國立交通大學

應用數學系

碩 士 論 文

不可約非負矩陣相似至正矩陣的問題探討

Irreducible nonnegative matrices that are similar to positive matrices

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中 華 民 國 一〇〇 年 七 月

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摘　要

我們所要探討的是不可約非負矩陣相似至正矩陣的問題，我們將探討二個主要問題。一是對於 n 階不可約非負矩陣且有 n+2 個零元素和對角線元素皆相異的條件之下，會相似至正矩陣。二是對於 n 階不可約非負矩陣且有 n+2 個非零元素的條件之下，會相似至正矩陣。

Irreducible nonnegative matrices that are similar to positive matrices

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ABSTRACT

We study the questions in which the irreducible nonnegative matrices are similar to positive matrices.We will solve the problems for n×n irreducible nonnegative matrices with exactly n+2 zeros, and all the entries of diagonal are distinct, and for n×n irreducible nonnegative matrices with exactly n+2 nonzero elements are similar to positive matrices.

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目　　　錄

1 Introduction

For a matrix $A = [a_{ij}]_{i,j=1}^{n}$, we say that A is **nonnegative** is all its entries a_{ij} are nonnegative. We say that A is **positive** ($A > 0$) if all its entries a_{ij} are positive. A nonnegative matrix A is called **reducible** matrix if there is a permutation matrix Q such that

$$Q^T A Q = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where B, C are square matrices, and it is called **irreducible** if it is not reducible.

We are interested in irreducible nonnegative matrices that are similar to positive matrices. This question was first studied in the paper of Borobia and Moro[1], where they obtained several results on the problem. Recently, in [2], Laffey, Loewy, and Šmigoc proved two main theorems:

LLS 1. Let A be an $n \times n, n \geq 4$, irreducible nonnegative matrix with exactly n zero elements. Then either A has trace zero or it is similar to a positive matrix.

LLS 2. Every irreducible $n \times n$ nonnegative matrix A with positive trace and exactly $n+1$ zeros is similar to a positive matrix for all $n \geq 3$.

In this paper, we consider the following problems:
Question 1. For $n \geq 4$, is every irreducible $n \times n$ nonnegative matrix A with positive trace and exactly $n+2$ zeros similar to a positive matrix ?
Question 2. Is every irreducible $n \times n$ nonnegative matrix A with positive trace and exactly $n+1$ or $n+2$ nonzeros similar to a positive matrix ?
In Question 2, if "exactly $n+1$ or $n+2$ nonzeros$''$ replaces by "exactly n+3 nonzeros$''$, then the answer is false.

Counterexample [1, Theorem 5]:
For $a \in (0, 1)$, $\begin{pmatrix} a & 0 & 1-a \\ 1-a & a & 0 \\ 0 & 1-a & a \end{pmatrix}$ is not similar to a positive matrix.

In section 3, we prove that for $n \geq 4$ and $n \neq 5$, if A is an $n \times n$ irreducible nonnegative matrix with exactly $n+2$ zeros and distinct diagonal entries, then A is similar to a positive matrix. In section 4, we solve Question 2.

2 Priliminaries

In the section, we list some Lemma, which will be used in the proofs of our main result. The following lemmas appeared in [2, Corollary 3.4, and Lemma 3.5].

Lemma 2.1 *Let A be an irreducible matrix with a positive row or column. Then A is similar to a positive matrix.*

Lemma 2.2 *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a nonnegative matrix such that A_{11} is similar to a positive matrix, $A_{12} > 0$, and $A_{21} > 0$. Then A is similar to a positive matrix.

Let P be a nonnegative pattern matrix (i.e. a matrix whose entries are either zeros or positive entries), and let $P^i, P^j (P_i, P_j)$ be two columns(rows) of P. We say that P^j *dominates* P^i(P_j *dominates* P_i), and we write $P^j \geq P^i (P_j \geq P_i)$, if $P^j(P_j)$ has positive entries in at least the same positions as $P^i(P_i)$. We denote by $P^i + P^j (P_i + P_j)$ the pattern column with stars in all positions expect in those where both P^i and P^j(P_i and P_j) have zeros.

The following two lemmas were proved in [1,Theorems 2 and 4]. We need them to derive our main results. For the sake of completion, we list their proofs below.

Lemma 2.3 *Let P be an irreducible pattern matrix with rows $P_1, \ldots, P_n$ and columns $P^1, \ldots, P^n$, and suppose $P^i \leq P^j (P_i \leq P_j)$ for a certain pair of indices i, j. Then any nonnegative matrix with pattern P is similar to a nonnegative irreducible matrix with pattern Q, where $Q_i = P_i + P_j$ and $Q_k = P_k$ for $k \neq i$ ($Q^i = P^i + P^j$ and $Q^k = P^k$ for $k \neq i$).*

Proof. Suppose that $i = 1, j = 2$, and $P^1 \leq P^2$(the proof in the row case is completely analogous). We consider the $n \times n$ matrix

$$X = \begin{bmatrix} Y & 0 \\ 0 & I_{n-2} \end{bmatrix}, \tag{2.1}$$

where

$$Y = \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}$$

and ϵ is positive. Let A be any nonnegative matrix with pattern P. We partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

conformally with (2.1). Then

$$XAX^{-1} = \begin{bmatrix} YA_{11}Y^{-1} & YA_{12} \\ A_{21}Y^{-1} & A_{22} \end{bmatrix}.$$

The block

$$A_{21}Y^{-1} = \begin{bmatrix} a_{31} & a_{32} - \epsilon a_{31} \\ \vdots & \vdots \\ a_{n1} & a_{n2} - \epsilon a_{n1} \end{bmatrix}$$

has, for ϵ small enough, the same pattern as A_{21}, since a_{k2} can only be zero if a_{k1} is zero as well. Note that A_{22} does not change, so the row $3,\ldots,n$ keep the same pattern as before the change of variable. The same applies to the second row, since

$$YA_{21}Y^{-1} = \begin{bmatrix} a_{11} + \epsilon a_{21} & a_{12} + \epsilon(a_{22} - a_{11} - \epsilon^2 a_{21}) \\ a_{21} & a_{22} - \epsilon a_{21} \end{bmatrix},$$

$$YA_{12} = \begin{bmatrix} a_{13} + \epsilon a_{23} & \cdots & a_{1n} + \epsilon a_{2n} \\ a_{23} & \cdots & a_{2n} \end{bmatrix}.$$

Finally, the first row can only increase its pattern, and this will only happen in those positions where $a_{1k} = 0, a_{2k} > 0$. In other words, the pattern of the first row becomes $P_1 + P_2$ and the rest remain invariant. ■

Lemma 2.4 *Let $A = [a_{ij}]_{i,j=1}^{n}$ be a nonnegative irreducible matrix with $a_{ij} = 0, i \neq j$, and suppose that $a_{kj} > 0$ for $k \neq i$ ($a_{ik} > 0$ for $k \neq j$).If $a_{ii} < a_{jj}$($a_{ii} > a_{jj}$).Then A is similar to a positive matrix.*

Proof. Suppose again that $i = 1, j = 2$, and satisfies $a_{k2} > 0$ for $k \neq 2$(the proof with $i = 1, j = 2$, and satisfies $a_{1k} > 0$ for $k \neq 1$ is carried out analogously). Hence, A is the form

$$A = \begin{bmatrix} a_{11} & 0 & \cdots \\ & a_{22} & \\ & a_{32} & \\ & \vdots & \\ & a_{n2} & \end{bmatrix}.$$

with $a_{12} = 0, a_{k2} > 0$ for $k \neq 1$, and $a_{11} < a_{22}$. We consider again the change of variables (2.1). One can check, as in the proof of Lemma 2.4, that for ϵ small enough the matrix XAX^{-1} is nonnegative and its pattern strictly dominates the pattern of A. Thus, XAX^{-1} is irreducible, and its second column

$$\begin{bmatrix} \epsilon(a_{22}-a_{11}-\epsilon a_{21}) \\ a_{22}-\epsilon a_{21} \\ a_{32}-\epsilon a_{31} \\ \vdots \\ a_{n2}-\epsilon a_{n1} \end{bmatrix}.$$

is positive. Hence, according to Lemma 2.2, XAX^{-1} is similar to a positive matrix. ■

3 Matrix with exactly $n+2$ zeros

Proposition 3.1 *For $n \geq 6$, let $A = [a_{ij}]_{i,j=1}^{n}$ be an irreducible $n \times n$ nonnegative matrix with exactly $n+2$ zeros and let the diagonal entries be all distinct . If $a_{n1} = a_{n2} = a_{n3} = 0$, then A is similar to a positive matrix.*

Proof. First, suppose that zeros in rows 1, 2, 3 does not appear in the same column. Since there is a column in the first three columns with exactly a zero, we may assume that it is the third column. Then $A^3 \geq A^1, A^3 \geq A^2$. By Lemma 2.3, we know that A is similar to a nonnegative irreducible matrix Q, where $Q_1 = A_1 + A_3, Q_2 = A_2 + A_3$, and $Q_k = A_k, k \neq 1, 2$. Moreover, Q has exactly k zeros, where $k < n+2$. Hence, we apply LLS 1 and LLS 2 to show that Q is similar to a positive matrix, and this implies that A is similar to a positive matrix.

Next, we consider the situation when zeros in rows 1, 2, 3 appear in the same column.

(1) If column n has only a zero, we may assume that $a_{14} = a_{24} = a_{34} = 0$. Then A is permutationally similar to the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a $k \times k$ matrix with the pattern

$$\begin{bmatrix} + & + & + & 0 & + & \cdots & + \\ + & + & + & 0 & + & \cdots & + \\ + & + & + & 0 & + & \cdots & + \\ + & + & + & + & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & + \\ + & + & + & + & \cdots & + & 0 \\ 0 & 0 & 0 & + & \cdots & \cdots & + \end{bmatrix},$$

A_{22} is an $(n-k)\times(n-k)$ matrix with $n-k$ zeros, and A_{12}, A_{21} are positive matrices. Since $a_{(k-1)k}=0$ with $a_{ik}>0, i\neq k-1$ and $a_{(k-1)j}>0, j\neq k$, by Lemma 2.4, we have A_{11} is similar to a positive matrix. By Lemma 2.2, this implies that A is similar to a positive matrix.

(2) Suppose that $a_{1n}=a_{2n}=a_{3n}=0$. Then A is permutationally similar to the form

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a 4×4 matrix with the pattern

$$\begin{bmatrix} + & + & + & 0 \\ + & + & + & 0 \\ + & + & + & 0 \\ 0 & 0 & 0 & + \end{bmatrix}$$

A_{22} is an $(n-4)\times(n-4)$ matrix with $n-4$ zeros, and A_{12}, A_{21} are positive matrices. If $n\geq 8$, then A_{22} is irreducible with nonzero trace. By LLS 1, A_{22} is similar to a positive matrix. Hence, by Lemma 2.2, A is similar to a positive matrix.

For $6\leq n<8$, there is an $a_{ik}=0$, $4<i,k\leq n$ and $i\neq k$, satisfying $a_{ij}>0$, $j\neq k$ and $a_{jk}>0$, $j\neq i$. By Lemma 2.4, A is similar to a positive matrix.

■

Theorem 3.2 *For $n\geq 4$ and $n\neq 5$, let $A=[a_{ij}]_{i,j=1}^{n}$ be an irreducible $n\times n$ nonnegative matrix with exactly $n+2$ zeros. If the diagonal entries are distinct, then A is similar to a positive matrix.*

Proof. By Lemma 2.1, we may assume that A has at least a zero in every row and column. It only need to consider that row n has three zeros or two zeros.

Case 1. Row n has three zeros:

From Proposition 3.1, we only need to consider the case $a_{n1}=a_{n2}=a_{nn}=0$. If there exist $a_{i_1j}=a_{i_2j}=a_{i_3j}=0$ and $a_{jj}>0$. Then we can replace A by A^T and use the arguments from Proposition 3.1 to finish the proof. Next, if zeros in rows 1, 2, n appear in the same column and a zero of them is on the main diagonal. Since the diagonal entries are distinct, we have $a_{1n}=a_{2n}=a_{nn}=0$.Then $A^1\geq A^n$. By Lemma 2.3, A is similar to a nonnegative irreducible matrix Q, where $Q_n=A_1+A_n$ and $Q_k=A_k$ for $k\neq n$. Moreover, Q has exactly k zeros, where $k<n+2$. Thus A is similar to a positive matrix. Finally, We consider the situation when zeros in rows 1, 2, n do not appear in the same column. In the situation, if $n\geq 6$, then there is an $a_{ij}=0, 3\leq j\leq 5$, $i\neq j$ such that

$a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By Lemma 2.4, A is similar to a positive matrix. For $n = 4$, applying Lemma 2.4, we only need to consider the pattern:

$$\begin{bmatrix} + & + & 0 & + \\ + & + & 0 & + \\ + & + & + & 0 \\ 0 & 0 & + & 0 \end{bmatrix}.$$

By Lemma 2.3, we know that it is similar to a nonnegative irreducible matrix with exactly k zeros, where $k < n + 2$. Hence it is similar to a positive matrix.

Case 2. Two zeros in row n are off the main diagonal. We may assume that $a_{n1} = a_{n2} = 0$:

1. There are two additional zeros in the first two columns:
   If $n \geq 6$, then there is an $a_{ij} = 0, 3 \leq j \leq 5$, $i \neq j$ such that $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By Lemma 2.4, A is similar to a positive matrix. For $n = 4$, using Lemma 2.4, we only need to consider the following patterns:

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & + \\ + & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix}, \begin{bmatrix} + & 0 & + & + \\ + & + & 0 & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}, \begin{bmatrix} 0 & + & + & + \\ + & + & 0 & 0 \\ + & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix},$$

$$\begin{bmatrix} + & 0 & + & + \\ 0 & + & + & + \\ + & + & 0 & 0 \\ 0 & 0 & + & + \end{bmatrix}.$$

   Four patterns above, by Lemma 2.3, we know that these are similar to nonnegative irreducible matrices with exactly k zeros, where $k < n + 2$. Hence these are similar to positive matrices.

2. There is an additional zero in the first two columns:
   We may assume that column 1 has two zeros. First, we consider the situation when zeros in rows 1, 2 do not appear in the same column. Since $A^2 \geq A^1$, by Lemma 2.3, A is similar to a nonnegative irreducible matrix Q, where $Q_1 = A_1 + A_2, Q_k = A_k$ for $k \neq 1$. Then Q has exactly m zeros, where $m < n + 2$. Hence, Q is similar to a positive matrix and this implies that A is similar to a positive matrix.

   Next, we consider the situation when zeros in rows 1, 2 appear in the same column. If $n \geq 6$, then there is an $a_{ij} = 0, 3 \leq j \leq 5$, $i \neq j$ such that $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By Lemma 2.4, A is similar to a positive matrix. For $n = 4$, by Lemmas 2.3 and 2.4, we only need to consider the following patterns:

$$A(1) = \begin{bmatrix} + & + & 0 & + \\ + & + & 0 & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}, A(2) = \begin{bmatrix} + & + & 0 & + \\ 0 & + & 0 & + \\ + & + & + & 0 \\ 0 & 0 & + & + \end{bmatrix}, A(3) = \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & 0 \\ + & + & 0 & + \\ 0 & 0 & + & + \end{bmatrix}.$$

All of the above patterns are discussed in Proposition 3.3.

3. The first two columns are both a zero:
First we consider the situation when zeros in rows 1, 2 do not appear in the same column. Since $A^2 \geq A^1$. By Lemma 2.3, A is similar to a nonnegative irreducible matrix Q such that Q has exactly k zeros, where $k < n+2$. Hence A is similar to a positive matrix. Next, suppose that $a_{1r} = a_{2r} = 0$. If $r = n$, from Lemmas 2.2 and 2.4, our analysis reduces to the situation:

$$B = \begin{pmatrix} + & + & 0 & + \\ + & + & 0 & + \\ 0 & 0 & + & + \\ + & + & + & 0 \end{pmatrix}.$$

By LLS 2, B is similar to a positive matrix. If $3 \leq r < n$, we may assume that $r = 3$ and $n \geq 6$. By the hypothesis, there is an $a_{ij} = 0$, $4 \leq j < n$, $3 \leq i < n$, $i \neq j$ such that $a_{kj} > 0$ for $k \neq i$ and $a_{il} > 0$ for $l \neq j$. By Lemma 2.4, A is similar to a positive.
Case 3. Row n has two zeros and a zero in row n is on the main diagonal: We may assume that $a_{n1} = a_{nn} = 0$. Then there exist $a_{i_1 j} = a_{i_2 j} = 0$ for some $1 \leq i_1, i_2 \leq n$ and $i_1, i_2 \neq j$. Replace A by A^T. By Case 2, we obtain the proof. ■

Proposition 3.3 *For $1 \leq i \leq 3$, $A(i)$ is defined as in the proof of Theorem 3.2. Then $A(i)$ is similar to a positive matrix.*

Proof. Let E_{ij} denote a $(0,1)$ matrix with element (i,j) equal to one and all other elements equal to zero. We define

$$F_{ij}(\epsilon) = I - \epsilon E_{ij}.$$

Clearly, $F_{ij}^{-1}(\epsilon) = F_{ij}(-\epsilon)$.
For $i = 1,2,3$, $A(i) = [a_{ij}]_{i,j=1}^4$, if $a_{22} < a_{44}$, $F_{42}(\epsilon)A(i)F_{42}^{-1}(\epsilon)$ is nonnegative and has positive column 2 for all sufficiently small $\epsilon > 0$. If $a_{22} > a_{44}$, $F_{42}(-\epsilon)A(i)F_{42}^{-1}(-\epsilon)$ is nonnegative and has positive column 2 for all sufficiently small $\epsilon > 0$. Thus $A(i)$ is similar to a positive matrix. ■

4 Matrix with exactly n+2 nonzero entries

Suppose that A is an $n \times n$ irreducible nonnegative matrix with $\mathrm{tr(A)} > 0$. In this section, we show that A is similar to a positive matrix when A has exactly $n+1$ or $n+2$ nonzeros entries.

Proposition 4.1 *Let A be an $n \times n$ irreducible nonnegative matrix with exactly $n+2$ positive elements and $\mathrm{tr(A)} > 0$. Suppose that A is permutationally similar to a matrix of the form*

$$\begin{bmatrix} 0 & & & & & & & & & & + \\ + & 0 & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \ddots & \ddots & & & & & & \\ & & & & + & 0 & & & & & \\ & & & & & + & \oplus & & & & \\ & & & & & & + & \oplus & & & \\ & & & & & & & + & 0 & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & \ddots & 0 \\ & & & & & & & & & & + \quad 0 \end{bmatrix},$$

where the positions of positive diagonal entries are $(i-1, i-1)th$ and $(i,i)th$, and we replace $+$ of diagonal entries by $\oplus$. Then A is similar to a positive matrix.

Proof. Obviously, $A^{i-1} \geq A^{i-2}$. By Lemma 2.3, A is similar to a nonnegative irreducible matrix B with pattern

$$B = \begin{bmatrix} 0 & & & & & & & & & & + \\ + & 0 & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \ddots & \ddots & & & & & & \\ & & & & + & + & + & & & & \\ & & & & & + & \oplus & & & & \\ & & & & & & + & \oplus & & & \\ & & & & & & & + & 0 & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & \ddots & 0 \\ & & & & & & & & & & + \quad 0 \end{bmatrix}.$$

Using similar step, we get that A is similar to a nonnegative irreducible matrix C with pattern

$$C = \begin{bmatrix} + & + & + & \cdots & + & + & + & & & & & + \\ + & + & + & \cdots & + & + & + & & & & & \\ & + & + & \cdots & + & + & + & & & & & \\ & & \ddots & \ddots & \vdots & \vdots & \vdots & & & & & \\ & & & + & + & + & + & & & & & \\ & & & & + & + & + & & & & & \\ & & & & & + & \oplus & & & & & \\ & & & & & & + & \oplus & & & & \\ & & & & & & & + & 0 & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \ddots & 0 & 0 \\ & & & & & & & & & & + & 0 \end{bmatrix},$$

by observing this pattern, we have $C^1 \geq C^n$, and then it is similar to a non-negative irreducible matrix P with pattern

$$P = \begin{bmatrix} + & + & + & \cdots & + & + & + & & & & & + \\ + & + & + & \cdots & + & + & + & & & & & \\ & + & + & \cdots & + & + & + & & & & & \\ & & \ddots & \ddots & \vdots & \vdots & \vdots & & & & & \\ & & & + & + & + & + & & & & & \\ & & & & + & + & + & & & & & \\ & & & & & + & \oplus & & & & & \\ & & & & & & + & \oplus & & & & \\ & & & & & & & + & 0 & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & + & 0 & \\ + & + & + & \cdots & + & + & + & 0 & \cdots & & + & + \end{bmatrix}$$

Using similar step, by Lemma 2.3, P is similar to a nonnegative irreducible matrix with pattern

$$\begin{bmatrix} + & + & + & \cdots & + & + & + & & & & + \\ + & + & + & \cdots & + & + & + & & & & \\ & + & + & \cdots & + & + & + & & & & \\ & & \ddots & \ddots & \vdots & \vdots & \vdots & & & & \\ & & & + & + & + & + & & & & \\ & & & & + & + & + & & & & \\ & & & & & + & \oplus & & & & \\ & & & & & & + & \oplus & & & \\ & & & & & & + & + & & & \\ & & & & & & \vdots & & \ddots & & \\ & & & & & & + & & & & \\ + & + & + & \cdots & + & + & + & 0 & \cdots & + & + \end{bmatrix},$$

where it has positive column $i-1$. By Lemma 2.1, it is similar to a positive matrix. ∎

From the proof of Proposition 4.1, the diagonal entry of position (i,i) can be zero. Hence we have the following theorem.

Theorem 4.2 *Let A be an $n \times n$ irreducible nonnegative matrix with exactly $n+1$ positive elements and $\mathrm{tr(A)} > 0$. Then A is similar to a positive matrix.*

Proof. From the hypothesis, A is permutationally similar to a matrix of the form

$$\begin{bmatrix} 0 & & & & & & & a_{n+1} \\ a_1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & a_{i-2} & a_{i-1} & & & & \\ & & & a_i & 0 & & & \\ & & & & a_{i+1} & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & a_n & 0 \end{bmatrix},$$

where $a_1, \cdots, a_n, a_{n+1} > 0$. From the proof of Proposition 4.1, we show that A is similar to a positive matrix. ∎

Proposition 4.3 *Let A be an $n \times n$ irreducible nonnegative matrix with exactly $n+2$ positive elements and $\mathrm{tr(A)} > 0$. If A is permutationally similar to a* *matrix of the form*

$$\begin{bmatrix}
0 & & & & & & & & & & & + \\
+ & 0 & & & & & & & & & & \\
 & \ddots & \ddots & & & & & & & & & \\
 & & + & \oplus & & & & & & & & \\
 & & & + & 0 & & & & & & & \\
 & & & & + & \ddots & & & & & & \\
 & & & & & \ddots & 0 & & & & & \\
 & & & & & & + & \oplus & & & & \\
 & & & & & & & + & 0 & & & \\
 & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & \ddots & 0 & \\
 & & & & & & & & & & + & 0
\end{bmatrix},$$

where the positions of positive diagonal entries are $(i,i)th$ *and* $(i+k,i+k)th$*, and we replace* $+$ *of diagonal entries by* $\oplus$*. Then* A *is similar to a positive matrix.*

Proof. Using the method of Proposition 4.1 and Lemma 2.3, A is similar to the following pattern matrix

$$P \equiv \begin{bmatrix}
+ & + & + & + & 0 & & & & & & & + \\
+ & + & + & + & 0 & & & & & & & \\
 & \ddots & \ddots & \vdots & \vdots & & & & & & & \\
 & & + & \oplus & 0 & & & & & & & \\
 & & & + & 0 & & & & & & & \\
 & & & & + & \ddots & & & & & & \\
 & & & & \vdots & \ddots & 0 & & & & & \\
 & & & 0 & 0 & & + & \oplus & & & & \\
 & & & + & 0 & & & + & 0 & & & \\
 & & & \vdots & \vdots & & & & \ddots & \ddots & & \\
 & & & + & 0 & & & & & \ddots & 0 & \\
+ & + & + & + & 0 & & & & & & + & +
\end{bmatrix}.$$

Since $P_i \geq P_{i+1}$, by Lemma 2.3, P is similar to the following pattern matrix

$$Q \equiv \left[\begin{array}{cccccccccccc}
+ & + & + & + & + & & & & & & & + \\
+ & + & + & + & + & & & & & & & \\
 & \ddots & \ddots & \vdots & \vdots & & & & & & & \\
 & & + & \oplus & + & & & & & & & \\
 & & & + & + & & & & & & & \\
 & & & & + & \ddots & & & & & & \\
 & & & & \vdots & \ddots & 0 & & & & & \\
 & & & & 0 & & + & \oplus & & & & \\
 & & & & + & & & + & 0 & & & \\
 & & & & \vdots & & & & \ddots & \ddots & & \\
 & & & & + & & & & & \ddots & 0 & \\
+ & + & + & + & + & & & & & & + & +
\end{array}\right].$$

By the discussion above, Q is similar to a nonnegative irreducible matrix with pattern

$$\left[\begin{array}{cccccccccccc}
+ & + & + & + & + & & + & & & & & + \\
+ & + & + & + & + & & + & & & & & \\
 & \ddots & \ddots & \vdots & \vdots & & \vdots & & & & & \\
 & & + & \oplus & + & & + & & & & & \\
 & & & + & + & & + & & & & & \\
 & & & & + & \ddots & + & & & & & \\
 & & & & \vdots & \ddots & + & & & & & \\
 & & & & 0 & & + & \oplus & & & & \\
 & & & & + & & + & + & 0 & & & \\
 & & & & \vdots & & \vdots & & \ddots & \ddots & & \\
 & & & & & & + & & & \ddots & 0 & \\
+ & + & + & + & + & & + & & & & + & +
\end{array}\right].$$

It has positive column $i+k-1$ and then A is similar to a positive matrix. ■

Proposition 4.4 *Let A be an $n \times n$ irreducible nonnegative matrix with exactly $n+2$ positive elements and $\mathrm{tr(A)} > 0$. If A satisfies the following properties:*

(*i*) *The diagonal entries have a nonzero element.*

(*ii*) *There is a positive (i,j) position with $i \neq j$, and ith row and jth column have an additional positive element respectively.*

Then A is similar to a positive matrix.

We need the following lemma to prove Proposition 4.4.

Lemma 4.5 *Let $A = [a_{i,j}]_{i,j=1}^{n}$ be an irreducible nonnegative matrix with exactly $n+1$ positive elements and $\operatorname{tr}(\mathrm{A}) = 0$. If A satisfies (ii) of Proposition 4.4. Then A is permutationally similar to a matrix of the form*

$$\begin{bmatrix} 0 & & & & & & & + \\ + & 0 & & & & & & \\ & + & \ddots & & & & & \\ & & \ddots & \ddots & & & & \\ \oplus & & & + & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \\ 0 & & & & & & + & 0 \end{bmatrix}.$$

Proof. Obviously. ■

The proof of Proposition 4.4:
Here we replace the nonzero diagonal entry (k,k) and positive (i,j) position by notation $\oplus$. By Lemma 4.5, we only need to consider the following two cases.
Case 1. $\oplus$ in the position $(i,1)$, where $3 \le i \le k$:

$$A \equiv \begin{bmatrix} 0 & 0 & 0 & & & & & & & & & + \\ + & 0 & 0 & & & & & & & & & \\ 0 & + & 0 & & & & & & & & & \\ & & + & \ddots & & & & & & & & \\ & & & \ddots & \ddots & & & & & & & \\ \oplus & & & & + & \ddots & & & & & & \\ & & & & & \ddots & 0 & & & & & \\ & & & & & & + & \oplus & & & & \\ & & & & & & & + & 0 & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \ddots & 0 & 0 \\ & & & & & & & & & & + & 0 \end{bmatrix}.$$

Since $A^k \ge A^{k-1}$, by Lemma 2.3, A is similar to a nonnegative irreducible matrix A' with $A'_{k-1} = A_k + A_{k-1}$ and $A'_r = A_r$ for $r \ne k-1$. Repeat this step, then A is similar to the following pattern

$$C \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & & & & + \\ + & + & + & \cdots & \cdots & \cdots & + & + & & & & \\ & + & + & \cdots & \cdots & \cdots & + & + & & & & \\ & & + & \ddots & \vdots & \vdots & \vdots & \vdots & & & & \\ & & & \ddots & \ddots & & & & & & & \\ \oplus & & & & + & \ddots & \vdots & \vdots & & & & \\ & & & & & \ddots & + & + & & & & \\ & & & & & & + & \oplus & & & & \\ & & & & & & & + & 0 & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \ddots & 0 & 0 \\ & & & & & & & & & & + & 0 \end{bmatrix}.$$

Since $C_k \geq C_{k+1}$, by Lemma 2.3, C is similar to a nonnegative irreducible matrix C' with $C'^{k+1} = C^k + C^{k+1}$ and $C'^r = C^r$ for $r \neq k+1$. Repeat this step, then C is similar to the following pattern

$$D \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & & & 0 & + \\ + & + & + & \cdots & \cdots & \cdots & + & + & & & + & + \\ & + & + & \cdots & \cdots & \cdots & + & + & & & + & + \\ & & + & \ddots & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ & & & \ddots & \ddots & & & & & & & \\ \oplus & & & & + & \ddots & \vdots & \vdots & & & & \vdots \\ & & & & & \ddots & + & + & & & & + \\ & & & & & & + & \oplus & & & & + \\ & & & & & & & + & + & & & + \\ & & & & & & & & \ddots & \ddots & & \vdots \\ & & & & & & & & & \ddots & + & + \\ & & & & & & & & & & + & + \end{bmatrix}.$$

From Lemma 2.1, D is similar to a positive matrix.

Case 2. $\oplus$ in the position $(i, 1)$, where $i > k$:

$$
A \equiv \begin{bmatrix}
0 & 0 & 0 & & & & & & & & & & + \\
+ & 0 & 0 & & & & & & & & & & \\
 & + & 0 & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & + & 0 & & & & & & & & \\
 & & & & + & \oplus & & & & & & & \\
 & & & & & + & 0 & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & \ddots & \ddots & & & & \\
\oplus & & & & & & & & + & 0 & & & \\
 & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & + & 0
\end{bmatrix}.
$$

Since $A^k \geq A^{k-1}$, by Lemma 2.3, A is similar to a nonnegative irreducible matrix A' with $A'_{k-1} = A_k + A_{k-1}$ and $A'_r = A_r$ for $r \neq k-1$. Repeat this step, A is similar to the following pattern

$$
Q \equiv \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & & & & & & + \\
+ & + & + & \cdots & + & + & 0 & & & & & & \\
 & + & + & \cdots & + & + & 0 & & & & & & \\
 & & \ddots & \ddots & \vdots & \vdots & \vdots & & & & & & \\
 & & & + & + & + & 0 & & & & & & \\
 & & & & + & \oplus & 0 & & & & & & \\
 & & & & & + & 0 & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & \ddots & \ddots & & & & \\
\oplus & & & & & & & & + & 0 & & & \\
 & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & + & 0
\end{bmatrix}.
$$

Since $Q_k \geq Q_{k+1}$, by Lemma 2.3, Q is similar to a nonnegative irreducible matrix Q' with $A'^{k+1} = Q^k + A^{k+1}$ and $Q'^r = Q^r$ for $r \neq k+1$. Repeat this step, Q is similar to the following pattern

$$R \equiv \left[\begin{array}{ccccccccccccc}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & & & + \\
+ & + & + & \cdots & + & + & + & \cdots & + & & & & \\
 & + & + & \cdots & + & + & + & \cdots & + & & & & \\
 & & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
 & & & + & + & + & + & \cdots & + & & & & \\
 & & & & + & \oplus & + & \cdots & + & & & & \\
 & & & & & + & + & \cdots & + & & & & \\
 & & & & & & \ddots & \ddots & \vdots & & & & \\
 & & & & & & & \ddots & + & & & & \\
\oplus & & & & & & & & + & 0 & & & \\
 & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & + & 0
\end{array}\right].$$

Since $R^{i-1} \geq R^1$, by Lemma 2.3, R is similar to a nonnegative irreducible matrix R' with $R'_1 = R_1 + R_{i-1}$ and $R'_r = R_r$ for $r \neq 1$. Repeat this step, R is similar to the following pattern

$$\left[\begin{array}{ccccccccccccc}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & + & & & & + \\
+ & 0 & 0 & \cdots & 0 & 0 & + & \cdots & + & & & & \\
0 & + & + & \cdots & + & + & + & \cdots & + & & & & \\
 & & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
 & & & + & + & + & + & \cdots & + & & & & \\
 & & & & + & \oplus & + & \cdots & + & & & & \\
 & & & & & + & + & \cdots & + & & & & \\
 & & & & & & \ddots & \ddots & \vdots & & & & \\
 & & & & & & & \ddots & + & & & & \\
\oplus & & & & & & & & + & 0 & & & \\
 & & & & & & & & \vdots & \ddots & \ddots & & \\
 & & & & & & & & + & & & 0 & 0 \\
 & & & & & & & & + & & & + & 0
\end{array}\right],$$

which is positive column $i-1$. From Lemma 2.1, A is similar to a positive matrix. ■

Proposition 4.6 *Let A be an $n \times n$ irreducible nonnegative matrix with exactly $n+1$ positive elements and* $\mathrm{tr(A)} = 0$. *If A doesn't satisfy (ii) of Proposition*

4.4. Then A is permutationally similar to a irreducible matrix

$$M(1) = \begin{bmatrix} 0 & + & + & & & & & & 0 \\ & \ddots & 0 & \ddots & & & & & \\ & & \ddots & \ddots & + & & & & \\ + & & & \ddots & 0 & & & & \\ & & & & 0 & + & & & \\ & & & & & \ddots & + & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \ddots & + \\ + & & & & & & & & 0 \end{bmatrix},$$

or

$$M(2) = \begin{bmatrix} 0 & + & + & & & & & & 0 \\ & \ddots & 0 & \ddots & & & & & \\ & & \ddots & \ddots & + & & & & \\ & & & \ddots & + & & & & \\ & & & & 0 & + & & & \\ & & & & & \ddots & + & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \ddots & + \\ + & & & & & & & & 0 \end{bmatrix},$$

or

$$M(3) = \begin{bmatrix} 0 & + & + & & & & & & & 0 \\ & \ddots & 0 & \ddots & & & & & & \\ & & \ddots & \ddots & + & & & & & \\ & & & \ddots & 0 & \ddots & & & & \\ & & & & \ddots & \ddots & + & & & \\ & & & & + & \ddots & 0 & & & \\ & & & & & & 0 & + & & \\ & & & & & & & 0 & \ddots & \\ & & & & & & & & \ddots & + \\ + & & & & & & & & & 0 \end{bmatrix}.$$

Proof. Since $A = [a_{ij}]_{i,j=1}^{n}$ is irreducible with exactly $n+1$ nonzero entries, there exist ith row and jth column such that $a_{ij_1}, a_{ij_2}, a_{i_1j}, a_{i_2j} > 0$. By the hypothesis, columns j_1, j_2 and rows i_1, i_2 are only a zero. Without loss of generality, we may assume that $i = 1, j_1 = 2$, and $j_2 = 3$. Hence if $j = 1$, then A is permutationally similar to $M(1)$. If $j \neq 1$, then A is permutationally similar to $M(2)$ or $M(3)$. ■

Proposition 4.7 *Let $A = [a_{ij}]_{i,j=1}^{n}$ be a nonnegative irreducible matrix with exactly $n+2$ positive elements and* $\mathrm{tr(A)} > 0$. *If A doesn't satisfy (ii) of Proposition 4.4 and the diagonal entries have a nonzero element, $\oplus$. Then A is similar to a positive matrix.*

Proof. By Proposition 4.6, we only need to consider the cases $M(i) + \oplus E_{kk}$, where $1 \leq i \leq 3, 1 \leq k \leq n$, and E_{kk} is defined as in Proposition 3.3.
The case $M(1) + \oplus E_{kk}$:
Suppose that $a_{i1} \neq 0$.
(1) For $k = 1, 2, 3$, by Lemma 2.3 (column), $M(1) + \oplus E_{kk}$ is similar to a nonnegative irreducible matrix with positive column k. Hence $M(1) + \oplus E_{kk}$ is similar to a positive matrix.
(2) For $4 \leq k \leq i+1$, if k is odd, by Lemma 2.3 (column), $M(1) + \oplus E_{kk}$ is similar to a nonnegative irreducible pattern matrix $B(1)$ with the form

$$B(1) = \begin{bmatrix}
0 & + & + & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & \\
 & & & & \ddots & 0 & + & + & & & & & & & \\
 & & & & & 0 & + & + & & & & & & & \\
 & & & & & & 0 & 0 & + & & & & & & \\
 & & & & & & & \oplus & 0 & + & & & & & \\
 & & & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & & 0 & 0 & + & & & \\
+ & & & & & & & & & & 0 & 0 & & & \\
 & & & & & & & & & & & 0 & + & & \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & 0 & + \\
+ & & & & & & & & & & & & & & 0
\end{bmatrix}.$$

$B(1)^k \geq B(1)^{k-1}$. Then $B(1)$ is similar a nonnegative irreducible pattern matrix $B(2)$ with the form

$$B(2) = \left[\begin{array}{ccccccccccccccc}
0 & + & + & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & \\
 & & & & \ddots & 0 & + & + & & & & & & & \\
 & & & & & 0 & + & + & & & & & & & \\
 & & & & & & 0 & + & + & + & & & & & \\
 & & & & & & & \oplus & 0 & + & & & & & \\
 & & & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & & 0 & 0 & + & & & \\
+ & & & & & & & & & & 0 & 0 & & & \\
 & & & & & & & & & & & 0 & + & & \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & 0 & + \\
+ & & & & & & & & & & & & & & 0
\end{array}\right].$$

$B(2)^k \geq B(2)^{k+1}$. Then $B(2)$ is similar to a nonnegative irreducible pattern matrix $B(3)$ with the form

$$B(3) = \left[\begin{array}{cccccccccccccccc}
0 & + & + & & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & & \\
 & & & & \ddots & 0 & + & + & & & & & & & & \\
 & & & & & 0 & + & + & & & & & & & & \\
 & & & & & & 0 & + & + & + & & & & & & \\
 & & & & & & & \oplus & 0 & + & & & & & & \\
 & & & & & & & + & 0 & + & + & & & & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & & & 0 & 0 & + & & & \\
+ & & & & & & & & & & & 0 & 0 & & & \\
 & & & & & & & & & & & & 0 & + & & \\
 & & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & & 0 & + \\
+ & & & & & & & & & & & & & & & 0
\end{array}\right].$$

$B(3)^k \geq B(3)^{k+2}$, and keep on repeating these steps. Then $B(3)$ is similar to a nonnegative irreducible pattern matrix $B(4)$ with the form

$$B(4) = \left[\begin{array}{cccccccccccccccc}
0 & + & + & & & & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & & \\
0 & 0 & 0 & \cdots & + & + & & & & & & & & & & \\
 & & & & \ddots & 0 & + & + & & & & & & & & \\
 & & & & & 0 & + & + & & & & & & & & \\
 & & & & & & 0 & + & + & + & & & & & & \\
 & & & & & & & \oplus & 0 & + & & & & & & \\
 & & & & & & & + & 0 & + & + & & & & & \\
 & & & & & & & \vdots & & \vdots & \ddots & \ddots & & & & \\
 & & & & & & & + & & + & 0 & 0 & + & & & \\
+ & & & & & & & + & & + & & 0 & 0 & & & \\
 & & & & & & & + & & + & & & 0 & + & & \\
 & & & & & & & \vdots & & \vdots & & & & \ddots & \ddots & \\
 & & & & & & & + & & + & & & & & 0 & + \\
+ & & & & & & & + & & + & & & & & & 0
\end{array}\right].$$

$B(4)^k \geq B(4)^1$, and keep on repeating these steps. Then $B(4)$ is similar to a nonnegative irreducible pattern matrix $B(5)$ with the form

$$B(5) = \left[\begin{array}{cccccccccccccccc}
0 & + & + & & & & & + & & + & & & & & & \\
 & & & \ddots & & & & \vdots & & \vdots & & & & & & \\
0 & 0 & 0 & \cdots & + & + & 0 & + & & + & & & & & & \\
0 & 0 & 0 & \cdots & + & + & 0 & + & & + & & & & & & \\
 & & & & \ddots & 0 & + & + & & 0 & & & & & & \\
 & & & & & 0 & + & + & & 0 & & & & & & \\
 & & & & & & 0 & + & + & + & & & & & & \\
 & & & & & & & \oplus & 0 & + & & & & & & \\
 & & & & & & & + & 0 & + & + & & & & & \\
 & & & & & & & \vdots & & \vdots & \ddots & \ddots & & & & \\
 & & & & & & & + & & + & 0 & 0 & + & & & \\
+ & & & & & & & + & & + & & 0 & 0 & & & \\
 & & & & & & & + & & + & & & 0 & + & & \\
 & & & & & & & \vdots & & \vdots & & & & \ddots & \ddots & \\
 & & & & & & & + & & + & & & & & 0 & + \\
+ & & & & & & & + & & + & & & & & & 0
\end{array}\right].$$

$B(5)$ has positive column k and than $B(5)$ is similar to a positive matrix.

Suppose that k is even. its proof will be omitted, since it is similar to the proof above.
(3) For $i+2 \leq k \leq n$, by Lemma 2.3 (column), $M(1) + \oplus E_{kk}$ is similar to a nonnegative irreducible pattern matrix P with the form

$$P = \begin{bmatrix} 0 & + & + & & & & & & & & & & & \\ 0 & 0 & 0 & + & 0 & & & & & & & & & \\ 0 & 0 & 0 & 0 & + & & & & & & & & & \\ & & & \ddots & \ddots & \ddots & & & & & & & & \\ & & & & 0 & 0 & + & & & & & & & \\ + & & & & & 0 & 0 & 0 & & & & & & \\ & & & & & & 0 & + & & & & & & \\ & & & & & & & \ddots & \ddots & & & & & \\ & & & & & & & & \ddots & \ddots & & & & \\ & & & & & & & & & 0 & + & & & \\ & & & & & & & & & & \oplus & + & & \\ & & & & & & & & & & \vdots & \ddots & \ddots & \\ & & & & & & & & & & + & + & + & + \\ + & & & & & & & & & & + & + & + & + \end{bmatrix}.$$

Since $P_n \geq P_{k-1}$, then P is similar to a nonnegative irreducible pattern matrix P', where $P'^{k-1} = P^n + P^{k-1}, P'^l = P^l$ for $l \neq k-1$. Repeat these steps, it is similar to a nonnegative irreducible pattern matrix with the form

$$\begin{bmatrix} 0 & + & + & & & & & & & & & & & \\ 0 & 0 & 0 & + & 0 & & & & & & & & & \\ 0 & 0 & 0 & 0 & + & & & & & & & & & \\ & & & \ddots & \ddots & \ddots & & & & & & & & \\ & & & & 0 & 0 & + & & & & & & & \\ + & & & & & 0 & 0 & 0 & & & & & & \\ & & & & & & 0 & + & & & & & & \\ & & & & & & & \ddots & \ddots & & & & & \\ & & & & & & & & \ddots & \ddots & & & & \\ & & & & & & & & & 0 & + & & & \\ & & & & & & & & & & \oplus & + & & \\ & & & & & & & & & & \vdots & \ddots & \ddots & \\ + & \cdots & & & & & & & \cdots & + & + & + & + & + \\ + & \cdots & & & & & & & \cdots & + & + & + & + & + \end{bmatrix}.$$

It has positive row n. Hence $M(1) + \oplus E_{kk}$ is similar to a positive matrix.
The cases $M(2) + \oplus E_{kk}$ and $M(3) + \oplus E_{kk}$:
Its proof will be omitted, since it is similar to the proof above. ■

From Propositions 4.1, 4.3, 4.4, and 4.7, we obtain the following theorem.

Theorem 4.8 *Let A be an $n \times n$ nonnegative irreducible matrix with exactly $n+2$ positive elements and* $\mathrm{tr}(\mathrm{A}) > 0$*. Then A is similar to a positive matrix.*

References

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[3] T. J. Laffey, Extreme nonnegative matrices, *Linear Algebra Appl.*, 275/276 (1998), *pp.* 349–357.

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