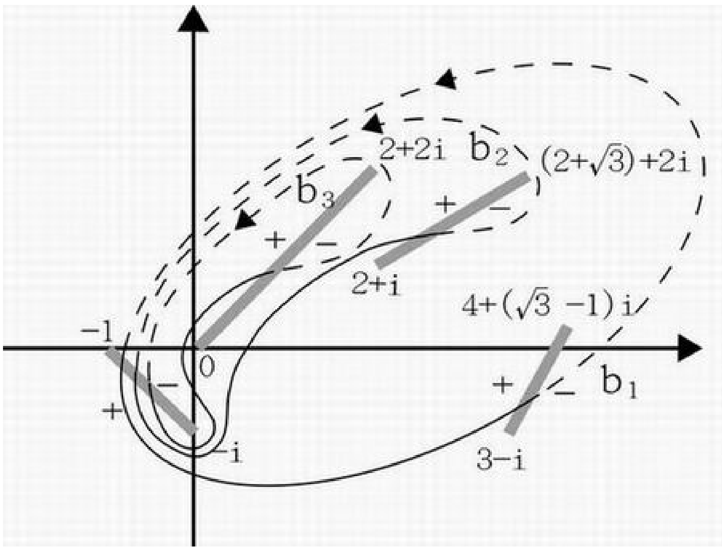

Figure 92: b cycles

4. b_3 : consider the equivalent path $b_3^* = b_{31}^* \bigcup b_{32}^* \bigcup b_{33}^* \bigcup b_{34}^* \bigcup b_{35}^* \bigcup b_{36}^*$ where b_{31}^* =the path along slant cut from $-\frac{1}{2} - \frac{1}{2}i$ to $-i$ on (+)edge of sheet-I, b_{32}^* =the path along slant cut from $-i$ to $-\frac{1}{2} - \frac{1}{2}i$ on (−)edge of sheet-I, b_{33}^* =the path along slant line from $-\frac{1}{2} - \frac{1}{2}i$ to 0 on sheet-I, b_{34}^* =the path along slant cut from 0 to $2+2i$ on (−)edge of sheet-II, b_{35}^* =the path along slant cut from $2+2i$ to 0 on (+)edge of sheet-II and b_{36}^* =the path along slant line from 0 to $-\frac{1}{2} - \frac{1}{2}i$ on sheet-II.

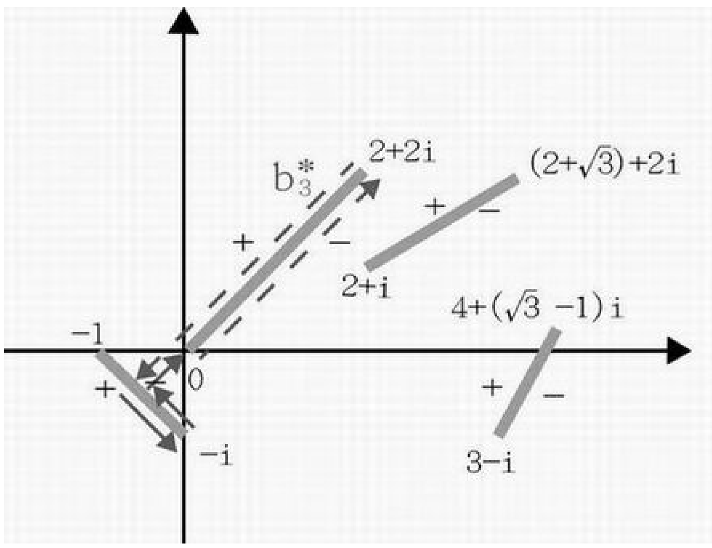

Figure 93: The equivalent path b_3^*

(1) b_{31}^*: Let $z = -i + (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r$, $r : \frac{1}{\sqrt{2}} \to 0$ and $dz = (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$
$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$
$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$
$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} -\sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{31}^*} f(z)dz \overset{Math.}{=} \int_{\frac{1}{\sqrt{2}}}^{0} \frac{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

(2) b_{32}^*: Let $z=-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r, r:0\to\frac{1}{\sqrt{2}}$ and $dz=(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$
$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$
$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$
$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{32}^*} f(z)dz \overset{Math.}{=} -\int_{0}^{\frac{1}{\sqrt{2}}} \frac{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

(3) b_{33}^*: Let $z=r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i), r:-\frac{1}{\sqrt{2}}\to 0$ and $dz=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$
$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$
$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$
$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{33}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-\frac{1}{\sqrt{2}}}^{0} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

(4) $b^*_{34} \equiv$ the path along slant cut from 0 to $2+2i$ on $(+)$edge of sheet-I

$b^*_{35} \equiv$ the path along slant cut from $2+2i$ to 0 on $(-)$edge of sheet-I

$$\int_{b^*_{34}\bigcup b^*_{35}} \frac{1}{f(z)}dz = -\int_{a^*_3} \frac{1}{f(z)}dz$$

(5) b^*_{36} : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{36} = the path along slant line from 0 to $-\frac{1}{2}-\frac{1}{2}i$ on sheet-I

Let $z = r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i), r: 0 \to -\frac{1}{\sqrt{2}}$ and $dz = (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b^*_{36}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{b^{**}_{36}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^{-\frac{1}{\sqrt{2}}} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr$$

By (1), (2),..., (5) and Cauchy Theorem

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)}dz &= \int_{b^*_3} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} 2\int_{\frac{1}{\sqrt{2}}}^{0} \frac{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r}dr \\
&\quad -2\int_{-\frac{1}{\sqrt{2}}}^{0} \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i}{f(r(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))}dr - \int_{a^*_3} \frac{1}{f(z)}dz \\
&= 0.0378157 - 0.00223065i
\end{aligned}$$

5. b_2 : consider the equivalent path $b^*_2 = b^*_{31}\bigcup b^*_{32}\bigcup b^*_{34}\bigcup a^*_{32}\bigcup b^*_{21}\bigcup b^*_{22}\bigcup b^*_{23}\bigcup b^*_{24}\bigcup b^*_{35}\bigcup b^*_{36}$ where b^*_{21} =the path along vertical line from $2+2i$ to $2+i$ on sheet-I,

b^*_{22} =the path along slant cut from $2+i$ to $2+\sqrt{3}+2i$ on $(-)$edge of sheet-II,

b^*_{23} =the path along slant cut from $2+\sqrt{3}+2i$ to $2+i$ on $(+)$edge of sheet-II and

b^*_{24} =the path along vertical line from $2+i$ to $2+2i$ on sheet-II.

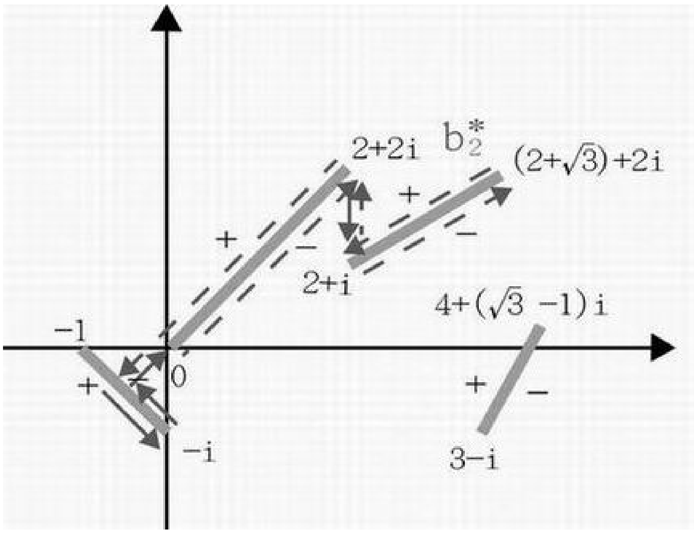

Figure 94: The equivalent path b_3^*

(1) b_{21}^* : Let $z = 2 + ri, r : 2 \to 1$ and $dz = idr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} -\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{21}^*} f(z)dz \overset{Math.}{=} -\int_2^1 \frac{i}{f(2+ri)}dr$$

(2) $b_{22}^* \equiv$ the path along slant cut from $2+i$ to $2+\sqrt{3}+2i$ on $(+)$edge of sheet-I and $b_{23}^* \equiv$ the path along slant cut from $2+\sqrt{3}+2i$ to $2+i$ on $(-)$edge of sheet-I.

$$\int_{b_{22}^* \cup b_{23}^*} \frac{1}{f(z)}dz = -\int_{a_2^*} \frac{1}{f(z)}dz$$

(3) b_{24}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider $b_{24}^{**} =$ the path along vertical line from $2+i$ to $2+2i$ on sheet-I

Let $z = 2 + ri, r : 1 \to 2$ and $dz = idr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} -\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$

$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{24}^*} \frac{1}{f(z)}dz = -\int_{b_{24}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^2 \frac{i}{f(2+ri)}dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{b_3^*} \frac{1}{f(z)}dz - 2\int_2^1 \frac{i}{f(2+ri)}dr \\
&\quad +2\int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+r(\frac{\sqrt{3}}{2}+\frac{1}{2}i))}dr \\
&= -0.169794 + 0.678808i
\end{aligned}$$

6. b_1 : Consider equivalent path $b_1^* = b_{31}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^* \bigcup b_{15}^* \bigcup b_{35}^* \bigcup b_{36}^*$ where

b_{11}^* = the path along horizontal line from $-i$ to $3-i$ on sheet-I,

b_{12}^* = the path along slant cut from $3-i$ to $4+(\sqrt{3}-1)i$ on (+)edge of sheet-I,

b_{13}^* = the path along slant line from $4+(\sqrt{3}-1)i$ to $3+\sqrt{3}+2i$ on sheet-II,

b_{14}^* = the path along horizontal line from $3+\sqrt{3}+2i$ to $2\sqrt{3}+2i$ on sheet-II and

b_{15}^* = the path along horizontal line from $2\sqrt{3}+2i$ to $2+2i$ on sheet-II.

(1) b_{11}^* : Let $z = r-i, r : 0 \to 3$ and $dz = dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} -\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$

$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$

$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$

$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{11}^*} f(z)dz \overset{Math.}{=} -\int_0^3 \frac{1}{f(r-i)}dr$$

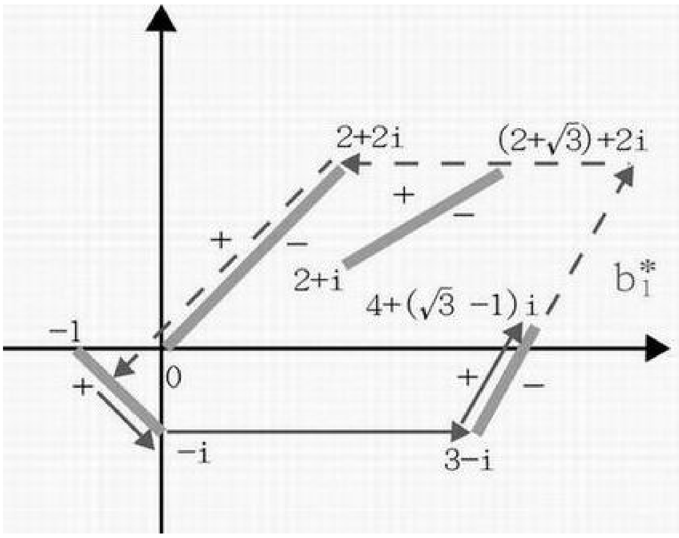

Figure 95: The equivalent path b_3^*

(2) $b_{12}^* \equiv -a_{11}^*$:

$$\int_{b_{12}^*} \frac{1}{f(z)} dz = -\int_{a_{11}^*} \frac{1}{f(z)} dz$$

(3) b_{13}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{13}^{**} = the path along horizon line from $-i$ to $3-i$ on sheet-I

Let $z = 3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i), r : 2 \to 2\sqrt{3}$ and $dz = \frac{1}{2} + \frac{\sqrt{3}}{2}idr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$

$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{13}^*} f(z)dz = -\int_{b_{13}^{**}} f(z)dz \overset{Math.}{=} \int_2^{2\sqrt{3}} \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{f(3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i))} dr$$

(4) b_{14}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{14}^{**} = the path along horizon line from $3 + \sqrt{3} + 2i$ to $2\sqrt{3} + 2i$ on sheet-I Let $z = r + 2i, r$: $3 + \sqrt{3} \to 2 + \sqrt{3}$ and then $dz = dr$

$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$

$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$

$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$
$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{14}^*} f(z)dz = -\int_{b_{14}^{**}} f(z)dz \overset{Math.}{=} \int_{3+\sqrt{3}}^{2+\sqrt{3}} \frac{1}{f(r+2i)}dr$$

(5) b_{15}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{15}^{**} = the path along horizontal line from $2\sqrt{3}+2i$ to $2+2i$ on sheet-I

Let $z = r+2i, r : 2+\sqrt{3} \to 2$ and then $dz = dr$

$$\sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i} \overset{Math.}{=} \sqrt{z-3+i}\sqrt{z-\sqrt{3}i+i}$$
$$\sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i} \overset{Math.}{=} \sqrt{z-2-i}\sqrt{z-2-\sqrt{3}-2i}$$
$$\sqrt{z}\sqrt{z-2-2i} \overset{Math.}{=} \sqrt{z}\sqrt{z-2-2i}$$
$$\sqrt{z+i}\sqrt{z+1} \overset{Math.}{=} \sqrt{z+i}\sqrt{z+1}$$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{15}^*} f(z)dz = -\int_{b_{15}^{**}} f(z)dz \overset{Math.}{=} \int_{2+\sqrt{3}}^{2} \frac{1}{f(r+2i)}dr$$

By (1),..., (5) and Cauchy Theorem

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)}dz &= \int_{b_1^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \frac{1}{2}\int_{b_3^*} \frac{1}{f(z)}dz - \int_0^3 \frac{1}{f(r-i)}dr \\
&\quad - \int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i))}dr - \int_2^{2\sqrt{3}} \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i))}dr \\
&\quad - \int_{3+\sqrt{3}}^{2+\sqrt{3}} \frac{1}{f(r+2i)}dr - \int_{2+\sqrt{3}}^{2} \frac{1}{f(r+2i)}dr \\
&= -0.0617421 - 0.145471i
\end{aligned}$$

Now we know how to modify the computation of the integral $\int \frac{1}{\sqrt{\prod_{k=1}^{n}(z-z_k)}}dz$ by Mathematica in any cuts. So we could use the statement above to modify in any situation

Example:

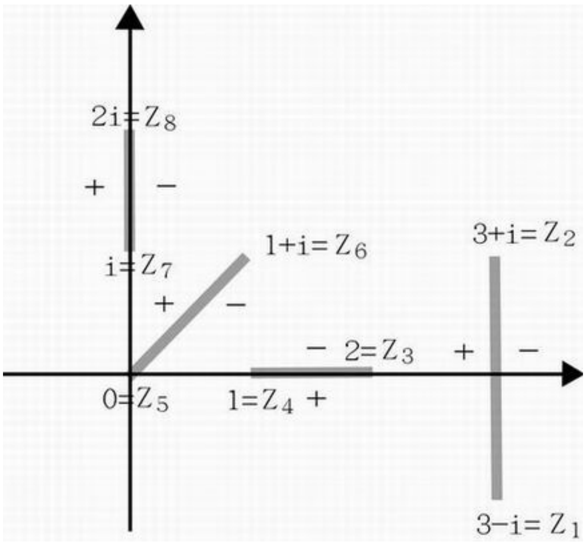

Figure 96: The cut plane of $f(z)$

Evaluate $\int \frac{1}{f(z)}dz$ over a_1, a_2, a_3, b_1, b_2 and b_3 cycles

$f(z) = \sqrt{(z-3-i)(z-3+i)(z-2)(z-1)(z-0)(z-1-i)(z-i)(z-3i)}$,

we let $z_1 = 3+i, z_2 = 3-i, z_3 = 2, z_4 = 1, z_5 = 0, z_6 = 1+i, z_7 = i, z_8 = 2i$.

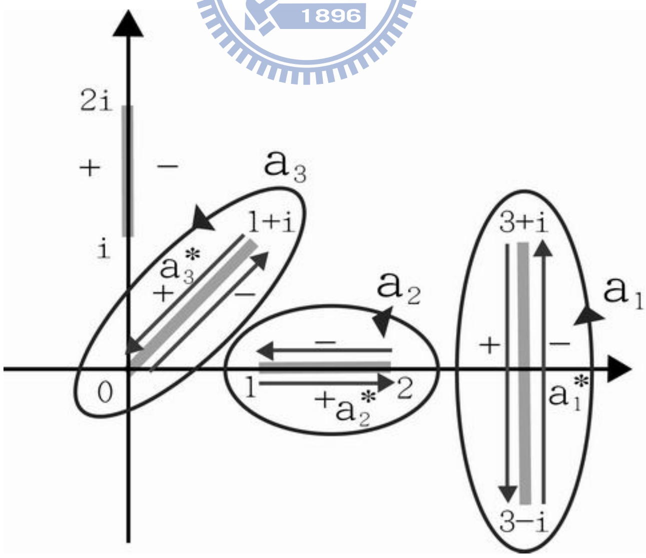

Figure 97: The cuts of f(z) in complex plane

1. a_1 : Consider the equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on the vertical cut from $3+i$ to $3-i$ on $(+)$edge of sheet-I and a_{12}^* = the path on the vertical cut from $3-i$ to $3+i$ on $(-)$edge of sheet-I.

(1) a_{11}^*: Let $z = 3 + ri, r : 1 \to -1, dz = idr$

$\arg(z-3+i) = -\frac{3}{2}\pi$ then $\sqrt{z-3+i} \stackrel{Math.}{=} -\sqrt{z-3+i}$

$\arg(z-3-i) = -\frac{1}{2}\pi$ then $\sqrt{z-3-i} \stackrel{Math.}{=} \sqrt{z-3-i}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k}$, k=3, 4, 5, 6, 7, 8

So we have

$$f(z) \stackrel{Math.}{=} -f(z)$$

$$\int_{a_{11}^*} \frac{1}{f(z)} dz \stackrel{Math.}{=} -\int_1^{-1} \frac{i}{f(3+ri)} dr$$

(2) a_{12}^*: Let $z = 3 + ri, r : -1 \to 1, dz = idr$

$\arg(z-3-i) = -\frac{1}{2}\pi$ then $\sqrt{z-3-i} \stackrel{Math.}{=} \sqrt{z-3-i}$

$\arg(z-3+i) = \frac{1}{2}\pi$ then $\sqrt{z-3+i} \stackrel{Math.}{=} \sqrt{z-3+i}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \stackrel{Math.}{=} \sqrt{z-z_k}$, k=3, 4, 5, 6, 7, 8

So we have

$$f(z) \stackrel{Math.}{=} f(z)$$

$$\int_{a_{12}^*} \frac{1}{f(z)} dz \stackrel{Math.}{=} \int_{-1}^{1} \frac{i}{f(3+ri)} dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}
\int_{a_1} \frac{1}{f(z)} dz &= \int_{a_1^*} \frac{1}{f(z)} dz \\
&\stackrel{Math.}{=} 2\int_{-1}^{1} \frac{i}{f(3+ri)} dr \\
&= -0.156969 + 0.225071i
\end{aligned}$$

2. a_2 : Consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^* \subset R$ where a_{21}^* = the path on the horizontal cut from 1 to 2 on (+)edge of sheet-I and a_{22}^* = the path on the horizontal cut from 2 to 1 on (−)edge of sheet-I.

(1) a_{21}^*:

$\arg(z-3+i)\in(-\frac{3}{2}\pi,-\pi)$ then $\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3+i}$

$\arg(z-2)=-\pi$ then $\sqrt{z-2}\overset{Math.}{=}-\sqrt{z-2}$

$\arg(z-1)=0$ then $\sqrt{z-1}\overset{Math.}{=}\sqrt{z-1}$

$\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}$, k=2,5,6,7,8

We have

$$f(z)\overset{Math.}{=}f(z)$$

$$\int_{a_{21}^*}\frac{1}{f(z)}dz\overset{Math.}{=}\int_1^2\frac{1}{f(z)}dz$$

(2) a_{22}^*:

$\arg(z-3+i)\in(-\frac{3}{2}\pi,-\pi)$ then $\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3+i}$

$\arg(z-2)=\pi$ then $\sqrt{z-2}\overset{Math.}{=}\sqrt{z-2}$

$\arg(z-1)=0$ then $\sqrt{z-1}\overset{Math.}{=}\sqrt{z-1}$

$\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}$, k=2,5,6,7,8

So we have

$$f(z)\overset{Math.}{=}-f(z)$$

$$\int_{a_{22}^*}\frac{1}{f(z)}dz\overset{Math.}{=}\int_2^1\frac{1}{f(z)}dz$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_2}\frac{1}{f(z)}dz &= \int_{a_2^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} 2\int_1^2\frac{1}{f(z)}dz\\ &= 1.07276-0.117388i\end{aligned}$$

3. a_3 : The equivalent path $a_3^*=a_{31}^*\bigcup a_{32}^*$ where $a_{31}^*=$ the path on the slant cut from 0 to $1+i$ on (+)edge of sheet-I and $a_{32}^*=$ the path on the slant cut from $1+i$ to 0 on $(-)$edge of sheet-I

(1) a_{31}^*: Let $z=r(1+i), r:0\to 1, dz=(1+i)dr$

$\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

$\arg(z) = -\frac{7}{4}\pi$ then $\sqrt{z} \overset{Math.}{=} -\sqrt{z}$

$\arg(z-1-i) = -\frac{3}{4}\pi$ then $\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z-1-i}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2,3,4,7,8

We have

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{31}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_0^1 \frac{1+i}{f(r(1+i))} dr$$

(2) a_{32}^*: Let $z = r(1+i), r: 1 \to 0, dz = (1+i)dr$

$\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

$\arg(z) = \frac{1}{4}\pi$ then $\sqrt{z} \overset{Math.}{=} \sqrt{z})$

$\arg(z-1-i) = -\frac{3}{4}\pi$ then $\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z-1-i}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2,3,4,7,8

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a_{32}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^0 \frac{1+i}{f(r+ri)} dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned} \int_{a_3} \frac{1}{f(z)} dz &= \int_{a_3^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_0^1 \frac{1+i}{f(r+ri)} dr = 1.27643 + 0.383835i \end{aligned}$$

4. b_3 : Consider the equivalent path $b_3^* = b_{31}^* \bigcup b_{32}^*$ where b_{31}^* = the path on the horizontal line i to $1+i$ on sheet-I and b_{32}^* = the path on the horizontal line from $1+i$ to i on sheet-II.

(1) b_{31}^*: Let $z = r+i, r: 0 \to 1, dz = dr$

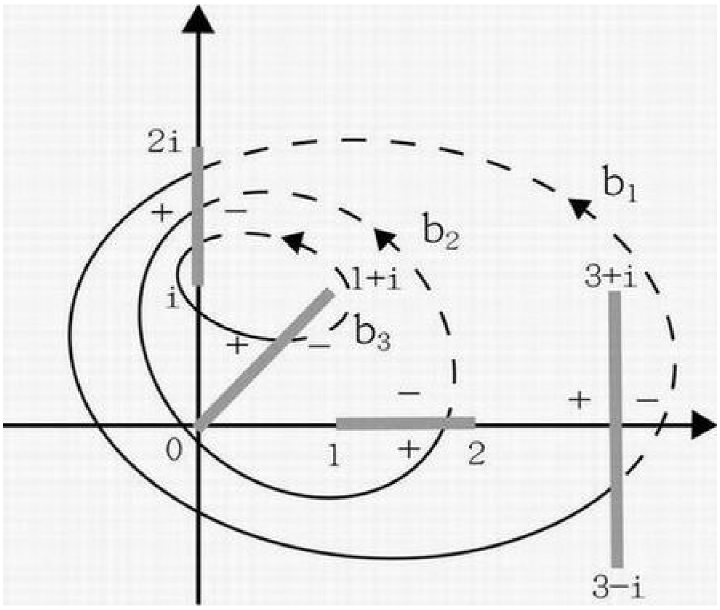

Figure 98: b-cycles

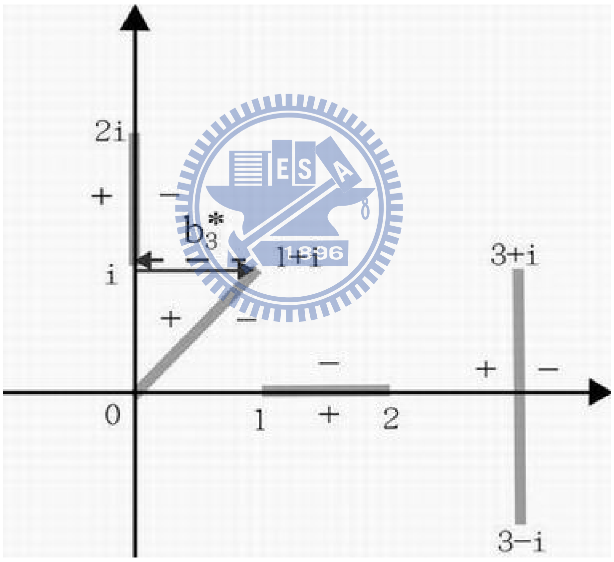

Figure 99: The equivalent path b_3^*

$\arg(z-3+i)\in(-\frac{3}{2}\pi,-\pi)$ then $\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3+i}$

$\arg(z)\in(-\frac{5}{4}\pi,-\pi)$ then $\sqrt{z}\overset{Math.}{=}-\sqrt{z}$

$\arg(z-3-i)=\arg(z-1-i)=-\pi$ then

$\sqrt{z-3-i}\overset{Math.}{=}-\sqrt{z-3-i}$ and $sqrtz-1-i\overset{Math.}{=}-\sqrt{z-1-i}$ $\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}$, k=3,4,7,8

So we have

$$f(z)\overset{Math.}{=}f(z)$$

$$\int_{b_{31}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0 \frac{1}{f(r+i)}dr$$

(2) b_{32}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{32}^{**} = the path on the horizontal line from $1+i$ to i on sheet-I

Let $z = r+i, r : 1 \to 0, dz = dr$

$\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

$\arg(z) \in (-\frac{5}{4}\pi, -\pi)$ then $\sqrt{z} \overset{Math.}{=} -\sqrt{z}$

$\arg(z-3-i) = \arg(z-1-i) = -\pi$ then

$\sqrt{z-3-i} \overset{Math.}{=} -\sqrt{z-3-i}$ and $sqrt z-1-i \overset{Math.}{=} -\sqrt{z-1-i}$ $\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=3,4,7,8

So we have

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{32}^*} \frac{1}{f(z)}dz = -\int_{b_{32}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0 \frac{1}{f(r+i)}dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{b_3} \frac{1}{f(z)}dz &= \int_{b_3^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_0^1 \frac{1}{f(r+i)}dr = 0.955186 - 0.204659i\end{aligned}$$

5. b_2 : Consider the equivalent path $b_2^* = b_3^* \bigcup b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup b_{32}^*$ where b_{21}^* = the path on the vertical line i to 0 on sheet-I, b_{22}^* = the path on the horizontal line 0 to 1 on sheet-I and b_{23}^* = the path on the vertical line 1 to $1+i$ on sheet-II.

(1) b_{21}^*: Let $z = ri$, $r : 1 \to 0$, $dz = idr$

$\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

$\arg(z) \in (-\frac{5}{4}\pi, -\pi)$ then $\sqrt{z} \overset{Math.}{=} -\sqrt{z}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2,3,4,6,7,8

So we have

$$f(z) \overset{Math.}{=} f(z)$$

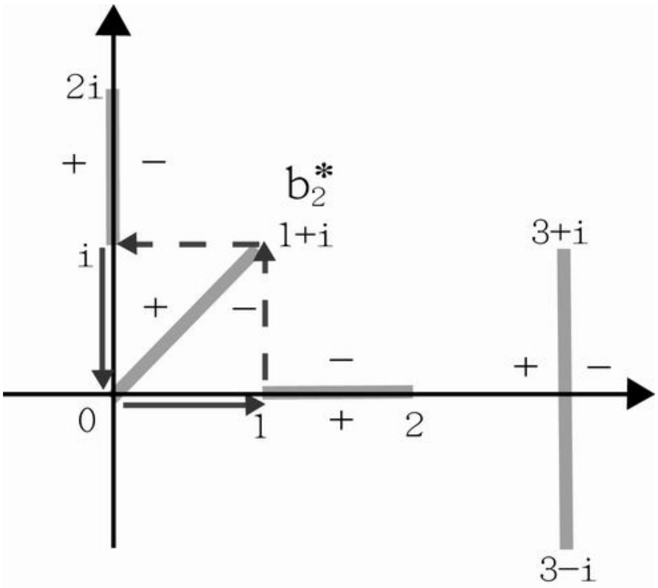

Figure 100: b_2^*

$$\int_{b_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0 \frac{1}{f(ri)}dr$$

(2) b_{22}^*:

$\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

$\arg(z-2) = -\pi$ then $\sqrt{z-2} \overset{Math.}{=} -\sqrt{z-2}$

$\arg(z-1) = -\pi$ then $\sqrt{z-1} \overset{Math.}{=} -\sqrt{z-1}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2,5,6,7,8

So we have

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^1 \frac{1}{f(z)}dz$$

(3) b_{23}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{23}^{**} = the path on the vertical line 1 to $1+i$ on sheet-I

Let $z = 1+ri$, $r: 0 \to 1$, $dz = idr$

$\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

$\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k = 2, \ldots, 8$

So we have

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{23}^*} \frac{1}{f(z)}dz = -\int_{b_{23}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^1 \frac{i}{f(1+ri)}dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_1^0 \frac{1}{f(ri)}dr - \int_0^1 \frac{1}{f(r)}dr \\ &\quad + \int_0^1 \frac{i}{f(1+ri)}dr - \int_1^0 \frac{1}{f(r+i)}dr \\ &= 0.406494 + 0.556588i\end{aligned}$$

6. b_1 : Consider the equivalent path $b_1^* = b_{21}^* \bigcup b_{22}^* \bigcup a_{21}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{24}^*$ where b_{11}^* = the path on the horizontal line 2 to 3 on sheet-I b_{12}^* = the path on the vertical cut 3 to $3+i$ on $(-)$edge of sheet-II b_{13}^* = the path on the horizontal line $3+i$ to $1+i$ on sheet-II

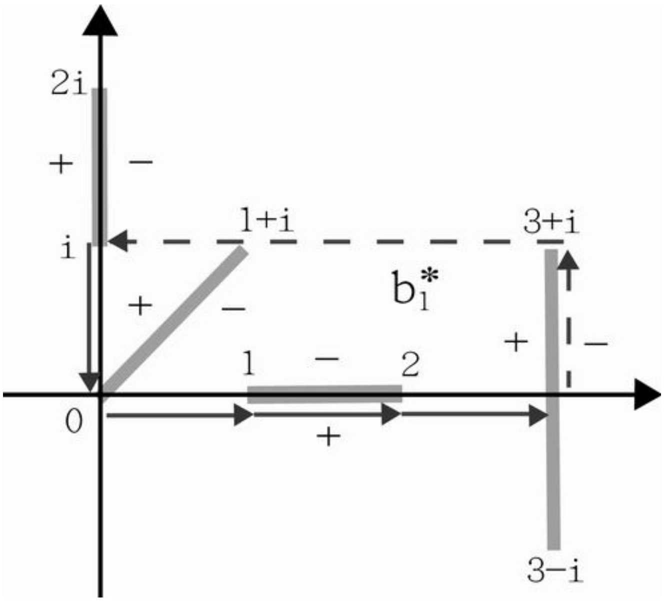

Figure 101: b_1^*

(1) b_{11}^*:

$\arg(z-3+i)\in(-\frac{3}{2}\pi,-\pi)$ then $\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3+i}$

$\arg(z-2)=0$ then $\sqrt{z-2}\overset{Math.}{=}\sqrt{z-2}$

$\arg(z-1)=0$ then $\sqrt{z-1}\overset{Math.}{=}\sqrt{z-1}$

$\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}$, k=2,5,6,7,8

So we have

$$f(z)\overset{Math.}{=}-f(z)$$

$$\int_{b*_{11}}\frac{1}{f(z)}dz\overset{Math.}{=}-\int_2^3\frac{1}{f(z)}dz$$

(2) $b^*_{12}\equiv$ the path on the vertical cut from 3 to $3+i$ on (+)edge of sheet-I

Let $z=3+ri, r:0\to 1, dz=idr$

$\arg(z-3+i)=-\frac{3}{2}\pi$ then $\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3+i}$

$\arg(z-3-i)=-\frac{1}{2}\pi$ then $\sqrt{z-3-i}\overset{Math.}{=}\sqrt{z-3-i}$

$\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}$, k=3,4,5,6,7,8

So we have

$$f(z)\overset{Math.}{=}-f(z)$$

$$\int_{b*_{12}}\frac{1}{f(z)}dz\overset{Math.}{=}-\int_0^1\frac{i}{f(3+ri)}dr$$

(3) b_{13} : We known that $f(z)|_{(I)}=-f(z)|_{(II)}$, so we can consider $b^{**}_{13}=$ the path on the horizontal line $3+i$ to $1+i$ on sheet-I

Let $z=r+i,\ r:3\to 1,\ dz=dr$

$\arg(z-3+i)\in(-\frac{3}{2}\pi,-\pi)$ then $\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3+i}$

$\arg(z-3-i)=-\pi$ then $\sqrt{z-3-i}\overset{Math.}{=}-\sqrt{z-3-i}$

$\arg(z-z_k)\in(-\pi,\pi)$ then $\sqrt{z-z_k}\overset{Math.}{=}\sqrt{z-z_k}, k=3,\ldots,8$

We have

$$f(z)\overset{Math.}{=}f(z)$$

$$\int_{b^*_{13}}\frac{1}{f(z)}dz=-\int_{b^*_{13}}\frac{1}{f(z)}dz\overset{Math.}{=}-\int_3^1\frac{1}{f(r+i)}dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}\int_{b_1}\frac{1}{f(z)}dz &= \int_{b_1^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} \int_1^0\frac{1}{f(ri)}dr-\int_0^1\frac{1}{f(z)}dz\\ &\quad -\int_1^0\frac{1}{f(r+i)}dr+\int_1^0\frac{1}{f(ri)}dr-\int_2^3\frac{1}{f(z)}dz\\ &\quad -\int_0^1\frac{i}{f(3+ri)}dr-\int_3^1\frac{1}{f(r+i)}dr\\ &= 0.14418-0.0612153i\end{aligned}$$

Using areas of domain to modify:

1. a_1 : Consider the equivalent path $a_1^*=a_{11}^*\bigcup a_{12}^*$ where $a_{11}^*=$ the path on the vertical cut from $3+i$ to $3-i$ on $(+)$edge of sheet-I and $a_{12}^*=$ the path on the vertical cut from $3-i$ to $3+i$ on $(-)$edge of sheet-I.

   (1) a_{11}^*: Let $z=3+ri, r:1\to -1, dz=idr$

   $\sqrt{z-3-i}\sqrt{z-3+i}\overset{Math.}{=}-\sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1}\overset{Math.}{=}\sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i}\overset{Math.}{=}\sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i}\overset{Math.}{=}\sqrt{z-i}\sqrt{z-2i}$

$$f(z)\overset{Math.}{=}-f(z)$$

$$\int_{a_{11}^*}\frac{1}{f(z)}dz\overset{Math.}{=}-\int_1^{-1}\frac{i}{f(3+ri)}dr$$

   (2) a_{12}^*: Let $z=3+ri, r:-1\to 1, dz=idr$

   $\sqrt{z-3-i}\sqrt{z-3+i}\overset{Math.}{=}\sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1}\overset{Math.}{=}\sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i}\overset{Math.}{=}\sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i}\overset{Math.}{=}\sqrt{z-i}\sqrt{z-2i}$

$$f(z)\overset{Math.}{=}f(z)$$

$$\int_{a_{12}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-1}^{1} \frac{i}{f(3+ri)}dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_{-1}^{1} \frac{i}{f(3+ri)}dr \\ &= -0.156969 + 0.225071i\end{aligned}$$

2. a^2 : Consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^* \subset R$ where a_{21}^* = the path on the horizon cut from 1 to 2 on (+)edge of sheet-I and a_{22}^* = the path on the horizon cut from 2 to 1 on (−)edge of sheet-I.

(1) a_{21}^*:

$$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$$
$$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} -\sqrt{z-2}\sqrt{z-1}$$
$$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$$
$$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$$
$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{a_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{1}^{2} \frac{1}{f(z)}dz$$

(2) a_{22}^*:

$$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$$
$$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$$
$$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$$
$$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$$
$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{a*22} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{2}^{1} \frac{1}{f(z)}dz$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_2} \frac{1}{f(z)}dz &= \int_{a_2^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_1^2 \frac{1}{f(z)}dz \\ &= 1.07276 - 0.117388i\end{aligned}$$

3. a_3 : Consider the equivalent path $a_3^* = a_{31}^* \bigcup a_{32}^*$ where a_{31}^* = the path on the slant cut from 0 to $1+i$ on (+)edge of sheet-I and a_{32}^* = the path on the slant cut from $1+i$ to 0 on (−)edge of sheet-I.

   (1) a_{31}^*: Let $z = r(1+i), r : 0 \to 1, dz = (1+i)dr$

   $$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$$
   $$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$$
   $$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} -\sqrt{z}\sqrt{z-1-i}$$
   $$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$$

   $$f(z) \overset{Math.}{=} f(z)$$

   $$\int_{a_{31}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^1 \frac{1+i}{f(r(1+i))}dr$$

   (2) a_{32}^*: Let $z = r(1+i), r : 1 \to 0, dz = (1+i)dr$

   $$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$$
   $$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$$
   $$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$$
   $$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$$

   $$f(z) \overset{Math.}{=} -f(z)$$

   $$\int_{a_{32}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0 \frac{1+i}{f(r+ri)}dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_3}\frac{1}{f(z)}dz &= \int_{a_3^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} 2\int_0^1\frac{1+i}{f(r+ri)}dr\\ &= 1.27643+0.383835i\end{aligned}$$

4. b_3 : Consider the equivalent path $b_3^* = b_{31}^*\bigcup b_{32}^*$ where b_{31}^* = the path on the horizontal line i to $1+i$ on sheet-I and b_{32}^* = the path on the horizontal line from $1+i$ to i on sheet-II.

(1) b_{31}^*: Let $z=r+i, r:0\to 1, dz=dr$

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} \sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{31}^*}\frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0\frac{1}{f(r+i)}dr$$

(2) b_{32}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{32}^{**} = the path on the horizontal line from $1+i$ to i on sheet-I

Let $z=r+i, r:1\to 0, dz=dr$

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} \sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{32}^*}\frac{1}{f(z)}dz = -\int_{b_{32}^{**}}\frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0\frac{1}{f(r+i)}dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{b_3} \frac{1}{f(z)}dz &= \int_{b_3^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} 2\int_0^1 \frac{1}{f(r+i)}dr \\ &= 0.955186 - 0.204659i\end{aligned}$$

5. b_2 : Consider the equivalent path $b_2^* = b_3^* \bigcup b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup b_{32}^*$ where b_{21}^* = the path on the vertical line i to 0 on sheet-I. b_{22}^* = the path on the horizontal line 0 to 1 on sheet-I. and b_{23}^* = the path on the vertical line 1 to $1+i$ on sheet-II.

   (1) b_{21}^*: Let $z = ri$, $r : 1 \to 0$, $dz = idr$

   $\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} -\sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0 \frac{1}{f(ri)}dr$$

   (2) b_{22}^*:

   $\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^1 \frac{1}{f(z)}dz$$

   (3) b_{23}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{23}^{**} = the path on the vertical line 1 to $1+i$ on sheet-I

Let $z = 1 + ri$, $r : 0 \to 1$, $dz = idr$

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b_{23}^*} \frac{1}{f(z)}dz = -\int_{b_{23}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^1 \frac{i}{f(1+ri)}dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_1^0 \frac{1}{f(ri)}dr - \int_0^1 \frac{1}{f(r)}dr \\ &\quad + \int_0^1 \frac{i}{f(1+ri)}dr - \int_1^0 \frac{1}{f(r+i)}dr \\ &= 0.406494 + 0.556588i\end{aligned}$$

6. b_1 : Consider the equivalent path $b_1^* = b_{21}^* \bigcup b_{22}^* \bigcup a_{21}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{24}^*$ where b_{11}^* = the path on the horizontal line 2 to 3 on sheet-I, b_{12}^* = the path on the vertical cut 3 to $3+i$ on $(-)$edge of sheet-II and b_{13}^* = the path on the horizontal line $3+i$ to $1+i$ on sheet-II.

(1) b_{11}^*:

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b*_{11}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_2^3 \frac{1}{f(z)} dz$$

(2) $b^*_{12} \equiv$ the path on the vertical cut from 3 to $3+i$ on (+)edge of sheet-I. Let

$z = 3 + ri, r : 0 \to 1, dz = idr$

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} -f(z)$$

$$\int_{b*_{12}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_0^1 \frac{i}{f(3+ri)} dr$$

(3) b_{13} : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{13} = the path on the horizontal line $3+i$ to $1+i$ on sheet-I

Let $z = r + i$, $r : 3 \to 1$, $dz = dr$

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} \sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

$$f(z) \overset{Math.}{=} f(z)$$

$$\int_{b^*_{13}} \frac{1}{f(z)} dz = -\int_{b^*_{13}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_3^1 \frac{1}{f(r+i)} dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)} dz &= \int_{b_1^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_1^0 \frac{1}{f(ri)} dr - \int_0^1 \frac{1}{f(z)} dz \\
&\quad - \int_1^0 \frac{1}{f(r+i)} dr + \int_1^0 \frac{1}{f(ri)} dr - \int_2^3 \frac{1}{f(z)} dz \\
&\quad - \int_0^1 \frac{i}{f(3+ri)} dr - \int_3^1 \frac{1}{f(r+i)} dr \\
&= 0.14418 - 0.0612153i
\end{aligned}$$

In fact, we can draw the areas of modify together, the region which need to change the sign, maybe overlapping. Like the cut if cross even times then it will be no change. Otherwise, odd times will change sign. Now we draw the region where need to modify of $\sqrt{z-3-i}\sqrt{z-3+i}$, $\sqrt{z-2}\sqrt{z-1}$, $\sqrt{z}\sqrt{z-1-i}$ and $\sqrt{z-i}\sqrt{z-2i}$ into one figure, we called it "figure of modify". The region needs change sign of $\sqrt{z-3-i}\sqrt{z-3+i}$ is $\{z = x+iy : x < 3, -1 \leq y < 1\} \bigcup \{(+)\text{edge of } \overline{z_1 z_2}\}$. The region needs change sign of $\sqrt{z-2}\sqrt{z-1}$ is $\{(+)\text{edge of } \overline{z_3 z_4}\}$. The region needs change sign of $\sqrt{z}\sqrt{z-1-i}$ is $\{z = x+iy : x - y < 0, 0 \leq y < 1\} \bigcup \{(+)\text{edge of } \overline{z_5 z_6}\}$ and $\sqrt{z-i}\sqrt{z-2i}$ is $\{z = x+iy : x < 0, 1 \leq y < 2\} \bigcup \{(+)\text{edge of } \overline{z_7 z_8}\}$.

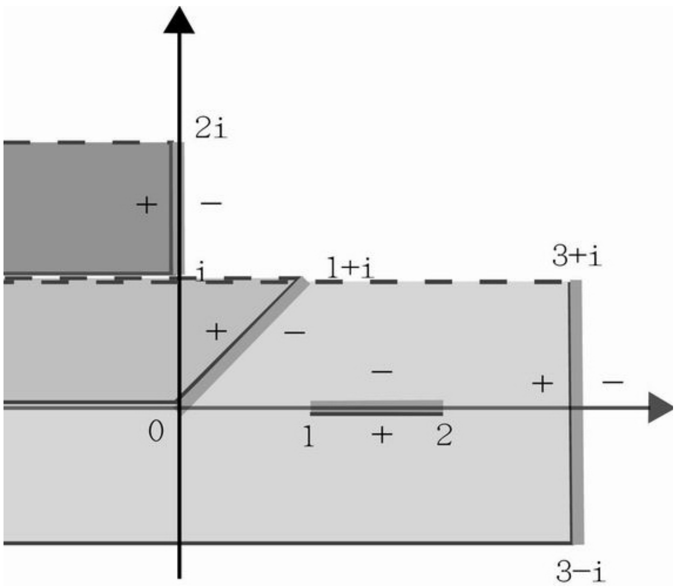

Figure 102: the regions need change sign of cuts

The region $\{z = x + iy : x - y < 0, 0 \leq y < 1\} \bigcup \{(+)\text{edge of } \overline{z_5 z_6}\}$ has twice change the sign, so no change here. We let (G) $= \{z = x + iy : x < 0, 1 \leq y < 2\} \bigcup \{(+)\text{edge of } \overline{z_7 z_8}\} \bigcup \{z = x + iy : x < 3, -1 \leq y < 1\} \bigcup \{(+)\text{edge of } \overline{z_1 z_2}\} \backslash \{z = x + iy : x - y < 0, 0 \leq y < 1\} \backslash \{(+)\text{edge of } \overline{z_5 z_6}\}$

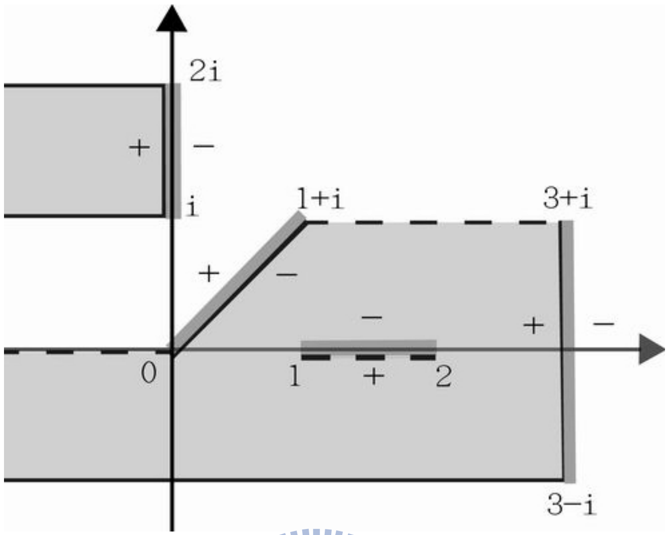

Figure 103: region of modify (G)

Solution:

1. a_1 : Consider the equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on the vertical cut from $3 + i$ to $3 - i$ on (+)edge of sheet-I and a_{12}^* = the path on the vertical cut from $3 - i$ to $3 + i$ on (−)edge of sheet-I.

   (1) $z \in a_{11}^*$: Let $z = 3 + ri, r : 1 \to -1, dz = idr$

   $z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result.

$$\int_{a_{11}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^{-1} \frac{i}{f(3 + ri)} dr$$

   (2) a_{12}^*: Let $z = 3 + ri, r : -1 \to 1, dz = idr$

   $z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result.

$$\int_{a_{12}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{-1}^{1} \frac{i}{f(3 + ri)} dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_1} \frac{1}{f(z)} dz &= \int_{a_1^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_{-1}^{1} \frac{i}{f(3+ri)} dr \\ &= -0.156969 + 0.225071i\end{aligned}$$

2. a_2 : Consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^* \subset R$ where a_{21}^* = the path on the horizon cut from 1 to 2 on (+)edge of sheet-I and a_{22}^* = the path on the horizon cut from 2 to 1 on (−)edge of sheet-I.

   (1) a_{21}^*: $z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result.

$$\int_{a_{21}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{1}^{2} \frac{1}{f(z)} dz$$

   (2) a_{22}^*: $z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result.

$$\int_{a*_{22}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{2}^{1} \frac{1}{f(z)} dz$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_2} \frac{1}{f(z)} dz &= \int_{a_2^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_{1}^{2} \frac{1}{f(z)} dz \\ &= 1.07276 - 0.117388i\end{aligned}$$

3. a_3 : Consider the equivalent path $a_3^* = a_{31}^* \bigcup a_{32}^*$ where a_{31}^* = the path on the slant cut from 0 to $1+i$ on (+)edge of sheet-I and a_{32}^* = the path on the slant cut from $1+i$ to 0 on (−)edge of sheet-I.

   (1) a_{31}^*: Let $z = r(1+i), r : 0 \to 1, dz = (1+i)dr$

   $z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result

$$\int_{a_{31}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{0}^{1} \frac{1+i}{f(r(1+i))} dr$$

(2) a_{32}^*: Let $z = r(1+i), r: 1 \to 0, dz = (1+i)dr$

$z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result

$$\int_{a_{32}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^0 \frac{1+i}{f(r+ri)} dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{a_3} \frac{1}{f(z)} dz &= \int_{a_3^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_0^1 \frac{1+i}{f(r+ri)} dr \\ &= 1.27643 + 0.383835i\end{aligned}$$

4. b_3 : Consider the equivalent path $b_3^* = b_{31}^* \bigcup b_{32}^*$ where b_{31}^* = the path on the horizontal line i to $1+i$ on sheet-I and b_{32}^* = the path on the horizontal line from $1+i$ to i on sheet-II.

(1) b_{31}^*: Let $z = r+i, r: 0 \to 1, dz = dr$

$z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result.

$$\int_{b_{31}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_1^0 \frac{1}{f(r+i)} dr$$

(2) b_{32}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{32}^{**} = the path on the horizontal line from $1+i$ to i on sheet-I.

Let $z = r+i, r: 1 \to 0, dz = dr$

$z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result.

$$\int_{b_{32}^*} \frac{1}{f(z)} dz = -\int_{b_{32}^{**}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^0 \frac{1}{f(r+i)} dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned}\int_{b_3} \frac{1}{f(z)} dz &= \int_{b_3^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} 2\int_0^1 \frac{1}{f(r+i)} dr \\ &= 0.955186 - 0.204659i\end{aligned}$$

5. b_2 : Consider the equivalent path $b_2^* = b_3^* \bigcup b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup b_{32}^*$ where b_{21}^* = the path on the vertical line i to 0 on sheet-I, b_{22}^* = the path on the horizontal line 0 to 1 on sheet-I and b_{23}^* = the path on the vertical line 1 to $1+i$ on sheet-II.

(1) b_{21}^*: Let $z = ri$, $r : 1 \to 0$, $dz = idr$

$z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result.

$$\int_{b_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0 \frac{1}{f(ri)}dr$$

(2) b_{22}^*: $z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result.

$$\int_{b_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^1 \frac{1}{f(z)}dz$$

(3) b_{23}^* : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{23}^{**} = the path on the vertical line 1 to $1+i$ on sheet-I.

Let $z = 1 + ri$, $r : 0 \to 1$, $dz = idr$

$z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result

$$\int_{b_{23}^*} \frac{1}{f(z)}dz = -\int_{b_{23}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^1 \frac{i}{f(1+ri)}dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_1^0 \frac{1}{f(ri)}dr - \int_0^1 \frac{1}{f(r)}dr \\
&\quad + \int_0^1 \frac{i}{f(1+ri)}dr - \int_1^0 \frac{1}{f(r+i)}dr \\
&= 0.406494 + 0.556588i
\end{aligned}$$

6. b_1 : Consider the equivalent path $b_1^* = b_{21}^* \bigcup b_{22}^* \bigcup a_{21}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{24}^*$where b_{11}^* = the path on the horizontal line 2 to 3 on sheet-I, b_{12}^* = the path on the vertical cut 3 to $3+i$ on $(-)$edge of sheet-II and b_{13}^* = the path on the horizontal line $3+i$ to $1+i$ on sheet-II.

(1) b^*_{11}: $z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result

$$\int_{b*11} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_2^3 \frac{1}{f(z)} dz$$

(2) $b^*_{12} \equiv$ the path on the vertical cut from 3 to $3+i$ on (+)edge of sheet-I

Let $z = 3 + ri, r : 0 \to 1, dz = idr$

$z \in (G)$ then $f(z) \overset{Math.}{=} -f(z)$ as same as above result

$$\int_{b*12} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_0^1 \frac{i}{f(3+ri)} dr$$

(3) b_{13} : We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b^{**}_{13} = the path on the horizontal line $3+i$ to $1+i$ on sheet-I

Let $z = r + i,\ r : 3 \to 1,\ dz = dr$

$z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$ as same as above result

$$\int_{b^*_{13}} \frac{1}{f(z)} dz = -\int_{b^*_{13}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_3^1 \frac{1}{f(r+i)} dr$$

By (1), (2), (3) and Cauchy Theorem

$$\begin{aligned} \int_{b_1} \frac{1}{f(z)} dz &= \int_{b^*_1} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_1^0 \frac{1}{f(ri)} dr - \int_0^1 \frac{1}{f(z)} dz \\ &\quad - \int_1^0 \frac{1}{f(r+i)} dr + \int_1^2 \frac{1}{f(r)} dr - \int_2^3 \frac{1}{f(z)} dz \\ &\quad - \int_0^1 \frac{i}{f(3+ri)} dr - \int_3^1 \frac{1}{f(r+i)} dr \\ &= 0.14418 - 0.0612153i \end{aligned}$$

Example 2. $f(z) = \sqrt{(z-3-i)(z-3+i)(z-2)(z-1)(z-0)(z-1-i)(z-i)(z-3i)}$. Evaluate $\int_\Gamma \frac{1}{f(z)} dz$. We let $z_1 = 3+i, z_2 = 3-i, z_3 = 2, z_4 = 1, z_5 = 0, z_6 = 1+i, z_7 = i, z_8 = 2i$.

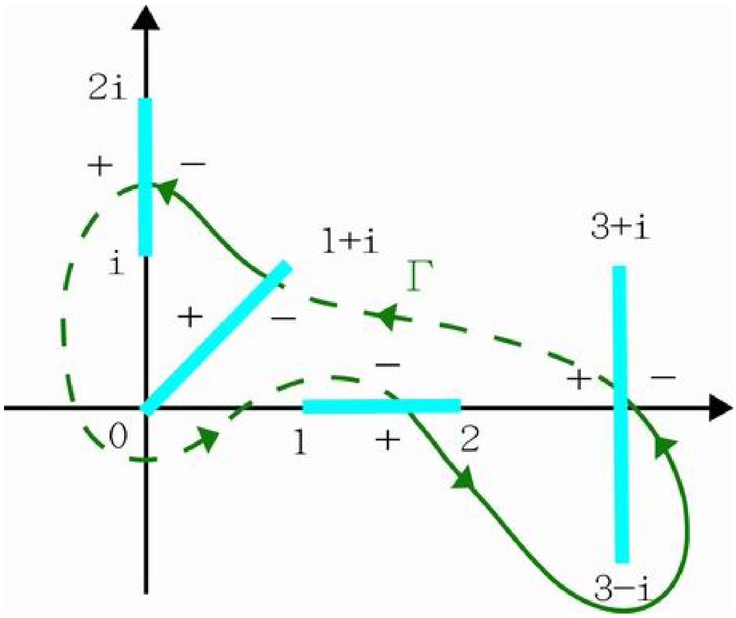

Figure 104: The curve Γ in the cut plane

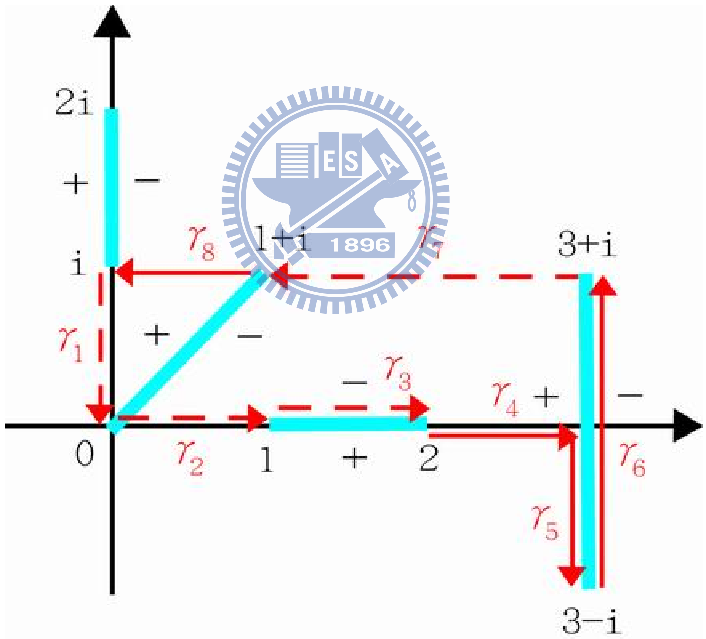

Figure 105: The equivalent path for Γ

Solve: Find the equivalent path which is easier write out its parameter type and using Mathematica obtain the value. $\Gamma \approx \bigcup_{k=1}^{8} \gamma_k$

1. γ_1 is the path on the vertical line from i to 0 on sheet-II. Since $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider a path on the vertical line from i to 0 on sheet-I, namely γ_1^*. Let $z = ri$, $r : 1 \to 0$, $dz = idr$

$\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

$\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

$\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} -\sqrt{z}\sqrt{z-1-i}$

$\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

From if z in γ_1^* then $f(z) \overset{Math.}{=} f(z)$, we have if z in γ_1 then

$$f(z) \overset{Math.}{=} -f(z)$$

2. γ_2 is the path on the horizontal line from 0 to 1 on sheet-II. Since $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider a path on the horizontal line from 0 to 1 on sheet-I, namely γ_2^*. Let $z = ri$, $r : 1 \to 0$, $dz = idr$

   $\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

   From if z in γ_2^* then $f(z) \overset{Math.}{=} -f(z)$, we have if z in γ_2 then

$$f(z) \overset{Math.}{=} f(z)$$

3. γ_3 is equiv to the path on the horizontal cut from 1 to 2 on (+)edge of sheet-I.

   $\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} -\sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

   So we have

$$f(z) \overset{Math.}{=} f(z)$$

4. γ_4 is the path on the horizontal line from 2 to 3 on sheet-I.

   $\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

   $\arg(z-z_k) \in (-\pi, \pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2,3,4,5,6,7,8

   So we have

$$f(z) \overset{Math.}{=} -f(z)$$

5. γ_5 is the path on the vertical cut from 3 to $3-i$ on (+)edge of sheet-I.

   Let $z = 3 + ri$, $r : 0 \to -1$, $dz = idr$

   $\arg(z-3+i) = -\frac{3}{2}\pi$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

   $\arg(z-z_k) \in (-\pi,\pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=2,3,4,5,6,7,8

   So we have

$$f(z) \overset{Math.}{=} -f(z)$$

6. γ_6 is the path on the vertical cut from $3-i$ to $3+i$ on $(-)$edge of sheet-I.

   Let $z = 3 + ri$, $r : -1 \to 1$, $dz = idr$

   $\arg(z-z_k) \in (-\pi,\pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=$1,\ldots,8$

   So we have

$$f(z) \overset{Math.}{=} f(z)$$

7. γ_7 is the path on the horizontal line from $3+i$ to $1+i$ on sheet-II. Since $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider a path on the horizontal line from i to 0 on sheet-I, namely γ_7^*. Let $z = r+i$, $r : 3 \to 1$, $dz = dr$ $\arg(z-3+i) \in (-\frac{3}{2}\pi, -\pi)$ then $\sqrt{z-3+i} \overset{Math.}{=} -\sqrt{z-3+i}$

   $\arg(z-3-i) = -\pi$ then $\sqrt{z-3-i} \overset{Math.}{=} -\sqrt{z-3-i}$

   $\arg(z-z_k) \in (-\pi,\pi)$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$, k=3,4,5,6,7,8

   From if z in γ_7^* then $f(z) \overset{Math.}{=} f(z)$, we have if z in γ_7 then

$$f(z) \overset{Math.}{=} -f(z)$$

8. γ_8 is the path on the vertical line from $1+i$ to i on sheet-I. Let $z = r+i$, $r : 1 \to 0$, $dz = dr$ $\sqrt{z-3-i}\sqrt{z-3+i} \overset{Math.}{=} \sqrt{z-3-i}\sqrt{z-3+i}$

   $\sqrt{z-2}\sqrt{z-1} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-1}$

   $\sqrt{z}\sqrt{z-1-i} \overset{Math.}{=} \sqrt{z}\sqrt{z-1-i}$

   $\sqrt{z-i}\sqrt{z-2i} \overset{Math.}{=} \sqrt{z-i}\sqrt{z-2i}$

   So we have

$$f(z) \overset{Math.}{=} f(z)$$

By (1), (2), ...,(8), we have

$$\begin{aligned}
\int_\Gamma \frac{1}{f(z)}dz &\overset{Math.}{=} -\int_1^0 \frac{r}{f(ri)}dr + \int_0^1 \frac{1}{f(r)}dr \\
&\quad -\int_2^1 \frac{1}{f(r)}dr - \int_3^2 \frac{1}{f(r)}dr \\
&\quad -\int_0^{-1} \frac{r}{f(3+ri)}dr + \int_{-1}^1 \frac{r}{f(3+ri)}dr \\
&\quad -\int_3^1 \frac{1}{f(r+i)}dr + \int_1^0 \frac{1}{f(r+i)}dr \\
&= -0.0980336 + 0.195762i
\end{aligned}$$

5 Applications of differential equations

Now discuss some example to apply above chapter of differential equations.

Example 1 : $u'' + \cos u = 0$ is a second order differential equation. We know that $\cos u \sim 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4$. Here we want to solve the equation,

$$u'' + 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 = 0.$$

So

$$\frac{1}{2}(u')^2 + u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 = E$$

where E is integral constant.

Let E=5, we have

$$\begin{aligned}(u')^2 &= 10 - 2u + \frac{2}{3!}u^3 - \frac{2}{5!}u^5 \\ &= R(u-4.54)(u-1.59-2.24i)(u-1.59+2.24i)(u+3.86-1.63i)(u+3.86+1.63i)\end{aligned}$$

R: constant. If $u' = \frac{du}{dt} = g(u)$ and then we have

$$\frac{1}{g(u)}du = dt$$

and

$$\int \frac{1}{g(u)}du = \int dt$$

So the function theory of solutions u of the equation involve $\sqrt{(u-4.54)(u-1.59-2.24i)(u-1.59+2.24i)(u+3.86-1.63i)(u+3.86+1.63i)}$ Let $f(u) = \sqrt{(u-4.54)(u-1.59-2.24i)(u-1.59+2.24i)(u+3.86-1.63i)(u+3.86+1.63i)}$ and compute $\int \frac{1}{f(u)}du$ over a,b cycles. Let $u_1 = 4.54$, $u_2 = 1.59 - 2.24i$, $u_3 = 1.59 + 2.24i$, $u_4 = -3.86 - 1.63i$, $u_5 = -3.86 + 1.63i$

Using region of modify to get result by Mathematica:

The region needs change sign of $\sqrt{u-4.54}$ is $\{u = x+iy : x < 4.54, 0 \le y\} \bigcup \{(+)\text{edge of the cut } [4.54, \infty)\}$. The region needs change sign of $\sqrt{u-1.59-2.24i}\sqrt{u-1.59+2.24i}$ is $\{u = x+iy : x < 1.59, -2.24 \le y < 2.24\} \bigcup \{(+)$edge of the cut $[1.59 - 2.24i, 1.59 +$

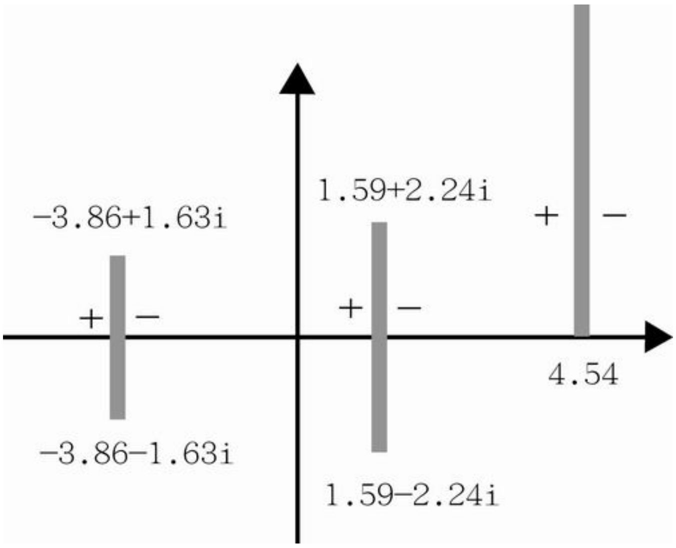

Figure 106: cut plane of f(u)

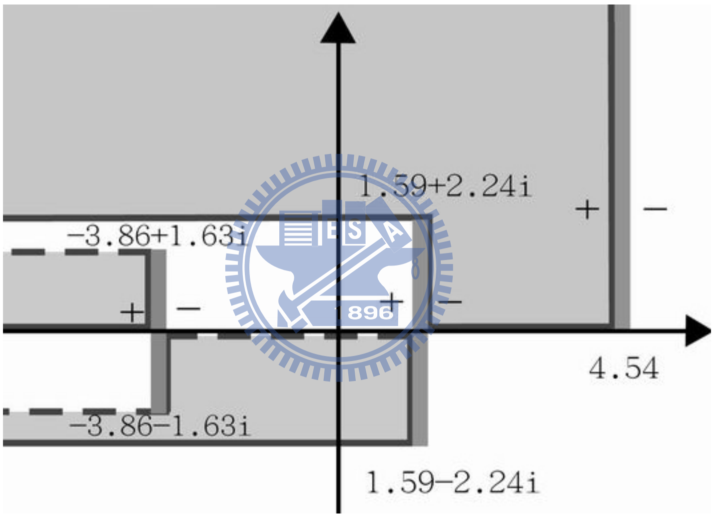

Figure 107: region of modify

$2.24i]\}$ and the region needs change sign of $\sqrt{u+3.86+1.63i}\sqrt{u+3.86-1.63i}$ is $\{u = x+iy : x < -3.86, -1.63 \leq y < 1.63\} \bigcup \{(+)$edge of the cut $[-3.86-1.63i, -3.86+1.63i]\}$. We let the region needs change sign of f(z) is (G)=$\{u = x+iy : x < 4.54, 2.24 \leq y\} \bigcup \{u = x+iy : 1.59 < x < 4.54, 2.24 \leq y\} \bigcup \{u = x+iy : -3.86 < x < 1.59, -2.24 \leq y < 2.24\}$ $\bigcup \{u = x+iy : x < -3.86, -2.24 \leq y < -1.63\} \bigcup \{u = x+iy : x < -3.86, 0 \leq y < 1.63\}$ $\bigcup \{(+)$edge of the cut $[4.54, \infty)\} \bigcup \{(+)$edge of the cut $[1.59-2.24i, 0]\}$ $\bigcup \{(-)$edge of the cut $[0, 1.59+2.24i]\}$ $\bigcup \{(+)$edge of the cut $[0, -3.86+1.63i]\}$ $\bigcup \{(-)$edge of the cut $[0, -3.86+1.63i]\}$

Solution:

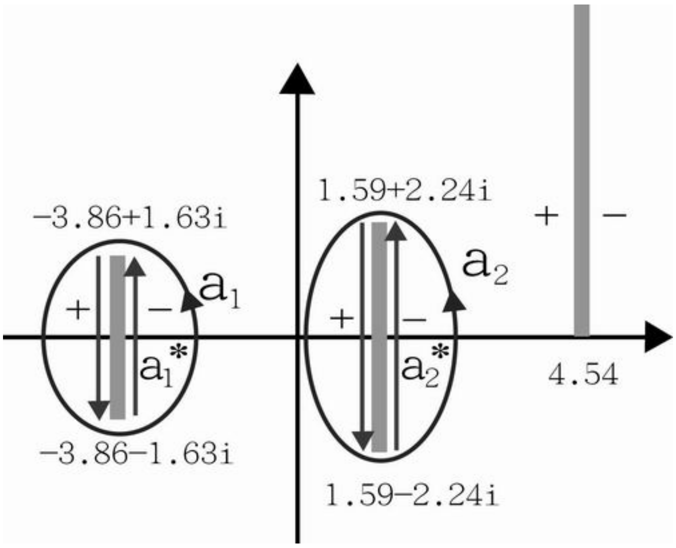

Figure 108: curve a_1, a_2 and their equivalent path

1. a_1 : Consider the equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^* \bigcup a_{13}^* \bigcup a_{14}^*$ a_{11}^* = the path along vertical cut from $-3.86 + 1.63i$ to 0 on (+)edge of sheet-I, a_{12}^* = the path along vertical cut from 0 to $-3.86 - 1.63i$ on (+)edge of sheet-I, a_{13}^* = the path along vertical cut from $-3.86 - 1.63i$ to 0 on (−)edge of sheet-I and a_{14}^* = the path along vertical cut from 0 to $-3.86 + 1.63i$ on (−)edge of sheet-I.

   (1) a_{11}^* : Let $u = -3.86 + ri, du = idr, r : 1.63 \to 0$

   $z \in (G)$ then $f(u) \overset{Math.}{=} -f(u)$

   $$\int_{a_{11}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{1.63}^{0} \frac{i}{f(-3.86 + ri)} dr$$

   (2) a_{12}^*: Let $u = -3.86 + ri, du = idr, r : 0 \to -1.63$

   $u \in C\backslash(G)$ then $f(u) \overset{Math.}{=} f(u)$

   $$\int_{a_{12}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{0}^{-1.63} \frac{i}{f(-3.86 + ri)} dr$$

   (3) a_{13}^*: Let $u = -3.86 + ri, du = idr, r : -1.63 \to 0$ $u \in (G)$ then $f(u) \overset{Math.}{=} -f(u)$

   $$\int_{a_{13}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.63}^{0} \frac{i}{f(-3.86 + ri)} dr$$

(4) a_{14}^*: Let $u = -3.86 + ri,\ du = idr\ ,\ r : 0 \to 1.63$

$z \in C\backslash(G)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{11}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_0^{1.63} \frac{i}{f(-3.86 + ri)} dr$$

By (1),(2),(3),(4) and Cauchy Theorem

$$\begin{aligned}\int_{a_1^*} \frac{1}{f(u)} du &\overset{Math.}{=} 2\int_0^{1.63} \frac{i}{f(-3.86 + ri)} dr - 2\int_{-1.63}^{0} \frac{i}{f(-3.86 + ri)} dr \\ &= 0.35043 - 5.55112 \times 10^{-17} i\end{aligned}$$

2. a_2: Consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^* \bigcup a_{23}^* \bigcup a_{24}^*$ a_{21}^* = the path along vertical cut from $1.59+2.24i$ to 0 on (+)edge of sheet-I, a_{22}^* = the path along vertical cut from 0 to $1.59 - 2.24i$ on (+)edge of sheet-I, a_{23}^* = the path along vertical cut from $1.59 - 2.24i$ to 0 on (−)edge of sheet-I and a_{24}^* = the path along vertical cut from 0 to $1.59 + 2.24i$ on (−)edge of sheet-I.

(1) a_{21}^* : Let $u = 1.59 + ri,\ du = idr,\ r : 2.24 \to 0$

$u \in C\backslash(G)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{11}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{2.24}^{0} \frac{i}{f(1.59 + ri)} dr$$

(2) a_{22}^*: Let $u = 1.59 + ri,\ dz = idr,\ r : 0 \to -2.24$

$u \in (G)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{22}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_0^{-2.24} \frac{i}{f(1.59 + ri)} dr$$

(3) a_{23}^*: Let $z = 1.59 + ri,\ du = idr,\ r : -2.24 \to 0$

$z \in C\backslash(G)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{a_{23}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-2.24}^{0} \frac{i}{f(1.59 + ri)} dr$$

(4) a_{24}^*: Let $u = 1.59 + ri,\ dz = idr,\ r : 0 \to 2.24$

$u \in (G)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{24}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_0^{2.24} \frac{i}{f(1.59 + ri)} dr$$

By (1),(2),(3),(4) and Cauchy Theorem

$$\int_{a_2^*} \frac{1}{f(u)}du \overset{Math.}{=} -2\int_0^{2.24} \frac{i}{f(1.59+ri)}dr + 2\int_{2.24}^{0} \frac{i}{f(1.59+ri)}dr$$
$$= -0.587776 + 2.77556\times 10^{-17} i$$

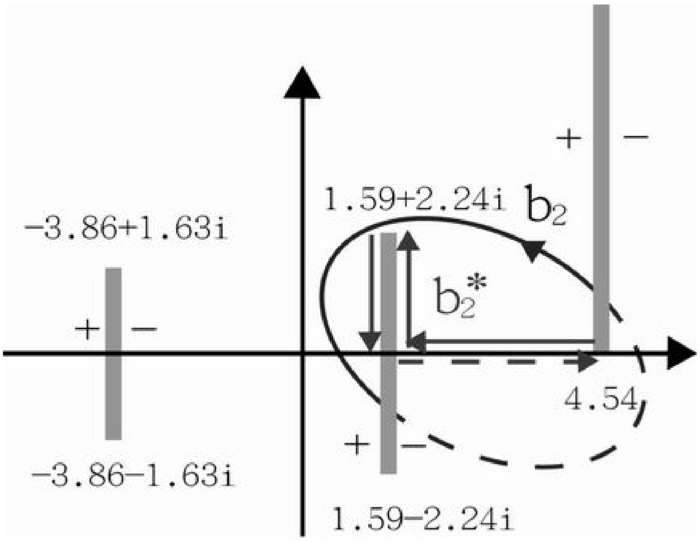

Figure 109: curve b_2 and its equivalent path

3. b_2: Consider the equivalent path $b_2^* = b_2^* = b_{21}^* \bigcup a_{24}^* \bigcup a_{21}^* \bigcup b_{22}^*$ where

   b_{21}^* = the path along horizon line from 4.54 to 1.59 on sheet-I and

   b_{22}^* = the path along horizon line from 1.59 to 4.54 on sheet-II.

   (1) b_{21}^*: $z \in (G)$ then $f(u) \overset{Math.}{=} -f(u)$

   $$\int_{b_{21}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{4.54}^{1.59} \frac{1}{f(u)}du$$

   (2) b_{22}^*: We known that $f(u)|_{(I)} = -f(u)|_{(II)}$, so we can consider b_{22}^{**} = the path along horizon line from 1.59 to 4.54 on sheet-I

   $u \in (G)$ then $f(u) \overset{Math.}{=} -f(u)$

   $$\int_{b_{22}^*} \frac{1}{f(u)}du = -\int_{b_{22}^{**}} \frac{1}{f(u)}dz \overset{Math.}{=} \int_{1.59}^{4.54} \frac{1}{f(u)}du$$

   By (1),(2) and Cauchy Theorem

   $$\int_{b_2^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_0^{2.24} \frac{i}{f(1.59+ri)}dr - 2\int_{4.54}^{1.59} \frac{1}{f(u)}du$$
   $$= -0.293888 - 0.170468i$$

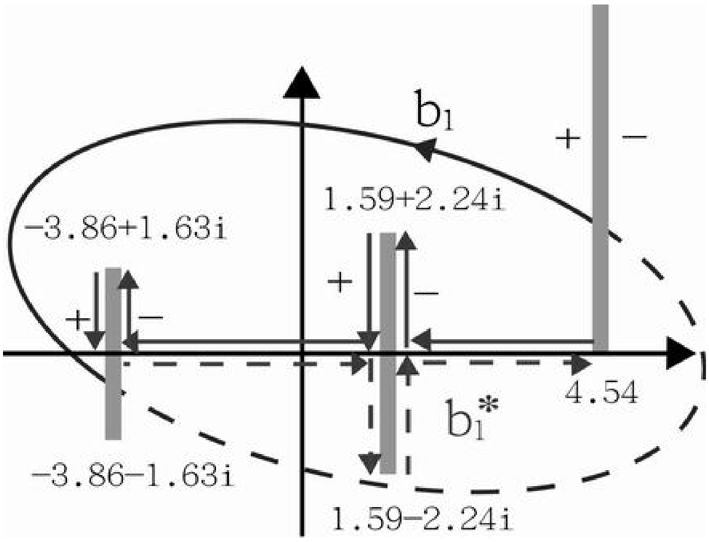

Figure 110: curve b_1 and its equivalent path

4. b_1 : Consider the equivalent path

$b_1^* = b_{21}^* \bigcup a_{24}^* \bigcup a_{21}^* \bigcup b_{11}^* \bigcup a_{14}^* \bigcup a_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^* \bigcup b_{22}^*$ where

$b_{11}^* =$ the path along horizontal line from 1.59 to -3.86 on sheet-I,

$b_{12}^* =$ the path along horizontal line from -3.86 to 1.59 on sheet-II,

$b_{13}^* =$ the path along vertical cut from 0 to $1.59 - 2.24i$ on $(+)$edge of sheet-II and

$b_{14}^* =$ the path along vertical cut from $1.59 - 2.24i$ to 0 on $(-)$edge of sheet-II.

(1) b_{11}^*: $z \in C\backslash(G)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b_{11}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{1.59}^{-3.86} \frac{1}{f(u)} du$$

(2) b_{12}^*: We known that $f(u)|_{(I)} = -f(u)|_{(II)}$, so we can consider $b_{12}^{**} =$ the path along horizon line from -3.86 to 1.59 on sheet-I

$z \in C\backslash(G)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b_{12}^*} \frac{1}{f(u)} du = -\int_{b_{12}^{**}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-3.86}^{1.59} \frac{1}{f(u)} du$$

(3) $b_{13}^* \equiv$ the path along vertical cut from 0 to $1.59 - 2.24i$ on $(-)$edge of sheet-I

$b_{14}^* \equiv$ the path along vertical cut from $1.59 - 2.24i$ to 0 on $(+)$edge of sheet-I

So

$$\int_{b_{13}^* \bigcup b_{14}^*} \frac{1}{f(u)} du = -\int_{a_{22}^* \bigcup a_{23}^*} \frac{1}{f(u)} du$$

By (1), (2), (3) and Cauchy Theorem

$$\int_{b_1^*} \frac{1}{f(u)} du \overset{Math.}{=} -2\int_{1.59}^{-3.86} \frac{1}{f(u)} du + 2\int_{-2.24}^{0} \frac{i}{f(1.59+ri)} dr$$
$$= 0.175215 - 0.12607i$$

Example 2 :

$u'' - 3u^5 + (\frac{35}{2} - 5i)u^4 - (20-36i)u^3 - (27+69i)u^2 + (47+6i)u - \frac{13}{2} + 36i = 0$. We have $(u')^2 - \frac{u^6}{2} + (\frac{7}{2} - i)u^5 - (5-9i)u^4 - (9+23i)u^3 + (\frac{47}{2} + 3i)u^2 - (\frac{13}{2} - 36i)u = E$ Let $F(u) = -\frac{u^6}{2} + (\frac{7}{2} - i)u^5 - (5-9i)u^4 - (9+23i)u^3 + (\frac{47}{2} + 3i)u^2 - (\frac{13}{2} - 36i)u$. Take E=60 and solve this second order equation.

So $\int \frac{1}{\sqrt{E-2F(u)}} du = \int \frac{1}{\sqrt{(u+1)(u+1+2i)(u+i)(u-2-i)(u-3)(u-4)}} du$

Let $g(u) = \sqrt{(u+1)(u+1+2i)(u+i)(u-2-i)(u-3)(u-4)}$. Compute $\int \frac{1}{g(u)} du$ over a,b cycles.

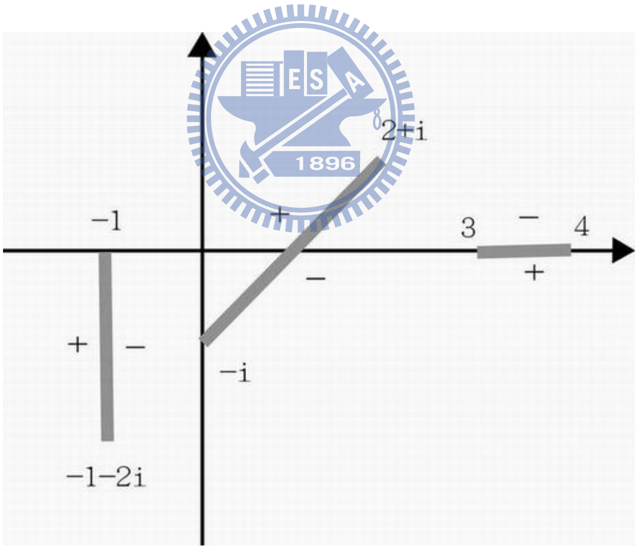

Figure 111: The cut plane

Using region of modify to get result by Mathematica. The region needs change sign of $\sqrt{u-3}\sqrt{u-4}$ is {(+)edge of the cut $[3,4]$}. The region needs change sign of $\sqrt{u+i}\sqrt{u-2-i}$ is (A)$\bigcup$(B)= $\{u = x+iy : x-y-1<0, -1 \le y < 1\} \bigcup$ {(+)edge of the cut $[-i, 1+2i]$} and the region needs change sign of $\sqrt{u+1}\sqrt{u+1+2i}$ is (B)$\bigcup$(C)= $\{u = x+iy : x < -1, -2 \le y < 0\} \bigcup$ {(+)edge of the cut $[-1, -1-2i]$}. Let (A)= $\{u = x+iy : x-y-1<0, -1 \le y < 1\} \bigcup$ {(+)edge of the cut $[-i, 1+2i]$}$\backslash \{u =$

$x + iy : x < -1, -1 \leq y < 0\} \backslash \{(+)$edge of the cut $[-1, -1-i]\}$, (B)=$\{u = x + iy : x < -1, -1 \leq y < 0\} \bigcup \{(+)$edge of the cut $[-1, -1-i]\}$ and (C)=$\{u = x+iy : x < -1, -2 \leq y < -1\} \bigcup \{(+)$edge of the cut $[-1, -1-i]\}$ So (B) does "Not" change the sign. We let the region needs change sign of g(u) is (G)=(A)$\bigcup$(C)$\bigcup\{(+)$edge of the cut $[3, 4]\}$

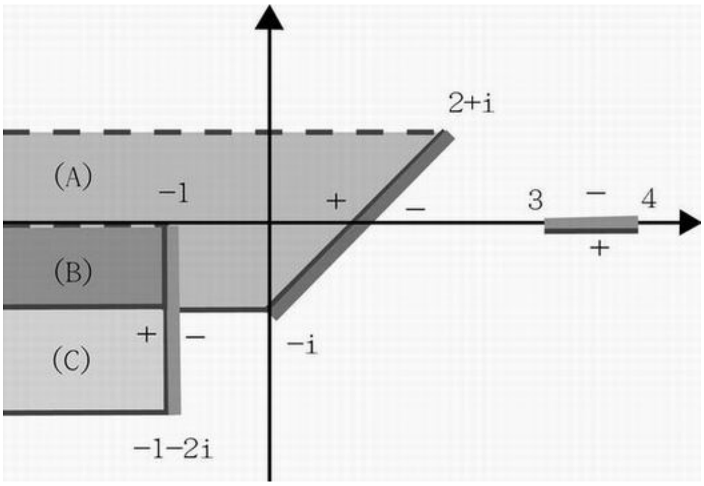

Figure 112: region of modify

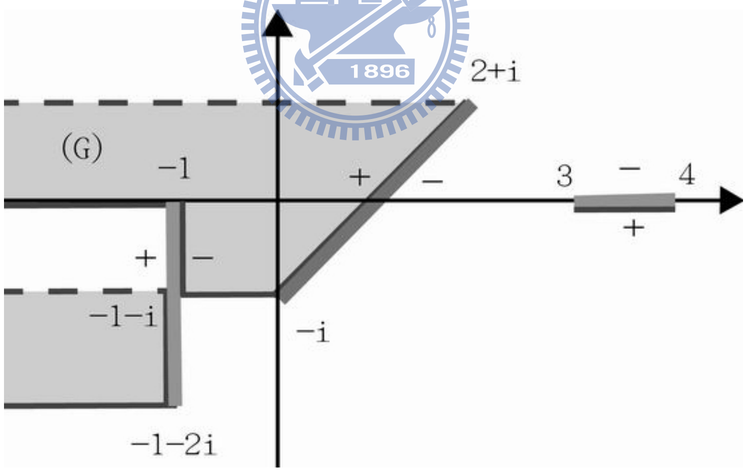

Figure 113: region of modify

Solution:

1. a_1 : Consider the equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on the horizontal cut from 3 to 4 on $(+)$edge of sheet-I and a_{12}^* = the path on the horizontal cut from 4 to 3 on $(-)$edge of sheet-I

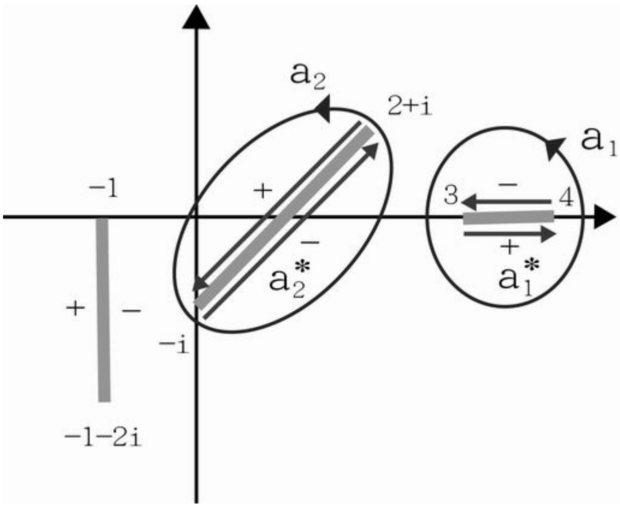

Figure 114: curve a_1, a_2 and their equivalent path

(1) $u \in a_{11}^*$: $u \in (G)$ then $g(u) \overset{Math.}{=} -g(u)$

$$\int_{a_{11}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_3^4 \frac{1}{g(u)} du$$

(2) a_{12}^*: $u \in C \backslash (G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{a_{12}^*} \frac{1}{g(u)} du \overset{Math.}{=} \int_4^3 \frac{1}{g(u)} du$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned} \int_{a_1} \frac{1}{g(u)} du &= \int_{a_1^*} \frac{1}{g(u)} du \\ &\overset{Math.}{=} 2\int_4^3 \frac{1}{g(u)} du \\ &= 0.022345 + 0.535123i \end{aligned}$$

2. a_2 : Consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^*$ where a_{21}^* = the path on the slant cut from $2+i$ to $-i$ on $(+)$edge of sheet-I and a_{22}^* = the path on the slant cut from $-i$ to $2+i$ on $(-)$edge of sheet-I

   (1) a_{21}^*: Let $u = -i + r(1+i)$, $r : 2 \to 0$, $du = (1+i)dr$

   $u \in (G)$ then $g(u) \overset{Math.}{=} -g(u)$

$$\int_{a_{21}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_2^0 \frac{1+i}{g(-i+r(1+i))} dr$$

(2) a^*_{22}: Let $u = -i + r(1+i)$, $r : 0 \to 2$, $du = (1+i)dr$

$u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{a*22} \frac{1}{g(u)} du \overset{Math.}{=} \int_0^2 \frac{1+i}{g(-i+r(1+i))} dr$$

By (1), (2) and Cauchy Theorem

$$\begin{aligned} \int_{a_2} \frac{1}{g(u)} du &= \int_{a_2^*} \frac{1}{g(u)} du \\ &\overset{Math.}{=} 2\int_0^2 \frac{1+i}{g(-i+r(1+i))} dr \\ &= -0.273396 - 1.04546i \end{aligned}$$

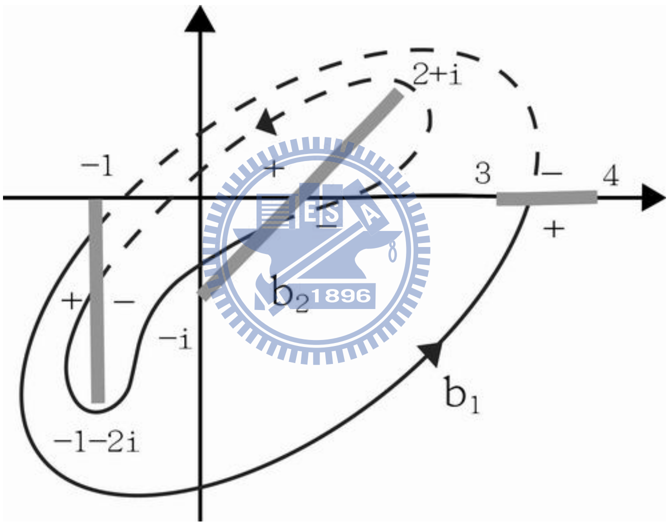

Figure 115: equivalent path b_2^* for b_2

3. b_2 : Consider the equivalent path b_2^* and separate it into many paths by figure of modify $b_2^* = b^*_{21} \bigcup b^*_{22} \bigcup b^*_{23} \bigcup b^*_{24} \bigcup b^*_{25} \bigcup b^*_{26}$ where b^*_{21} = the path on the vertical cut $-1-i$ to $-1-2i$ on (+)edge of sheet-I, b^*_{22} = the path on the vertical cut $-1-2i$ to $-1-i$ on (−)edge of sheet-I, b^*_{23} = the path on the horizontal line $-1-i$ to $-i$ on sheet-I, b^*_{24} = the path on the slant cut $-i$ to $2+i$ on (−)edge of sheet-II, b^*_{25} = the path on the slant cut $2+i$ to $-i$ on (+)edge of sheet-II and b^*_{26} = the path on the horizontal line $-i$ to $-1-i$ on sheet-II.

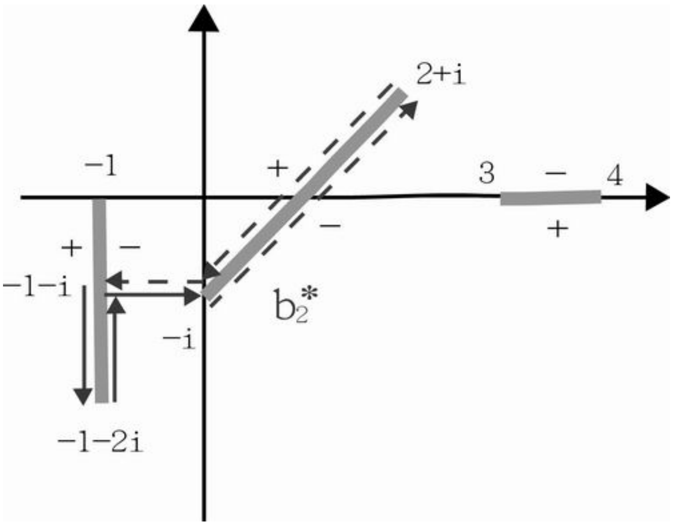

Figure 116: equivalent path b_2^* for b_2

(1) b_{21}^*: Let $u = -1 + ri$, $r : -1 \to -2$, $du = idr$ $u \in (G)$ then $g(u) \overset{Math.}{=} -g(u)$

$$\int_{b_{21}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_{-1}^{-2} \frac{i}{g(-1+ri)} dr$$

(2) b_{22}^*: Let $u = -1+ri$, $r : -2 \to -1$, $\Rightarrow du = idr$ $u \in C \backslash (G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{22}^*} \frac{1}{g(u)} du \overset{Math.}{=} \int_{-2}^{-1} \frac{i}{g(-1+ri)} dr$$

(3) b_{23}^*: Let $u = r - i$, $r : -1 \to 0$, $du = dr$ $u \in (G)$ then $g(u) \overset{Math.}{=} -g(u)$

$$\int_{b_{23}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_{-1}^{0} \frac{1}{g(r-i)} dr$$

(4) $b_{24}^* \equiv -a_{21}^*$ the path on the slant cut $-i$ to $2+i$ on $(+)$edge of sheet-I and $b_{25}^* \equiv -a_{22}^*$ the path on the slant cut $2+i$ to $-i$ on $(-)$edge of sheet-I

$$\int_{b_{23}^* \bigcup b_{24}^*} \frac{1}{g(u)} du = -\int_{a_2^*} \frac{1}{g(u)} du$$

(5) b_{26}^*: We known that $f(z)|_{(I)} = -f(z)|_{(II)}$, so we can consider b_{26}^{**} = the path on the horizontal line $-i$ to $-1-i$ on sheet-I

Let $u = r - i$, $r : 0 \to -1$, $du = dr$ $u \in (G)$ then $g(u) \overset{Math.}{=} -g(u)$

$$\int_{b_{26}^*} \frac{1}{g(u)} du - \int_{b_{26}^{**}} \frac{1}{g(u)} du \overset{Math.}{=} \int_{0}^{-1} \frac{1}{g(r-i)} dr$$

By (1), (2), (3), (4), (5) and Cauchy Theorem

$$\int_{b_2} \frac{1}{f(z)} dz = \int_{b_2^*} \frac{1}{f(z)} dz$$
$$\overset{Math.}{=} 2\int_{-2}^{-1} \frac{i}{g(-1+ri)} dr - 2\int_{-1}^{0} \frac{1}{g(r-i)} dr$$
$$-2\int_{0}^{2} \frac{1+i}{g(-i+r(1+i))} dr =$$

$0.902328 + 0.939835i$

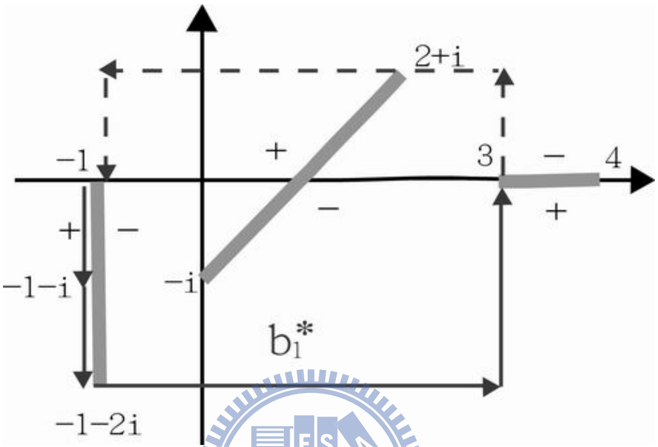

Figure 117: equivalent path b_1^* for b_1

4. b_1 : Consider the equivalent path $b_1^* = b_{21}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^* \bigcup b_{15}^* \bigcup b_{16}^*$ where b_{11}^* = the path on the horizontal line from $-1-2i$ to $3-2i$ on sheet-I, b_{12}^* = the path on the vertical line from $3-2i$ to 3 on sheet-I, b_{13}^* = the path on the vertical line from 3 to $3+i$ on sheet-II, b_{14}^* = the path on the horizontal line from $3+i$ to $-1+i$ on sheet-II, b_{15}^* = the path on the vertical cut from $-1+i$ to -1 on sheet-II and b_{16}^* = the path on the vertical cut from -1 to $-1-i$ on (+)edge of sheet-I.

(1) b_{11}^*: Let $u = r - 2i$, $r : -1 \to 3$, $du = dr$ $u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{11}^*} \frac{1}{g(u)} du \overset{Math.}{=} \int_{-1}^{3} \frac{1}{g(r)} dr$$

(2) b_{12}^*: Let $u = 3 + ri, r : -2 \to 0, du = idr$ $u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{12}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_{-2}^{0} \frac{i}{g(3+ri)} dr$$

(3) b_{13}^*: We known that $g(u)|_{(II)} = -g(u)|_{(I)}$, so we can consider b_{13}^{**} = the path on the vertical line from 3 to $3+i$ on sheet-I Let $u = 3+ri, r: 0 \to 1, dz = idr$ $u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{13}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_{b_{13}^{**}} \frac{1}{g(u)} du \overset{Math.}{=} -\int_0^1 \frac{i}{g(3+ri)} dr$$

(4) b_{14}^*: We known that $g(u)|_{(II)} = -g(u)|_{(I)}$, so we can consider b_{14}^{**} = the path on the horizontal line from $3+i$ to $-1+i$ on sheet-I

Let $u = r+i, r: 3 \to -1, du = dr$ $u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{14}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_{b_{14}^{**}} \frac{1}{g(u)} du \overset{Math.}{=} -\int_3^{-1} \frac{1}{g(r+i)} dr$$

(5) b_{15}^*: We known that $g(u)|_{(II)} = -g(u)|_{(I)}$, so we can consider b_{15}^{**} = the path on the vertical cut from $-1+i$ to -1 on sheet-I

Let $u = -1+ri, r: 1 \to 0, du = idr$ $u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{15}^*} \frac{1}{g(u)} du \overset{Math.}{=} -\int_{b_{15}^{**}} \frac{1}{g(u)} du \overset{Math.}{=} -\int_1^0 \frac{i}{g(-1+ri)} dr$$

(6) b_{16}^*: Let $u = -1+ri, r: 0 \to -1, du = idr$

$u \in C\backslash(G)$ then $g(u) \overset{Math.}{=} g(u)$

$$\int_{b_{16}^*} \frac{1}{g(u)} du \overset{Math.}{=} \int_0^{-1} \frac{i}{g(-1+ri)} dr$$

By (1), (2), (3), (4), (5), (6) and Cauchy Theorem

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)} dz &= \int_{b_1^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_{-1}^{-2} \frac{i}{g(-1+ri)} dr + \int_{-1}^{3} \frac{1}{g(r)} dr - \int_{-2}^{0} \frac{i}{g(3+ri)} dr \\
&\quad -\int_0^1 \frac{i}{g(3+ri)} dr - \int_3^{-1} \frac{1}{g(r+i)} dr \\
&\quad -\int_1^0 \frac{i}{g(-1+ri)} dr + \int_0^{-1} \frac{i}{g(-1+ri)} dr \\
&= -0.0494252 + 0.215845i
\end{aligned}$$

6 Conclusion

In this paper, we construct Riemann surfaces and use Mathematica to compute the integrals over a, b cycles of the algebraic curve of $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$. First, when the cut on the line and the slope of line is $m = \tan\alpha$, $0 < \alpha \leq \pi$. Define that

$$z = \begin{cases} re^{i\theta}, \theta \in [\alpha - 2\pi, \alpha) & \text{if z in sheet-I} \\ re^{i\theta}, \theta \in [\alpha, \alpha + 2\pi) & \text{if z in sheet-II} \end{cases}$$

the cut in each sheet has two edges, label the edge of the left of cut with "+" and the right of cut with "−". To get right value of integrals, we conclude the way to modify the computation to get right value. If the cut with z_k on the line where the slope of line is $m = \tan\alpha$, $0 < \alpha \leq \pi$. If z in sheet-I then

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} \sqrt{z - z_k} & \text{if } \arg(z - z_k) \in [\alpha - 2\pi, -\pi], \\ -\sqrt{z - z_k} & \text{if } \arg(z - z_k) \in (-\pi, \alpha) \end{cases}$$

Under above basis, we can use the endpoints of cuts to help us determine how to modify. If the endpoints of cut is z_k and z_{k+1} and the slope of line is $m = \tan\alpha$, $0 < \alpha \leq \pi$ where the cut on. If z in sheet-I then the area of modify with $\sqrt{z - z_k}\sqrt{z - z_{k+1}}$ is $(B) = \{z = x + yi : (x, y) \in C \text{ and } y_{2j-1} \leq y < y_{2j}\} \bigcup \{$ the cut of (+) edge of cut $[z_k, z_{k+1}]\}$ where C is a condition (x, y) on the left of the line which we defined above. Then

$$\sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \begin{cases} \sqrt{z - z_k}\sqrt{z - z_{k+1}} & \text{if } z \in \mathbb{C}\backslash(B), \\ -\sqrt{z - z_k}\sqrt{z - z_{k+1}} & \text{if } z \in (B) \end{cases}$$

We also can draw the region of modification on domain to determine.

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